Prelude to a Well-Integrable Function Theory

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Abstract

Quite often in physics we encounter a question about nature, which can only be answered by taking an integral. A formalism for writing such integrals does not guarantee quality answers nor appreciable progress. Difficulties abound, especially when working with function-valued integrals, whose integrands involve one or more auxiliary parameters. Yet such parameters allow differentiation under the integral sign, so can be turned into an advantage. In many cases, a difficult-looking integral function is also the solution to a relatively simple ordinary differential equation. Playing through a few fundamental problems about ellipses and elliptic curves, we begin to hear intertwined themes from physics and mathematics. These themes will recur in more substantial followup works.

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I. HISTORY AND INTRODUCTION

Lest we look all the way back to the geometric works of antiquity (circa 200-300BC), it seems unlikely that we could find a better starting place than the musical works of Johannes Kepler (1571-1630). Kepler advanced the heliocentric theory by refining it to maximum-available precision. He did not do so by over-specializing in data analysis, rather by accomplishing superlative mastery of the *quadrivium*—a generalist curriculum of medieval Europe, one that placed arithmetic, geometry, astronomy, and music on even footing. As continental Europe transitioned into the brutal thirty-years war (1618-1648), Kepler published his brilliant assay in two parts, first in *Astronomia Nova* (1609) and subsequently in *Harmonices Mundi* (1619). Despite hundreds of years elapsed, Kepler's three laws are still remembered today[1]:

- **I.** The orbit of a planet is an ellipse with the Sun at one of the two foci.
- **II.** A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
- **III.** The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

Kepler's original "proof" of the three laws relied upon a beautiful but doubtful musical analogy. More development was both desirable and necessary, so Kepler's laws gave way to the *Kepler Problem*. It asks for a derivation of the three laws from a more fundamental physical theory, and subsequently for an adherent solution of the time-variant planetary equations of motion. This task can not be accomplished by harmony alone. It requires another paradigm change, which first came about during the European enlightenment[2].

With the publication of *Principia Mathematica* (1687), Isaac Newton (1642-1727) made a significant contribution toward the initiation of European Enlightenment. In this effort to defeat the specter of irrational religiosity, Newton's work was like a clarion, calling all subsequent generations to the front lines of scientific research. Newton's three laws are also remembered to this day:

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[1] These are quoted verbatim from Wikipedia, see: "Kepler's Laws of planetary motion".

[2] Do not confuse European and Asian enlightenment! At the very least, they happened at different times, in different geographical regions. (When considering whole and indivisible spacetime, confuse freely away!)

I. Absent of an external force, an object in motion stays in motion, while an object at rest stays at rest.

II. A net force $\mathbf{F}$ applied to a massive object causes an acceleration $\mathbf{a}$. The two dynamical variables are linearly proportional by the mass m, i.e. $\mathbf{F} = m\mathbf{a}$.

III. For every force from one body to another, there is an equal and opposite response force from the later body to the former (It is often written, $\mathbf{F}_{21} = -\mathbf{F}_{12}$).

To these three, Newton also gave an important addendum regarding the particular case of gravitating bodies, the *Universal Law of Gravitation*,

G. The attractive force between two point masses is directly proportional to the product of masses, and is inversely proportional to the square of the distance between them.

If m_1 and m_2 are the masses, and $\mathbf{r}$ the distance vector, the gravitational force vector $\mathbf{F}$ is usually written $\mathbf{F} = G\frac{m_1 m_2}{\mathbf{r}\cdot\mathbf{r}}\hat{r}$, with gravitational constant G. The adjective "universal" indicates that law **G** applies to the orbits of planets, to the orbit of the moon, to the tides between the moon and the oceans, as well to the oscillation of various types of mechanical pendulums. In fact, universal law **G** applies to any pair of gravitating bodies, anywhere in the universe[3]. Accepting **I**, **II**, **III**, and **G** as all valid and applicable, Kepler's laws can be proven mathematically using only the geometrical techniques of Newton's day and age. Richard Feynman (1918-1988) took this as a challenge when he gave a lecture on planetary motion, March 13, 1964. The lecture stands on its own as an active and imaginative contribution to the history of science, and it is quite different from anything that we would readily recognize as a typical solution to the Kepler problem.

Famously, Newton wrote "if I have seen further it is by standing on the shoulders of Giants." In so doing he became a part of the gigantic scientific enterprise, as did his follower Leonhard Euler (1707-1783). Perhaps no one worked more than Euler to raise this giant into its present-day stance. Encyclopedic works typically credit Euler for originating (or at least co-originating) the first abstract definition of what a function is, and for giving the first

[3] We are not disregarding Einstein's theory of general relativity, so must also say that Newton's universal laws are not exactly universal. In some parts of the universe, they completely fail, e.g. black holes.

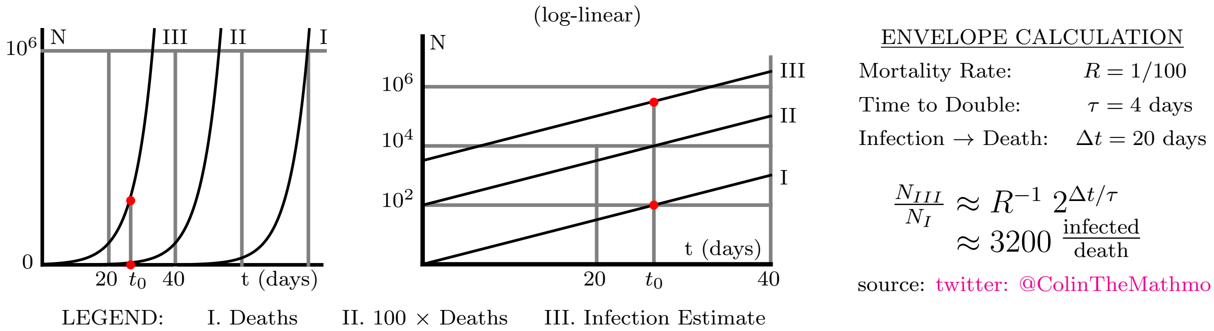

FIG. 1. Onset of a pandemic: exponential curves $N(t) = N_0 e^{\log(2)\frac{t}{\tau}}$ plotted over time t.

important examples[4]. Most noteably, the functions e^x, $\cos(x)$, and $\sin(x)$, were written by Euler in series expansion,

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 4}x^4 + \frac{1}{2\cdot 3\cdot 4\cdot 5}x^5 + etc.$$
$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{2\cdot 3\cdot 4}x^4 - \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6}x^6 + etc.$$
$$\sin(x) = x - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 4\cdot 5}x^5 - \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + etc.$$

with "$+etc.$" indicating continuation of the numerical pattern to infinity. From this definition it is straightforward to infer all the following annihilating relations,

$$\partial_x e^x - e^x = 0, \qquad \partial_x^2 \cos(x) + \cos(x) = 0, \qquad \text{and} \qquad \partial_x^2 \sin(x) + \sin(x) = 0.$$

The important composition identity $e^x = \cos(ix) - i\sin(ix)$, where $i^2 = -1$, also follows, as does the beautiful and profitable *Euler's identity* that $e^{i\pi} = -1$. It is apparent from his collected works that Euler understood the practical value of transcendental functions, and intended for subsequent generations to use these tools to continue solving new and interesting problems. The three functions e^x, $\sin(x)$ and $\cos(x)$ are among the best specialized tools a scientist ever receives. When used together with statistical analysis, these tools are often enough to predicate an entire career, even in practical disciplines or the so-called "real world". The infographic Fig. 1, gives one example related to the COVID-19 pandemic of 2020. Meanwhile, sine and cosine contribute an essential part to subsequent analyses.

[4] Be careful if studying Wikipedia. The article "History of the function concept" *at least* needs a section on pre-history, starting with compass and straightedge, the functional implements of antiquity. Also, don't forget to read primary source documents. Hundreds of Euler's works are available online through the Euler archive[cite], see E101 Ch. 7-8 for early definitions of e^x, $\cos(x)$, and $\sin(x)$ and more.

Euler was also interested in calculus as a theory, regardless of the material or the mundane. He thought abstractly, made numerical analogies, and ventured into lesser known realms of mathematics to find and analyze other important functions. The Euler archive records early series definitions for elliptic integrals,

$$\frac{2}{\pi}E(x) = 1 - \frac{1}{2^2}x - \frac{1^2\cdot 3}{2^2\cdot 4^2}x^2 - \frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6^2}x^3 - etc.,$$
$$\frac{2}{\pi}K(x) = 1 + \frac{1^2}{2^2}x + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}x^2 + \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}x^3 + etc.,$$

under entry numbers E028 and E503 respectively[5]. These basic examples eventually led Euler to an early discovery of the general hypergeometric series, in his notation,

$$s = 1 + \frac{ab}{1\cdot c}x + \prod\frac{(a+1)(b+1)}{2\cdot(c+1)}x^2 + \prod\frac{(a+2)(b+2)}{2\cdot(c+2)}x^3 + etc.,$$

where recursive symbol $\prod$ stands for multiplication by the previous series coefficient. This equation appears verbatim in *Specimen transformationis singularis serierum*, archive entry E710, alongside its defining differential equation "$0 = x(1-x)\partial\partial s + [c-(a+b+1)x]\partial s - abs$". We no longer use Euler's notation or ordering, and instead write an annihilating operator,

$$\mathcal{A}_F = z(1-z)\partial_z^2 + \big(c-(a+b+1)z\big)\partial_z - ab$$
$$\text{such that}\quad \mathcal{A}_F\circ F = z(1-z)\partial_z^2 F + \big(c-(a+b+1)z\big)\partial_z F - abF = 0,$$

which constrains all possible solutions. The putative simplest series solution,

$${}_2F_1\left[\begin{matrix}a,b\\c\end{matrix}\middle|z\right] = \sum_{n\geq 0} f_n z^n \quad\text{with}\quad f_0 = 1 \quad\text{and}\quad (n+1)(n+c)f_{n+1} = (n+a)(n+b)f_n,$$

introduces a concise notation where, for example, elliptic integrals are easy to define,

$$\frac{2}{\pi}E(z) = {}_2F_1\left[\begin{matrix}-\frac{1}{2},\frac{1}{2}\\1\end{matrix}\middle|z\right] \qquad\text{and}\qquad \frac{2}{\pi}K(z) = {}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix}\middle|z\right].$$

However nice it may be to get rid of Euler's "*etc.*", the simple hypergeometric solution is not a unique or final definition. Sections IV and V of this work will explore alternative definitions of E and K, targeted toward precise and efficient calculation.

Reversing the order of presentation, we mean to portray the hypergeometric differential equation as more fundamental than any one particular solution[6]. This reversal raises a

[5] The notation here is similar, not identical, to notation used originally by Euler. Standard usage of letters K and E is a more recent development attributed to A.M. Legendre (1752-1833).

[6] In fact, the second-order H.D.E. must have a solution-space with two degrees of freedom.

question about procedure: if ${}_2F_1$ is to follow from $\mathcal{A}_F$, what shall precede $\mathcal{A}_F$? For special values (a, b, c) it is possible that $\mathcal{A}_F$ has a natural geometric origin. This is the case for functions E and K, which may also be written as,

$$E(z) = \int_0^{\pi/2} \sqrt{1 - z\sin(\phi)^2}\, d\phi \qquad \text{and} \qquad K(z) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - z\sin(\phi)^2}}\, d\phi.$$

Euler already knew how to derive these integral forms (or similar) from geometry and/or Newtonian physics, and he took them as a fundamental starting place. However, Euler did not have a rigorous procedure for analyzing partial derivatives of the integrands, so he could not derive the corresponding cases of $\mathcal{A}_F$ without resorting to series expansion methods.

In present times, the related fields of Creative Telescoping and Holonomic Functions can add rigor where it may be missing[7]. Algorithms from these theories help to analyze the sort of integrals typified by elliptic E and K. For simplicity sake, let us take the one-dimensional case, where $I(\alpha) = \oint_{\mathcal{X}(\alpha)} \frac{dI}{dt} dt$ over domain $\mathcal{X}(\alpha)$, an algebraic plane curve, also a Jordan curve[8]. Integral $I(\alpha)$ is sensitive to how the shape of curve $\mathcal{X}(\alpha)$ depends on the auxiliary parameter α. If $\mathcal{X}(\alpha)$ and dI/dt are both sufficiently simple, then there will exist an annihilating operator $\mathcal{A}_I \in \mathbb{Q}[\![\alpha, \partial_\alpha]\!]$ (also called a "telescoper"), which satisfies $\mathcal{A}_I \circ I(\alpha) = 0$ because $\mathcal{A}_I \circ \frac{dI}{dt} = \frac{d}{dt}(\Xi_I^t)$. An annihilator $\mathcal{A}_I$ and its certificate Ξ_I^t can sometimes be calculated concurrently using only a combination of partial-fraction decomposition and the Ostrogradsky-Hermite reduction. This is the case for elliptic E and K, as well for many other geometries to appear in our sustained research effort.

An expression such as $\mathcal{A}_I \circ I(\alpha) = 0$ tells us that $I(\alpha)$ is the solution of an ordinary differential equation. How should we understand certificates such as Ξ_I^t? Can we calculate certificates, and should we? If so, how? These are motivating questions for the present work. Using an empirical, example-driven style, we will go from Kepler's laws and Newton's laws in sections II and III, to ellipses, elliptic curves, and elliptic integrals in sections IV and V, while stopping only briefly to solve a few problems in section VI. Finally in section VII, we take a closer look at certificate geometry. As with any prelude, the progression from start to start is a right of passage, a test of technical skill, and ultimately only a hint of what is to come next. Rather than concluding entirely, section VIII gives the prospectus to a dissertation where physical and mathematical themes will cipher 'round again.

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[7] For broad summaries, see xxx.

[8] See also Mathworld: Algebraic Curve, Jordan Curve.

