Bradley Mont
CS 161
Disc 1C
804-993-030

Homework 7

1

Prove: $Pr(\alpha_1, \ldots, \alpha_n | \beta) = Pr(\alpha_1 | \alpha_2, \ldots, \alpha_n, \beta) \, Pr(\alpha_2 | \alpha_3, \ldots, \alpha_n, \beta) \ldots Pr(\alpha_n | \beta)$
by Induction

Base Case

$n=1$:

$Pr(\alpha_1 | \beta) = Pr(\alpha_1 | \beta)$ ✓

$n=2$: $Pr(\alpha_1, \alpha_2 | \beta)$

Using Bayes Conditioning Rule,

$Pr(\alpha_1, \alpha_2 | \beta) = Pr(\alpha_1 | \alpha_2 | \beta) \cdot Pr(\alpha_2 | \beta)$

$Pr(\alpha_1, \alpha_2 | \beta) = Pr(\alpha_1 | \alpha_2, \beta) \cdot Pr(\alpha_2 | \beta)$ ✓

Inductive Step

- assume statement is true for $1 \ldots (n-1)$

$Pr(\alpha_1, \ldots, \alpha_n | \beta) = Pr((\alpha_1, \ldots, \alpha_{n-1}), \alpha_n | \beta)$

- Using Bayes Conditioning Rule again,

$Pr(\alpha_1, \ldots, \alpha_n | \beta) = Pr(\alpha_1, \ldots, \alpha_{n-1} | \alpha_n, \beta) \cdot Pr(\alpha_n | \beta)$

- Using the assumption for $n-1$,

$Pr(\alpha_1, \ldots, \alpha_{n-1} / \alpha_n, \beta) = Pr(\alpha_1 | \alpha_2, \ldots, \alpha_n, \beta) \cdot Pr(\alpha_2 | \alpha_3, \ldots, \alpha_n | \beta) \cdot$
$\ldots \cdot Pr(\alpha_{n-1} | \alpha_n, \beta)$

$\therefore Pr(\alpha_1, \ldots, \alpha_n | \beta) = Pr(\alpha_1 | \alpha_2, \ldots, \alpha_n, \beta) \, Pr(\alpha_2 | \alpha_3, \ldots, \alpha_n, \beta) \ldots Pr(\alpha_n | \beta)$ ✓

2

O = oil is present

N = natural gas is present

P = test gives positive result

We know

- $Pr(O) = .5$, $Pr(\neg O) = .5$
- $Pr(N) = .2$, $Pr(\neg N) = .8$
- $Pr(\neg O \wedge \neg N) = .3$, $Pr(O \vee N) = .7$
- $Pr(O \wedge N) = 0$
- $Pr(P \mid O) = .9$, $Pr(\neg P \mid O) = .1$
- $Pr(P \mid N) = .3$, $Pr(\neg P \mid N) = .7$
- $Pr(P \mid \neg O \wedge \neg N) = .1$, $Pr(\neg P \mid \neg O \wedge \neg N) = .9$

- We want $Pr(O \mid P)$

$$\left.Pr(O \mid P) = \frac{Pr(P \mid O)\, Pr(O)}{Pr(P)}\quad\right\} \text{Bayes Rule}$$

- find $Pr(P)$ using Case Analysis:
- mutually exclusive and exhaustive:
  - $B_1 = O$ (oil present)
  - $B_2 = N$ (natural gas present)
  - $B_3 = \neg O \wedge \neg N$ (neither present)
  - they can't be present at the same time, so $B_1 \ldots B_3$ is mutually exclusive and exhaustive

