Geometric Interpretation of a few Hypergeometric Series

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Abstract

Ramanujan's article "Modular equations and approximations to π" draws special attention to three curious hypergeometric series. In mathematical physics, each series determines periods along the elliptic level curves of an integrable Hamiltonian surface. We perform a combinatorial search of Hamiltonian function space and find a diverse collection of closely related species, some of which may have been unknown previously. The search is made rigorous by here defined diagnostic algorithms `ExpToODE`, `HyperelipticToODE` and `DihedralToODE`. All three take an input integral function and output the relevant ordinary differential equation. Examples growing out of Ramanujan's theory have intrinsic value, but they are not the only interesting use cases of the diagnostic algorithms. What we can learn from this "walk through Ramanujan's garden" will ultimately help us to extend systematic analysis elsewhere.

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I. HISTORY AND INTRODUCTION

The aphorism "history tends to favor those who wrote it" applies no less in particular to history of science. Ideally, science should be an apolitical process of finding out the truth by reason and experimentation. Practically, we can always expect to hear another anecdote of unfair practice and needless exclusion, even if we only listen to the inner workings of the dominant western system. Eurocentrism and Anglocentrism present an even more significant problem to the world. Western scientists can, and often do, ignore developments and contributions of non-western origin. Our discussion of classical science thus far repeats the mistake of exclusive biasing by focusing solely on European lineages [1, 2]. We will now begin to correct for this mistake by discussing a few earlier developments of India.

The biased[1] slogan *Asian ideas matter* is especially relevant when we grapple with even a small part of the theory of Srinivasa Ramanujan (1887-1920). From what we know about Ramanujan's heritage, we can easily understand that his life and work built upon thousands of years of Indian culture[2]. To buck Anglocentrism, we can look at Ramanujan's notebooks, and then ask, which of his basic ideas and techniques were already known in India before the invasion of the British East India Trading Company? Less than two decades after Ramanujan's untimely death, A.N. Singh published an imminently useful, English-language review, "On the use of series in Hindu mathematics"[3][3]. This article clearly shows that Ramanujan *was not* the first Indian scientist to take an interest in binomial coefficients, series expansions, or the transcendental constant π. That being said, Ramanujan *was* among the first to work directly with the western scientific world. Two histories need to be compared.

The binomial coefficients are an irrevocably important set of numbers denoted $\binom{n}{k}$, usually for integer n and k with $0 \leq k \leq n$. In combinatorics, $\binom{n}{k}$ determines the number of ways to choose k distinct elements from a set of cardinality n, and the identity $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ counts out the answer in terms of the factorial function $n! = n \times (n-1) \times (n-2) \times \ldots \times 1$ and $0! = 1$. Algebra gives an alternative, additive definition, $\binom{n}{0} = \binom{n}{n} = 1$ otherwise $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, which follows from the expansion $(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k = (1+z) \sum_{k=0}^{n-1} \binom{n-1}{k} z^k$. It is quite natural to view the binomial coefficients as elements of a triangular array, as in Fig. 1. When

[1] Compare with "Hsin Shin Ming", "Nagarjuna's four propositions and zen", available at https://kansaszencenter.org/resources/.

[2] Robert Kanigel has written a biography of Ramanujan's life, *The man who knew infinity*. The book was recently adapted to full-length motion picture under the same title.

[3] A scan is available through Erv Wilson's electronic library, http://www.anaphoria.com/library.html.

the integers are arranged in this particular way, each row determines each next row by the addition rule. In the west, such an arrangement goes by the name "Pascal's triangle". A.N. Singh tells us that the same tabulation was derived in India under a poetic heading, *Meru Prastāra*, "the Steps of Meru", possibly as early as Pingala's time circa 200 BC. In parts of the old Indian cosmological system, Meru plays a role similar to that of Parnassus in the old Greek cosmology—It is the center of the poetic universe. Pingala asked the question, given a verse of n syllables divided into k *lagu*, meaning light, and $n-k$ *guru*, meaning heavy, how many orderings are possible? Not only did Pingala know the answer in terms of *Meru Prastāra*, he also knew the row sums $\sum_{k=0}^{n}\binom{n}{k}=2^n$[3]. This was an important discovery for Indian science, and thereafter Sanskrit prosody became a standard format.

Not much is known about the life of Pingala, but the fact that he wrote high-minded Sanskrit suggests a northern origin, possibly in one of the larger cities on the Ganges river. The implicit allusion of *Meru* to the Himalayan mountains seems to support this hypothesis; however, conclusive evidence is lacking. During second and third century BC, the Maurya empire included much of the Indian subcontinent, but did not extend all through the southern peninsula to the Indian ocean. Excluded territory to the far south roughly aligns with two present-day states, Kerala and Tamil Nadu. More is known about Pingala's follower, Halayudha. During X Century, he lived in Karnataka, and later moved 1,000 kilometers northward to Ujjain, where he could accomplish his important exposition of Pingala's earlier work on *Meru Prastāra* (again, see [3]). This gives some idea of the extent to which ideas could move on a hundred year time scale. After another few centuries, an important mathematical school, versed in Sanskrit prosody, would finally arise in the southernmost state of Kerala.

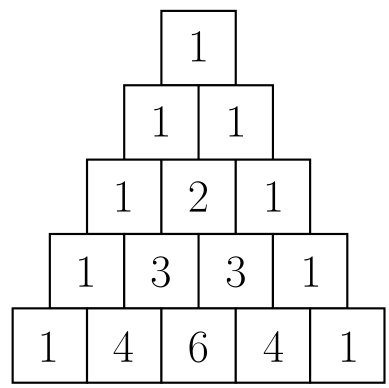

FIG. 1. Pingala's *Meru Prastāra*.

Founding of the Kerala school of Astronomy and Mathematics is associated with the life and work Madhava of Sangamagrama (circa 1340-1425). It is now commonly accepted that Madhava originated the poetic definition of series expansions for sine, cosine[4]. At the time, the standard educational procedure was to transmit knowledge mainly by spoken word. The teacher would recite his verses directly to the students, and rigor would only be done if the listeners could think of an objection to the master's teaching (S.P. Singh told me as much by email). Due to this practice, the written record is not as helpful as we would like; although, over time, more details became available through the notes of subsequent students working in the lineages at Kerala[5]. Their accomplishment is too profound to belittle by imposing an exogenous western bias. Without any notion of calculus, Indian mathematicians at Kerala preceeded the greatest western scientists at the central task of exactly defining the transcendental constant π in terms of an infinite series expansion[4].

Given that knowledge was already sufficiently mobile and moving southward, it is entirely probable that Hindu science had reached Tamil Nadu by XIX Century. Ramanujan was born in Tamil Nadu on December 22, 1887, and attended primary school there. Were benevolent Hindu nationalists able to influence Ramanujan's early development as a mathematician? We can only say that Ramanujan, as a member of the Brahmin caste, had a right to receive Hindu science, not that he actually did. At this time, the traditions of India became undermined by British interventionism, which included a commandeering of the educational system. During the same years, Mahatma Gandhi (1869-1948) developed the strategy of *Satyagraha*[4], which would eventually win back Indian independence, though not during Ramanujan's short lifetime. Sadly, we do not know if Ramanujan was aware of Pingala or Madhava by the start of his research into western mathematics. Regardless of what resources he could access in the material world, Ramanujan became the natural successor of India's earlier number theorists. Instead of linguistic poetry, he wrote poetic equations. His Indian notebooks are filled with thousands of cryptic entries, which have resisted attempts to translate or decipher.

The story about Ramanujan's ascendance is, as Hardy called it, romantic. It provides a stark contrast to other goings-on around Europe and the rest of the world. A few months after Ramanujan's arrival in England, the Archduke Franz Ferdinand of Austria was assassinated on 28 June, 1914 by Gavrillo Princip, a murderous agitator fighting on behalf of the Bosnian independence movement. There certainly were Hindu nationalists who advocated violent rebellion against colonial forces (as did Americans during the earlier American revolution), but Ramanujan was not one of them. Having learned from the books

[4] In his monumental "I Have a Dream" speech, Martin Luther King Jr. echoed Ghandi by saying *"We must not allow our creative protest to degenerate into physical violence. Again and again we must rise to the majestic heights of meeting physical force with soul force."*

of Cambridge mathematicians, and with encouragement from his countrymen, Ramanujan sought out approval from the Cambridge Dons. The Royal Society answered Ramanujan's letters through the person of G.H. Hardy, himself an eccentric genius of relatively modest upbringing. Ramanujan was invited to attend Cambridge, and after three days meditating at a temple in Namakkal, he received permission to accept the invitation. His life thereafter is more well documented. Some of his ideas were properly published, though not as many as could have been. Posisbly during a brief stint as a mendicant, Ramanujan is thought to have contracted a long lasting form of Dysentery. He later died a tragic and untimely death at the young age of 32.

Regrettably, the effort to understand Ramanujan's educational circumstances started only after his death, with the obituary of Seshu Aiyer and Ramachandra Rao, and Hardy's short biography "The Indian mathematician Ramanujan"[6, 7]. Neither article says anything about Pingala or Madhava, but both mention Euler in connection with Ramanujan's early interest in sine and cosine functions. Hardy's account also claims that "Carr's *Synopsis*...first aroused Ramanujan's full powers". We don't know if this Hardy assertion is true, but it sounds possibly overstated due to Anglocentric bias. Nevertheless, it is an important historical fact that Ramanujan studied Carr, because *Synopsis* defines figurate numbers next to hypergeometric function ${}_2F_1$ in items 289-292. In fact, Item 290 gives a tabulation, in a slightly different form, of the first few rows of Pascal's Triangle. Carr, an Englishman, mentions neither Pascal nor Pingala. Even if Ramanujan was unaware of his own nation's historical contributions to science, he would have found the same ideas in Carr; admittedly, with wrong attribution.

For a period of time, interest in Ramanujan languished. In the late 1970's, Bruce Berndt and coworkers revived the mission of editing Ramanujan's notebooks. An overview of their work is given in [8]. It shows that Ramanujan was interested in many topics, and that he paid quite a lot of attention to the theory of elliptic functions. According to a bibliography of Ramanujan's primary sources[9], he had access to A.G. Greenhill's treatise on elliptic functions[5], and at first, found some interest in the period function of the simple pendulum, the complete elliptic integral of the first kind[6], here denoted by symbol $K(\alpha)$. Legendre found an identity, $K\big(\sin(5\pi/12)^2\big) = \sqrt{3}K\big(\sin(\pi/12)^2\big)$, which piqued Ramanujan's interest

[5] Scans are available online via archive.org, including Carr's *Synopsis* and Greenhill's *Elliptic Functions*.

[6] We have already rigorously defined $K(\alpha)$ in [1, 2]. Another certificate is given in Section IV.

and led him to record a few original results in his notebooks[10] (and see also item 13 of Hardy's obituary [7]). Not only did Ramanujan at times exceed the preexisting theory of elliptic integrals, he also broke completely free of its confines. Section 11 of Berndt's overview pertains to Ramanujan's theory of alternative bases, which is also described in two full length articles [11, 12]. An important takeaway from these accounts is that entries of Ramanujan's notebooks show he had already discovered existence of the three alternative theories *while living in India*. This both does and does not help to explain how he could rapidly publish the article "Modular Equations and approximations to π", much less than one year after arriving at Cambridge University[7][13].

Following the poetry of Freeman Dyson (1923-2020), Berndt also wrote a lovely invitation to catalogue the "Flowers which we cannot yet see growing in Ramanujan's garden of hypergeometric series, elliptic functions, and q's"[8] [14]. In this work of outlook (or is it actually introspection?), mathematical loose ends of Ramanujan's theory of alternative bases take eminent position in section 2. There it says,

> In his famous paper [Modular equations and approximations to π], Ramanujan records several elegant series for $1/\pi$ and asserts "There are corresponding theories in which q is replace by one or other of the functions"
>
> $$q_r := q_r(x) := \exp\left(-\pi\csc(\pi/r)\frac{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;1-x)}{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;x)}\right),$$
>
> where $r = 3, 4$, or 6 and where ${}_2F_1$ denotes the classical Gaussian hypergeometric function.

Neither in the original notebooks, nor in the journal article, nor in subsequent editing did explicit mention of elliptic curves play a very significant role. However, Berndt and colleagues took a significant step in this direction by changing Ramanujan's Euler-inspired notation,

$$\begin{aligned}
K_1 &= 1 + \frac{1\cdot 3}{4^2}k^2 + \frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}k^4 + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{4^2\cdot 8^2\cdot 12^2}k^6 + \dots,\\
K_2 &= 1 + \frac{1\cdot 2}{3^2}k^2 + \frac{1\cdot 2\cdot 4\cdot 5}{3^2\cdot 6^2}k^4 + \frac{1\cdot 2\cdot 4\cdot 5\cdot 7\cdot 8}{3^2\cdot 6^2\cdot 9^2}k^6 + \dots,\\
K_3 &= 1 + \frac{1\cdot 5}{6^2}k^2 + \frac{1\cdot 5\cdot 7\cdot 11}{6^2\cdot 12^2}k^4 + \frac{1\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17}{6^2\cdot 12^2\cdot 18^2}k^6 + \dots,,
\end{aligned}$$

[7] A complete bibliography is available online at https://www.imsc.res.in/~rao/ramanujan/.

[8] Our own thoughts are along the lines of "Parametric curves / which cannot yet be seen to grow / in a garden of Mystery", but we will defer to historical precedent until a better consensus can be reached.

to that of the standard hypergeometric theory. Since we know that the hypergeometric function, whether it is due to Gauss or Euler, is the solution of a second-order ordinary differential equation(Cf. [15], Section 2), we can immediately hypothesize that Ramanujan's original assertion is equivalent to an assertion that

> *There are corresponding theories in which the underlying elliptic curve geometry is replaced by one or other of the curve families $\mathcal{X}_3(\alpha)$, $\mathcal{X}_4(\alpha)$, $\mathcal{X}_6(\alpha)$.*

In this hypothesis the unknowns $\mathcal{X}_s(\alpha)$ are Riemann tori over the complex numbers. As their shape varies with α, the real and complex periods must be solutions of the hypergeometric differential equation[9],

$$\mathcal{A}_s \circ T_s(\alpha) = 0, \quad \text{where} \quad \mathcal{A}_s = (s-1) - s^2(1-2\alpha)\partial_\alpha - s^2\alpha(1-\alpha)\partial_\alpha^2.$$

For $s = 2, 4$ and 6 geometric models were known classically, and maybe a model for the difficult case $s = 3$ was also known to a small cadre of European cognescenti(?). Nevertheless, a systematic exposition was missing from the literature, until the issue was taken up by L.C. Shen. In a series of articles [16–19] he revealed that the relevant geometries could be obtained from the Chebyshev polynomials, amazing! We may not be too surprised to find that the mysterious garden contains even more as yet unseen. The unmentioned alternatives have different symmetries, different genera, and perhaps even different fragrances, so they are also deserving of correct and individual diagnosis.

Presently the theory of Creative Telescoping gives us new tools for systematizing and automating analysis of integrals taken along deformable curves[10]. In a most general form, as recently discussed by Bostan and Lairez, the complications are many[21, 22]. However, when it is possible to reduce a geometric integral to a univariate form, simple Hermite reduction can be used effectively. The goal is to take a curve, say $\mathcal{X}(\alpha) = \{(p,q) : 2H(p,q) = \alpha\}$, write the period integral $T(\alpha) = \oint_{\mathcal{X}} dt$, and instead of evaluating $T(\alpha)$ directly, to find an annihilating operator $\mathcal{A} \in \mathbb{Q}[\![\alpha, \partial_\alpha]\!]$ (also called a telescoper) such that $\mathcal{A} \circ T(\alpha) = 0$. Annihilation of the period function happens when $\mathcal{A} \circ dt - \partial_t \Xi = 0$, with certificate Ξ a function of t. If the annihilator and its certificate are known, the task of evaluating the

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[9] Our notation is different: r is reserved for radius, s is for signature, and α is the default expansion parameter. For a detailed working of case $s = 2$, see [1] Section V and [2].

[10] For a list of relevant references, trackback from the list given in [1] and see also [20].

integral $T(\alpha)$ can be changed for a similar task of solving an ordinary differential equation. This is a much easier approach and allows us to efficiently search for special $\mathcal{X}(\alpha)$.

