Numeric HW3

Boyu Yang: question 4
Julius Gruber and Ziyuan Li: question 2 & 3
Yicheng Sun: general question, question 1

General Questions

(1)

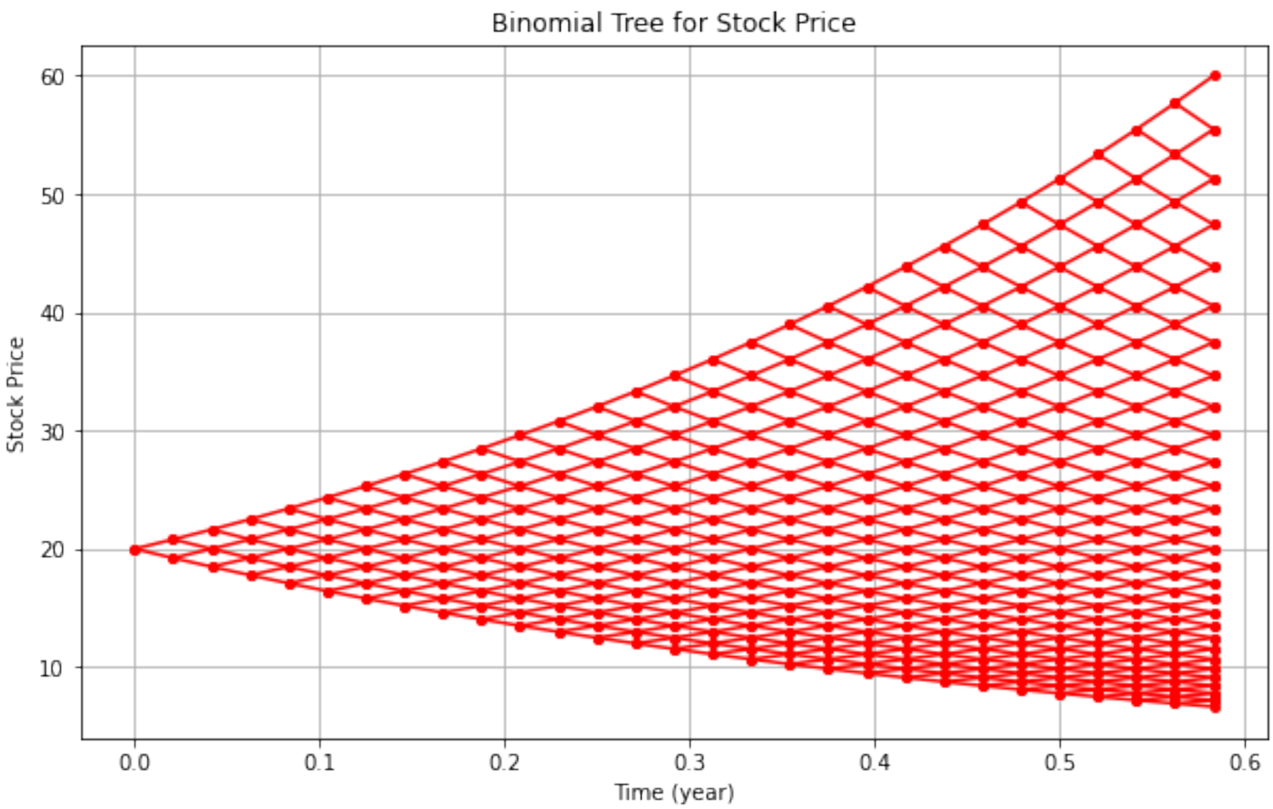
Construct the following portfolio:
i) long $e^{-q\delta t}$ unit of S; ii) short $S(0)e^{-q\delta t}$ unit of cash
Then at time 0, we have $V(0) = e^{-q\delta t}S(0) - S(0)e^{-q\delta t} = 0$.

