§11.9–Representations of Functions as Power Series

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Spring Semester
2015

Outline

The geometric series

Recall that, for $-1 < u < 1$,

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots = \sum_{n=0}^{\infty} u^n.$$

This gives us a representation of $1/(1-u)$ as a power series.

Problem

Give power series representations centered at 0 for the following related functions:

1. $\dfrac{1}{1+x}$
2. $\dfrac{1}{1-x^2}$
3. $\dfrac{3}{5-x}$

Theorem

Let the power series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ have radius of convergence $R > 0$. Then f can be differentiated and integrated term-by-term (that is, just like a polynomial) for $x \in (a-R, a+R)$.

Problem

Starting from

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n,$$

show that

$$\left(\frac{1}{1-x}\right)^2 = \sum_{n=1}^{\infty} nx^{n-1}, \quad \text{for } |x| < 1.$$

Problem

Use power series to express the integral $\int_0^{1/10} \frac{1}{1+x^5} dx$ *as a series. Use the first four terms of the series to approximate the integral and estimate the error in the approximation.*

Problem

Show that

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} x^{n+1}$$

for $|x| < 1$.

Problem

Show that $\ln(2) = \sum_{n=0}^{\infty} \frac{1}{(n+1)2^{n+1}}$

Simple examples The calculus of power series The natural log and arctangent functions

Problem

Show that

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}, \quad \textit{for } |x| < 1.$$