In two previous articles (or chapters), we have already discussed the standard case $s = 2$ in detail[1, 2]. The appropriate period function, $T_2(\alpha) = 4K(\alpha) = 2\pi\, {}_2F_1\left[\begin{smallmatrix}\frac{1}{2},\frac{1}{2}\\1\end{smallmatrix}\middle|\alpha\right]$, can be written in terms of the central binomial coefficients $T_2(\alpha) = 2\pi \sum_{n\geq 0} \frac{1}{16^n}\binom{2n}{n}^2\alpha^n$. If we throw away the denominators $\frac{1}{16^n}$, and search the Online Encyclopedia of Integer Sequences (OEIS)[23] for the integer expansion coefficients, then we find entry A002894. Clicking through the cross references, we might also find A006480 and A000897, and both entries cross reference to A113424. Though historically unusual[11], this is still a great and easy way to find out more about Ramanujan's alternative series,

$$T_3(\alpha) = {}_2F_1\left[\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\middle|\alpha\right] = \sum_{n\geq 0}\frac{1}{27^n}\binom{3n}{n}\binom{2n}{n}\alpha^n,$$

$$T_4(\alpha) = {}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{3}{4}\\1\end{matrix}\middle|\alpha\right] = \sum_{n\geq 0}\frac{1}{64^n}\binom{2n}{n}\binom{4n}{2n}\alpha^n,$$

$$\text{and}\qquad T_6(\alpha) = {}_2F_1\left[\begin{matrix}\frac{1}{6},\frac{5}{6}\\1\end{matrix}\middle|\alpha\right] = \sum_{n\geq 0}\frac{1}{432^n}\binom{3n}{n}\binom{6n}{3n}\alpha^n,$$

in Ramanujan's notation K_2, K_1, and K_3 respectively. Written in this way, the dependence on Pingala is clear, but mystery remains. Why should these particular binomial products matter at all, and why only these three? These questions are motivation enough for the present work, but a little more will be said about π and the Kerala connection.

When calculating and storing proof data on a computer, we are not limited by paper shortages. Nothing but time will prevent us from recording as many results as we can. The combinatorics of Section II can be skipped, but it leads quite naturally into section III where most of the rigorous analysis is done by a few different implementations of an algorithm `EasyCT`. Requiring concordance between alternative implementations, not only do we easily find three relevant families of elliptic curves, we find that these three are somehow the lone minimal examples (except for a few higher-dimensional reflections). New geometric definitions allow quick and easy proof of Legendre-style identities, which gets us into Chapter 1 of "π and the AGM"[24]. It is tempting to keep going in a pure maths direction, but by the end, we will need to steer ourselves back toward physics calculations. Again, mathematical loose ends may be appreciated for what they are worth.

[11] When, before now, have we ever made such interesting maths discoveries by clicking a hyperlink?

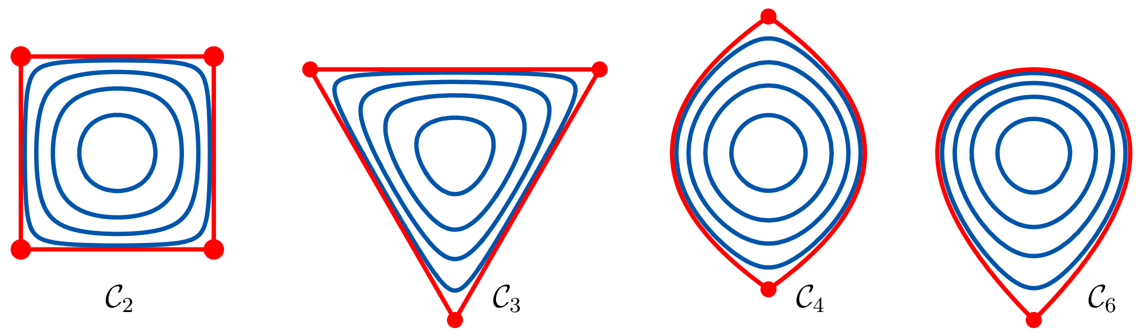

FIG. 2. Elliptic oscillation disks, with $T_s(\alpha) = 2\pi\, {}_2F_1\left[\begin{smallmatrix} \frac{1}{s}, \frac{s-1}{s} \\ 1 \end{smallmatrix} \middle| \alpha\right]$ for $s = 2, 3, 4, 6$.

II. CREATIVE COMBINATORICS

We have already shown that a few different Hamiltonian functions determine the simple pendulum's libration behavior, and that transformation theory accounts for equivalence between the alternative forms[2]. Now it is due time to continue developing Hamiltonian mechanics by considering oscillation disks in more generality. As a breif reminder, an *oscillation disk* is a topological disk taken from the phase plane, which is bounded at its center by a circular point, and bounded on its outer edge(s) by a separatrix curve and at least one hyperbolic point[12]. Figure 2 shows four examples with blue level curves,

$$\mathcal{C}_s(\alpha) = \Big\{ (p, q) : \alpha = 2H_s(p, q), \quad s = 2, 3, 4, \text{ or } 6 \Big\} \qquad \text{with} \qquad \alpha \in (0, 1).$$

The four geometries differ in symmetry; however, any curve $\mathcal{C}_s(\alpha)$ is an elliptic curve of genus one. In fact, the curves $\mathcal{C}_2(\alpha)$ to far left are determined by the physicist's version of Edwards's normal form[25], $2H_2 = p^2 + q^2 - p^2q^2$. For other examples, the assertion of genus 1 can be proven a few different ways using the standard theory of elliptic curves[13].

Let us start with the more familiar cases where kinetic energy and potential energy are separable, i.e. cases that have $H(p, q) = \frac{1}{2}p^2 + V(q)$ in units where $m = 1$. Choosing the potential energy function $V(q) = \frac{1}{2}q^2 + \frac{1}{2}\sum_{n=3}^{N} v_n q^n$ forces a circular limit, $2H(p, q) \approx p^2 + q^2$, around the origin. The harmonic period at the circular point is $T_0 = 2\pi$, so the angular frequency scale is $\omega_0 = 1$ for the entire oscillation disk. Two exceedingly simple examples of

[12] For more definitions and theory, refer back to [2] section III.

[13] All non-singular cubic plane curves are elliptic due to the existence of chord-and-tangent addition rule [26, 27]. The other quartic H_4 is birationally equivalent to a cubic function, read on for more details.

this form are the cubic and quartic anharmonic oscillators,

$$2H_6(p,q) = p^2 + q^2 - \frac{2\sqrt{3}}{9}q^3, \qquad 2H_4(p,q) = p^2 + q^2 - \frac{1}{4}q^4.$$

Coefficients $v_3 = -\frac{2\sqrt{3}}{9}$ and $v_4 = -\frac{1}{4}$ are chosen so that $2V(\sqrt{3}) = 1$ for the cubic function, and $2V(\pm\sqrt{2}) = 1$ for the quartic. Values $q = \sqrt{3}$ and $q = \pm\sqrt{2}$ determine the only local maxima of either potential. These conventions must have a separatrix curve at $\alpha = 1$ and an oscillation disk with domain $\alpha \in [0,1)$. To distinguish that H_6 and H_4 are not equivalent by canonical transformation, we need only calculate distinct period functions.

In action-angle coordinates the Hamiltonian function of the quartic anharmonic osicllator is written as $2H_4(\lambda,\phi) = 2\lambda - \lambda^2 \sin(\phi)^4$. Solving $\alpha = 2H_4(\lambda,\phi)$ for $\lambda = \frac{1-\sqrt{1-\alpha\sin(\phi)^4}}{\sin(\phi)^4}$ then allows us to write the period integral, $T_4(\alpha) = \oint 2(\partial_\alpha \lambda) d\phi = \oint \frac{d\phi}{\sqrt{1-\alpha\sin(\phi)^4}}$. We should already know how to solve this integral in series expansion,

$$T_4(\alpha) = \oint \frac{d\phi}{\sqrt{1-\alpha\sin(\phi)^4}} = \sum_{n=0}^{\infty} \frac{1}{4^n}\binom{2n}{n} \oint \sin(\phi)^{4n} d\phi = \sum_{n=0}^{\infty} \frac{2\pi}{64^n}\binom{2n}{n}\binom{4n}{2n}.$$

As with the previous cases $E(\alpha)$ and $K(\alpha)$, the expansion coefficients, call them f_n, satisfy a hypergeometric recursion,

$$f_0 = 1,\ (n+1)^2 f_{n+1} = (n+\tfrac{1}{4})(n+\tfrac{3}{4}) f_n \iff f_n = \frac{1}{64^n}\binom{2n}{n}\binom{4n}{2n},$$

which determines a standard expression, $T_4(\alpha) = 2\pi\, {}_2F_1\left[\begin{smallmatrix}\frac{1}{4},\frac{3}{4}\\ 1\end{smallmatrix}\middle|\alpha\right]$. It is instructive to compare $T_4(\alpha)$ with the earlier $T_2(\alpha) = 4K(\alpha)$ by writing the Hadamard products,

$${}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix}\middle|\alpha\right] = {}_2F_1\left[\begin{matrix}\frac{1}{2},\cdot\\ \cdot\end{matrix}\middle|\alpha\right] \star {}_2F_1\left[\begin{matrix}\frac{1}{2},\cdot\\ \cdot\end{matrix}\middle|\alpha\right] \text{ and } {}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{3}{4}\\ 1\end{matrix}\middle|\alpha\right] = {}_2F_1\left[\begin{matrix}\frac{1}{2},\cdot\\ \cdot\end{matrix}\middle|\alpha\right] \star {}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{3}{4}\\ \frac{1}{2}\end{matrix}\middle|\alpha\right],$$

where the $\star$ operator indicates multiplication of expansion coefficients[14],

$$F(\alpha) \star G(\alpha) = \sum_{n=0}^{\infty} f_n\, \alpha^n \star \sum_{n=0}^{\infty} g_n\, \alpha^n = \sum_{n=0}^{\infty} f_n\, g_n\, \alpha^n.$$

The shared form, $2H = 2\lambda - \lambda^2\Phi$, determines the equivalent first factor, and they differ on the second factor $\sum \alpha^n \oint \Phi^n d\phi$. This observation suggests a first combinatorial foray.

The condition of separable potential and kinetic energy requires $\Phi \propto \sin(\phi)^4$. If we loosen this condition, then Φ need only be a trigonometric polynomial of homogeneous degree 4,

[14] When F and G are both hypergeometric functions: join upper parameters, join lower parameters with an additional 1, and finally pairwise cancel parameters occurring in both upper and lower sets.

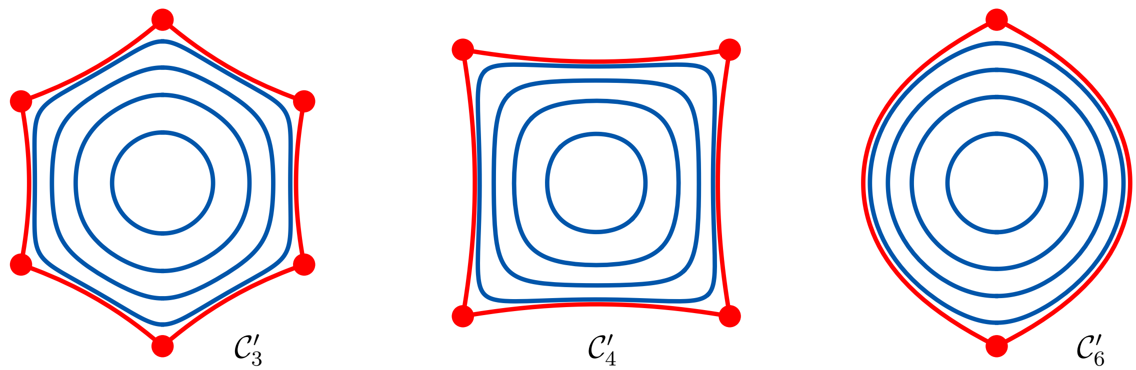

FIG. 3. Higher genus oscillation disks with $T(\alpha) = T_s(\alpha^2) = 2\pi\, {}_2F_1\left[\begin{smallmatrix} \frac{1}{s}, \frac{s-1}{s} \\ 1 \end{smallmatrix} \middle| \alpha^2\right]$ for $s = 3, 4, 6$.

i.e. must have the form $\Phi = \sum_{n=0}^{4} c_n \cos(\phi)^n \sin(\phi)^{4-n}$. By searching the range of valid perturbations Φ, it is possible to find at least one more well-related case,

$$2H_4' = p^2 + q^2 - \frac{1}{4}(p^2+q^2)^2 + 2p^2q^2 \iff \Phi = \cos(\phi)^4 - 6\cos(\phi)^2\sin(\phi)^2 + \sin(\phi)^4.$$

This case stands out because perturbing term reduces to $\Phi = \cos(4\phi)$. The oscillation disk takes the shape of a square with concave edges, as in the center of Fig. 3. On the disk, time is measured by period function $T_4(\alpha^2) = 2\pi\, {}_2F_1\left[\begin{smallmatrix} \frac{1}{4}, \frac{3}{4} \\ 1 \end{smallmatrix} \middle| \alpha^2\right]$. To see why this coincidence should occur, observe that odd powers of Φ integrate to zero, while $\oint \cos(4\phi)^{2n} d\phi \propto \binom{2n}{n}$. Relative to the case $\Phi = \sin(\phi)^4$, left and right Hadamard factors of the period function transpose—an impressive dance of parameters and exponents! We will encounter similar symmetry and similar maneuvers as we continue to study simple period functions.

The cubic oscillator is terribly more difficult to solve. In action-angle coordinates, the definition $2H_6 = 2\lambda - \frac{4\sqrt{6}}{9}\lambda^{3/2}\cos(\phi)^3$ is a special case of $2H = 2\lambda - \lambda^{3/2}\Phi$. Constraint $\alpha = 2H$ implies generally that $\alpha\Phi^2 = 2x - x^{3/2}$, $x = \lambda\Phi^2$. Exact root-solving is too unwieldy a process, so we have no better recourse than series reversion (cf. A214377). Here we will drop rigor, skip the expansion of λ, and instead proceed directly to assert that,

$$T_6(\alpha) = \sum_{n=0}^{\infty} \frac{1}{8^n}\binom{3n}{n} \oint \left(\frac{4\sqrt{6}}{9}\cos(\phi)^3\right)^{2n} d\phi = 2\pi \sum_{n=0}^{\infty} \frac{1}{432^n}\binom{3n}{n}\binom{6n}{3n}.$$

A relatively easy proof will be given in the next section. For now, let us note the similarities to previous examples. The standard expression $T_6(\alpha) = 2\pi\, {}_2F_1\left[\begin{smallmatrix} \frac{1}{6}, \frac{5}{6} \\ 1 \end{smallmatrix} \middle| \alpha\right]$ follows from,

$$f_0 = 1,\ (n+1)^2 f_{n+1} = (n+\tfrac{1}{6})(n+\tfrac{5}{6}) f_n \iff f_n = \frac{1}{432^n}\binom{3n}{n}\binom{6n}{3n}.$$

As before, Hadamard decomposition ${}_2F_1\left[\begin{smallmatrix}\frac{1}{6},\frac{5}{6}\\1\end{smallmatrix}\middle|\alpha\right] = {}_2F_1\left[\begin{smallmatrix}\frac{1}{3},\frac{2}{3}\\\frac{1}{2}\end{smallmatrix}\middle|\alpha\right] \star {}_3F_2\left[\begin{smallmatrix}\frac{1}{6},\frac{1}{2},\frac{5}{6}\\\frac{1}{3},\frac{2}{3}\end{smallmatrix}\middle|\alpha\right]$ combines a left factor determined from the general cubic constraint, $\alpha\Phi^2 = 2\lambda - \lambda^{3/2}$, with a right factor determined by integrating powers of Φ. Again we can search valid choices of Φ to find another interesting case, $2H_3 = p^2 + q^2 - (\frac{4}{27})^{\frac{1}{2}}\,(q^3 - 3p^2q)$, with $\Phi = \frac{4\sqrt{6}}{9}\sin(3\phi)$. The period function can be calculated simply by changing the right factor of the Hadamard product. That is, $T_3(\alpha) = 2\pi\,{}_2F_1\left[\begin{smallmatrix}\frac{1}{3},\frac{2}{3}\\1\end{smallmatrix}\middle|\alpha\right]$ follows from ${}_2F_1\left[\begin{smallmatrix}\frac{1}{3},\frac{2}{3}\\1\end{smallmatrix}\middle|\alpha\right] = {}_2F_1\left[\begin{smallmatrix}\frac{1}{3},\frac{2}{3}\\\frac{1}{2}\end{smallmatrix}\middle|\alpha\right] \star {}_2F_1\left[\begin{smallmatrix}\frac{1}{2},\cdot\\\cdot\end{smallmatrix}\middle|\alpha\right]$.

