# Mathematical and Computational Analysis of Stochastic Car and Drone Sharing Network Models

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**Abstract**

The first part of this paper describes car sharing systems, as constructed by Fricker [1], and observes the differences caused by allowing cars to return to the same station, highlighting convergence to steady state probabilities. The next part describes how a computational simulation of car sharing can be changed to resemble drone sharing and the results of that change. Applications of these models are suggested.

Car sharing systems are important in today's world, both to give economic opportunity to those who cannot afford a car and to limit the environmental impact of car production. Drone transportation may also become important in electrifying more transportation and thus reducing carbon emissions.

It is assumed that customers would arrive to partake of these networks in a stochastic fashion, where the time of the next customer's arrival is not affected by the previous customer's arrival but the customers do arrive according to a probability distribution. The time a given trip takes is similarly determined.

First, say there is a car sharing system where customers arrive at a rate according to the exponential distribution with mean $\lambda$ and trips are also completed at a rate according to the exponential distribution with mean $\mu$. There are a given number of cars and stations in the system with a given number of parking spaces at each station. Customers must claim a car and reserve a parking space at their destination before departing.

This model can be analyzed mathematically, through the construction of matrices, and computationally, through setting the random, exponentially distributed rates that customers arrive and trips are completed, and analyze which event happens next in a simulation in order to construct a progression of states.

In the computer simulation, if the next event is a customer arrival, a destination is determined, a car is taken, and a spot is reserved at the destination station. The time of this transition is recorded. Otherwise, the next event is a car being returned, the car comes to occupy the reserved space, which ceases to be reserved as it is then occupied, and the time of this transition is recorded.

For the 2 cars, 2 stations (with 2 parking spaces at each) system, there are 6 unique states. The states are as follows in terms of (k, l) where k is the number of

cars parked at a station (and thus taking up a parking space) and l being the number of cars in transit to the station (and thus having reserved a parking space), where the 2 stations are interchangeable (i.e. (2,0); (0,0) = (0,0); (2,0)):
1: (2,0); (0,0)
2: (1,1); (0,0)
3: (1,0); (0,1)
4: (1,0); (1,0)
5: (0,1); (0,1)
6: (0,2); (0,0)

However, there are two different policies on where cars can go. They can either go to any station except their origin station, as described in Fricker's paper, or they can go to any station including the origin station, possibly more relevant when tours of the area are a possibility. The latter system is, from here onward, referred to as the "my way" system.

The "Fricker's way" (not allowing a car to immediately return to its origin station) system yields the following matrix:

$$\begin{pmatrix} -\lambda & 0 & \lambda & 0 & 0 & 0 \\ \mu & -(\mu+\lambda) & 0 & 0 & \lambda & 0 \\ 0 & 0 & -(\mu+\lambda) & \mu & 0 & \lambda \\ 0 & 2\lambda & 0 & -2\lambda & 0 & 0 \\ 0 & 0 & 2\mu & 0 & -2\mu & 0 \\ 0 & 2\mu & 0 & 0 & 0 & -2\mu \end{pmatrix}$$

Where, for example, $\frac{dP_1}{dt} = -\lambda\, P_1 + \mu\, P_2$

A $\lambda$ can be factored out of the matrix such that the matrix is only in terms of $r = \frac{\mu}{\lambda}$

Through taking the transpose of the matrix and using row reduction and substitution (with the assistance of WolframAlpha), the steady states of the system can be found. The normalized steady states of the "Fricker's way" system are:

$$P_1 = \frac{2r^2}{3r^2+4r+2}$$

$$P_2 = \frac{2r}{3r^2+4r+2}$$

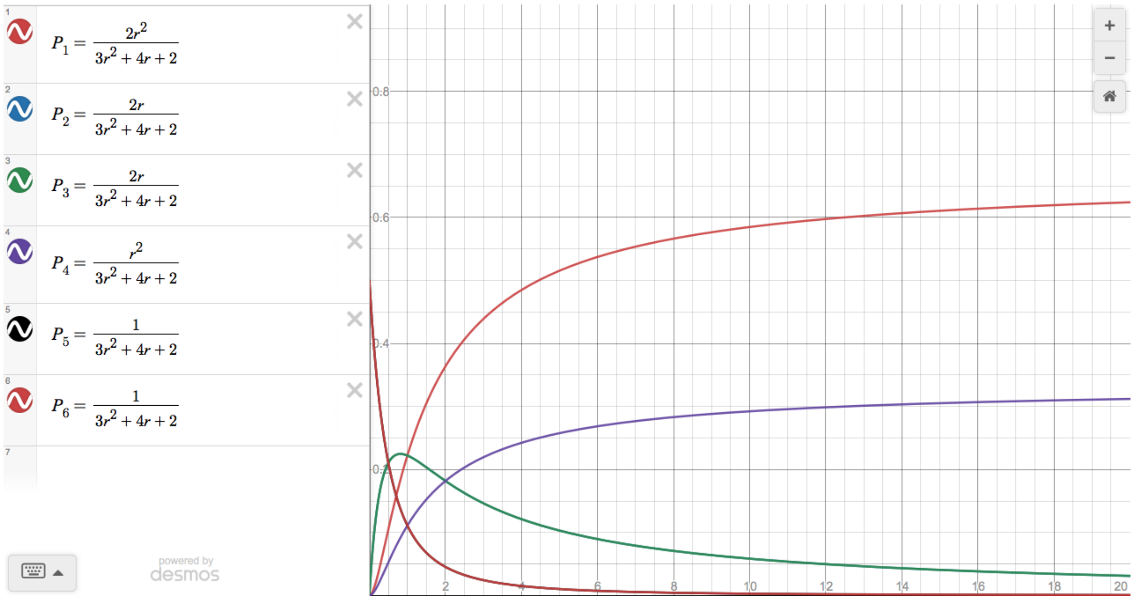
$$P_3 = \frac{2r}{3r^2+4r+2}$$

$$P_4 = \frac{r^2}{3r^2+4r+2}$$

$$P_5 = \frac{1}{3r^2+4r+2}$$

$$P_6 = \frac{1}{3r^2+4r+2}$$

This allows the steady state distribution to be graphed with varying r:




