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CS 161
Disc 1C
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Homework 5

1.

- $P \Rightarrow Q$, $\neg Q \Rightarrow \neg P$

Say Δ: $P \Rightarrow Q$ $\qquad$ α: $\neg Q \Rightarrow \neg P$

Show $\Delta \Leftrightarrow \alpha$

	P	Q	Δ	α
ω1	t	t	✓	✓
ω2	t	f	×	×
ω3	f	t	✓	✓
ω4	f	f	✓	✓

$M(\Delta) = \{\omega 1, \omega 3, \omega 4\}$ $\quad$ $M(\alpha) = \{\omega 1, \omega 3, \omega 4\}$

Since $M(P \Rightarrow Q) = M(\neg Q \Rightarrow \neg P)$, the two sentences are equivalent

- $P \Leftrightarrow \neg Q$, $((P \wedge \neg Q) \vee (\neg P \wedge Q))$

Say $\Delta : P \Leftrightarrow \neg Q$

$\alpha : ((P \wedge \neg Q) \vee (\neg P \wedge Q))$

- Show $\Delta \Leftrightarrow \alpha$

	P	Q	Δ	α
$\omega 1$	t	t	✗	✗
$\omega 2$	t	f	✓	✓
$\omega 3$	f	t	✓	✓
$\omega 4$	f	f	✗	✗

$M(\Delta) = \{\omega 2, \omega 3\}$ $M(\alpha) = \{\omega 2, \omega 3\}$

Since $M(P \Leftrightarrow \neg Q) = M(((P \wedge \neg Q) \vee (\neg P \wedge Q)))$, the two sentences are equivalent.

2

- $(Smoke \Rightarrow Fire) \Rightarrow (\neg Smoke \Rightarrow \neg Fire)$

Say $\Delta: (Smoke \Rightarrow Fire) \Rightarrow (\neg Smoke \Rightarrow \neg Fire)$

	Smoke	Fire	Δ
$\omega 1$	t	t	✓
$\omega 2$	t	f	✓
$\omega 3$	f	t	✗
$\omega 4$	f	f	✓

$M(\Delta) = \{\omega 1, \omega 2, \omega 4\}$

- Since $\omega 2 \not\models \Delta$, $M(\Delta) \neq$ all worlds, so Δ is not valid.
- Since $M(\Delta) \neq \emptyset$, there is at least one world where Δ holds, so Δ is satisfiable.

- $(Smoke \Rightarrow Fire) \Rightarrow ((Smoke \lor Heat) \Rightarrow Fire)$

Say Δ: $(Smoke \Rightarrow Fire) \Rightarrow ((Smoke \lor Heat) \Rightarrow Fire)$

	Smoke	Fire	Heat	Δ
$\omega 1$	t	t	t	✓
$\omega 2$	t	t	f	✓
$\omega 3$	t	f	t	✓
$\omega 4$	t	f	f	✓
$\omega 5$	f	t	t	✓
$\omega 6$	f	t	f	✓
$\omega 7$	f	f	t	✗
$\omega 8$	f	f	f	✓

$M(\Delta) = \{\omega 1, \omega 2, \omega 3, \omega 4, \omega 5, \omega 6, \omega 8\}$

- Since $\omega 7 \not\models \Delta$, $M(\Delta) \neq$ all worlds, so Δ is not valid.
- Since $M(\Delta) \neq \emptyset$, there is at least one world where Δ holds, so Δ is satisfiable.

• $((Smoke \land Heat) \Rightarrow Fire) \Leftrightarrow ((Smoke \Rightarrow Fire) \lor (Heat \Rightarrow Fire))$

Say $\Delta : ((Smoke \land Heat) \Rightarrow Fire) \Leftrightarrow ((Smoke \Rightarrow Fire) \lor (Heat \Rightarrow Fire))$

	Smoke	Fire	Heat	Δ
$\omega 1$	t	t	t	✓
$\omega 2$	t	t	f	✓
$\omega 3$	t	f	t	✓
$\omega 4$	t	f	f	✓
$\omega 5$	f	t	t	✓
$\omega 6$	f	t	f	✓
$\omega 7$	f	f	t	✓
$\omega 8$	f	f	f	✓

$M(\Delta) = \{\omega 1, \omega 2, \omega 3, \omega 4, \omega 5, \omega 6, \omega 7, \omega 8\}$

• Since $M(\Delta)$ = all worlds, Δ is valid.

