§11.10–Taylor and Maclaurin Series

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Spring Semester
2015

Taylor's Theorem When does a function equal its Taylor series? Important examples: e^x, $\sin(x)$, $\cos(x)$ Newton's binomial theorem

Outline

Overview

In the last section we learned that *some* functions can be expressed as power series. In this section we explore a general method for expressing a function as a power series:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a)^1 + c_2(x-a)^2 + \cdots$$

Some questions:

- Which functions have power series representations?
- If a function can be represented by a power series, what are the coefficients $\{c_n\}$?
- Why is it helpful for a function to be represented by a power series?

Problem (The form of the series)

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a)^1 + c_2(x-a)^2 + \cdots$$

in some symmetric interval about the point a. Show that the coefficients must be of the form

$$c_n = \frac{f^{(n)}(a)}{n!}, \quad n \geqslant 0.$$

Definition

- Given a function f and a point a, the series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

  is called a Taylor series for f centered at a.
- When $a = 0$, the series is called a Maclaurin series.

Problem

Find the Maclaurin series for e^x, $\sin(x)$, *and* $\cos(x)$. *In each case, find the radius of convergence of the resulting series.*

Definition

- For each $N \geqslant 0$, let
$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
This is called the *Taylor polynomial of order N for* $f(x)$.
- Let $R_N(x) = f(x) - T_N(x)$. The function R_N is called the *remainder*.
- It follows that
$$f(x) = \underbrace{T_N(x)}_{\text{Taylor polynomial}} + \underbrace{R_N(x)}_{\text{remainder}}$$

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Example

$$e^x = \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}}_{T_4(x)} + R_4(x) \quad \text{or}$$
$$e^x = \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}}_{T_5(x)} + R_5(x).$$

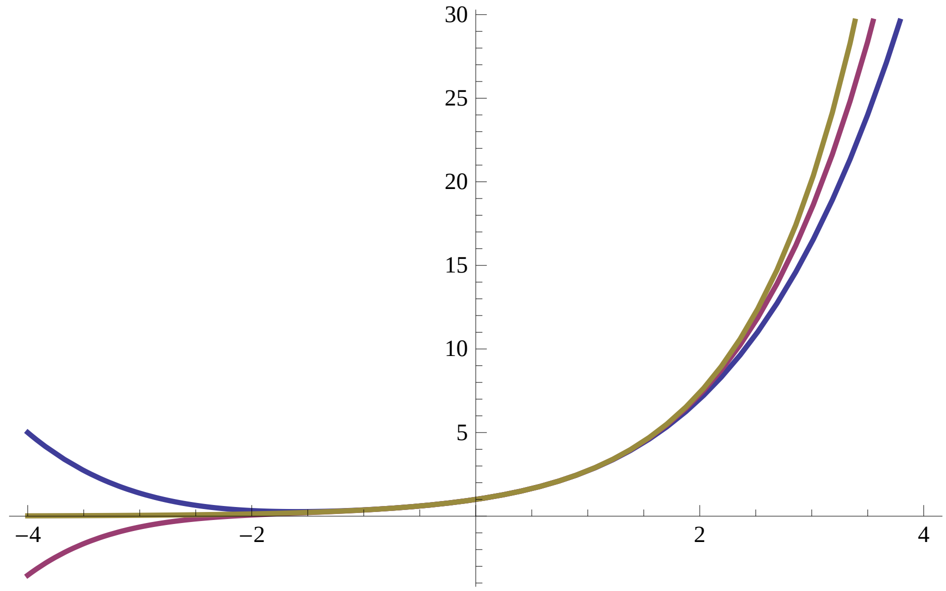

Figure : The graphs of e^x (gold), $T_4(x)$ (blue), and $T_5(x)$ (red).

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Theorem

If $R_N(x) \to 0$ as $N \to \infty$, then

$$f(x) = \lim_{N\to\infty} T_N(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

that is, the function f is equal to its Taylor series at x.

Estimating the remainder

Theorem (Taylor's Inequality)

- *Let $R_N(x)$ be the Nth remainder for the function f and center a.*
- *Let M be any number such that*

$$|f^{(N+1)}(z)| \leqslant M$$

  for all z between a and x.
- *Then*

$$|R_N(x)| \leqslant \frac{M}{(N+1)!}|x-a|^{n+1}.$$

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Problem

Let $R_N(x)$ be the Nth remainder for $f(x) = \sin(x)$ centered at $a = 0$. Show that

$$|R_N(x)| \leqslant \frac{|x|^{N+1}}{(N+1)!}.$$

for any real number x.

Problem

Show that the same is true for $f(x) = \cos(x)$ centered at $a = 0$.

A helpful result!

Theorem

For each $x \in \mathbb{R}$, $\dfrac{x^n}{n!} \to 0$ as $n \to \infty$.

Problem

Show that e^x, $\sin(x)$ and $\cos(x)$ are represented by their Maclaurin series for all x.

Problem

In each case, find the sum of the series:

- $S = \sum_{n=0}^{\infty} \frac{1}{n!}$
- $S = 5 - 5^3/6 + 5^5/120 - 5^7/5040 + \cdots$
- $S = \frac{2}{1!} + \frac{4}{2!} + \frac{8}{3!} + \frac{16}{4!} + \cdots.$

Definition (Falling factorial)

Given $x \in \mathbb{R}$ and an integer $n \geqslant 1$, let

$$(x)_n = \underbrace{(x)(x-1)(x-2)\cdots(x-n+1)}_{n \text{ terms}}$$

Let $(x)_0 = 1$ for completeness.

Problem

Compute $(5/3)_4$ *and* $(-1/2)_3$.

Definition (Binomial coefficients)

Given $x \in \mathbb{R}$ and an integer $n \geqslant 0$, let

$$\binom{x}{n} = \frac{(x)_n}{n!}$$

Problem

Evaluate $\binom{-1/2}{3}$.

Problem

Let $k \in \mathbb{R}$ and let $f(x) = (1+x)^k$. Show that the Maclaurin series for f is

$$\sum_{n=0}^{\infty} \binom{k}{n} x^n$$

Find the radius of convergence of the series.

Theorem (Newton's binomial theorem)

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n.$$

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Problem

Find the Maclaurin series for $\sqrt{1-x}$. For which x does the series represent the function?

Problem

Find the Maclaurin series for $1/\sqrt{9+x}$. *For which* x *does the series represent the function?*

Problem

Express $I = \int_0^1 e^{-x^2} dx$ *as an infinite series.*

Approximate the integral to within an error of $.001$.