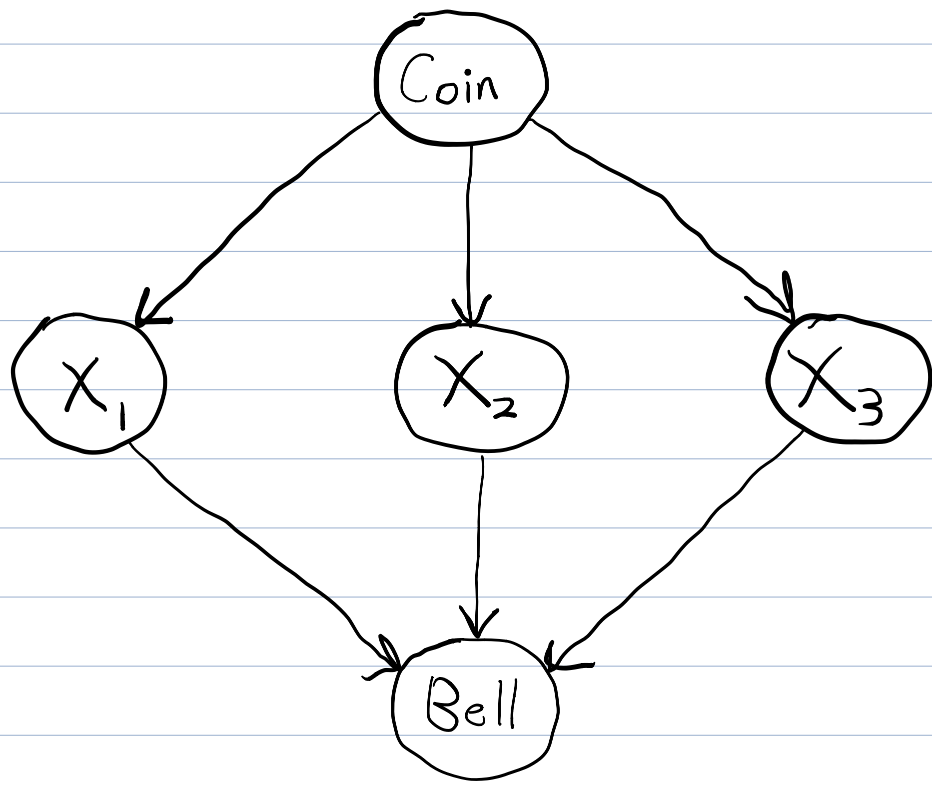
$$Pr(P) = Pr(P|O)\,Pr(O) + Pr(P|N)\,Pr(N) + Pr(P|\neg O \wedge \neg N)\,Pr(\neg O \wedge \neg N)$$
$$Pr(P) = (.9)(.5) + (.3)(.2) + (.1)(.3)$$
$$Pr(P) = .54$$

$$Pr(O|P) = \frac{Pr(P|O)\,Pr(O)}{Pr(P)} = \frac{(.9)(.5)}{.54}$$

$$Pr(O|P) = .833 = 83.3\%$$

3

- Bayesian network:



Coin = {a, b, c}
X_1 = {heads, tails}
X_2 = {heads, tails}
X_3 = {heads, tails}
Bell = {on, off} (on = the bell rings)

CPTs:

Coin	θ_{Coin}
a	1/3
b	1/3
c	1/3

Coin	X_1	$\theta_{X_1 \mid Coin}$
a	heads	.2
a	tails	.8
b	heads	.4
b	tails	.6
c	heads	.8
c	tails	.2

Coin	X_2	$\theta_{X_2 \mid Coin}$
a	heads	.2
a	tails	.8
b	heads	.4
b	tails	.6
c	heads	.8
c	tails	.2

Coin	X_3	$\theta_{X_3 \mid Coin}$
a	heads	.2
a	tails	.8
b	heads	.4
b	tails	.6
c	heads	.8
c	tails	.2

X_1	X_2	X_3	Bell	$\theta_{Bell \mid x_1, x_2, x_3}$
heads	heads	heads	on	1
heads	heads	heads	off	0
heads	heads	tails	on	0
heads	heads	tails	off	1
heads	tails	heads	on	0
heads	tails	heads	off	1
heads	tails	tails	on	0
heads	tails	tails	off	1
tails	heads	heads	on	0
tails	heads	heads	off	1
tails	heads	tails	on	0
tails	heads	tails	off	1
tails	tails	heads	on	0
tails	tails	heads	off	1
tails	tails	tails	on	1
tails	tails	tails	off	0