By the most expedient, intuitive analysis, we have already uncovered a secret that is well within the reaches of what Ramanujan himself could have known and calculated:

> *The three alternatives to $K(\alpha)$ all obey a Hadamard decomposition to two hypergeometric factors. One factor is determined by the general choice of a degree, either quartic or cubic. The other factor is then determined by the special choice of a trigonometric polynomial, homogeneous in the chosen degree.*

We will never know exactly how Ramanujan found K_1, K_2, and K_3, but his notebooks do evidence a propensity for exhaustive searches. In any case, the statement is not lacking insight, but certainly needs more rigor. We can do incrementally better by using Creative Telescoping to print and verify Table I. Here, right factors are written as $I(\alpha) = \oint \frac{d\phi}{1-\alpha\Phi}$ and given alongside an annihilator $\mathcal{A}$ such that $\mathcal{A} \circ I(\alpha) = 0$. When annihilator $\mathcal{A}$ determines a hypergeometric $I(\alpha)$, the coefficient recursion can be found by the Frobenius method. We will now give a brief, easy example of how this works in practice.

The most simple choice, $\Phi = 4\sin(\phi)^2$, recalls earlier calculations[15] of $E(\alpha)$ and $K(\alpha)$. The integral function $I(\alpha) = \oint \frac{d\phi}{1-4\alpha\sin(\phi)^2}$ has no square root, so details work out with even less effort. The identity,

$$\mathcal{A} \circ \frac{dI}{d\phi} - \partial_\phi \Xi = \Big(2 - (1-4\alpha)\partial_\alpha\Big) \circ \frac{1}{1-4\alpha\sin(\phi)^2} - \partial_\phi\left(\frac{2\cos(\phi)\sin(\phi)}{1-4\alpha\sin(\phi)^2}\right) = 0.$$

utilizes certificate Ξ to prove that $\mathcal{A}$ almost annihilates $\frac{dI}{d\phi}$. Again, exact differentials integrate to zero on a closed contour, thus $\mathcal{A}$ completely annihilates $I(\alpha)$, or $\mathcal{A} \circ I(\alpha) = 0$. When solving for a coefficient recursion, it is also useful to view $\mathcal{A}$ as a coefficient matrix, $\mathcal{A} \sim \left(\begin{smallmatrix}2 & -1\\0 & 4\end{smallmatrix}\right)$, where powers of α increase by row, and powers of ∂_α increase by column. The

[15] Cf. [1] sec. 1-2. In fact, up to scale of α, the same matrix invariants can be used again.

TABLE I. Integral series $I(\alpha) = \oint \frac{d\phi}{1-\alpha\Phi} = \sum_{n=0}^{\infty} \oint (\alpha\Phi)^n d\phi$ must satisfy $\mathcal{A} \circ I(\alpha) = 0$.

s	$\Phi =$	$\mathcal{A} =$	$I(z)$	${}_pF_q$?
2	P^2Q^2	$2-(1-4\alpha)\partial_\alpha$	A000984	yes
2	I_2Q^2	same as for $\Phi = P^2Q^2$	A000984	yes
4	Q^4	$6-(1-64)\alpha\partial_\alpha - 2\alpha(1-16\alpha)\partial_\alpha^2$	A001448	yes
4	$(Q_4)^2$	same as for $\Phi = P^2Q^2$	A000984	yes
6	$(PQ^3)^2$	$30-3(1-452\alpha)\partial_\alpha - 8\alpha(4-243\alpha)\partial_\alpha^2$	A211419	yes
	(cont.)	$-16\alpha^2(1-27\alpha)\partial_\alpha^3$		
	P_2P^2	$2(1+4\alpha)^2-(1-32\alpha-112\alpha^2-160\alpha^3)\partial_\alpha$	A288470	no
	(cont.)	$-2\alpha(1+\alpha)(1+4\alpha)(1-8\alpha)\partial_\alpha^2$		
	P_3P	$12\alpha(4+9\alpha)-2(1-13\alpha-84\alpha^2-270\alpha^3)\partial_\alpha$	A092765	no
	(cont.)	$-\alpha(1-4\alpha)(1+6\alpha)(4+9\alpha)\partial_\alpha^2$		
	$(P_3Q)^2$	$12(10-126\alpha+729\alpha^2)$	nAn	no
	(cont.)	$-6(5-1093\alpha+7308\alpha^2-13122\alpha^3)\partial_\alpha$		
	(cont.)	$-\alpha(320-11043\alpha+85806\alpha^2-69984\alpha^3)\partial_\alpha^2$		
	(cont.)	$-2\alpha^2(5-54\alpha)(16-207\alpha+108\alpha^2)\partial_\alpha^3$		
3	$(Q_3)^2$	same as for $\Phi = P^2Q^2$	A000984	yes
3	$(I_2Q)^2$	same as for $\Phi = P^2Q^2$	A000984	yes
4	$(PQ^2)^2$	$24-6(1-136\alpha)\partial_\alpha - 18\alpha(3-64\alpha)\partial_\alpha^2$	A005810	yes
	(cont.)	$-\alpha^2(27-256\alpha)\partial_\alpha^3$		
6	$(Q^3)^2$	$40-2(1-904\alpha)\partial_\alpha - 18\alpha(1-144\alpha)\partial_\alpha^2$	A066802	yes
	(cont.)	$-9\alpha^2(1-64\alpha)\partial_\alpha^3$		
	$(P_2Q)^2$	$24(1-16\alpha)^2$	A005721	no
	(cont.)	$-6(1-248\alpha+2688\alpha^2-9216\alpha^3)\partial_\alpha$		
	(cont.)	$-6\alpha(1-16\alpha)(9-276\alpha+512\alpha^2)\partial_\alpha^2$		
	(cont.)	$-\alpha^2(1-16\alpha)^2(27-32\alpha)\partial_\alpha^3$		
	I_2Q^4	same as for $\Phi = Q^4$	A001448	yes
3	$I_2P^2Q^2$	same as for $\Phi = P^2Q^2$	A000984	yes
	$(P^5Q)^2$	$7560-30(3-184504\alpha)\partial_\alpha - 30\alpha(459-1076000\alpha)\partial_\alpha^2$	nAn	yes
	(cont.)	$-423\alpha^2(99-80000\alpha)\partial_\alpha^3 - 32\alpha^3(729-312500\alpha)\partial_\alpha^4$		
	(cont.)	$-4\alpha^4(729-200000\alpha)\partial_\alpha^5$		
6	$(P_2Q_2^2)^2$	same as for $\Phi = (PQ^2)^2$	A005810	yes
$\vdots$	$\vdots$		$\vdots$	$\vdots$

$P_n = 2\cos(n\phi),\ Q_n = 2\sin(n\phi),\ P = P_1,\ Q = Q_1,\ I_2 = \frac{1}{4}(P^2+Q^2) = 1.$
See Appendix A for computer proofs.

matrix encoding of $\mathcal{A}$ allows algorithmic streamlining of the Frobenius method,

$$\frac{a_{n+1}}{a_n} = \frac{(2,4)\cdot(1,n)}{(1)\cdot(1+n)} = 4\frac{(n+\frac{1}{2})}{(n+1)} \implies I(\alpha) = \oint \frac{d\phi}{1-4\alpha\sin(\phi)^2} = {}_2F_1\left[\begin{matrix}\frac{1}{2},\cdot\\ \cdot\end{matrix}\middle|4\alpha\right].$$

In case it is not already clear, dots in the arguments of a ${}_2F_1$ function indicate canceled parameters, so the evaluation is more precisely to a ${}_1F_0$ function.

The main virtue of Table I is *systematism*. Rather than searching by intuition through the space of valid Φ, we develop a comprehensive list of monomials, and use the diagnostic `ExpToODE` to distinguish cases. We include any monomial of the variables $Q_n = 2\sin(n\phi)$, $P_n = 2\cos(n\phi)$, and $I_2 = \frac{1}{4}(P^2+Q^2) = 1$, which satisfies the degree constraint that $d = \sum$ subscript $\times$ exponent, with $d=4$ or $d=3$, and summing over all multiplicands. For the quartic case, we find only 6 distinct cases, and only 4 distinct cases for cubic functions. These appear in the first and second divisions of Table I. Perturbations Φ appear with an extra square when odd powers would otherwise integrate to zero. This is true for all cubic perturbations, and also for a few of the quartic perturbations, as we have already seen.

Given the data of Table I, matrix encodings of each annihilator can be inspected to determine whether or not $I(\alpha)$ is a hypergeometric function. If the matrix has non-zero values only on the central diagonal and the first upper diagonal, then it is hypergeometric and gets a "yes" in the last column. Otherwise if the matrix form of $\mathcal{A}$ contains non-zero values on $k>2$ diagonals, the Frobenius recursion will relate $a_n, a_{n+1}, \ldots, a_{n+k-1}$, so it cannot be hypergeometric.[16] We should not be too surprised to find a few more interesting models for the series called by Ramanujan K_1, K_2 and K_3. However it is somewhat amazing that all "yes" hits lead to one of these periods.

The asymmetric alternatives are also worth a look. Reading down the table, we first find the pertubation $\Phi = I_2 Q^2 = Q^2$, and can recognize the corresponding Hamiltonian function, $2H_\varphi = (p^2+q^2)(1-\frac{1}{4}q^2)$, as the algebraic form that describes simple pendulum libration. The first unknown is $\Phi = (PQ^3)^2$, or in the full notation, $2H''_6 = p^2+q^2-\frac{4\sqrt{3}}{9}pq^3$. The period function, $T_6(\alpha^2) = 2\pi\,{}_2F_1\left[\begin{matrix}\frac{1}{6},\frac{5}{6}\\ 1\end{matrix}\middle|\alpha^2\right]$, follows from the Hadamard decomposition,

$${}_2F_1\left[\begin{matrix}\frac{1}{6},\frac{5}{6}\\ 1\end{matrix}\middle|\alpha^2\right] = {}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{3}{4}\\ \frac{1}{2}\end{matrix}\middle|\alpha^2\right] \star {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{1}{2},\frac{5}{6}\\ \frac{1}{4},\frac{3}{4}\end{matrix}\middle|\alpha^2\right].$$

[16] There is a sometimes a caveat about "minimal telescopers", but all "no" cases of Table I have been double checked for the minimal property by guess and check, and by referencing with OEIS.

Comparison with the similar cubic period reveals more than dancing parameters,

$$\binom{4n}{2n}\sum_{k=0}^{n}\binom{6n}{k}\binom{5n-k-1}{n-k}=\binom{3n}{n}\binom{6n}{3n}.$$

a seemingly improbable, nonetheless true, binomial identity! Another way to prove the identity is to observe that the shear transformation $p \to p + \frac{2\sqrt{3}}{9}q^3$ takes quartic H_6'' to sextic $2H_6' = p^2 + q^2 - \frac{4}{27}q^6$. All such shears have Jacobian determinant 1, so they are canonical transformations, thus preserve periods of oscillation. As it must, H_6' also follows from $\Phi = (Q^3)^2$ applied to the general sextic form. Another similar example for signature 4 is found from $H_4'' = p^2 + q^2 - pq^2$, by applying the shear $p \to p + \frac{1}{2}p^2$. This canonical transformation again obtains H_4, thus $\binom{3n}{n}\binom{4n}{n} = \binom{2n}{n}\binom{4n}{2n}$, another unexpected identity!

The only cases left are those marked as $s = 3$ in Table I. Applying an Abel-Wick rotation, $p \to ip, q \to \sqrt{3} - q$ to $1 - H_3$ obtains $\widetilde{H}_3 = (3p^2 + q^2)(1 - \frac{2\sqrt{3}}{9}q)$. Up to a factor $\sqrt{3}$ on the frequency scale ω_0, $\widetilde{H}_3 \approx (p^2 + q^2)(1 - \frac{2\sqrt{3}}{9}q)$, what we obtain by applying $\Phi = I_2Q$ to the general cubic form. Isoperiodicity follows from the fact that $\alpha \to 1 - \alpha$ acts invariantly on the annihilator $\mathcal{A}_s$. The last essential case is $\Phi = Q_3^2$, with Hamiltonian form, $2H_3' = p^2 + q^2 - \frac{4}{27}(q^3 - 3p^2q)^2$. By now, the derivation H_3' and its period function $T_3(\alpha^2)$ should be quite obvious. If not, we assert that sextic anharmonic oscillators of the form $\alpha = 2H(\lambda, \phi) = 2\lambda - \lambda^3\Phi$ are measured by the period function,

$$T(\alpha) = \sum_{n=0}^{\infty}\frac{1}{8^n}\binom{3n}{n}\alpha^{2n}\oint \Phi^n d\phi = {}_2F_1\left[\begin{matrix}\frac{1}{3},\frac{2}{3}\\ \frac{1}{2}\end{matrix}\middle|\alpha^2\right] \star \oint \frac{d\phi}{1-\frac{27}{32}\alpha^2\Phi}.$$

Consequently, when a cubic function $\alpha = 2H = 2\lambda - \lambda^{\frac{3}{2}}\Phi$ has period $T(\alpha)$, the corresponding sextic function $\alpha = 2H = 2\lambda - \lambda^3\Phi^2$ has period $T(\alpha^2)$.

Having done so much more analysis, we can now strengthen the earlier theorem:

> *Assuming either form, $\alpha = 2H = 2\lambda - \lambda^{\frac{3}{2}}\Phi$ or $\alpha = 2H = 2\lambda - \lambda^2\Phi$, a valid choice of monomial Φ determines a hypergeometric period $T(\alpha)$ if and only if $\mathcal{A}_s \circ T(\alpha) = 0$ for $s = 2, 3, 4$ or 6. Canonical models H_s all have elliptic level curves $\mathcal{C}_s(\alpha)$ for $\alpha \in (0, 1)$. Except for $s = 2$, each H_s has a higher-genus analog, and a canonical set of H_s' for $s = 3, 4, 6$ can be chosen to maximize symmetry.*

Take a step back and think about what the strengthened result says. Not only is it possible to find Ramanujan's set K_1,K_2, and K_3 by searching a space of geometric models; if we

TABLE II. Sextic Hamiltonians and their period functions, $T(\alpha) = {}_4F_3\left[\begin{matrix}\frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{5}{6}\\ \frac{1}{4},\frac{1}{2},\frac{3}{4}\end{matrix}\middle|\alpha^4\right] \star I(\alpha)$.

$\alpha = 2H =$	$I(\alpha) =$	$T(\alpha) =$
$p^2+q^2-\frac{8}{9}(p^5q-\frac{10}{3}p^3q^3+q^5p)$	$2\pi\,{}_2F_1\left[\begin{matrix}\frac{1}{2},\cdot\\ \cdot\end{matrix}\middle|\alpha^4\right]$	$2\pi\,{}_4F_3\left[\begin{matrix}\frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{5}{6}\\ \frac{1}{4},\frac{3}{4},1\end{matrix}\middle|\alpha^4\right]$
$p^2+q^2-\frac{4\sqrt{3}}{9}pq(p^2-q^2)^2$	$2\pi\,{}_3F_2\left[\begin{matrix}\frac{1}{4},\frac{1}{2},\frac{3}{4}\\ \frac{1}{3},\frac{2}{3}\end{matrix}\middle|\alpha^4\right]$	$2\pi\,{}_2F_1\left[\begin{matrix}\frac{1}{6},\frac{5}{6}\\ 1\end{matrix}\middle|\alpha^4\right]$
$p^2+q^2-\frac{4}{27}(p^2-q^2)^3$	$2\pi\,{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{1}{2},\frac{5}{6}\\ \frac{1}{3},\frac{2}{3}\end{matrix}\middle|\alpha^4\right]$	$2\pi\,{}_4F_3\left[\begin{matrix}\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}\\ \frac{1}{4},\frac{3}{4},1\end{matrix}\middle|\alpha^4\right]$
$\left(p^2+q^2\right)\left(1-\frac{64\sqrt{3}}{243}pq^3\right)$	$2\pi\,{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{1}{2},\frac{5}{6}\\ \frac{1}{4},\frac{3}{4}\end{matrix}\middle|\alpha^4\right]$	$2\pi\,{}_6F_5\left[\begin{matrix}\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{5}{6},\frac{5}{6}\\ \frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},1\end{matrix}\middle|\alpha^4\right]$
$p^2+q^2-\frac{32\sqrt{5}}{125}pq^5$	$2\pi\,{}_5F_4\left[\begin{matrix}\frac{1}{10},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{9}{10}\\ \frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{5}{6}\end{matrix}\middle|\alpha^4\right]$	$2\pi\,{}_4F_3\left[\begin{matrix}\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}\\ \frac{1}{4},\frac{3}{4},1\end{matrix}\middle|\alpha^4\right]$

place mild constraints on the search space, we will only find the set K_1,K_2, and K_3! In that sense Ramanujan's presentation is *complete* if not *comprehensive*. This curious case of good insight blurs the line between fortune and genius, but not between apathy and work. Don't forget, Ramanujan did not have a computer to generate and check results!

