§7.8–Improper Integrals

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Outline

An overview

Type I integrals: unbounded domains

Type II integrals: unbounded integrands

What makes an integral improper?

Recall that the definite integral $\int_a^b f(x)dx$ is only defined for a bounded function f on a bounded domain $[a, b]$. Explain why each integral below is improper:

- $\displaystyle\int_0^{\infty} e^{-x} dx$
- $\displaystyle\int_0^{1} x^{-1/2} dx$
- $\displaystyle\int_0^{\infty} \frac{1}{x^{4/3} + x^{2/3}} dx$

Type I: unbounded domains

When the domain of integration is unbounded, we must solve the problem by limits:

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx,$$

whenever this limit exists.

Likewise

$$\int_{-\infty}^{a} f(x)dx = \lim_{b\to-\infty} \int_b^a f(x)dx$$

Problem

Evaluate $\int_1^{\infty} x^{-2} dx$

Problem

Evaluate $\int_1^{\infty} x^{-1} dx$

Terminology

When this limit exists, we say that the improper integral $\int_a^\infty f(x)dx$ converges. When this limit does not exist, we say that the integral diverges.

Problem

Evaluate the following integrals:

- $\displaystyle\int_{-\infty}^{0} e^x dx$
- $\displaystyle\int_{0}^{\infty} x^2 e^{-x} dx$
- $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$ *How should we define this?*
- $\displaystyle\int_{0}^{\infty} \frac{1}{x^2+3x+2} dx$

Type II integrals: unbounded integrands

- If f is unbounded as $x \to a^+$, then we define

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx,$$

provided the limit exists.

- If f is unbounded as $x \to b^-$, then we define

$$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx,$$

provided the limit exists.

Problem

Evaluate $\int_0^1 x^{-1/2} dx$.

Problem

- $\int_{-8}^{1} x^{-2/3} dx$ *How should we define this?*
- $\int_{0}^{1} \frac{x^2}{\sqrt{1-x^2}} dx$

Problem

Here is an integral of Type I and Type II. Evaluate
$\int_{0}^{\infty} \frac{1}{x^{4/3}+x^{2/3}} dx$