§ 11.6

#8. $\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!}{10^{n+1}} \frac{10^n}{n!} = \frac{n+1}{10} \longrightarrow +\infty$

The series diverges by the ratio test

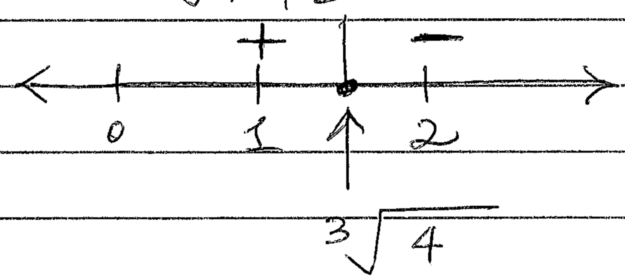
#10. Let $T = \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+2}}$, $S = \sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+2}}$

Notice that

$$\frac{\left(\frac{n}{\sqrt{n^3+2}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \frac{n^{3/2}}{n^{3/2}\sqrt{1+\frac{2}{n^3}}} = \frac{1}{\sqrt{1+2/n^3}} \longrightarrow 1$$

Since $0 < 1 < \infty$, the comparison is valid. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (PST, $p = 1/2$); thus, the series T diverges by LCT.

$$f(x) = \frac{x}{\sqrt{x^3+2}} \qquad f'(x) = \frac{4-x^3}{2(2+x^3)^{3/2}}$$



Thus f is decreasing for $x \geq \sqrt[3]{4}$.

$$\frac{n}{\sqrt{n^3+2}} = \frac{1}{\sqrt{n+\frac{2}{n^2}}} \longrightarrow 0$$

Thus S converges (A.S.T.) and S converges cond.

#20 $\sqrt[n]{|a_n|} = \sqrt[n]{\dfrac{2^n}{n^n}} = \dfrac{2}{n} \longrightarrow 0$

Thus the series converges absolutely by the root test.

#30. $\dfrac{|a_{n+1}|}{|a_n|} = \dfrac{2^{n+1}(n+1)!}{5\cdot 8\cdot 11\cdots(3n+2)(3n+5)} \cdot \dfrac{5\cdot 8\cdots(3n+2)}{2^n n!}$

$$= \frac{2\cdot(n+1)}{(3n+5)} \longrightarrow \frac{2}{3}$$

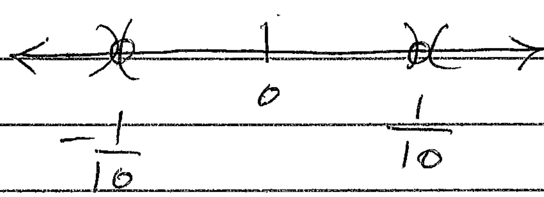
Thus the series converges absolutely by the ratio test.

§11.8

#10. $\dfrac{10^{n+1}|x|^{n+1}}{(n+1)^3} \cdot \dfrac{n^3}{10^n|x|^n}$

$$= 10\,|x| \left(\frac{n}{n+1}\right)^3 \longrightarrow 10|x| \begin{cases} <1 & \text{conv. ab} \\ =1 & ? \\ >1 & \text{div.} \end{cases}$$

$$|x| \begin{cases} < \frac{1}{10} & \text{conv. abs.} \\ = \frac{1}{10} & ? \\ > \frac{1}{10} & \text{diverges.} \end{cases}$$



$R = 1/10$.

We need to test the endpoints:

$x = \frac{1}{10}$: $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n^3}\left(\frac{1}{10}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^3}$

converges by PST, $p=3$.

$x = \frac{-1}{10}$: $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n^3}\left(\frac{-1}{10}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

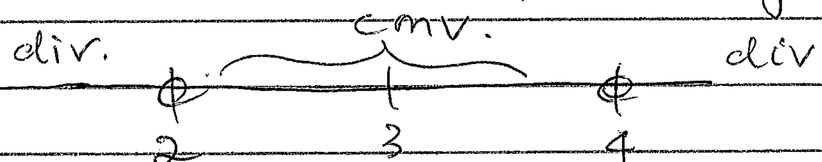
converges absolutely by PST., $p=3$.

Interval of convergence $= \left[-\frac{1}{10}, \frac{1}{10}\right]$

Radius of convergence $R = 1/10$.

