§6.2*–The Natural Logarithm Function

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Outline

Properties of the base 10 logarithm function

Here are some familiar properties of the base 10 logarithm function.

- $\log_{10}(1) = 0$, $\log_{10}(10) = 1$, $\log_{10}(100) = 2$.
- $\log_{10}(ab) = \log_{10}(a) + \log_{10}(b)$, $a, b > 0$.
- $\log_{10}(1/a) = -\log_{10}(a)$, $a > 0$.
- $\log_{10}(a/b) = \log_{10}(a) - \log_{10}(b)$.
- $\log_{10}(a^r) = r\log_{10}(a)$, $a > 0$, r rational.

Definition (The natural logarithm function)

For $x > 0$, let

$$\ln(x) = \int_1^x \frac{1}{t} dt.$$

This function is called the *natural logarithm.*

Theorem (Elementary properties of ln)

- $\ln(1) = 0$
- $\ln(x) < 0$ *if* $0 < x < 1$
- $\ln(x) > 0$ *if* $x > 1$
- $D_x \ln(x) = 1/x$. *In particular,* $\ln(x)$ *is an increasing function.*

Problem

Show that $.5 \leqslant \ln(2) \leqslant 1$.

Theorem

- $\ln(ab) = \ln(a) + \ln(b)$, $a, b > 0$.
- $\ln(a/b) = \ln(a) - \ln(b)$.
- $\ln(a^p) = p\ln(a)$, *for* $p > 0$, p *rational. (Homework)*

Theorem

- $\ln$ *is increasing and concave down.*
- *As* $x \to +\infty$, $\ln(x) \to +\infty$.
- *As* $x \to 0^+$, $\ln(x) \to -\infty$.

Problem

Sketch the graph of $\ln(x)$.

Definition

- The range of $\ln(x)$ is $(-\infty, \infty)$. Since $\ln(x)$ is increasing, there exists a unique number e such that

$$\ln(e) = 1.$$

- The number e is called *Euler's number*. Note that

$$e \approx 2.71828$$

- Since $\ln(e) = 1$, e is called the *base* of the natural logarithm function.

Theorem (The chain rule)

If f is a positive, differentiable function, then

$$\frac{d}{dx} \ln f(x) = \frac{1}{f(x)} f'(x).$$

Problem

Find $\frac{dy}{dx}$ in each case:

- $y = \ln(x^2)$
- $y = x^2\big(\ln(x^2+1)\big)^3$
- $y = \ln(|x|)$

Theorem

$$\int \frac{1}{x} dx = \ln(|x|) + C.$$

Problem

Evaluate the following integrals:

- $\int \frac{x^2}{x^3+1} dx$
- $\int_{-3}^{-2} (x+1)^{-1} dx$
- $\int \tan(x) dx$

Problem

Use the logarithm function and its properties to evaluate the derivative of

$$f(x) = \frac{x^2(x-4)^3}{(x^2+1)^4}.$$