# Excerpt: Catalan Numbers are Fun(ctional)!

Bradley Klee*

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* bradklee@gmail.com

A maximally permissive set of function compositions on one head $F$ may be written as,

$$\mathbb{CATF}_n = \left\{ F(f_1, f_2, \ldots, f_m) : f_k \in \mathbb{CATF}_{n_k - 1}, n = \sum_1^m n_k \right\}, \qquad \mathbb{CATF}_0 = \{F_0\},$$

This set includes all function compositions with $n+1$ symbols drawn from $\{F, F_0\}$. Functions have a bilateral symmetry, which ultimately translates nicely into geometry. Using $R$ for open function, and $L$ for close function, and $RL$ for $F_0$, the set $\mathbb{CATF}$ becomes instead,

$$\mathbb{CATF}'_n = \left\{ R s_1 s_2 \ldots s_m L : s_k \in \mathbb{CATF}'_{n_k - 1}, n = \sum_1^m n_k \right\}, \qquad \mathbb{CATF}'_0 = \{RL\}.$$

Another, often preferable representation portrays the words of $\mathbb{CATF}'$ as walks along an integer lattice. In this interpretation, the letters are also vectors in the complex plane, $R = 1 + i$ and $L = -1 + i$. Partial summation of the vector sequence determines a path across a subset of the Gaussian integers. If the walk starts from $z_0 = -1 - i$ the next step goes to the origin $z_1 = 0$, and subsequent steps $z_n$ have positive imaginary part, $\Im(z_n) > 0$. As every function opens before closing, points $z_n$ also have positive real part, $\Re(z_n) > 0$.

A redefinition tuned to lattice walking,

$$\begin{aligned}
\mathbb{CBCW}_n &= \left\{ (s_1, s_2, \ldots, s_{2n}) : s_k \in \{1+i, -1+i\}, (2n)i = \sum_1^{2n} s_k \right\}, \\
\mathbb{CATW}_n &= \left\{ (s_1, s_2, \ldots, s_{2n}) \in \mathbb{CBCW}_n : 0 \geq \Re\left( \sum_1^m s_k \right), 1 \leq m \leq 2n \right\},
\end{aligned}$$

removes one superfluous pair of letters or vectors. Then a walk reaches location $z_m$ after $m$ steps away from the origin $z_0 = 0$. More importantly, the inclusion of $\mathbb{CATW}_n$ in $\mathbb{CBCW}_n$ almost immediately implies a good upper bound. The origin-returning condition requires $n$ left steps, $n$ right steps, and a count

$$|\mathbb{CBCW}_n| = \frac{(2n)!}{(n!)^2} = \binom{2n}{n}.$$

Every element of $\mathbb{CBCW}_n$ has a reflected image, $1 + i \longleftrightarrow -1 + i$, which places an even tighter bound, $C_n = |\mathbb{CATW}_n| < \binom{2n}{n}/2$.