So long as we don't pay too much attention to the particulars of series reversion, there is no reason to cease the search at quartic degree. Given that H_3' and H_6' are sextic functions, it is natural to wonder: what happens generally for quintics and higher? Using `ExpToODE` we are able to perform an exhaustive search up to octic degree, over 200 choices of Φ. Most frequently the annihilators are not hypergeometric, but we also found many hypergeometric cases, including those of Table II. In the subset of hypergeometric search results, we do not find any surprising examples where $T(\alpha^n) = {}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix}\middle|\alpha^n\right]$ with $(a,b) \neq (s, \frac{s-1}{s})$ or $c \neq 1$. For $s = 6$, the search uncovers another sextic model, though with period $T_6(\alpha^4)$. In a narrow search above $d = 8$, we find $\alpha = 2H = p^2 + q^2 - \frac{64}{27}p^4q^6$, again with period $T_6(\alpha^4)$.

The two extra occurrences of $T_6(\alpha)$ both entail freak-accident parameter cancellations. More commonly, Hadamard products involve little-to-no cancellation. For $d = 7$ or $d = 8$ it is already possible to find periods of the form ${}_{10}F_8$ or ${}_{10}F_9$. The search results show a clear average pattern: as degree increases, complexity of the period function increases, and simple examples become sparsely distributed, if at all. We could continue to make observations and hypotheses, but again these would depend on unproven assertions. To prove claims more surely, we will now delve deeper into the algorithmic theory of Creative Telescoping.

Algorithm 1. Simple Creative Telescoping via Hermite-Ostrogradsky reduction.

Input: An integrand $dI\big(\alpha, z(\alpha,t)\big)$, denominator ρ, and differentials $\partial_\alpha z$, and $\partial_t z$.

Output: Annihilator $\mathcal{A}$ and certificate Ξ such that $\mathcal{A} \circ \frac{dI}{dt} - \partial_t \Xi = 0$.

function $\textsc{EasyCT}(dI,\rho,\partial_\alpha z,\partial_t z)$: `#(Ha, ha, ha!  Have Fun!)>`
  $d \leftarrow \textsc{Deg}(\rho)$; $\Delta \leftarrow \textsc{Deg}(\partial_t z) - 1$;
  $u \leftarrow \sum_{n=0}^{d+\Delta-1} u_n z^n$; $v \leftarrow \sum_{n=0}^{d-1} v_n z^n$;
  $\mathbf{G} \leftarrow \textsc{CoefficientMatrix}(\rho u - (\partial_t z)(\partial_z \rho) v, 2d+\Delta)$;
  if $\textsc{Det}(\mathbf{G}) = 0$ **then return** "Error: no inverse for $\mathbf{G}$.";
  else $(\mathbf{U},\mathbf{V}) \leftarrow$ decomposition of $\mathbf{G}^{-1}$; $\dot{\mathbf{V}} \leftarrow (\partial_t z)\partial_z \mathbf{V}$;
  $n \leftarrow 0$; $x_0 \leftarrow \frac{dI}{dt}$; $\mathbf{x}_0 \leftarrow \textsc{CoefficientVector}(\rho x_0, d+\Delta)$;
  $\mathbf{A} \leftarrow \textsc{NullSpace}(\{\mathbf{x}_0\}^T)$;
  while $\textsc{Empty}(\mathbf{A})$ **do** $n \leftarrow n+1$; $\mathbf{x}_n = 0$;
    $x_n \leftarrow \partial_\alpha x_{n-1} + (\partial_\alpha z)\partial_z x_{n-1}$;
    if $!\textsc{Reducible}(\rho, x_n, n)$ **then**
      return "Error: ρ/dI mismatch.";
    $\mathbf{w} = \{w_0, w_1, \dots, w_n\} \leftarrow \textsc{PartialFractions}(x_n, \rho)$;
    for $m = 0, 1, \dots, n$ **do**
      $\mathbf{w}_m \leftarrow \textsc{CoefficientVector}(w_m, d+\Delta)$;
      $\mathbf{x}_n \leftarrow \mathbf{x}_n + \textsc{VectorReduce}(\mathbf{U},\mathbf{V},\dot{\mathbf{V}}, m, \mathbf{w}_m, d-1)$;
    $\mathbf{A} \leftarrow \textsc{NullSpace}(\{\mathbf{x}_0, \dots, \mathbf{x}_n\}^T)$;
  $\Xi \leftarrow \textsc{Reap}()$; **return** $\big(\mathbf{A} \cdot \{1, \partial_\alpha, \dots, \partial_\alpha^n\}, \mathbf{A} \cdot \textsc{SubTotal}(\Xi)\big)$;
function $\textsc{VectorReduce}(\mathbf{U},\mathbf{V},\dot{\mathbf{V}}, k, \mathbf{w}, l)$:
  while $k > 0$ **do**
    $\textsc{Sow}(\frac{1}{k\rho^k}\{1, z, \dots, z^l\} \cdot \mathbf{V} \cdot \mathbf{w})$;
    $\mathbf{w} \leftarrow (\mathbf{U} + \frac{1}{k}\dot{\mathbf{V}}) \cdot \mathbf{w}$;
    $k \leftarrow k - 1$;
  return $\mathbf{w}$;

III. DIAGNOSTIC ALGORITHMS

Creative Telescoping is an algorithmic theory that aims to assist in the process of redefining functions according to the ordinary differential equations they satisfy. We have already made use of `ExpToODE`, with input/output map $I(\alpha) \to \mathcal{A}$. Soon we will introduce two more algorithms, `hyperellipticToODE` and `DihedralToODE`. Despite differing input domains, all three algorithms follow the same pseudocode, as written in Alg. 1. This pseudocode is a template for a range of C.T. algorithms that rely on degree-bounded Hermite-Ostrogradsky reduction[28]. Such algorithms are the simplest possible, while still meeting minimum rigor. They make a good starting place for curious beginners.

Hermite-Ostrogradsky reduction is essentially limited to univariate cases; however, for the sake of versatility, it is useful to assume a chain-rule structure. The integral $I(\alpha) = \oint \frac{dI}{dt} dt$ requires a period T such that $t \to t + T$ leaves $\frac{dI}{dt}$ invariant, so most likely, t will not make a good basis for reductions. We need an algebraic variable $z(\alpha, t)$. A good choice for z is often determined by inspection of ρ, for typically $\rho \in \mathbb{Q}[\![i, \alpha, z]\!]$. In fact, the algorithm assumes ρ is a degree-d polynomial of z, i.e. $\rho = \sum_{n=0}^{d} c_n z^n$, and the c_n themselves are usually rational functions or polynomials in the variable α. Subsequently we must also require $\dot{\rho} = (\partial_t z) \sum_{n=0}^{d} n c_n z^{n-1} = \sum_{n=0}^{d+\Delta} c_n' z^n$, i.e. that $\dot{\rho}$ is a z-polynomial. We will discuss in detail the obvious examples $z = e^{it}$ and $z(t) = q(t)$, as well as a non-obvious, more difficult case, $z(t) = \lambda(t)$. Since these three cases are all strongly related, it should help to start with an overview of Alg. 1 including what it does, why it works, and when it can possibly fail.

The first few lines 2-4 determine the dimensionality of the reduction process, but do not guarantee success. Once variables are chosen, a feasibility analysis needs to be performed relative to the central statement of Hermite-Ostrogradsky reduction from w to $[w]$,

$$\frac{[w]}{\rho^m} = \left(u - \frac{1}{m}\dot{v}\right)\frac{1}{\rho^m} = \frac{w}{\rho^{m+1}} - \partial_t\left(\frac{v}{m\rho^m}\right) \quad \Longleftrightarrow \quad w = \rho u - \dot{\rho} v.$$

That such a reduction exists is itself an improbable assumption. The variables u, v, and w must be polynomials in the variable z with consistent degrees, i.e. $\deg(w) = \deg(u) + \deg(\rho)$ and $\deg(w) = \deg(v) + \deg(\dot{\rho})$. Polynomials u and v are treated as unknowns with many degrees of freedom, $\mathrm{dof}(v) = \deg(v) + 1$ and $\mathrm{dof}(u) = \deg(u) + 1$. Solving the degree bound, $\deg(w) = \mathrm{dof}(u) + \mathrm{dof}(v)$, we obtain that $\deg(u) = \deg(\dot{\rho}) - 1$ and $\deg(v) = \deg(\rho) - 1$. We can also state that $\deg(u) = \deg(\dot{v}) = d + \Delta - 1$ in terms of $d = \deg(\rho)$ and $\Delta = \deg(\partial_t z) - 1$. This restricts all functions $w(z)$ with implicit $\deg(w) = 2d + \Delta - 1$ to have non-zero w_n only from $n = 0$ to $n = d + \Delta - 1$. When all of these conditions are met, it is possible to proceed to line 5 of Alg. 1, where the first critical error could possibly occur.

The $\mathbf{G}$ matrix is constructed relative to spanning vectors $\mathbf{z} = \{1, z, z^2, \ldots, z^{2d+\Delta}\}$ and $\mathbf{uv} = \{u_0, u_1, \ldots, u_{d+\Delta-1}, v_0, v_1, \ldots, v_{d-1}\}$. It encodes the essentials of the right hand side of $w = \rho u - \dot{\rho} v$. Introducing another coefficient vector $\mathbf{w}$ such that $w = \mathbf{z} \cdot \mathbf{w}$, we can rewrite the reduction constraint as $\mathbf{w} = \mathbf{G} \cdot \mathbf{uv}$. If and only if $\det(\mathbf{G}) \neq 0$, the linear equation is uniquely solveable. If $\det(\mathbf{G}) = 0$, an error is thrown on line 5 and Alg. 1 fails. For some choices of z and ρ, it is possible to prove that $\det(\mathbf{G})$ never equals to zero, and this marks an important distinction between *proveable* and *effective* algorithms. Proveable algorithms are preferable because they will always work, whereas effective algorithms can only be guaranteed case-by-case. Absolute rigor is never necessary, and as long as possible errors are well-understood, it is not always desirable[17].

Line 5 is a major milestone for the algorithm, and passing it strongly suggests that the a positive result will be obtained upon halting. To explain subsequent line 6, we must split apart vector $\mathbf{uv}$ to $\mathbf{u}$ and $\mathbf{v}$, lengths $d+\Delta$ and d respectively, while dropping zeros of $\mathbf{w}$ to obtain $\mathbf{w}'$, another vector of length $d+\Delta$. Then we can define matrices $\mathbf{U}$ and $\mathbf{V}$ such that $\mathbf{u} = \mathbf{U}\cdot\mathbf{w}'$ and $\mathbf{v} = \mathbf{V}\cdot\mathbf{w}'$. Given that $\mathbf{uv} = \mathbf{G}^{-1}\cdot\mathbf{w}$, matrices $\mathbf{U}$ and $\mathbf{V}$ must be submatrices of $\mathbf{G}^{-1}$, so the first part of line 6 is no more complex than matrix inversion. Once the matrix $\mathbf{V}$ is known, the function $v(w)$ is known, as is $\dot{v}(w)$, as is the matrix $\dot{\mathbf{V}}$. Both $\mathbf{U}$ and $\dot{\mathbf{V}}$ are square matrices, so we can finally achieve the goal of writing Hermite-Ostrogradsky reduction in terms of linear algebra, $[\mathbf{w}'] = (\mathbf{U}+\frac{1}{m}\dot{\mathbf{V}})\cdot\mathbf{w}'$, as in line 22 of Alg. 1. The halting condition is already in sight, but there is at least one more significant chance of failure.

Lines 11-12 validate that ρ is well-chosen with regard to integrand dI. As the length of $\mathbf{x}_0$ is $d+\Delta$, we can anticipate needing to calculate no more than $D = d+\Delta$ reductions, before the set $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_D\}$ must contain at least one linear dependency. Depending on symmetry, the first linear dependency may occur for $D'+1$ vectors, with $D' < D$. For this reason, we compute and validate derivatives on the fly. To reach $n = D'$ and exit the loop after line 17, we must always find, for each $n \le D'$, that $\partial_\alpha^n \frac{dI}{dt} = \sum_{m=0}^{n} \frac{w_m}{\rho^{m+1}}$. In other words, the partial fraction decomposition of $\partial_\alpha^n \frac{dI}{dt}$ must yield no more than $n+1$ degree-bounded numerators $w_m = \sum_{k=0}^{d+\Delta} w_{m,k} z^k$. During each loop, if the check on line 11 passes, then the numerators w_m are calculated on line 13. The algorithm enters a second loop on line 14, and calculates the coefficients $w_{m,k}$ on line 15. A third loop is entered on line 20 via line 16, and fourth loops occur on lines 21 and 22 as matrix multiplication. When this process repeats without error, the algorithm approaches halting on success. Eventually, on line 17, a linear dependence between the $\mathbf{x_n}$ will be found and set to $\mathbf{A}$. The loop breaks, and shortly thereafter, the function returns a positive result.

The subfunction defined on line 19 deserves a closer look. It manages the iterative part of Hermite-Ostrogradsky reduction by taking a $\mathbf{w}_m \sim \frac{w_m}{\rho^{m+1}}$ from its initial form to its minimal form $\mathbf{w}_f \sim \frac{w_f}{\rho}$. Lines 21-22 involve $2m$ matrix multiplications, only m of which are strictly

[17] Our view is that extra rigor must sometimes be sacrificed to explore more deeply.

necessary. The sow step on line 21 keeps track of partial certificates by hiding them in the memory. Later, on line 18, they can be retrieved and summed. When subtotaled by index n and dotted with $\mathbf{A}$, the partial certificates $\boldsymbol{\Xi}$ determine a total certificate Ξ such that $\mathcal{A} \circ \frac{dI}{dt} - \partial_t \Xi = 0$. If the reduction reaches large recursion depth, speed and memory usage can suffer, so sometimes we choose to run the algorithm without calculating certificates. However, if there is any doubt or misunderstanding, having a certificate assures quality.

In practice, we also care about the complexity of Alg. 1. How does it scale relative to the degree bound D? Naively, the answer is that Alg. 1 is polynomial-time, roughly $\mathcal{O}(D^{3+\omega})$, where $1 < \omega < 3$ is the complexity cost of matrix multiplication. In many cases three loops are not necessary, and the figure without partial fraction decomposition $\mathcal{O}(D^{2+\omega})$ looks even better[18]. The problem is that matrix multiplication generates larger and larger polynomial data as recursion depth of the k-loop increases. This causes slow-down during the reduction phase, and eventually makes nullspace calculation prohibitively slow. If $2 < \omega' < 3$ is the complexity of finding a null vector, then $\mathcal{O}(D^{1+\omega'}) \sim \mathcal{O}(D^{2+\omega})$, and we could expect this step to dominate time dynamics. In many calculations, indeed it does. Heuristically, the complexity of Alg. 1 is much better described by an exponential or super exponential figure. If necessary, more accurate estimates can follow from testing on random inputs. Of course, this requires an implementation for some particular choice of inputs.

