Practice Test #2. #1

7.8

#20 By integration by parts,

$$\int y e^{-3y} dy = -e^{-3y}\left(\frac{1}{9}+\frac{y}{3}\right).$$

$$\int_2^\infty y e^{-3y} dy = \lim_{b\to\infty} \int_2^b y e^{-3y} dy$$

$$= \lim_{b\to\infty} \left(-e^{-3b}\left(\frac{1}{9}+\frac{b}{3}\right) + e^{-6}\left(\frac{1}{9}+\frac{2}{3}\right)\right)$$

$$= 0 + \frac{7}{9}e^{-6} = \frac{7}{9}e^{-6}$$

↑ by l'hopital's rule.

#28 $$\int \frac{1}{\sqrt{3-x}} dx = \int (3-x)^{-\frac{1}{2}} dx$$

Let $u = 3-x$, $du = -dx$

$$= -\int u^{-1/2} du$$

$$= -2u^{1/2} + C$$

$$= -2(3-x)^{1/2} + C$$

$$\int_2^3 \frac{1}{\sqrt{3-x}} dx = \lim_{b\to 3^-} \int_2^b \frac{1}{\sqrt{3-x}} dx$$

$$= \lim_{b\to 3^-} \left[-2(3-b)^{1/2} + 2(1)^{1/2}\right] = 2.$$

§ 8.1

#10. $f(y) = \frac{y^4}{8} + \frac{1}{4}y^{-2}$, $1 \le y \le 2$

$$f'(y) = \frac{y^3}{2} - \frac{1}{2y^3}$$

$$(f'(y))^2 = \frac{y^6}{4} - \frac{1}{2} + \frac{1}{4y^6}$$

$$\sqrt{(f'(y))^2 + 1} = \sqrt{\frac{y^6}{4} + 1 + \frac{1}{4y^6}}$$

$$= \frac{y^3}{2} + \frac{1}{2y^3}$$

$$\text{arc length} = \int_1^2 \left(\frac{y^3}{2} + \frac{1}{2}y^{-3}\right) dy$$

$$= 33/16.$$

§ 11.1

#12 $a_3 = -1$, $a_4 = -2$, $a_5 = -1$, $a_6 = 1$, etc.

#30. $a_n = \sqrt{\frac{n+1}{9n+1}} = \sqrt{\frac{1+1/n}{9+1/n}} \longrightarrow \sqrt{\frac{1}{9}}$

$$= \frac{1}{3}.$$

#42 $a_n = \ln(n+1) - \ln(n)$

$$= \ln\left(\frac{n+1}{n}\right) = \ln\left(1+\frac{1}{n}\right)$$

$$\longrightarrow \ln(1+0) = 0.$$

#46 $a_n = \dfrac{\cos(n\pi)}{2^n} = \dfrac{(-1)^n}{2^n}$

Method #1

$$\frac{-1}{2^n} \leq \frac{(-1)^n}{2^n} \leq \frac{1}{2^n}$$

But $\dfrac{1}{2^n} \longrightarrow 0$ and $\dfrac{-1}{2^n} \longrightarrow 0$ $\therefore$

$\dfrac{(-1)^n}{2^n} \longrightarrow 0$ by the squeeze thm.

Method #2

$$\left|\frac{(-1)^n}{2^n}\right| = \frac{1}{2^n} \longrightarrow 0$$

Since $|a_n| \to 0$, $a_n \longrightarrow 0$.