II. ELLIPSE AREA INTEGRALS

Kepler's second law asks for the area swept out by a point moving on the circumference of an ellipse, from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$. We choose all points from the ellipse,

$$\mathcal{E} = \{(x, y) : (1 - e^2)(x + ae)^2 + y^2 = a^2(1 - e^2)\},$$

with eccentricity $e \in [0, 1)$ and semi-major axis length typically set to $a = 1$ (without loss of generality). These conventions for ellipse $\mathcal{E}$ place one focus at the origin and another at $x = -2e$, as in Fig. 2. By the integral property that $\int_{P_1}^{P_2} dA = \int_{P_0}^{P_2} dA - \int_{P_0}^{P_1} dA$, a standard reference point P_0 can be chosen to simplify analysis. Either the apogee or perigee is a natural choice. Between the two, we choose the perigee at $P_0 = (x_0, y_0) = (1 - e, 0)$. In Cartesian coordinates, the trigonometric area integral,

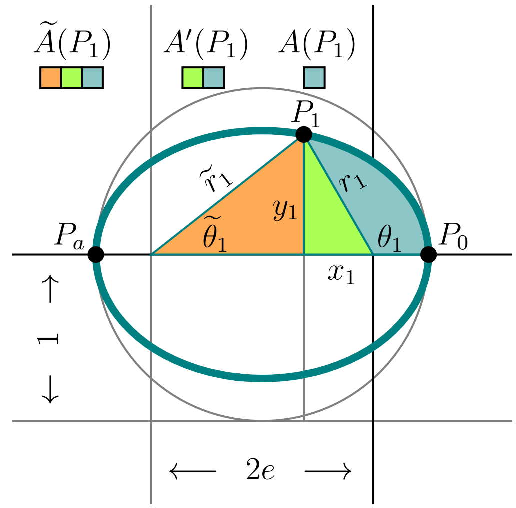

FIG. 2. An Ellipse $\mathcal{E}$ with $e = 2/3$.

$$A^x_{\mathcal{E}}(P_1) = \int_{P_0}^{P_1} dA^x_{\mathcal{E}} = \int_{x_1}^{1-e} y\, dx = \int_{x_1}^{1-e} \sqrt{(1 - e^2)(1 - (x + e)^2)}\, dx,$$

has a simple and well-known[9] closed-form,

$$A^x_{\mathcal{E}}(P_1) = \frac{1}{2}\Big(\sqrt{1 - e^2} \arccos(e + x_1) - (e + x_1)y_1\Big).$$

However, area $A^x_{\mathcal{E}}(P_1)$ is the *Cartesian area*, so does not immediately help with the Kepler problem. Instead we need to calculate the *sectorial area*,

$$A^\theta_{\mathcal{E}}(P_1) = \int_{P_0}^{P_1} dA^\theta_{\mathcal{E}} = \frac{1}{2}\int_0^{\theta_1} r^2 d\theta = \frac{1}{2}\int_0^{\theta_1} \Big(\frac{1 - e^2}{1 + e\cos(\theta)}\Big)^2 d\theta,$$

in polar coordinates where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The integrand of $A^\theta_{\mathcal{E}}(P_1)$ is prohibitively complicated[10]. From Fig. 2, sectorial area equals to Cartesian area plus or minus the area of a triangle with base length $|x_1|$ and height $|y_1|$, $A^\theta_{\mathcal{E}}(P_1) = A^x_{\mathcal{E}}(P_1) + \frac{1}{2}x_1y_1$

[9] Mathematica produces this evaluation automatically and within a few seconds.
[10] Mathematica takes a long time thinking, and returns an over-complicated answer.

(x_1 is negative to the left of the origin). There is another way to derive this identity, but without using trigonometry. The following exercise in calculus may seem superfluous, but it is worthwhile training when the goal is to progress to more complicated integrals, as in subsequent sections of this paper.

Choosing P_1 as the apogee point $P_a = (x_a, y_a) = (-1-e, 0)$, the total ellipse area is written, $A(e) = 2A^\theta_{\mathcal{E}}(P_a) = \pi\sqrt{1-e^2}$. Complete area $A(e)$ is an algebraic function, which satisfies $\mathcal{A}_A \circ A(e) = \big((1-e^2)\partial_e + e\big) \circ A(e) = (1-e^2)\partial_e A(e) + eA(e) = 0$. Moving the annihilating operator $\mathcal{A}_A = (1-e^2)\partial_e + e$ under the integral sign of either $A^\theta_{\mathcal{E}}(P_1)$ or $A^x_{\mathcal{E}}(P_1)$ we obtain two checks on the validity of $\mathcal{A}_A$,

$$\mathcal{A}_A \circ \frac{dA^\theta_{\mathcal{E}}}{d\theta} = \frac{d\Xi^\theta_A}{d\theta} = -\Big(\tfrac{3}{2}e + \cos(\theta) + \tfrac{1}{2}e^2\cos(\theta)^2\Big)\frac{(1-e^2)^2}{(1+e\cos(\theta))^3},$$
$$\mathcal{A}_A \circ \frac{dA^x_{\mathcal{E}}}{dx} = \frac{d\Xi^x_A}{dx} = \frac{-(1-e^2)^2(x+e)}{\sqrt{(1-e^2)(1-(x+e)^2)}}.$$

Certificate functions Ξ^θ_A and Ξ^x_A must exist and contribute to an exact differential[11] so that $\mathcal{A}_A \circ \oint dA^\theta_{\mathcal{E}}$ and $\mathcal{A}_A \circ \oint dA^x_{\mathcal{E}}$ will equal zero, as necessary. The certificate functions follow from indefinite integration[12],

$$\Xi^\theta_A = \int \Big(\frac{d\Xi^\theta_A}{d\theta}\Big)d\theta = -\sin(\theta)\big(1 + \tfrac{e}{2}\cos(\theta)\big)\Big(\frac{1-e^2}{1+e\cos(\theta)}\Big)^2,$$
$$\Xi^x_A = \int \Big(\frac{d\Xi^x_A}{dx}\Big)dx = (1-e^2)\sqrt{(1-e^2)(1-(x+e)^2)}.$$

Upon another integration to P_1, in terms of $\Delta A(e) = A^\theta_{\mathcal{E}}(P_1) - A^x_{\mathcal{E}}(P_1)$, we have that

$$(1-e^2)\partial_e\Delta A(e) + e\Delta A(e) = \int_0^{\theta_1}\frac{d\Xi^\theta_A}{d\theta}d\theta - \int_{x_1}^{1-e}\frac{d\Xi^x_A}{dx}dx.$$

Terms on the right-hand side are evaluations of Ξ^θ_A and Ξ^x_A,

$$\int_0^{\theta_1}\frac{d\Xi^\theta_A}{d\theta}d\theta = \Xi^\theta_A(P_1) = -(r_1 + \tfrac{e}{2}x_1)y_1, \quad \text{and} \quad \int_{x_1}^{1-e}\frac{d\Xi^x_A}{dx}dx = \Xi^x_A(P_1) = -(r_1 + ex_1)y_1,$$

while the term $\partial_e\Delta A(e)$ may safely be ignored as equal to zero[13]. Putting it all together, $\Delta A(e) = \big(\Xi^\theta_A(P_1) - \Xi^x_A(P_1)\big)/e = \frac{1}{2}x_1y_1$, we find again the product $\frac{1}{2}x_1y_1$. Taken separately the certificates seem like nothing too special. Combined via subtraction, they are a circuitous means to determine the green triangular area of Fig. 2. We will return to this idea in Section VII, but presently need to continue solving.

[11] Exact differentials integrate to zero on any complete cycle, $\oint df = 0$ implies df exact.

[12] If these integrals are too difficult by the usual deductive procedures, try a guess-and-check strategy.

[13] Point P_1 is fixed, thus boundaries x_1 and θ_1 are fixed. Variation of e by de allows for disagreement on another triangular area, $\Delta A(e+de) - \Delta A(e) \propto de^2$, thus $\partial_e\Delta A(e) = 0$. See also Section VII and Fig. 11.

Yet another important coordinate system exists, the Keplerian coordinates,

$$x = \cos(\vartheta) - e \quad \text{and} \quad y = \sqrt{1-e^2}\,\sin(\vartheta),$$

written in terms of the *eccentric anomaly* ϑ, also the polar angle of Fig. 3. Introducing the *mean anomaly*, $\Theta(P_1) = 2A^{\theta}_{\mathcal{E}}(P_1)/\sqrt{1-e^2}$, and combining various equations, we finally arrive at Kepler's equation $\Theta = \vartheta - e\sin(\vartheta)$. If instead we measure sectorial area from the second focus, Kepler's equation would read $\widetilde{\Theta} = \widetilde{\vartheta} + e\sin(\widetilde{\vartheta})$. Choosing $\widetilde{\Theta} = -\Theta$, the points $P(\Theta)$ and $P(\widetilde{\Theta})$ fall onto the intersection of ellipse $\mathcal{E}$ with a sine wave,

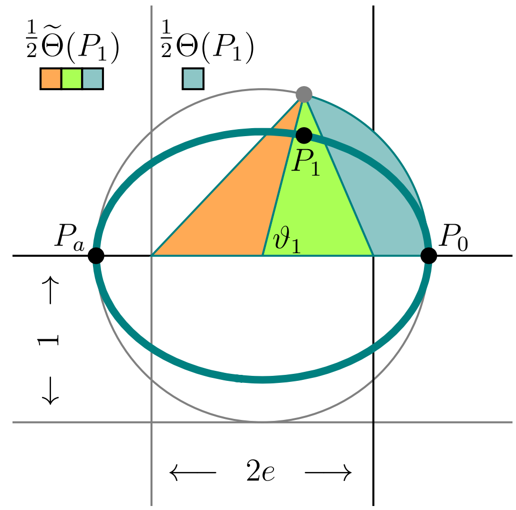

FIG. 3. Keplerian Coordinates.

$$\mathcal{W}(\Theta) = \left\{ \left(\cos\left(\frac{e\,y}{\sqrt{1-e^2}} + \Theta \right) - e, y \right) : y \in \mathbb{R} \right\},$$

traveling at constant velocity $v_y \propto dy/d\Theta$ along the y-axis, as in Fig. 4 (for more details, see ref. []). This geometric fact, though neat, is not entirely satisfying from a physicist's point of view.

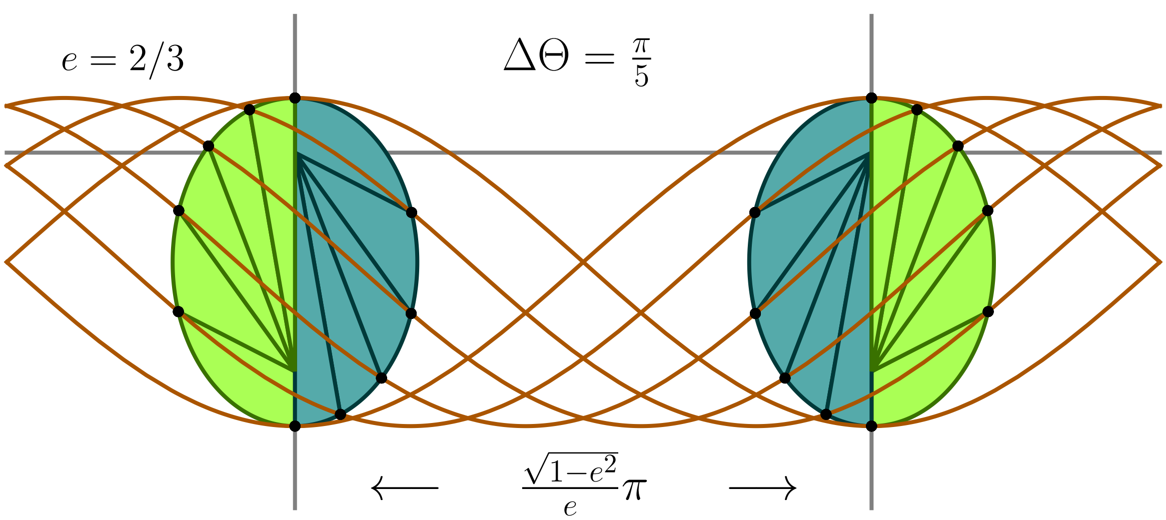

FIG. 4. Asymmetric Intersection Geometry for $\{P(\Theta), P(\widetilde{\Theta})\} = \mathcal{W}(\Theta) \cap \mathcal{E}$.

III. THE KEPLER PROBLEM

Classical mechanics determines planetary orbits according to a gravitational force field, which sums over contributions from all masses within a particular region of space. As the masses move through space, generally the gravitational force field changes with time. However, when one body dominates the gravitational field we may assume a time-independent force field where the dominant body has zero velocity. In the Kepler problem, a star of mass M generates a fixed gravitational field, which determines the classical orbit of a planet whose mass m satisfies $m \ll M$, as in Fig. 5. We choose a system of cylindrical coordinates which places the star at the origin, and then we can solve the Kepler problem by combining Newton's law of gravitation, $\mathbf{F} = -\frac{GMm}{r^2}\hat{r}$, with his second law of motion, $\mathbf{F} = \frac{d}{dt}\mathbf{p}$.

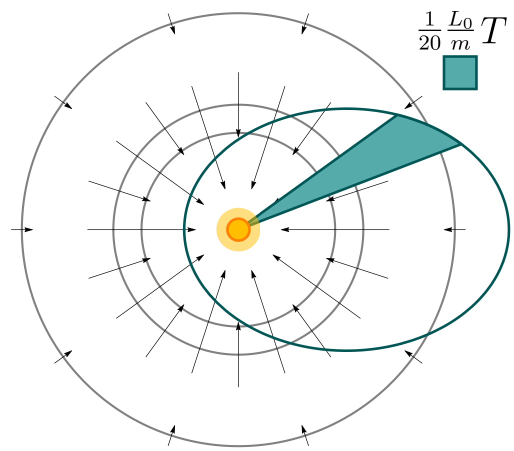

FIG. 5. A Strong Gravitational Force Field.