In the previous section, we have already revealed that functions $\sin(t)$ and $\cos(t)$ make a good basis for an application of Creative Telescoping. The choice $z(t) = \cos(t) + i \sin(t)$ is even better because $\partial_t z = iz$. If we allow negative powers of z by requiring that $\rho \in Q[\![i, \alpha, z, z^{-1}]\!]$ then we can represent $\sin(t)$ and $\cos(t)$ as $\frac{1}{2i}(z - \frac{1}{z})$ and $\frac{1}{2}(z + \frac{1}{z})$ respectively. When ρ is chosen as $\rho = 1 - \alpha\Phi$ with $\Phi \in Q[\![i, z, z^{-1}]\!]$, and when $\frac{dI}{dt} = \frac{1}{\rho}$ we can make slight edits to Alg. 1 and arrive at the algorithm `ExpToODE`. In fact, we need only make changes to account for negative powers. Assuming that $\Phi = \sum_{\tilde{d}}^{d} c_n z^n$ then $\text{bideg}(\rho) = (\tilde{d}, d)$. As $\Delta = 0$, we can set $\text{bideg}(u) = \text{bideg}(v) = \text{bideg}(\rho)$, which implies $\text{bideg}(w) = (2\tilde{d}, 2d)$ and $\text{dof}(u) + \text{dof}(v) = 2(d + \tilde{d}) + 1$ when v has no constant term. Thus the reduction constraint $w = \rho u - \dot{\rho} v$ leads to a square matrix $\mathbf{G}$ of size $2(d + \tilde{d}) + 1$. After line 5 the algorithm is much the same, except that the degree bound is changed to $D = d + \tilde{d} + 1$.

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[18] Both complexity estimates are comparable to estimates for similar algorithms [20].

$$\mathbf{G} = \left[\begin{array}{c|c} & \\ \hline \mathbf{I} & \\ \hline & \end{array}\right] + \alpha \left[\begin{array}{c|c} & LT \\ \hline & \\ \hline & UT \end{array}\right], \quad \mathbf{G} = \left[\begin{array}{c|c} LT & LT \\ \hline UT & UT \end{array}\right] + \alpha \left[\begin{array}{c|c} \mathbf{I} & \\ \hline & \end{array}\right]$$

UT for upper triangle, LT for lower triangular.

FIG. 4. Forms for $\mathbf{G}$ matrices: `ExpToODE` (left), `HyperellipticToODE` (right).

The form $\rho = 1 - \alpha\Phi$ is simple enough to prove that `ExpToODE` halts on success for any valid input Φ. As in Fig. III, decompose $\mathbf{G}$ by $\mathbf{G} = \mathbf{G}_0 + \alpha\mathbf{G}_\alpha$, with both $\mathbf{G}_0$ and $\mathbf{G}_\alpha$ members of $\mathbb{Q}[\![i]\!]$. Under its first $\mathrm{dof}(u) = d + \tilde{d} + 1$ columns, matrix $\mathbf{G}_0$ contains an identity submatrix, $\mathbf{I}$, and these are the only non-zero values of $\mathbf{G}_0$. Expanding the determinant by minors determines a bound that $\det(\mathbf{G}) = \det(\mathbf{D}) + \mathcal{O}(\alpha^{d+\tilde{d}+1})$, with $\mathbf{D}$ a $\mathbf{G}$-submatrix on the space complement to that of $\mathbf{I}$. In the $\mathbf{D}$ subspace, $\mathbf{G}_0 = 0$, so $\mathbf{D} \subset \alpha\mathbf{G}_\alpha$. Matrix $\mathbf{D}$ has only one non-zero diagonal, and it is possible to identify that $\det(\mathbf{D}/\alpha) = d^d\tilde{d}^{\tilde{d}}ab$ when $\Phi = az^{-\tilde{d}} + \ldots + bz^d$. Finally we obtain $\det(\mathbf{G}) \propto \alpha^{d+\tilde{d}}$ or $\det(\mathbf{G}) \neq 0$, thus an inversion error is never thrown. As for derivatives, they can be written naively in terms of the central binomial coefficients $\partial_\alpha^n \frac{dI}{dt} = \binom{2n-2}{n-1}\frac{\Phi^n}{\rho^{n+1}}$. A valid partial fraction decomposition must always exist, and a mismatch error between dI and ρ is never thrown.

With a little more effort, it is possible to obtain a closed form, $\partial_\alpha^n \frac{dI}{dt} = \frac{n!}{\alpha^n}\sum_{k=0}^n \binom{n}{k}\frac{(-1)^{n+k}}{\rho^{k+1}}$, for the partial fraction expansion. This particular form is an amazing fortuity because it allows elimination of the loop on line 14. Once we have computed the reduction of $\frac{1}{\rho^n}$, we do not need to calculate it again during subsequent iteration of the n loop. In the optimized algorithm, each $\mathbf{x}_n$ can be calculated as the reduction of $\frac{1}{\rho^n}$. Once a linear dependency is found, it must exist in any linear transformation $\mathbf{x}' = \mathbf{M} \cdot \mathbf{x}$ so long as $\det(\mathbf{M}) \neq 0$. In this case, $\mathbf{M}$ is a lower triangular matrix with non-zero elements $m_{n,k} = n!\binom{n}{k}\frac{(-1)^{n+k}}{\alpha^n}$, so $\det(\mathbf{M}) \neq 0$ always holds true. The coefficients $\mathbf{A}$ can then be found from the nullspace of $\mathbf{x}'$ and returned as usual. This is a nice optimization that helps `ExpToODE` to be an extremely successful workhorse, as we have already seen in the preceding section.

The other easy choice, not so obvious, is that $z(t) = q(t)$ with $\partial_t z(t) = p(t)$, where $q(t)$ and $p(t)$ are solutions to Hamilton's equations given a hyperelliptic form $\alpha = 2H = p^2 + 2V(q)$, and $V(q) = \sum_{n=1}^d c_n q^n$ The period is not a single number, but a function $T(\alpha) = \oint dt = \oint \frac{dq}{p}$, with derivatives $\partial_\alpha^n \frac{dI}{dt} = p\,\partial_\alpha^n \frac{1}{p} = \frac{(2n-1)!!}{(-2)^n}\frac{1}{p^{2n}}$, as is proven using the chain rule with $\partial_\alpha p = \frac{1}{2p}$.

Even though p is not a polynomial of q we can choose $\rho = p$ if we rewrite that,

$$\frac{[w]}{\rho^{2n}} = \left(u - \frac{1}{2n+1}\partial_q v\right)\frac{1}{\rho^{2n}} = \frac{w}{\rho^{2n+2}} - \partial_t\left(\frac{v}{(2n+1)\rho^{2n+1}}\right) \quad \iff \quad w = \rho^2 u - \dot{\rho} v.$$

This statement is not as straightforward. After applying chain rule, $\partial_t v = (\partial_t q)\partial_q v = \rho\partial_q v$, the factor ρ cancels with the extra $+1$ power in the denominator. The degree bounding behaves as if $\Delta = -1$, and all reductions must be carried out to the zeroth power of ρ. Also, the denominators in lines 21-22 need to be changed to replace k with $2k+1$. Other than these few minor changes, the resulting algorithm `HyperellipticToODE` exactly follows the template of Alg. 1.

For essentially the same reason as before, `HyperellipticToODE` is also a rigorously proveable algorithm. However, in this case the decomposition $\mathbf{G} = \mathbf{G}_0 + \alpha\mathbf{G}_\alpha$ has its identity submatrix in $\mathbf{G}_\alpha$, consequently $\det(\mathbf{G}) = \alpha^{d-1}+$ smaller powers of α. The case $d = 1$ is of no interest, and the case $d = 2$ always results in $\mathcal{A} = \partial_\alpha$, as it must. In the quadratic case, the Riemann surface associated to $2H = p^2 + aq + bq^2$ is the genus zero harmonic hyperboloid, with $T(\alpha) = T_0 =$ constant. For $d = 3$ and $d = 4$, usually the level curves of H are elliptic and $\mathcal{A} \circ T(\alpha) = 0$ has two solutions, corresponding to orthogonal periods along real and complex time dimensions. For $d > 4$, $\mathcal{A}$ typically contains $2g+1$ terms, with genus $g = \lfloor (d-1)/2 \rfloor > 1$. These cases are the proper hyperelliptic curves, whose Riemann surfaces admit g distinct real periods and g distinct complex periods. In this sense, the proof of algorithm `HyperellipticToODE` is also a reproof of Fuchs's theorem on the periods of hyperelliptic curves[29], though with slightly different conventions.

Last but not least, we have the case of `DihedralToODE`, a very effective algorithm, but not entirely proveable. For inputs we assume a form,

$$\alpha = 2H = \left(\sum_{n=1}^{d_1} c_{1,n}\lambda^n\right) + \left(\sum_{n=0}^{d_2} c_{2,n}\lambda^n\right)\lambda^{\frac{m}{2}}\cos(m\phi) = h_1 + h_2\cos(m\phi),$$

so chosen because it can easily be solved to eliminate ϕ dependence, $\cos(m\phi) = \frac{\alpha - h_1}{h_2}$. Relative to Alg. 1, the choice $z = \lambda$ is somewhat surprising, but it follows follows from the observation that $\dot{\lambda}^2$ and $\ddot{\lambda}$ can both be written as λ polynomials,

$$\dot{\lambda}^2 = \frac{-m^2}{4}(\alpha - h_1 - h_2)(\alpha - h_1 + h_2) \qquad \text{and} \qquad \ddot{\lambda} = \frac{m^2}{4}\big((\alpha - h_1)(\partial_\lambda h_1) + h_2(\partial_\lambda h_2)\big).$$

Similarly, the angular derivative $\dot{\phi} = \big(h_2(\partial_\lambda h_1) + (\alpha - h_1)(\partial_\lambda h_2)\big)/(2h_2)$ works out to be a rational function of λ polynomials, and we can usually choose $\rho = \big(h_2(\partial_\lambda h_1) + (\alpha - h_1)(\partial_\lambda h_2)\big)$.

Again, the basic assumption of Hermite-Ostrogradsky reduction must be rewritten,

$$\frac{[w]}{\rho^n} = \left(u - \frac{1}{n}(\ddot{\lambda} + \dot{\lambda}^2\partial_q v)\right)\frac{1}{\rho^n} = \frac{w}{\rho^{n+1}} - \partial_t\left(\frac{\dot{\lambda}v}{n\rho^n}\right) \qquad \Longleftrightarrow \qquad w = \rho u - \dot{\lambda}^2 v(\partial_\lambda \rho).$$

As before $\frac{dI}{dt} = 1$ and subsequent derivatives are calculated as $\partial_\alpha^n \frac{dI}{dt} = \dot{\phi}\partial_\alpha^n \dot{\phi}^{-1}$ via the chain rule with $\partial_\alpha \lambda = (2\dot{\phi})^{-1}$. Degree bounding is possible with $d = \deg(\rho)$ and $\Delta = \deg(\dot{\lambda}^2) - 1$. Thereafter, the best practice is to check for errors while computing and to require *post hoc* quality analysis on the outputs. When $\mathcal{A} \circ \frac{dI}{dt} - \partial_t \Xi = 0$, minimum rigor is still achieved.

Something very interesting happens with the I/O map of `DihedralToODE`. For a typical input with $d_1 = \lfloor m/2 \rfloor$, $d_2 = 0$, and $d = \lfloor (m-2)/2 \rfloor$, the ouput $\mathcal{A}$ has $2d+1$ terms. We are tempted to identify $g' = d = \lfloor (m-2)/2 \rfloor$ as the genus of the Riemann surfaces associated with $\alpha = 2H$. This would foolishly contradict old and well-known theorems, which typically predict genus as a quadratic function of degree. Instead, what appears to happen is that the choice of dihedral symmetry gives the Riemann surface unexpected isoperiodicities, thus the number of distinct periods is fewer than the number of distinct homology classes. Since the dimension of $\mathcal{A}$ counts periods, it cannot yield the genus without a complementary symmetry analysis. We do not necessarily need homology/cohomology to complete the proof of Section II, but will continue working toward these advanced topics anyways.

IV. FINISHING THE PROOF

Recall that in Section II, we treated the problem of series reversion as too difficult, and simply accepted assertions for period function left factors. How can we be sure that such a factorization even exists? Consider the general form $\alpha = 2H = 2\lambda - \lambda^{d/2}\Phi$, or if you like, $\alpha\Phi^{\frac{2}{d-2}} = 2x + x^{d/2}$ with $x = \lambda\Phi^{\frac{2}{d-2}}$. The formal solution is either,

$$x = \alpha\Phi^{\frac{2}{d-2}}\left(\frac{1}{2} + \sum_{n=1}^{\infty} c_n(\alpha^{\frac{d-2}{2}}\Phi)^n\right) \quad \text{or} \quad \lambda = \alpha\left(\frac{1}{2} + \sum_{n=1}^{\infty} c_n(\alpha^{\frac{d-2}{2}}\Phi)^n\right),$$

so that chosen exponents $j = \frac{d-2}{2}$ and $k = 1$ satisfy $\alpha^{1+jn}\Phi^{\frac{2}{d-2}+kn} = \alpha^{\frac{d}{2}+j(n-1)}\Phi^{\frac{d}{d-2}+k(n-1)}$. Then we can solve either constraint on coefficients of $\alpha^{\frac{d-2}{2}}\Phi$ to obtain c_n as a function of c_{n-1} with terminal c_0. It is possible, not guaranteed that the c_n obey a hypergeometric recursion, but that property is not necessary to proceed. We only need to understand that the c_n *do not* depend on choice of Φ and that they *do* depend on a choice of d. This allows us to

TABLE III. Period analysis of a few simple hyperelliptic Hamiltonians.

$2V(q) =$	$\mathcal{A} =$	$T(\alpha) =$	s
$-\frac{2\sqrt{3}}{3}q^3$	$5 - 36(1-2\alpha)\partial_\alpha - 36\alpha(1-\alpha)\partial_\alpha^2$	$2\pi\, {}_2F_1\left[\begin{smallmatrix}\frac{1}{6},\frac{5}{6}\\ 1\end{smallmatrix}\middle\vert\alpha\right]$	6
$-\frac{1}{4}q^4$	$3 - 16(1-2\alpha)\partial_\alpha - 16\alpha(1-\alpha)\partial_\alpha^2$	$2\pi\, {}_2F_1\left[\begin{smallmatrix}\frac{1}{4},\frac{3}{4}\\ 1\end{smallmatrix}\middle\vert\alpha\right]$	4
$-\frac{6\sqrt{15}}{125}q^5$	$1701 - 2000(10-207\gamma)\partial_\gamma$	$2\pi\, {}_4F_3\left[\begin{smallmatrix}\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}\\ \frac{1}{3},\frac{2}{3},1\end{smallmatrix}\middle\vert\alpha^3\right]$	
(cont.)	$-\ldots-\ldots-90000\gamma^3(1-\gamma)\partial_\gamma^4$		
$-\frac{4}{27}q^6$	$5 - 36(1-2\beta)\partial_\beta - 36\beta(1-\beta)\partial_\beta^2$	$2\pi\, {}_2F_1\left[\begin{smallmatrix}\frac{1}{6},\frac{5}{6}\\ 1\end{smallmatrix}\middle\vert\alpha^2\right]$	6
$-\frac{50\sqrt{35}}{2401}q^7$	$12065625 - 57624(3136-815625\epsilon)\partial_\epsilon$	$2\pi\, {}_6F_5\left[\begin{smallmatrix}\frac{1}{14},\frac{3}{14},\frac{5}{14},\frac{9}{14},\frac{11}{14},\frac{13}{14}\\ \frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1\end{smallmatrix}\middle\vert\alpha^5\right]$	
(cont.)	$-\ldots-\ldots-\ldots-\ldots$		
(cont.)	$-4705960000\epsilon^5(1-\epsilon)\partial_\epsilon^6$		
$-\frac{27}{256}q^8$	$945 - 256(32-675\gamma)\partial_\gamma$	$2\pi\, {}_4F_3\left[\begin{smallmatrix}\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}\\ \frac{1}{3},\frac{2}{3},1\end{smallmatrix}\middle\vert\alpha^3\right]$	
(cont.)	$-\ldots-\ldots-36864\gamma^3(1-\gamma)\partial_\gamma^4$		

Dots indicate dropped terms. Variables are $\beta = \alpha^2, \gamma = \alpha^3, \epsilon = \alpha^5$.

produce a factored from for the period function,

$$T(\alpha) = \oint 2\partial_\alpha \lambda d\phi = \oint \left(\sum_{n=0}^{\infty} a_n (\alpha^{\frac{d-2}{2}}\Phi)^n\right) d\phi = L_d\Big(\alpha\Big) \star I_d\Big(\alpha\Big),$$

with $I_d(\alpha) = \oint (1-\alpha^{\frac{d-2}{2}}\Phi)^{-1} d\phi$. For odd powers of d, odd powers of Φ integrate to zero, thus $T(\alpha)$ always expands in whole powers of α. The right factor is analyzable via `ExpToODE`. We have already seen that $\Phi = Q^2$ leads to hypergeometric $I_d(\alpha)$, and we can also calculate the hypergeometric parameters of the $I(\alpha)$ arising from $\Phi = Q^{2n}$, for any n. This allows proof by concordance between `ExpToODE` and `HyperellipticToODE`. If $2H = p^2 + q^2 - c_d q^d$ has hypergeometric $T(\alpha)$, then the left factor L must also be hypergeometric. We can find its parameters easily by comparing the parameters of $I_d(\alpha)$ and $T(\alpha)$. Admittedly, the strategy is convoluted, but it trades the difficulties of series reversion for a routine algorithmic calculation. Ultimately, the main reason to prefer this constructive, data driven proof is that it allows us to gain more familiarity with the algorithms of Section III.

