MTH 9821 Numerical Methods for Finance

Fall 2023

Homework 2

Assigned: September 4; Due: September 11

This homework is to be done as a group. Each team will hand in one homework solution, and each member of the team should write at least one problem. On the cover page of the homework, please indicate the members of the team and who wrote each problem.

Details on generating random numbers for this homework:

Generate N independent samples from the standard normal distribution as follows:

Generate N_0 independent samples from the uniform distribution on $[0,\ 1]$ by using the Linear Congruential Generator

$$\begin{aligned} x_{i+1} &= a x_i \ + \ c \ (\mathrm{mod}\ k) \\ u_{i+1} &= \frac{x_{i+1}}{k}, \end{aligned}$$

with $x_0 = 1$, $a = 39373$, $c = 0$, and $k = 2^{31} - 1$ to generate $u_1,\ u_2,\ \ldots,\ u_{N_0}$.

Use the Box–Muller Method to generate independent samples from the standard normal distribution using the Marsaglia–Bray algorithm below:

while $X > 1$
    Generate $u_1, u_2 \in U([0,1])$
    $u_1 = 2u_1 - 1$; $u_2 = 2u_2 - 1$
    $X = u_1^2 + u_2^2$
end
$Y \ = \ \sqrt{-2\frac{\ln(X)}{X}}$
$Z_1 = u_1 Y$; $Z_2 = u_2 Y$
return Z_1, Z_2

Use the samples u_i, $i = 1 : N_0$, from the uniform distribution generated previously. Make sure N_0 is large enough in order to generate the required number N independent samples of the standard normal distribution.

Variance Reduction Techniques for Monte Carlo Pricing of European Options

Consider a nine months European put option with strike 54 on a non–dividend–paying underlying asset with spot price 56 following a lognormal distribution with volatility 27%. Assume that the risk-free rate is constant at 2%.

Compute the Black–Scholes value V_{BS} of the option.

Use the $n = 10,000 \cdot 2^9$ independent samples z_i, $i = 1 : n$, from the standard normal distribution obtained above.

Control Variate Technique

We control for the value of the put option using the value of the spot price of the underlying asset. Given z_i, $i = 1 : n$, samples from the standard normal distribution, let

$$\begin{aligned} S_i &= S(0) \exp\left(\left(r - \frac{\sigma^2}{2}\right)T \; + \; \sigma\sqrt{T}z_i\right) \; i = 1 : n; \\ V_i &= e^{-rT} \max(K - S_i, 0), \; i = 1 : n. \end{aligned}$$

Note that $E[S_i] = e^{rT} S(0)$, for all $i = 1 : n$.
Let

$$\begin{aligned} \widehat{S}(n) &= \frac{1}{n}\sum_{i=1}^{n} S_i; \\ \widehat{V}(n) &= \frac{1}{n}\sum_{i=1}^{n} V_i. \end{aligned}$$

Let

$$W_i \; = \; V_i \; - \; \widehat{b}\left(S_i - e^{rT} S(0)\right), \; i = 1 : n,$$

where

$$\widehat{b} \; = \; \frac{\sum_{i=1}^{n}(S_i - \widehat{S}(n))(V_i - \widehat{V}(n))}{\sum_{i=1}^{n}(S_i - \widehat{S}(n))^2}.$$

Report the approximate values

$$\widehat{V}_{CV}(n) \; = \; \widehat{W}(n) \; = \; \frac{1}{n}\sum_{i=1}^{n} W_i,$$

for $n = 10,000 \cdot 2^k$, $k = 1 : 9$, and the corresponding approximation errors in the table below:

n	$\widehat{V}_{CV}(N)$	$\lvert V_{BS} - \widehat{V}_{CV}(N)\rvert$
10,000		
20,000		
...		
5,120,000		

Antithetic Variables

Given $z_{i,1}$, $i = 1 : n$, samples from the standard normal distribution, let

$$z_{i,2} \; = \; -z_{i,1}, \; i = 1 : n.$$

Let

$$\begin{aligned}
S_{i,1} &= S(0)\,\exp\left(\left(r-\frac{\sigma^2}{2}\right)T \;+\; \sigma\sqrt{T}z_{i,1}\right),\ i=1:n;\\
S_{i,2} &= S(0)\,\exp\left(\left(r-\frac{\sigma^2}{2}\right)T \;+\; \sigma\sqrt{T}z_{i,2}\right),\ i=1:n;\\
V_{i,1} &= e^{-rT}\max(K-S_{i,1},0),\ i=1:n;\\
V_{i,2} &= e^{-rT}\max(K-S_{i,2},0),\ i=1:n.
\end{aligned}$$