The unknown differences,

$$\Delta_n = \frac{1}{2}\binom{2n}{n} - C_n,$$

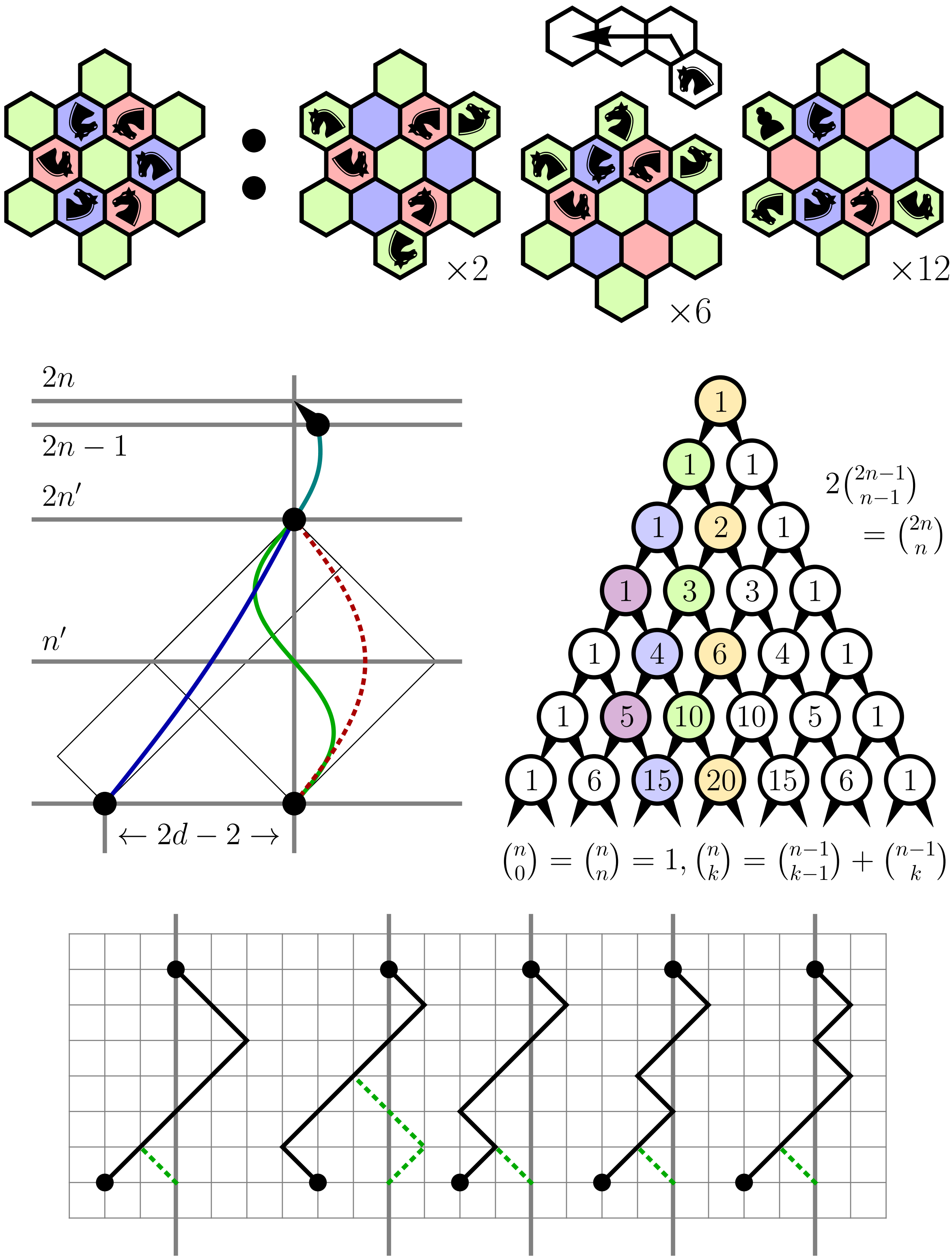


FIG. 1. Up top, diagram of a game for finding that six choose three equals twenty; on the left, a template for mapping between lattice walks across the axis $\Re(z) = 0$; on the right, a few rows from Pascal's triangle of binomial coefficients; and on bottom, a drawing of $\mathbb{DIFFW}_3$ with $d = 2$.

propose to count exactly half of the worldlines going between the first and second quadrants during the odyssey from $z_0 = 0$ to $z_{2n} = (2n)i$. Such walks start and end on the dividing line between quadrants one and two. This fact adds to difficulty. We make an Ansatz,

$$\mathbb{DIFFW}_n = \left\{ (s_1, s_2, \dots, s_{2n-1}) : s_k \in \{1+i, -1+i\}, 2d - 1 + (2n-1)i = \sum_1^{2n-1} s_k \right\},$$

that moves laterally by $2d-1$ steps. From an origin $z_0 = 2-2d$, every $\mathbb{DIFFW}_n$ walk moves right by $n+d-1$ steps, left by $n-d$ steps, and finally reaches $z_{2n} = (2n)i$ after one extra left step, $s_{2n} = -1+i$. All such paths have the desired property of crossing the axis $\Re(z) = 0$. The hypothesis is that an integer $d$ exists to satisfy

$$\Delta_n = |\mathbb{DIFFW}_n| = \binom{2n-1}{n-d}.$$

There is hope to prove this identity, if only a bijective function exists from $\mathbb{DIFFW}_n$ to a half of $\mathbb{LRW}_n = \mathbb{CBCW}_n / (\mathbb{CATW}_n \cup \widetilde{\mathbb{CATW}}_n)$, with $\widetilde{\mathbb{CATW}}_n$ the reflected image of $\mathbb{CATW}_n$ across axis $\Re(z) = 0$ .

Walks that cross the central axis potentially do so numerous times. Let us assume that walks from $\mathbb{DIFFW}_n$ and $\mathbb{LRW}_n$ agree after $z_{2n'} = (2n')i$, and that $\Re(z_m) \geq 0$ when $2n' \leq m \leq 2n$. A bijection between walk sets then requires that,

$$\binom{2n'}{n'-d+1} = \binom{2n'}{n'} - C_{n'},$$

i.e. that all paths from $\mathbb{CBCW}_{n'}/\mathbb{CATW}_{n'}$ have an image in the subset of $\mathbb{DIFFW}_n$ through $z_{2n'} = (2n')i$. Rewriting $n' \to n$, produces a second equation in the three unknowns. Eliminating $C_n$ and $\Delta_n$, we are left with

$$\binom{2n}{n-d+1} = \frac{1}{2}\binom{2n}{n} + \binom{2n-1}{n-d}.$$

Is there an integer $d$ that satisfies this equality regardless of $n$? If yes, then the hypothesis holds true and also determines a count of $\mathbb{CATW}_n$. The conjectured identity already appears strikingly similar to the addition rule for Pascal's wonderful triangle of binomial coefficients. Indeed it is exactly the same for $d = 2$! A few more manipulations produce a final count of the Catalan numbers,

$$C_n = |\mathbb{CATW}_n| = \binom{2n-1}{n-1} - \binom{2n-1}{n-2} = \frac{1}{n+1}\binom{2n}{n}.$$

At last, if one needs an explicit bijection between walks, there are a few distinct, equally valid recipes for partial reflection across the line $\Re(z) = -1$.