3

If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.

Propositional Symbols

A = mythical

B = immortal

C = mammal

D = horned

E = magical

a. knowledge base: $[A \Rightarrow B] \wedge [\neg A \Rightarrow (\neg B \wedge C)] \wedge [(B \vee C) \Rightarrow D] \wedge [D \Rightarrow E]$

6. convert to CNF

• Step 1: get rid of all connectives besides $\wedge, \vee, \neg$

$[\neg A \vee B] \wedge [A \vee (\neg B \wedge C)] \wedge [\neg(B \vee C) \vee D] \wedge [\neg D \vee E]$

• Step 2: push negations inwards

$[\neg A \vee B] \wedge [A \vee (\neg B \wedge C)] \wedge [(\neg B \wedge \neg C) \vee D] \wedge [\neg D \vee E]$

• Step 3: distribute $\vee$ over $\wedge$:

$[\neg A \vee B] \wedge [(A \vee \neg B) \wedge (A \vee C)] \wedge [(\neg B \vee D) \wedge (\neg C \vee D)]$
$\wedge [\neg D \vee E]$

CNF:

$(\neg A \vee B) \wedge (A \vee \neg B) \wedge (A \vee C) \wedge (\neg B \vee D) \wedge$
$(\neg C \vee D) \wedge (\neg D \vee E)$

C.

- Can we use the knowledge base to prove that the unicorn is mythical?

$\Delta: (\neg A \vee B) \wedge (A \vee \neg B) \wedge (A \vee C) \wedge (\neg B \vee D) \wedge (\neg C \vee D) \wedge (\neg D \vee E)$

$\alpha: A$

$\Delta \models \alpha$?

Is $\Delta \wedge \neg\alpha$ unsatisfiable?

• Use Resolution:

- write clauses of $\Delta \wedge \neg\alpha$ in a numbered list:

0. $\neg A \vee B$
1. $A \vee \neg B$
2. $A \vee C$
3. $\neg B \vee D$
4. $\neg C \vee D$
5. $\neg D \vee E$
6. $\neg A$
7. $\neg B$ 1, 6 $A \vee \neg B, \neg A / \neg B$
8. C 2, 6 $A \vee C, \neg A / C$
9. D 4, 8 $\neg C \vee D, C / D$
10. E 5, 9 $\neg D \vee E, D / E$

We can no longer apply rules, and we have found no contradictions, so $\Delta \wedge \neg\alpha$ is satisfiable. Therefore, $\Delta \not\models \alpha$, so we cannot use the knowledge base to prove that the unicorn is mythical.

- Can we use the knowledge base to prove that the unicorn is magical?

$\Delta: (\neg A \vee B) \wedge (A \vee \neg B) \wedge (A \vee C) \wedge (\neg B \vee D) \wedge$
$(\neg C \vee D) \wedge (\neg D \vee E)$

$\alpha: E$

$\Delta \models \alpha$?

Is $\Delta \wedge \neg\alpha$ unsatisfiable?

• Use Resolution:

- write clauses of $\Delta \wedge \neg\alpha$ in a numbered list:

0. $\neg A \vee B$
1. $A \vee \neg B$
2. $A \vee C$
3. $\neg B \vee D$
4. $\neg C \vee D$
5. $\neg D \vee E$
6. $\neg E$
7. $\neg D$ 5, 6 $\neg D \vee E, \neg E / \neg D$
8. $\neg C$ 4, 7 $\neg C \vee D, \neg D / \neg C$
9. A 2, 8 $A \vee C, \neg C / A$
10. B 0, 9 $\neg A \vee B, A / B$
11. D 3, 10 $\neg B \vee D, B / D$

There is a contradiction between 7. ($\neg D$) and 11. (D), so $\Delta \wedge \neg\alpha$ is unsatisfiable. Therefore, $\Delta \models \alpha$, and we can use the knowledge base to prove that the unicorn is magical.

