MTH9831_Homework11_Group1

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MTH 9831. HW 11.

2. i) $M(t) = N(t) - \lambda t \Rightarrow M^2(t) = N^2(t) - 2\lambda t N(t) + \lambda^2 t^2$

11.1. So $E[M^2(t)] < \infty$.

by conditional Jensen's inequality $f(x) = x^2$ is convex.

$E[M^2(t) | F(s)] \geq E[M(t) | F(s)]^2 = M^2(s)$. for $\forall\ s \leq t$.

$\Rightarrow M^2(t)$ is submartingale.

ii). $M(t)$ has independent and stationary increment.

$\Rightarrow E[M^2(t) - M^2(s) | F(s)] = E[(M_t - M_s)^2 + 2M_s(M_t - M_s) | F_s]$

$= E[M^2_{t-s}] + 2M_s E[M_{t-s}] = \lambda(t-s)$ for $\forall\ s \leq t$.

So $E[M_t^2 - \lambda t | F_s] = E[M_s^2 - \lambda s]$.

$\Rightarrow M^2(t) - \lambda t$ is martingale.

11.3. $E[S_t | F_u] = E[S_u \cdot \exp\{(N(t) - N(u)) \log(1+\sigma) - \lambda\sigma(t-u)\} | F_u]$

$= S_u \cdot E[\exp\{N_{t-u} \cdot \log(1+\sigma)\}] \cdot \exp\{-\lambda\sigma(t-u)\}$.

using GMF of $N(t)$.

for $\forall\ u \leq t$. $= S_u \cdot \exp\{-\lambda\sigma(t-u)\} \cdot \exp\{\lambda(t-u)(e^{\log(1+\sigma)} - 1)\} = S_u$.

$\Rightarrow S_t$ is martingale.

3. $E[h(Q_T) | F_t] = E[h(Q_t + Q_T - Q_t) | F_t] = E[h(x + Q_{T-t})]\Big|_{x = Q_t}$

$= g(t, x)$. $g(t,x) = E[h(Q_{T-t} + x)]$.

4. a) $\displaystyle\int_0^t N(s)\,dN(s) = \sum_{0<u<t} N(u)\,\Delta N(u) = \frac{(1+N(t))\,N(t)}{2}.$

b). $\displaystyle\int_0^t N(s-)\,dN(s) = \sum_{0<u<t} N(u-)\,\Delta N(u) = \frac{(-1+N(t))\,N(t)}{2}.$

c). $\displaystyle M^2(t) = M^2(0) + 2\int_0^t M(u)\,d(-\lambda t) + \sum_{0<u\le t}\left(M^2(u)-M^2(u-)\right)$

$\displaystyle = 2\int_0^t M(u-)(-\lambda)\,du + \sum_{0<u\le t}(2M(u-)+1)\,\Delta N(u)$

$\displaystyle = 2\int_0^t M(u-)\,dM(u) + N(t)$

$\displaystyle \Rightarrow \int_0^t M(u-)\,dM(u) = \frac{1}{2}\left(M^2(t) - N(t)\right).$

d). $\displaystyle M^2(t) = 2\int_0^t M(u)(-\lambda)\,du + \sum_{0<u\le t}(2M(u)-1)\,\Delta N(u).$

$\displaystyle = 2\int_0^t M(u)\,dM(u) - N(t).$

$\displaystyle \Rightarrow \int_0^t M(u)\,dM(u) = \frac{1}{2}\left(M^2(t) + N(t)\right).$

5. a). $Z(t) = Z(0) f(X(t))$. $\quad f(x) = f'(x) = e^x$.

$X(t) = (\lambda - \tilde{\lambda})t + \ln(\tilde{\lambda}/\lambda) N(t)$.

$\Rightarrow dX(t) = (\lambda - \tilde{\lambda})dt + \ln(\tilde{\lambda}/\lambda) dN(t)$. then by Itô's formula.

$Z(t) = Z(0) + (\lambda-\tilde{\lambda})\int_0^t Z(u)du + \sum_{0<u\le t} (Z(u) - Z(u-))$.

$Z(u) - Z(u-) = Z(0) \exp\{(\lambda-\tilde{\lambda})u\} (\frac{\tilde{\lambda}}{\lambda})^{N(u-)} [(\frac{\tilde{\lambda}}{\lambda})^{N(u)-N(u-)} - 1]$.

$= Z(u-)(\frac{\tilde{\lambda}}{\lambda} - 1)\Delta N(u)$

$\Rightarrow \sum_{0<u\le t} Z(u) - Z(u-) = \frac{\tilde{\lambda}-\lambda}{\lambda}\int_0^t Z(u-)dN(u)$.

$Z(t) = Z(0) + (\lambda-\tilde{\lambda})\int_0^t Z(u-)du + \frac{\tilde{\lambda}-\lambda}{\lambda}\int_0^t Z(u-)dN(u)$

$= Z(0) + \frac{\tilde{\lambda}-\lambda}{\lambda}\int_0^t Z(u-)d\bar{M}(u)$. $\quad \Rightarrow dZ(t) = \frac{\tilde{\lambda}-\lambda}{\lambda} Z(t-)dM(t)$.

b). $X(t) = X(0) + \int_0^t \Gamma(s)dW(s) + \int_0^t \theta(s)ds + J(t)$.

Define $Y(t) = \exp\{\int_0^t \Gamma(s)dW(s) + \int_0^t \theta(s)ds - \frac{1}{2}\int_0^t \Gamma^2(s)ds\}$.

$= \exp\{X^c(t) - \frac{1}{2}[X^c, X^c](t)\}$.