Report the approximate values

$$\widehat{V}_{AV}(n) \;=\; \frac{1}{n}\sum_{i=1}^{n}\frac{V_{i,1}+V_{i,2}}{2},$$

for $n = 10,000 \cdot 2^k$, $k = 1:9$, and the corresponding approximation errors in the table below:

n	$\widehat{V}_{AV}(N)$	$\lvert V_{BS} - \widehat{V}_{AV}(N)\rvert$
10,000		
20,000		
...		
5,120,000		

Moment Matching

Given z_i, $i = 1:n$, samples from the standard normal distribution, let

$$S_i \;=\; S(0)\,\exp\left(\left(r-\frac{\sigma^2}{2}\right)T \;+\; \sigma\sqrt{T}z_i\right),\ \forall\ i=1:n,$$

and let

$$\widehat{S}(n) \;=\; \frac{1}{n}\sum_{i=1}^{n} S_i.$$

Note that $E[S_i] = e^{rT}S(0)$, for all $i = 1:n$.
Let

$$\widetilde{S}_i \;=\; S_i \cdot \frac{e^{rT}S(0)}{\widehat{S}(n)},\ \forall\ i=1:n,$$

and let

$$\widetilde{V}_i \;=\; e^{-rT}\max(K-\widetilde{S}_i,0),\ \forall\ i=1:n.$$

Compute

$$\widehat{V}_{MM}(n) \;=\; \widehat{\widetilde{V}}(n) \;=\; \frac{1}{n}\sum_{i=1}^{n}\widetilde{V}_i.$$

Report the approximate values

$$\widehat{V}_{MM}(n) \;=\; \widehat{\widetilde{V}}(n) \;=\; \frac{1}{n}\sum_{i=1}^{n}\widetilde{V}_i,$$

for $n = 10,000 \cdot 2^k$, $k = 1:9$, and the corresponding approximation errors in the table below:

Simultaneous Moment Matching and Control Variates

n	$\widehat{V}_{MM}(N)$	$\lvert V_{BS} - \widehat{V}_{MM}(N)\rvert$
10,000		
20,000		
...		
5,120,000		

Given z_i, $i = 1 : n$, samples from the standard normal distribution, let

$$S_i \;=\; S(0)\,\exp\left(\left(r - \frac{\sigma^2}{2}\right)T \;+\; \sigma\sqrt{T}z_i\right),\ \forall\, i = 1 : n,$$

and let

$$\widehat{S}(n) \;=\; \frac{1}{n}\sum_{i=1}^{n} S_i.$$

Note that $E[S_i] = e^{rT}S(0)$, for all $i = 1 : n$.

Let

$$\widetilde{S}_i \;=\; S_i \cdot \frac{e^{rT}S(0)}{\widehat{S}(n)},\ \forall\, i = 1 : n,$$

and let

$$\widetilde{V}_i \;=\; e^{-rT}\max(K - \widetilde{S}_i, 0),\ \forall\, i = 1 : n.$$

Let

$$\widehat{V}(n) \;=\; \frac{1}{n}\sum_{i=1}^{n} \widetilde{V}_i.$$

Note that

$$\widehat{S}(n) \;=\; \frac{1}{n}\sum_{i=1}^{n} \widetilde{S}_i \;=\; e^{rT}S(0).$$

Let

$$W_i \;=\; \widetilde{V}_i \;-\; \widehat{b}\left(\widetilde{S}_i - e^{rT}S(0)\right),\ i = 1 : n,$$

where

$$\widehat{b} \;=\; \frac{\sum_{i=1}^{n}(\widetilde{S}_i - e^{rT}S(0))(\widetilde{V}_i - \widehat{V}(n))}{\sum_{i=1}^{n}(\widetilde{S}_i - e^{rT}S(0))^2}.$$

Report the approximate values

$$\widehat{V}_{CV,MM}(n) \;=\; \widehat{W}(n) \;=\; \frac{1}{n}\sum_{i=1}^{n} W_i,$$

for $n = 10,000 \cdot 2^k$, $k = 1 : 9$, and the corresponding approximation errors in the table below; include the Monte Carlo values previously obtained using Control Variates and Moment Matching separately:

n	$\widehat{V}_{CV,MM}(N)$	$\lvert V_{BS} - \widehat{V}_{CV,MM}(N)\rvert$	$\widehat{V}_{CV}(N)$	$\lvert V_{BS} - \widehat{V}_{CV}(N)\rvert$	$\widehat{V}_{MM}(N)$	$\lvert V_{BS} - \widehat{V}_{MM}(N)\rvert$
10,000						
20,000						
...						
5,120,000						

Monte Carlo Pricing for Basket Options

We want to price European basket options on two underlying assets following a lognormal model with known correlation using Monte Carlo simulation.

