§7.3–Trigonometric Substitution

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Outline

Quadratic forms

We will learn techniques for solving integrals that contain roots of quadratic functions of x. There are three types:

Example	Type
$\int (9-x^2)^{1/2}dx$	$\sqrt{a^2-x^2}$
$\int_0^1 \frac{1}{(1+x^2)^2}dx$	$\sqrt{a^2+x^2}$
$\int \frac{1}{\sqrt{x^2-16}}dx$	$\sqrt{x^2-a^2}$

Sine substitutions

- For integrals containing the form $\sqrt{a^2-x^2}$, make the substitution

$$x = a\sin(\theta), \quad -\pi/2 \leqslant \theta \leqslant \pi/2.$$

- Under this substitution, $dx = a\cos(\theta)d\theta$ and

$$\begin{aligned}\sqrt{a^2-x^2} &= \sqrt{a^2-a^2\sin^2(\theta)}\\ &= \sqrt{a^2(1-\sin^2(\theta))}\\ &= a\cos(\theta).\end{aligned}$$

Problem

Evaluate the following integrals using trigonometric substitutions:

- $\int (9-x^2)^{1/2} dx$
- $\int_0^{1/2} x^3\sqrt{1-4x^2}dx$
- $\int \frac{x}{(9-x^2)^{5/2}} dx$

Problem

Find the area enclosed by the ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Tangent substitutions

- For integrals containing the form $\sqrt{a^2+x^2}$, make the substitution

$$x = a\tan(\theta), \quad -\pi/2 \leqslant \theta \leqslant \pi/2.$$

- Under this substitution, $dx = a\sec^2(\theta)d\theta$ and

$$\begin{aligned}\sqrt{a^2+x^2} &= \sqrt{a^2+a^2\tan^2(\theta)}\\ &= \sqrt{a^2(1+\tan^2(\theta))}\\ &= a\sec(\theta).\end{aligned}$$

Problem

Solve the following integrals:

- $\displaystyle\int \frac{1}{\sqrt{x^2+4}}dx$
- $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}dx$

Secant substitutions

- For integrals containing the form $\sqrt{x^2 - a^2}$, make the substitution

$$x = a\sec(\theta), \quad 0 < \theta < \pi/2 \quad \text{or} \quad \pi < \theta < 3\pi/2.$$

- Under this substitution, $dx = a\sec(\theta)\tan(\theta)d\theta$ and

$$\begin{aligned}\sqrt{x^2 - a^2} &= \sqrt{a^2\sec^2(\theta) - a^2} \\ &= \sqrt{a^2(\sec^2(\theta) - 1)} \\ &= a\tan(\theta).\end{aligned}$$

Problem

Solve the integrals:

- $\displaystyle\int \frac{1}{\sqrt{x^2 - 16}}dx$
- $\displaystyle\int_3^6 \frac{\sqrt{x^2 - 9}}{x}dx$