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OPTIMAL CONTROL IN WORLD HEALTH PROBLEMS: ONCHOCERCIASIS CONTROL*

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Abstract. In some regions of the world black flies are the vector of serious endemic diseases such as Onchocerciasis or river blindness. The object of this paper is to present theoretical and numerical solutions to the control of black fly larvae in running waters. The problem is modelled by a diffusion-transport partial differential equation with impulse controls and state constraint. We present the solutions for the one-dimensional.

1. INTRODUCTION.

Black flies (Simulium damnosum) are known not only for their nuisance causing economic losses in different areas of human activities, but also for transmission of pathogens and parasites to man and animals. In some areas, black flies are vectors of a filarial worm (Onchocerca Volvulus) which causes a serious endemic disease whose final stage is known as river blindness. "Onchocerciasis, or river blindness, is one of the major endemic, parasitic diseases which in addition to causing untold human suffering is a major obstacle to socioeconomic development. It is found in the Americas, in the south-western part of the Arabian peninsula and in East, Central and West Africa. It is estimated that between 20 and 30 million people are infected by onchocerciasis throughout the world."

The strategy chosen was to break the chain of transmission by destroying the vector at its most vulnerable state, that is, the larval state. To control black fly larvae in running waters special products have been developped with targeted toxic effects. Helicopters are used to periodically spray the rivers at prescribed sites over very large geographical areas. To reduce the costs of operations, it is important to determine the amounts of product and locations of the injection sites to minimize the total quantity of sprayed larvicide while maintaining a given level of mortality along the river to be treated.

Black fly larvae are found at specific breeding sites in rivers. Since we have to control the mortality level over long distances, we have to take into account the transport, the diffusion and the decay of the larvicide. The behaviour of the concentration of larvicide along the river can be modelled by a diffusion-transport partial differential equation. Biologists have established that for the types of larvicides used the rate of mortality is proportional to the "dose": the time integral of the concentration up to infinity. In this problem the state variable is the spacial distribution of the dose which is the solution of a partial differential equation over the river. The injections of insecticide appear as impulse controls at points along the river (one dimensional model) or along lines corresponding to the paths followed by the helicopter (two dimensional model). The one-dimensional model is used to treat a segment of river while the two-dimensional model is used for a complex site.

The control problem consists in finding the best locations to inject the larvicide while maintaining a minimum dose and minimizing the total amount of larvicide sprayed in the river. So it is an impulse control problem for partial differential equations with state constraint. On top of this there is an interesting logistic problem to schedule helicopters and manage fuel and insectide caches over a large network of rivers.

The object of this paper is to present a modelisation of the problem and a numerical solution of the optimal control problem in dimension one. A mixed finite element method (LESAINT-RAVIART [1] and P.-A. RAVIART [1]) has been used to solve the diffusion-transport equations. A special technique has also been developped to get around the combinatorial problem which naturally arises from the discretization of the impulse control problem. Numerical results will be presented using real data.

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The modelisation of this type of problem is related to the problem of the localization of industrial sites or plants along rivers where the objective is to minimize ecological damages to the environment. For more details the reader is referred to G.I. MARCHUK [1] and his team.

2. ONE DIMENSIONAL EQUATIONS AND SUSCEPTIBILITY MODEL.

2.1. Equations for the concentration.

First consider a one-dimensional river. Denote by $c(x,t)$ (kg/m^3) the concentration of larvicide at time $t>0$ and at point x $(meter)$ downstream of the origin 0. It is the solution of the diffusion-transport equation (G.I. TAYLOR [1], R. ARIS [1], P. KHALIG [1], G.I. MARCHUK[1])

$$\frac{\partial c}{\partial t}+V(x)\frac{\partial c}{\partial x}-\frac{\partial}{\partial x}\left(E(x)\frac{\partial c}{\partial x}\right)+R(x)c = \sum_{i=1}^{N}\frac{m_i}{A(x_i)}\delta_i(t)\delta_i(x),\ x\in\mathbf{R},\ t>0, \tag{2.1}$$

where
$V(x)$ is the "mean velocity" of the product in m/sec,
$A(x)>0$ is the cross-sectional area of the river in m^2,
$E(x)\geq 0$ is the dispersion coefficient in m^2/sec,
$R(x)\geq 0$ is the loss coefficient in sec^{-1},
$N>0$ is the number of sites to be visited,
$x_1, x_2, ..., x_N$ are the locations of the sites such that

$$x_0 \stackrel{\text{def}}{=} 0 \leq x_1 \leq x_2 \leq ... \leq x_N \leq x_{N+1} \stackrel{\text{def}}{=} L, \tag{2.2}$$

for some finite $L>0$,
$t_1, t_2, ..., t_N$ are the times at which the sites are visited

$$t_0 \stackrel{\text{def}}{=} 0 \leq t_1 \leq t_2 \leq ... \leq t_N \leq t_{N+1} \stackrel{\text{def}}{=} T, \tag{2.3}$$

for some finite $T>0$,
and $\delta_i(t)$ and $\delta_i(x)$ are the Dirac delta functions at $t=t_i$ and $x=x_i$, respectively. In each site x_i the helicopter sprays m_i kg of product. We assume that the mixing of the product is instantaneous and produces a uniform concentration in the vertical section $A(x_i)$ of the river at x_i. The initial condition is

$$c(x,0)=c^0(x),\quad x\in\mathbf{R}. \tag{2.4}$$

2.2. Susceptibility model.

Laboratory and field experiments have established a direct relation between the mortality rate of larvae and the dose $u(x)$ which is the time-integral of the concentration of larvicide over $t>0$

$$u(x) = \int_0^\infty c(x,t)\,dt \tag{2.5}$$

for the biological larvicide B.t.i. (GUILLET-ESCAFFRE ET AL. [1], GUILLET-HOUGARD ET AL. [1]). So to obtain a given mortality rate of P%, it is sufficient to specify a minimum level u_P and to require that at each point in the river

$$u(x) \geq u_P,\ x\in[0,L]. \tag{2.6}$$

The total mass $M[a,b]$ of larvicide going through the segment $[a,b]$ is given by

$$M[a,b]=\int_a^b u(x)A(x)V(x)\,dx \tag{2.7}$$

where

$$Q(x) = V(x)A(x), \tag{2.8}$$

is the flow in m^3/sec.

2.3. Equations for the dose.

Assuming that the concentration $c(x,t)$ in each point x goes to 0 as t goes to ∞, we obtain the following equation for the dose $u(x)$

$$-\frac{d}{dx}\left(E(x)\frac{du}{dx}\right) + V(x)\frac{du}{dx} + R(x)u \;=\; c^0(x) + \sum_{i=1}^{N} \frac{m_i}{A(x_i)}\delta_i(x),\ x \in \mathbf{R}. \tag{2.9}$$

It is readily seen that the times t_i's completely disappear in the above equation. This considerably simplifies the formulation. We now have a problem on $\mathbf{R}$.

2.3.1. The problem without diffusion.

When there is no diffusion (that is, $E = 0$), we assume that the boundary condition

$$c(0,t) = c_0(t), \quad t > 0, \tag{2.10}$$

is given at $x = 0$. Equation (2.9) reduces to

$$V(x)\frac{du}{dx} + R(x)u \;=\; c^0(x) + \sum_{i=1}^{N} \frac{m_i}{A(x_i)}\delta_i(x),\ x \in \mathbf{R},, \tag{2.11}$$

with the boundary condition

$$u(0) = u_0 \;\stackrel{\text{def}}{=}\; \int_0^\infty c_0(t)\, dt. \tag{2.12}$$

For $R \in L^2(0,L)$ and $V^{-1} \in H^1(0,L)$ system (2.11)-(2.12) has a unique solution given by

$$\begin{aligned} u(x) &= u_0\, e^{-\int_0^x \frac{R(y)}{V(y)}\, dy} + \int_0^x e^{-\int_z^x \frac{R(y)}{V(y)}\, dy}\ \frac{c^0(z)}{V(z)}\, dz \\ &+ \sum_{\substack{i=1 \\ x \geq x_i}} e^{-\int_{x_i}^x \frac{R(y)}{V(y)}\, dy}\ \frac{m_i}{A(x_i)V(x_i)}. \end{aligned} \tag{2.13}$$

It will also be convenient to give a variational formulation of system (2.11)-(2.12) on a fixed segment $[0,L]$, $L > 0$ finite, and consider more general right hand sides. For this purpose we introduce the following continuous bilinear form:

$$\begin{aligned} b_0 &: L^2(0,L) \times H^1(0,L) \;\rightarrow\; \mathbf{R} \\ b_0((u,u_L),v) &= \int_0^L u\left[-\frac{d}{dx}(Vv) + Rv\right] dx + u_L V(L)v(L). \end{aligned} \tag{2.14}$$

THEOREM 2.1. *Assume that $R \in L^2(0,L)$ and $V^{-1} \in H^1(0,L)$. Then for any ℓ in $H^1(0,L)'$ the variational equation*

$$\exists (u,u_L) \in L^2(0,L) \times \mathbf{R},\ \forall v \in H^1(0,L), \quad b_0((u,u_L),v) = \langle \ell, v\rangle_{H^1}, \tag{2.15}$$

has a unique solution. □

System (2.11)-(2.12) corresponds to

$$\langle \ell, v\rangle = \int_0^L c^0(x)v(x)\, dx + \sum_{i=1}^{N} \frac{m_i}{A(x_i)}v(x_i) + u_0 V(0)v(0). \tag{2.16}$$

2.3.2. The problem with diffusion.

When $E(x) > 0$, equation (2.9) has a unique solution in the Sobolev space $H^1(\mathbf{R})$ under the following conditions.

THEOREM 2.2. *Assume that $c^0 \in L^2(\mathbf{R})$, E and R in $L^\infty(\mathbf{R})$ and V in $W^{1,\infty}(\mathbf{R})$ verify the conditions*

$$\forall x \in \mathbf{R},\ \exists \alpha > 0, \quad R(x) - \frac{1}{2}\frac{dV}{dx}(x) \geq \alpha,\ E(x) \geq \alpha. \tag{2.17}$$

Then equation (2.9) has a unique solution in $H^1(\mathbf{R})$. □

Under the above hypotheses the bilinear form

$$a(u,v) = \int_{\mathbf{R}} E(x)\frac{dy}{dx}\frac{dv}{dx} + V(x)\frac{dy}{dx}v + R(x)uv\, dx \tag{2.18}$$

is bilinear continuous and coercive on $H^1(\mathbf{R})$ and equation (2.9) is equivalent to the problem: to find u in $H^1(\mathbf{R})$ such that for all v in $H^1(\mathbf{R})$

$$a(u,v) = \int_{\mathbf{R}} c^0(x)v(x)\, dx + \sum_{i=1}^{N} \frac{m_i}{A(x_i)}v(x_i). \tag{2.19}$$

However for operational and numerical reasons, the user is generally interested in simulating only a finite segment $[0,L]$, $L > 0$ of the river. To do that we have to specify in $x = 0$ and $x = L$ "transparent conditions" which do not perturb too much the physics of the problem. So, in addition to the hypotheses of Theorem 2.2, we assume that E and R are continuous and that

$$\left\{\begin{array}{llll} E(x) = E(0), & V(x) = V(0), & R(x) = R(0) \geq 0, & \forall x \leq 0, \\ E(x) = E(L), & V(x) = V(L), & R(x) = R(L) \geq 0, & \forall x \geq L. \end{array}\right\} \tag{2.20}$$

So upstream of $x = 0$ and downstream of $x = L$ the river is uniform with constant parameters E, V and R. Then on the part $]-\infty, 0]$ (resp. $[L, \infty[$), we have the asymptotic condition $u(-\infty) = 0$ (resp. $u(+\infty) = 0$) and it is easy to verify that at $x = 0$ (resp. $x = L$) the following identity holds

$$-E(0)\frac{du}{dx}(0) + \beta(0)u(0) \;=\; C_0, \quad (\text{resp. } E(L)\frac{du}{dx}(L) + \beta(L)u(L) \;=\; C_L) \tag{2.21}$$

where

$$\begin{aligned} C_0 &= \int_{-\infty}^0 e^{\frac{1}{2E(0)}\left[\sqrt{V(0)^2+4E(0)R(0)}-V(0)\right]y} c^0(y)\, dy \\ C_L &= \int_L^\infty e^{-\frac{1}{2E(L)}\left[\sqrt{V(L)^2+4E(L)R(L)}+V(L)\right](y-L)} c^0(y)\, dy \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} \beta(0) &= \frac{1)}{2}\left[\sqrt{V(0)^2 + 4E(0)R(0)} + V(0)\right], \\ \beta(L) &= \frac{1}{2}\left[\sqrt{V(L)^2 + 4E(L)R(L)} - V(L)\right]. \end{aligned} \tag{2.23}$$

The idea is to consider equation (2.9) on $]0, L[$ with the boundary conditions (2.21). The following theorem now gives the connection between the solution of (2.9) on $\mathbf{R}$ and (2.9) $]0, L[$ with the boundary conditions (2.21).

THEOREM 2.3. *Assume that, in addition to the hypotheses of Theorem 2.2, E and R are continuous and verify assumptions (2.20). Then system*

$$\begin{aligned} &-\frac{d}{dx}\left(E(x)\frac{du}{dx}\right) + V(x)\frac{du}{dx} + R(x)u \\ &= c^0(x) + \sum_{i=1}^{N} \frac{m_i}{A(x_i)}\delta_i(x),\ x \in [0,L], \\ &-E(0)\frac{du}{dx}(0) + \beta(0)u(0) \;=\; C_0, \qquad E(L)\frac{du}{dx}(L) + \beta(L)u(L) \;=\; C_L \end{aligned} \tag{2.24}$$

has a unique solution in $H^1(0,L)$ which coincides with the restriction to $[0,L]$ of the solution of equation (2.9) on $\mathbf{R}$. □

PROOF. (i) System (2.24) is equivalent to the following variational problem: to find u in $H^1(0,L)$ such that for all v in $H^1(0,L)$

$$b(u,v) = \int_0^L c^0(x)v(x)\, dx + \sum_{i=1}^{N} \frac{m_i}{A(x_i)}v(x_i) + C_0 v(0) + C_L v(L), \tag{2.25}$$

where b is the coercive continuous bilinear form

$$\begin{aligned} b(u,v) = \int_0^L & E(x)\frac{du}{dx}\frac{dv}{dx} + V(x)\frac{du}{dx}v + R(x)uv\, dx \\ & + \beta(L)u(L)v(L) + \beta(0)u(0)v(0). \end{aligned} \tag{2.26}$$

(ii) Next we show that if we put together (2.9) on $]-\infty, 0]$, $[0,L]$ and $[L, \infty[$, and assume the continuity of u at 0 and L, it is a solution to the variational equation (2.18) on $\mathbf{R}$. Then the result follows by uniqueness of solution. □

REMARK 2.1. In system (2.1)-(2.2) we put all the injection points x_i in $]0, L[$ and avoided the points $x = 0$ and $x = L$. However this is not a limitation in the variational formulations (2.18) and (2.25). Such injections will not appear in equation (2.9), but rather in the boundary conditions (2.21):

$$\begin{aligned} -E(0)\frac{du}{dx}(0) + \beta(0)u(0) &= \frac{m_0}{A(0)} + C_0, \\ E(L)\frac{du}{dx}(L) + \beta(L)u(L) &= \frac{m_L}{A(L)} + C_L. \quad \square \end{aligned} \tag{2.27}$$

3. OPTIMAL CONTROL OF THE ONE-DIMENSIONAL MODEL.

In this section we assume that the initial condition $c^0(x)$ is zero. So $C_0 = C_L = 0$ in the results of section 2.3.2.

3.1. Problem formulation.

Consider a segment $[0,L]$, $L > 0$, of river to be treated in $N > 0$ ordered sites x_i's and verifying equation (2.9) with the appropriate boundary conditions: (2.12) without diffusion and (2.21) with diffusion.

The following three problems will be discussed.

PROBLEM 1. Given the number of sites N and the positions $\vec{x}_N = \{x_i : 1 \le i \le N\}$, to find the masses $\vec{m}_N = \{m_i : m_i \ge 0, 1 \le i \le N\}$ which minimize the total mass of sprayed product

$$M(N, \vec{x}_N, \vec{m}_N) = \sum_{i=1}^{N} m_i, \tag{3.1}$$

under the state constraint

$$u(x) \ge u_P, \quad 0 \le x \le L, \tag{3.2}$$

where u is the solution of the state equation (2.9) with its appropriate boundary conditions. □

PROBLEM 2. Given the number of sites N to find the positions $\vec{x}_N = \{x_i : 1 \le i \le N\}$ and the masses $\vec{m}_N = \{m_i : m_i \ge 0, 1 \le i \le N\}$ which minimize the total mass of sprayed product (3.1) under the state constraint (3.2), where u is the solution of the state equation (2.9) with its appropriate boundary conditions. □

PROBLEM 3. To find the number of sites N, the positions $\vec{x}_N = \{x_i : 1 \le i \le N\}$ and the masses $\vec{m}_N = \{m_i : m_i \ge 0, 1 \le i \le N\}$ which minimize the total mass of sprayed product (3.1) under the state constraint (3.2), where u is the solution of the state equation (2.9) with its appropriate boundary conditions. □

If there is no upper bound on the number of sites to be treated, the optimal solution is not given by a finite vector of pairs (x_i, m_i) but a measure on $]0, L[$. There the masses m_i go to zero as the number N goes to infinity and it can also be verified that in each point

$$u(x) = u_P, \quad 0 \le x \le L. \tag{3.3}$$

Of course this is a simplified version of the problem. In practice the helicopter can only treat a finite number of site and there is an upper bound on the number of sites and a lower bound on the amount of product that is sprayed at each site. Several other constraints are present: compulsory site, upper bound on the dose to minimize damages to the environment, etc... However Problem 3 will provide a lower bound on the total amount of product necessary to treat a given segment of river.

3.2. Solution of the optimization problems.

In this section we introduce sets of hypotheses under which the optimization problems 1 to 3 have a solution. In some cases we even give the exact form of the solution which will help in testing the numerical algorithm.

3.2.1. The case without diffusion.

When E=0, we have seen in section 2.3.1 that the problem has an explicit solution given by (2.13). We assume that at time 0 there is no larvicide in the segment of river ($c^0 = 0$, $u_0 = 0$). We have the following explicit solution.

THEOREM 3.1. *Assume that the hypotheses of Theorem 2.1 are verified. Let* $E = 0$, $V(x) > 0$, $R(x) \ge 0$, $A(x) > 0$ *be continuous and*

$$\forall i = 0, ..., N, \quad Q(x_{i+1}) \le Q(x_i) e^{\int_{x_i}^{x_{i+1}} \frac{R(y)}{V(y)} dy}, \tag{3.4}$$

then the solution

$$\begin{aligned} m_i &= u_P Q(x_i) \, [e^{\int_{x_i}^{x_{i+1}} \frac{R(y)}{V(y)} dy} - 1], \quad 2 \le i \le N, \\ m_1 &= u_P Q(x_1) \, e^{\int_{x_1}^{x_2} \frac{R(y)}{V(y)} dy}, \quad x_1 = 0, \end{aligned} \tag{3.5}$$

is a minimizing solution of Problem 1. Moreover

$$u(x_i^-) = u_P, \; 1 \le i \le N+1 \tag{3.6}$$

and the total mass of sprayed larvicide is

$$M_N = \sum_{i=1}^{N} m_i = u_P \{Q(0) + \sum_{i=1}^{N} Q(x_i) [e^{\int_{x_i}^{x_{i+1}} \frac{R(y)}{V(y)} dy} - 1]\}. \tag{3.7}$$

If conditions (3.4) are verified with strict inequalities, the solution (3.5) is the unique minimizing solution. □

Hypothesis (3.4) is verified when the volume of water per second is not increasing as we go downstream. If there is an important new inflow of water at one point which significantly changes the concentration, then the problem has to be set up as a system of two or more segments of river with appropriate mixing conditions at the points of junction.

It is also possible to find the best site of the N sites.

THEOREM 3.2. *Assume that the assumptions of Theorem 3.1 are verified and that*

$$Q(x)' - \frac{R(x)}{V(x)} Q(x) \; < \; 0, \; \forall x. \tag{3.8}$$

Then the solution of Problem 2 is given by the following set of equations

$$\begin{aligned} p_{i-1} - 1 = &\frac{a(x_i)Q(x_i) - Q'(x_i)}{a(x_i)Q(x_i)} (p_i - 1) \\ &+ \frac{Q(x_i) - Q(x_{i-1})}{Q(x_{i-1})}, \quad 2 \le i \le N, \end{aligned} \tag{3.9}$$

$$\prod_{i=1}^{N} p_i = e^{\int_0^L a(y)\, dy}, \tag{3.10}$$

where

$$a(y) = \frac{R(y)}{V(y)}, \quad p_i = e^{\int_{x_i}^{x_{i+1}} a(y)\, dy}, \quad 1 \le i \le N. \; \square \tag{3.11}$$

When Q is constant we obtain

$$\int_{x_i}^{x_{i+1}} a(y)\, dy = \frac{1}{N} \int_0^L a(y)\, dy, \quad 1 \le i \le N. \tag{3.12}$$

If, in addition, a is constant then

$$x_{i+1} - x_i = \frac{L}{N}, \quad 1 \le i \le N. \tag{3.13}$$

REMARK 3.1. For a fixed $N \ge 1$ the total mass of product is

$$M_N = \sum_{i=1}^{N} m_i = u_P \{Q(0) + \sum_{i=1}^{N} Q(x_i) \Big[e^{\int_{x_i}^{x_{i+1}} \frac{R(y)}{V(y)} dy} - 1 \Big] \} \tag{3.14}$$

and necessarily for $N' > N$

$$\inf_{m,x} M_{N'} \; \le \; \inf_{m,x} M_N. \tag{3.15}$$

As N goes to ∞

$$\begin{aligned} M_N &\simeq u_P \{Q(0) + \sum_{i=1}^{N} Q(x_i) \frac{R(x_i)}{V(x_i)} (x_{i+1} - x_i)\} \\ &\simeq u_P \{Q(0) + \int_0^L Q(x) \frac{R(x)}{V(x)} dx\} \stackrel{\text{def}}{=} M_\infty. \end{aligned} \tag{3.16}$$

This is a lower bound on the amount necessary to satisfy the constraint (2.6). □

Going back to formula (3.5) we obtain as $N \to \infty$ and $|x_{i+1} - x_i| \to 0$

$$\begin{aligned} m_i &\simeq u_P Q(x_i) \frac{R(x_i)}{V(x_i)} (x_{i+1} - x_i), \quad i \ge 2, \\ m_1 &\simeq u_P \Big[Q(0) \frac{R(0)}{V(0)} (x_2 - x_1) + Q(0) \Big], \end{aligned}$$

and

$$dm(x) = u_P \Big[Q(x) \frac{R(x)}{V(x)} dx + Q(0) \Big].$$

The optimal control becomes a measure with density

$$\frac{dm}{dx}(x) = u_P \Big[Q(x) \frac{R(x)}{V(x)} + Q(0) \delta_0 \Big]. \tag{3.17}$$

In this case the constraint is saturated:

$$u(x) = u_P, \quad \forall x \in [0, L]. \tag{3.18}$$

This essentially provides the solution to Problem 3, but we can make this statement more precise.

As suggested by the previous asymptotic estimate, it is wise to enlarge the space of controls for Problem 3 to the space of positive measures: that is elements m in $C([0, L])'$ such that

$$\forall v \ge 0, \; v \in C([0, L]), \quad \int_0^L v(x)\, dm(x) \ge 0. \tag{3.19}$$

The state (u, u_L) is now a solution of the variational problem: to find (u, u_L) in $L^2(0,L) \times \mathbf{R}$ such that for all v in $H^1(0,L)$

$$b_0((u, u_L), v) = \int_0^L \frac{v(x)}{A(x)}\, dm(x), \tag{3.20}$$

where b_0 is the bilinear form (2.14). The associated cost function becomes

$$M(m) = \int_0^L dm(x). \tag{3.21}$$

Since $H^1(0,L) \subset C([0,L])$, problem (3.20) is well-posed and has a unique solution (u, u_L) in $L^2(0,L) \times \mathbf{R}$ and Problem 3 can be formulated as follows

$$\inf_{\substack{0 \leq m \in C([0,L])' \\ u(x) \geq u_P,\, u_L \geq u_P}} \int_0^L dm(x). \tag{3.22}$$

For $A \in C([0,L])$ it is easy to verify that the solution $u(x) = u_P$ and $u_L = u_P$ corresponds to the measure m_P defined by

$$\int_0^L v(x)\, dm_P(x) = b_0((u_P, u_P), Av). \tag{3.23}$$

where

$$b_0((u_P, u_P), Av) = u_P\left[\int_0^L R(x)A(x)v(x)\, dx + V(0)A(0)v(0)\right]. \tag{3.24}$$

But $A \geq 0$ and $R \geq 0$. So for $V(0) \geq 0$, m_P is positive. Therefore $(u, u_L) = (u_P, u_P)$ is a feasible solution which turns out to be minimizing under reasonable hypotheses on A.

THEOREM 3.3. *Assume that the hypotheses of Theorem 2.1 are verified, $A \in C([0,L])$, $A \geq 0$, $V(0) \geq 0$, and $R \geq 0$, and that*

$$\forall (u, u_L) \in L^2(0,L) \times \mathbf{R}, \quad u \geq 0,\ u_L \geq 0 \Rightarrow b((u, u_L), A) \geq 0. \tag{3.25}$$

Then $(u, u_L) = (u_P, u_P)$ is a minimizing solution of Problem 3, the distribution m_P of larvicide is given by the measure

$$\int_0^L w\, dm_P = b_0((u_P, u_P), Aw), \quad \forall w \in C([0,L]),\ . \tag{3.26}$$

and the total amount of product used by

$$\int_0^L dm_P = b_0((u_P, u_P), A), \tag{3.27}$$

where

$$b_0((u_P, u_P), A) = u_P\left\{\int_0^L R(x)A(x)\, dx + V(0)A(0)\right\}. \tag{3.28}$$

If condition (3.25) is verified with a strict inequality for all positive non-zero (u, u_L) ($u \geq 0$, $u_L \geq 0$ and $(u, u_L) \neq (0,0)$) the above solution is unique. □

COROLLARY. *(i) Condition (3.25) is equivalent to*

$$-\frac{d}{dx}(VA) + RA \ \geq\ 0, \quad \forall x \in [0,L], \quad V(L)A(L)\ \geq 0. \tag{3.29}$$

which is equivalent to

$$V(x) \geq 0,\ Q(x) \geq Q(L)e^{-\int_x^L \frac{R(y)}{V(y)}\, dy}\ \geq\ 0,\ \forall x \in [0,L]. \tag{3.30}$$

(ii) If, in addition to the hypotheses of Theorem 3.3, A is constant, then (3.25) is verified. If A, R and V are positive constants, then

$$\int_0^L dm_P = u_P A[LR + V]. \quad \square \tag{3.31}$$

REMARK 3.2. Condition (3.29) is verified for a river where there is no inflow of water:

$$Q(x)\ \geq\ 0, \quad \frac{dQ}{dx}(x)\ \leq\ 0, \quad \forall x \in [0,L].$$

This means that losses of water are permitted, but an important inflow of water at a point would have to be explicitly modelled by an appropriate balance condition. It is also interesting to notice that condition (3.8) used in Theorem 3.2 for Problem 2 is precisely condition (3.29) for Problem 3. Similarily condition (3.4) in Theorem 3.1 for Problem 1 is the pointwise version of condition (3.8). □

3.2.2. The case with diffusion.

When E is not zero, we no longer have an explicit solution and Problems 1 and 2 are best solved numerically. Problem 3 can be solved by the method of the previous section. So we choose as space of controls for Problem 3 the space of positive measures m in $C([0,L])'$. The state u is now the solution of the variational problem: to find u in $H^1(0,L)$ such that

$$b(u, v) = \int_0^L \frac{v(x)}{A(x)}\, dm(x),\ \forall v \in H^1(0,L), \tag{3.32}$$

where b is the bilinear form (2.26). The associated cost function is expression (3.21). Problem 3 can be formulated as follows:

$$\inf_{\substack{0 \leq m \in C([0,L])' \\ u(x) \geq u_P}} \int_0^L dm(x). \tag{3.33}$$

As in section 3.2.1 it is easy to verify that the solution $u(x) = u_P$ corresponds to the measure m_P defined by

$$\int_0^L v(x)\, dm_P(x) = b(u_P, Av), \tag{3.34}$$

where

$$\begin{aligned} b(u_P, Av) = u_P\Big\{ &\int_0^L R(x)A(x)v(x)\, dx \\ &+ \beta(L)A(L)v(L) + \beta(0)A(0)v(0)\Big\}. \end{aligned} \tag{3.35}$$

If $R \geq 0$ and $A \geq 0$, it is positive for all $v \geq 0$. Therefore $u = u_P$ is a feasible solution which turns out to be minimizing under reasonable hypotheses on A.

THEOREM 3.4. *Assume that the hypotheses of Theorem 2.3 are verified, $R \geq 0$ and $A \geq 0$, and that*

$$\forall w \in H^1(0,L), \quad w \geq 0 \Rightarrow b(w, A) \geq 0. \tag{3.36}$$

Then $u = u_P$ is a minimizing solution of Problem 3, the distribution m_P of larvicide is given by the positive measure

$$\int_0^L w\, dm_P = b(u_P, Aw), \quad \forall w \in C([0,L]), \tag{3.37}$$

and the total amount of product used by

$$\int_0^L dm_P = b(u_P, A), \tag{3.38}$$

where

$$\begin{aligned} b(u_P, A) = u_P\Big\{ &\int_0^L R(x)A(x)\, dx \\ &+ \frac{1}{2}[\sqrt{V(L)^2 + 4E(L)R(L)} - V(L)]A(L) \\ &+ \frac{1}{2}[\sqrt{V(0)^2 + 4E(0)R(0)} + V(0)]A(0)\Big\}. \end{aligned} \tag{3.39}$$

If condition (3.36) is verified with a strict inequality for all $w > 0$, the above solution is unique. □

COROLLARY. *(i) Condition (3.36) is equivalent to*

$$\begin{aligned} &\forall w \in H^1(0,L), \quad w \geq 0, \\ &b(w, A) = \int_0^L E(x)\frac{dw}{dx}\frac{dA}{dx} + V(x)\frac{dw}{dx}A + R(x)wA\, dx \\ &\quad + \beta(L)w(L)A(L) + \beta(0)w(0)A(0)\ \geq\ 0. \end{aligned} \tag{3.40}$$

(ii) If, in addition to the hypotheses of Theorem 3.4, A is constant and

$$-\frac{dQ}{dx} + \frac{R}{V}Q\ \geq\ 0,\ \forall x \in [0,L], \tag{3.41}$$

then (3.40) is verified. In particular if A, E and V are positive constants, (3.41) is verified and

$$\int_0^L dm_P = u_P A[LR + \sqrt{V^2 + 4ER}]. \quad \square \tag{3.42}$$

REMARK 3.3. If we set $E = 0$ in expression (3.39), we recover expression (3.27)–(3.28). $\square$

4. NUMERICAL APPROXIMATION.

4.1. Approximation of the state equation.

4.1.1. The case without diffusion.

When $E = 0$ we have the explicit solution (2.13) and there is no need for an approximation of system (2.11)–(2.12).

4.1.2. The case with diffusion.

Problem 3 has an explicit solution. Problems 1 and 2 require a numerical method to solve the diffusion-transport equation (2.24) with impulsive controls. When the transport dominates it is well-known that classical finite elements do not produce good results. Mixed finite elements combined with the techniques of LESAINT-RAVIART [1] produce better results by introducing the right amount of numerical dissipation. Their generalization to dimension 2 uses the elements of Raviart-Thomas (cf. P.-A. RAVIART [1]).

Start from the first equation (2.24) and introduce the new variable p through the equation

$$E^{-1}(x)p = \frac{du}{dx}. \tag{4.1}$$

The boundary conditions become

$$-p(0) + \beta(0)u(0) = 0, \quad p(L) + \beta(L)u(L) = 0 \tag{4.2}$$

and equation (2.24) reduces to

$$-\frac{dp}{dx} + V(x)\frac{du}{dx} + R(x)u = \sum_{i=1}^{N} \frac{m_i}{A(x_i)}\delta_i(x), \; x \in [0, L]. \tag{4.3}$$

Let $\{\xi_j : 0 \le j \le J\}$ be a uniform partition of the interval $[0, L]$,

$$\xi_j = jh, \; 0 \le j \le J, \; h = \frac{L}{J}, \tag{4.4}$$

for some integer $J > 0$ and consider the system of equations (4.1), (4.2) and

$$-\frac{dp}{dx} + V(x)\frac{du}{dx} + R(x)u = f + \sum_{j=0}^{J} c_j\delta_j, \tag{4.5}$$

for some coefficients c_j in $\mathbf{R}$, Dirac delta functions δ_j at ξ_j and a function f in $C([0, L])$. This new problem is a mesh-dependent problem for which we now give a discontinuous mesh-dependent formulation.

We introduce the traces $\{U_j : 0 \le j \le J\}$ at each node ξ_j. On each interval $I_j = [\xi_{j-1}, \xi_j]$ we write equation (4.1) in weak form

$$\int_{I_j}\left\{E^{-1}(x)pq + u\frac{dq}{dx}\right\}dx + U_{j-1}q(\xi_{j-1}^+) - U_jq(\xi_j^-) = 0, \; \forall q \in H^1(I_j), \tag{4.6}$$

with boundary and jump conditions

$$-p(0^+) + \beta(0)U_0 = c_0, \quad p(\xi_J^-) + \beta(L)U_J = c_J \tag{4.7}$$

$$p(\xi_j^-) - p(\xi_j^+) = c_j, \quad 1 \le j \le J - 1. \tag{4.8}$$

Equation (4.5) is also written in weak form but the term in u is treated "à la LESAINT-RAVIART [1]":

$$\begin{aligned}
&\int_{I_j}\{[-\frac{dp}{dx} + R(x)u]w - u\frac{d}{dx}[V(x)w]\}\,dx \\
&+ u(\xi_j^-)V(\xi_j)w(\xi_j^-) - u(\xi_{j-1}^-)V(\xi_{j-1})w(\xi_{j-1}^+) \\
&= \int_{I_j} fw\,dx, \quad \forall w \in H^1(I_j),
\end{aligned} \tag{4.9}$$

where $u(0^-) = U_0$. So the solution (u, p) can be discontinuous at each ξ_j and

$$u_{|I_j} \text{ and } p_{|I_j} \text{ in } H^1(I_j). \tag{4.10}$$

Now choose an approximation such that

$$u_{|I_j}, \; w_{|I_j} \in P^0(I_j), \; p_{|I_j}, \; q_{|I_j} \in P^1(I_j). \tag{4.11}$$

and use the trapeze formula to evaluate the integrals over I_j, where P^k is the space of polynomials of degree less or equal to $k \ge 0$. When the c_j's are zero, there are no jumps and p is continuous at each node ξ_j. This corresponds to the elements of Raviart-Thomas (cf. P.-A. RAVIART [1]) with a continuous normal trace. So we have a relaxation of the continuity to a prescribed jump condition given by (4.8). This can be handled by relaxing the continuity of the normal trace and introducing a Lagrange multiplier to specify the jump. The complete weak formulation is given by

$$\begin{aligned}
&\sum_{j=1}^{J}\Big\{\int_{I_j}\left[E^{-1}(x)pq + u\frac{dq}{dx}\right]dx \; - U_jq(\xi_j^-) + U_{j-1}q(\xi_{j-1}^+) \\
&\quad + \int_{I_j}\left[-\frac{dp}{dx} + Ru\right]w - u\frac{d}{dx}[V(x)w] - fw\,dx \\
&\quad + \; u(\xi_j^-)V(\xi_j)w(\xi_j^-) - u(\xi_{j-1}^-)V(\xi_{j-1})w(\xi_{j-1}^+)\Big\} \\
&+\sum_{j=1}^{J-1}\,[p(\xi_j^-) - p(\xi_j^+) - c_j]W_j \\
&\quad + [-p(0^+) + \beta_0U_0 - c_0]W_0 + [p(\xi_J^-) + \beta_JU_J - c_J]W_J,
\end{aligned} \tag{4.12}$$

where, in addition to (4.11), the W_j's, $0 \le j \le J$, are multipliers. It is important to understand that (4.12) is not only a weak formulation of (4.1), (4.2), (4.5), but also a weak formulation of the adjoint state equation when u, p and the U's play the role of test functions and w, q and the W's constitute the adjoint state.

The final algorithm for the state can be reduced to a system of linear equations of the form

$$\Lambda\vec{u} = h\vec{f} + B\vec{c} \tag{4.13}$$

for the $(J + 2)$-vectors $\vec{u} = \{U_0, u_1, ..., u_J, U_J\}$, $\vec{f} = \{0, f_1, ..., f_J, 0\}$, and $(J + 1)$-vector $\vec{c} = \{c_0, c_1, ..., c_J\}$, and all the other variables can be obtained by explicit formulae. The matrices Λ and B are specified by the following system of equations:

$$\begin{aligned}
&\left[\frac{2E_0}{h} + \beta_0\right]U_0 - \frac{2E_0}{h}u_1 = c_0 \\
&-\left[\frac{2E_0}{h} + V_0\right]U_0 + \left[\frac{2E_0 + E_1}{h} + hR_1 + V_0\right]u_1 - \frac{E_1}{h}u_2 = hf_1 + \frac{c_1}{2} \\
&\begin{cases} -\left[\frac{E_{j-1}}{h} + V_{j-1}\right]u_{j-1} + \left[\frac{E_{j-1}+E_j}{h} + hR_j + V_{j-1}\right]u_j - \frac{E_j}{h}u_{j+1} \\ = hf_j + \frac{c_{j-1}+c_j}{2}, \quad 2 \le j \le J - 1, \end{cases} \\
&\begin{cases} -\left[\frac{E_{J-1}}{h} + V_{J-1}\right]u_{J-1} + \left[\frac{E_{J-1}+2E_J}{h} + hR_J + V_{J-1}\right]u_J - \frac{2E_J}{h}U_J \\ = hf_J + \frac{c_{J-1}}{2}, \end{cases} \\
&-\frac{2E_J}{h}u_J + \left[\frac{2E_J}{h} + \beta_J\right]U_J = c_J,
\end{aligned} \tag{4.14}$$

where $\beta_J = \beta(L)$, $\beta_0 = \beta(0)$,

$$\begin{cases} E_j = E(\xi_j), & V_j = V(\xi_j), \quad 0 \le j \le J, \\ R_j = \frac{R(\xi_{j-1})+R(\xi_j)}{2}, & f_j = \frac{f(\xi_{j-1})+f(\xi_j)}{2}, \quad 1 \le j \le J. \end{cases} \tag{4.15}$$

The other variables can be computed with the following formulae

$$\begin{aligned}
&U_j = \frac{u_{j+1} + u_j}{2} + \frac{1}{4E_j}c_j, \quad 1 \le j \le J - 1, \\
&\begin{cases} p_0 = 2\frac{E_0}{h}(u_1 - U_0), \quad p_J = \frac{2E_J}{h}[U_J - u_J], \\ p_j = \frac{E_j}{h}[u_{j+1} - u_j] + \frac{1}{2}c_j, \qquad 1 \le j \le J - 1. \end{cases}
\end{aligned} \tag{4.16}$$

The right hand side of the final system (4.13) is precisely the Raviart-Thomas system (cf. P.-A. RAVIART[1]) where the jumps appear on the left hand side. An internal jump ξ_j, $1 \le j \le J - 1$, is linearily distributed between the two neighbouring nodes ξ_{j-1} and ξ_j.

In our analysis we do not want to put the injection points at the discretization nodes $\{\xi_j\}$ and solve a combinatorial problem. When the injection m_i occurs at a point x_i inside the interval $I_j = [\xi_{j-1}, \xi_j]$, we distribute the impulsion between the two nodes ξ_{j-1} and ξ_j by introducing a weighting function

$$B(\zeta) = \begin{cases} \hat{B}(\zeta), & |\zeta| \le 1 \\ 0, & \text{otherwise}, \end{cases} \tag{4.17}$$

where for instance $\hat{B}(\zeta)$ can be chosen as

$$\hat{B}(\zeta) = \frac{1}{2}[1 + cos\pi\zeta], \quad (1 - |\zeta|)^2(1 + 2|\zeta|), \quad \text{or } 1 - |\zeta|. \tag{4.18}$$

The first two choices are continuously differentiable on $]-1, 1[$ and $\mathbf{R}$, respectively. To make the connection between the original equation (4.3) and the mesh dependent formulation (4.5) assign the coefficients

$$c_j = \sum_{i=1}^{N} \frac{m_i}{A(x_i)}B\left(\frac{\xi_j - x_i}{h}\right), \; 0 \le j \le J. \tag{4.19}$$

Notice that for each i

$$\sum_{j=0}^{J} B\left(\frac{\xi_j - x_i}{h}\right) = 1.$$

With this type of formulation the state now depends on the position of the injection point inside a given interval I_j.

4.2. Approximation of the optimization problem.

4.2.1. The case without diffusion.

Again here everything is explicit and can be solved directly. We shall use the optimal solution for $E = 0$ as an initial guess for the optimization problem with diffusion.

4.2.2. The case with diffusion.

Consider Problem 2 and introduce the penalized cost function

$$\begin{aligned} M_\epsilon(\vec{m}_N, \vec{x}_N) = \sum_{i=1}^{N} m_i + \frac{1}{2\epsilon_1}\sum_{i=1}^{N}([m_i]^-)^2 \\ + \frac{1}{2\epsilon_2}\sum_{j=0}^{J}([u_P - U_j]^+)^2 + \frac{1}{2\epsilon_3}\int_0^L ([u_P - u(x)]^+)^2\,dx, \end{aligned} \tag{4.20}$$

for $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$. Since N is fixed we shall drop the subscript N. For simplicity we shall use a gradient method. The partial derivative of M_ϵ with respect to m_i and x_i can be obtained by using the Lagrangian

$$L(\vec{m}, \vec{x}, \vec{u}, \vec{p}, \vec{U}, \vec{w}, \vec{q}, \vec{W}) \tag{4.21}$$

which is equal to the right hand side of (4.19) plus the weak formulation (4.12). This yields the adjoint system

$$\Lambda^* \vec{w} = B\vec{d} + h\vec{g} \tag{4.22}$$

with the vectors $\vec{w} = (W_0, w_1, \ldots, w_J, W_J)$, $\vec{g} = (0, g_1, \ldots, g_{J-1}, g_J, 0)$, and $\vec{d} = (d_0, d_1, \ldots, d_{J-1}, d_J)$, and the matrices Λ and B from system (4.13), where

$$\begin{cases} d_j = \frac{1}{\epsilon_2}[u_P - U_j]^+, \quad 0 \le j \le J, \\ g_j = \frac{1}{\epsilon_3 h}\int_{I_j}[u_P - u(x)]^+\,dx, \quad 1 \le j \le J. \end{cases} \tag{4.23}$$

Moreover

$$\begin{aligned} & W_j = \frac{w_j + w_{j+1}}{2} - \frac{h}{4E_j}d_j, \quad 1 \le j \le J-1, \\ & \begin{cases} q_0 = \frac{2E_0}{h}[W_0 - w_1], \quad q_J = \frac{2E_J}{h}[w_J - W_J], \\ q_j = \frac{E_j}{h}[w_j - w_{j+1}] + \frac{d_j}{2}, \quad 1 \le j \le J-1. \end{cases} \end{aligned} \tag{4.24}$$

The partial derivatives of M_ϵ are

$$\frac{\partial M_\epsilon}{\partial m_i} = 1 + \frac{1}{\epsilon_1}[m_i]^- - \frac{1}{A(x_i)}\sum_{j=0}^{J} W_j B\left(\frac{\xi_j - x_i}{h}\right), \quad 1 \le i \le N, \tag{4.25}$$

$$\frac{\partial M_\epsilon}{\partial x_i} = \frac{m_i}{hA(x_i)}\sum_{j=0}^{J} W_j B'\left(\frac{\xi_j - x_i}{h}\right), \quad 1 \le i \le N, \tag{4.26}$$

where $B'(\zeta)$ is the derivative of the function $B(\zeta)$.

5. NUMERICAL EXPERIMENTATION IN DIMENSION ONE.

We have chosen a segment of the Amoutchou river in Togo (West Africa). The estimated parameters are:

$$V = 0.25m/sec, \quad R = 1.1\ 10^{-4}sec^{-1}, \quad E = 10m^2/sec, \quad A = 6.8m^2,$$
$$u_P = 0.5kg \times sec/m^3, \quad L = 12\ 10^3m, \quad J = 120, \quad h = 100m.$$

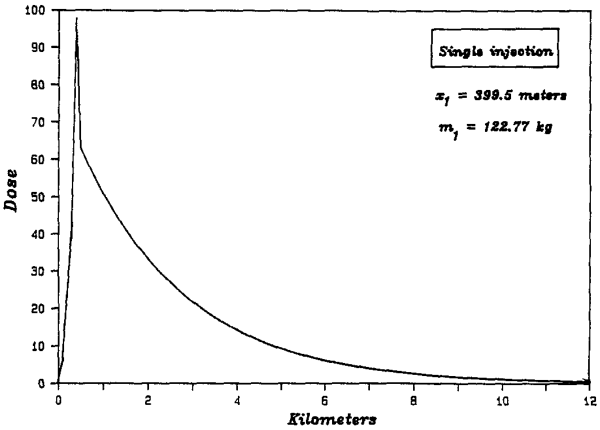
For $E = 0$ the theoretical optimal masses are

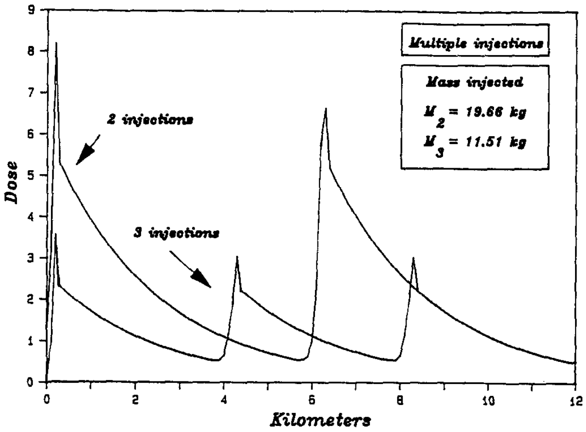
N	m_1	x_1	m_2	x_2	m_3	x_3	$M(\vec{m})$
1	166.9kg	0m					166.9kg
2	11.91kg	0m	11.06kg	6000m			22.97kg
3	4.941kg	0m	4.090kg	4000m	4.090kg	8000m	13.12kg
20	1.107kg	0m	0.2568kg	600m	0.2568kg	1200m	5.986kg
∞							5.34kg

For a diffusion $E = 10m^2/sec$ we have computed optimal masses and site locations for N =1, 2 and 3 sites, and the theoretical optimal total mass for the asymptotic solution ($N = \infty$). With a penalization $\epsilon = 10^{-3}$ the results are

N	*iterations*	m_1	x_1	m_2	x_2	m_3	x_3	$M(\vec{m})$
1	850	122kg	399m					122kg
2	800	10.3kg	199m	9.40kg	6250m			19.6kg
3	500	4.41kg	199m	3.55kg	4279m	3.55kg	8275m	11.5kg
∞								5.37kg

The distributions of the dose for N=1, 2 and 3 are shown below.





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