Now we just turn on the computer, instantiate the algorithms, calculate a few differential equations, and list them in Table III. They are all hypergeometric, and they should[19] exactly match previously asserted forms. As an example to show how the Hadamard factoring actually works, we use the output of `ExpToODE`$[\sin(\phi)^{10}]$ to determine that

$$I_5(\alpha) = \oint \frac{d\phi}{1-\alpha^3\sin(\phi)^{10}} = 2\pi\,{}_5F_4\left[\begin{matrix}\frac{1}{10},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{9}{10}\\ \frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\end{matrix}\middle|\alpha^3\right] = 2\pi\sum_{n=0}^{\infty}\frac{1}{2^{10n}}\binom{10n}{5n}\alpha^{3n}.$$

The complement $\{\frac{1}{10},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{9}{10}\}/\{\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}\} = \{\frac{1}{2}\}$ determines a lower parameter $\frac{1}{2}$ to go along with $\{\frac{1}{3},\frac{2}{3}\}$ from $T(\alpha)$. The extra parameter 1 we get for free, which leaves the upper parameters only. Since no cancellation occurs, they will just be the lower parameters of $I_5(\alpha)$. Now better than assertion, for $d=5$, we derive that

$$L_d(\alpha) = {}_4F_3\left[\begin{matrix}\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\\ \frac{1}{3},\frac{1}{2},\frac{2}{3}\end{matrix}\middle|\alpha^3\right] = \sum_{n=0}^{\infty}\left(\frac{2^2 3^3}{5^5}\right)^n\binom{5n}{2n}\alpha^{3n}.$$

A similar explicit calculation proves for any odd d that $a_n \propto \binom{dn}{2n}$ and for any even d that $a_n \propto \binom{dn/2}{n}$. It would be nice to have a second proof (perhaps using series reversion) especially as calculating Annihilators becomes time intensive as d increases. However, it is no bother to do the work up to $d=8$. We have done this work, so we can assure the wide results of Section II.

The outstanding case $\alpha = 2H = p^2+q^2-\frac{64}{27}p^4q^6$ should not be forgotten. It involves a difficult decic perturbation, which we will deal with circuitously. Again, start instead with a simple hyperelliptic decic, $\alpha = 2H = p^2+q^2-\frac{256}{2135}q^{10}$. The computer tells us via `HyperellipticToODE` that the period function is annihilated by

$$\mathcal{A} = 189 - 125(15-368\delta)\partial_\delta - 125\delta(335-1144\delta)\partial_\delta^2 - 10^4\delta^2(5-8\delta)\partial_\delta^3 - 10^4\delta^3(1-\delta)\partial_\delta^4,$$

with $\delta=\alpha^4$, and we can check the certificate if doubt about $\mathcal{A}$ persists[20]. The hypergeometric parameters of $T(\alpha)$ are $\{\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}\}$ upper and $\{\frac{1}{4},\frac{3}{4},1\}$ lower. Meanwhile, the parameters of right factor I_{10} are the same as I_5, so by cancellation, we calculate parameters for the shared left factor L_{10}. They are $\{\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\}$ upper and $\{\frac{1}{4},\frac{1}{2},\frac{3}{4}\}$ lower. According to `ExpToODE` with $\Phi = (z+\frac{1}{z})^5 z^3$ these parameters also determine the L_{10} expansion coefficients $a_n \propto \binom{5n}{n}$, up to choice of scale for α. Writing the right factor as,

$$I(\alpha) = \oint \frac{d\phi}{1-\alpha^4(2^{10}\cos(\phi)^6\sin(\phi)^4)} = \sum_{n=0}^{\infty}\sum_{k=0}^{4n}\binom{6n}{n+k}\binom{4n}{k}(-1)^k\alpha^{4n},$$

[19] Unfortunately when typing out long equations typos are often made. Write the author if you notice any!
[20] The certificate sums terms up to q^{35} in its numerator, so a computer will probably be necessary.

TABLE IV. Period analysis of a few simple dihedral Hamiltonians.

m	c_m	$\mathcal{A} =$	$T(\alpha) =$	s
		$\alpha = 2H = 2\lambda - c_m \lambda^{m/2}\cos(m\phi)$		
3	$\frac{4\sqrt{6}}{9}$	$2 - 9(1-2\alpha)\partial_\alpha - 9\alpha(1-\alpha)\partial_\alpha^2$	$2\pi\, {}_2F_1\left[\begin{smallmatrix}\frac{1}{3},\frac{2}{3}\\ 1\end{smallmatrix}\middle\vert\alpha\right]$	3
4	1	$3 - 16(1-2\beta)\partial_\beta - 16\beta(1-\beta)\partial_\beta^2$	$2\pi\, {}_2F_1\left[\begin{smallmatrix}\frac{1}{4},\frac{3}{4}\\ 1\end{smallmatrix}\middle\vert\alpha^2\right]$	4
5	$\frac{24\sqrt{30}}{125}$	$216 - 250(5-108\gamma)\partial_\gamma$	$2\pi\, {}_4F_3\left[\begin{smallmatrix}\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\\ \frac{1}{3},\frac{2}{3},1\end{smallmatrix}\middle\vert\alpha^3\right]$	
	(cont.)	$-\ldots-\ldots-5625\gamma^3(1-\gamma)\partial_\gamma^4$		
6	$\frac{32}{27}$	$40 - 9(27-680\delta)\partial_\delta$	$2\pi\, {}_4F_3\left[\begin{smallmatrix}\frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{5}{6}\\ \frac{1}{4},\frac{3}{4},1\end{smallmatrix}\middle\vert\alpha^4\right]$	
	(cont.)	$-\ldots-\ldots-1296\delta^3(1-\delta)\partial_\delta^4$		
7	$\frac{400\sqrt{70}}{2401}$	$450000 - 8232(343-93750\epsilon)\partial_\epsilon$	$2\pi\, {}_6F_5\left[\begin{smallmatrix}\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7},\frac{6}{7}\\ \frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1\end{smallmatrix}\middle\vert\alpha^5\right]$	
	(cont.)	$-\ldots-\ldots-\ldots-\ldots$		
	(cont.)	$-73530625\epsilon^5(1-\epsilon)\partial_\epsilon^6$		
8	$\frac{27}{16}$	$25515 - 160(1024-341901\zeta)\partial_\zeta$	$2\pi\, {}_6F_5\left[\begin{smallmatrix}\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{5}{8},\frac{3}{4},\frac{7}{8}\\ \frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{5}{6},1\end{smallmatrix}\middle\vert\alpha^6\right]$	
	(cont.)	$-\ldots-\ldots-\ldots-\ldots$		
	(cont.)	$-5308416\zeta^5(1-\zeta)\partial_\zeta^6$		
		$\alpha = 2H = 2\lambda - c_m\lambda^m\big(1+\cos(2m\phi)\big)$		
2	$\frac{1}{2}$	$1 - 4(1-2\alpha)\partial_\alpha - 4\alpha(1-\alpha)\partial_\alpha^2$	$2\pi\, {}_2F_1\left[\begin{smallmatrix}\frac{1}{2},\frac{1}{2}\\ 1\end{smallmatrix}\middle\vert\alpha\right]$	2
3	$\frac{16}{27}$	$2 - 9(1-2\beta)\partial_\beta - 9\beta(1-\beta)\partial_\beta^2$	$2\pi\, {}_2F_1\left[\begin{smallmatrix}\frac{1}{3},\frac{2}{3}\\ 1\end{smallmatrix}\middle\vert\alpha\right]$	3
4	$\frac{27}{32}$	$27 - 8(16-351\gamma)\partial_\gamma$	$2\pi\, {}_4F_3\left[\begin{smallmatrix}\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4}\\ \frac{1}{3},\frac{2}{3},1\end{smallmatrix}\middle\vert\alpha^3\right]$	
	(cont.)	$-\ldots-\ldots-576\gamma^3(1-\gamma)\partial_\gamma^4$		
5	$\frac{4096}{3125}$	$384 - 375(5-128\delta)\partial_\delta$	$2\pi\, {}_4F_3\left[\begin{smallmatrix}\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\\ \frac{1}{4},\frac{3}{4},1\end{smallmatrix}\middle\vert\alpha^4\right]$	
	(cont.)	$-\ldots-\ldots-10000\delta^3(1-\delta)\partial_\delta^4$		

Dots indicate dropped terms. Variables are $\beta=\alpha^2, \gamma=\alpha^3, \delta=\alpha^4, \epsilon=\alpha^5, \zeta=\alpha^6$.
See Appendix A for computer proofs.

TABLE V. Canonical models H_s for which $\mathcal{A}_s = (s-1) - s^2(1-2\alpha)\partial_\alpha - s^2\alpha(1-\alpha)\partial_\alpha^2$.

$2H_2 = p^2 + q^2 - p^2q^2$	$T_2(\alpha) = \ 2\pi\ {}_2F_1\left[\begin{smallmatrix}\frac{1}{2};\frac{1}{2}\\1\end{smallmatrix}\middle\vert\alpha\right]$	genus 1
$2H_3 = p^2 + q^2 - (\frac{4}{27})^{\frac{1}{2}}\ (q^3 - 3p^2q)$	$T_3(\alpha) = \ 2\pi\ {}_2F_1\left[\begin{smallmatrix}\frac{1}{3};\frac{2}{3}\\1\end{smallmatrix}\middle\vert\alpha\right]$	genus 1
$2H_4 = p^2 + q^2 - \frac{1}{4}q^4$	$T_4(\alpha) = \ 2\pi\ {}_2F_1\left[\begin{smallmatrix}\frac{1}{4};\frac{3}{4}\\1\end{smallmatrix}\middle\vert\alpha\right]$	genus 1
$2H_6 = p^2 + q^2 - \frac{2\sqrt{3}}{9}q^3$	$T_6(\alpha) = \ 2\pi\ {}_2F_1\left[\begin{smallmatrix}\frac{1}{6};\frac{5}{6}\\1\end{smallmatrix}\middle\vert\alpha\right]$	genus 1
$2H_3' = p^2 + q^2 - \frac{4}{27}(q^3 - 3p^2q)^2$	$T_3(\alpha^2) = 2\pi\ {}_2F_1\left[\begin{smallmatrix}\frac{1}{3};\frac{2}{3}\\1\end{smallmatrix}\middle\vert\alpha^2\right]$	genus 4
$2H_4' = p^2 + q^2 - \frac{1}{4}(p^2+q^2)^2 + 2p^2q^2$	$T_4(\alpha^2) = 2\pi\ {}_2F_1\left[\begin{smallmatrix}\frac{1}{4};\frac{3}{4}\\1\end{smallmatrix}\middle\vert\alpha^2\right]$	genus 3
$2H_6' = p^2 + q^2 - \frac{4}{27}q^6$	$T_6(\alpha^2) = 2\pi\ {}_2F_1\left[\begin{smallmatrix}\frac{1}{6};\frac{5}{6}\\1\end{smallmatrix}\middle\vert\alpha^2\right]$	genus 2

Transforming $\alpha \to \beta = \alpha^2$ reduces $\mathcal{A}_s' = 4(s-1) - s^2(1-3\alpha^2)\partial_\alpha - s^2\alpha(1-\alpha^2)\partial_\alpha^2 \to \mathcal{A}_s$.

allows us to uncover yet another unexpected binomial identity,

$$\binom{3n}{n}\binom{6n}{3n} = \binom{5n}{n}\sum_{k=0}^{4n}\binom{6n}{n+k}\binom{4n}{k}(-1)^k.$$

This identity essentially explains why $\alpha = 2H = p^2 + q^2 - \frac{64}{27}p^4q^6$ should have period $T_6(\alpha^4)$. If there is any doubt about veracity, the identity can be double checked using Zielberger's algorithm for hypergeometric summations.

The general strategy of concordance is effective though convoluted. We would like to have more direct proofs, especially for the seven cases depicted in Fig. 2 and Fig. 3. These cases divide neatly into two classes: simple hyperelliptic or simple dihedral. Table III already contains three annihilators with identifiable signature. This is no surprise, as we already know that H_6, H_4, and H_6' have the required hyperelliptic form. The remaining H_2, H_3, H_3' and H_4' all have simple dihedral symmetry, so their period functions can be proven more directly using `DihedralToODE`. We instantiate an implementation, map across a small search space, and in Table IV list the four hits alongside a few other negatives. Since `DihedralToODE` is only an effective algorithm, we also list certificates,

$$\Xi_3 = \frac{4\sqrt{6}\lambda^{5/2}\sin(3\phi)}{(3\alpha-2\lambda)^3},\quad \Xi_4' = \frac{\lambda^3\sin(4\phi)}{4(\alpha-\lambda)^3},\quad \Xi_2 = \frac{\lambda^3\sin(4\phi)}{8(\alpha-\lambda)^3},\quad \text{and}\quad \Xi_3' = \frac{32\lambda^4\sin(6\phi)}{3(3\alpha-4\lambda)^3},$$

for H_2, H_4', H_2 and H_3' respectively. The results of `DihedralToODE` listed in Table V can be checked against these certificates. By judicious use of the chain rule, we simply by calculate a zero value, $\mathcal{A}_s \circ dt \pm \partial_t \Xi_s = 0$, where choice of either $+$ or $-$ accounts for the possibility of a sign error (whether accidental or not, sign errors do sometimes happen). The other three results of `HyeperellipticToODE` involve longer certificates, here omitted. Those annihilation relations can also be checked post computation if necessary (cf. Appendix A).

The last step of the proof involves verifying the genus figures quoted in Table V, column three. According to well-known genus degree bounds, we expect $g = \lfloor (d-1)/2 \rfloor$ for hyperelliptic cases and $g = \frac{1}{2}(d-1)(d-2)$ for dihedral cases. The upper bound is only met for H_3, H_4, H_6, H_4', and H_6'. The other models, H_2, and H_3', involve hidden symmetries, which require a corrective term $\Delta = -\sum_n \frac{1}{2} r_n(r_n+1)$. The correction Δ characterizes behavior of singular points at infinity. As early as 1884, Max Noether gave a procedure for calculating Δ[30]. The same basic idea of counting genus as $g = \frac{1}{2}(d-1)(d-2) + \Delta$ is nowadays built into standard algorithms of algebraic geometry[31]. In this case, we do not need to reinvent the wheel, so simply input $2H_s - \frac{1}{2} = 0$ or $2H_s' - \frac{1}{2} = 0$ into Singular software, and use the built-in genus function to output the correct integer[32]. In principle, this I/O approach only tells us the genus of a curve $\mathcal{C}_s(\frac{1}{2})$ or $\mathcal{C}_s'(\frac{1}{2})$. The domain of the oscillation disk is unobstructed by critical points, so it is safe to assume[21] that $g\big(\mathcal{C}(\frac{1}{2})\big) = g\big(\mathcal{C}(\alpha)\big)$ when $\alpha \in (0,1)$. The strong statement of Section II stands true.

