MTH9831_Homework10_Group1

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Question1 - Question6

1. $r=0 \Rightarrow D(t)=1.$

$dX(t) = \gamma(t)\,dS(t) = d(\gamma(t)S(t)) - S(t)\,d\gamma(t)$

integrate from $T-c$ to T

$$X(T) = X(T-c) + \gamma(T)S(T) - \gamma(T-c)S(T-c) - \int_{T-c}^{T} S(u)\gamma'(u)\,du.$$

$$\Rightarrow \begin{cases} X(T-c) + \gamma(T)S(T) - \gamma(T-c)S(T-c) = -K \\ -\gamma'(u) = \frac{1}{c}. \quad \text{for } T-c \le u \le T. \end{cases}$$

Similarly $\gamma(T)=0 \quad \Rightarrow \gamma(t) = -\frac{1}{c}t + \frac{1}{c}T. \quad$ for $T-c \le t \le T$

$\gamma(T-c) = 1.$

$$\Rightarrow \gamma(t) = \begin{cases} 1. & 0 \le t \le T-c \\ -\frac{1}{c}t + \frac{1}{c}T. & T-c < t \le T. \end{cases}$$

$$X(t) = \begin{cases} S(0) - K. & 0 \le t \le T-c. \\ (-\frac{1}{c}t + \frac{1}{c}T)S(t) - K + \frac{1}{c}\int_{T-c}^{t} S(u)\,du. & T-c < t \le T. \end{cases}$$

2.
$$\left.\begin{array}{l} -c\gamma'(t)e^{r(T-t)} = 1. \\ e^{rc}X(T-c) + \gamma(T)S(T) - e^{rc}\gamma(T-c)S(T-c) = -S(T). \end{array}\right\}.$$

$$\Rightarrow \gamma(t) = \begin{cases} -1 + \frac{1}{rc}(1-e^{-rc}). & 0 \le t \le T-c \\ -1 + \frac{1}{rc}(1-e^{-r(T-t)}). & T-c < t \le T. \end{cases}$$

$X(0) = [-1 + \frac{1}{rc}(1-e^{-rc})]\,S(0).$

Theorem. For $0 \le t \le T$. $V(t)$ for Asian call option paying $(\frac{1}{c}\int_{T-c}^{T} S(t)\,dt - S(T))_+$ is given by

$$V(t) = S(t)\,g(t, \frac{X(t)}{S(t)}).$$

$g(t,x)$ satisfies $g_t(t,x) + \frac{1}{2}\sigma^2(\gamma(t)-x)^2 g_{xx}(t,x) = 0. \quad 0 \le t \le T.$

$g(T,x) = x_+. \quad \lim_{x\to-\infty} g(t,x) = 0. \quad \lim_{x\to+\infty} g(t,x) = 0. \quad 0 \le t \le T.$

$X(t)$ is a portfolio process. with $\gamma(t) = \begin{cases} -1 + \frac{1}{rc}(1-e^{-rc}) & 0 \le t \le T-c \\ -1 + \frac{1}{rc}(1-e^{-r(T-t)}). & T-c \le t \le T. \end{cases}$

3. $V_L(x) = (K-L)\cdot(\frac{x}{L})^{-2r/\sigma^2}$. $\quad x \geq L$.

$\Rightarrow V_L'(x) = (K-L)\cdot(-\frac{2r}{\sigma^2})\cdot(\frac{x}{L})^{-2r/\sigma^2-1}\cdot\frac{1}{L} \Rightarrow V_L'(L+) = -\frac{2r}{\sigma^2}\cdot\frac{K-L}{L}$.

$V_L'(L-) = V_L'(L+) \Leftrightarrow -\frac{2r}{\sigma^2}\frac{K-L}{L} = -1 \Leftrightarrow L = \frac{2rK}{2r+\sigma^2} = L^*$.

4. (i). $V(x) = x^p \Rightarrow x^p(r - pr - \frac{1}{2}\sigma^2 p(p-1)) = 0$. $\Rightarrow p=1$ or $p = -\frac{2r}{\sigma^2}$.

general Solution: $C_1 x + C_2 x^{-2r/\sigma^2}$. $\quad C_1, C_2$ are Constants.