Isotropic symmetry around the sun immediately suggests a few well-known shortcuts. Throughout time, an isolated, classically orbiting planet falls into a plane $\{(x, y, z) : z = 0\}$ with normal vector $\hat{z} \propto \mathbf{r} \times \dot{\mathbf{r}}dt$. The necessary condition, $\dot{z} = 0$, follows from conservation of momentum along the vertical, $F_z = 0 = \frac{d}{dt}p_z$. The in-plane angular component F_θ of gravitational force $\mathbf{F}$ also equals zero, thus conservation of angular momentum, $rF_\theta = 0 = \frac{d}{dt}L_\theta$, constrains angular motion by $L_\theta = mr^2\dot{\theta} = L_0$. The vector identity $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \dot{\mathbf{r}}) = L_0\hat{z}$ conceals a hint to Kepler's second law. Vector $\dot{\mathbf{r}}\ dt$ translates $\mathbf{r}$ along a tangent line, $\mathbf{r}_2 = \mathbf{r}_1 + \dot{\mathbf{r}}_1 dt$. In the infinitesimal limit, sectorial area takes a triangular shape such that $dA\ \hat{z} = \frac{1}{2}\mathbf{r}_1 \times \mathbf{r}_2 = \frac{1}{2}\mathbf{r} \times \dot{\mathbf{r}}\ dt = \frac{1}{2}\frac{L_0}{m}\ dt\ \hat{z}$, or $\frac{dA}{dt} = \frac{1}{2}\frac{L_0}{m}$.

Newton's law along the radial dimension, $F_r = -\frac{GMm}{r^2} + mr\dot{\theta}^2 = m\ddot{r}$, has an extra term for the fictitious force of centripetal acceleration. A standard approach substitutes for $\dot{\theta}$ and changes variables by $r \to u = 1/r$ and $dt \to d\theta = \frac{L_0}{m}u^2dt$. Then we obtain a recognizable form, $\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{L_0^2} = c_0$, essentially the defining differential equation for $\sin(\theta)$ and $\cos(\theta)$.

The general solution is written as $u = c_0 + c_1\cos(\theta) + c_2\sin(\theta)$, with initial conditions c_1 and c_2. In terms of radial coordinate r the solution becomes $1 = r\big(c_0 + c_1\cos(\theta) + c_2\sin(\theta)\big)$, or in Cartesian coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$, $(1 - c_1 x - c_2 y)^2 = c_0^2(x^2 + y^2)$. This x-y constraint equation has no terms higher than quadratic, so its locus of points determines a conic section. Of the conic sections, only the circle and the ellipse will bind the orbiting planet to the sun. A rotation of the ellipse is chosen by setting $c_2 = 0$. Upon rearranging terms we finally reach an almost-canonical form, $c_0^2 = (c_1 + (c_0^2 - c_1^2)x)^2 + c_0^2(c_0^2 - c_1^2)y^2$.

Section II already goes through sufficient detail on how to calculate ellipse area integrals, so completing the Kepler solution only requires a bit of dimensional analysis. Comparing ellipse constraints determines integral constants in terms of eccentricty, $1/c_0 = 1 - e^2$ and $e/c_1 = 1 - e^2$. In units of length and time where $GM = 1$ and $a = 1$, constant c_0 entirely determines sectorial velocity $\frac{dA}{dt} = \frac{1}{2}\frac{L_0}{m} = \frac{1}{2}c_0^{-1/2} = \frac{1}{2}(1-e^2)^{1/2}$. The yearly period does not depend on eccentricity, for $Y = \frac{dt}{dA}A(e) = 2\pi$. According to Kepler's third law, yearly period does depend on semi-major axis length, $Y(a) = 2\pi a^{3/2}$. This is exactly the result we find by restoring scale a^2 to total area, and $a^{1/2}$ to sectorial velocity. With the three laws proven, all that's left is to invert Kepler's equation and make another plot or two.

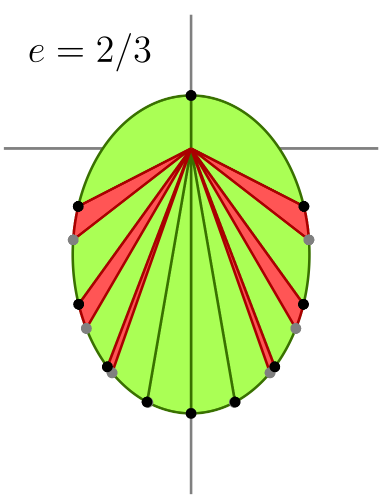

FIG. 6. Error of $\vartheta_1(\Theta)$.

Whenever ϑ is an integer multiple of π the perturbing term $e\sin(\vartheta)$ equals to zero and $\vartheta = \Theta$. The slope of the inverse function, $d\vartheta/d\Theta$ is then easy to determine, $d\vartheta/d\Theta = (1-e)^{-1}$ for even n, $(1+e)^{-1}$ for odd n. These boundary conditions are sufficient data to build a decent *ad hoc* approximation. Identity $\vartheta(n\pi) = \Theta$ suggests the form $\vartheta(\Theta) = \Theta + f(\Theta)\sin(\Theta)$, with either

$$f(\Theta) = \frac{e}{1-e}\left(1 - \frac{2e}{(1+e)}\frac{\Theta(2\pi - \Theta)}{\pi^2}\right),$$

$$\text{or} \qquad f(\Theta) = \frac{e}{1-e}\left(\frac{1}{1+e} + \frac{e}{1+e}\cos(\Theta)\right),$$

chosen to fit the slopes. The former approximation (with Θ evaluated modulo 2π) achieves 99% accuracy[14] for any eccentricity satisfying $e \leq 0.5$. Both approximations have better than 99.5% accuracy when $e \leq 0.25$ and

[14] Here accuracy is defined as $\frac{100}{2\pi}|\vartheta_{Approximate} - \vartheta_{Exact}|$, in percentages of circular circumference.

worse than 5% error when $e > 0.7$. In our solar system, Mercury has the highest eccentricity at $e \approx 0.2$, so either *ad hoc* approximation will work just fine. Calculation of planetary orbits to much greater accuracy would require the entire physical theory to be reworked with fewer simplifying assumptions. Kepler's laws do not hold out in general, and even Newton's laws, famously, have trouble with Mercury.

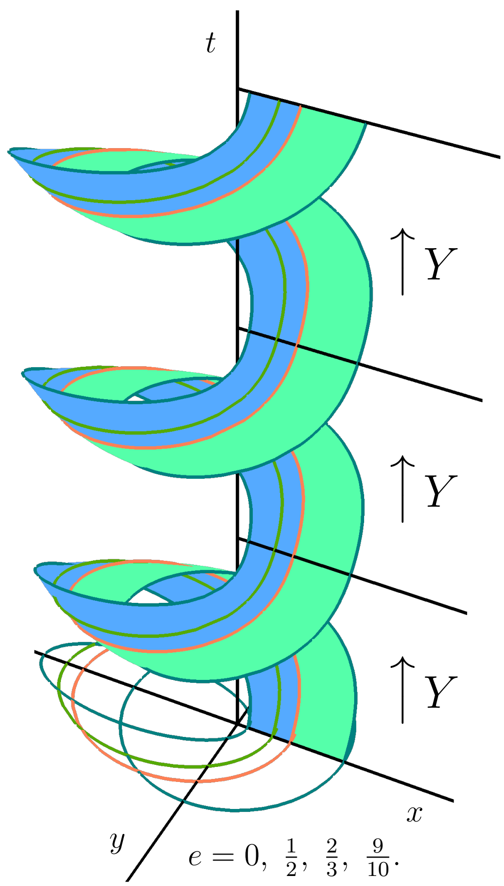

FIG. 7. Kepler Orbits in Spacetime.

In ranges of eccentricity where *ad hoc* approximations begin to fail, a higher precision solution to Kepler's equation is desirable and necessary. Assuming Θ a fixed constant, Newton's iterative method will find $\vartheta(\Theta)$ as root of $\Theta - (\vartheta - e\sin(\vartheta)) = 0$, according to the recursive equation,

$$\vartheta_{i+1} = \vartheta_i + \frac{\Theta - (\vartheta_i - e\sin(\vartheta_i))}{1 - e\cos(\vartheta_i)}, \quad \vartheta_0 = \Theta.$$

A first approximation ϑ_1 satisfies the boundary conditions above, and reaches about the same accuracy as either *ad hoc* approximation. Iteration to higher values of i converges the estimate $\vartheta_i(\Theta)$ toward it's actual value, such that $\Theta - (\vartheta_i - e\sin(\vartheta_i))$ approaches zero, but only at the cost of increased complexity for the functions $\vartheta_i(\Theta)$. Other references such as [cite] discuss convergence and error analysis, here we are content simply to use function $\vartheta_4(\Theta)$ and plot a few spacetime trajectories in Fig. 7.

The spacetime diagram is a graphical solution of the Kepler problem. It shows isoperiodic orbits of varying eccentricity, all with fixed length scale $a = 1$. Solutions with $a \neq 1$ simply scale the vertical axis by a factor $a^{3/2}$, so have entirely similar shapes. When $e = 0$, the worldline is a circular helix. For any other $e \in (0, 1)$, the worldline is an almost-helix with anisotropic vertical stretching via $\vartheta(\Theta)$. Green or blue coloring indicates where an *ad hoc* solution of $\vartheta(\Theta)$ will or will not work decently well. A dividing wordline of $e = 1/2$ appears in red (also compare green $e = 2/3$ worldline with errors of Fig. 6).

SPECIMEN
CONSTRVCTIONE AEQVATIONVM DIFFERENTIALIVM
SINE INDETERMINATARVM SEPARATIONE

FIG. 8. Transcription of the Latin title to Euler's E28.

IV. ELLIPSE CIRCUMFERENCE

Euler's difficult but comprehensive approach to mathematics and physics is useful even when asking seemingly simple questions, for example: What is the average velocity $\bar{v}$ of a Kepler orbit? In the special case of a circular orbit, instantaneous velocity v is a constant of motion, for $L_0 = m\ a\ v$, and this implies $\bar{v} = v = a^{-1/2}$ (again in units where $GM = 1$). The $a^{-1/2}$ scaling of average velocity also follows from its definition as distance-over-time, $\bar{v} = C_0/Y$, with circular circumference $C_0 = 2\pi a$. The circumference $C(e^2)$ of a general ellipse, of course, depends intricately on eccentricity e, so too must average velocity, $\bar{v}(e) = C(e^2)/Y$. Thus to answer the deceptively simple question about average velocity, we must follow Euler and find the function $C(e^2)$ by arclength integration. It is not too easy a task, and perhaps not worthwhile if the Kepler problem is our only motivation. We entreat the wary reader to keep in mind the richness of nature, and to have faith that mathematics will continue to prevail in other interesting circumstances.

We choose coordinates[15] and redefine that $\mathcal{E} = \{(p,q) : p^2 + (1-\alpha)q^2 = 1-\alpha\}$ with $\alpha = e^2$ and a parametric solution $q = \sin(\varphi)$ and $p = \sqrt{1-\alpha}\cos(\varphi)$. The arclength integral, already assuming $a = 1$, takes a concise form in terms of angle φ,

$$C(\alpha) = \oint dl = \oint \sqrt{dp^2 + dq^2} = \oint \sqrt{\left(\frac{dp}{d\varphi}\right)^2 + \left(\frac{dq}{d\varphi}\right)^2} d\varphi = \oint \sqrt{1 - \alpha \sin(\varphi)^2} d\varphi.$$

Over a complete domain, $\varphi \in [0, 2\pi]$, term-by-term integration of the α-series expansion yields a solution,

$$C(\alpha) = \sum_{n \geq 0} \frac{1}{1-2n} \binom{2n}{n} \left(\frac{\alpha}{4}\right)^n \oint \sin(\varphi)^{2n} d\varphi = \sum_{n \geq 0} \frac{2\pi}{1-2n} \binom{2n}{n}^2 \left(\frac{\alpha}{16}\right)^n.$$

This is not the only solution of $C(\alpha)$, nor even the best. Practically speaking, values of $C(\alpha)$ become difficult to calculate at large α where convergence of the series expansion slows to a crawl. Fortunately, there is a stronger analysis, one that owes back to Euler himself.

---

[15] Compare with Keplerian coordinates by $\varphi = \pi/2 - \vartheta$ and $q = x + e$.

Euler was among the first to realize that the function $C(\alpha)$ could be defined as the solution of an ordinary differential equation. Though it produces the correct answer, his intuitive method of solution leaves some doubt and room for improvement. A more rigorous approach starts by observing that the first two α-derivatives of the arclength element dl can be written in terms of the trigonometric polynomial $\Phi = \left(\frac{dl}{d\varphi}\right)^2 = 1 - \alpha\sin(\varphi)^2$,

$$\partial_\alpha dl = \frac{1}{2\alpha}\left(\Phi^{\frac{1}{2}} - \Phi^{-\frac{1}{2}}\right)d\varphi, \qquad \partial_\alpha^2 dl = -\frac{1}{4\alpha^2}\left(\Phi^{\frac{1}{2}} - 2\Phi^{-\frac{1}{2}} + \Phi^{-\frac{3}{2}}\right)d\varphi,$$

after decomposing to partial fractions[16]. Each term is of the form $w\ \Phi^{n/2}$, with odd n and w a ratio of polynomials in variable α. For every such integrand, the technique of Hermite reduction produces a canonical least form by the addition of exact φ-differentials. With u, v, and w all undetermined functions of angle φ, a first reduction $[w]$ of w is written as,

$$\frac{[w]}{\Phi^{m-1}} = \left(u - \frac{dv/d\varphi}{m-1}\right)\frac{1}{\Phi^{m-1}} = \frac{w}{\Phi^m} - \frac{d}{d\varphi}\left(\frac{v}{(m-1)\Phi^{m-1}}\right).$$

This equation can be iterated to find successive reductions of w, but only when u,v, and w satisfy a closure requirement. The closure requirement follows from analysis of the consequential identity $w = \Phi u - \frac{d\Phi}{d\varphi}v$. Notice that $\Phi = 1 - \alpha\sin(\varphi)^2$ is a quadratic polynomial of $\sin(\varphi)$ and an even function, while $\frac{d\Phi}{d\varphi} = -2\alpha\sin(\varphi)\cos(\varphi)$ is quadratic and odd[17]. When u and w are even, v must be odd. Imposing a degree bound d, we have that,

$$\sum_{n=0}^{d} w_n \sin(\varphi)^{2n} = \Phi\sum_{n=0}^{d} u_n \sin(\varphi)^{2n} - \frac{d\Phi}{d\varphi}\sum_{n=1}^{d} v_n \cos(\varphi)\sin(\varphi)^{2n-1},$$

with $d+2$ coefficients to powers of $\sin(\varphi)^2$ and $2d+1$ undetermined coefficients on the right hand side. Choosing $d = 1$, the system of linear equations is exactly solvable for u_0, u_1 and v_1 in terms of w_0 and w_1.