The matrix for "my way" (allowing a car to immediately return to its origin station) is:

$$\begin{pmatrix} -\lambda & \frac{\lambda}{2} & \frac{\lambda}{2} & 0 & 0 & 0 \\ \mu & -(\mu+\lambda) & 0 & 0 & \frac{\lambda}{2} & \frac{\lambda}{2} \\ 0 & 0 & -(\mu+\lambda) & \mu & \frac{\lambda}{2} & \frac{\lambda}{2} \\ 0 & \lambda & \lambda & -2\lambda & 0 & 0 \\ 0 & 0 & 2\mu & 0 & -2\mu & 0 \\ 0 & 2\mu & 0 & 0 & 0 & -2\mu \end{pmatrix}$$

Where, for example, $\frac{dP_1}{dt} = -\lambda\, P_1 + \mu\, P_2$

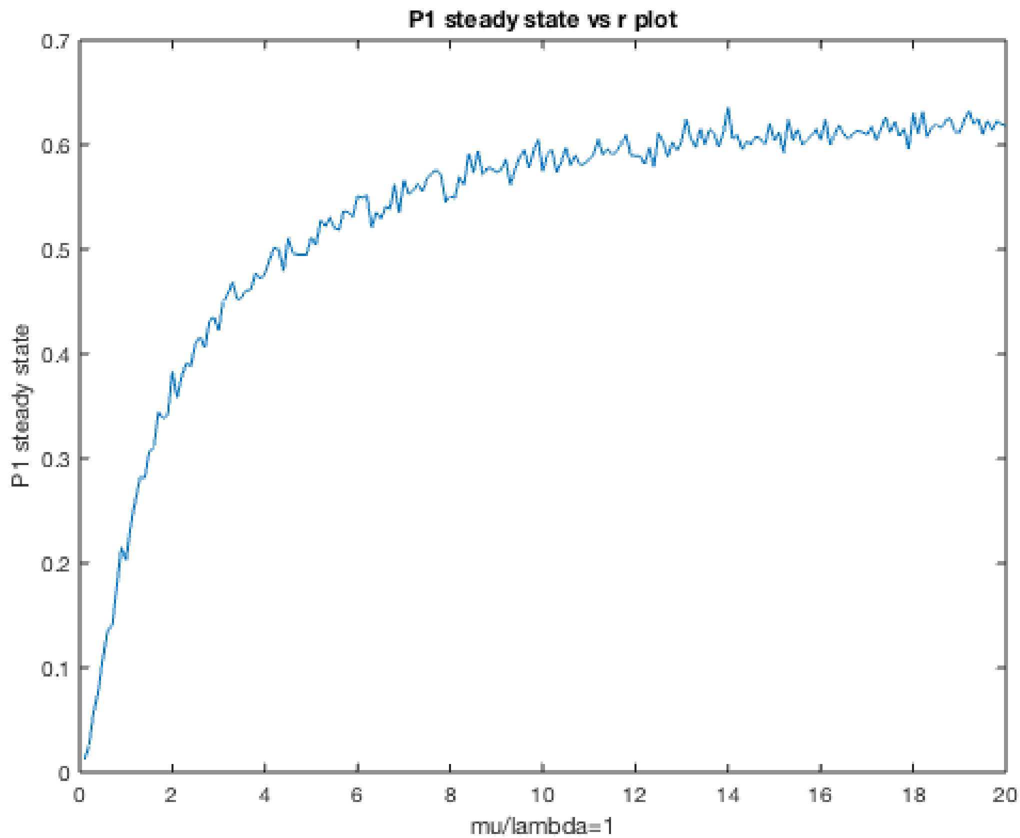
Again, a $\lambda$ can be factored out of the matrix such that the matrix is only in terms of $r = \frac{\mu}{\lambda}$

Through taking the transpose of the matrix and using row reduction and substitution, the steady states of the system can be found. The normalized steady states of the "my way" system are . . . the same as the "Fricker's way" system!

