710a Metrics Notes

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Stats

Geometric series sum

$\sum_{j=0}^{n} r^j = \frac{1-r^{n+1}}{1-r}$

Conditional expectation

$E[Y] = \sum_l E[Y|Z=l]Pr(Z=l)$

Block inversion formula

M is invertible iff $A - BD^{-1}C$ is invertible, and $M^{-1} =$

$$\begin{bmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1}+D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1} \end{bmatrix}$$

Sherman-Morrison formula

$A + uv'$ is invertible iff $1 + v'A^{-1}u \neq 0$, and

$$(A+uv')^{-1} = A^{-1} - \frac{A^{-1}uv'A^{-1}}{1+v'A^{-1}u}$$

Partial and average partial effect

- Partial effect (at $X = x$): $\frac{\partial}{\partial x} E[Y|X=x]$
- Average partial effect: $E[\frac{\partial}{\partial X} E[Y|X]]$

Inequalities

- Chebyshev's: $Pr(|X - E[X]| \geq \epsilon) \leq \frac{Var(X)}{\epsilon^2}$ for all $\epsilon > 0$
- Markov: $Pr(X \geq \epsilon) \leq \frac{E[X]}{\epsilon}$ for all $\epsilon > 0$
- Jensen's: For convex function $g(\cdot)$, $g(E[X]) \leq E[g(X)]$
- Cauchy Schwarz: $E[XY]^2 \leq E[X^2]E[Y^2]$

Martingale CLT

If $\{Z_t\}_{t=1}^T$ satisfy

- $\{Z_t\}_{t=1}^T$ is strictly stationary
- $E[Z_1^2] < \infty$
- $E[Z_t|Z_{t-1}, Z_{t-2}, \ldots, Z_1] = 0$
- $\frac{1}{T}\sum_{t=1}^T Z_t^2 \xrightarrow{p} E[Z_1^2]$

then $\frac{1}{\sqrt{T}} \sum_{t=1}^T Z_t \xrightarrow{d} N(0, E[Z_1^2])$ as $T \to \infty$

IV

Model: $Y_i = X_i'\beta + U_i$

Assumptions:

- Validity of instrument
  - Exogeneity: $E[U|Z] = 0$
  - Relevance: $E[ZX']$ and $\sum_{i=1}^n Z_i X_i'$ are invertible
- $E[|Y|^2 + ||X||^2 + ||Z||^2] < \infty$
- $\{(Y_i, X_i', Z_i')\}$ are i.i.d.

IV estimator: $\hat{\beta}_{IV} = (\sum_{i=1}^n Z_i X_i')^{-1}(\sum_{i=1}^n Z_i Y_i')$

Small sample property

Model: $Y_i = \beta_0 + X_i\beta_1 + U_i$

$X_i = \pi_0 + Z_i\pi_1 + V_i$

$$\begin{pmatrix} U_i \\ V_i \end{pmatrix} \Bigg| \; Z_i \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{pmatrix}\right)$$

$\Rightarrow E[\hat{\beta}_{IV}|\mathbb{X}, \mathbb{Z}] = \beta + \frac{\sigma_{uv}}{\sigma_v^2} \frac{\nu}{\frac{\pi_1}{se(\hat{\pi}_1)} + \nu}$, where $\nu \sim N(0,1)$

Large sample property

Large sample approximation is reasonable depends on both

- sample size n (the bigger, the better), and
- the covariance between the instrument and endogenous regressor π_1 (the bigger, the better)

Which means there's a need of testing whether the instrument is relevant (whether $\pi_1 \neq 0$)

Model: $X_1 = Z_1\pi_1 + X_2'\pi_2 + V$. Let $\hat{\pi}_1$ be OLS estimate of π_1.
Then the F-statistic is $F = \frac{(\hat{\pi}_1 - 0)^2}{se(\hat{\pi}_1)^2} = \frac{\hat{\pi}_1^2}{se(\hat{\pi}_1)^2}$
Empirically, if $F \geq 10$, then some rough insurance with nominal 95% confidence intervals have actual coverage of at least 80%.

Weak instrument

Model: $Y = X_1\beta_1 + U$. Weak instrument is Z_1.