- Can we use the knowledge base to prove that the unicorn is horned?

$\Delta: (\neg A \lor B) \land (A \lor \neg B) \land (A \lor C) \land (\neg B \lor D) \land (\neg C \lor D) \land (\neg D \lor E)$

$\alpha: D$

$\Delta \models \alpha$?

Is $\Delta \land \neg\alpha$ unsatisfiable?

- Use Resolution:

- write clauses of $\Delta \wedge \neg\alpha$ in a numbered list:

0. $\neg A \vee B$
1. $A \vee \neg B$
2. $A \vee C$
3. $\neg B \vee D$
4. $\neg C \vee D$
5. $\neg D \vee E$
6. $\neg D$
7. $\neg C$ 4, 6 $\neg C \vee D, \neg D / \neg C$
8. A 2, 7 $A \vee C, \neg C / A$
9. B 0, 8 $\neg A \vee B, A / B$
10. D 3, 9 $\neg B \vee D, B / D$

There is a contradiction between 6. ($\neg D$) and 10. (D), so $\Delta \wedge \neg\alpha$ is unsatisfiable. Therefore, $\Delta \models \alpha$, and we can use the knowledge base to prove that the unicorn is horned.

4.

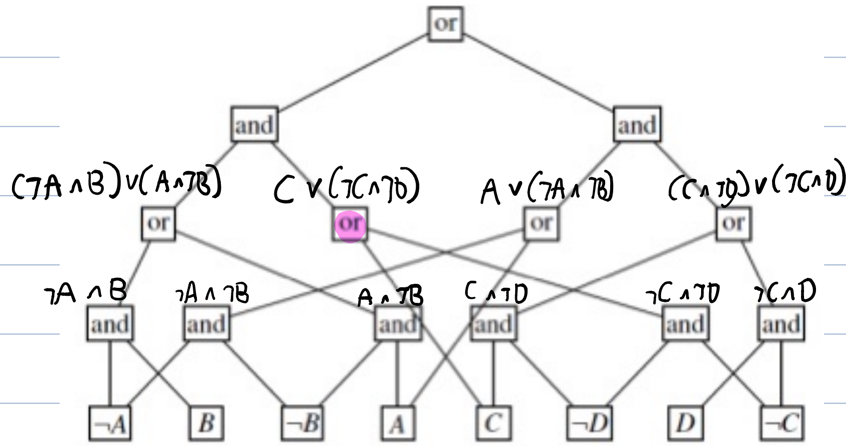

Figure 1

- The circuit is decomposable since the subcircuits feeding into each and-gate don't share any variables.

- The circuit is not smooth. A counterexample is that for the or-gate highlighted pink, one of inputs contains only C, while its other input contains C and D. So for all $\alpha \vee \beta$, $var(\alpha) = var(\beta)$ is not always true in this circuit.

- The circuit is not deterministic because the top or-gate does not have at most 1 true input under any circuit input. For example, when A = true, B = false, C = true, and D = false, both inputs to the top or-gate will be true, so the circuit is not deterministic.