The existence of a degree bounded Hermite reduction guarantees an annihilating operator $\mathcal{A}_E$ for $C(\alpha)$ with no more than three terms. We will use a matrix method to calculate this operator directly from the solution,

$$u = w_0 + \frac{w_0\alpha + w_1}{1-\alpha}\sin(\varphi)^2, \qquad v = -\frac{w_0\alpha + w_1}{2(1-\alpha)}\sin(\varphi)\cos(\varphi).$$

[16] By inspection, $\partial_\alpha^n dl = -(\frac{-1}{2\alpha})^n\left((2n-3)!!\right)\sum_{m=0}^{n}(-1)^m\binom{n}{m}\Phi^{1/2-m}$, see also OEIS: A330797.
[17] Even functions satisfy $f(\varphi) = f(-\varphi)$; odd functions satisfy $f(\varphi) = -f(-\varphi)$.

Functions u and v determine a set of invariants for Φ, which collect in reduction matrices,

$$\mathbf{U} = \begin{bmatrix} 1 & 0 \\ \frac{\alpha}{1-\alpha} & \frac{1}{1-\alpha} \end{bmatrix}, \qquad \mathbf{V}' = \begin{bmatrix} \frac{-\alpha}{2(1-\alpha)} & \frac{-1}{2(1-\alpha)} \\ \frac{\alpha}{1-\alpha} & \frac{1}{1-\alpha} \end{bmatrix}.$$

These two matrices allow us to simplify the reductive process to mere matrix multiplication, $[\mathbf{w}] = (\mathbf{U} - \frac{1}{m-1}\mathbf{V}') \cdot \mathbf{w} = \mathbf{R}(m) \cdot \mathbf{w}$, with column vector $\mathbf{w} = [w_0, w_1]^T$. In terms of $\mathbf{w}(dl) = [1, 0]^T$, derivatives $\partial_\alpha dl$ and $\partial_\alpha^2 dl$ reduce according to,

$$\begin{aligned} [\mathbf{w}(\partial_\alpha dl)] &= \tfrac{1}{2\alpha}\Big(\mathbf{I} - \mathbf{R}\big(\tfrac{1}{2}\big)\Big) \cdot \mathbf{w}(dl) &&= \left[\frac{1}{2(1-\alpha)}, \frac{-3}{2(1-\alpha)}\right]^T, \\ [[\mathbf{w}(\partial_\alpha^2 dl)]] &= -\tfrac{1}{4\alpha^2}\Big(\mathbf{I} - 2\,\mathbf{R}\big(\tfrac{1}{2}\big) + \mathbf{R}\big(\tfrac{1}{2}\big) \cdot \mathbf{R}\big(\tfrac{3}{2}\big)\Big) \cdot \mathbf{w}(dl) &&= \left[\frac{-3}{4(1-\alpha)\alpha}, \frac{3}{2(1-\alpha)\alpha}\right]^T, \end{aligned}$$

with 2×2 identity matrix $\mathbf{I}$. Three column vectors with two components each must admit at least one zero-sum. In this case, the identity,

$$[0,0]^T = \mathbf{w}(dl) + 4(1-\alpha)\ [\mathbf{w}(\partial_\alpha dl)] + 4(1-\alpha)\alpha\ [[\mathbf{w}(\partial_\alpha^2 dl)]].$$

reveals an annihilator $\mathcal{A}_E = 1 + 4(1-\alpha)\partial_\alpha + 4(1-\alpha)\alpha\partial_\alpha^2$ such that $\mathcal{A}_E \circ \frac{dl}{d\varphi} = \frac{d\Xi_E^\varphi}{d\varphi}$ and consequentially $\mathcal{A}_E \circ E(\alpha) = 0$. Certificate function Ξ_E^φ need not be calculated; however, when known, it provides a worthwhile quality check on $\mathcal{A}_E$. Indefinite integration,

$$\Xi_E^\varphi = \int \left(\mathcal{A}_E \circ \frac{dl}{d\varphi}\right) d\varphi = \int \frac{1 - 2\sin(\varphi)^2 + \alpha \sin(\varphi)^4}{\big(1 - \alpha\sin(\varphi)^2\big)^{3/2}} d\varphi = \frac{\cos(\varphi)\sin(\varphi)}{\sqrt{1 - \alpha\sin(\varphi)^2}},$$

after careful bookkeeping, must agree with a total of exact differentials. The row vector,

$$\mathbf{V}(m) = \frac{-\cos(\varphi)\sin(\varphi)}{2(m-1)\Phi^{m-1}}\left[\frac{\alpha}{1-\alpha}, \frac{1}{1-\alpha}\right],$$

of function v determines the certificate by a recursive calculation,

$$\Xi_E^\varphi = \left(\frac{-2(1-\alpha)}{\alpha}\mathbf{V}(\tfrac{1}{2}) + \frac{1-\alpha}{\alpha}\Big(2\mathbf{V}(\tfrac{1}{2}) - \mathbf{V}(\tfrac{3}{2}) - \mathbf{V}(\tfrac{1}{2}) \cdot \mathbf{R}(\tfrac{3}{2})\Big)\right) \cdot \mathbf{w}(dl),$$

which follows from the reductions above. The zero sum, $\mathcal{A}_E \circ \frac{dl}{d\varphi} - \frac{d\Xi_E^\varphi}{d\varphi} = 0$, is easy to check, and verifies $\mathcal{A}_E$ against Ξ_E^φ. Although the preceding derivation looks formidable, it is actually an easy, $n = 2$ case of a general n-dimensional method. Such calculations are not usually carried out by hand. In practice, a computer algebra system such as Mathematica routinely automates the details (Cf. Appendix). If there is any doubt about the veracity of an algorithmic derivation, the annihilating relation can be checked again on the output.

After centuries of development, analysis and solution of $\mathcal{A}_E$ now follows a widely-known, standard schedule: "The regular singular points of $\mathcal{A}_E$ are correctly aligned, so that it is possible to read out hypergeometric parameters $(a, b, c) = (-1/2, 1/2, 1)$, which define a general solution around $\alpha = 0$". That solution[18],

$$C(\alpha) = (C_0)\, {}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}\middle|\alpha\right] + (C_1)\, {}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}\middle|\alpha\right] \int \alpha^{-1}\, {}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}\middle|\alpha\right]^{-2} d\alpha,$$

agrees with the earlier term-by-term expansion when $C(0) = C_0 = 2\pi$ and $C_1 = 0$. The function ${}_2F_1\left[\begin{smallmatrix} -\frac{1}{2}, \frac{1}{2} \\ 1 \end{smallmatrix}\middle|\alpha\right] = \sum f_n \alpha^n$ sums over $n \geq 0$, with coefficients f_n defined according to a hypergeometric recursion,

$$f_0 = 1,\ (n+1)^2 f_{n+1} = (n - \tfrac{1}{2})(n + \tfrac{1}{2}) f_n \iff f_n = \frac{1}{1-2n}\binom{2n}{n}^2\left(\frac{1}{16}\right)^n.$$

Yet nothing much is gained by changing notation. The solution, so far, has not diversified enough to avoid convergence difficulty around the regular singular point at $\alpha = 1$. We will forge a way forward by taking advantage of flexibility inherent to the operator $\mathcal{A}_E$.

Change of variables $\alpha \to \reflectbox{$\alpha$} = 1 - \alpha$ produces another, reversed annihilating operator, $\mathcal{A}_E \to {}_{\reflectbox{$\scriptstyle E$}}\reflectbox{$\mathcal{A}$} = 1 - 4\reflectbox{α}\partial_{\reflectbox{$\scriptstyle\alpha$}} + 4(1-\reflectbox{α})\reflectbox{α}\partial_{\reflectbox{$\scriptstyle\alpha$}}^2$, with a differing solution $(\reflectbox{$\alpha$})\reflectbox{$C$}$ around $\reflectbox{$\alpha$} = 0$. Operator ${}_{\reflectbox{$\scriptstyle E$}}\reflectbox{$\mathcal{A}$}$ is again hypergeometric, but it is not as easy to solve. The parameters $(a, b, c) = (-1/2, 1/2, 0)$ set c equal to zero, and consequently $a_1 = a_0/0$, utter nonsense. We resort to a second solution, similar to the first above,

$$(\reflectbox{α})\reflectbox{C} = \reflectbox{C}_1\reflectbox{α}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 2 \end{matrix}\middle|\reflectbox{α}\right] + \reflectbox{C}_0\reflectbox{α}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 2 \end{matrix}\middle|\reflectbox{α}\right] \int \frac{-1}{1-\reflectbox{α}}\left(\reflectbox{α}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 2 \end{matrix}\middle|\reflectbox{α}\right]\right)^{-2} d\reflectbox{α}.$$

In this case, $\reflectbox{$\alpha$} = 0$ corresponds to a completely collapsed ellipse with $(0)\reflectbox{$C$} = 4$, thus the second term can not be ignored. Rather than go into detail repeating a proof from *Mathworld*, let us derive the same solution using Frobenius's method. An Ansatz that,

$$(\reflectbox{α})\reflectbox{C} = \reflectbox{C}_0 + \left(\reflectbox{C}_1 + \frac{3}{8}\reflectbox{C}_0\right)\reflectbox{α} - \frac{\reflectbox{C}_0}{4}\log(\reflectbox{α})\reflectbox{α}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 2 \end{matrix}\middle|\reflectbox{α}\right] + \sum_{n>1} \reflectbox{C}_n\, \reflectbox{α}^n,$$

allows two degrees of freedom by $\reflectbox{$C$}_0$ and $\reflectbox{$C$}_1$, while the other $\reflectbox{$C$}_n$ coefficients with $n > 1$ are entirely constrained by the differential equation ${}_{\reflectbox{$\scriptstyle E$}}\reflectbox{$\mathcal{A}$} \circ (\reflectbox{$\alpha$})\reflectbox{$C$} = 0$, as in table I. Choosing that $\reflectbox{$C$}_0 = 4$ and $\reflectbox{$C$}_1 = 4\log(2) - 5/2$ defines an α-reversed circumference function such that $(\reflectbox{$\alpha$})\reflectbox{$C$} = C(\alpha)$ over the domain $\reflectbox{$\alpha$} = 1 - \alpha \in [0, 1]$.

[18] Mathworld: Hypergeometric Function, Second-Order ODE Second Solution.

The appearance of log(2) in $Ɔ_1$ is an unresolved mystery of this presentation. In practice, the zero sum $C(\frac{1}{2}) - (\frac{1}{2})Ɔ = 0$ determines $Ɔ_1$ to an arbitrary precision, which depends on N, the number of summed terms. Choosing a large value such as $N = 100$, we calculate that $Ɔ_1 \approx 0.27258872223978123766892848583271$, with error creeping in only on the very last digit. Such precision is overkill for many use cases. Instead, the choice of N should be tuned to specific precision goals. Any value of the piecewise function,

TABLE I. Constraints on $(ɒ)Ɔ$

$_{\text{Ǝ}}\text{Ɐ} \circ (ɒ)Ɔ = \sum c_n ɒ^n, n \geq 0$	
n	$0 = c_n =$
0	0
1	$32Ɔ_2 - 12Ɔ_1 - Ɔ_0$
2	$512Ɔ_3 - 320Ɔ_2 + 7Ɔ_0$
$\vdots$	$\vdots$
n	determines $Ɔ_{n+1}$

$$\underset{pw}{C}(\alpha) = \begin{cases} C(\alpha) & \alpha \leq 1/2 \\ (1-\alpha)Ɔ & \alpha > 1/2 \end{cases},$$

is expected to reach roughly the same precision as $C(\frac{1}{2}) = (\frac{1}{2})Ɔ$. An $N = 60$ approximation already reaches double precision of $\mathcal{A}_E \circ \underset{pw}{C}(\alpha) < 10^{-16}$ on the domain $\alpha \in (0, 1)$. This check assures the quality of $\underset{pw}{C}(\alpha)$, which we can now begin to use in calculations about average orbital velocity or whatever else. More importantly, the process of finding an answer has introduced concepts and techniques that we will have occasion to use again, when building computable realizations for other similar integral functions.

Thus far we have deliberately avoided standard nomenclature by neglecting to mention the tautology that $C(\alpha) = 4E(\alpha)$, in terms of $E(\alpha)$, the *complete elliptic integral of the second kind*. In so doing, we might have skipped over another integral function, $K(\alpha)$, the *complete elliptic integral of the first kind*. There is no deductive reason why one should precede the other, for it is possible to define that either $K(\alpha) = (1 - 2\alpha\partial_\alpha) \circ E(\alpha)$ or $E(\alpha) = \big((1-\alpha) + 2(1-\alpha)\alpha\partial_\alpha\big) \circ K(\alpha)$. The reason for nomenclature to ignore historical ordering is apparently more subtle. In the modern theory of elliptic curves and elliptic functions, as well in the theory of pendulum motion, function $K(\alpha)$ is a period not too dissimilar from Kepler's orbital period $Y(a)$. Neither are these periods too similar. Again $K(\alpha)$ is hypergeometric whereas $Y(a)$ is only algebraic. Before we get a chance to classify in more detail, we will show how elliptic integral $K(\alpha)$ measures a family of elliptic curves.