(ii). assume the interval $[x_1, x_2]$ exists. and

$V(x) = C_1 x + C_2 x^{-2r/\sigma^2} \not\equiv 0$.

if $\exists\ x_0 \in [x_1, x_2]$ s.t. $V(x_0) = V'(x_0) = 0$.

then by the uniqueness of ODE $\Rightarrow V(x) \equiv 0$. Contradiction.

so $0 < x_1 < x_2 < K$.

$$\Rightarrow \begin{cases} C_1 x_1 + C_2 x_1^{-2r/\sigma^2} = K - x_1 & ① \\ C_1 x_2 + C_2 x_2^{-2r/\sigma^2} = K - x_2 & ② \\ C_1 - \frac{2r}{\sigma^2} C_2 x_1^{-2r/\sigma^2-1} = -1 & ③ \\ C_1 - \frac{2r}{\sigma^2} C_2 x_2^{-2r/\sigma^2-1} = -1 & ④ \end{cases}$$

from ③④. $x_1 \neq x_2 \Rightarrow C_2 = 0$. $C_1 = -1$.

plug into ①② $\Rightarrow K = 0$. Contradiction.

$\Rightarrow$ the interval $[x_1, x_2]$ do not exist unless $V(x) \equiv 0$.

(iii). assume $x_2 > 0$ exist. $\Rightarrow V(x) = C_1 x + C_2 x^{-2r/\sigma^2}$.

$\Rightarrow \lim_{x\to 0} V(x) = 0 < (K-0)^+$. Contradiction.

$\Rightarrow rV - rxV' - \frac{1}{2}\sigma^2 x^2 V'' > 0$.

(iv). as shown in (iii), it will contradict $V(0) \geq (K-0)^+$.

(v). if $V(x) = (K-x)^+$. then V do not have Continuous derivative at $x = K$. Contradiction.

(vi).
$$v(x) = \begin{cases} (K-x)^+ & x \le x_1 \\ c_1 x + c_2 x^{-2r/\sigma^2} & x > x_1. \end{cases}$$

$\Rightarrow$ at x_1. $v(x)$ and $v'(x)$ are continuous

$$\Rightarrow \begin{cases} K - x_1 = (K-x_1)^+ = c_1 x_1 + c_2 x_1^{-2r/\sigma^2} \\ -1 = c_1 - \frac{2r}{\sigma^2} c_2 \cdot x_1^{-2r/\sigma^2 - 1} \end{cases}.$$

v is bounded $\Rightarrow$ $c_1 = 0$ plug in.

$$\Rightarrow x_1 = \frac{2r}{2r+\sigma^2} K = L^*. \qquad v(x) = \begin{cases} K - x & x \le L_* \\ (K - L^*) \cdot \left(\frac{x}{L^*}\right)^{-2r/\sigma^2} & x > L_*. \end{cases}$$

$$c_2 = (K - L^*) \cdot L^{*\,2r/\sigma^2}.$$

5. (i). $S(t) = S(0) \cdot \exp\{\sigma \tilde{W}(t) + (r - a - \frac{1}{2}\sigma^2) t\}$.

$S(0) = x$. $S(t) = L$ $\Rightarrow$ $-\tilde{W}(t) - \frac{1}{\sigma}(r - a - \frac{1}{2}\sigma^2) t = \frac{1}{\sigma} \log \frac{x}{L}$.

let $\gamma = \frac{1}{\sigma^2}(r - a - \frac{1}{2}\sigma^2) + \frac{1}{\sigma}\sqrt{\frac{1}{\sigma^2}(r - a - \frac{1}{2}\sigma^2)^2 + 2r}$

$$\Rightarrow E[e^{-r\tau_L}] = e^{-\gamma \log x/L} = \left(\frac{x}{L}\right)^{-\gamma}$$

$$\Rightarrow \text{payoff} \quad v_L(x) = \begin{cases} K - x & 0 \le x \le L \\ (K - L) \cdot \left(\frac{x}{L}\right)^{-\gamma}. & x > L. \end{cases}$$