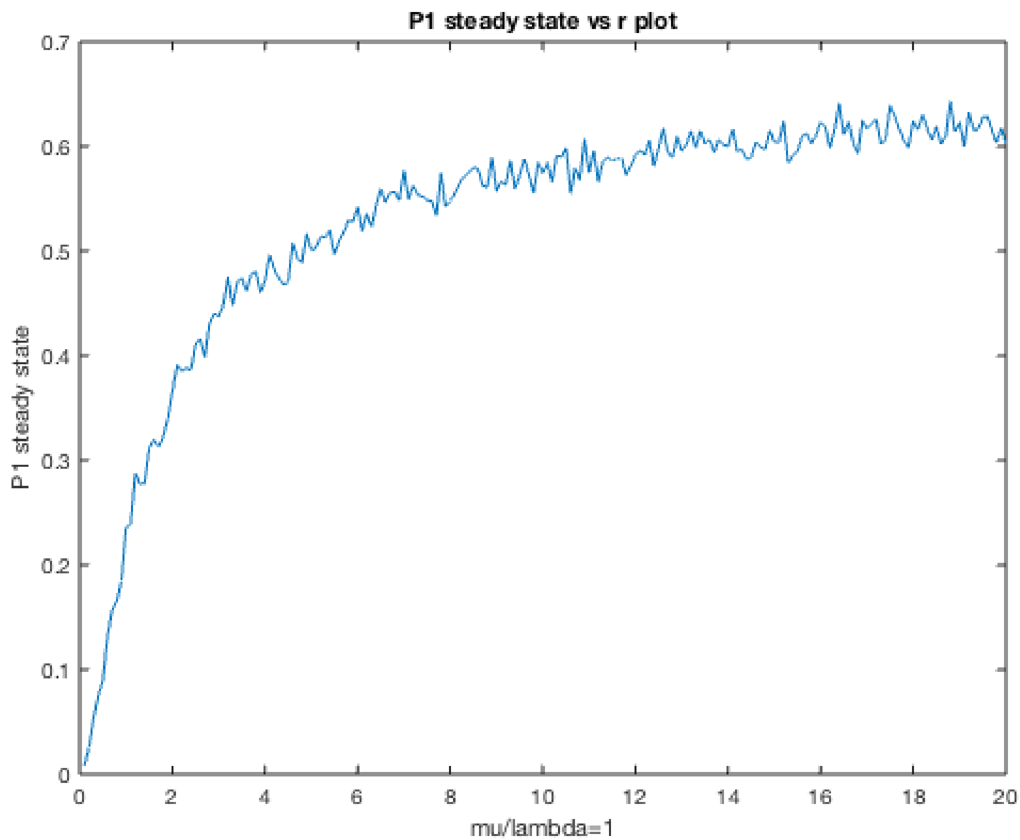
This is not obvious since the "my way" system includes several transitions that the "Fricker's way" system does not include directly (happening in one instant).

Graphs generated via MATLAB computer simulation, as described above, of the $P_1$ steady state with varying r confirms this similarity.

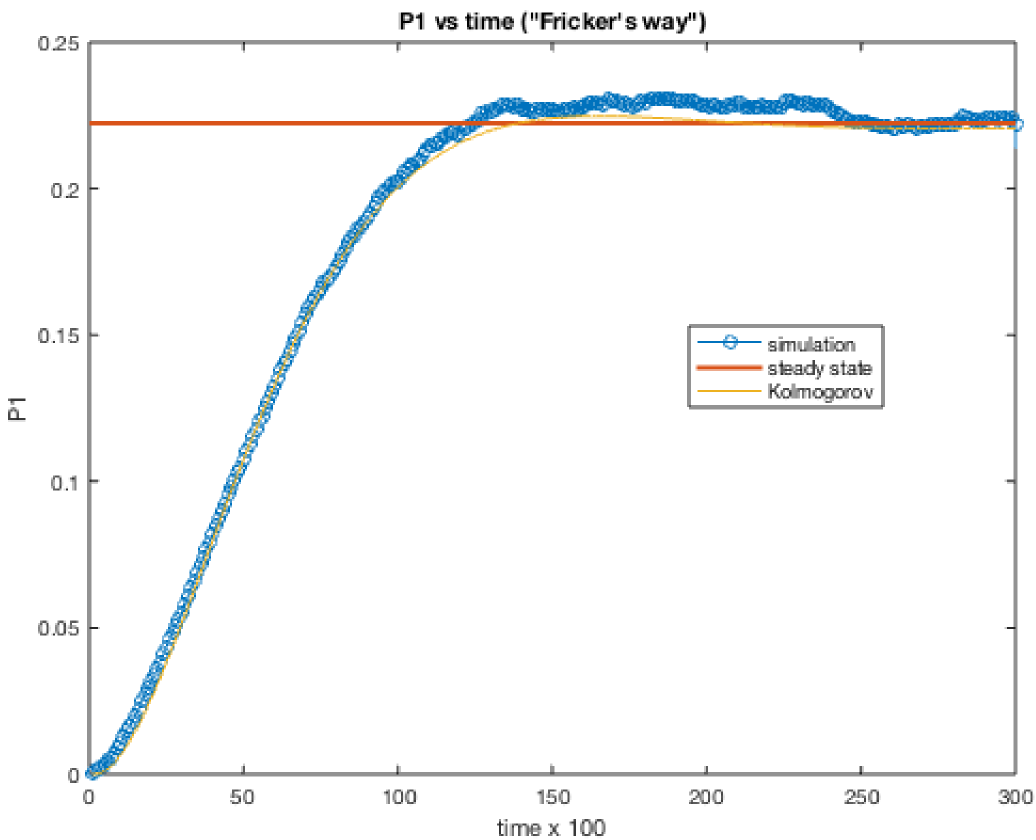
We have the graph of $P_1$ using "Fricker's way":


P1 steady state vs r plot
P1 steady state
mu/lambda=1
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
12
14
16
18
20