V. ELLIPTIC CURVES

Another worthwhile geometric problem asks for the total area $S(\alpha)$ within a deformable, closed elliptic curve $\mathcal{C}(\alpha)$. An answer to this problem contributes a key fact to the construction of elliptic functions, or sometimes, even to an exact solution of the simple pendulum's motion [cite]. Yet various acceptable choices of $\mathcal{C}(\alpha)$ are not exactly equivalent from a metrical perspective. In particular, enclosed area $S(\alpha)$ depends explicitly on the shape of curve $\mathcal{C}(\alpha)$. Pursuant to finding the integral function $K(\alpha)$, we will use a variant of Edwards's normal form and select square-symmetric curves,

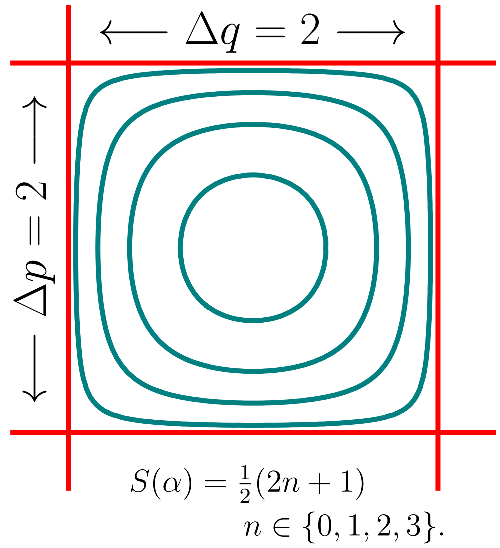

FIG. 9. A few Elliptic Curves $C(\alpha)$.

$$\mathcal{C}(\alpha) = \Big\{(p, q) : \alpha = p^2 + q^2 - p^2 q^2\Big\},$$

with $\alpha \in [0, 1)$. A few of these curves are depicted in Fig. 9. The selection constraint, $\alpha = p^2 + q^2 - p^2 q^2$, must be solved to obtain integrands of the area integrals,

$$\begin{aligned}
\text{either} \qquad S^p_{\mathcal{C}}(P_1) &= \int_{P_1}^{P_0} dS^p_{\mathcal{C}} = \int_{p_1}^{\sqrt{\alpha}} q\; dp = \int_{p_1}^{\sqrt{\alpha}} \sqrt{\frac{\alpha - p^2}{1 - p^2}}\; dp \\
\text{or} \qquad S^q_{\mathcal{C}}(P_1) &= \int_{P_0}^{P_1} dS^q_{\mathcal{C}} = \int_{0}^{q_1} p\; dq = \int_{0}^{q_1} \sqrt{\frac{\alpha - q^2}{1 - q^2}}\; dq \\
\text{or} \qquad S^\phi_{\mathcal{C}}(P_1) &= \int_{P_0}^{P_1} dS^\phi_{\mathcal{C}} = \int_{0}^{\phi_1} \lambda\; d\phi = \int_{0}^{\phi_1} \frac{1 - \sqrt{1 - \alpha \sin(2\phi)^2}}{\sin(2\phi)^2}\; d\phi,
\end{aligned}$$

with boundaries $P_0 = (p_0, q_0) = (\sqrt{\alpha}, 0)$ and $P_1 = (p_1, q_1) = \Big(\sqrt{2\lambda_1}\cos(\phi_1), \sqrt{2\lambda_1}\sin(\phi_1)\Big)$. The third alternative is written in action-angle coordinates[19] (λ, ϕ), defined relative to Cartesian (p, q) by $p = \sqrt{2\lambda}\cos(\phi)$, $q = \sqrt{2\lambda}\sin(\phi)$, or relative to polar coordinates (r, ϕ), by $\lambda = \frac{1}{2}r^2$, ϕ identical. After perigee P_0, the next nearest apogee P_a falls onto a diagonal line of symmetry where $p_a = q_a = \sqrt{1 - \sqrt{1 - \alpha}}$ or $\phi_a = \pi/4$. According to dihedral symmetry, this choice determines the total area $S(\alpha) = 8S^p_{\mathcal{C}}(P_a) = 8S^q_{\mathcal{C}}(P_a) = 8S^\phi_{\mathcal{C}}(P_a)$.

---

[19] Letters p, q, and λ allude to momentum, position, and action quantities of Hamiltonian mechanics.

Differentiating the third area function once with respect to α produces a period integral in action-angle coordinates,

$$T_{\mathcal{C}}^{\phi}(P_1) = \int_{P_0}^{P_1} dt = \int_0^{\phi_1} 2(\partial_\alpha \lambda)\, d\phi = \int_0^{\phi_1} \frac{1}{\sqrt{1-\alpha\sin(2\phi)^2}}\, d\phi,$$

where $T(\alpha) = 8T_{\mathcal{C}}^{q}(P_a) = 8T_{\mathcal{C}}^{\phi}(P_a) = 4K(\alpha)$. After scaling ϕ and t by factors of two, we obtain a more comparable integrand, $dt/d\phi = \Phi^{-1/2}$ with $\Phi = 1-\alpha\sin(\phi)^2$. Again, the matrices $\mathbf{R}(m)$ can be used to reduce the first two α-derivatives,

$$\partial_\alpha dt = -\frac{1}{2\alpha}\left(\Phi^{-\frac{1}{2}} - \Phi^{-\frac{3}{2}}\right)d\phi, \qquad \partial_\alpha^2 dt = \frac{3}{4\alpha^2}\left(\Phi^{-\frac{1}{2}} - 2\Phi^{-\frac{3}{2}} + \Phi^{-\frac{5}{2}}\right)d\phi,$$

As above, let $\mathbf{w}(dt) = [1,0]^T$. Canonical, least coefficient vectors may be written out by recursion of Hermite reduction,

$$\begin{aligned}
[\mathbf{w}(\partial_\alpha dt)] &= -\tfrac{1}{2\alpha}\Big(\mathbf{I} - \mathbf{R}\big(\tfrac{3}{2}\big)\Big)\cdot\mathbf{w}(dt) &&= \left[\frac{1}{2(1-\alpha)}, \frac{-1}{2(1-\alpha)}\right]^T, \\
[[\mathbf{w}(\partial_\alpha^2 dt)]] &= \tfrac{3}{4\alpha^2}\Big(\mathbf{I} - 2\,\mathbf{R}\big(\tfrac{3}{2}\big) + \mathbf{R}\big(\tfrac{3}{2}\big)\cdot\mathbf{R}\big(\tfrac{5}{2}\big)\Big)\cdot\mathbf{w}(dt) &&= \left[\frac{-(1-3\alpha)}{4(1-\alpha)^2\alpha}, \frac{1-2\alpha}{2(1-\alpha)^2\alpha}\right]^T.
\end{aligned}$$

In this next case the zero sum,

$$[0,0]^T = \mathbf{w}(dt) - 4(1-2\alpha)\,[\mathbf{w}(\partial_\alpha dt)] - 4(1-\alpha)\alpha\,[[\mathbf{w}(\partial_\alpha^2 dt)]],$$

determines an annihilating operator. $\mathcal{A}_K = 1-4(1-2\alpha)\partial_\alpha - 4(1-\alpha)\alpha\partial_\alpha^2$, The corresponding certificate function,

$$\begin{aligned}
\Xi_K^{\phi} &= \int\left(\mathcal{A}_E \circ \frac{dt}{d\phi}\right)d\phi = \int \frac{1-2\sin(\phi)^2 + \alpha\sin(\phi)^4}{\big(1-\alpha\sin(\phi)^2\big)^{3/2}}d\phi = \frac{\cos(\phi)\sin(\phi)}{\sqrt{1-\alpha\sin(\phi)^2}} \\
&= \left(\frac{-2(1-2\alpha)}{\alpha}\mathbf{V}\big(\tfrac{3}{2}\big) + \frac{3(1-\alpha)}{\alpha}\Big(2\mathbf{V}\big(\tfrac{3}{2}\big) - \mathbf{V}\big(\tfrac{5}{2}\big) - \mathbf{V}\big(\tfrac{3}{2}\big)\cdot\mathbf{R}\big(\tfrac{5}{2}\big)\Big)\right)\cdot\mathbf{w}(dt),
\end{aligned}$$

allows verification of the necessary zero sum, $\mathcal{A}_K \circ \frac{dt}{d\phi} - \frac{d\Xi_K^{\phi}}{d\phi} = 0$. It is easy to check this identity when the derivation is unavailable, misunderstood, or otherwise in doubt.

Hypergeometric annihilator $\mathcal{A}_K$, with parameters $(a,b,c) = (\frac{1}{2},\frac{1}{2},1)$, bears at least a superficial similarity to $\mathcal{A}_E$. Even qualitatively, the functions $E(\alpha)$ and $K(\alpha)$ differ at their limits, with $K(\alpha)$ diverging to infinity on approach to the singular point $\alpha = 1$. From the geometric standpoint, the analogy is more clear between perimeter function $C(\alpha)$ and area function $S(\alpha)$. The existence of $\mathcal{A}_K$ implies existence of another, similar annihilator,

$\mathcal{A}_S = 1 - 4(1-\alpha)\alpha\partial_\alpha^2$, whose exact form can be calculated by solving a simple system of linear equations. By inspection, we can immediately see that $\mathcal{A}_S$ is hypergeometric with parameters $(a,b,c) = (-\frac{1}{2}, -\frac{1}{2}, 0)$, and that $\mathcal{A}_S$ admits reflection around $\alpha = 1/2$ as an invariant transformation. The fact that $c = 0$ strengthens the analogy to $\mathcal{A}_E$ and suggests that piecewise construction of $S(\alpha)$ will involve no new difficulties.

A general form for the solution is written out as,

$$\begin{aligned} S(\alpha) &= S_1 \alpha\, {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 2\end{matrix}\middle|\alpha\right] - S_0 \alpha\, {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 2\end{matrix}\middle|\alpha\right] \int \left(\alpha\, {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 2\end{matrix}\middle|\alpha\right]\right)^{-2} d\alpha \\ &= S_0 + \left(S_1 + \frac{1}{8} S_0\right)\alpha + \frac{S_0}{4}\log(\alpha)\alpha\, {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 2\end{matrix}\middle|\alpha\right] + \sum_{n>1} S_n\, \alpha^n. \end{aligned}$$

As $S(\alpha)$ satisfies a second-order ODE, the coefficients S_n with $n > 1$ are entirely determined by the choice of S_0 and S_1. Table II lists the first few constraints. According to reflection symmetry, the reversed function $\reflectbox{$S(\alpha)$}$ has the same formal expansion, but with integral constants $\reflectbox{$S$}_0$ and $\reflectbox{$S$}_1$. The harmonic limit toward $\alpha = 0$ requires that $S(\alpha) = \pi\alpha$, thus $S_0 = 0$ and $S_1 = \pi$. The opposite and strongly anharmonic limit toward $\alpha = 1$ determines $\reflectbox{$S$}_0 = 4$, the area of a 2×2 square, but leaves $\reflectbox{$S$}_1$ undetermined. Rogue constant $\log(2)$ returns to cause more trouble, and we find by reference that $\reflectbox{$S$}_1 = -4\log(2) - 3/2$, again without a decently intelligible explanation. Proceeding pragmatically instead, we match functions at $\alpha = \reflectbox{$\alpha$} = \frac{1}{2}$, sum to cutoff $N = 100$, and calculate numerically that $\reflectbox{$S$}_1 \approx -4.2725887222397812376689284858327063$. As in the previous case of $\underset{pw}{E}$, we do not need to increase precision, and instead expand a piecewise solution $\underset{pw}{S}(\alpha)$ only to $N = 60$. Already the approximation $\underset{pw}{S}(\alpha)$ reaches double precision of $\mathcal{A}_S \circ \underset{pw}{S}(\alpha) < 10^{-16}$ everywhere on the domain $\alpha \in [0,1]$. In subsequent analyses, we will put computable functions $\underset{pw}{S}(\alpha)$ and $\underset{pw}{C}(\alpha)$ to good use, but for now we are satisfied to have shown, by explicit calculation, exactly how solution techniques generalize from one specimen to the next.

TABLE II. Constraints on $S(\alpha)$

$\mathcal{A}_E \circ S(\alpha) = \sum c_n \alpha^n, n \geq 0$	
n	$0 = c_n =$
0	0
1	$32S_2 - 4S_1 - 3S_0$
2	$512S_3 - 192S_2 - 3S_0$
$\vdots$	$\cdots$
n	determines S_{n+1}

(and same for $S_n \to \reflectbox{$S$}_n$)

VI. EXAMPLE CALCULATIONS

Having built both computable functions $\underset{pw}{C}(\alpha)$ and $\underset{pw}{S}(\alpha)$, we can also test relative convergence according to the mutual definitions of $E(\alpha)$ and $K(\alpha)$,

$$10^{-16} > (1 - 2\alpha\partial_\alpha) \circ \underset{pw}{C}(\alpha) - 2\partial_\alpha \underset{pw}{S}(\alpha),$$
$$10^{-16} > \big((1-\alpha) + 2(1-\alpha)\alpha\partial_\alpha\big) \circ (2\partial_\alpha \underset{pw}{S}(\alpha)) - \underset{pw}{C}(\alpha),$$

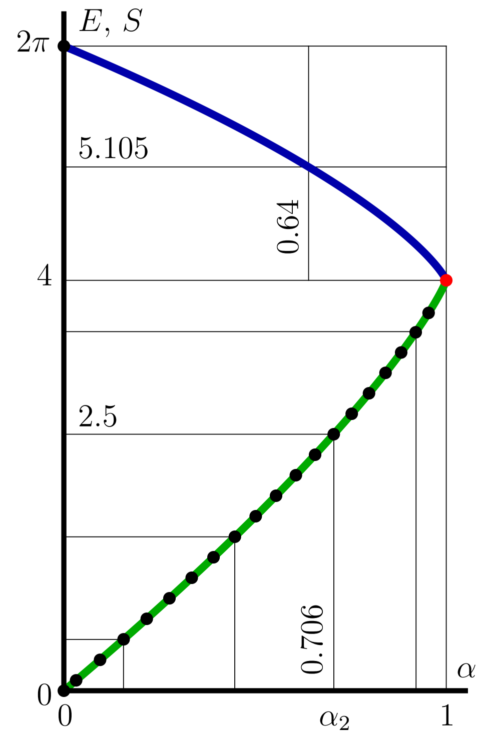

FIG. 10. $E(\alpha)$ and $S(\alpha)$.

where the inequality holds for $\alpha \in [0,1]$. As is reasonable to expect after Sections IV & V, these quality-assurance calculations reach double precision after summing only up to $\mathcal{O}(\alpha^{60})$. In fact, the sums above exactly equal zero on every coefficient of α^n when $C_0 - 2S_1 = 0$ and $\text{Ɔ}_1 + \text{Ƨ}_1 = -\text{Ɔ}_0 = -\text{Ƨ}_0$. Somewhat strangely, the geometric interpretation of identity $\text{Ɔ}_0 = \text{Ƨ}_0 = 4$ says that a linear distance equals an area, as does identity $C_0 = 2S_1 = 2\pi$ between circular circumference and area. More to the point, precise verification of interrelations between $C(\alpha)$ and $S(\alpha)$ allows us to choose just one function to investigate in detail. Derived function $\underset{pw}{T}(\alpha) = 2\partial_\alpha \underset{pw}{S}(\alpha)$ is the best to work with, because there exists another, computationally-distinct means to calculate particular values.

