§7.4–Partial Fractions

Tom Lewis

Spring Semester
2015

Outline

Problem

Let

$$f(x) = \frac{x+1}{2x^2+7x+6}.$$

1. *Show that*

$$f(x) == \frac{1}{x+2} - \frac{1}{2x+3}.$$

   The expression on the right is called the partial fraction decomposition.

2. *Use this decomposition to evaluate* $\int f(x)dx$.

Definition

A rational function is any function of the form $f(x) = P(x)/Q(x)$, where P and Q are polynomials. The rational function f is said to be proper if $\deg P < \deg Q$.

Example

Here are two examples:

- $f(x) = \dfrac{x+1}{2x^2+7x+6}$ is proper.
- $g(x) = \dfrac{x^3+2x+4}{x^2-1}$ is not proper.

Long division

Using long division, an improper rational function can be written as a sum of a polynomial and a proper rational function.

The method illustrated Terminology Factoring Polynomials Partial fraction decompositions Further examples

Problem

Express

$$\frac{x^4 - 4x^2 + 3x + 4}{x^2 - 4}$$

as the sum of a polynomial plus a proper rational function.

Definition (Types of factors)

There are two types of factors:

1. A factor of the form $Ax + B$ is called linear.
2. A factor of the form $Ax^2 + Bx + C$ for which $B^2 - 4AC < 0$ is called an irreducible quadratic.

Check!

Is $2x^2 - x - 36$ irreducible?

Problem

1. *Factor* $P(x) = x^3 + 2x^2 - 4x - 8$. *Identify the factors and their multiplicities.*
2. *Factor* $Q(x) = x^4 - 1$. *Identify the factors and their multiplicities.*

Theorem (The Fundamental Theorem of Algebra)

Every polynomial can be expressed as a product of powers of linear factors $(ax+b)^m$ *and powers of irreducible quadratic factors* $(ax^2+bx+c)^n$.

Theorem (Partial Fraction Decompositions)

Assume that the rational function $\dfrac{P(x)}{Q(x)}$ *is proper.*

- *Each factor of* Q *will generate terms of the partial fraction decomposition of* P/Q.
- *To each linear factor* $(ax+b)^m$ *of* Q, *the decomposition of* P/Q *will contain the terms*

$$\frac{D_1}{(ax+b)^1}+\cdots+\frac{D_m}{(ax+b)^m}.$$

- *To each irreducible quadratic factor* $(ax^2+bx+c)^n$ *of* Q, *the decomposition will contain the terms*

$$\frac{E_1x+F_1}{(ax^2+bx+c)^1}+\cdots+\frac{E_nx+F_n}{(ax^2+bx+c)^n}.$$

Problem

Find the partial fraction decomposition of
$f(x) = (x+1)/(2x^2+7x+6)$ *and evaluate* $\int f(x)dx$.

Problem

Find the partial fraction decomposition of
$f(x) = (3x^2+3x-2)/(x^3+2x^2-4x-8)$ *and evaluate* $\int f(x)dx$.

Problem

Evaluate $\int \frac{x^6 + 2x^4 + x^3 - 2x^2 - x - 5}{x^4 - 1}\, dx.$