The payoff of the basket option is

$$\max(S_1(T) + S_2(T) - K, 0),$$

where $S_1(T)$ and $S_2(T)$ are the spot prices of the two underlying assets at maturity T.

Assume the risk-free rate is constant $r = 0.025$, the options mature in six months, i.e., $T = 0.5$, the prices today are $S_1(0) = 26$ and $S_2(0) = 29$, the volatilities of the underlying assets are $\sigma_1 = 0.31$ and $\sigma_2 = 0.21$, and the correlation of the prices of the two underlying assets is $\rho = 0.3$.

Since the payoff of the option depends only on the price of the underlying assets at maturity, we do not need to generate entire paths for the evolution of the underlying assets, just their prices at maturity.

Generate $2N$ independent samples $z_1, z_2, \ldots, z_{2N}$ from the standard normal distribution using the linear congruential generator and the Box–Muller Method.

We want to price a basket option with strike $K = 50$ and payoff

$$\max(S_1(T) + S_2(T) - K, 0).$$

Given $z_1, z_2, \ldots, z_N$ obtained above, let

$$\begin{aligned} S_{1,j}(T) &= S_1(0)\exp\left(\left(r - \frac{\sigma_1^2}{2}\right)T + \sigma_1\sqrt{T}z_{2j+1}\right), \quad \forall\ j = 0:(N-1) \\ S_{2,j}(T) &= S_2(0)\exp\left(\left(r - \frac{\sigma_2^2}{2}\right)T + \sigma_2\sqrt{T}\ \left(\rho z_{2j+1} + \sqrt{1-\rho^2}z_{2j+2}\right)\right), \quad \forall\ j = 0:(N-1) \end{aligned}$$

and let

$$V_j = e^{-rT}\max(S_{1,j}(T) + S_{2,j}(T) - K, 0).$$

Compute an approximate value

$$\widehat{V}(N) = \frac{1}{N}\sum_{j=1}^{N} V_j.$$

Do this for $N = 10,000 \cdot 2^k$, $k = 0:8$. Comment on the convergence of the Monte Carlo pricer.

Monte Carlo Pricing for Path–Dependent Basket Options

Consider now a path dependent basket options, e.g., a lookback basket call option whose payoff is

$$\max\left(\max_{0\leq t\leq T}\left(S_1(t)+S_2(t)\right)-K,\ 0\right).$$

Use the same assets and assumptions as before ($S_1(0)=26$, $S_2(0)=29$, $\sigma_1=0.31$, $\sigma_2=0.21$, $\rho=0.3$, $r=0.025$). The basket option is a six months option struck at 50.

We simulate the risk neutral random paths of the two assets on n different paths, each one discretized by m time steps corresponding to $\delta t=T/m$. To do this, generate $N=2nm$ independent samples of the standard normal distribution using the linear congruential generator and the Box–Muller method as before.

To generate n different paths for the evolution of the two underlying assets, discretize the time to maturity using m time steps corresponding to $\delta t=\frac{T}{m}$. Let $t_j=j\delta t$, for $j=0:m$. Use the multiplicative formula for the evolution of the underlying.

For every $2m$ standard normal samples denoted, e.g., by $z_1,\ z_2,\ \ldots,\ z_{2m}$, generate one possible path for the two underlying assets as follows:

$$\begin{aligned}
S_1(t_{j+1}) &= S_1(t_j)\,\exp\left(\left(r-\frac{\sigma_1^2}{2}\right)\delta t\ +\ \sigma_1\sqrt{\delta t}z_{2j+1}\right), \quad \forall\ j=0:(m-1)\\
S_2(t_{j+1}) &= S_2(t_j)\,\exp\left(\left(r-\frac{\sigma_2^2}{2}\right)\delta t\ +\ \sigma_2\sqrt{\delta t}\ \left(\rho z_{2j+1}\ +\ \sqrt{1-\rho^2}z_{2j+2}\right)\right), \quad \forall\ j=0:(m-1)
\end{aligned}$$

Use the N independent samples from the standard normal distribution to generate n different paths for the joint evolution of the two underlying assets as above. Let V_k, $k=1:n$, the value of the option provided the two underlying assets evolves along the path k.