The *arithmetic-geometric mean*[20], $\text{agm}(a_0, b_0) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, is a recursive function of two variables, which rapidly converges on a number between successive arithmetic and geometric means, $a_{n+1} = \frac{1}{2}(a_n + b_n)$ and $b_{n+1} = \sqrt{a_n b_n}$ respectively. Elsewhere it is proven that $T(\alpha)\,\text{agm}(1+\sqrt{\alpha}, 1-\sqrt{\alpha}) - 2\pi = 0$ (Ref. []). This identity defines a numerical reference function $\underset{agm}{T}(\alpha)$, with convergence dependent upon recursion depth $M = n_{max}$. As

[20] Another definition is that $1/\text{agm}(x,y) = \frac{2}{\pi}\int_0^{\pi/2} d\theta \big(x^2\cos(\theta)^2 + y^2\sin(\theta)^2\big)^{-1/2}$.

a first test, let us calculate for a difficult value $\alpha = 1/2$ that,

$$\frac{1}{2\pi}\underset{pw}{T}\left(\tfrac{1}{2}\right) = 1.180340599016096226036 3, \qquad (N = 60)$$
$$\frac{1}{2\pi}\underset{agm}{T}\left(\tfrac{1}{2}\right) = 1.1803405990160962260453, \qquad (M = 7)$$

with error beginning to show around the 20^{th} digit (Cf. OEIS: A175574). For the same termination parameters, $N = 60$ and $M = 7$, we find that $10^{-20} > |\underset{pw}{T}(\alpha)/\underset{agm}{T}(\alpha) - 1|$ over the domain $\alpha \in [0, 1)$. Considering that uncertainties tend to worsen as they propagate through calculations, it is not at all surprising to observe that a direct test of function values yields a tighter bound on the error due to series truncation. Ignoring more conservative tests, a sum to $N = 50$ already allows $\underset{pw}{T}(\alpha)$ to reach double precision. To show off the utility of double-precision computable functions $\underset{pw}{C}(\alpha)$ and $\underset{pw}{S}(\alpha)$, we will now go through two short example calculations, only at a superficial level of detail.

A problem in high-school physics asks for the magnetic field at the center of a circular loop of radius a. The answer, found by a simple Biot-Savart integral, is that $\mathbf{B}_\circ = \frac{\mu_0 I_\circ}{2a}\hat{z}$, with current $I_\circ$ and $\hat{z}$ normal to the plane of the loop. A generalization of this question concerns the magnetic field at the center of a charge conducting ellipse of eccentricity e and semi-major axis a. The field is directed along the vertical, and its strength depends linearly on current strength I_0 according to another not-too-difficult integral,

$$\mathbf{B}_0 = \mathbf{B}_\circ \frac{I_0}{I_\circ}\frac{a}{2\pi}\oint \frac{d\theta}{r} = \mathbf{B}_\circ \frac{I_0}{I_\circ}\frac{a}{2\pi}\oint \frac{dl}{a^2} = \mathbf{B}_\circ \frac{I_0}{I_\circ}\frac{C(e^2)}{2\pi}.$$

The magnetic field at the origin can be canceled to zero by superimposing left and right handed currents. For example, cancellation occurs between two fields $\mathbf{B}_\circ$ and $\mathbf{B}_0$ when $\mathbf{B}_0 \cdot \mathbf{B}_\circ = -\mathbf{B}_\circ \cdot \mathbf{B}_\circ$, or equivalently when $I_0/I_\circ = -(2\pi)/C(e^2)$. Say that we choose to work with an ellipse of eccentricity $e = 4/5, \alpha = 16/25$. Field cancellation requires a ratio $I_0/I_\circ \approx -(2\pi)/\underset{pw}{C}(16/25) \approx -1.23$ (and we could get more digits of precision if necessary).

In semi-classical quantum mechanics, another problem asks for an estimate of quantum pendulum energy eigenvalues. The period function of a simple pendulum is $T(\alpha)$, and its action function is the corresponding $S(\alpha)$. Eigenvalues α_n are found by solving a qunatization condition[21] such as $S(\alpha_n) = \frac{1}{2}(2n + 1)$, with $n = 0, 1, 2, 3$. To find the "quantum values" $\alpha_n = S^{-1}\big(\frac{1}{2}(2n + 1)\big)$, an inverse problem needs to be solved. Function $S^{-1}(s)$ can be found

[21] For more explanation, see: xxx.

TABLE III. Semiclassical quantization of the elliptic curves $\mathcal{C}(\alpha)$.

n	$S(\alpha_n) = \frac{1}{2}(2n+1)$	Eigenvalues	Percent Difference
0	$0.1559223091638732\ldots$	$0.15627\ldots$	0.223%
1	$0.4469484490110412\ldots$	$0.44719\ldots$	0.056%
2	$0.7057110691134417\ldots$	$0.70573\ldots$	0.003%
3	$0.9212998367788911\ldots$	$0.92011\ldots$	0.129%

by series reversion, but this is not the most sensible approach. As we do not need an entire function, it is more expedient to simply apply a root-solving method to the zero sum $\underset{pw}{S}(\alpha_n) - \frac{1}{2}(2n+1) = 0$. In so doing, we calculate the numerical values in the second column of Tab. III, and these values are used to plot the four teal blue curves of Fig. 9.

Alternatively, constraint $\alpha = p^2 + q^2 - p^2q^2$ suggests the form of a quantum mechanical Hamiltonian matrix H, which may be written by exchanging coordinate variables p and q for their corresponding matrix representations [22]. Due to non-commutation of p and q matrices, quantization is a non-unique procedure. Consequently, there are many different Hamiltonian matrices, whose eigenvalues overlap $S(\alpha)$ within a similar range of error. We have chosen, somewhat arbitrarily, the matrix H with elements:

$$h_{i,j} = h_{j,i} = \begin{cases} \frac{1-2i-2i^2+40(2i+1)\pi}{400\pi^2} & i = j \\ \frac{1}{400\pi^2}\sqrt{24\binom{i}{4}} & i = j+4 \\ \frac{1}{400\pi^2}\sqrt{24\binom{j}{4}} & j = i+4 \\ 0 & \text{otherwise} \end{cases},$$

A few of the eigenvalues[23] of H determine equally-spaced black points on the green curve of Fig. 10, four of which are written in the second column of Tab. III. Although enumeration of roots α_n to arbitrary precision shows off computational prowess, comparison with matrix eigenvalues gives apprehension as to when such efforts would actually be necessary. If the task is to approximate quantum pendulum eigenvalues to 99% accuracy, the expansion of $\underset{pw}{S}(\alpha_n)$ needs far fewer than $N = 60$ terms. Instead of having a bikeshed digression about significant figures, let us get back to analysis of the theory itself.

[22] see also: xxx.

[23] We calculate H as a 100×100 matrix with 100 eigenvalues, and select only 19 of the lowest lying. Due to duplicate values, accurate selection requires a criterion in terms of eigenvector elements.

VII. COMPARING CERTIFICATES

Recall from Section II that alternative area integrals $A^x_{\mathcal{E}}(P_1)$ and $A^\theta_{\mathcal{E}}(P_1)$ relate to one another according the difference between certificate functions Ξ^x_A and Ξ^θ_A. To continue developing the theory of certificates by an inductive process, we ask: does the derived identity $\Delta A(e) = (\Xi^\theta_A - \Xi^x_A)/e$ have any analog for the elliptic integrals discussed in Sections IV & V? And if so, how do these analogs differ from $\Delta A(e)$ of the first example? The answers have curious nuances, so deserve a close look.

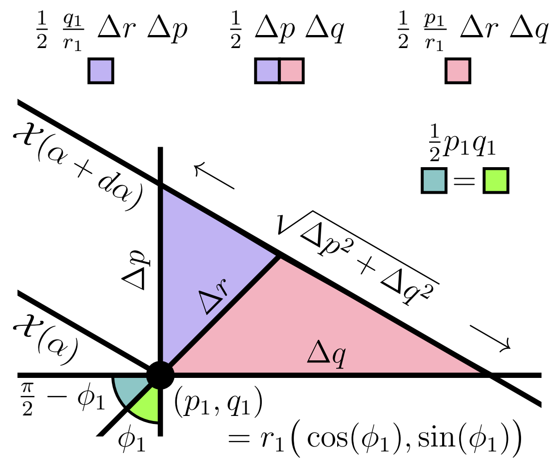

FIG. 11. Tangent Geometry of curve $\mathcal{X}(\alpha)$.

An incomplete arclength integral along $\mathcal{E}$, from initial point $P_0 = (p_0, q_0) = (\sqrt{1-\alpha}, 0)$ to final point $P_1 = (p_1, q_1) = \big(\sqrt{1-\alpha}\cos(\varphi_1), \sin(\varphi_1)\big)$ is written as,

$$\begin{aligned}
\text{either}\quad C^p_{\mathcal{E}}(P_1) &= \int_{P_1}^{P_0} dC^p_{\mathcal{E}} = \int_{p_1}^{\sqrt{1-\alpha}} \sqrt{\frac{(1-\alpha)^2 + \alpha p^2}{(1-\alpha)(1-\alpha-p^2)}}\, dp \\
\text{or}\quad C^q_{\mathcal{E}}(P_1) &= \int_{P_0}^{P_1} dC^q_{\mathcal{E}} = \int_0^{q_1} \sqrt{\frac{1-\alpha q^2}{1-q^2}}\, dq \\
\text{or}\quad C^\varphi_{\mathcal{E}}(P_1) &= \int_{P_0}^{P_1} dC^\varphi_{\mathcal{E}} = \int_0^{\varphi_1} \sqrt{1-\alpha\sin(\varphi)^2}\, d\varphi.
\end{aligned}$$

Applying change of coordinates $(\cos(\varphi), \sin(\varphi)) \to (\sqrt{1-q^2}, q)$ and $d\varphi \to dq/\cos(\varphi)$, the later two of these integrals are proven equal, with $\frac{dq}{d\varphi}\frac{dC^q_{\mathcal{E}}}{dq} = \frac{dC^\varphi_{\mathcal{E}}}{d\varphi}$. Differential $\frac{dq}{d\varphi} = \cos(\varphi)$ depends not on α, thus ∂_α commutes with $\frac{dq}{d\varphi}$, and $\frac{dq}{d\varphi}\partial^n_\alpha \frac{dC^q_{\mathcal{E}}}{dq} = \partial^n_\alpha \frac{dC^\varphi_{\mathcal{E}}}{d\varphi}$. Consequently, integrals are not merely equal. They are also identical under α-differentiation, $\partial^n_\alpha C^q_{\mathcal{E}}(P_1) = \partial^n_\alpha C^\varphi_{\mathcal{E}}(P_1)$, $n \geq 0$. Next, certificates must equate, $\mathcal{A}_E \circ \big(C^q_{\mathcal{E}}(P_1) - C^\varphi_{\mathcal{E}}(P_1)\big) = \Xi^q_E - \Xi^\varphi_E = 0$. After changing coordinates, $\Xi^\varphi_E \to \Xi^q_E = q(dq/dC^q_{\mathcal{E}}) = q\sqrt{(1-q^2)/(1-\alpha q^2)}$, we can verify that $\mathcal{A}_E \circ \big(\frac{dC^q_{\mathcal{E}}}{dq}\big) - \frac{d\Xi^q_E}{dq} = 0$ by explicit calculation (again, using a computer algebra system).

Had we first chosen p rather than q, the calculation would have been much worse, for $dp/d\varphi = -\sqrt{1-\alpha}\sin(\varphi)$, and certificates do not equate, $\Xi^p_E \neq \Xi^\varphi_E$. The situation is not much better when comparing p and q, except that the Pythagorean theorem directly determines the hypotenuse length $\Delta C(\alpha + d\alpha) = \sqrt{\Delta p^2 + \Delta q^2}$. The series expansion[24],

$$\Delta C(\alpha + d\alpha) = \sqrt{\left(\frac{\partial p}{\partial \alpha}\right)^2 + \left(\frac{\partial q}{\partial \alpha}\right)^2} \left(d\alpha + \frac{1}{2} \frac{\frac{\partial p}{\partial \alpha}\frac{\partial^2 p}{\partial \alpha^2} + \frac{\partial q}{\partial \alpha}\frac{\partial^2 q}{\partial \alpha^2}}{\left(\left(\frac{\partial p}{\partial \alpha}\right)^2 + \left(\frac{\partial q}{\partial \alpha}\right)^2\right)} d\alpha^2 \right) + \mathcal{O}(d\alpha^3),$$

follows the variational geometry of Fig. 11, after defining $\Delta C(\alpha) = C^q_{\mathcal{E}}(P_1) - C^p_{\mathcal{E}}(P_1)$, with point P_1 held independent of $d\alpha$. Partial derivatives in the first line can all be written as polynomial ratios,

$$\frac{\partial p}{\partial \alpha} = \frac{q^2 - 1}{2\, p}, \quad \frac{\partial^2 p}{\partial \alpha^2} = -\frac{(1-q^2)^2}{4\, p^3} \quad \text{and} \quad \frac{\partial q}{\partial \alpha} = -\frac{(1-q^2)^2}{2\, p^2\, q}, \quad \frac{\partial^2 q}{\partial \alpha^2} = \frac{(q^2-1)^3(1+3q^2)}{4\, p^4\, q^3}.$$

Comparison of the explicit series with a formal expansion,

$$\Delta C(\alpha + d\alpha) = \Big(\partial_\alpha \Delta C(\alpha)\Big) d\alpha + \frac{1}{2}\Big(\partial_\alpha^2 \Delta C(\alpha)\Big) d\alpha^2 + \mathcal{O}(d\alpha^3),$$

determines first and second variations of the arclength difference,

$$\partial_\alpha \Delta C(\alpha) = \frac{(1-q^2)\sqrt{(1-q^2)^2 + p^2 q^2}}{2\, p^2\, q}$$
$$\text{and} \quad \partial_\alpha^2 \Delta C(\alpha) = \frac{(1-q^2)^2(1+q^2-5q^4+p^2q^4+3q^6)}{4\, p^4\, q^3\, \sqrt{(1-q^2)^2+p^2q^2}}.$$

These data are what we need to determine the certificate difference,

$$\mathcal{A}_E \circ \Delta C(\alpha) = \Xi^q_{\mathcal{E}} - \Xi^p_{\mathcal{E}} = \frac{(1-q^2)^3(1+3q^2) - p^2(1-q^2-2q^4+2q^6) + p^4q^4}{p^2\, q^3\, \sqrt{(1-q^2)^2+p^2q^2}},$$

and subsequently, the missing certificate function,

$$\Xi^p_{\mathcal{E}} = \Xi^q_{\mathcal{E}} - \mathcal{A}_E \circ \Delta C(\alpha) = \frac{p(1-\alpha)^4 + 2p^3(3-\alpha)(1-\alpha)\alpha - p^5(1+3\alpha-\alpha^2)}{\sqrt{(1-\alpha)^3(1-\alpha-p^2)^3\big((1-\alpha^2)+\alpha p^2\big)}},$$

a truly monstrous expression! Despite gruesome details, the zero sums $\mathcal{A}_E \circ \left(\frac{dC^p_{\mathcal{E}}}{dp}\right) - \frac{d\Xi^p_E}{dp} = 0$ can be checked via symbolic computation.