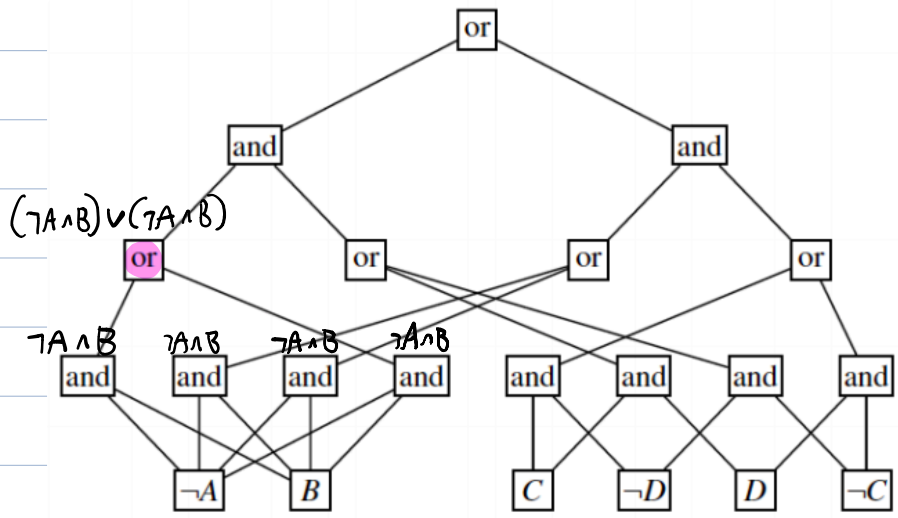

Figure 2

• The circuit is decomposable since the subcircuits feeding into each and-gate don't share any variables.

• The circuit is smooth since for every or-gate $\alpha \vee \beta$, $var(\alpha) = var(\beta)$.

• The circuit is not deterministic because every or-gate does not have at most 1 true input under any circuit input. For example, for the or-gate highlighted in pink, when $A = \text{false}$ and $B = \text{true}$, both inputs to that or-gate will be true, so the circuit is not deterministic.

5

- Note: a model is a complete variable assignment that satisfies the propositional formula

a.

	A	B	$(\neg A \wedge B) \vee (\neg B \wedge A)$
$\omega 1$	t	t	f
$\omega 2$	t	f	t
$\omega 3$	f	t	t
$\omega 4$	f	f	f

- models: $\{\omega 2, \omega 3\}$

$$
\begin{aligned}
WMC &= \omega(A, \neg B) + \omega(\neg A, B) \\
&= \omega(A)\omega(\neg B) + \omega(\neg A)\omega(B) \\
&= (.2)(.6) + (.8)(.4) \\
&= .12 + .32
\end{aligned}
$$

$\boxed{WMC = .44}$

b.

• first, find the count on the root:

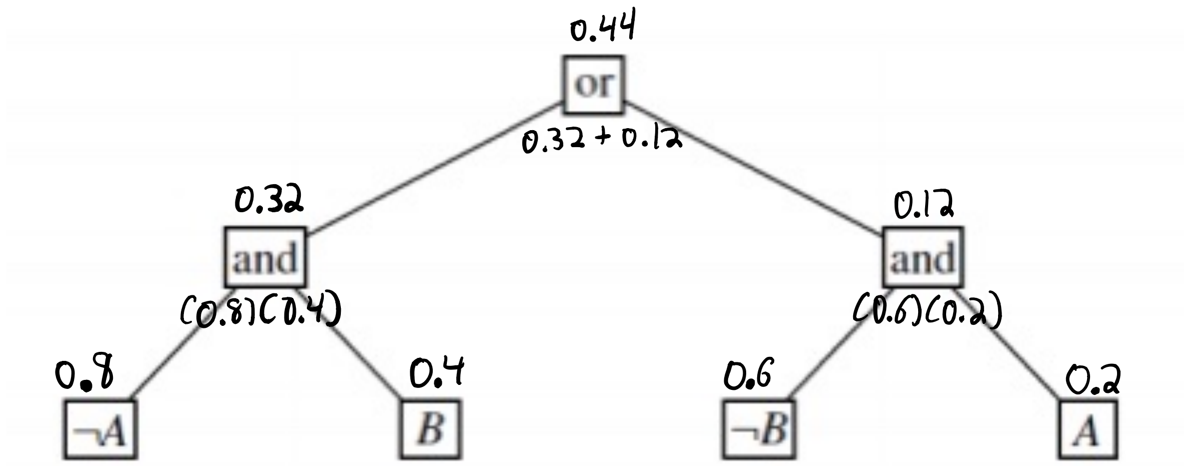

Figure 3

• the count on the root is .44.