Assumptions on Z_1:

- Exogeneity: $E[U|Z_1] = 0$
- Allow for relevance
- $E[Z_1^2]$ and $\frac{1}{n}\sum_i Z_{1i}^2$ are both non-zero (i.e. invertible)

Under weak instrument, want to test

$$H_0: \beta_1 = c \qquad H_1: \beta_1 \neq c$$

Identification under H_0: $E[Z_1(Y - X_1c)] = 0$

$\Rightarrow T = \frac{1}{n}\sum_i Z_{1i}(Y_i - X_{1i}c)$. Reject H_0 when $|T|$ is large

Anderson-Rubin Test:

$AR := \frac{\sqrt{n}T}{S} \xrightarrow{d} N(0,1)$
where $S^2 = \frac{1}{n} Z_{1i}^2 (Y_i - X_{1i}c)^2 \xrightarrow{p} E[Z_1^2 U^2]$
With a size of 5%, reject H_0 when $|AR| > 1.96$
Confidence set: $\{c \in \mathbb{R} : |AR(c)| \leq 1.96\}$

Optimal instrument

Say optimal instrument is $h^*(Z)$. Identification yields from $E[h(Z)U] = 0$. So $\hat{\beta}_{IV}^h = (\frac{1}{n}\sum_i h(Z_i)X_i')^{-1} \frac{1}{n}\sum_i h(Z_i)Y_i$

Under homoskedasticity, asymptotic variance (AVAR) is

$$\Omega^h = \frac{E[h(Z)^2]}{E[h(Z)X]^2}\sigma_u^2 = \frac{E[h(Z)^2]}{E[h(Z)E[X|Z]]^2}\sigma_u^2$$
$$\geq \frac{E[h(Z)^2]}{E[h(Z)]^2 E[E[X|Z]^2]}\sigma_u^2 = \frac{\sigma_u^2}{E[E[X|Z]^2]}$$

To achieve the lower bound of AVAR, $h^*(Z) = E[X|Z]$

Random coefficient model

Model: $Y = X'\beta_0 \cdot U = X'\beta_0 + X'\beta_0 \cdot (U-1)$

Assumptions:

- $E[U|Z] = 1$, and $E[ZX']$ invertible
- Z independent of Y
- Let $X = (1, X)'$, $Z = (1, Z)'$, $Z \in \{0,1\}$, $X \in \{0,1\}$

$$\hat{\beta}_{IV} \xrightarrow{p} \frac{Cov(Z,Y)}{Cov(Z,X)} = \frac{E[Y|Z=1] - E[Y|Z=0]}{E[X|Z=1] - E[X|Z=0]}$$

For binary X, its response to Z given unobservable U is $X_U(Z)$

	$X_U(0) = 0$	$X_U(0) = 1$
$X_U(1) = 0$	Never taking	Defying
$X_U(1) = 1$	Complying	Always taking

Assume no defyers, and $Pr(\text{Complying}) > 0$, then

$$\begin{aligned} &E[X|Z=1] - E[X|Z=0] = E[X_U(1) - X_U(0)] \\ &= Pr(X_U(1) - X_U(0) = 1) - Pr(X_U(1) - X_U(0) = -1) \\ &= Pr(\text{Complying}) \\ &E[Y|Z=1] - E[Y|Z=0] = E[Y_U(1)X_U(1) + Y_U(0)(1 - X_U(1))] \\ &\quad - E[Y_U(1)X_U(0) + Y_U(0)(1 - X_U(0))] \\ &\qquad\qquad (\text{since } Y = Y_U(1)X + Y_U(0)(1-X)) \\ &= E[(Y_U(1) - Y_U(0))(X_U(1) - X_U(0))] \\ &= E[(Y_U(1) - Y_U(0))|X_U(1) - X_U(0) = 1] \times Pr(\text{Complying}) \end{aligned}$$

Thus, $\hat{\beta}_{IV} \xrightarrow{p} E[\underbrace{(Y_U(1) - Y_U(0))}_{\text{mean response of } Y \text{ to } X} \,|\, \underbrace{X_U(1) - X_U(0) = 1}_{\text{for complyers}}]$

Time Series

TS Models

- Static: $Y_t = \alpha_0 + X_t'\delta_0 + U_t$
- FDL(s): $Y_t = \alpha_0 + X_t'\delta_0 + \ldots + X_{t-s}'\delta_s + U_t$
- AR(p): $Y_t = \alpha_0 + Y_{t-1}\rho_1 + \ldots + Y_{t-p}\rho_p + U_t$
- MA(q): $Y_t = \varepsilon_t + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q}$
- Trend:
  - Linear time trend: $Y_t = \beta_0 + \beta_1 t + U_t$
  - Exponential trend: $\log(Y_t) = \beta_0 + \beta_1 t + U_t$
- Seasonality: $Y_t = \alpha_0 + \alpha_1 1_{\{t/12 \text{ is an integer}\}} + U_t$

Stationarity

- Strict: $(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_k}) \sim (Y_{t_1+l}, Y_{t_2+l}, \ldots, Y_{t_k+l})$
  (Joint distribution is t independent)
- Weak (Covariance): For all t, $E[Y_t]$ and $\gamma(k) = Cov(Y_t, Y_{t+k})$ ($\leftarrow$ autocovariance function) are both independent of t