§11.2–Series

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Spring Semester
2015

Outline

Prologue

Precise definition

Geometrics

Telescoping

Harmonic

Divergence

Theorems

Problem

Evaluate the following infinite sums:

- $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$
- $S = 1 + 2 + 4 + 8 + \cdots$
- $S = 1 + (-1) + 1 + (-1) + \cdots$

Commentary

The preceding examples underscore the need for a proper definition of the sum of an infinite series.

Definition (Sequence of partial sums)

Given a sequence $\{a_n\}$ of real numbers, we form a new sequence as follows:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \end{aligned}$$

The sequence $\{s_n\}$ is called the sequence of partial sums.

Definition (Convergence of a series)

If $s_n \to L$, then we say that the infinite series $\sum_{k=1}^{\infty} a_k$ converges and we write

$$\sum_{k=1}^{\infty} a_k = L.$$

If $\{s_n\}$ diverges, then we say that the infinite series $\sum_{k=1}^{\infty} a_k$ diverges.

Problem

Determine whether or not the series $\sum_{k=1}^{\infty}(-1)^{k+1}$ converges by examining the sequence of partial sums.

Definition (Geometric series)

Let a and r be nonzero numbers. Any series of the form

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar^1 + ar^2 + ar^3 + \cdots$$

is called a geometric series. The number r is called the common ratio of the series.

Theorem

$$\sum_{k=1}^{\infty} ar^{k-1} \begin{cases} = \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geqslant 1 \end{cases}$$

Problem

Evaluate the series $\sum_{k=1}^{\infty} 5(1/3)^{k+2}$.

Problem (Zeno's superball)

A certain super ball has the property that it will always return to 60 percent of the maximum height of the previous bounce. If a ball is dropped from 20 feet, how far will it travel (up and down) before it comes to rest?

Problem (Telescoping series)

Use a partial fraction decomposition to find an expression for the partial sums of the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+k}$. *Find the limit (sum) of the series.*

Problem (Harmonic series)

Show that the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges.

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ converges, then $|a_n| \to 0$.

Theorem (nth Term Test for Divergence)

If $|a_n| \not\to 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Problem

Does the series $\sum_{k=1}^{\infty}(1+k^{-1})^k$ *converge?*

Theorem

Suppose that $\sum a_n$ *and* $\sum b_n$ *are convergent series and let* c *be a constant. Then*

- $\sum(a_n + b_n) = \sum a_n + \sum b_n$
- $\sum(a_n - b_n) = \sum a_n - \sum b_n$
- $\sum ca_n = c \sum a_n$

Problem

Find the sum of the series $\displaystyle\sum_{k=1}^{\infty}\left(\frac{5}{k^2+k}+6(.4)^{k+3}\right)$