Assignment #2 **Name** Answer Key

23 January 2015

1. Following our proofs from class, prove the following identities:

   (a) $\ln(x^p) = p\ln(x)$ for p rational and $x > 0$.

Let $f(x) = \ln(x^p) - p\ln(x)$.

$$f'(x) = \frac{1}{x^p} p x^{p-1} - p\frac{1}{x}$$

$$= \frac{p}{x} - \frac{p}{x} = 0$$

Thus $f(x) = c$.

But $f(1) = \ln(1) - p\ln(1) = 0$

Thus $f(x) = 0$ and $\ln(x^p) = p\ln(x)$.

   (b) $e^{b-a} = e^b/e^a$ for real numbers a and b.

$\ln(e^{b-a}) = b-a$ and

$\ln(e^b/e^a) = \ln e^b - \ln e^a = b-a$

$\therefore e^{b-a} = \frac{e^b}{e^a}$, since $\ln(x)$ is one-to-one.

2. Evaluate the integrals:

   (a) $\int xe^{-x^2}dx$

Let $u = -x^2$ $du = -2x\,dx$.

$$= -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + c$$

$$= -\frac{1}{2}e^{-x^2} + c.$$

   (b) $\int_e^{e^5} \frac{1}{x\ln(x)}dx$

Let $u = \ln(x)$ $du = \frac{1}{x}dx$.

$$= \int_1^5 \frac{1}{u}du = \ln(|u|)\Big|_1^5 = \ln(5) - \ln(1)$$

$$= \ln(5)$$

3. Let $y = \dfrac{x^3\sqrt{x^2+4}}{(x^4+16)^4}$. Use logarithmic differentiation to find y'.

$$\ln(y) = 3\ln(x) + \frac{1}{2}\ln(x^2+4) - 4\ln(x^4+16).$$

$$\frac{1}{y}y' = \frac{3}{x} + \frac{1}{\cancel{2}}\frac{1}{x^2+4}(\cancel{2}x) - 4\frac{1}{x^4+16}(4x^3).$$

$$y' = \frac{x^3\sqrt{x^2+4}}{(x^4+16)^4}\left\{\frac{3}{x} + \frac{x}{x^2+4} - \frac{16x^3}{x^4+16}\right\}.$$

4. Let $f(x) = e^x - e^{-x}$. Show that f is one-to-one and evaluate $(f^{-1})'(0)$.

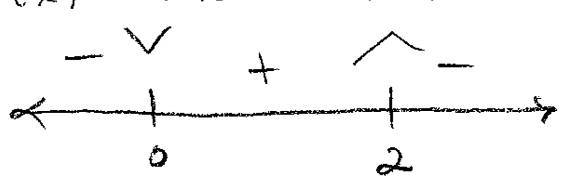
$f'(x) = e^x + e^{-x} > 0$. Since f is increasing, f is one-to-one.

$e^x - e^{-x} = 0$ for $x = 0$ $\therefore$

$$(f^{-1})'(0) = \frac{1}{f'(0)} = \frac{1}{2}.$$

5. Let $f(x) = x^2e^{-x}$. Find the intervals on which f is increasing; find the intervals on which f is decreasing; find and classify the local extreme values of f.

$$f'(x) = 2x\,e^{-x} + x^2(-1)e^{-x} = x\,e^{-x}(2-x).$$



f is decreasing on $(-\infty, 0)$ and $(2, +\infty)$.

f is increasing on $(0, 2)$.

$f(0) = 0$ is a local min.

$f(2) = 4e^{-2}$ is a local max.