4

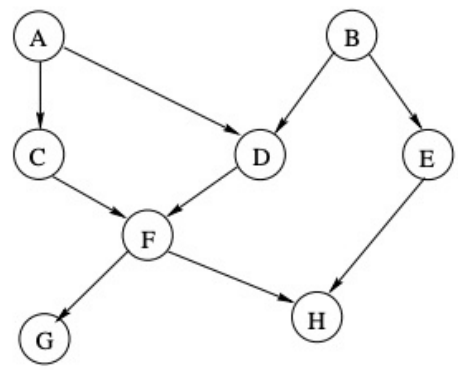


Figure 1: The DAG of a Bayesian network.

a. Markovian assumptions:

$I(A, \emptyset, BE)$

$I(B, \emptyset, AC)$

$I(C, A, BDE)$

$I(D, AB, CE)$

$I(E, B, ACDFG)$

$I(F, CD, ABE)$

$I(G, F, ABCDEH)$

$I(H, EF, ABCDG)$

b.

- dsep(A, F, E) ?

Z = {F} (known)

3 paths from A to E:

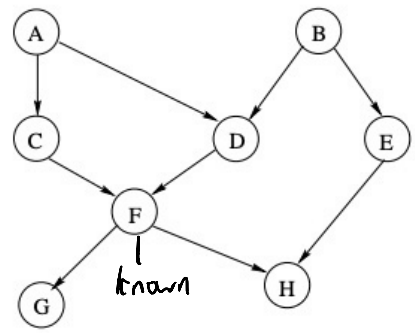


Figure 1: The DAG of a Bayesian network.

Path 1: A - D - B - E

- valve D (convergent): open
- valve B (divergent): open

∴ Path 1 is not blocked

Since Path 1 is not blocked by F, not every path between A and E is blocked by F, so A and E are not d-separated given F. → FALSE

• dsep(G, B, E)?

Z = {B} (known)

3 paths from G to E:

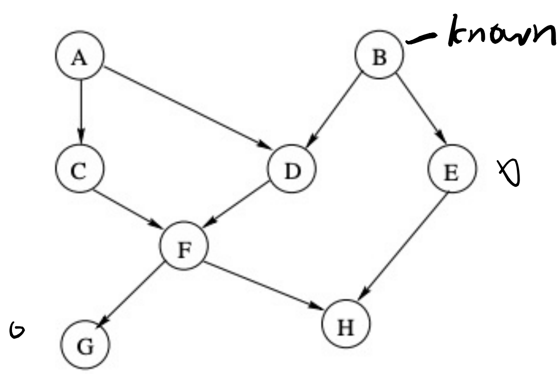


Figure 1: The DAG of a Bayesian network.

Path 1: G - F - H - E

• valve F (divergent): open
• valve H (convergent): closed
∴ Path 1 is blocked

Path 2: G - F - D - B - E

• valve F (sequential): open
• valve D (sequential): open
• valve B (divergent): closed
∴ Path 2 is blocked

Path 3: G - F - C - A - D - B - E

• valve B (divergent): closed (now we know the path is blocked)
∴ Path 3 is blocked

Since every path between G and E is blocked by B,
G and E are d-separated given B. → TRUE

• dsep(AB, CDE, GH)?

$Z = \{CDE\}$ (known)
$X = \{AB\}$
$Y = \{GH\}$

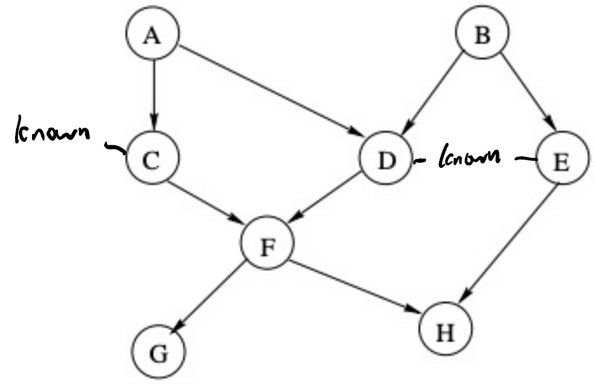


Figure 1: The DAG of a Bayesian network.