Compute an approximate value

$$\widehat{V}(n)\ =\ \frac{1}{n}\sum_{i=1}^{n}V_i.$$

Use a small time interval when discretizing the path, i.e., one day. This corresponds approximately to $m=150$. Use $n=50\cdot 2^k$ paths, where $k=0:9$, i.e., $N=2\cdot 7,500\cdot 2^k$, $k=0:9$. Comment on the convergence of the Monte Carlo pricer.

Monte Carlo Simulation for the Heston Model

The Heston Model for the evolution of an asset with mean–reverting stochastic volatility is

$$\begin{aligned} dS &= \mu(t)S(t)dt \;+\; \sqrt{V(t)}S(t)dX_1; \\ dV &= -\lambda(V(t)-\overline{V})dt \;+\; \eta\sqrt{V(t)}dX_2, \end{aligned}$$

where $V(t)$ is the instantaneous variance of the asset returns, and λ, $\overline{V}$, and η are positive constants:

- $\overline{V}$ is the long term variance mean;
- λ is the speed of mean–reversion;
- η is the standard deviation of the asset variance.

The correlation between the Weiner processes $X_1(t)$ and $X_2(t)$ from the Heston Model is

$$< dX_1, dX_2 > = \; \rho dt.$$

The Heston Model can be written in the following form suitable for numerical discretization:

$$\begin{aligned} d(\ln(S)) &= \left(\mu(t) - \frac{V(t)}{2}\right)dt \;+\; \sqrt{V(t)}dX_1; \\ dV &= -\lambda(V(t)-\overline{V})dt \;+\; \eta\sqrt{V(t)}dX_2. \end{aligned}$$

Using an Euler discretization with absorbing volatility over a time step $\delta t = \frac{T}{m}$, and assuming a constant risk–neutral drift $\mu(t) = r$, we find that

$$\begin{aligned} S_{j+1} &= S_j \exp\left(\left(r - \frac{V_j^+}{2}\right)\delta t \;+\; \sqrt{V_j^+}\sqrt{\delta t}z_j^{(1)}\right); \\ V_{j+1} &= V_j^+ - \lambda(V_j^+ - \overline{V})\delta t \;+\; \eta\sqrt{V_j^+}\sqrt{\delta t}\left(\rho z_j^{(1)} + \sqrt{1-\rho^2}z_j^{(2)}\right), \end{aligned}$$

for $j = 0 : (m-1)$, where $V_j^+ = \max(V_j, 0)$ and $z_j^{(1)}$ and $z_j^{(2)}$ are independent random samples from the standard normal distribution.

Consider an asset with spot price \$50 and spot volatility 30% (i.e., with intital variance $V_0 = 0.09$), and assume that the price of the asset follows a Heston Model with three month reversion time scale to a long–term standard deviation mean of 35% and with 25% standard deviation of the variance, i.e.,

$$\lambda \;=\; 4; \quad \sqrt{\overline{V}} \;=\; 0.35; \quad \eta \;=\; 0.25.$$

Assume a correlation $\rho = -0.15$ between the Wiener processes.

Consider a six months at–the–money put on this option. Let $T = 0.5$ and $K = 50$. Assume the risk-free rate is constant $r = 0.05$.

Simulate n random paths for the evolution of the price of the asset as above, each one discretized by m time steps corresponding to $\delta t = T/m$. Generate $N = 2nm$ independent samples of the standard normal distribution using the linear congruential generator and the Box–Muller method as before.

Use a small time interval when discretizing the path, i.e., half a day, in order to capture the mean reversion of the volatility. This corresponds approximately to $m = 175$. Use $n = 500 \cdot 2^k$ paths, where $k = 0 : 5$, i.e., $N = 2 \cdot 87,500 \cdot 2^5$.

Let V_k be the value of the option corresponding to the asset evolving along the path k, for $k = 1 : n$, and let

$$\widehat{V}(n) \;=\; \frac{1}{n}\sum_{i=1}^{n} V_i.$$

For each value of $\widehat{V}(n)$ also compute the corresponding Black–Scholes implied volatility.

Report the results in the table below:

n	$\widehat{V}(n)$	$\sigma_{imp,BS}(n)$
500		
1,000		
$\dots$		
16,000		