# §11.3–The Integral Test and Estimates of Sums

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## Outline

### Theorem

*A series $\sum a_n$ composed of nonnegative terms converges if and only if the sequence of partial sums is bounded above.*



### Theorem

*If $f(x)$ is continuous, nonnegative, and decreasing on the interval $[1, \infty)$, then*

$$\sum_{n=1}^{\infty} f(n) \qquad \textit{and} \qquad \int_1^{\infty} f(x)dx$$

*converge or diverge together.*

## Problem

*Use the integral test to determine the convergence or divergence of the following series:*

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{k}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{k^2}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{k\big(\ln(k)\big)^2}$



# A very important family of series!

## Definition

Let $p > 0$. A $p$-series is any series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p}.$$

## Theorem (The $p$-Series Test (PST))

*The p-series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} \textit{converges} & \textit{if } p > 1 \\ \textit{diverges} & \textit{if } p \leqslant 1 \end{cases}$$



## Partial sum plus remainder

Write

$$\underbrace{\sum_{k=1}^{\infty} \frac{1}{k^2}}_{\text{full sum}} = \underbrace{\sum_{k=1}^{1000} \frac{1}{k^2}}_{\text{partial sum}} + \underbrace{\sum_{k=1001}^{\infty} \frac{1}{k^2}}_{\text{remainder}}$$

- The full sum is what we want.
- The partial sum is our estimate of the full sum.
- The remainder is the error in our estimate.

## The question

How can we estimate the remainder?

## Theorem (Estimating a remainder)

*Let $f$ be a continuous, nonnegative, and monotone decreasing. Then*

$$\underbrace{\int_{N+1}^{\infty} f(x)dx}_{\text{lower estimate}} \leqslant \sum_{k=N+1}^{\infty} f(k) \leqslant \underbrace{\int_{N}^{\infty} f(x)dx}_{\text{upper estimate}}$$



## Problem

*Let $S = \sum_{n=1}^{\infty} ne^{-n}$. Estimate the error in approximating $S$ by summing the first four terms of the series.*