For most purposes, the first four rows of Table V are a good-enough starting place. If not, a next step into the territory of higher genus takes into consideration the models listed in the final three rows. We have already found more exotic species, and are not quite done with the task of cataloging. One glaring omission on our part is that we have yet to mention $2H = p^2 + 12q^2 + 8q^3 - 36q^4 - 48q^5 - 16q^6$, a genus 2 hyperelliptic model with period $T(\alpha) \propto T_3(\alpha)$. It belongs to an infinite class of hyperelliptic models, indexed by degree $d = 3, 4, 5...$, each having a potential $V_d(q) = \frac{1}{4}\big(1 - \mathcal{T}_d(q)\big)$, where $\mathcal{T}_d(q)$ stands for the d^{th} Chebyshev polynomial of the first kind[22]. Using `HyperellipticToODE` it is possible to prove that $\mathcal{A}_d = \frac{d^2-4}{4d^2} - (1-2\alpha)\partial_\alpha - \alpha(1-\alpha)\partial_\alpha^2$, say for the first ten or twenty cases. It would be interesting to determine a d-dependent form for certificates Ξ_d, but not strictly necessary since one proof of $\mathcal{A}_d$ has already been given. This family of Chebyshev potentials is truly amazing in terms of hidden symmetry, but it doesn't yield any new cases with $g = 1$ after $d = 4$, nor does it yield any more cases of identifiable signature after $d = 6$.

[21] Perhaps this point could use more rigor eventually, but presently we take it as common sense.

[22] There are many different kinds of Chebyshev polynomials, here we mean $\mathcal{T}_n\big(\cos(x)\big) = \cos(nx)$.

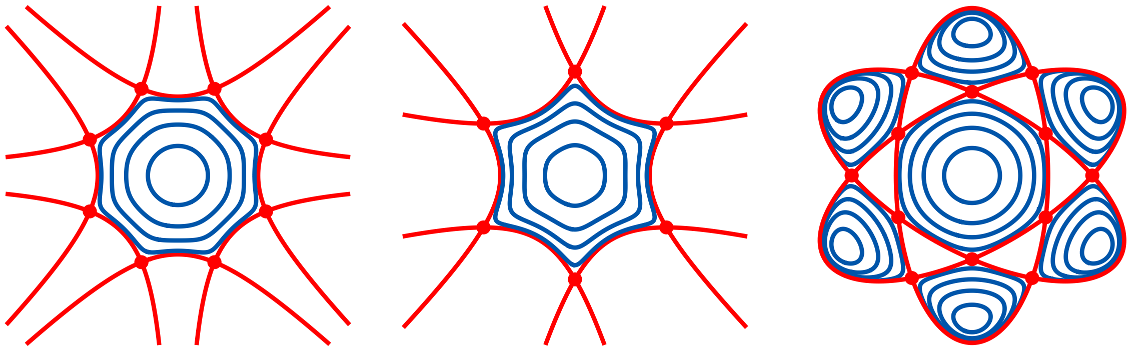

TABLE VI. A few more use cases for `DihedralToODE`.

Input: $2H =$	Output: $\mathcal{A} =$	s
$\lambda^2 - \frac{1}{8}\lambda^4\big(1-\cos(8\phi)\big)$	$(4-15\alpha)+16\alpha(2-3\alpha)\partial_\alpha$ $+16\alpha^2(1-\alpha)\partial_\alpha^2$	4
$\lambda^2 - \frac{2\sqrt{3}}{9}\lambda^3\cos(6\phi)$	$(9-32\alpha)+36\alpha(2-3\alpha)\partial_\alpha$ $+36\alpha^2(1-\alpha)\partial_\alpha^2$	6
$2\lambda - \frac{32}{27}\lambda^2 + \frac{16}{729}\lambda^3\big(9-\cos(6\phi)\big)$	$8(18-25\alpha)-9(27-104\alpha+75\alpha^2)\partial_\alpha$ $-9\alpha(1-\alpha)(27-25\alpha)\partial_\alpha^2$	

One last construction is worth mentioning. Both H_3' and H_4' feature a factorizable separatrix that decomposes to a product of hyperbolas. Generalizing on this form and performing analysis via `DihedralToODE`, we find a few more interesting cases to take note of, as in Table VI and depicted above. The first two rows list Hamiltonians with identifiable signature. The corresponding curve geometries are not oscillation disks in the strict sense because leading term λ^2 does not allow for a harmonic limit at the origin. Yet period functions still exist, and they can be written as $T(\alpha) = T_s(\alpha)/\sqrt{\alpha}$ for $s = 4$ or 6. In the third row, the annihilator is not hypergeometric, but it does have the special property[23] that $\partial_\alpha(\alpha(1-\alpha)(27-25\alpha)) = (27-104\alpha+75\alpha^2)$. Remarkably, this operator also shows up in a completely different combinatorial search performed by Bostan et al. [33].

Even more waits to be discovered and explained. It would be nice to develop a constructive theory for explaining all higher-genus models in terms of the genus 1 examples given in Table V, but we cannot do so presently. Instead of digging deeper into questions of existence and equivalence, next we will return to the practical task of using period functions to calculate numerical values.

[23] Compare with the "well-integrable" geometries of the Prospectus [1].

V. PERIODS AND SOLUTIONS

Physicists recognize hyperelliptic Hamiltonians without any qualms because the assumed form requires separability, with kinetic energy $\frac{1}{2}p^2$. Indeed, the simplest cases, H_6 and H_4, are textbook examples often found under the heading of anharmonic oscillation or perturbation theory[34]. As models for data, H_6 and H_4 work well in situations where a period varies linearly around a harmonic limit. In this context, local analysis of H_6 and H_4 generalizes the simple harmonic approximation to the *simple anharmonic approximation*. If the data is precise enough, more terms can be added to the potential $V(q)$ until the corresponding function $T(\alpha)$ sufficiently matches data[24]. Sometimes it may be possible to measure a period function over an entire oscillation disk. These circumstances motivate us to develop an entire theory for solving period functions. An important question is: what can we exclude?

Our own prejudices may tempt us to discard any functions not conforming to the seperable form $H = \frac{1}{2}p^2 + V(q)$. This would be a crime of oversimplification and an unnecessary handicap to analysis in general. Hamiltonian H_2 is a perfect example in support of inclusiveness, because the corresponding oscillation disk exactly models the libration motion of a simple pendulum. The role of transformation theory can not be ignored. It allows us to change a seemingly malformed perturbing term, p^2q^2, into something more familiar, $V(q) = \sin(q)^2$. We have no catalogue of all such transforms, but expect more to be discovered soon. The point is that even though dihedral models look funny compared to their hyperelliptic counterparts, they may eventually facilitate analysis. As nature produces a variety of forms, so should we.

When formulating a space of viable models for anharmonic oscillation, a permissive attitude allows us to include any oscillation disk that we can manage to integrate. Algorithms of the previous sections allow a great many integral period functions to be calculated as solutions to output ordinary differential equations. Instead of evaluating an integral every time we need a value of function $T(\alpha)$, we only need to calculate integrals for a small set of initial data, say $T_1 = T(\alpha_1),\ T_2 = T(\alpha_2),\ \ldots,\ T_n = T(\alpha_n)$. When n equals to the order of $\mathcal{A}$, the system of equations should be uniquely solvable for $T(\alpha)$ as a proper function. Our preference is for piecewise series solutions, but completely numerical solutions also work.

[24] Although, there is one caveat. Period function $T(\alpha)$ does not uniquely determine potential $V(q)$ [34].

To show how solutions are carried out in practice, we will now solve $T_s(\alpha) = \partial_\alpha S_s(\alpha)$ for each of $s = 2, 3, 4$ and 6. The four cases are similar enough to proceed with variable s, except when determining initial data. Recall that action function $S_s(\alpha)$ has a natural geometric interpretation as the area enclosed by curve $\mathcal{C}_s(\alpha)$. For this reason, $S_s(\alpha)$ makes a better starting place than $T_s(\alpha)$, but first we need to transform the period annihilator into an action annihilator. Assume that $\mathcal{A}_S \circ S(\alpha) = 0$, then $\partial_\alpha \circ \big(\mathcal{A}_S \circ S(\alpha)\big) = 0$ can be rewritten as $\mathcal{A}_T \circ T(\alpha) = 0$, and the identity $\mathcal{A}_T = \mathcal{A}_s$ determines that $\mathcal{A}_S = (s-1) - s^2\alpha(1-\alpha)\partial_\alpha^2$. It is again hypergeometric, but the missing middle term causes lower parameter c to equal zero. The usual coefficient recursion would have $f_1 \propto f_0/0$. This solution does not work, so another must be found. In general, $\mathcal{A}_S \circ S_s(\alpha) = 0$ is solved by[25],

$$
\begin{aligned}
S_s(\alpha) &= C_1 \alpha \, {}_2F_1\left[\begin{matrix} \frac{1}{s}, \frac{s-1}{s} \\ 2 \end{matrix} \middle| \alpha \right] - C_0 \alpha \, {}_2F_1\left[\begin{matrix} \frac{1}{s}, \frac{s-1}{s} \\ 2 \end{matrix} \middle| \alpha \right] \int \left(\alpha \, {}_2F_1\left[\begin{matrix} \frac{1}{s}, \frac{s-1}{s} \\ 2 \end{matrix} \middle| \alpha \right] \right)^{-2} d\alpha \\
&= C_0 \bigg(1 + \Big(\frac{s-1}{2\, s^2} \Big) \alpha \bigg) + \bigg(C_1 + C_0 \Big(\frac{s-1}{2\, s^2} \Big) \log(\alpha) \bigg) \alpha \, {}_2F_1\left[\begin{matrix} \frac{1}{s}, \frac{s-1}{s} \\ 2 \end{matrix} \middle| \alpha \right] + \sum_{n>1} C_n \, \alpha^n .
\end{aligned}
$$

The first line is perfectly valid, but may be doubted. The second line gives an Ansatz, which is already correct up to a set of undetermined coefficients. The condition,

$$
\begin{aligned}
0 = \mathcal{A}_S \circ S_s(\alpha) = \sum_{n>0} a_n \alpha^n = &\Big((s-1)(s^2 - s + 1) C_0 - 2 s^4 C_2 \Big) \alpha \\
&+ \Big((s-1)^2 (5 - 5s + 8s^2) C_0 + 12 s^4 (1+s)(2s-1) C_2 - 72 s^6 C_3 \Big) \alpha^2 + \ldots ,
\end{aligned}
$$

sets up a recursion on the coefficients a_n. As the summation continues to higher powers of α, the pattern continues, and the first appearance of C_n always occurs in coefficient a_{n-1}. The system of equations can be solved sequentially to determine $C_n \propto C_0$, but they are just the expansion coefficients of the integral on the preceding line.

The expansion around $\alpha = 0$ requires $C_0 = 0$, for $\partial_\alpha(\alpha \log(\alpha)) = 1 + \log(\alpha)$, but $T(\alpha)$ is finite valued at the origin. The harmonic limit $T(0) = 2\pi$ also determines $C_1 = \pi$, so we need only one initial condition rather than two. With the solution $S_s(\alpha) = \pi \alpha \, {}_2F_1\left[\begin{matrix} \frac{1}{s}, \frac{s-1}{s} \\ 2 \end{matrix} \middle| \alpha \right]$, very slow convergence becomes a problem as α approaches 1. Fortunately, symmetry allows for a second expansion, which converges rapidly where the first diverges and *vice versa*. Annihilator $\mathcal{A}_S$ transforms invariantly by $\alpha \to \reflectbox{$\alpha$} = 1 - \alpha$, so we can write a reversed expansion $(\reflectbox{$\alpha$})_{s}\reflectbox{$S$}$ around $\alpha = 1$ by simply taking $S_s(\alpha)$ and reversing $C_n \to \reflectbox{$C$}_n$ and $\alpha \to \reflectbox{$\alpha$}$.

[25] Recall [1] Sec. IV-V, or see again: Hypergeometric Function, Second-Order ODE Second Solution.

TABLE VII. Initial data of reversed action functions.

s	$\reflectbox{$\mathfrak{C}$}_0 =$	$\reflectbox{$\mathfrak{C}$}_1 =$	
2	4	$-4.2725887222397812\ldots$	$= -\frac{3}{2} - 4\log(2)$
3	$9\frac{\sqrt{3}}{4}$	$-4.1533165583656958\ldots$	$= -\frac{3}{4}\sqrt{3}\big(1+2\log(3)\big)$
4	$8\frac{\sqrt{2}}{3}$	$-4.0014346021854628\ldots$	$= -\frac{3}{4}\sqrt{2}\big(1+4\log(2)\big)$
6	$\frac{18}{5}$	$-3.7842127941220551\ldots$	$= -\frac{1}{4}\big(3+8\log(2)+6\log(3)\big)$

The identity $\mathcal{A}_S \circ \reflectbox{$S_s(\alpha)$} = 0$ holds through reversal of α, thus the solution is correct up to determination of $\reflectbox{$\mathfrak{C}$}_0$ and $\reflectbox{$\mathfrak{C}$}_1$. These initial data depend on the behavior near $\alpha = 1$, which in turn depends on the choice of s. The zeroth coefficient is just the area enclosed by the separatrix, $\reflectbox{$\mathfrak{C}$}_0 = \text{area}\big(\mathcal{C}_s(1)\big)$. Determination of $\reflectbox{$\mathfrak{C}$}_0$ requires case by case integration. Curves $\mathcal{C}_2(1)$ and $\mathcal{C}_3(1)$ bound equilateral polygons, so the corresponding interior areas can be calculated proportional to squared side lengths. Separatrices $\mathcal{C}_4(1)$ and $\mathcal{C}_3(1)$ are not polygon boundaries, but the area can be calculated by $2\int p\, dq$ across the maximum width of the oscillation disk. When $\reflectbox{$\mathfrak{C}$}_0$ is known exactly, then $\reflectbox{$\mathfrak{C}$}_1$ can be determined to arbitrary precision by solving $S_s(\alpha) = \reflectbox{$S_s(\alpha)$}$ at the midpoint $\alpha = \reflectbox{$\alpha$} = \frac{1}{2}$. Table VII collects the results of our numerical calculations.

The two solutions are equivalent, so they can be stitched together at $\alpha = \reflectbox{$\alpha$} = \frac{1}{2}$ to form a piecewise solution $\underset{pw}{S_s}(\alpha)$, which equals $S_s(\alpha)$ when $0 \le \alpha \le \frac{1}{2}$, otherwise it equals $\reflectbox{$S_s(\alpha)$}$ when $\frac{1}{2} < \alpha \le 1$. As a *computable function*, $\underset{pw}{S_s}(\alpha)$ implicitly relies on a truncation parameter N, which controls how many terms are summed in both expansions. Choosing $N = 60$ is good enough to guarantee double precision, according to the error bound $\mathcal{A}_s \circ \underset{pw}{S_s}(\alpha) < 10^{-16}$ when $\alpha \in [0,1]$. Precision can always be increased by retaining more summation terms, but don't forget, a reconstruction of $\underset{pw}{S_s}(\alpha)$ with more terms also requires recalculation of the coefficient $\reflectbox{$\mathfrak{C}$}_1$. This is only a slight annoyance, one we must live with for now. We still don't know of a reasonable method to obtain exact solutions for the four values $\reflectbox{$\mathfrak{C}$}_1$, but strongly suspect the guesses[26] in Table VII are correct. As error reaches a maximum at the boundary, a less conservative error estimate is obtained by subtracting the numerical value from the (assumed correct) exact value. When this is done for the $N = 60$ approximation, we can

[26] The computer program RIES accepts numerical values as input and returns probable closed forms as output. It is free software and available online at https://mrob.com/pub/ries/.

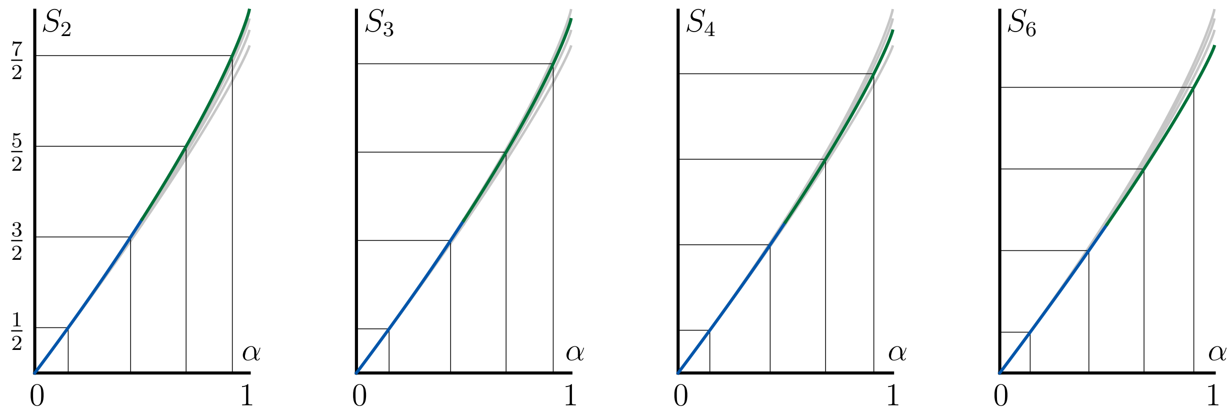

FIG. 5. Root solving computable functions, $\underset{pw}{S_s}(\alpha_n) = \frac{1}{4}(n+\frac{1}{2})\underset{pw}{S_s}(1)$ with $n = 0, 1, 2$, and 3.