Transferring analysis from ellipses to elliptic curves, we can guess that dependence of differential dp/dq on parameter α causes area integrals $S^p_{\mathcal{C}}(P_1)$ and $S^q_{\mathcal{C}}(P_1)$ to have non-identical certificates. As in Fig. 11, the interior area is rectangular, with area $\Delta S(\alpha) = S^q_{\mathcal{C}}(P_1) - S^p_{\mathcal{C}}(P_1) = qp$, while the exterior triangle has area,

$$\Delta S(\alpha + d\alpha) - \Delta S(\alpha) = -\frac{1}{2}\frac{\partial p}{\partial \alpha}\frac{\partial q}{\partial \alpha} d\alpha^2 + \mathcal{O}(d\alpha^3) = -\frac{d\alpha^2}{8(1-p^2)(1-q^2)pq} + \mathcal{O}(d\alpha^3),$$

[24] The same expressions apply to any valid endpoint, so we omit subscript 1 on p and q variables.

to second order in $d\alpha$. Comparison with a formal expansion determines $\partial_\alpha^2 \Delta S(\alpha)$, and next,

$$\mathcal{A}_S \circ \Delta S(\alpha) = \frac{p^2 + q^2}{pq} = \frac{p}{q} + \frac{q}{p}.$$

Since the curve $\mathcal{C}(\alpha)$ transforms invariantly by $(p, q) \to (q, p)$, it is quite obvious to guess that either $\Xi_S^p = 1/\Xi_S^q = p/q$ or $\Xi_S^p = 1/\Xi_S^q = q/p$. In fact, the first alternative is correct, and the two zero sums, $\mathcal{A}_s \circ \left(\frac{dS_\mathcal{C}^p}{dp}\right) - \frac{d\Xi_S^p}{dp} = 0$ and $\mathcal{A}_s \circ \left(\frac{dS_\mathcal{C}^q}{dq}\right) - \frac{d\Xi_S^q}{dq} = 0$, are relatively easy to check thereafter. A third certificate Ξ_S^ϕ can be found by applying a similar analysis to triangular areas of Fig. 11. This analysis is left as an exercise for the interested reader.

Having gone through a few examples of Creative Telescoping in thorough detail, the notion of a certificate function can not be as foreign. Let us now offer a summary:

1. If Ostrogradsky-Hermite reduction closes, invariant matrices $\mathbf{U}$ and $\mathbf{V}'$ determine annihilator $\mathcal{A}_I$, without needing to calculate the corresponding certificate Ξ_I^t.

2. When certificate Ξ_I^t can be efficiently calculated, it is useful for quality analysis. Verification of $\mathcal{A}_I \circ \frac{dI}{dt} - \frac{d}{dt}\Xi_I^t = 0$ implies that $\mathcal{A}_I \circ I(\alpha) = 0$.

3. Two certificates Ξ_I^t and Ξ_I^u can be identical under change of coordinates, but generally they are not equal, and depend on choice of coordinates.

4. If the integral is geometric, then the certificate difference $\Xi_I^t - \Xi_I^u$ can be calculated by trigonometric means, after expanding the tangent geometry in powers of $d\alpha$.

The first two observations are already well known theorems in Creative Telescoping, while the last two, if not novel, are at least lesser known. It would be interesting to generalize upon the geometric interpretation and to promote points 3 and 4 to proper theorems; however, this is outside of our present scope.

We have a pragmatic perspective, and can agree that certificates are essential only at the level of extra rigor. Following point 1, knowledge of certificate Ξ_I^t is not necessary when constructing a function $I(\alpha)$, nor when evaluating $I(\alpha)$ at a particular value. In physics, it is often the case that a complete period $I(\alpha)$ is much easier to measure than any partial integral $\int_{t_0}^{t_1} dI/dt$. A few theorists will need to derive and verify $\mathcal{A}_I$. Once $\mathcal{A}_I$ is known as fact, any scientist can use the well-developed theory of ordinary differential equations to construct convergence-rated approximations to $I(\alpha)$. The business of calculating values to $I(\alpha)$ then requires much less thought on the user's end. How fortunate for them!

VIII. PROSPECTUS

During XVIII century, Latin was a *lingua franca* between scientific researchers throughout Europe. According to the Euler archive, Euler wrote at least nine articles under titles starting with the word *specimen*, two of which are immediate to the analysis above. Around the same time, Carl Linnaeus (1707-1788) began to publish *Systema Naturæ* (1735-1758), one of the founding documents of the modern taxonomic system in biology. Presently, the word specimen is more familiar in the Linnaen context, where it usually refers to a particular plant or animal, as collected from the wild. In mathematics, translation of specimen to "example" is now a ubiquitous preference, but we hope that the imperfect analogy between math and biology will not be entirely forgotten, and that collection and analysis of specimens will continue to contribute an important part to scientific research.

Ideas evolve as do plants and animals, though with entirely different constraints and rates of change. As it turns out, on planet Earth, a time sequence of accidental occurrences leads just as well to a tree of knowledge as to a tree of life. The locution that "every new answer, leads to a few new questions" is itself a suggestion of branching structure implied by the tree-of-knowledge metaphor. Should we attempt to transfer the techniques of phylogeny—the study of evolutionary relationships—from biology to domains of pure idea? There are reasonable arguments yes and no. Before dismissing the idea as completely impossible, let us attempt to justify an evolutionary diagram such as in Fig. 12. This abbreviated, evolutionary flow chart attempts to trace back the existence of elliptic integrals to a momentous idea of Archimedes of Syracuse (circa 287-212 BC). His specimen of a circle bounded by two, regular 96-sided polygons is an ancestral milestone and an immense progenitor. Without it, perhaps this current work would not have come fully into existence.

During antiquity, the ratio of a circle's circumference to it's diameter, the irrational number π, caused much frustration and eluded reasonable description. Finally after centuries of Greek thought, Circa 250 BC, the analysis of Archimedes's *Measurement of a circle* determined that $\frac{223}{71} < \pi < \frac{22}{7}$. This error bounded result can be refined to higher precision by choosing circumscribing polygons with more than 96 sides; however, the task of doing so is prohibitively complicated. By the time of Newton, Archimedes's technique eventually evolved to become arclength integration, so can be considered an

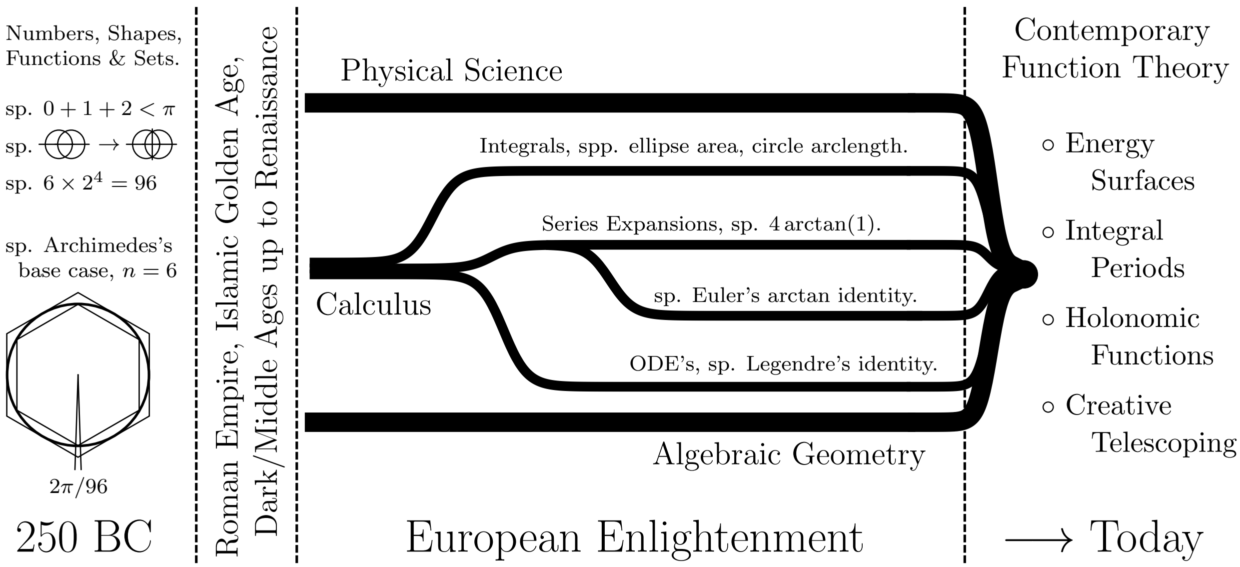

FIG. 12. Evolution of ideas about π (above). A drawer of closely related specimens (below).

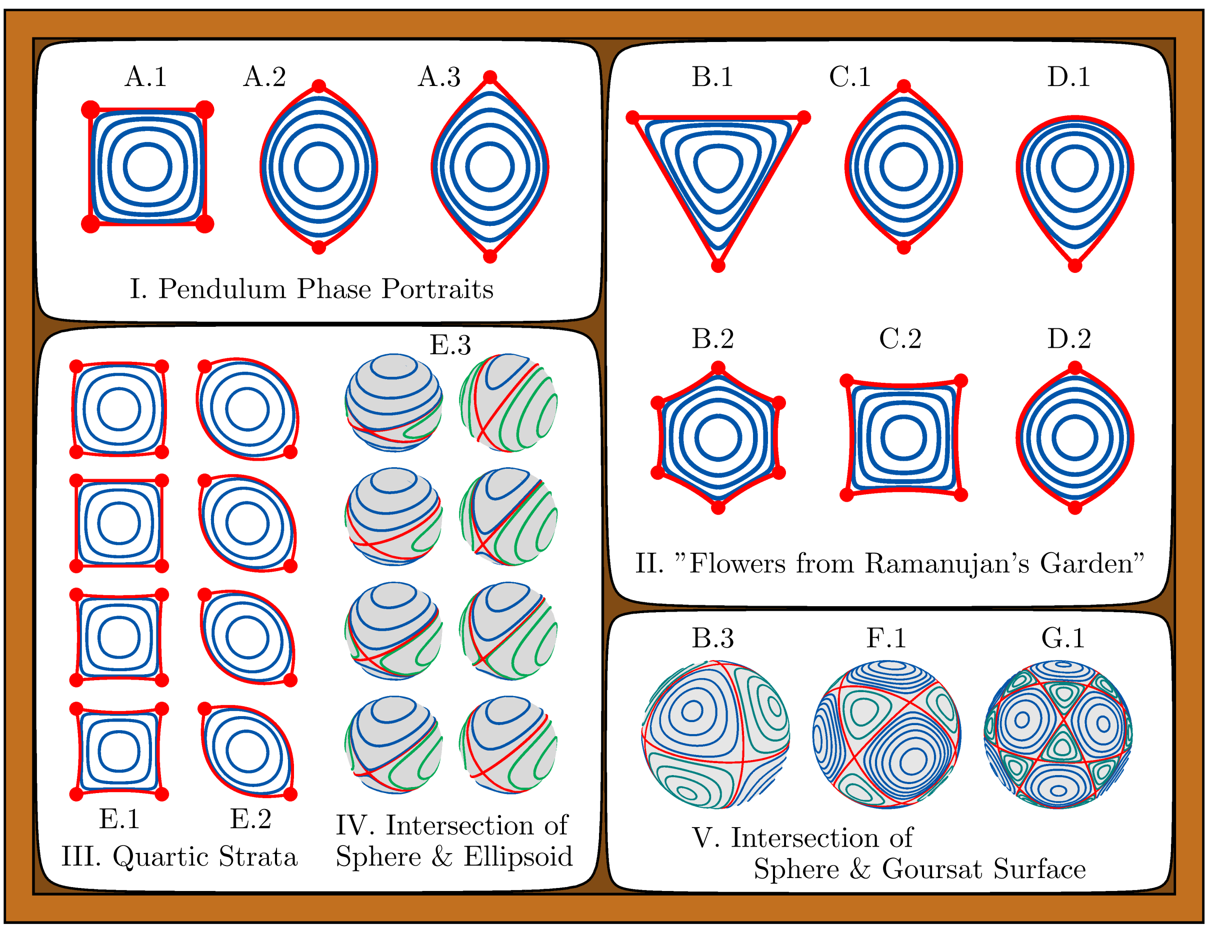

TABLE IV. Well-Integrable Hamiltonian Energy Functions and their Period Constraints.