At time δt, we have $V(\delta t) = S(\delta t) - S(0)e^{(r-q)\delta t}$
According to the no arbitrage condition, we obtain

$$\begin{aligned} V_{up}(\delta t) &= S(0)(u - e^{(r-q)\delta t}) > 0 \\ V_{down}(\delta t) &= S(0)(d - e^{(r-q)\delta t}) < 0 \end{aligned} \tag{1}$$

so we get the result: $d < e^{(r-q)\delta t} < u$

(2)



(3)

Plug in $u=e^{\sigma\sqrt{\delta t}}, d=e^{-\sigma\sqrt{\delta t}}$, when $\delta t\to 0$, we obtain

$$\begin{aligned}
\frac{e^{(r-q)\delta t}-d}{u-d}-\frac{1}{2} &= \frac{e^{(r-q)\delta t}-e^{-\sigma\sqrt{\delta t}}}{e^{\sigma\sqrt{\delta t}}-e^{-\sigma\sqrt{\delta t}}}-\frac{1}{2}\\
&= \frac{1+(r-q)\delta t+O(\delta^2t^2)-(1-\sigma\sqrt{\delta t}+1/2*\sigma^2\delta t)+O((\delta t)^{3/2})}{(1+\sigma\sqrt{\delta t}+1/2*\sigma^2\delta t)-(1-\sigma\sqrt{\delta t}+1/2*\sigma^2\delta t)+O((\delta t)^{3/2})}-\frac{1}{2}\\
&= \frac{\sigma\sqrt{\delta t}+(r-q-1/2*\sigma^2)\delta t+O((\delta t)^{3/2})}{2\sigma\sqrt{\delta t}+O((\delta t)^{3/2})}-\frac{1}{2}\\
&= \frac{1}{2\sigma}(r-q-\frac{1}{2}\sigma^2)\sqrt{\delta t}+O(\delta t)
\end{aligned}$$

That is

$$\frac{e^{(r-q)\delta t}-d}{u-d}=\frac{1}{2}+\frac{1}{2}(\frac{r-q}{\sigma}-\frac{\sigma}{2})\sqrt{\delta t}+O(\delta t) \tag{2}$$

(4)

Consider S moves k steps up and $N-k$ steps down, we have

$$\begin{aligned}
&S_N(T)=S_0u^kd^{N-k}=S_0u^{2k-N}\\
&log(\frac{S_N}{S_0})=(2k-N)log(u)=(2k-N)\sqrt{T/N\sigma}
\end{aligned}$$

Therefore

$$Z=log(\frac{S_N}{S_0})/\sigma\sqrt{T}=\frac{2k-N}{\sqrt{N}} \tag{3}$$

Calculating the GMF of Z, as $N\to\infty$, we obtain

$$\begin{aligned}
\mathbb{E}(e^{tZ}) &= \sum_{k=0}^{N}e^{\frac{2k-N}{\sqrt{N}}t}C_N^kp^k(1-p)^{N-k}\\
&= e^{-\sqrt{N}t}\sum_{k=0}^{N}C_N^k(pe^{\frac{2t}{\sqrt{N}}})^k(1-p)^{N-k}\\
&= e^{-\sqrt{N}t}(1+(e^{\frac{2t}{\sqrt{N}}}-1)p)^N\\
&= e^{-\sqrt{N}t}(1+(\frac{1}{2}+\frac{1}{2}(\frac{r-q}{\sigma}-\frac{\sigma}{2})\sqrt{\delta t}+O(\delta t))(\frac{2t}{\sqrt{T}}\sqrt{\delta t}+\frac{2t^2}{T}\delta t+O((\delta t)^{3/2})))^N\\
&= (1-\frac{t}{\sqrt{N}}+\frac{t^2}{2N}+O(\frac{1}{N^{3/2}}))^N(1+\frac{t}{\sqrt{N}}+(t^2+(\frac{r-q}{\sigma}-\frac{\sigma}{2})t\sqrt{T})\frac{1}{N}+O(\frac{1}{N^{3/2}}))^N\\
&= (1+(\frac{t^2}{2}+(\frac{r-q}{\sigma}-\frac{\sigma}{2})t\sqrt{T})\frac{1}{N})^N\\
&= exp\{\frac{t^2}{2}+(\frac{r-q}{\sigma}-\frac{\sigma}{2})t\sqrt{T}\}
\end{aligned}$$

so we have

$$\mathbb{E}(e^{tlog(\frac{S_N}{S_0})}) = \mathbb{E}(e^{t\sigma\sqrt{T}Z}) = exp\{\frac{\sigma^2T^2}{2}t^2 + (r - q - \frac{\sigma^2}{2})Tt\}$$

by which we conclude that

$$\lim_{N\to\infty} log(\frac{S_N}{S_0}) \sim \mathbb{N}((r - q - \frac{\sigma^2}{2})T, \sigma^2T^2) \tag{4}$$