§6.4*–General Logarithmic and Exponential Functions

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General exponential functions The derivative The logarithm to the base a. Solving equations e as a limit

Outline

Problem

Let $a > 0$ be a real number and let r be a rational number. Show that

$$a^r = \exp\big(\ln(a^r)\big) = \exp\big(r\ln(a)\big).$$

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Definition

For $a > 0$, it make sense to define

$$a^x = \exp\big(x\ln(a)\big).$$

for all $x \in \mathbb{R}$.

Problem

Suppose that your calculator has an e^x and $\ln x$ buttons, but no other buttons for exponentials and logarithms to other bases. How can we calculate $(\sqrt{2})^\pi$?

The laws of exponents

The *laws of exponents* are inherited from exp:

$$a^{x+y} = a^x a^y \qquad a^{x-y} = \frac{a^x}{a^y} \qquad (a^x)^y = a^{xy}.$$

In addition, we have the following two identities:

$$(ab)^x = a^x b^x \qquad (a/b)^x = a^x/b^x.$$

Problem

Sketch the graph of $y = a^x$ *in each case:* $a = 1$, $a > 1$, *and* $0 < a < 1$.

Theorem

- $\dfrac{d}{dx} a^x = a^x \ln(a)$
- $\displaystyle\int a^x \, dx = \frac{a^x}{\ln(a)}$

Problem

Find y' in each case:

- $y = x2^{x^3}$
- $y = \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$

Problem

- $\displaystyle\int x^2 3^{x^3}\,dx$
- $\displaystyle\int \frac{1}{1+2^{-x}}\,dx$

Definition

For $a > 0$, we will use $\log_a$ to denote the inverse of a^x. Thus

$$\log_a(x) = y \quad \text{iff} \quad a^y = x.$$

Theorem (Change of base formula)

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Theorem

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

Problem

Find y' for each of the following:

- $y = \log_{10}(x^2 + 3x + 5)$
- $y = x^x$

Problem

In each case, solve for x:

- $2^x = \sqrt{2}/8$
- $2^x - 35 \cdot 2^{-x} = 2$

Theorem

$$\lim_{h\to 0}\frac{\ln(1+h)}{h}=1$$

Corollary

$$\lim_{h\to 0}(1+h)^{1/h}=e.$$

Problem

Show that

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n=e^2.$$