Assignment #7 **Name** Answer Key

Due 8 March 2015

1. Evaluate the following integrals:

(a) $\int_0^\infty x^2 e^{-x} dx$

$I = \int x^2 e^{-x} dx \qquad u = x^2 \quad dV = e^{-x} dx$

$du = 2x dx \quad V = -e^{-x}$

$I = -x^2 e^{-x} + 2\int x e^{-x} dx \qquad u = x \quad dV = e^{-x} dx$

$du = dx \quad V = -e^{-x}$

$I = -x^2 e^{-x} - 2x e^{-x} + 2\int e^{-x} dx$

$I = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$

$\int_0^\infty x^2 e^{-x} dx = \lim_{b\to\infty} \int_0^b x^2 e^{-x} dx =$

$\lim_{b\to\infty} \left(\frac{-b^2}{e^b} - \frac{2b}{e^b} - \frac{2}{e^b} + 0 + 0 + 2 \right) = 2.$

(b) $\int_0^1 x \ln(x) dx = \lim_{c\to 0^+} \int_c^1 x \ln(x) dx.$

side calc.

$I = \int x \ln x \, dx$

$u = \ln x \quad dV = x$

$du = \frac{1}{x} dx \quad V = \frac{x^2}{2}$

$I = \frac{x^2}{2}\ln(x) - \int \frac{x}{2} dx$

$= \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C$

$= \lim_{c\to 0^+} \left(\frac{x^2}{2}\ln(x) - \frac{x^2}{4} \right) \Big|_c^1$

$= \lim_{c\to 0^+} \left(\frac{1}{2}\ln(1) - \frac{1}{4} - \frac{c^2}{2}\ln(c) - \frac{c^2}{4} \right)$

$\lim_{c\to 0^+} c^2 \ln(c) = \lim_{c\to 0} \frac{\ln(c)}{c^{-2}} = 0$

Check: $\lim_{c\to 0^+} \frac{\frac{1}{c}}{-2c^{-3}} = \lim_{c\to 0^+} \frac{-c^2}{2} = 0$

$\therefore \int_0^1 x \ln(x) dx = \frac{1}{2}\ln(1) - \frac{1}{4} - 0 - 0 = -\frac{1}{4}$

(c) $\int_0^1 \frac{x^3}{\sqrt{1-x^2}}dx$

$$I = \int \frac{x^3}{\sqrt{1-x^2}}dx \qquad x = \sin\theta, \quad dx = \cos\theta \, d\theta$$

$$= \int \frac{\sin^3\theta \cancel{\cos\theta} \, d\theta}{\cancel{\cos(\theta)}} = \int \sin^3(\theta)\, d\theta$$

$$= \int (1-\cos^2(\theta)) \sin(\theta)\, d\theta. \qquad u = \cos\theta, \quad du = -\sin\theta\, d\theta$$

$$= \int (1-u^2)(-du) = -\left(u - \frac{u^3}{3}\right) + C$$

$$= \frac{\cos^3(\theta)}{3} - \cos(\theta) + C$$

$$= \frac{(1-x^2)^{3/2}}{3} - (1-x^2)^{1/2} + C.$$

$$\int_0^1 \frac{x^3}{\sqrt{1-x^2}}dx = \lim_{c\to 1^-} \int_0^c \frac{x^3}{\sqrt{1-x^2}}dx$$

$$= \lim_{c\to 1^-} \left\{ \frac{(1-c^2)^{3/2}}{3} - (1-c^2)^{1/2} - \frac{1}{3} + 1 \right\}$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$