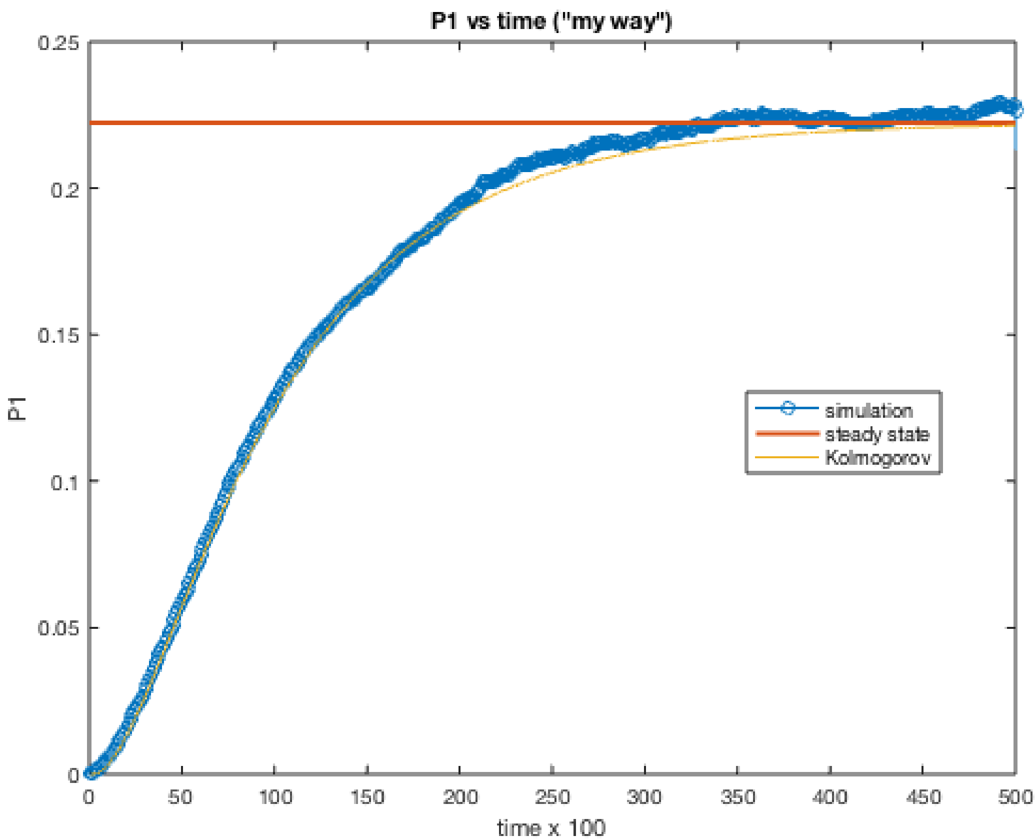
And the graph of $P_1$ using "my way":



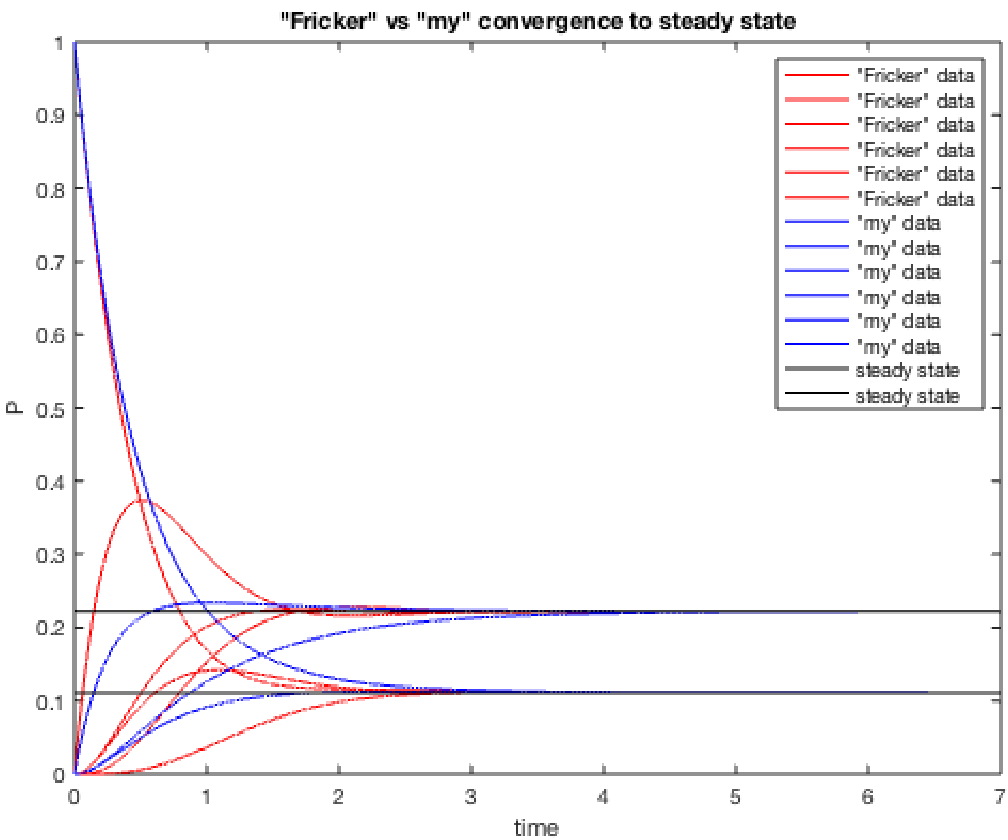
It can also be shown that the MATLAB simulation matches the Kolmogorov equation graphs for time evolution. The Kolmogorov equation graphs were mathematically derived from the matrices above (generated via MATLAB, but using the above matrices rather than exponential distributions). For the following graphs, $\mu = \lambda = 1$. The "Fricker's way" system generates the following graph:

