§6.1–Inverse Functions

Tom Lewis

Spring Semester
2015

Outline

Definition

A relation in the plane is a set of ordered pairs (a, b) in the plane.

Relations defined by equations

Typically we are interested in relations defined through equations.

- For example, consider the set

$$\{(x, y) : x = y^3 + 3y^2 + 2y\}$$

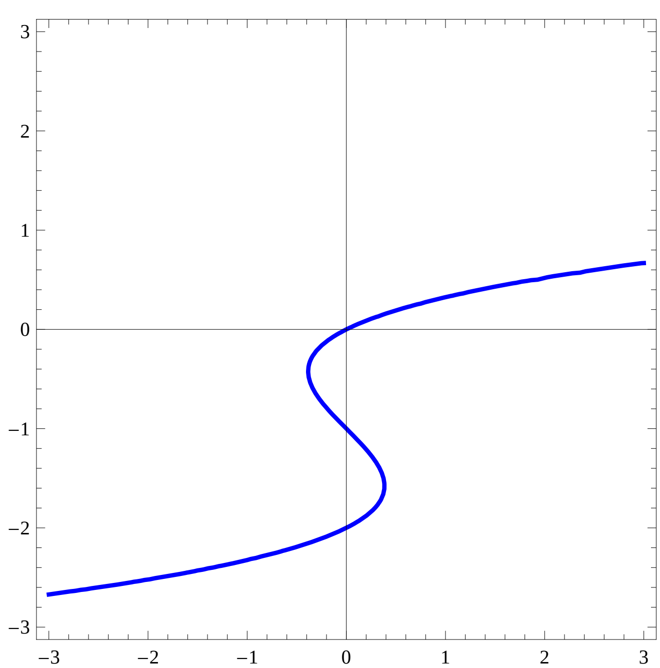
  This defines a relation, a set of ordered pairs, in the plane.

- It is customary to refer to this relation by specifying the equation only, namely,

$$x = y^3 + 3y^2 + 2y$$

The graph of a relation

We can graph the relation specified by $x = y^3 + 3y^2 + 2y$ by simply plotting the ordered pairs.



Definition

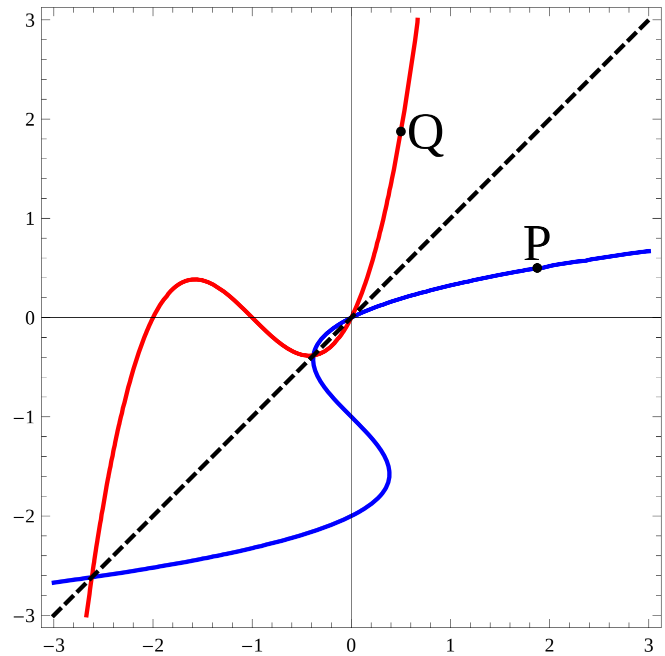
The inverse of a relation R in the plane is the relation R^{-1} obtained by transposing (swapping) the x and y-coordinates of each point of R. Thus (b, a) is in R^{-1} if and only if (a, b) is in R.

Problem

- *What is the inverse of the relation* $R = \{(2,3),(3,3),(5,2)\}$. *Graph* R *and its inverse; what is the geometric relationship between* R *and its inverse?*
- *If a relation is given by the equation* $x = y^3 + 3y^2 + 2y$, *what equation gives the inverse relation?*

The graph of a relation and its inverse

Here are the graphs of the relation $x = y^3 + 3y^2 + 2y$ (in blue) and its inverse (in red). The point P and its inverse Q are shown.



Problem

Consider the following two examples of functional relationships among the ordered pairs:

$$f = \{(1, -1), (2, 1), (3, 2), (4, 0)\}$$

and

$$g = \{(1, 1), (2, 3), (3, 1), (4, 2)\}$$

- *In each case, find the inverse relations.*
- *Are the resulting inverse relations functions?*
- *How are the domains and ranges of the functions and their inverse relations related?*
- *By what tests can we tell whether a function will have an inverse function?*

Definition (One-to-one)

- A function f with domain D is called *one-to-one* if distinct elements of D have distinct images. In other words,

$$f(s) = f(t) \quad \text{if and only if} \quad s = t.$$

- Said another way, a function is called one-to-one if it never takes on the same value more than once.

Problem

Which of the specified functions is one-to-one?

1. $f(x) = x^2$.
2. $f(x) = x/(1+x)$

Theorem (Horizontal line test)

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Theorem (Increasing and decreasing)

If a function is either increasing or decreasing on an interval domain, then it is one-to-one.

Problem

Show that $f(x) = 2x + \sin(x)$ *is one-to-one on* $(-\infty, \infty)$.

Problem

Explain how to restrict the domain of the function $f(x) = x^2$ to make it one-to-one.

Definition (Inverse function)

- Let f be one-to-one with domain A and range B. The inverse function of f, denoted by f^{-1}, has domain B and range A.
- f^{-1} maps y to x if and only if f maps x to y.
- Equivalently, for any $y \in B$,

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y$$

Theorem (Cancellation equations)

Let f be one-to-one with domain A and range B.

- $f^{-1}(f(x)) = x$ *for each* $x \in A$.
- $f(f^{-1}(y)) = y$ *for each* $y \in B$.

Problem

Let $f(x) = \sqrt{x-4}$ *on the interval* $[4, \infty)$. *Find* $f^{-1}(x)$.

Problem

Let $f(x) = x^2 + 1$ *on the interval* $[0, \infty)$*. Show that* f *is invertible and find* f^{-1}*.*

Theorem

If f *is a one-to-one, differentiable function with inverse function* f^{-1}*, if* (a, b) *is on the graph of* f*, and if* $f'(a) \neq 0$*, then* f^{-1} *is differentiable at* b *and*

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}$$

Problem

Let $f(x) = x^3 + 3x$. *Find* $(f^{-1})'(4)$.

The inverse of a relation One-to-one functions Inverse functions Finding inverse functions The calculus of inverse functions

Problem

Let $f(x) = 2x + \sin(x)$. *Find* $(f^{-1})'(4\pi)$.