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Test #3 $\frac{3}{2}(\text{raw}) + 25$ **Name** answer key

Show all of your work

5 1. Evaluate $\int_0^4 \frac{1}{\sqrt{4-x}}dx$.

$$= \lim_{c \to 4^-} \int_0^c \frac{1}{\sqrt{4-x}}dx$$

$$= \lim_{c \to 4^-} \left(-2\sqrt{4-c} + 4\right)$$

$$= 0 + 4 = 4$$

$$\int (4-x)^{-1/2}dx \qquad u = 4-x \quad du = -dx$$

$$= -\int u^{-1/2}du$$

$$= -2u^{1/2} + C = -2\sqrt{4-x} + C$$

5 2. Find the length of the curve given by $f(x) = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$ from $x = 1$ to $x = 2$.

$$f'(x) = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$$

$$(f'(x))^2 = \frac{x^4}{4} - \frac{1}{2} + \frac{x^{-4}}{4}$$

$$1 + (f'(x))^2 = \frac{x^4}{4} + \frac{1}{2} + \frac{x^{-4}}{4} = \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^2$$

$$\sqrt{1 + f'(x)^2} = \frac{x^2}{2} + \frac{x^{-2}}{2}$$

$$\int_1^2 \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)dx = \frac{x^3}{6} - \frac{1}{2x}\Big|_1^2$$

$$= \left(\frac{8}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{2}\right)$$

$$= 17/12$$

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3. Let $L = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k10^k}$ and let s_n denote the nth partial sum of the series. Compute s_3, expressing it as a ratio of integers. Estimate the error in approximating L by s_3.

5 $$S_3 = \frac{1}{10} - \frac{1}{200} + \frac{1}{3000} = \frac{143}{1500} = .095\overline{3}33$$

The error is no larger than $\frac{1}{40000}$

$= 2.5 \times 10^{-5}$

4. (a) Evaluate $\int xe^{-x^2/2}dx$.

Let $u = -x^2/2$ $\quad du = -2x/2\, dx = -x\, dx$

3 $$= -\int e^u du = -e^u + C = -e^{-x^2/2} + C$$

(b) Evaluate $\int_2^{\infty} xe^{-x^2/2}dx$. $= \lim_{b\to\infty} \int_2^b x e^{-x^2/2} dx$

3 $$= \lim_{b\to+\infty}\left(-e^{-b^2/2} + e^{-2}\right) = e^{-2}$$

(c) Show that the series $\sum_{k=2}^{\infty} ke^{-k^2/2}$ converges.

3 $f(x) = x e^{-x^2/2}$

① $f'(x) = e^{-x^2/2} + x e^{-x^2/2}(-x) = e^{-x^2/2}(1-x^2)$

$-$ | $+$ | $-$ on the number line at -1, 0, $+1$ — decreasing on $[1,\infty)$.

② $f(x) \geq 0$, being the product of positive terms.

Since $\int_2^{+\infty} x e^{-x^2/2} dx$ converges, $\sum_{k=2}^{\infty} k e^{-k^2/2}$ converges by the integral test.

5. In each case, find the **sum** of the series S:

(a) $S = \sum_{k=1}^{\infty} 4(-1/3)^{k-2}$ $= -12 + 4 - \frac{4}{3} + \cdots$

3 geometric $a = -12$, $r = -1/3$.

$$S = \frac{-12}{1-(-\frac{1}{3})} = \frac{-12}{(4/3)} = -9.$$

(b) $S = \sum_{k=1}^{\infty} \frac{1}{k^2+3k+2}$ $\frac{1}{k^2+3k+2} = \frac{1}{k+1} - \frac{1}{k+2}$

$(k+1)(k+2)$

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$$S = \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \cancel{\cdots}$$

$$S = \frac{1}{2}$$

6 6. A sequence is defined recursively by $a_1 = 8$ and, for $n \geq 2$, $a_n = .5a_{n-1} + 8$.

(a) Find a_2 and a_3.

$a_2 = .5(8) + 8 = 12$, $a_3 = .5(12) + 8 = 14$

(b) Show that $\{a_n\}$ is monotone increasing.

$a_2 - a_1 \geq 0$ because $12 - 8 = 4 > 0$

assume $a_n - a_{n-1} \geq 0$. Then

$a_{n+1} - a_n = (.5a_n + 8) - (.5a_{n-1} + 8) = .5(a_n - a_{n-1})$

≥ 0. QED.

(c) Show that $\{a_n\}$ is bounded above by 16.

clearly $a_1 \leq 16$, since $8 \leq 16$.

assume $a_n \leq 16$. Then

$a_{n+1} = .5a_n + 8 \leq .5(16) + 8 = 8 + 8 = 16.$

Q.E.D.

7. Let $a_n = \frac{2n^2+5n}{n^2+9} + \frac{n^2+2^n}{e^n}$. Find the limit of the sequence. Explain your reasoning.

5

$$a_n = \frac{2+\frac{5}{n}}{1+\frac{9}{n^2}} + \frac{n^2}{e^n} + \left(\frac{2}{e}\right)^n \longrightarrow 2+0+0=2$$

$$\frac{2+\frac{5}{n}}{1+\frac{9}{n^2}} \longrightarrow \frac{2}{1}, \quad \frac{n^2}{e^n} \to 0 \text{ (l'hopital)}, \quad \left(\frac{2}{e}\right)^n \to 0 \text{ because } \left|\frac{2}{e}\right| < 1$$

8. In each case determine whether the series converges or diverges; indicate which test or tests your are using to make your conclusion.

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(a) $\sum_{n=1}^{\infty} \frac{3+3^n}{2+5^n}$

$$\frac{\frac{3+3^n}{2+5^n}}{\frac{3^n}{5^n}} = \left(\frac{3+3^n}{2+5^n}\right)\left(\frac{5^n}{3^n}\right) = \frac{\frac{3}{3^n}+1}{\frac{2}{5^n}+1} \longrightarrow \frac{1}{1} = 1. \text{ (valid)}$$

since $\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$ converges (geometric series)

the original series converges by LCT.

(b) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+8}}$

$$\frac{\frac{n}{\sqrt{n^4+8}}}{\frac{1}{n}} = \frac{\cancel{n^2}}{\cancel{n^2}\sqrt{1+8/n^4}} \longrightarrow 1. \text{ (valid)}$$

since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

thus the original series diverges.

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

$\frac{1}{\sqrt{n}}$ is monotone decreasing and converges to 0; thus, the series converges by A.S.T.