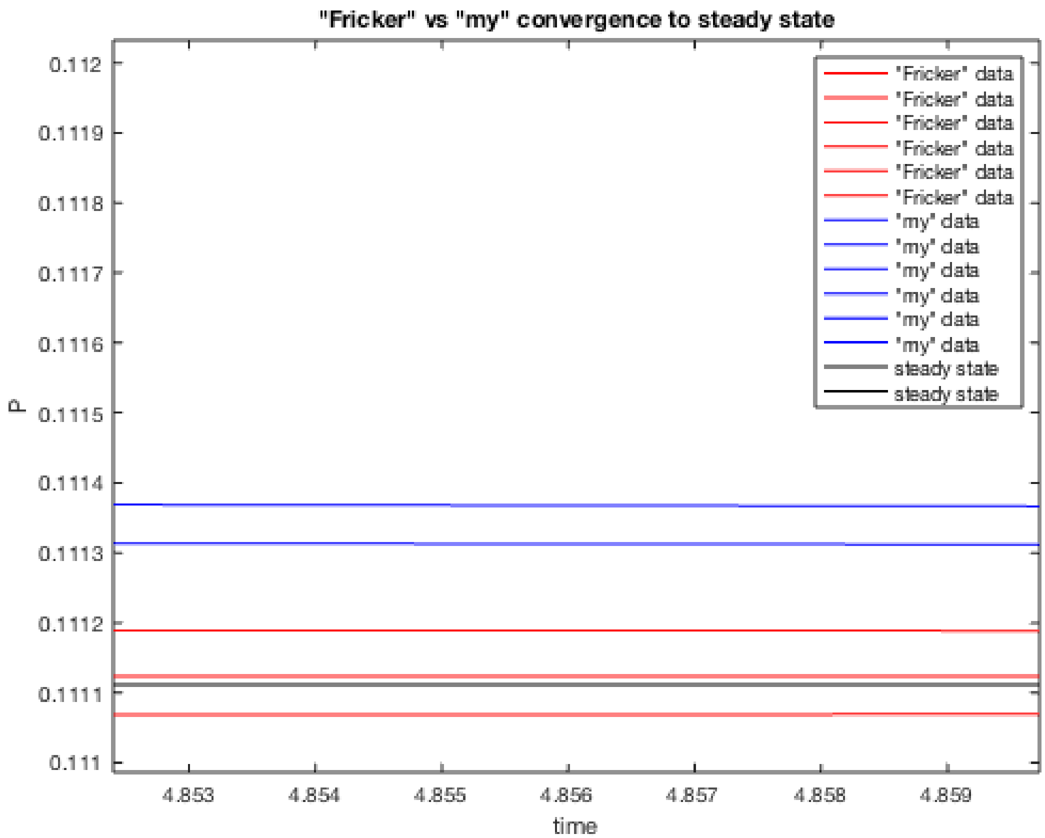


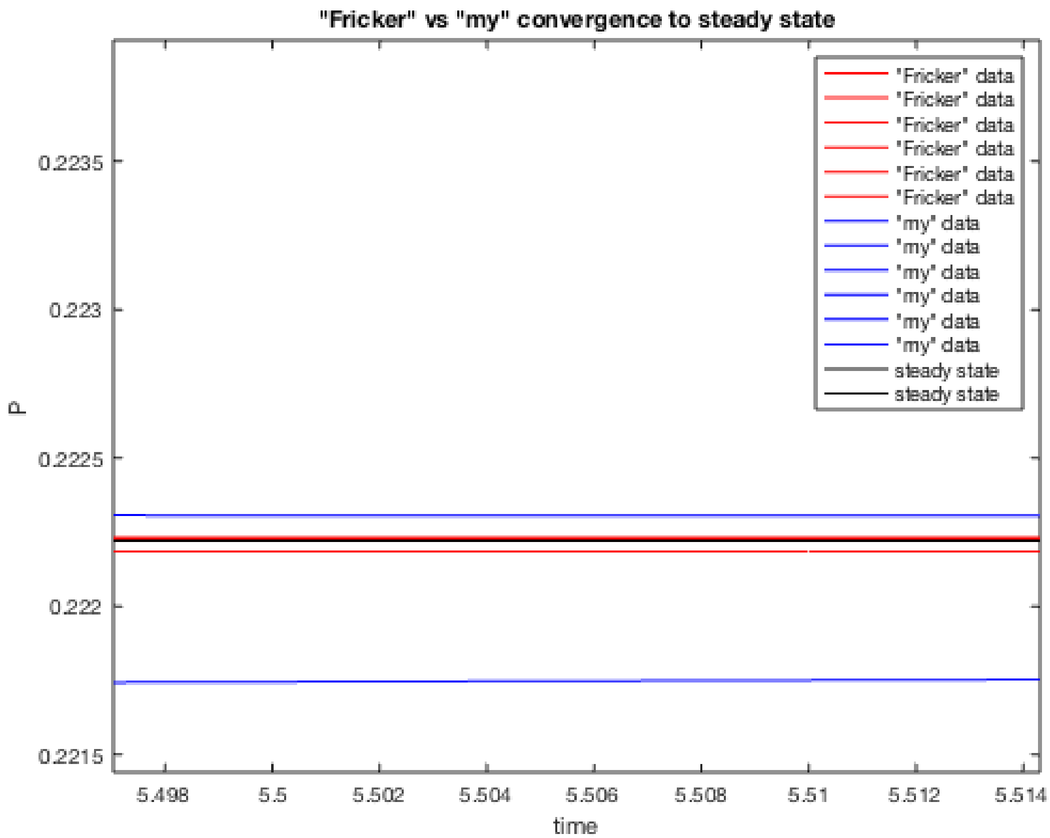
The “my way” system generates the following graph:



The “Fricker’s way” system converges to its steady state distribution faster than the “my way” system, as shown by the Kolmogorov equation graphs:




"Fricker" vs "my" convergence to steady state
0.112
0.1119
0.1118
0.1117
0.1116
0.1115
0.1114
0.1113
0.1112
0.1111
0.111
P
4.853
4.854
4.855
4.856
4.857
4.858
4.859
time
"Fricker" data
"Fricker" data
"Fricker" data
"Fricker" data
"Fricker" data
"Fricker" data
"my" data
"my" data
"my" data
"my" data
"my" data
"my" data
steady state
steady state



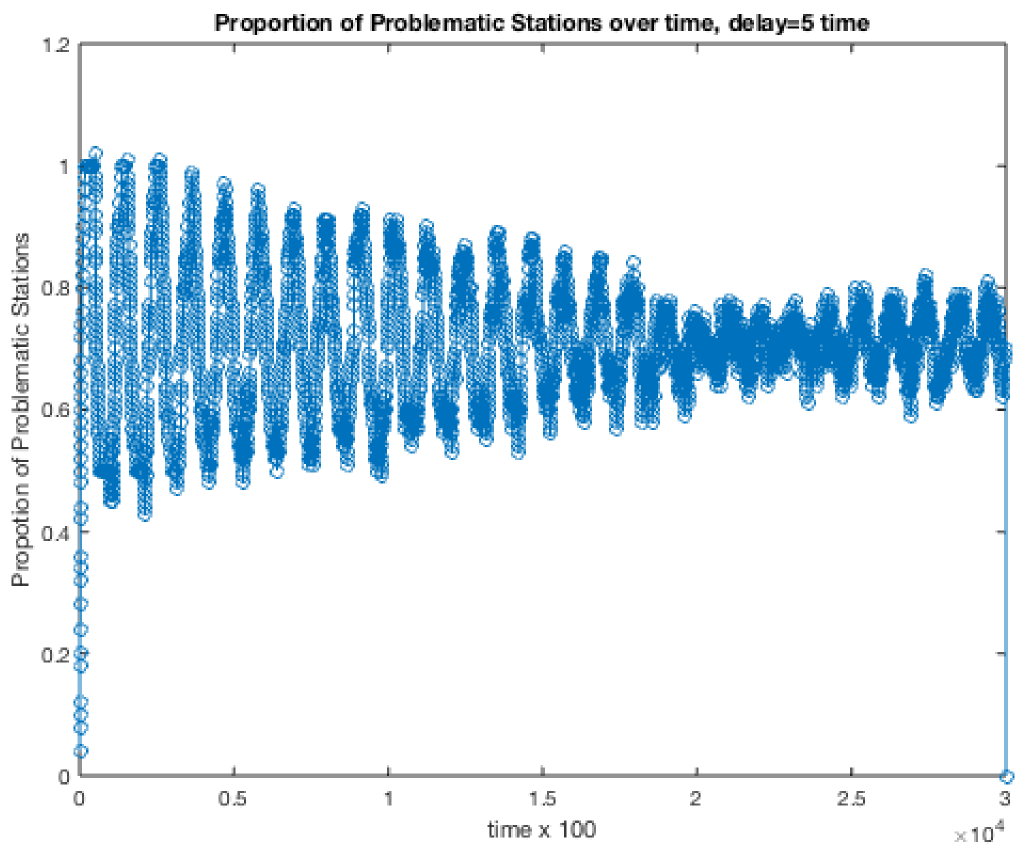

This can be understood intuitively, since “Fricker's way” forces more “mixing” whereas “my way” allows the persistence of a given state since “my way” allows for cars to return to the station they just departed from. This is valuable information since it can now be said that an approximation of the steady state distribution can be achieved faster computationally using “Fricker's way”. Further, the steady state distribution of any system with given $\mu$, $\lambda$, number of cars, number of stations, and number of spots at each station can be found more quickly using “Fricker's way”.

Now, the code used for the simulation (used to make the above “simulation” graphs) can be used to determine what proportion of stations will be “problematic”, as defined by Fricker in her paper (stations that are full or empty), but this is beyond the scope of this paper as it is not as mathematically interesting as analyzing steady state convergence. It does, however, have obvious applications for car sharing networks and an area of future research might be whether “my way” or “Fricker's way” produces a smaller proportion of problematic stations for given initial conditions, $\mu$, $\lambda$, number of cars, number of stations, and number of parking spaces at each station.

The key differences between a car sharing and drone sharing network, for the purposes of this paper, is that drones land on landing pads and are then taken to a parking lot. They are also taken from a parking lot to a landing pad when a customer wants one. For the purposes of this paper, these trips to and from the parking lot are assumed to require a constant amount of time and the parking lot's capacity, as well as the number of drones in the parking lot, is assumed to be infinite. This paper also assumes that a customer would only want to travel via drone to a different station, and not immediately back to the origin station.