• Now we check if all paths between a node in X and a node in Y are blocked by Z

Path 1: A - C - F - G

• valve C (sequential): closed
∴ Path 1 is blocked

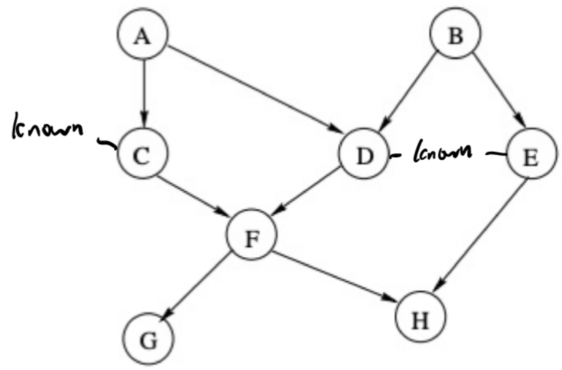
Path 2: A - D - F - G

• valve D (sequential): closed
∴ Path 2 is blocked

Path 3: A - D - B - E - H - F - G

• valve D (convergent): open
• valve B (divergent): open
• valve E (sequential): closed
∴ Path 3 is blocked

from $A \to G$



Path 4: A-C-F-H

- valve C (sequential): closed

∴ Path 4 is blocked

Path 5: A-D-F-H

- valve D (sequential): closed

∴ Path 5 is blocked

Path 6: A-D-B-E-H

- valve E (sequential): closed

∴ Path 6 is blocked

Path 7: A-C-F-D-B-E-H

- valve E (sequential): closed

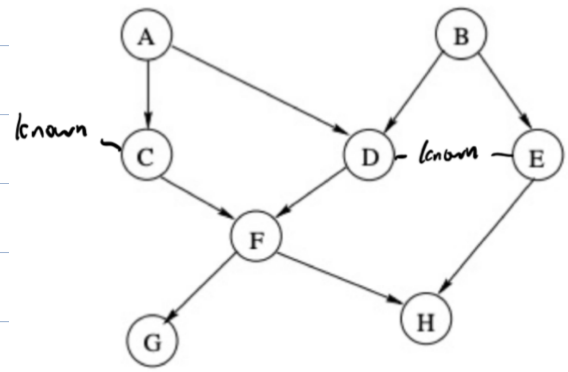
∴ Path 7 is blocked

from $A \to H$

Path 8 : B-D-F-G

• value D(sequential): closed

∴ Path 8 is blocked



Path 9: B-D-A-C-F-G

• value D (convergent): open

• value A (divergent): open

• value C(sequential): closed

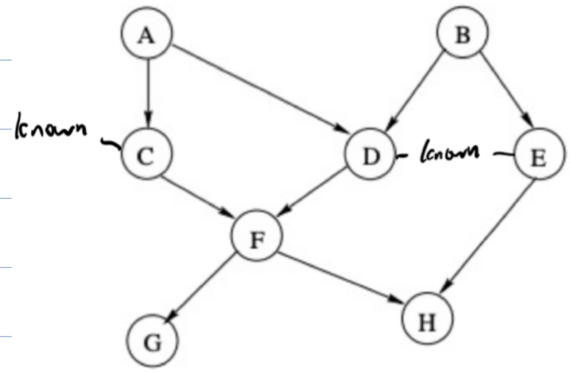
∴ Path 9 is blocked

Path 10: B-E-H-F-G

• value E(sequential): closed

∴ Path 10 is blocked

B → G



Path 11: B - E - H

- valve E (sequential): closed

∴ Path 11 is blocked

Path 12: B - D - F - H

- valve D (sequential): closed

∴ Path 12 is blocked

Path 13: B - D - A - C - F - H

- valve C (sequential): closed

∴ Path 13 is blocked

$B \to H$

Since every path between a node in $X = \{AB\}$ and a node in $Y = \{GH\}$ is blocked by $Z = \{CDE\}$, X and Y are d-separated given Z. → TRUE c.

