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Test #1 **Name** Answer Key

Show all of your work

1. Prove that $\ln(xa) = \ln(x) + \ln(a)$ for all positive real numbers a and x.

Let $f(x) = \ln(xa) - \ln(x) - \ln(a)$.

3 $f'(x) = \frac{1}{xa} \cdot a - \frac{1}{x} - 0 = 0 \quad \therefore f(x) = c.$

$f(1) = \ln(a) - \ln(1) - \ln(a) = 0 \quad \therefore c = 0$

Thus $\ln(x a) = \ln(x) + \ln(a)$.

2. The half-life of radium-226 is 1590 years.

(a) What is the decay constant k?

3 $k = \frac{\ln(.5)}{1590} = -.000435942$

(b) How much of a 100g sample of radium-226 will be left after 1000 years?

3 $A(1000) = 100\, e^{k(1000)}$

$= 64.67\text{ g}$

3. \$1000.00 is invested into an account with an annual interest rate of 5%.

(a) What is the value of the account in 10 years if interest is compounded monthly?

3 $A(10) = 1000\left(1 + \frac{.05}{12}\right)^{12(10)} = 1647.01$

(b) What is the value of the account in 10 years if interest is compounded continuously?

3 $A(10) = 1000\, e^{(.05)(10)} = 1648.72$

(c) If the interest is compounded continuously, then in how many years will the account triple in value?

3 $3000 = 1000\, e^{(.05)t}$

$\ln(3) = (.05)\, t \qquad t = \frac{\ln(3)}{.05}$

$= 21.97$ years.

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4. Find all solutions to each of the following equations:

(a) $\log_4(x) + \log_4(5) = 2$

$\log_4(5x) = 2$

3

$5x = 2^4 = 16$

$x = 16/5$

(b) $3e^{2x} = 8e^x + 3$

$3e^{2x} - 8e^x - 3 = 0$ $u = e^x$

3

$e^x = -\frac{1}{3}$ no solution

$3u^2 - 8u - 3 = 0$

$e^x = 3$ or $x = \ln 3$

$(3u+1)(u-3) = 0$

$u = -1/3, u = 3$

5. Find a formula for the inverse function of $f(x) = \dfrac{1}{e^x + 1}$.

5 $x = \dfrac{1}{e^y + 1}$ $e^y = \frac{1}{x} - 1$

$(e^y + 1) = \frac{1}{x}$ $y = \ln\left(\frac{1}{x} - 1\right)$.

6. Let $f(x) = 4x + \ln(2x)$ for $x > 0$.

(a) Show that f is one-to-one on its domain.

3 $f'(x) = 4 + \frac{1}{2x}(2) = 4 + \frac{1}{x}$

if $x > 0$, then $f'(x) > 0$ ∴ increasing and one-to-one.

(b) $(f^{-1})'(2)$.

3 $f(x) = 4x + \ln(2x) = 2$

$x = \frac{1}{2}$

$(f^{-1})'(2) = \dfrac{1}{f'(\frac{1}{2})} = \dfrac{1}{4+2} = \dfrac{1}{6}$.

7. Differentiate the following functions. Use any proper method.

(a) $y = \ln\left(\dfrac{(2+3x)^2}{(x^4+1)^{1/2}}\right) = 2\ln(2+3x) - \frac{1}{2}\ln(x^4+1)$

$$y' = 2\,\frac{1}{2+3x}(3) - \frac{1}{2}\,\frac{1}{x^4+1}(4x^3).$$

3

(b) $y = x^6 + 7^x$

$$y' = 6x^5 + 7^x \ln(7)$$

3

(c) $y = x^{1/x} = e^{\ln(x^{\frac{1}{x}})} = e^{\frac{\ln(x)}{x}}$.

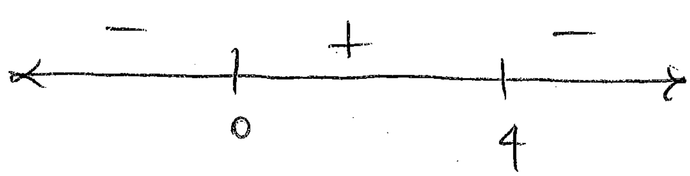
3 $$y' = e^{\frac{\ln(x)}{x}}\left\{\frac{x\frac{1}{x} - \ln(x)}{x^2}\right\}$$

$$= e^{\frac{\ln(x)}{x}}\left(\frac{1-\ln(x)}{x^2}\right)$$

8. Let $f(x) = e^{-x/2}x^2$. On what interval(s) is f increasing? On what interval(s) is f decreasing?

5 $$f'(x) = e^{-x/2}\left(-\frac{1}{2}\right)x^2 + e^{-x/2}(2x)$$

$$= x e^{-x/2}\left\{-\frac{x}{2} + 2\right\} = \frac{1}{2}x(4-x)e^{-x/2}$$



f is increasing on $(0, 4)$

f is decreasing on $(-\infty, 0)$ and $(4, +\infty)$.

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9. Evaluate the following integrals:

(a) $\int_3^9 \frac{1}{x\log_3(x)}dx$

3

Let $u = \log_3(x)$

$$du = \frac{1}{x\ln(3)}dx$$

$$= \ln(3)\int_1^2 \frac{1}{u}du = \ln(3)\ln(|u|)\Big|_1^2$$

$$= \ln(3)\left(\ln(2) - \ln(1)\right)$$

$$= \ln(3)\ln(2).$$

(b) $\int xe^{-x^2}dx$

3

Let $u = -x^2$

$$du = -2x\,dx.$$

$$-\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$$

(c) $\int_{\pi/3}^{\pi/2} \frac{\cos(x)}{1-2\sin(x)}dx$

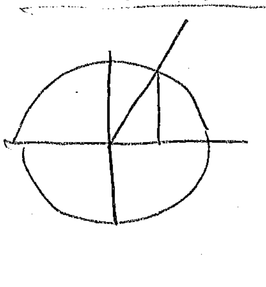
3

Let $u = 1 - 2\sin(x)$

$$du = -2\cos(x)$$

$$-\frac{1}{2}\int_{1-\sqrt{3}}^{-1} \frac{du}{u} = -\frac{1}{2}\ln(|u|)\Big|_{1-\sqrt{3}}^{-1} = -\frac{1}{2}\ln(1) + \frac{1}{2}\ln(|1-\sqrt{3}|)$$

$$= \frac{1}{2}\ln(\sqrt{3}-1)$$



$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ $\sin\left(\frac{\pi}{2}\right) = 1$