II. NFW PROFILES

The NFW density profile can be written as:

$$\rho_{\rm NFW}(R) = \begin{cases} \frac{\rho_s}{\frac{R}{R_s}(1+\frac{R}{R_s})^2} & \text{for } R < R_{\rm max}\,, \\ 0 & \text{for } R > R_{\rm max}\,, \end{cases} \tag{21}$$

where we have included an explicit truncation at $R = R_{\rm max}$.

The concentration of the halo is defined as $c = R_{\rm max}/R_s$. Writing $x = R/R_{\rm max}$, then the dimensionless NFW density profile $\varrho = \rho/\rho_s$ can be written as:

$$\varrho_{\rm NFW}(x) = \begin{cases} \frac{1}{c\,x\,(1+c\,x)^2} & \text{for } x < 1\,, \\ 0 & \text{for } x > 1\,. \end{cases} \tag{22}$$

The enclosed mass for $x < 1$ can be calculated as:

$$\begin{aligned} \mathcal{M}_{\rm enc}(x) &= 4\pi \int_0^x x^2 \varrho_{\rm NFW}(x)\,{\rm d}x && (23)\\ &= \frac{4\pi}{c^3}\left[\log(cx+1) + \frac{1}{cx+1} - 1\right] && (24)\\ &= 4\pi \int_0^x x^2 \varrho_{\rm NFW}(x)\,{\rm d}x && (25)\\ &= \frac{4\pi}{c^3}\left[\log(cx+1) - \frac{cx}{cx+1}\right]\,. && (26) \end{aligned}$$

The total enclosed mass for $x \geq 1$ is then:

$$\mathcal{M}_{\rm enc}(x \geq 1) = \frac{4\pi}{c^3}\left[\log(c+1) - \frac{c}{c+1}\right] \equiv \mathcal{M}_{\rm tot}\,. \tag{27}$$

We'll also define $f_{\rm NFW}(c) = \log(1+c) - c/(1+c)$, such that $\mathcal{M}_{\rm tot} = 4\pi f_{\rm NFW}(c)/c^3$.

We define the potential as $\psi = -\Psi/\Psi_0$, where $\Psi_0 = GM_{\rm tot}/R_{\rm max}$. This can be written as:

$$\psi_{\rm NFW}(x) = -\frac{1}{\mathcal{M}_{\rm tot}} \int_\infty^x \frac{\mathcal{M}_{\rm enc}(x)}{x^2}\,{\rm d}x\,. \tag{28}$$

For $x \geq 1$, the NFW profile just looks like a point mass with $\mathcal{M}_{\rm enc} = \mathcal{M}_{\rm tot}$, such that:

$$\psi_{\rm NFW}(x \geq 1) = 1/x\,. \tag{29}$$

For $x < 1$, we have:

$$\psi_{\rm NFW}(x) = 1 - \frac{1}{\mathcal{M}_{\rm tot}} \int_1^x \frac{\mathcal{M}_{\rm enc}(x)}{x^2}\,{\rm d}x\,, \tag{30}$$

where the first term is the contribution for moving a particle from infinity to $x = 1$ (in the gravitational field of a point-like mass) and the second term is for moving a particle from $x = 1$ to x through the NFW potential. Explicitly computing the second term, we obtain:

$$\psi_{\rm NFW}(x) = 1 + \frac{1}{f_{\rm NFW}(c)}\left[\frac{\log(cx+1)}{x} - \log(c+1)\right]\,. \tag{31}$$