$Pr(a, b, c, d, e, f, g, h) = ?$

- We use the distribution a Bayesian network:
- multiply the CPT parameter value of each variable
- (each variable depends on its parents)

$$Pr(a, b, c, d, e, f, g, h) = Pr(a) \cdot Pr(b) \cdot Pr(c|a) \cdot Pr(d|a,b) \cdot Pr(e|b) \cdot Pr(f|c,d) \cdot Pr(g|f) \cdot Pr(h|e,f)$$

d. $Pr(A=1, B=1)$

- We know A and B are independent from the Markovian assumption: $I(A, \emptyset, BE)$
- Since A and B are independent (given $\emptyset$), $Pr(A=1, B=1) = Pr(A=1)Pr(B=1) = (.2)(.7)$

$Pr(A=1, B=1) = 0.14$

$Pr(E=0 | A=0)$

Using Bayes Conditioning,

$$Pr(E=0|A=0) = \frac{Pr(E=0, A=0)}{Pr(A=0)}$$

- We know A and E are independent from the Markovian assumption: $I(A, \emptyset, BE)$

- so $Pr(E=0, A=0) = Pr(E=0)Pr(A=0)$

$$Pr(E=0|A=0) = \frac{Pr(E=0)\cancel{Pr(A=0)}}{\cancel{Pr(A=0)}} = Pr(E=0)$$

- Using Case Analysis,

$Pr(E=0) = Pr(E=0|B=0)Pr(B=0) + Pr(E=0|B=1)Pr(B=1)$

$Pr(E=0) = (.1)(.3) + (.9)(.7) = .03 + .63 = .66$

$Pr(E=0|A=0) = .66$

5

$\alpha : A \Rightarrow B$

	A	B	$Pr(A,B)$	$A \Rightarrow B$
w_0	T	T	0.3	✓
w_1	T	F	0.2	×
w_2	F	T	0.1	✓
w_3	F	F	0.4	✓

Table 1: A joint probability distribution.

a. $M(\alpha) = \{\omega_0, \omega_2, \omega_3\}$

b. $Pr(\alpha) = \sum_{\omega \models \alpha} Pr(\omega)$

$Pr(\alpha) = Pr(\omega_0) + Pr(\omega_2) + Pr(\omega_3)$

$Pr(\alpha) = 0.3 + 0.1 + 0.4$

$Pr(\alpha) = 0.8$

c. Compute $Pr(A, B \mid \alpha)$

$$Pr(\omega \mid \alpha) = \begin{cases} 0 & \text{if } \omega \models \neg\alpha \\ Pr(\omega)/Pr(\alpha) & \text{if } \omega \models \alpha \end{cases}$$

	A	B	$Pr(A,B \mid \alpha)$
ω_0	T	T	$.3/.8 = .375$
ω_1	T	F	0
ω_2	F	T	$.1/.8 = .125$
ω_3	F	F	$.4/.8 = .5$

d. $Pr(A \Rightarrow \neg B \mid \alpha)$

- Say Δ is $A \Rightarrow \neg B$
- $M(\Delta) = \{\omega_1, \omega_2, \omega_3\}$

$Pr(\Delta|\alpha) = \sum_{\omega \models (\Delta|\alpha)} Pr(\Delta|\alpha)$

$Pr(\Delta|\alpha) = Pr(\omega_1) + Pr(\omega_2) + Pr(\omega_3)$

$Pr(\Delta|\alpha) = 0 + .125 + .5$

$Pr(A \Rightarrow \neg B \mid \alpha) = .625$