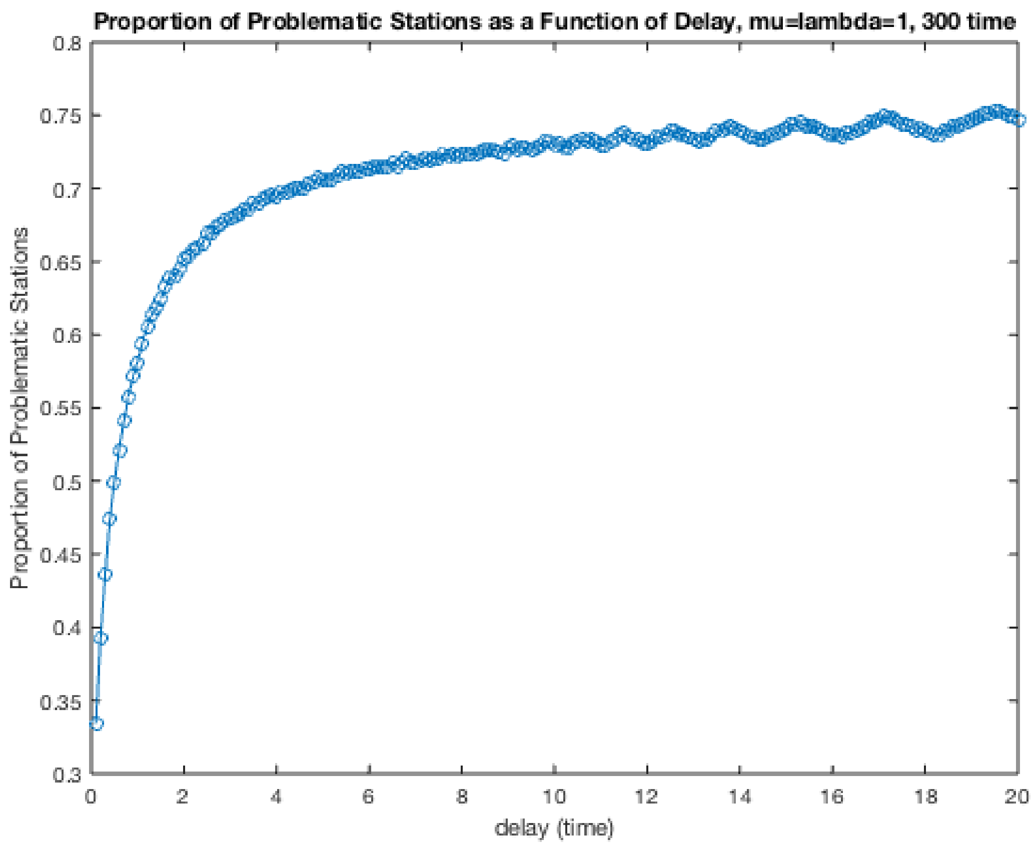
Also, the concept of "problematic stations" is useful in this paper for understanding a network of drone transportation. However, in the case of drones a "problematic station" is defined as one that has no free spaces, which would be reserved by an arriving drone or be used by a departing drone.

The proportion of problematic stations oscillates and starts to die, as shown in the following graph of problematic stations, where there are 2 stations with 2 landing pads at each, $\mu = \lambda = 1$, and the delay is 5 time:

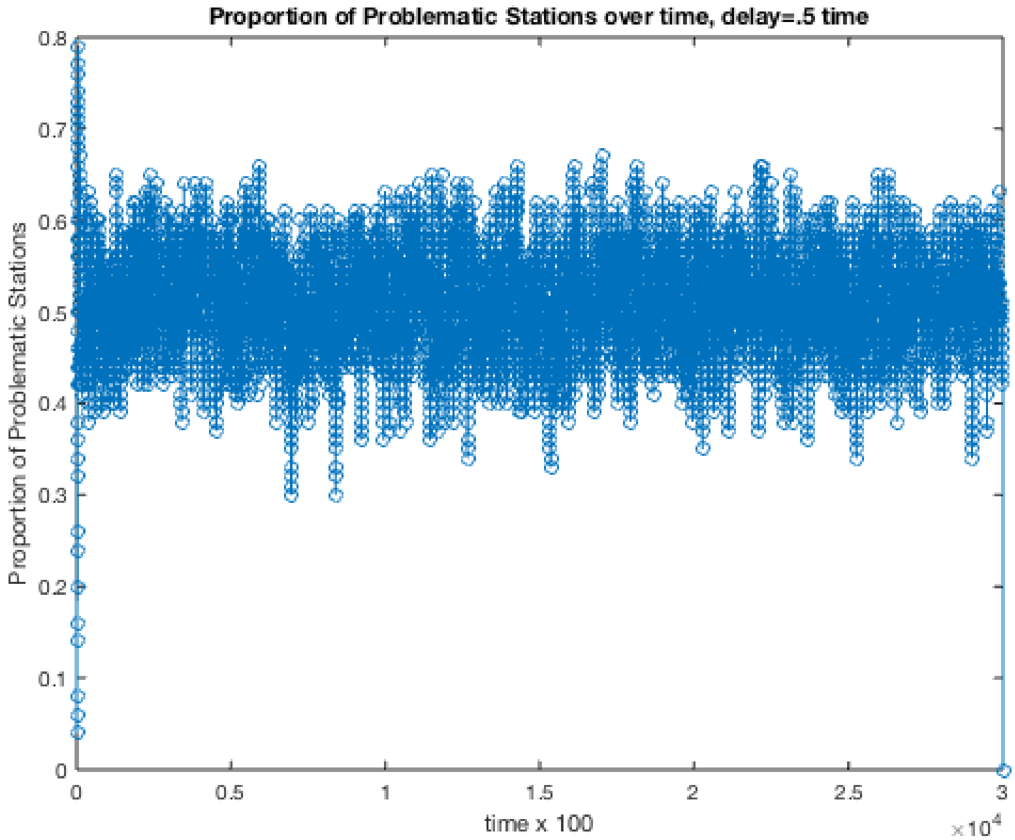


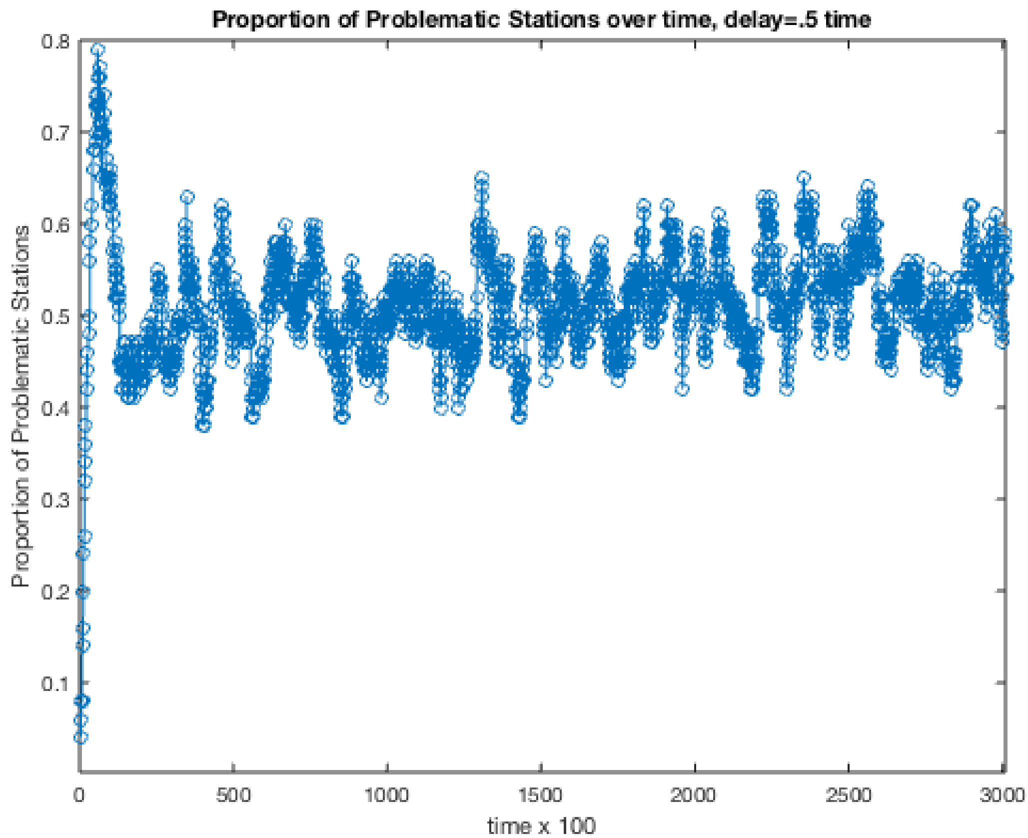
This can be understood intuitively in that drones would start as taking off and landing at the same time, and due to the exponential distribution would grow to be out of sync with each other.

One can also see how the average proportion of problematic stations increases fairly quickly as a function of delay through the following graph, where there are 2 stations with 2 landing pads at each, $\mu = \lambda = 1$, and the total time is still 300 time:



Indeed, when we instead set the delay to be $\frac{1}{2}$ time, there is a much lesser proportion of problematic stations:






Thus, the system for analyzing a car sharing network can also be used to analyze a drone sharing network with minimal alterations.

So, as was previously mentioned, areas of further research may include applications to car sharing networks, including whether "my way" or "Fricker's way" causes fewer problematic stations for a given system. The simulation for drones might also be updated to include the possibility of landing pads being used when it is known that a drone will use it in the future but is not currently using it. For example, a drone could use a landing pad to take off when another drone is in transit to that pad but will land long after the former drone has departed. This current simulation, however, might be optimal for situations where the trip lengths are relatively unpredictable.

The simulation might also include the possibility for different stations to have different numbers of landing pads or different amounts of demand. Similarly, car sharing might also be analyzed in this way, with differing numbers of parking spots and demand at each station.

It is important to keep furthering the understanding of these systems, as they provide economic opportunity and reduce carbon emissions. With a better understanding of the way these systems behave, networks can create incentives. These incentives might cause the choosing of a destination that has fewer cars located at it (as outlined by Fricker), in the case of cars, or more free spaces, in the case of drones.

This would be especially possible when multiple destinations are preferable, and a smaller proportion of problematic stations arise. Another effort to reduce problematic stations might also include changing the number of parking spots/landing pads in the case of uneven demand.

Whatever the case, these networks are here to stay and proper analyses can help them run efficiently and profitably.

# References

[1] Christine Fricker, Cedric Bourdais. *A Stochastic Model for Car-Sharing Systems.* April, 2015.