

Assignment #9

Name Answer Key

Due 25 March 2015

1. Consider the infinite series $\sum_{k=1}^{\infty} k e^{-k}$.

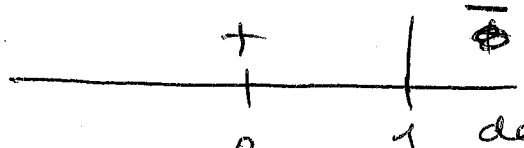
3 (a) Evaluate $\int x e^{-x} dx$. $u = x \quad dv = e^{-x} dx$
 $du = dx \quad v = -e^{-x}$

$$\begin{aligned} \int x e^{-x} dx &= -x e^{-x} + \int e^{-x} dx \\ &= -x e^{-x} - e^{-x} + C \\ &= -e^{-x}(x+1) + C \end{aligned}$$

- (b) Show that the two conditions of the integral test are satisfied by the series and show that the series converges. Let $f(x) = x e^{-x}$.

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① f is non-negative because $x \geq 1$ and $e^{-x} > 0$.

② $f'(x) = (1-x)e^{-x}$ 
 f is decreasing on $[1, \infty)$.

$$\int_1^{\infty} x e^{-x} dx = \lim_{b \rightarrow \infty} \int_1^b x e^{-x} dx = \lim_{b \rightarrow \infty} \left(\frac{-(b+1)}{e^b} + \frac{2}{e} \right)$$

$= 0 + 2 = 2$, by l'hôpital.
 Since $\int_1^{\infty} x e^{-x} dx$ converges, $\sum_{k=1}^{\infty} k e^{-k}$ converges.

- (c) Let L denote the sum of the series and let $s_n = \sum_{k=1}^n k e^{-k}$ be the n th partial sum of the series. Compute s_4 , expressing your answer as a decimal.

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$$s_4 = 1e^{-1} + 2e^{-2} + 3e^{-3} + 4e^{-4} \\ \approx .861174$$

Let $R = L - s_4$, the remainder.

- (d) Find upper and lower bounds for the error in approximating L by s_4 .

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$$\int_5^{\infty} x e^{-x} dx \leq R \leq \int_4^{\infty} x e^{-x} dx$$

$$\text{But } \int_4^{\infty} x e^{-x} dx = \lim_{b \rightarrow \infty} \left(-\frac{(b+1)}{e^b} + \frac{5}{e^4} \right) = \frac{5}{e^4}$$

$$\int_5^{\infty} x e^{-x} dx = \lim_{b \rightarrow \infty} \left(-\frac{(b+1)}{e^b} + \frac{6}{e^5} \right) = \frac{6}{e^5}$$

$$\frac{6}{e^5} \leq R \leq \frac{5}{e^4}$$

$$.0404 \leq R \leq .0915$$