#81. $a_1 = 1 \quad a_{n+1} = 3 - \dfrac{1}{a_n}$

$a_2 = 3 - \dfrac{1}{1} = 3-1 = 2 \quad \therefore \quad a_1 \leq a_2.$

Suppose $a_n \leq a_{n+1}$. Then

$$\frac{1}{a_n} \geq \frac{1}{a_{n+1}}$$

or $\quad -\dfrac{1}{a_n} \leq -\dfrac{1}{a_{n+1}}$

thus $3 - \frac{1}{a_n} \le 3 - \frac{1}{a_{n+1}}$

$a_{n+1} \le a_{n+2}$

§ 11.2

#20. $2 + .5 + .125 + .03125 + \cdots$

$$r = \frac{.5}{2} = \frac{1}{4}$$

Since $-1 < r < 1$, the series converges. The sum is $\frac{2}{1-\frac{1}{4}} = \frac{8}{3}$

#22 $\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}} = 10 + \frac{100}{(-9)} + \frac{1000}{81} + \cdots$

$$r = \frac{\left[\frac{100}{(-9)}\right]}{10} = \frac{100}{-9} \times \frac{1}{10} = \frac{10}{-9}$$

since $|r| = \frac{10}{9} \ge 1$, the series diverges.

#32. $\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}$

Note that $\lim_{n\to\infty} \frac{1+3^n}{2^n} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n + \left(\frac{3}{2}\right)^n$

$= +\infty$. Since $a_n \not\to 0$, the series diverges.

#45. $$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+3}\right]$$

$$S_n = \left[\frac{1}{1} - \cancel{\frac{1}{4}}\right] + \left[\frac{1}{2} - \frac{1}{5}\right] + \left[\frac{1}{3} - \frac{1}{6}\right]$$

$$+ \left[\cancel{\frac{1}{4}} - \frac{1}{7}\right] + \left[\frac{1}{5} - \frac{1}{8}\right] + \left[\frac{1}{6} - \frac{1}{9}\right] +$$

$$\cdots + \left[\frac{1}{n} - \frac{1}{n+3}\right]$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$S_n \to 1 + \frac{1}{2} + \frac{1}{3} = 11/6.$$

#52. $$.\overline{46} = .46\,46\,46 \ldots$$

$$= \frac{46}{100} + \frac{46}{100^2} + \frac{46}{100^3} + \cdots$$

$$= \frac{\frac{46}{100}}{1 - \frac{1}{100}} = \frac{46}{99}.$$

§ 11.3 #6. $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$$

$$f(x) = \frac{1}{\sqrt{x+4}} \qquad f'(x) = \frac{-1}{2(x+4)^{3/2}} < 0$$

Thus f is decreasing. Also $f(x) \geq 0$.

Check

$$\int_1^\infty \frac{1}{\sqrt{x+4}} = \lim_{b\to\infty} \int_1^b (x+4)^{-1/2} dx$$

$$= \lim_{b\to\infty} \left[2\sqrt{4+b} - 2\sqrt{5} \right] = +\infty$$

#20 is below

Because the integral diverges, the series diverges.

#24. $\sum_{n=3}^{\infty} n^2 e^{-n}$. Let $f(x) = x^2 e^{-x}$.

Notice that $f(x) \geq 0$ and

$f'(x) = -e^{-x} x(x-2)$

$$\underset{0}{\overset{-}{\longleftarrow}}\Big|\overset{+}{\quad}\Big|\underset{2}{\overset{-}{\longrightarrow}}$$

Thus f is decreasing on $[2, +\infty)$.

Check: $$\int_3^{+\infty} x^2 e^{-x} dx = \lim_{b\to\infty} \int_3^b x^2 e^{-x} dx$$

$$= \lim_{b\to\infty} \left[-\frac{(b^2+2b+2)}{e^b} + \frac{17}{e^3} \right]$$

$$= \frac{17}{e^3}$$ Since the integral converges, the series converges.

#20 $$\sum_{n=1}^{\infty} \frac{1}{n^2+6n+13}$$

$$f(x) = \frac{1}{x^2+6x+13}$$

$$f'(x) = \frac{-2(x+3)}{(x^2+6x+13)^2}$$

$+ \quad | \quad -$

$-3 \qquad$ dec.

f is decreasing on $[-3, +\infty)$.

Also, $f(x) \geq 0$.

