§6.8–Indeterminate Forms and L'Hôpital's Rule

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Outline

Definition (Indeterminate Forms)

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then we say that

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is of the form $0/0$. The ∞/∞ form is defined analogously.

Note

In this context, a can be any of $a^+, a^-, +\infty$, or $-\infty$.

Theorem (L'Hôpital's Rule)

If $\lim_{x\to a} f(x)/g(x)$ *is an ideterminate form of type* $0/0$ *or* ∞/∞ *and if*

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

Problem

Evaluate the following limits:

- $\lim_{x\to 0} \dfrac{\sin(x)}{x}$
- $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$
- $\lim_{x\to \infty} \dfrac{x^4}{e^x}$.

Problem

$$\lim_{x\to 0^+} x\ln(x)$$

is called a $0\cdot\infty$ form. Evaluate this limit by converting it into either one of the forms $0/0$ or ∞/∞.

Strategy for the $0 \cdot \infty$ form

Suppose that the limit $\lim_{x \to a} f(x)g(x)$ is of the form $0 \cdot \infty$. This form can be converted into either a 0/0 or an $\infty/infty$ form by algebra:

$$f(x)g(x) = \frac{g(x)}{1/f(x)} \quad \text{or} \quad f(x)g(x) = \frac{f(x)}{1/g(x)}.$$

Now the limit can be attacked by the previous methods.

Problem

Evaluate the following limits:

- *Find* $\lim_{x \to \infty} xe^{-x}$.
- *Find* $\lim_{x \to \infty} x\big(\pi/2 - \tan^{-1}(x)\big)$.

Problem

$$\lim_{u\to 0^+}\left(\frac{1}{1-e^{-u}}-\frac{1}{u}\right)$$

is called an $\infty-\infty$ form. Evaluate this limit converting it into one of the previous forms.

The basic strategy for indeterminate differences

- An *indeterminate difference* is any limit of the form

$$\lim_{x\to a}\big(f(x)-g(x)\big)$$

  in which f and g simultaneously approach $+\infty$ or $-\infty$.
- To handle an $\infty-\infty$ form, use algebra to convert this form into one of the other forms.

The basic strategy for indeterminate powers

- An *indeterminate power* is any limit of the form

$$\lim_{x \to a} f(x)^{g(x)}$$

resulting in 0^0, ∞^0 and 1^∞.

- In each of these cases, first write

$$f(x)^{g(x)} = \exp\big(g(x) \ln f(x)\big).$$

The exponent, $g(x) \ln f(x)$, will be in one the preceding forms and can be handled by those methods.

Problem

- *Find* $\displaystyle\lim_{x \to \infty} \left(1 + \frac{1}{x^2}\right)^x$.
- *Find* $\lim_{x \to \infty} x^{1/x}$.
- *Find* $\lim_{x \to 0^+} x^{\sin(x)}$.