TABLE VIII. Semiclassical quantum values for $h = S_s(1)$ and $N = 4$, as in Fig. 5.

s	$\alpha_0 =$	$\alpha_1 =$	$\alpha_2 =$	$\alpha_3 =$
2	0.15592230 . . .	0.44694844 . . .	0.70571106 . . .	0.92129983 . . .
3	0.15232502 . . .	0.43916833 . . .	0.69785371 . . .	0.91761719 . . .
4	0.14788266 . . .	0.42934933 . . .	0.68764827 . . .	0.91260116 . . .
6	0.14176761 . . .	0.41544690 . . .	0.67263195 . . .	0.90470726 . . .

estimate the error of $\underset{pw}{S_s}(\alpha)$ less than 10^{-20} on the entire oscillation disk. Having defined a set of computable functions with error bounds, we can now begin to calculate.

Plots of functions $S_s(\alpha)$, as in Fig. 5, allow us to see with our eyes how similar the alternatives are. All four have the same small-amplitude limit $S_s(\alpha) \approx \pi\alpha$, but begin to diverge from one another as α approaches 1. Maximum difference occurs at $\alpha = 1$. An easy bound, $\frac{2(4-18/5)}{(4+18/5)} = \frac{2}{19}$, says that, at most, the value of one action function differs from the value of another by about 10%. In addition to the graphs, we also want to calculate a few special "quantum values"[27]. In principle, the action functions could be inverted to find energy $\alpha = S_s^{-1}(S)$ as a function of action S. However, when only a few values α_n are needed, it is much more efficient to simply root solve $\underset{pw}{S_s}(\alpha_n) - h_n = 0$, say by Newton's method. Calculating the so-called quantum values requires special choice of a density N/h, which in turn determines $h_n = \frac{h}{N}(n + \frac{1}{2})$ for $n = 0, 1, \ldots, N-1$. In Figure 5 we set the action scale as $h = \text{Ↄ}_0$, choose $N = 4$. The values of Table VIII then determine the blue

[27] It is left as an exercise: find matrices with comparable eigenvalues, see also [1] Sec. VI.

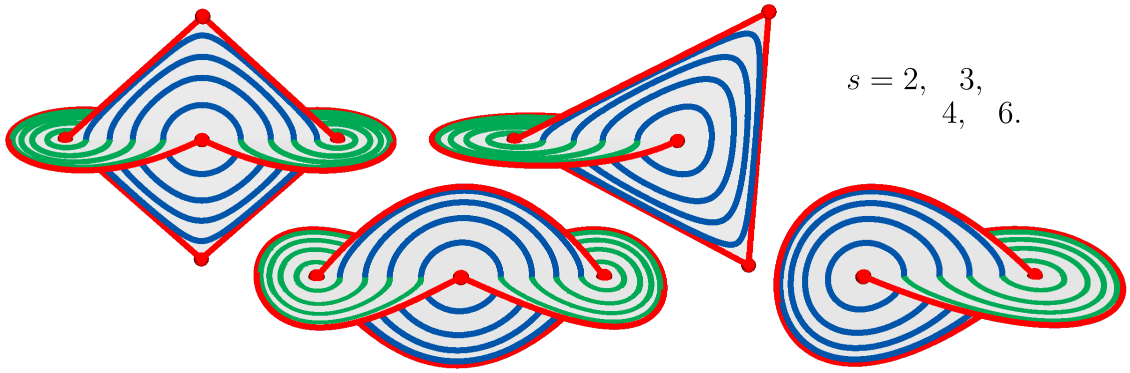

FIG. 6. Toric cross sections with real/complex contours in blue/green.

curves of Fig. 2. In those graphs, the area contained by any central $n = 0$ blue curve is $\frac{1}{8}\Im_0$, as is the area between any $n = 3$ curve and the separatrix. The area between any two consecutive blue curves equals to $\frac{1}{4}\Im_0$. Thus the total area enclosed by the separatrix is $\left(\frac{1}{8} + \frac{1}{8} + 3\frac{1}{4}\right)\Im_0 = \Im_0$ as necessary. The quantum theory will contribute more significantly to followup articles. Before then, we have a few other classical calculations to discuss.

Computable instances of the alternative period functions, allow us to calculate alternative nomes, the q's mentioned in the introduction. Yet the q's seem to stand for *questions unanswered*, and we are left to wonder. What should we make of them? How are they used in the theory of elliptic curves? They are not the q's of Hamiltonian mechanics, rather, they are *nomes*, which conform to the general expression $q = e^{i\tau}$, with $\tau = T_{\mathfrak{I}}/T_{\mathfrak{R}}$, the ratio between real and complex periods. Thus far, we have only dealt with real periods, but for the set of basic H_s the real periods are the complex periods, up to a change of Harmonic frequency and inversion of the energy scale $\alpha \to 1 - \alpha$. The annihilator $\mathcal{A}_s$ transforms invariantly by $\alpha \to 1 - \alpha$, so we can write $T_{\mathfrak{I}} = i(\omega_s/\widetilde{\omega}_s)T_s(1 - \alpha)$. To obtain the frequency ratios, simply apply an Abel-Wick rotation[28] from $H_s \to \widetilde{H}_s$, as in Fig. 6, and find the harmonic frequency around a circular point of $\widetilde{H}_s$. For signatures $s = 2, 3, 4, 6$, we calculate the corresponding complex-time frequencies $\widetilde{\omega}_s$ as $\widetilde{\omega}_2 = 2$, $\widetilde{\omega}_3 = \sqrt{3}$, $\widetilde{\omega}_4 = \sqrt{2}$, and $\widetilde{\omega}_6 = 1$. By convention, $\omega_s = 1$, then follows $\omega_s/\widetilde{\omega}_s = \frac{1}{2}\csc(\pi/s)$, as quoted in the introduction. In the case $s = 2$, we have already shown how the nome q contributes to an exact solution using Harold Edwards's alternative theory [25]. More work needs to be done using other q's to solve the time parameterization problem on other H_s, especially case $s = 3$.

[28] Refer back to [2] Sec. IV. A rotational axis must be chosen to intersect a hyperbolic point on the separatrix.

VI. A FEW BINOMIAL SERIES FOR π

A very special feature of $S_s(\alpha)$ and $T_s(\alpha)$ is that they satisfy a Legendre-style identity, $S_s(\alpha)T_s(\reflectbox{\wp}) + S_s(\reflectbox{\wp})T_s(\alpha) = 2\pi\ \reflectbox{\mathcal{C}}_0$. It follows from $T_s(\alpha) = 2\partial_\alpha S_s(\alpha)$ and $\mathcal{A}_S \circ S_s(\alpha) = 0$, after taking either limit $\alpha \to 0$ or $\alpha \to 1$. First observe,

$$\partial_\alpha\big(S_s(\alpha)T_s(\reflectbox{\wp}) + S_s(\reflectbox{\wp})T_s(\alpha)\big) = -S_s(\alpha)S_s''(\reflectbox{\wp}) + S_s(\reflectbox{\wp})S_s''(\alpha) = 0,$$

because solving either $\mathcal{A}_S \circ S_s(\alpha) = 0$ or $\mathcal{A}_S \circ S_s(\reflectbox{\wp}) = 0$ produces $S_s''(z) = \frac{s-1}{s^2\alpha(1-\alpha)} S_s(z)$ with either $z = \alpha$ or $z = \reflectbox{\wp}$. Next take the limit,

$$\lim_{\alpha\to 0} S_s(\alpha)T_s(\reflectbox{\wp}) + S_s(\reflectbox{\wp})T_s(\alpha) = \text{constant} \times \lim_{\alpha\to 0}\big(\alpha \times \log(1-\alpha)\big) + \reflectbox{\mathcal{C}}_0 \times 2\pi = 2\pi\ \reflectbox{\mathcal{C}}_0,$$

and the identity is already proven, no problem! Choosing $\alpha = \reflectbox{\wp} = \frac{1}{2}$ and dividing both sides by $2\pi^2$, we obtain a few summations for $\reflectbox{\mathcal{C}}_0/\pi$,

$$\begin{aligned}
\frac{4}{\pi} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{32^n}\frac{1}{(k+1)}\binom{2(n-k)}{n-k}^2\binom{2k}{k}^2, \\
\frac{9\sqrt{3}}{4\pi} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{54^n}\frac{1}{(k+1)}\binom{3(n-k)}{n-k}\binom{2(n-k)}{n-k}\binom{3k}{k}\binom{2k}{k}, \\
\frac{8\sqrt{2}}{3\pi} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{128^n}\frac{1}{(k+1)}\binom{2(n-k)}{n-k}\binom{4(n-k)}{2(n-k)}\binom{2k}{k}\binom{4k}{2k}, \\
\frac{18}{5\pi} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{864^n}\frac{1}{(k+1)}\binom{3(n-k)}{n-k}\binom{6(n-k)}{3(n-k)}\binom{3k}{k}\binom{6k}{3k}.
\end{aligned}$$

After dropping denominators, the corresponding integer sequences,

$$s = 2 : 1, 6, 56, 620, 7512, 96208, 1279168, 17471448, 243509720, 3447792656, \ldots$$

$$s = 3 : 1, 9, 138, 2550, 51840, 1116612, 24999408, 575368596, 13518747000, \ldots$$

$$s = 4 : 1, 18, 632, 27300, 1306200, 66413424, 3515236032, 191434588488, \ldots$$

$$s = 6 : 1, 90, 20280, 5798100, 1854085464, 632693421360, 225235329359040, \ldots$$

are not found in OEIS as of June 16, 2020. These are not exactly what Ramanujan had in mind for Section 14 of [13], but they appear similar. Perhaps they are too straightforward or not rapidly convergent enough, but at least they are easy to derive. We would like to carry this idea farther, but simply do not have time or space. Interested readers are refered to Jonathan and Peter Borwein's "Pi and the AGM"[24].

VII. CONCLUSION

Famously, G.N. Watson (1886-1965) compared Ramanujan with the Italian renaissance by saying,

> *Ramanujan's formula gave me a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuova of the Capille Midicee and see before me the austere beauty of the four statues representing Day, Night, Evening, and Dawn, which Michelangelo has set over the tombs of the Medicis.*

We don't want to take away Watson's imminent and lordly right to a personal opinion. This sort of high praise is appropriate, but is it another example of Eurocentric bias from a leader of an Anglocentric society? Even to this day, the question of how we should view Ramanujan and his work remains persistently difficult. An antithesis is to compare with the temple architecture at Namakkal. This could be more appropriate considering the profound importance of place in Ramanujan's life. Such a comparison misses the point of Watson's appraisal, which is to say that Ramanujan became a solid contributor to Western culture.

There is a middle way and a synthesis of perspectives, perhaps not known only to the author, where Ramanujan's works are described as a type of music that came into existence before it could even be properly heard. Although Pandit Ravi Shankar and Ustad Zakir Hussain do not hail from Tamil Nadu, they are also natural Indian citizens who used their work to transcend national boundaries. In Watson's day the idea of a world harmony was only a promise, one made amidst the machinations of World War. Later it did come to pass that Ravi Shankar and Zakir Hussain helped to invent the audible genre of world music. The rest is on records, tapes, CDs, DVDs and probably a few youtube videos[29].

The musicality of Ramanujan's mathematics bears some relation to the prosody of earlier Indian scientists. First the Pingala wrote sayings for binomial coefficients, then Madhava for trigonometric functions, and after Ramanujan, now there is even a saying for elliptic integrals. Is Ramanujan's famous assertion of alternatives K_1, K_2, and K_3 a product of the East or of the West? And whose hearing is it meant to reach? The chosen language is not Sanskrit, nor Latin. Nor are alternatives K_1, K_2, and K_3 otherwise hidden from lower classes or less

[29] Listen also: *West Meets East Volumes I & II*, Ravi Shankar and Yehudi Menuhin; *Passages*, Ravi Shankar and Phillip Glass; *Remember Shakti Live at Jazz a Vienne*, Zakir Hussain, John McLaughlin et al. And here's one more, *Tala Matrix*, Zakir Hussain, Bill Laswell, et al.

privileged castes. Mahatma Ghandi and his followers won their engagements worldwide. The future is not the exclusive property of any trading company, and the Brahmin's exclusive right to practice Hindu science has also been released. It has never been easier for anyone to find out about Ramanujan's theory, or any theory thereafter!

How can we all come to appreciate Ramanujan? We don't know, but continue to spend our efforts unfolding what we can from ideas he left behind. The ideas are difficult, but as we have seen, computing power makes them more readily accessible. Already, billions and trillions of digits of π have been computed using his insight. Too bad we do not have time to develop more about $1/\pi$, but it is not the only place where Ramanujan's presence can be felt. The preceding analysis of geometric models means to open up a whole new plot in the Mysetrious garden. What else grows in this pleasant and peaceful space? Are there perhaps beetles and butterflies practicing symbiosis with the flora? Poetics aside, at least we have a program for extending systematic analysis[30].

Science is a process of developing bias, and in the end, science will defeat itself unless the practice of *branching out* is maintained. One weakness of the present article is that we have focused mostly on hypergeometric cases in order to suggest that only four achieve a minimal form. This point is at once worthwhile and irrelevant. We should, and we will, spend more time analyzing geometries such as case 3 of Table VI. It *is not* a hypergeometric case, so the Hadamard factor analysis does not even apply! Yet the periods of this geometry, and of other similar geometries, are still differentiably-finite. That is, the period and action functions are the solutions of ordinary differentiable equations. The analysis is not much more difficult, and we can make more progress easily.

In Section V, Table VIII is really more profound than it seems. The semiclassical eigenvalues do more than enable harmonic proportions in drawn figures. In physics, they provide a bridge between the classical theory and the quantum theory. Similar values can be calculated by matrix methods, but tunneling between states also needs to be accounted for. This is one focus of a planned followup article, where we will also include a brief explanation as to how differential equations can be used for classical data analysis. This effort will build upon algorithms defined in this article, so we have already completed much of the work necessary. Physics and math, practice non-duality and the way is clear.

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[30] Again refer back to the Prospectus of [1].

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Appendix A SUPPORTING DOCUMENTS

The three implementations for `EasyCT` discussed above were given ahead of schedule as supplementary material for the dissertation prelude, and in fact, all results of Table V have been proven correct since the release of the Wolfram Demonstration A few more geometries after Ramanujan. Similarly, analysis of right factors has been possible since the earlier release of, Approximating Pi with Trigonometric-Polynomial Integrals. For Section II, a bit of extra programming was done to create a large search space and analyze results found. Another notebook containing this search is available via github. The search is fast enough to rerun in under an hour(?). We have also created an extensive set of I/O records, each of which is trivial to validate using the CheckCert notebook. For section IV, more searching was done using `HyperellipticToODE` and `DihedralToODE`. Those searches are also given (with certificate checking turned on) in another supplementary notebook at github.

As is typically the case, this article was slightly delayed by another branching-out opportunity that we decided to take seriously. An argument on the [math-fun] mailing list eventually led us all to consider how simple knots such as the trefoil might be represented as algebraic varieties. Following a suggestion of Cris Moore, we quickly found a seemingly minimal solution for the trefoil, and also managed to integrate its natural period function, essentially using an `EasyCT` algorithm. The certified result, along with a few plots, was then published on Wolfram Demonstrations under the title, Algebraic Family of Trefoil Curves. It is relatively easy to generalize the calculation to other similar torus knots, so there is certainly room for growth and more comparative analysis. After all, why should we spend all of our thoughts and efforts on integrating unknots?