Check: $$\int_1^{\infty} \frac{1}{x^2+6x+13}\,dx$$

$$= \lim_{b \to \infty} \int_1^b \frac{1}{(x+3)^2+4}\,dx$$

$$\int \frac{1}{(x+3)^2+4}\,dx = \int \frac{1}{u^2+4}\,du$$

$u = x+3 \qquad u = 2\tan(\theta)$

$du = dx \qquad du = 2\sec^2(\theta)\,d\theta$

$$= \int \frac{2\cancel{\sec^2\theta}\,d\theta}{4\cancel{\sec^2(\theta)}} = \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$$

$$\lim_{b\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{b+3}{2}\right)-\frac{1}{2}\tan^{-1}(2)\right]$$

$$=\frac{1}{2}\frac{\pi}{2}-\frac{1}{2}\tan^{-1}(2)$$

Since the integral converges, the series converges.

#36 (a). $S_{10}=1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+\frac{1}{10^4}$

Let $r=\sum_{n=11}^{\infty}\frac{1}{n^4}$. Then

$$\int_{11}^{\infty}x^{-4}\,dx\le r\le\int_{10}^{\infty}x^{-4}\,dx$$

$$\frac{1}{3993}\le r\le\frac{1}{3000}$$

$$2.5\times10^{-4}\le r\le 3.4\times10^{-4}.$$

§11.4

#14 $$\frac{\left(\frac{\sqrt{n}}{n-1}\right)}{\left(\frac{1}{\sqrt{n}}\right)}=\frac{n}{n-1}=\frac{1}{1-\frac{1}{n}}\to1$$

Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges (PST, $p=\frac{1}{2}$),

$\sum_{n=2}^{\infty}\frac{\sqrt{n}}{n-1}$ diverges.

#20.
$$\frac{\left(\frac{n+4^n}{n+6^n}\right)}{\left(\frac{4}{6}\right)^n} = \left(\frac{n+4^n}{n+6^n}\right)\left(\frac{6^n}{4^n}\right)$$

$$= \frac{\left(\frac{n}{4^n}+1\right)\cancel{4^n}}{\left(\frac{n}{6^n}+1\right)\cancel{6^n}} \cdot \frac{\cancel{6^n}}{\cancel{4^n}} \longrightarrow 1 \quad \text{(l'hôpital's rule).}$$

Since $\sum_{n=1}^{\infty}\left(\frac{4}{6}\right)^n$ converges (ratio test) $\sum_{n=1}^{\infty}\frac{n+4^n}{n+6^n}$ converges.

#26.
$$\frac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}} = \frac{n^2}{n\sqrt{n^2-1}}$$

$$= \frac{1}{\sqrt{1-\frac{1}{n^2}}} \longrightarrow \frac{1}{\sqrt{1}} = 1.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges,

$\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n^2-1}}$ converges.

§ 11.5.

#8. $f(x) = \frac{x}{\sqrt{x^3+2}}$ ① $f(x) \longrightarrow 0$ as $x \to \infty$.

② $f'(x) = \frac{4-x^3}{2(2+x^3)^{3/2}}$ $\quad \longleftarrow \overset{}{\underset{4^{1/3}}{|}} \longrightarrow$ dec.

The sequence is decreasing for $n \geq 2$. Thus the alt. series test applies, and the series converges.

#12. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$

$f(x) = x e^{-x} = \dfrac{x}{e^{x}} \longrightarrow 0$ as $x \to +\infty$

$f'(x) = e^{-x}(1-x)$ $\longleftarrow \; + \; | \; - \; \longrightarrow$

1 dec.

f is decreasing and converging to 0 $\therefore$ the alternating series converges by AST.

#14. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \arctan(n)$

$\left| (-1)^{n-1} \arctan(n) \right| = \arctan(n) \longrightarrow \dfrac{\pi}{2}$ as $n \to \infty$.

Since $a_n \not\to 0$, the series diverges.

#24. $s = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n 5^n}$ when is $\dfrac{1}{n 5^n} < 10^{-4}$?

$\dfrac{1}{5 \cdot 5^5} \cong 6.4 \times 10^{-5}$

$n = 5$

Thus $s_4 \approx s$ to the required tolerance.