Assignment #12 **Name** Answer Key

Due 27 April 2015

1. Find the Maclaurin series for $\sinh(x)$ and $\cosh(x)$.

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$f(x) = \sinh(x)$ $f(0)=0$
$f'(x) = \cosh(x)$ $f'(0)=1$
$f''(x) = \sinh(x)$ $f''(0)=0$
$f'''(x) = \cosh(x)$ $f'''(0)=1$
$\vdots$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$f(x) = \cosh(x)$ $f(0)=1$
$f'(x) = \sinh(x)$ $f'(0)=0$
$f''(x) = \cosh(x)$ $f''(0)=1$
$f'''(x) = \sinh(x)$ $f'''(0)=0$
$\vdots$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

2. Show that the Maclaurin series $\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$ converges $\cos(x)$ for each x by showing that $|R_N(x)| \to 0$ as $N \to \infty$.

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Let $f(x) = \cos(x)$. Then $f^{(N+1)}(x) = \begin{cases} \cos(x) \\ -\cos(x) \\ \sin(x) \\ -\sin(x) \end{cases}$

Thus $|f^{(N+1)}(z)| \le 1$ for z between 0 and x. Thus

$$|R_N(x)| \le \frac{1 \cdot |x|^{N+1}}{(N+1)!}$$

Since $\frac{|x|^{N+1}}{(N+1)!} \longrightarrow 0$ as $N \to \infty$, $|R_N(x)| \longrightarrow 0$ as $N \to \infty$.

3. In each case, find the *exact* sum of the series:

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(a) $\sum_{n=1}^{\infty}(-1)^n \frac{3^n}{n!}$ $= \sum_{n=1}^{\infty} \frac{(-3)^n}{n!} = e^{-3} - 1$

(b) $\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!}$ $= \sum_{n=0}^{\infty} (-1)^n \frac{(\pi/2)^{2n+1}}{(2n+1)!} = \sin\left(\frac{\pi}{2}\right) = 1.$

(c) $\sum_{n=2}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n)!}$ $= \left(\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}\right) - 1 + \frac{\pi^2}{2!}$

$= \cos(\pi) - 1 + \frac{\pi^2}{2!} = -2 + \frac{\pi^2}{2}$

4. In each case, find the Maclaurin series for the given function.

(a) $x^5 e^{3x}$. $= x^5 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^{n+5}$

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(b) $(1 + x^2)\sin(x)$ $= \sin x + x^2 \sin x$

$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) + \left(x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \dots\right)$

$= x + x^3\left(1 - \frac{1}{3!}\right) - x^5\left(\frac{1}{3!} - \frac{1}{5!}\right) + x^7\left(\frac{1}{5!} - \frac{1}{7!}\right)$

$= x + \sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)!} - \frac{1}{(2n+1)!}\right) x^{2n+1}$