Random CMB Stuff

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References: arXiv:0802.3688

The C_ℓ's can be defined as:

$$C_\ell = 4\pi \int \mathrm{d}\ln k \, j_\ell^2(kD_*) \, \Delta_T^2(k) \,. \tag{1}$$

The *dimensionless* power spectrum Δ^2 is defined from the power spectrum $P(k)$ as:

$$\Delta_T^2(k) = \frac{k^3}{2\pi^2} P(k) \,. \tag{2}$$

The power spectrum is *defined* from the two-point function as:

$$\langle \Theta(\mathbf{k})^* \, \Theta(\mathbf{k}') \rangle = (2\pi)^3 \, \delta\left(\mathbf{k} - \mathbf{k}'\right) P(k) \,. \tag{3}$$

Let's write

$$\Theta(\eta, \mathbf{k}) = \Theta(0, \mathbf{k}) \tilde{\Theta}(\eta, \mathbf{k}) \,, \tag{4}$$

such that $\tilde{\Theta}(0, \mathbf{k}) = 1$. Then we have:

$$\langle \Theta(\mathbf{k})^* \, \Theta(\mathbf{k}') \rangle = \left\langle \Theta(0, \mathbf{k})^* \tilde{\Theta}(\eta, \mathbf{k})^* \Theta(0, \mathbf{k}') \tilde{\Theta}(\eta, \mathbf{k}') \right\rangle \,. \tag{5}$$

The $\tilde{\Theta}$ are deterministic and independent of the initial conditions, so we can remove these from the correlator, which is an ensemble average over realisations of $\Theta(\mathbf{k})$.

$$\langle \Theta(\mathbf{k})^* \, \Theta(\mathbf{k}') \rangle = \langle \Theta(0, \mathbf{k})^* \Theta(0, \mathbf{k}') \rangle \, \tilde{\Theta}(\eta, \mathbf{k})^* \tilde{\Theta}(\eta, \mathbf{k}') \,. \tag{6}$$

The initial temperature fluctuations can be related to the initial curvature perturbations ζ as

$$\Theta(0, \mathbf{k}) = -\frac{1}{5} \zeta(\mathbf{k}) \,. \tag{7}$$

$$\langle \Theta(\mathbf{k})^* \, \Theta(\mathbf{k}') \rangle = \langle \Theta(0, \mathbf{k})^* \Theta(0, \mathbf{k}') \rangle \, \tilde{\Theta}(\eta, \mathbf{k})^* \tilde{\Theta}(\eta, \mathbf{k}') \tag{8}$$

$$= \frac{1}{25} \langle \zeta(\mathbf{k}) \zeta(\mathbf{k}')^* \rangle \, \tilde{\Theta}(\eta, \mathbf{k})^* \tilde{\Theta}(\eta, \mathbf{k}') \tag{9}$$

The power spectrum of initial curvature perturbations is:

$$\langle \zeta(\mathbf{k}) \zeta(\mathbf{k}')^* \rangle = (2\pi)^3 \frac{2\pi^2}{k^3} \Delta_\zeta^2(k) \delta\left(\mathbf{k} - \mathbf{k}'\right) \,. \tag{10}$$

This gives:

$$\langle \Theta(\mathbf{k})^* \, \Theta(\mathbf{k}') \rangle = \frac{(2\pi)^3}{25} \frac{2\pi^2}{k^3} \Delta_\zeta^2(k) \delta \left(\mathbf{k} - \mathbf{k}' \right) \tilde{\Theta}(\eta, \mathbf{k})^* \tilde{\Theta}(\eta, \mathbf{k}') \, . \tag{11}$$

Comparison with the definition of $P(k)$ for the temperature fluctuations gives:

$$P(k) = \frac{\Delta_\zeta^2(k)}{25} \frac{(2\pi)^3}{4\pi k^3} \left| \tilde{\Theta}(\eta, \mathbf{k}) \right|^2 \, . \tag{12}$$

This gives a dimensionless power spectrum which is:

$$\Delta_T^2 = \frac{\Delta_\zeta^2(k)}{25} \left| \tilde{\Theta}(\eta, \mathbf{k}) \right|^2 \, . \tag{13}$$

We then just evaluate at recombination $\eta = \eta_*$.

Let's define the transfer function which takes us from initial temperature fluctuations to observed temperature fluctuations:

$$T(k) = \left| \tilde{\Theta}(\eta_*, \mathbf{k}) \right|^2 \, . \tag{14}$$

Thus, the CMB temperature anisotropy power spectrum is:

$$C_\ell = \frac{4\pi}{25} \int \mathrm{d} \ln k \, j_\ell^2(k D_*) \, \Delta_\zeta^2(k) \, T(k) \, . \tag{15}$$

The final ingredient is the spectrum of initial curvature perturbations:

$$\Delta_\zeta^2(k) = A_s \left(\frac{k}{k_*} \right)^{n_s - 1} \, , \tag{16}$$

where $A_s \approx 2.196 \times 10^{-9}$, $k_* = 0.05 \, \mathrm{Mpc}^{-1}$ and $n_s \approx 0.9603$.