• The formula is $(\neg A \wedge B) \vee (\neg B \wedge A)$. We found the WMC for this formula in part a, and its value is .44.

• The count on the root (.44) and the WMC for the formula (.44) are the same.

C.

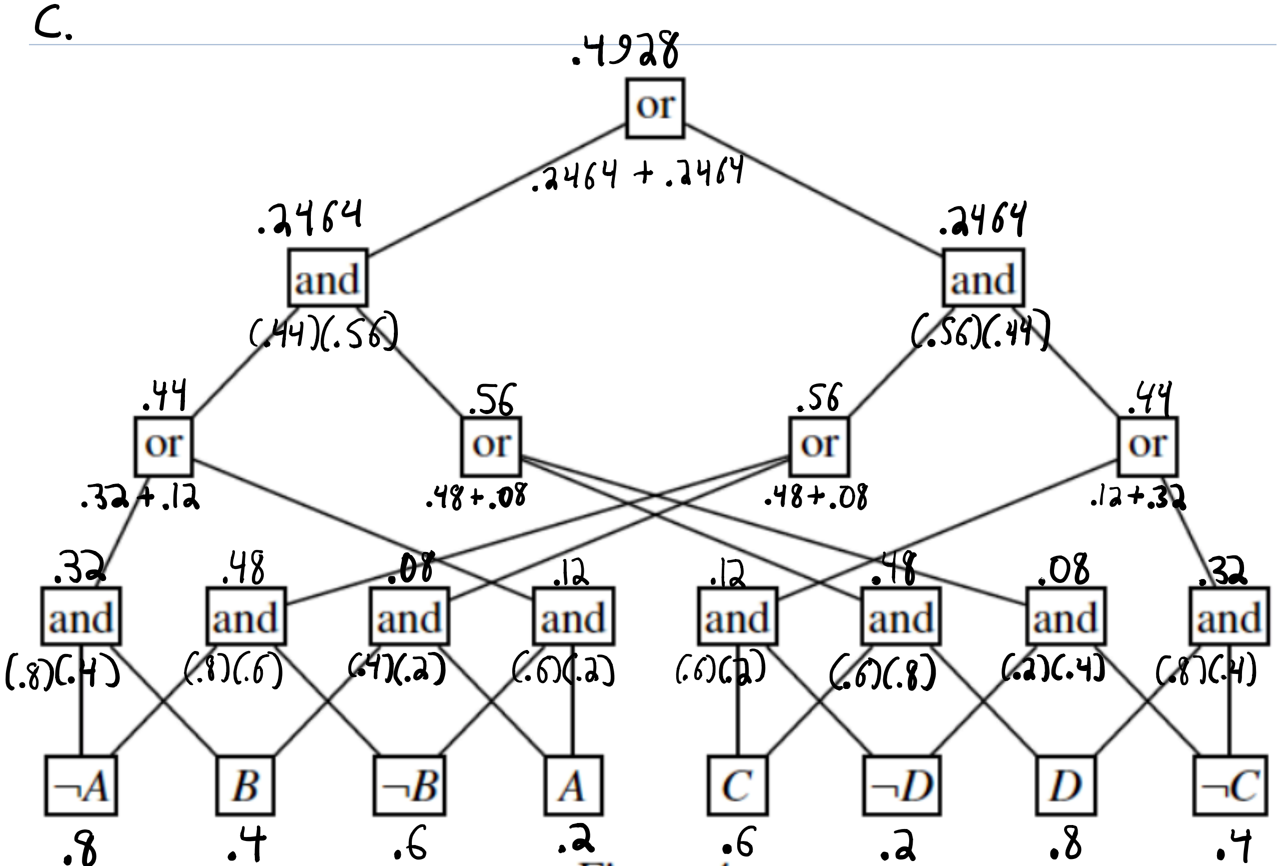

Figure 4

WMC = 0.4928