$\Rightarrow dY(t) = Y(t)dX^c(t) = Y(t-)dX^c(t)$.

Define $K(t) = \prod_{0<s\le t}(1+\Delta X(s))$. $\quad K(0) = 1$.

$K(t) = K(t-)\cdot(1+\Delta X(t)) \Rightarrow \Delta K(t) = K(t-)\Delta X(t)$.

$Y(t)$ is continuous. $K(t)$ is pure jump process $\Rightarrow [Y, K](t) \equiv 0$

$\Rightarrow Z(t) = Y(t)K(t) = Y(0) + \int_0^t K(s-)dY(s) + \int_0^t Y(s-)dK(s)$

$= 1 + \int_0^t K(s-)Y(s-)dX^c(s) + \sum_{0<s\le t} Y(s-)K(s-)\Delta X(s)$

$= 1 + \int_0^t Z(s-)dX(s)$

$\Rightarrow$ satisfies $dZ(t) = Z(t-)dX(t)$ with $Z(0) = 1$.

why martingale: since the integrand is left-continuous.

6. Applying Itô's formula to

$S(t) = S(0) f(X(t))$. where $f(x) = f'(x) = e^x$.

$X(t) = -\lambda\sigma t + \ln(1+\sigma) N(t)$.

$\Rightarrow \quad S(t) = S(0) - \lambda\sigma \int_0^t S(u)\,du + \sum_{0<u\le t} (S(u) - S(u-))$

$$\left.\begin{array}{l} \int_0^t S(u)\,du = \int_0^t S(u-)\,du. \\ S(u) - S(u-) = S(u-)\sigma(N(u) - N(u-)) = S(u-)\sigma \Delta N(u) \end{array}\right\} \Rightarrow$$

$$S(t) = S(0) - \lambda\sigma \int_0^t S(u-)\,du + \sigma \int_0^t S(u-)\,dN(u)$$

$$= S(0) + \sigma \int_0^t S(u-)\,dM(u).$$

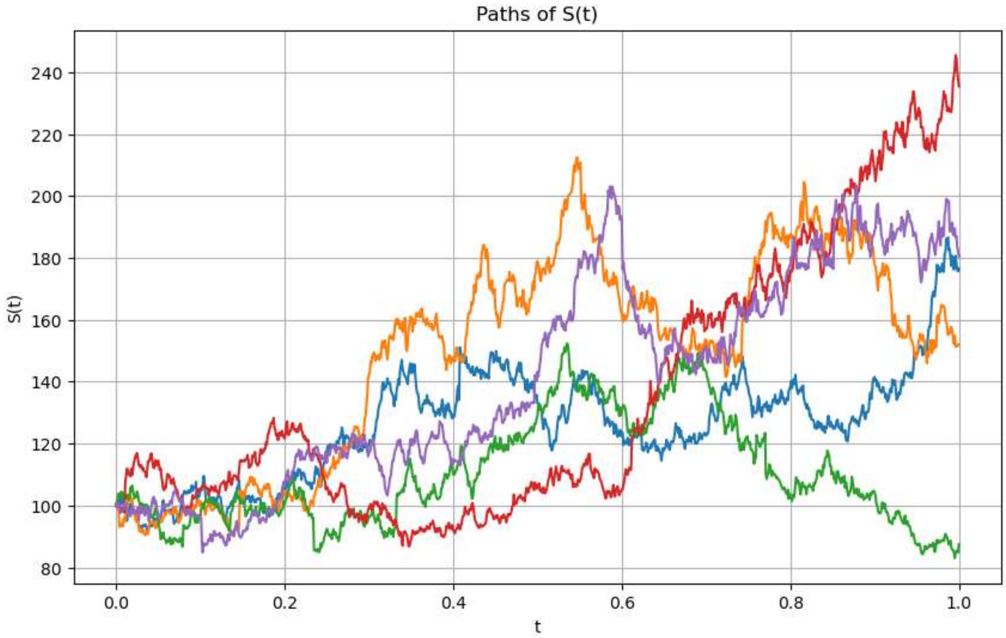
So. $S(t) = S(0) + \sigma \int_0^t S(u-)\,dM(u)$

$$= S(0) e^{-\lambda\sigma t} (1+\sigma)^{N(t)}.$$

In [ ]:

```
# Merton's jump diffusion model
import numpy as np
import matplotlib.pyplot as plt

# setup coefficients
mu = 0.2
sigma = 0.4
lbd = 10
h = 0.001
t_steps = 1000
num_paths = 5
```

```
# generate S(t)
def generate_path():
    X = np.zeros(t_steps+1)
    S = np.zeros(t_steps+1)
    X[0] = 0

    for i in range(t_steps):
        Z = np.random.normal(0, 1)
        N = np.random.poisson(h * lbd)

        if N == 0:
            M = 0
        else:
            Y = np.random.uniform(-0.1, 0.1, N)
            M = np.sum(Y)

        X[i+1] = X[i] + mu * h + np.sqrt(h) * sigma * Z + M
        S[i+1] = 100 * np.exp(X[i+1])

    S[0] = 100

    return S

# Define the time steps
t = np.linspace(0, 1, t_steps + 1)
# Generate and plot multiple paths of S(t)
plt.figure(figsize=(10, 6))
for _ in range(num_paths):
    S = generate_path()
    plt.plot(t, S)

plt.xlabel('t')
plt.ylabel('S(t)')
plt.title('Paths of S(t)')
plt.grid(True)
plt.show()
```



Paths of S(t)
S(t)
t
240
220
200
180
160
140
120
100
80
0.0
0.2
0.4
0.6
0.8
1.0