When is One Enough?

https://github.com/bradyb/quantum/

Lately I've been interested in Grover's Algorithm after I came across `http://pyquil.readthedocs.io/en/latest/start.html#exercise-3-grover-s-algorithm` while reading through the pyquil docs. The name of the example function they provide implies that this problem can be solved after one application of the Grover Iterator, so I tried to see what happen if we used the example input with one application of the Grover Iterator.

So, we start off with the state $|00\rangle$ and apply $H^{\otimes 2}$ to get the state:

$$|\psi\rangle = H^{\otimes 2}|00\rangle = \frac{1}{2} \cdot (|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

Now we query the quantum oracle to perform the phase shift on the states we are searching for (in this case just the second index or $|10\rangle$). Therefore, our system is now in the state:

$$\frac{1}{2} \cdot (|00\rangle + |01\rangle - |10\rangle + |11\rangle) = |\psi\rangle - |10\rangle$$

Now we apply our (only!) Grover Iterator:

$$\begin{aligned}
(2|\psi\rangle\langle\psi| - I)(|\psi\rangle - |10\rangle) &= 2|\psi\rangle\langle\psi|\psi\rangle - |\psi\rangle - 2|\psi\rangle\langle\psi|10\rangle + |10\rangle \\
&= 2|\psi\rangle - |\psi\rangle - 2|\psi\rangle\langle\psi|10\rangle + |10\rangle \\
&= 2|\psi\rangle - |\psi\rangle - 2|\psi\rangle \cdot \frac{1}{2} + |10\rangle \\
&= |10\rangle
\end{aligned} \tag{1}$$

So when we measure our system, the only possible outcome is $|10\rangle$, the index of the one in our array!

Lets generalize the point that this exercise is driving home. Consider a state space S of size 2^n and a function $f : S \to \{0, 1\}$ such that $|f^{-1}(1)| = \frac{1}{4} \cdot 2^n$ (implying that $|f^{-1}(0)| = \frac{3}{4} \cdot 2^n$). We want to find a state in $f^{-1}(1)$ with one application of the Grover Iterator. As before, we start with the 0 state and apply $H^{\otimes n}$ to create the maximally entangled state on n qubits.

$$|\psi\rangle = H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle$$

Querying our quantum oracle gives the state $|\psi\rangle - \frac{2}{\sqrt{2^n}} \sum_{i\in f^{-1}(1)} |i\rangle$. Again, we apply the Grover Iterator to the phase-flipped system:

$$\begin{aligned}
(2|\psi\rangle\langle\psi| - I)(|\psi\rangle - \frac{2}{\sqrt{2^n}} \sum_{i\in f^{-1}(1)} |i\rangle) &= 2|\psi\rangle\langle\psi|\psi\rangle - |\psi\rangle - 2|\psi\rangle\langle\psi| \frac{2}{\sqrt{2^n}} \sum_{i\in f^{-1}(1)} |i\rangle + \frac{2}{\sqrt{2^n}} \sum_{i\in f^{-1}(1)} |i\rangle \\
&= 2|\psi\rangle - |\psi\rangle - \frac{4}{\sqrt{2^n}} \sum_{i\in f^{-1}(1)} |\psi\rangle\langle\psi|i\rangle + \frac{2}{\sqrt{2^n}} \sum_{i\in f^{-1}(1)} |i\rangle \\
&= |\psi\rangle - \frac{4}{\sqrt{2^n}} \sum_{i\in f^{-1}(1)} |\psi\rangle \cdot \frac{1}{\sqrt{2^n}} + \frac{2}{\sqrt{2^n}} \sum_{i\in f^{-1}(1)} |i\rangle \\
&= |\psi\rangle - \frac{4}{\sqrt{2^n}} \cdot \frac{1}{4} \cdot 2^n \cdot \frac{1}{\sqrt{2^n}} |\psi\rangle + \frac{2}{\sqrt{2^n}} \sum_{i\in f^{-1}(1)} |i\rangle \\
&= \frac{2}{\sqrt{2^n}} \sum_{i\in f^{-1}(1)} |i\rangle
\end{aligned} \tag{2}$$

So the only states, x, left with non-zero amplitudes are those with $f(x) = 1$! Therefore, a measurement will result in a solution state.