§11.1–Sequences

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Outline

Definition (Informal)

A *sequence* is a list of real numbers in a definite order: $a_1, a_2, a_3 \ldots$. The sequence $a_1, a_2, a_3, \ldots$ will be denoted by $\{a_n\}$.

Definition (Formal)

A *sequence* is a function from the natural numbers to the real numbers, $a : \mathbb{N} \to \mathbb{R}$.

Problem

Let $a_n = 2/(n^2 + n + 1)$, $n \geqslant 1$. *List the first four terms of* $\{a_n\}$.

Definition

A triangle of dots is created by placing one less dot in each successive row. The number of dots in the resulting triangle is called a *triangular number*. The first four triangular numbers are 1, 3, 6, and 10. The corresponding triangles are pictured below.

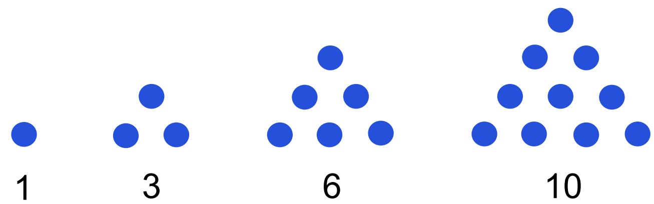

Figure : The first four triangular numbers

Problem

Let T_n denote the nth triangular number. Find a formula for T_n.

A recursively defined sequence

Problem

Let f_n count the number of distinct ways to write $n-1$ as a sum of 1s and 2s. We will set $f_1 = 1$.

1. *Find f_2, f_3, and f_4 by inspection.*
2. *Find a recursive formula for f_n. Use this recursive formula to compute f_8.*

Problem

Graph the sequence $\{a_n\}$ with $a_n = \dfrac{n}{n+1}$ by two different methods:

1. *By plotting a_n on the real line.*
2. *By plotting (n, a_n) in the plane.*

Problem

What is the tendency of the sequence $\{a_n\}$ as $n \to \infty$.

Problem

In each case, determine the limit of the sequence $\{a_n\}$.

1. $a_n = \dfrac{n^2+1}{n^3+4n+9}$
2. $a_n = ne^{-n}$.

Theorem

If $a_n \to a$ and $b_n \to b$ as $n \to \infty$, and if c is a constant, then

- $(a_n + b_n) \to a + b$ as $n \to \infty$;
- $(a_n - b_n) \to a - b$ as $n \to \infty$;
- $ca_n \to ca$ as $n \to \infty$;
- $a_n b_n \to ab$ as $n \to \infty$;
- $a_n / b_n \to a/b$ as $n \to \infty$, provided $b \neq 0$;
- $(a_n)^p \to a^p$ as $n \to \infty$, provided $a \geqslant 0$ and $p > 0$.

Problem

Find the limit, if it exists, of the sequence $\{c_n\}$ where

$$c_n = \frac{n^2 + 2n + 5}{4n^2 + 8} + \frac{n^2}{3^n}.$$

Theorem (Squeeze Theorem)

If there exists an integer m such that $a_n \leqslant b_n \leqslant c_n$ *for* $n \geqslant m$ *and if* $a_n \to L$ *and* $c_n \to L$ *as* $n \to \infty$, *then* $b_n \to L$ *as* $n \to \infty$.

Problem

Let $\{a_n\}$ *be the sequence whose nth term is* $a_n = \dfrac{\sin(n)}{n}$. *Show that* $a_n \to 0$ *as* $n \to \infty$.

Theorem

If $|a_n| \to 0$ *as* $n \to \infty$, *then* $a_n \to 0$ *as* $n \to \infty$.

Theorem

Let r be a real number.

$$r^n \begin{cases} \to 0 & \text{if } -1 < r < 1, \\ \to 1 & \text{if } r = 1, \\ \text{diverges} & \text{if otherwise.} \end{cases}$$

Problem

Let $a_n = \dfrac{2^n}{3^n + 8}$. *Evaluate* $\lim_{n\to\infty} a_n$.

Definition

- A sequence $\{a_n\}$ is said to be *bounded above* if there exists a number M such that $a_n \leqslant M$ for each $n \geqslant 1$.
- A sequence $\{a_n\}$ is said to be *bounded below* if there exists a number M such that $a_n \geqslant M$ for each $n \geqslant 1$.
- A sequence $\{a_n\}$ is said to be *bounded* if it is bounded above and below.

Problem

Show that the sequence $a_n = \dfrac{n}{n+1}$ *is bounded.*

Definition

- A sequence $\{a_n\}$ is called *monotone increasing* if $a_n \leqslant a_{n+1}$ for all $n \geqslant 1$.
- A sequence $\{a_n\}$ is called *monotone decreasing* if $a_n \geqslant a_{n+1}$ for all $n \geqslant 1$.
- A sequence $\{a_n\}$ is called *monotone* if it is either monotone increasing or decreasing.

Problem

Let $a_n = \dfrac{n}{n+1}$. *Show that the sequence* $\{a_n\}$ *is monotone.*

Theorem

- *If a sequence is monotone increasing and bounded above, then it converges to a limit.*
- *If a sequence is monotone decreasing and bounded below, then it converges to a limit.*

Problem

Let $a_1 = 1$ *and let* $a_n = 20 + \frac{1}{3}a_{n-1}$ *for* $n \geqslant 2$. *Show that* $\{a_n\}$ *is monotone increasing and bounded above. Find the limit of the sequence.*