cross_section.png

Figure 1: Cross-Section of Dendrite

NOTE: To extract the LaTeX source of this PDF and the supporting files, execute:

```
pdftk basic_writeup.pdf unpack_files output .
```

We want to parameterize the three shapes so we can easily understand what is happening. We assume the top of the sphere is $0.3\mu m$ away from the horizontal cylinder, the sphere has a radius of $0.1\mu m$, the vertical cylinder (dendrite) has diameter $0.1\mu m$ and the “main line” horizontal cylinder (that all the dendrites come out of has a diameter) of $1.0\mu m$. We can visualize a cross-section of the top part via

```
H = 0.5;
theta = 0:0.01:(2*pi);
sphereIn2D = [0.1*cos(theta); 0.1*sin(theta) + 0.2 + H];

vertY = H:0.001:(H + 0.15);
leftPoints = [-0.05 * ones(size(vertY)); vertY];
rightPoints = [0.05 * ones(size(vertY)); vertY];
verticalCylinder = [leftPoints, rightPoints];

horizontalX = -0.3:0.001:0.3;
horizontalCylinder = [horizontalX; H * ones(size(horizontalX))];

allPoints = [sphereIn2D, verticalCylinder, horizontalCylinder];
scatter(allPoints(1, :), allPoints(2, :));
axis equal;

axis([-0.5, 0.5, 0.3, 1])
```

and see in Figure **??** the general cross-sectional geometry.

We need to find intersection points, but first get a basic idea of the equations describing them:

- Top-Sphere: $x^2 + y^2 + (z - 0.7)^2 = 0.1^2$. This has a diameter of $0.2\mu m$ and has to be $0.3\mu m$ above the main (horizontal) cylinder which is already $0.5\mu m$ above $z = 0$, hence we assume the center lies at $(0, 0, 0.5 + 0.3 - 0.1) = (0, 0, 0.7)$ (since the radius is 0.1).
- Vertical Cylinder: $x^2 + y^2 = \left(\frac{0.1}{2}\right)^2 = 0.05^2$. This is simply a circle of diameter 0.1 that extends infinitely in the z-direction. This is assumed to be centered around $(x, y) = (0, 0)$ but may change the x position of the center as we consider dendrites on the left and/or right.
- Horizontal Cylinder: $y^2 + z^2 = \left(\frac{1.0}{2}\right)^2 = 0.5^2$. This is simply a circle of diameter 0.1 that extends infinitely in the x-direction. This is the “main line” that all dendrites sprout out of.

Intersection points: To find places where our geometries intersect we need to find simultaneous solutions to the equations defining the surfaces. For the intersection of the sphere and the dendrite cylinder, we have

$$0.1^2 = x^2 + y^2 + (z - 0.7)^2 = 0.05^2 + (z - 0.7)^2 .$$

This gives us

$$(z - 0.7)^2 = 0.0075 \Rightarrow z = 0.7 \pm \sqrt{0.0075}.$$

However, the top one of these points corresponds to an intersection of our cylinder that is irrelevant to our geometry hence we can say that $z = 0.7 - \sqrt{0.0075}$ uniquely. This gives us the first rule for determining if a point changes geometries:

$$\boxed{\text{if } z \geq 0.7 - \sqrt{0.0075} \text{ the point is on the sphere, else it below}}$$

To determine the other intersection, we want to find the curve that is the intersection of the surfaces given by

$$x^2 + y^2 = 0.05^2 \quad \text{and} \quad y^2 + z^2 = 0.5^2$$

cylinder_intersection_pringle.png

(a) Pringle

cylinder_intersection_domain.png

(b) z-agnostic domain

Figure 2: Intersection of Cylinders

We can think of this as a parametric curve where z is the parameter. On the first surface we must have $0 \leq x, y \leq 0.05$ hence in the second we see that

$$0.2475 = 0.5^2 - 0.05^2 \leq z^2 = 0.5^2 - y^2 \leq 0.5^2 - 0^2 = 0.25.$$

Since our cylinders only intersect in the top half of the plane (i.e. the dendrites don't sprout on both sides) we only consider the positive z-values:

$$\sqrt{0.2475} \leq z \leq 0.5.$$

Given such a z-value we have $y^2 = 0.25 - z^2$ and $x^2 = 0.0025 - y^2 = z^2 - 0.2475$. Hence our parameterization results in four curves:

$$\left(\pm\sqrt{z^2 - 0.2475}, \pm\sqrt{0.25 - z^2}, z\right).$$

Plotting this via

```
z = sqrt(1/4 - 1/400):0.00001:sqrt(1/4);
x = sqrt(z.*z - 99/400);
y = sqrt(1/4 - z.*z);
pts1 = [x; y; z];
pts2 = [-x; y; z];
pts3 = [x; -y; z];
pts4 = [-x; -y; z];
pts = [pts1, pts2, pts3, pts4];

otherx = -sqrt(1/400):0.001:sqrt(1/400);
othery = sqrt(max(1/400 - otherx.*otherx, 0));
otherpts1 = [otherx; othery; zeros(size(otherx))];
otherpts2 = [otherx; -othery; zeros(size(otherx))];
otherpts = [otherpts1, otherpts2];

scatter3(pts(1, :), pts(2, :), pts(3, :));
figure;
scatter3(otherpts(1, :), otherpts(2, :), otherpts(3, :));
```

we see in Figure **??** that the intersection is a "pringle" shape.

From this it's clear that points with $z > 0.5$ (and also below $0.7 - \sqrt{0.0075}$, where the sphere ends) must be on the dendrite. Also points with $z < \sqrt{0.2475}$ must be on the "main line". However points in

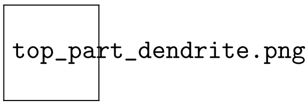


Figure 3: Dendrite above Main Line

between can be on either one. For such points, we must also know the x, y values. If $x^2+y^2 > 0.05^2$, then the point must be on the main line. If $x^2 + y^2 < 0.05^2$, then it must be thrown away — this would be a part of the "main line" cylinder covered up by the interior of the dendrite cylinder. If $x^2 + y^2 = 0.05^2$ then it lies on the dendrite. For points on the dendrite, if $y^2 + z^2 < 0.5^2$, then we discard the point.

Using

```
theta = 0:0.05:(2*pi);
r = 0.05;
xPoints = r * cos(theta);
yPoints = r * sin(theta);
zPoints = sqrt(0.2475):0.0001:0.505;
duplicateX = transpose(xPoints) * ones(size(zPoints));
duplicateY = transpose(yPoints) * ones(size(zPoints));
duplicateZ = ones(size(transpose(xPoints))) * zPoints;

duplicateX = reshape(duplicateX, 1, numel(duplicateX));
duplicateY = reshape(duplicateY, 1, numel(duplicateY));
duplicateZ = reshape(duplicateZ, 1, numel(duplicateZ));

cylinderPoints = [duplicateX; duplicateY; duplicateZ];
ysqPluszSq = (cylinderPoints.^2)(2, :) + (cylinderPoints.^2)(3, :);
goodIndices = (ysqPluszSq > 0.5^2);
cylinderPoints = cylinderPoints(:, goodIndices);

scatter3(cylinderPoints(1, :), cylinderPoints(2, :), cylinderPoints(3, :));
```

we see in Figure **??** the shape of the dendrite at the boundary using this rule.

We similarly crop the main line part using

```
theta = -pi/10:0.005:pi/10;
r = 0.5;
xPoints = -0.06:0.003:0.06;
yPoints = r * sin(theta);
zPoints = r * cos(theta);

duplicateX = ones(size(transpose(yPoints))) * xPoints;
duplicateY = transpose(yPoints) * ones(size(xPoints));
duplicateZ = transpose(zPoints) * ones(size(xPoints));

duplicateX = reshape(duplicateX, 1, numel(duplicateX));
duplicateY = reshape(duplicateY, 1, numel(duplicateY));
duplicateZ = reshape(duplicateZ, 1, numel(duplicateZ));

mainCylinderPoints = [duplicateX; duplicateY; duplicateZ];
xsqPlusySq = (mainCylinderPoints.^2)(1, :) + (mainCylinderPoints.^2)(2, :);
goodIndices = (xsqPlusySq > 0.05^2);
mainCylinderPoints = mainCylinderPoints(:, goodIndices);

scatter3(mainCylinderPoints(1, :), mainCylinderPoints(2, :), ...
         mainCylinderPoints(3, :));
```

and produce Figure **??**.

Using

Figure 4: Cropped Main Line

whole_phenomenon.png

Figure 5: Complete Picture

```
theta = pi/2:0.1:pi;
phi = 0:0.15:(2*pi);
r = 0.1;

x = r * transpose(sin(theta)) * cos(phi);
x = reshape(x, 1, numel(x));

y = r * transpose(sin(theta)) * sin(phi);
y = reshape(y, 1, numel(y));

z = r * transpose(cos(theta)) * ones(size(phi));
z = 0.7 + reshape(z, 1, numel(z));

goodIndices = (z > 0.7 - sqrt(0.0075));
x = x(goodIndices);
y = y(goodIndices);
z = z(goodIndices);

spherePts = [x;y;z];

%% Do the dendrite
theta = 0:0.1:(2*pi);
r = 0.05;
xPoints = r * cos(theta);
yPoints = r * sin(theta);
zPoints = sqrt(0.2475):0.007:(0.7 - sqrt(0.0075));
duplicateX = transpose(xPoints) * ones(size(zPoints));
duplicateY = transpose(yPoints) * ones(size(zPoints));
duplicateZ = ones(size(transpose(xPoints))) * zPoints;

duplicateX = reshape(duplicateX, 1, numel(duplicateX));
duplicateY = reshape(duplicateY, 1, numel(duplicateY));
duplicateZ = reshape(duplicateZ, 1, numel(duplicateZ));

cylinderPoints = [duplicateX; duplicateY; duplicateZ];
ysqPluszSq = (cylinderPoints.^2)(2, :) + (cylinderPoints.^2)(3, :);
goodIndices = (ysqPluszSq > 0.5^2);
cylinderPoints = cylinderPoints(:, goodIndices);

%% Do the Main Line
theta = -pi/6:0.05:pi/6;
r = 0.5;
xPoints = -0.15:0.02:0.15;
yPoints = r * sin(theta);
zPoints = r * cos(theta);

duplicateX = ones(size(transpose(yPoints))) * xPoints;
duplicateY = transpose(yPoints) * ones(size(xPoints));
duplicateZ = transpose(zPoints) * ones(size(xPoints));
```

standard_orientation.png

Figure 6: Standard Orientation

```
duplicateX = reshape(duplicateX, 1, numel(duplicateX));
duplicateY = reshape(duplicateY, 1, numel(duplicateY));
duplicateZ = reshape(duplicateZ, 1, numel(duplicateZ));

mainCylinderPoints = [duplicateX; duplicateY; duplicateZ];
xsqPlusySq = (mainCylinderPoints.^2)(1, :) + (mainCylinderPoints.^2)(2, :);
goodIndices = (xsqPlusySq > 0.05^2);
mainCylinderPoints = mainCylinderPoints(:, goodIndices);

%% Plot the whole thing
allPts = [spherePts, cylinderPoints, mainCylinderPoints];
scatter3(allPts(1, :), allPts(2, :), allPts(3, :));
axis equal;
```

we are able to bring this all together as in Figure **??**.

Moving Particles To move on the sphere we need to flatten out the surface **around** our current point p_0.

To cover this we first consider the unit sphere centered at the origin. We consider ourselves to be standing in front of the point $p_0 = (x_0, y_0, z_0)$ looking at it and have our feet on "the ground" so that $z > 0$ still corresponds to up. If $p_0 = (0, 0, 1)$ is the north pole, we can't stand in front and have our feet on the ground so we consider that later as a special case. This is also true for the south pole but in our geometry the bottom of the sphere is cut off so we ignore the south pole.

Given p_0, we want to think of our scene as we would if we were standing at $p_0 = (0, -1, 0)$. Using

```
points = [
  0, 0, 0, 1;
  1, -1, 0, 0;
  0, 0, 1, 0
]
colors = [
   1, 0, 0;
   1, 1, 0;
   0, 1, 0;
   0, 0, 1;
]
scatter3(points(1, :), points(2, :), points(3, :), [], colors);
axis([-1, 1, -1, 1, -1, 1]);
xlabel('x'); ylabel('y'); zlabel('z');
```

we create Figure **??**. In it, we see $(0, -1, 0)$ as our red viewpoint and $(0, 1, 0)$ as the yellow opposite. Given this, the green $(0, 0, 1)$ is the "up" point and the blue $(1, 0, 0)$ is the right point. We seek to find equivalent points for any given p_0 (except for the north pole).

To find the up point, we want to find a point in the same vertical plane that p_0 is in. Since a vertical plane is independent of z, this plane is simply determined by the line $y_0 x - x_0 y = 0$ in 3-space. Given such a point $p_1 = (a, b, c)$ we also it to be perpendicular to p_0 and of course want it to lie on the sphere. All together this gives three conditions:

$$\begin{aligned} y_0 a - x_0 b &= 0 \\ a x_0 + b y_0 + c z_0 &= 0 \\ a^2 + b^2 + c^2 &= 1 \end{aligned}$$

Solving this in Mathematica

```
soln = Simplify[Solve[a Subscript[x, 0] + b Subscript[y, 0] + c Subscript[z, 0] == 0 &&
                      a^2 + b^2 + c^2 == 1 &&
                      a Subscript[y, 0] == b Subscript[x, 0],
                      {a, b, c}],
                Subscript[x, 0]^2 + Subscript[y, 0]^2 + Subscript[z, 0]^2 == 1]
a = soln[[1, 1, 2]]
b = soln[[1, 2, 2]]
c = soln[[1, 3, 2]]
Subscript[p, 0] = {Subscript[x, 0], Subscript[y, 0], Subscript[z, 0]}
Subscript[p, 1] = {a, b, c}
```

It turns out the two solutions of the system are (a, b, c) and $(-a, -b, -c)$ but since we are standing on the ground and we want the "up" point (the green point) we pick the solution with positive z-coordinate. This corresponds to

$$p_1 = \left(-\frac{x_0 z_0}{\sqrt{x_0^2 + y_0^2}}, -\frac{y_0 z_0}{\sqrt{x_0^2 + y_0^2}}, \sqrt{x_0^2 + y_0^2}\right).$$

To find the "right" point (the blue point) we use the cross product $p_2 = p_1 \times p_0$ since by the right-hand rule we want the "index finger" to point at us, which is the direction of p_0, hence it is the second element. Actually computing this we have

```
Subscript[p, 2] = Cross[Subscript[p, 1], Subscript[p, 0]]
```

which gives

$$p_2 = \left(-\frac{y_0}{\sqrt{x_0^2 + y_0^2}}, \frac{x_0}{\sqrt{x_0^2 + y_0^2}}, 0\right).$$

Moving a distance $L > 0$ at an angle of θ from the positive x-axis, we know we have a displacement of $(L \cos \theta, L \sin \theta)$. This is really

$$L \cos \theta \cdot e_{\text{right}} + L \sin \theta \cdot e_{\text{up}}.$$

where $e_{\text{right}}, e_{\text{up}}$ are vectors in the right and up direction. The same sort of thing happens for our points, which move in a flattend out version of the sphere except we have $e_{\text{right}} = p_2$ and $e_{\text{up}} = p_1$.

The distance along the sphere from p_0 to p_1 (or any other pair) is a quarter of the circumference since p_1 lies half way between p_0 and $-p_0$ along an equator. Thus, cutting the sphere at the equator containing p_1 and p_2 we get a circle with center at p_0 and a radius of $\frac{2\pi R}{4} = \frac{\pi}{2}$ (since we have $R = 1$ on the unit sphere). If we move in a direction of θ, we could cross the point

$$p_\theta = (\cos \theta)\, p_2 + (\sin \theta)\, p_1$$

along the equator through p_1 and p_2. (This is true independent of R.) To travel a distance of L along the equator from p_0 to p_θ, we'd be moving along the sphere, so would need to determine how much rotation that would involve. Thus we solve $\frac{\theta'}{2\pi} = \frac{L}{2\pi R}$ (partial angle should be the same as partial circumference) which in this case gives a rotation of $\theta' = \frac{L}{R} = L$. As with rotating from e_{right} towards e_{up} gives the cosine term to e_{right}, since we are rotating from p_0 towards p_θ this would have us arrive at the point

$$p_{0,\text{new}} = (\cos \theta')\, p_0 + (\sin \theta')\, p_\theta.$$

Expanding, this becomes

$$p_{0,\text{new}} = \left(\cos \frac{L}{R}\right) p_0 + \left(\sin \frac{L}{R}\right) \left[(\cos \theta)\, p_2 + (\sin \theta)\, p_1\right].$$

We have already accounted for an arbitrary radius R but not an arbitrary center c. However, by shifting the global coordinates by $-c$, we would have a sphere centered at the origin, hence we can change the above to

$$(p_{0,\text{new}} - c) = \left(\cos \frac{L}{R}\right) (p_0 - c) + \left(\sin \frac{L}{R}\right) \left[(\cos \theta)\, (p_2 - c) + (\sin \theta)\, (p_1 - c)\right].$$

This also will change the way we compute p_1 and p_2. Since our center is $c = (0, 0, 0.7)$ we instead have

$$p_1 = \left(\frac{x_0(0.7 - z_0)}{\sqrt{x_0^2 + y_0^2}}, \frac{y_0(0.7 - z_0)}{\sqrt{x_0^2 + y_0^2}}, 0.7 + \sqrt{x_0^2 + y_0^2}\right)$$

and

$$p_2 = \left(-\frac{0.1y_0}{\sqrt{x_0^2 + y_0^2}}, \frac{0.1x_0}{\sqrt{x_0^2 + y_0^2}}, 0.7\right).$$

NOTE: Most mathematicians use the term "great circle" instead of equator.

To determine if we need to go from the sphere to the vertical dendrite, we need to determine the minimal distance from p_0 to the intersection of the two surfaces, which occurs on the line $z = 0.7 - \sqrt{0.0075}$ and of course $x^2 + y^2 = 0.05^2$ on the cylinder. This minimal distance will occur in the vertical plane containing p_0 and p_1 (the "up" point). This plane is given by $y_0 x - x_0 y = 0$ so this minimal distance occurs at one of two points

$$\left(\frac{0.05x_0}{\sqrt{x_0^2 + y_0^2}}, \frac{0.05y_0}{\sqrt{x_0^2 + y_0^2}}, 0.7 - \sqrt{0.0075}\right), \left(-\frac{0.05x_0}{\sqrt{x_0^2 + y_0^2}}, -\frac{0.05y_0}{\sqrt{x_0^2 + y_0^2}}, 0.7 - \sqrt{0.0075}\right).$$

To find the distance from these to p_0, we need to find the angles along the sphere. Clearly the one with the same signs as $p_0 = (x_0, y_0, z_0)$ in the x, y components is closest hence we have angle

$$\begin{aligned}
\cos(\theta_+) &= \frac{(p_0 - (0,0,0.7)) \cdot \left[\left(\frac{0.05x_0}{\sqrt{x_0^2+y_0^2}}, \frac{0.05y_0}{\sqrt{x_0^2+y_0^2}}, 0.7 - \sqrt{0.0075}\right) - (0,0,0.7)\right]}{\|p_0 - (0,0,0.7)\|_2 \left\|\frac{0.05x_0}{\sqrt{x_0^2+y_0^2}}, \frac{0.05y_0}{\sqrt{x_0^2+y_0^2}}, -\sqrt{0.0075}\right\|_2} \\
&= \frac{0.05\sqrt{x_0^2 + y_0^2} + (z_0 - 0.7)(-\sqrt{0.0075})}{0.1 \cdot 0.1} = 5\sqrt{x_0^2 + y_0^2} + (0.7 - z_0)\sqrt{75}
\end{aligned}$$

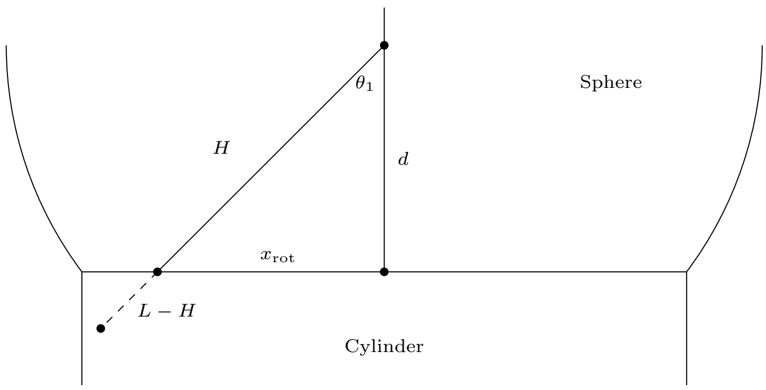
Can verify with Mathematica

```
Subscript[x, s] =
 0.05 Subscript[x, 0]/Sqrt[Subscript[x, 0]^2 + Subscript[y, 0]^2]
Subscript[y, s] =
 0.05 Subscript[y, 0]/Sqrt[Subscript[x, 0]^2 + Subscript[y, 0]^2]
Subscript[z, s] = 0.7 - Sqrt[0.0075]
Simplify[Subscript[x, s]^2 + Subscript[y, s]^2 + (Subscript[z, s] - 0.7)^2]
Subscript[\[CapitalDelta], 1] = {Subscript[x, 0], Subscript[y, 0],
   Subscript[z, 0]} - {0, 0, 0.7}
Subscript[\[CapitalDelta], 2] = {Subscript[x, s], Subscript[y, s],
   Subscript[z, s]} - {0, 0, 0.7}
Simplify[(Subscript[\[CapitalDelta], 1]).(Subscript[\[CapitalDelta], 1]),
         Subscript[x, 0]^2 + Subscript[y, 0]^2 + (Subscript[z, 0] - 0.7)^2 == 0.1^2]
Simplify[(Subscript[\[CapitalDelta], 2]).(Subscript[\[CapitalDelta], 2])]
Simplify[(Subscript[\[CapitalDelta], 1]).(Subscript[\[CapitalDelta], 2])]
```

To get the distance we multiply the angle by the radius of the sphere (since $\frac{\theta_+}{2\pi} = \frac{\text{dist}}{2\pi R}$). Thus p_0 is a distance of

$$d = 0.1 \arccos\left(5\sqrt{x_0^2 + y_0^2} + (0.7 - z_0)\sqrt{75}\right)$$

from the boundary. Given an angle $-\pi \leq \theta \leq \pi$, if $\theta \geq 0$ then (given an assumption about the maximal displacement) we know we stay in the sphere. If $\theta < 0$, then we can compute the angle $\theta_1 = \theta + \frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and determine how far we can travel within the sphere along this angle:



Given θ_1 we have a maximum hypotenuse H given by $\cos\theta_1 = \frac{d}{H} \Rightarrow H = \frac{d}{\cos\theta_1}$. If our displacement L exceeds H then we first rotate the point

$$p_{\text{cyl}} = \left(\frac{0.05x_0}{\sqrt{x_0^2+y_0^2}}, \frac{0.05y_0}{\sqrt{x_0^2+y_0^2}}, 0.7-\sqrt{0.0075}\right)$$

by a distance $x_{\text{rot}} = H\sin\theta_1$ and then move within the dendrite by a vertical displacement of

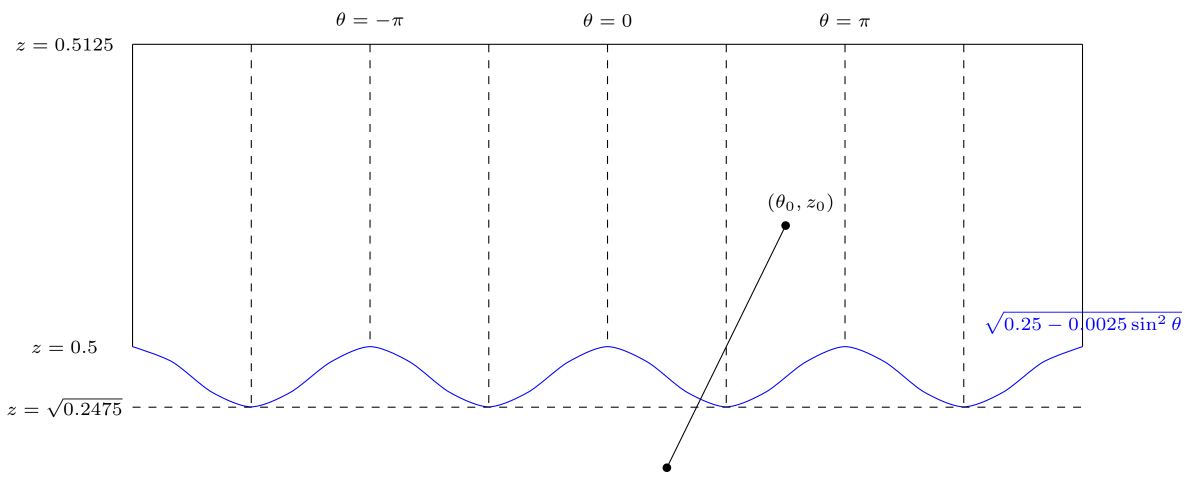
$$\boxed{-(L-H)\cos\theta_1}$$

and a horizontal displacement (via rotation) of

$$\boxed{(L-H)\sin\theta_1}$$

(the sign of $\sin\theta_1$ will determine if we go left or right).

To deal with the intersection of the two dendrites, we flatten out the top cylinder and examine the (cyclic) boundary of intersection:



In order to find this we can crudely intersect the line

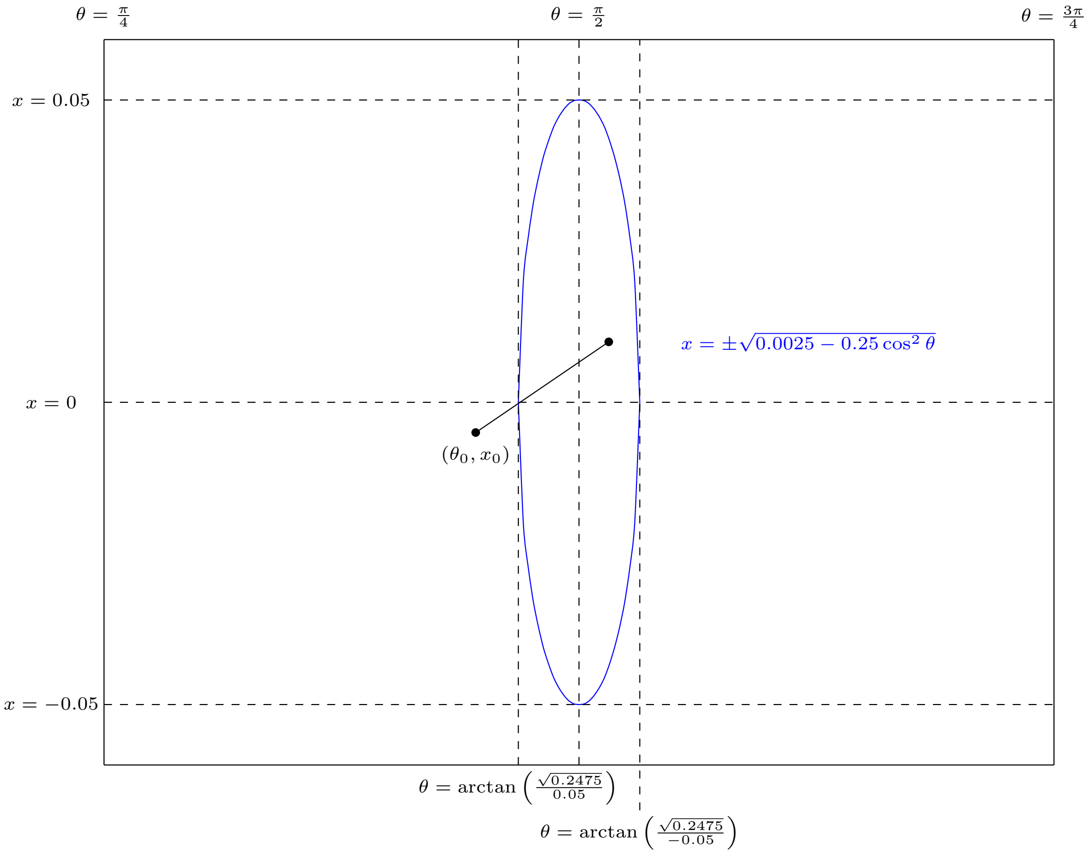
$$\frac{z-z_0}{\theta-\theta_0} = \frac{y_{\text{normal}}}{x_{\text{normal}}}$$

with the curve

$$z = \sqrt{0.25 - 0.0025\sin^2\theta}.$$

Once we've done this, we simply account for the fraction of the curve we've used and scale the $k\cdot\mathcal{N}(0,1)$ distributed vector $(x_{\text{normal}}, y_{\text{normal}})$ by the remaining part.

When going from the horizontal dendrite to the vertical dendrite, we are in a similar situation since we want to find where the cylinder $x^2+y^2 = 0.05^2$ intersects the cylinder $y^2+z^2 = 0.5^2$. Since we're writing $y = 0.5\cos\theta, z = 0.5\sin\theta$ this becomes $x = \pm\sqrt{0.0025 - 0.25\cos^2\theta}$:



As before, we want to intersect the line

$$\frac{x - x_0}{\theta - \theta_0} = \frac{y_{\text{normal}}}{x_{\text{normal}}}$$

with the above and then will use the segment after there.

It's also worth mentioned that we can reflect this ellipse-like figure across the x-axis (vertical here) since $\cos\theta$ is an even function, but that would correspond to the bottom of the horizontal dendrite, so we have no issues there.

Also, since

$$\arctan\left(\frac{\sqrt{0.2475}}{0.05}\right) \approx 1.4706, \arctan\left(\frac{\sqrt{0.2475}}{-0.05}\right) \approx 1.6710, \text{ and } \frac{\pi}{2} \approx 1.5710$$

we don't worry about points on the bottom of the cylinder $z < 0$. (These points correspond to $\theta < 0$ since $z = 0.5\sin\theta$.) This is because 0 and π are too far away from $\frac{\pi}{2}$ to matter.