**Assignment #8** Name Answer Key

**Due 20 March 2015**

1. Let $a_1 = 6$ and, for $n \geq 2$, let $a_n = \dfrac{2}{3}a_{n-1} + 4$.

   (a) Find $a_2$, $a_3$, and $a_4$.

   $$a_2 = \frac{2}{3}(6) + 4 = 8$$

   $$a_3 = \frac{2}{3}(8) + 4 = \frac{28}{3} \approx 9.33$$

   $$a_4 = \frac{2}{3}\left(\frac{28}{3}\right) + 4 = \frac{92}{9} \approx 10.22$$

   (b) Show that $\{a_n\}$ is monotone increasing.

   $a_2 - a_1 = 8 - 6 \geq 0.$

   assume $a_n - a_{n-1} \geq 0$. Then

   $$a_{n+1} - a_n = \left(\frac{2}{3}a_n + 4\right) - \left(\frac{2}{3}a_{n-1} + 4\right) = \frac{2}{3}(a_n - a_{n-1}) \geq 0.$$

   Thus $\{a_n\}$ is monotone increasing.

   (c) Show that $\{a_n\}$ is bounded above.

   $a_1 \leq 30$, because $6 \leq 30$.

   assume $a_n \leq 30$. Then

   $$a_{n+1} = \frac{2}{3}a_n + 4 = \frac{2}{3}(30) + 4 = 24 \leq 30.$$

   Thus $\{a_n\}$ is bounded by 30

   (d) Find the limit of the sequence.

   $a_n \to L$, for some $L$.

   $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{2}{3}a_{n-1} + 4\right)$$

   $$L = \frac{2}{3}L + 4 \quad \therefore \quad L = 12$$

2. In each case, find the limit of the sequence $\{a_n\}$:

(a) $a_n = \dfrac{\sin(n)}{n^2} + \dfrac{4^n + 2^n}{4^n + e^n}$

① $\dfrac{-1}{n^2} \le \dfrac{\sin(n)}{n^2} \le \dfrac{1}{n^2}$

Since $\dfrac{1}{n^2} \to 0$ and $\dfrac{-1}{n^2} \to 0$, $\dfrac{\sin(n)}{n^2} \to 0$ by the squeeze thm.

② $\dfrac{4^n + 2^n}{4^n + e^n} = \dfrac{1 + \left(\frac{2}{4}\right)^n}{1 + \left(\frac{e}{4}\right)^n} \longrightarrow \dfrac{1+0}{1+0} = 1.$

Thus $a_n \longrightarrow 0 + 1 = 1.$

(b) $a_n = n^2 e^{-n} + \dfrac{n^2 - 3}{n^2 + 4}$

① $n^2 e^{-n} = \dfrac{n^2}{e^n} \longrightarrow 0$ (by l'hopital's rule).

② $\dfrac{n^2 - 3}{n^2 + 4} = \dfrac{1 - \frac{3}{n^2}}{1 + \frac{4}{n^2}} \longrightarrow \dfrac{1-0}{1+0} = 1$

Thus $a_n \longrightarrow 0 + 1 = 1.$