MTH 9821 Numerical Methods for Finance

Fall 2023

Homework 4
Assigned: September 18; Due: September 25

This homework is to be done as a group. Each team will hand in one homework solution, and each member of the team should write at least one problem. On the cover page of the homework, please indicate the members of the team and who wrote each problem.

Trinomial Tree Methods for European Options

Throughout this homework, the following parameterizations will be used for trinomial tree methods:

$$(1) \qquad u \ = \ e^{\sigma\sqrt{3\delta t}}; \qquad d \ = \ e^{-\sigma\sqrt{3\delta t}};$$

$$(2) \qquad p_u \ = \ \frac{1}{6} + (r - q - \frac{\sigma^2}{2})\sqrt{\frac{\delta t}{12\sigma^2}};$$

$$(3) \qquad p_m \ = \ \frac{2}{3};$$

$$(4) \qquad p_d \ = \ \frac{1}{6} - (r - q - \frac{\sigma^2}{2})\sqrt{\frac{\delta t}{12\sigma^2}}.$$

Consider a one year European put and a one year American put, both with strike \$39, on an asset with spot price \$41 paying dividends continuously at rate 0.5%, and following a lognormal process with volatility 25%. Assume the risk free interest rates are constant at 3%.

Compute the Black–Scholes option value V_{BS}, and the following Greeks: Δ_{BS}, Γ_{BS}, and Θ_{BS}.

Price the European put option using the following tree methods:

- Trinomial Tree with $N \in \{10, 20, 40, \ldots, 1280\}$ time steps;

- Trinomial Black–Scholes with $N \in \{10, 20, 40, \ldots, 1280\}$ time steps;

- Trinomial Black–Scholes with Richardson Extrapolation, with $N \in \{10, 20, 40, \ldots, 1280\}$ time steps.

For each method, record the first six decimals of the following values in the solution template file hw_sol_template-TRINOMIAL-European.xls:

- $V(N)$, the value given by the tree method with N time steps;
- $|V(N) - V_{BS}|$, the approximation error of the tree method;
- $N\,|V(N) - V_{BS}|$ and $N^2\,|V(N) - V_{BS}|$, terms that indicate whether the convergence of the tree method is linear or quadratic;
- the following approximations for the Delta, the Gamma, and the Theta of the option:

$$(5) \qquad \Delta_{approx} \ = \ \frac{V_{1,0} - V_{1,2}}{S_{1,0} - S_{1,2}};$$

$$(6) \qquad \Gamma_{approx} \ = \ \frac{\frac{V_{2,0}-V_{2,2}}{S_{2,0}-S_{2,2}} - \frac{V_{2,2}-V_{2,4}}{S_{2,2}-S_{2,4}}}{(S_{2,0} - S_{2,4})/2};$$

$$(7) \qquad \Theta_{approx} \ = \ \frac{V_{1,1} - V_{0,0}}{\delta t},$$

and the approximation errors $|\Delta_{approx} - \Delta_{BS}|$, $|\Gamma_{approx} - \Gamma_{BS}|$, and $|\Theta_{approx} - \Theta_{BS}|$.

Rank the methods in terms of convergence speed and comment on the order of the convergence.

Trinomial Tree Methods for American Options

Compute the value of an American Put with the same parameters by using an average binomial tree with $10,000$ and $10,001$ time steps and denote it by V_{exact}. *Do not use variance reduction.*

Price the American put option using the following tree methods:

- Trinomial Tree with $N \in \{10, 20, 40, \ldots, 1280\}$ time steps;

- Trinomial Black–Scholes with $N \in \{10, 20, 40, \ldots, 1280\}$ time steps;

- Trinomial Black–Scholes with Richardson Extrapolation, with $N \in \{10, 20, 40, \ldots, 1280\}$ time steps.

For each method, record the first six decimals of the following values in the solution template file hw_sol_template-TRINOMIAL-American.xls:

- $V(N)$, the value given by the tree method with N time steps;
- $|V(N) - V_{exact}|$, the approximation error of the tree method;
- $N\,|V(N) - V_{exact}|$ and $N^2\,|V(N) - V_{exact}|$, terms that indicate whether the convergence of the tree method is linear or quadratic;
- compute the approximations Δ_{approx}, Γ_{approx}, and Θ_{approx}, for the Delta, the Gamma, and the Theta of the option, respectively, and the approximation errors $|\Delta_{approx} - \Delta_{exact}|$, $|\Gamma_{approx} - \Gamma_{exact}|$, and $|\Theta_{approx} - \Theta_{exact}|$, where, e.g.,

$$\Gamma_{exact} = \frac{\Gamma_{American}(10,000) + \Gamma_{American}(10,001)}{2}.$$

Repeat the process by using Variance Reduction for each method.

Rank the methods in terms of convergence speed and comment on the order of the convergence.

Implied Volatility Computation with Trinomial Trees

A four months American put with strike \$50 on an asset with spot price \$48 paying dividends continuously at rate 1% is worth \$4.08. The risk free interest rates are constant at 2%.

(i) Identify the number of time steps N_{fixed} in the trinomial tree pricer which gives the value of a four months put option within 10^{-6} tolerance:

Start with 10 time steps for the given option and double the time steps until two consecutive approximations are within tolerance of each other. The finest tree considered corresponds to N_{fixed} time steps.

(ii) Use the secant method with initial volatility approximations 5% volatility and 100% volatility, i.e., $\sigma_0 = 0.05$ and $\sigma_1 = 1$. Use tolerance 10^{-4}. Report all the implied volatility approximations from the secant method recursion as well as the corresponding trinomial tree approximate values of the option.