

703 Problem Set 4

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26. To prove that $f : X \rightarrow \mathbb{R}$ is continuous, we want to show that for any $x, y \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ if $d(x, y) < \delta$.

Since d is a metric, by triangle inequality,

$$\begin{aligned}d(a, x) &\leq d(a, y) + d(x, y) \\d(a, x) - d(a, y) &\leq d(x, y)\end{aligned}$$

Similarly,

$$\begin{aligned}d(a, y) &\leq d(a, x) + d(x, y) \\d(a, y) - d(a, x) &\leq d(x, y)\end{aligned}$$

Since $d(a, x) - d(a, y) \leq d(x, y)$ and $d(a, y) - d(a, x) \leq d(x, y)$ both hold, we have

$$|d(a, x) - d(a, y)| \leq d(x, y)$$

Since $f(x) = d(a, x)$, the above inequality is equivalent to $|f(x) - f(y)| \leq d(x, y)$.

Thus, for $d(x, y) < \delta$, set $\varepsilon = \delta$, then $\forall \varepsilon > 0$, $d(x, y) < \delta$ implies $|f(x) - f(y)| < \varepsilon$, proving that $f(x)$ is continuous.