Index	Hamiltonian Function	Diagnostic	Period Constraint
I.A.1	$2H = p^2 + q^2 - p^2 q^2$	Q D	$0 = T - \partial_\alpha\big(4\,\alpha(1-\alpha)T'\big)$
I.A.2	$2H = (p^2+q^2)(1-\frac{1}{4}q^2)$	Q D	$0 = T - \partial_\alpha\big(4\,\alpha(1-\alpha)T'\big)$
I.A.3	$2H = p^2 - \sin(q)^2$		$0 = T - \partial_\alpha\big(4\,\alpha(1-\alpha)T'\big)$
II.B.1	$2H = p^2 + q^2 + (\frac{4}{27})^{\frac{1}{2}}\,(3p^2 q - q^3)$	D	$0 = 2T - \partial_\alpha\big(9\,\alpha(1-\alpha)T'\big)$
II.C.1	$2H = p^2 + q^2 - \frac{1}{4}q^4$	Q H	$0 = 3T - \partial_\alpha\big(16\,\alpha(1-\alpha)T'\big)$
II.D.1	$2H = p^2 + q^2 - (\frac{4}{27})^{\frac{1}{2}} q^3$	H	$0 = 5T - \partial_\alpha\big(36\,\alpha(1-\alpha)T'\big)$
II.B.2	$2H = p^2 + q^2 - \frac{4}{27}(p^2+q^2)^3 + \frac{4}{27}p^2(p^2-3q^2)^2$	D	$0 = 8\,\alpha\,T - \partial_\alpha\big(9\,\alpha(1-\alpha^2)T'\big)$
II.C.2	$2H = p^2 + q^2 - \frac{1}{4}(p^2+q^2)^2 + 2p^2q^2$	Q D	$0 = 3\,\alpha\,T - \partial_\alpha\big(4\,\alpha(1-\alpha^2)T'\big)$
II.D.2	$2H = p^2 + q^2 - \frac{4}{27}q^6$	H	$0 = 5\,\alpha\,T - \partial_\alpha\big(9\,\alpha(1-\alpha^2)T'\big)$
III.E.1	$2H = p^2 + q^2 - \frac{1}{4}(p^2+q^2)^2 + \epsilon\,p^2q^2$	Q D	$0 = \big(3\,\alpha(\epsilon-1) - \epsilon + 2\big)T$
III.E.2	$2H = p^2 + q^2 - \frac{1}{4}(p^2+q^2)^2 + \frac{1}{4}\epsilon(p^2+q^2)q^2$	Q D	$-\partial_\alpha\big(4\,\alpha\,(1-\alpha)(1-\alpha+\alpha\epsilon)T'\big)$
IV.E.3	$H = a\,J_x^2 + b\,J_y^2 + c\,J_z^2$	D	$0 = (a+b+c-3\alpha)T$
(cont.)			$+\partial_\alpha\big(4(a-\alpha)(b-\alpha)(c-\alpha)T'\big)$
V.B.3	$H = J_z^3 + \frac{\sqrt{2}}{2}(J_x^3 - 3J_xJ_y^2) - \frac{3}{2}(J_x^2J_z + J_y^2J_z)$	D	$0 = 8\,\alpha\,T - \partial_\alpha\big(9\,\alpha(1-\alpha^2)T'\big)$
V.F.1	$H = 4(J_xJ_y)^2 + 4(J_yJ_z)^2 + 4(J_zJ_x)^2$	D	$0 = 9\,(4-5\,\alpha)\,T$
(cont.)			$-\partial_\alpha\big(16\,\alpha(1-\alpha)(4-3\,\alpha)T'\big)$
V.G.1	$H = J_z^6 - 5(J_x^2+J_y^2)J_z^4 + 5(J_x^2+J_y^2)^2J_z^2$	D	$0 = 5\,(5-21\,\alpha)\,T$
(cont.)	$-2(J_x^4 - 10J_x^2J_y^2 + 5J_y^4)J_xJ_z$		$+\partial_\alpha\big(4\,\alpha(1-\alpha)(5+27\,\alpha)T'\big)$
Hotaru	$H = \frac{1}{3}(J_x^4 + J_y^4 + J_z^4) - (J_xJ_yJ_z)^2$	D	$0 = 5(1190 - 13149\,\alpha + 18954\,\alpha^2)T$
mirabilis	$-\frac{2}{27}(J_x^2 + J_y^2 + J_z^2)$		$-\partial_\alpha\big(36\,\alpha(1-\alpha)(7-27\,\alpha)(5-54\,\alpha)T'\big)$

Diagnostic Key: `Q`=`QuarticToODE`, `H`=`HyperellipticToODE`, `D`=`DihedralToODE`. See also Appendix xxx.

ancestor to the more familiar $\pi = \int_{-1}^{1}(1-x^2)^{-1/2}dx$. In the first generation of calculus, it was also possible to expand π in series, $\pi = 4\sum_{n\geq 0}\frac{(-1)^n}{2n+1}$, or alternatively to write $\pi = 4\arctan(1)$. These definitions, though correct, are not very fit in the sense that they converge too slowly. Euler subsequently gave a series with improved convergence, $\pi = 4\Big(4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{70}\right) + \arctan\left(\frac{1}{99}\right)\Big)$; however, even this approach was quickly out-competed by another. The Legendre identity, $K(\alpha)E(\wp)+K(\wp)E(\alpha)-K(\wp)K(\alpha) = \frac{\pi}{2}$, underpins the current best-practice for computing digits of π, so falls on another branch of the evolutionary diagram in the upper portion of Fig. 12.

We choose to describe Legendre's identity as a consequence of differential equations, because the annihilators $\mathcal{A}_E, \mathcal{A}_K$, and $\mathcal{A}_S$ are fundamental to its definition and proof. In our notation, it is better to replace E and K by S and T, which puts Legendre's identity into a more symmetric form, $S(\alpha)T(\wp) + S(\wp)T(\alpha) = 8\pi$. First, the left-hand side is determined constant, $\partial_\alpha\big(S(\alpha)T(\wp)+S(\wp)T(\alpha)\big) = 0$. This follows from the chain rule with $T(\alpha) = 2\partial_\alpha S(\alpha)$, $S(\alpha) = 2\alpha(1-\alpha)T(\alpha)$. Second, a limit is evaluated,

$$\lim_{\alpha\to 0} S(\alpha)T(\wp) + S(\wp)T(\alpha) = \lim_{\wp\to 1}\Big(\wp \times 2\log(\wp)\Big) + 4\times 2\pi = 8\pi,$$

by canceling the first term to zero, and then by multiplying harmonic frequency 2π times square area 4. The proof is complete, so we can calculate an approximation of π,

$$\pi = \frac{8\pi^2}{\underset{pw}{T}(\frac{1}{2})\underset{pw}{S}(\frac{1}{2})} = 4\left({}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix}\middle|\frac{1}{2}\right]{}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\\2\end{matrix}\middle|\frac{1}{2}\right]\right)^{-1} \approx 3.14159265358979323848...,$$

with only the last digit incorrect due to truncation of the series after 60 terms. Choosing $\alpha = 1/100$ improves the approximation to accuracy beyond the fiftieth decimal place. If more precision is necessary, it is usually better to calculate $T(\alpha)$ and $S(\alpha)$ via the arithmetic-geometric mean (Cf. []).

The lineage of π is also a good example for showing a fallacy of the analogy between evolutionary trees. In zoology, it is never the case that a specimen of Coleoptera can identically equal a specimen of Lepidoptera, preposterous! Although both orders are arthropods and hexapods, beetles have hard shells and hidden wings, while butterflies are more delicate with obvious, exposed wings. At the finer level of classification by species, the ultimate test is whether or not two individuals can successfully mate. In science, capability to mate is not at all a criterion for identity, nor for correctness. One purpose for mathematical proofs is to show that seemingly different expressions are indeed identical, or *isomorphisms*. Expressions equal to π are in isomorphism to one another, and this fact contradicts the basic idea of an evolutionary diagram. In summary, the descent of biological species only diverges, while the descent of ideological species diverges and converges with comparable rates. This is not a theorem, but an important observation nonetheless.

When scientific ideas coalesce originally, a new field is born. Experimentation within the field leads to all sorts of novelties: examples, diagnostics, hypotheses, and eventually to theorems and proofs. This is currently happening with the field of *Integral Functions*. Here, Creative Telescoping algorithms are leading the way. We have already seen a few nice examples of historical importance, the well-known elliptic integrals. These are not indicative of the entire range of possibilities. According to Lairez and others, Creative Telescoping applies to any rational integral, after generalizing Ostrogradsky-Hermite reduction to its multivariate form, the Griffiths-Dwork reduction. Development of broadly general methods counts as progress made toward answering big questions such as the Hodge conjecture or the Bombieri-Dwork conjecture[25]. Yet general progress comes at the expense of lost accessibility, and the general is not always preferable to the special. For the purposes of widespread and public enlightenment, it is also worthwhile to ask: what are the most interesting examples where the simple Ostrogradsky-Hermite reduction suffices? We will continue to explore this question through a dissertation in mathematical physics.

So far we have not defined the phrase ”well-integrable”, which appears in the title of this article. It is not a precise scientific term, rather a witticism about XVIII century history and language. Contemporary to Euler and Linnaeus lived Johan Sebastian Bach (1685-1750), the famous Baroque composer. Bach's masterwork *Das wohltemperierte Klavier* (1722) is a source for ”præludia und Fugen durch alle Tone und Semitonia”[26]. The German word ”wohltemperierte” means well-tempered. It is a diagnostic term describing quality and completeness of a particular musical sample. There is no reason to worry about the exact meaning of either term ”well-tempered” or ”well-integrable”. Instead, we give the examples of Fig. 12 and Tab. IV, and allow the observer to form his or her own opinions. It is tempting to wonder, do the letters A-G have something to do with key signatures in western

[25] totaro ref?

[26] Almost beyond belief, András Schiff plays the entire Book I by memory on youtube. This is good way to experience music that inspired many subsequent fugues, including Hofstatder's *Godel, Escher, Bach.*

music? In good humor, the answer is yes, in sober scientific explanation, no.

Let us briefly explain one difference between musical and scientific indexing schemes. According to the order of sharps, the musical keys have a second arrangement, FCGDAEB, which is more telling than the alphabetic arrangement. In this arrangement, the second "Crystal Clear" key of C is usually thought of as a starting place. Indeed, Bach chooses C Major as they key of the first prelude and fugue in *Das wohltemperierte Klavier*. In our presentation of the few well-integrable geometries, the initial example, chosen as A, is followed immediately by similar examples B,C, and D, subsequently generalized by the examples of E[27]. Only then is it possible for us to exceed the learning curve and complete skills development by transferring analysis to examples F and G. Unlike the musical system, the letter indices are neither finite nor cyclic. There could be an VI.H.1 (not depicted in Fig. 12) for the mysterious geometry *Hotaru mirabilis*. We might have listed a few more in Tab. IV, but simply ran out of room on the page.

All jokes aside, our chosen examples A-G must belong together. Many identities between them support appropriateness of the arrangement, and relative completeness can be explored through combinatorial analysis. The most obvious similarity is that, according to the diagnostic algorithms `HyperellipticToODE` and `DihedralToODE`, all derived period constraints are second-order ordinary differential equations. Examples A-D are all of the hypergeometric type, while examples E,F, and G are Heun equations, i.e. they have exactly four regular singular points. The cohesion is even stronger, because annihilators of the respective period functions all have the form $\mathcal{A}_T = a_0(\alpha) + \big(\partial_\alpha a_2(\alpha)\big)\partial_\alpha + a_2(\alpha)\partial_\alpha^2$, with just two polynomial coefficients $a_0(\alpha)$ and $a_2(\alpha)$. This allows factoring of the corresponding ODE, as in Tab. IV. Previously, Fritz Beukers and Don Zagier explored this form in connection with Apéry's proof of the irrationality of $\zeta(3)$. Zagier also found the set A-D as complete, but the limited scope of his massive search precluded the possibility of finding any of the subsequent examples from Tab. IV. All of the geometries listed involve "integrality miracles", most notably *Hotaru mirabilis*[28].

Yet we must doubt our own bias, and act as potential naysayers to ourselves by asking: Why the results of Fig. 12 and Tab. IV? Granted, some importance derives from relation

[27] The letters A-D were also used by Almkvist and von Straten. Letter E is the first of Euler's family name. Euler was among the first to study rigid body rotations, leading eventually to IV.E.3.

[28] Nomenclature: This east-meets-west name commemorates discovery "by the light of the fireflies", and that the period ODE begets integrality miracles around all four finite-valued, regular singular points.

to leading mathematical theories, such as the KZ-Period theory, Creative Telescoping, Cohomology, etc. These theories are only part of the picture, high and far away from the concerns of most scientists. Ultimately our answer why derives from relation to leading physical theories, where integral periods are also laboratory observables. This is true throughout Hamiltonian classical mechanics, and more specifically in the subsequent theory of semi-classical quantum mechanics. Up until the present analysis it has not been very well understood that it is possible to calculate a semi-classical level spectrum by root-solving the solution to an ODE. This is a central "theorem" that will arise out of the subsequent exposition. By developing the examples E, F, and G, we can improve upon the rigor of the original analysis by Harter (cf. [] [] []), and set up the theory of Rotational Energy Surfaces to grow in new directions parallel to those of pure mathematics. This will require a lot of hard work and effort on the readers part, but the payoff is immense. Not only are the results useful in physics calculations, they are also beautiful in their own right!

This prelude already gives detailed analysis of I.A.1, and in the next article, we will start the development of Hamiltonian mechanics in order to prove I.A.2 and I.A.3 equivalent to I.A.1. The development of Hamiltonian mechanics continues with an analysis of examples II.B.1, II.C.1, and II.D.1, which finds that the periods of each obey an analog of the Legendre identity associated to I.A.1. We will also show a remarkable set of identities, that the period constraints of II.B.2, II.C.2 and II.D.2 relate to those of their predecessors by a change of variables $\alpha \to \alpha^2$. Before verifying V.B.3 isoperiodic to II.B.2, it is pertinent to introduce the E examples, where another amazing isoperiodicity waits to be found. Under transformation $\epsilon \to \frac{c-a}{c-b}$, the period constraint of examples III.E.1 and III.E.2 transforms to that of IV.E.3. Example IV.E.3 is the first case of sphere curves rather than plane curves, so we will have to explain exactly how sphere curves relate rigid body or semi-rigid body rotation. More symmetrical sphere curves are given in examples V.F.1 and V.G.1. These geometries should be easy to master once the progression from A-E is complete. It will take more time and effort. At the end of it, with time expiring, we will have to sit and watch the small mystery of *Hotaru mirabilis* drift away into the darkness around. Having even mentioned the fireflies—actually they are usually beetles of the family Lampyridae—we are already falling behind schedule! Let the voices of mathematics and physics intertwine. Without further ado...