§6.2*–The Natural Logarithm Function

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Outline

The log base 10

The natural logarithm

Further properties of the natural logarithm

The graph of the log function

The number e, Euler's number

Further derivative problems

Integration

Logarithmic differentiation

Properties of the base 10 logarithm function

Here are some familiar properties of the base 10 logarithm function.

- $\log_{10}(1) = 0$, $\log_{10}(10) = 1$, $\log_{10}(100) = 2$.
- $\log_{10}(ab) = \log_{10}(a) + \log_{10}(b)$, $a, b > 0$.
- $\log_{10}(1/a) = -\log_{10}(a)$, $a > 0$.
- $\log_{10}(a/b) = \log_{10}(a) - \log_{10}(b)$.
- $\log_{10}(a^r) = r\log_{10}(a)$, $a > 0$, r rational.

Definition (The natural logarithm function)

For $x > 0$, let

$$\ln(x) = \int_1^x \frac{1}{t}\,dt.$$

This function is called the *natural logarithm.*

Theorem (Elementary properties of ln)

- $\ln(1) = 0$
- $\ln(x) < 0$ *if* $0 < x < 1$
- $\ln(x) > 0$ *if* $x > 1$
- $D_x \ln(x) = 1/x$. *In particular,* $\ln(x)$ *is an increasing function.*

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Problem

Show that $.5 \leqslant \ln(2) \leqslant 1$.

Theorem

- $\ln(ab) = \ln(a) + \ln(b)$, $a, b > 0$.
- $\ln(a/b) = \ln(a) - \ln(b)$.
- $\ln(a^p) = p\ln(a)$, *for* $p > 0$, p *rational. (Homework)*

Theorem

- $\ln$ *is increasing and concave down.*
- *As* $x \to +\infty$, $\ln(x) \to +\infty$.
- *As* $x \to 0^+$, $\ln(x) \to -\infty$.

Problem

Sketch the graph of $\ln(x)$.

Definition

- The range of $\ln(x)$ is $(-\infty, \infty)$. Since $\ln(x)$ is increasing, there exists a unique number e such that

$$\ln(e) = 1.$$

- The number e is called *Euler's number*. Note that

$$e \approx 2.71828$$

- Since $\ln(e) = 1$, e is called the *base* of the natural logarithm function.

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Theorem (The chain rule)

If f is a positive, differentiable function, then

$$\frac{d}{dx} \ln f(x) = \frac{1}{f(x)} f'(x).$$

Problem

Find $\dfrac{dy}{dx}$ *in each case:*

- $y = \ln(x^2)$
- $y = x^2\big(\ln(x^2+1)\big)^3$
- $y = \ln(|x|)$

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Theorem

$$\int \frac{1}{x}\,dx = \ln(|x|) + C.$$

Problem

Evaluate the following integrals:

- $\int \frac{x^2}{x^3+1} dx$
- $\int_{-3}^{-2} (x+1)^{-1} dx$
- $\int \tan(x) dx$

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Problem

Use the logarithm function and its properties to evaluate the derivative of

$$f(x) = \frac{x^2(x-4)^3}{(x^2+1)^4}.$$