(ii).
$$\frac{\partial v_L(x)}{\partial L} = \left(-1 + \frac{\gamma(K-L)}{L}\right) \cdot \left(\frac{x}{L}\right)^{-\gamma} = 0$$

$$\Rightarrow L^* = \frac{\gamma}{\gamma+1} K.$$

6. (a). $P(N(t) - N(s) = k) = \exp\{-\lambda(t-s)\} \cdot \frac{\lambda^k (t-s)^k}{k!}$.

$$E(N(t) - N(s)) = \sum_{k=0}^{\infty} k \cdot P(N(t) - N(s) = k) = e^{-\lambda(t-s)} \cdot \lambda(t-s) \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}(t-s)^{k-1}}{(k-1)!}$$

$$= e^{-\lambda(t-s)} \cdot \lambda(t-s) \cdot e^{\lambda(t-s)} = \lambda(t-s).$$

$$Var(N(t) - N(s)) = E[(N(t) - N(s))^2] - E[N(t) - N(s)]^2$$

$$= \sum_{k=0}^{\infty} k^2 \cdot P(N(t) - N(s) = k) - \lambda^2 (t-s)^2$$

$$= \sum_{k=0}^{\infty} k \cdot P(N(t)-N(s)=k) + \sum_{k=1}^{\infty} k(k-1) \cdot P(N(t)-N(s)=k) - \lambda^2(t-s)^2$$

$$= \lambda(t-s) + e^{-\lambda(t-s)} \cdot \lambda^2(t-s)^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2}(t-s)^{k-2}}{(k-2)!} - \lambda^2(t-s)^2 = \lambda(t-s)$$

$$E(e^{u(N(t)-N(s))}) = \sum_{k=0}^{\infty} e^{uk} \cdot P(N(t)-N(s)=k)$$

$$= e^{-\lambda(t-s)} \cdot \sum_{k=0}^{\infty} \frac{[e^u \cdot \lambda(t-s)]^k}{k!} = e^{-\lambda(t-s)} \cdot e^{e^u \cdot \lambda(t-s)} = e^{\lambda(t-s)(e^u-1)}.$$

(b). $X_1 \sim \text{poisson}(\lambda_1)$. $X_2 \sim \text{poisson}(\lambda_2)$. independent.

$$P(X_1+X_2=n) = \sum_{k=0}^{n} P(X_1=k) P(X_2=n-k).$$

$$= e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} = e^{-(\lambda_1+\lambda_2)} \cdot \frac{1}{n!} \sum_{k=0}^{n} C_n^k \lambda_1^k \lambda_2^{n-k}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^n}{n!} \Rightarrow X_1+X_2 \sim \text{Poisson}(\lambda_1+\lambda_2).$$

(c). $N_1(t)$ is poisson process with λ_1. $N_2(t)$ is poisson process with λ_2.

here we verify the definition of poisson process. $N(t) = N_1(t)+N_2(t)$.

① $N(0) = N_1(0)+N_2(0) = 0$.

② for $0 = t_0 < t_1 < \dots < t_m$. $1 \le i \le m$.

$$N(t_i)-N(t_{i-1}) = (N_1(t_i)-N_1(t_{i-1})) + (N_2(t_i)-N_2(t_{i-1})).$$

Since $N_1(t)$ and $N_2(t)$ are independent process.

$\Rightarrow$ $N(t_1)-N(t_0)$, ... $N(t_m)-N(t_{m-1})$ are independent.

③. $P(N(t+s)-N(s)=n) = P((N_1(t+s)-N_1(s)) + (N_2(t+s)-N_2(s)) = k)$

using (b) $= e^{-(\lambda_1+\lambda_2)t} \cdot \dfrac{(\lambda_1+\lambda_2)^n t^n}{n!}$

$\Rightarrow$ $N(t) = N_1(t)+N_2(t)$ is Poisson process with density $\lambda_1+\lambda_2$.

In [ ]: