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Math Behind Diffusion as a PDE

NOTE: To extract the LaTeX source of this PDF and the supporting files, execute:

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pdftk kolmogorov_formulation.pdf unpack_files output .
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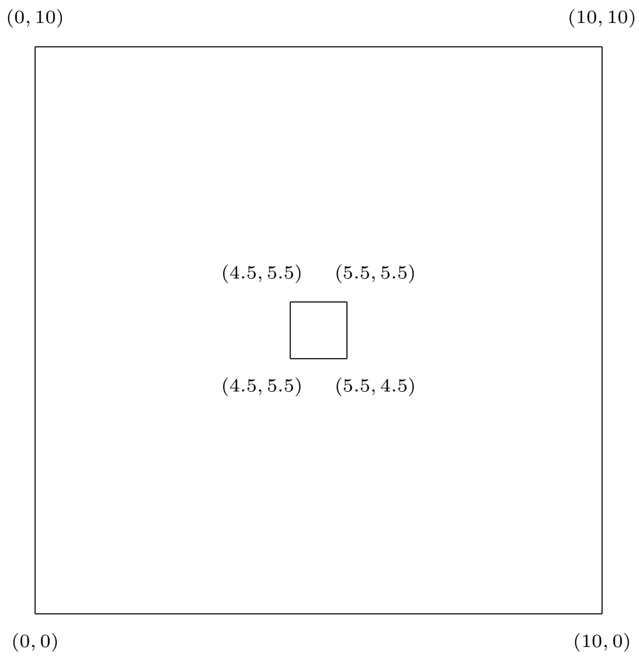
1 Initial Formulation

The forward **Kolmogorov forward equation (diffusion)** or more commonly known as the Fokker-Planck equation describes a process which dictates the time evolution of the probability density function for a random process.

We have a two-dimensional random process

$$\mathbf{X}_t = \left[\begin{array}{c} X_t \\ Y_t \end{array} \right]$$

which diffuses via a Wiener/Brownian motion $\mathbf{W}_t$. This diffusion has no drift but has a piecewise constant diffusion rate in both space directions, parameterized by the following region:



Given this description, the distribution satisfies the stochastic process

$$d\mathbf{X}_t = \boldsymbol{\sigma}\left(\mathbf{X}_t, t\right) d\mathbf{W}_t$$

where the diffusion is given by

$$\boldsymbol{\sigma}\left(\mathbf{X}_t, t\right) = \left[\begin{array}{cc} \sigma\left(\mathbf{X}_t, t\right) & 0 \\ 0 & \sigma\left(\mathbf{X}_t, t\right) \end{array} \right]$$

and the scalar piecewise diffusion $\sigma\left(\mathbf{X}_t, t\right)$ is given by

$$\sigma\left(\mathbf{X}_t, t\right) = a + (b - a) \cdot \chi\left(\mathbf{X}_t\right)$$

(note this does not depend on time, only on position $\mathbf{X}_t$). In the above

- χ is the indicator function for the small region in the middle $4.5 \leq x, y \leq 5.5$
- a is the constant diffusion rate in the large region
- b is the constant diffusion rate in the small region

2 Defining the PDE

Letting $f(\mathbf{x}, t)$ be the probability density function for $\mathbf{x} \in [0, 10] \times [0, 10]$, the theory gives us a PDE from the stochastic process defined above.

To define the PDE, we need to compute the diffusion tensors

$$D_{ij}(\mathbf{x}, t) = \sum_{k=1}^{2} \sigma_{ik}(\mathbf{x}, t)\sigma_{jk}(\mathbf{x}, t)$$

for $i, j \in \{1, 2\}$. Since $\sigma_{12} = \sigma_{21} = 0$ the quantities are only non-zero if $i = j$. In either case

$$D_{11}(\mathbf{x}, t) = D_{22}(\mathbf{x}, t) = \sigma(\mathbf{x}, t)^2 = \sigma(\mathbf{x})^2$$

(we recall from above that σ does not depend on time). Given this, the PDE is

$$\boxed{\frac{\partial f}{\partial t} = \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2}{\partial x_i \, \partial x_j} [D_{ij} \cdot f] = \frac{1}{2} \left[\frac{\partial^2}{\partial x^2} [D_{11} \cdot f] + \frac{\partial^2}{\partial y^2} [D_{22} \cdot f] \right]}.$$

Since $D_{11} = D_{22}$ we recognize this as the Laplace operator

$$\frac{\partial f}{\partial t} = \frac{1}{2} \nabla^2 \left[\sigma(\mathbf{x})^2 \cdot f \right].$$

3 Determining the Steady State

We seek to find the equilibrium (or steady state) behavior of this system. In other words we seek to find the density function:

$$g(\mathbf{x}) = \lim_{t \to \infty} f(\mathbf{x}, t)$$

defined over the region. Passing to the limit in our PDE above and utilizing the fact that σ does not depend on time, this gives

$$0 = \frac{1}{2} \nabla^2 \left[\sigma(\mathbf{x})^2 \cdot g \right].$$