#6. $\displaystyle\frac{|x-3|^{n+1}}{2n+3} \cdot \frac{2n+1}{|x-3|^n} = |x-3|\left(\frac{2n+1}{2n+3}\right)$

$\longrightarrow |x-3| \begin{cases} <1 & \text{conv. abs.} \\ =1 & ? \\ >1 & \text{diverges.} \end{cases}$



$x=4$: $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ converges by A.S.T.

$x=2$: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1}$ diverges (LCT. with $\displaystyle\sum_{n=\cancel{0}}^{\infty} \frac{1}{n}$)

Int. of conv. $= (2, 4]$

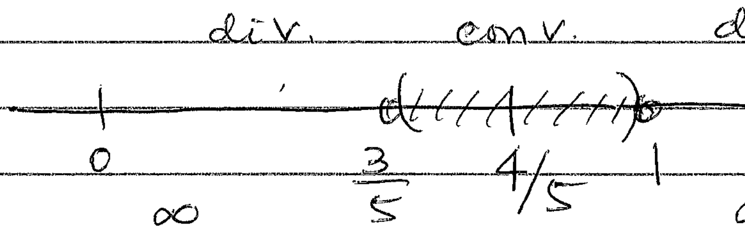
Radius of conv. $= 1$.

#25 $\displaystyle\sum_{n=1}^{\infty} \frac{5^n (x - 4/5)^n}{n^3}$

$$\frac{5^{n+1} \left| x - 4/5 \right|^{n+1}}{(n+1)^3} \cdot \frac{n^3}{5^n |x - 4/5|^n}$$

$$= \left(\frac{n}{n+1}\right)^3 |x - 4/5| \cdot 5 \to 5 \, |x - 4/5|$$

$$|x - 4/5| \begin{cases} < 1/5 & \text{conv. abs} \\ = 1/5 & ? \\ > 1/5 & \text{diverges.} \end{cases}$$



$x = 1$: $\displaystyle\sum_{n=1}^{\infty} \frac{5^n (1/5)^n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$

converges PST, $p = 3$.

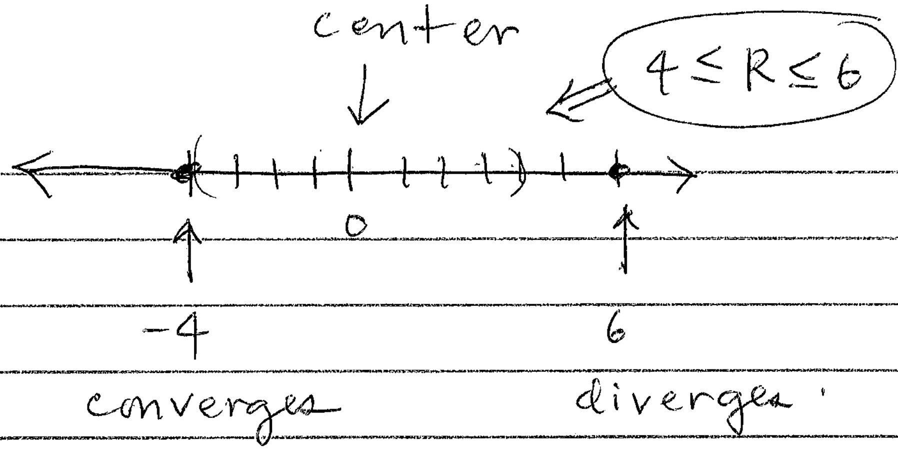
$x = 3/5$: $\displaystyle\sum_{n=1}^{\infty} \frac{5^n (-1/5)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

converges absolutely, PST, $p = 3$.

Int of conv. $= [3/5, 1]$

radius of conv. $= 1/5$.

#30.



(a) $\sum_{n=0}^{\infty} c_n \leftarrow x=1 \therefore$ converges
$(R \geq 4)$

(b) $\sum_{n=0}^{\infty} c_n 8^n \leftarrow x=8 \therefore$ diverges
(radius is at most 6)

(c) $\sum_{n=0}^{\infty} c_n (-3)^n \leftarrow x=-3$ converges.
$(R \geq 4)$.

(d) $\sum_{n=0}^{\infty} (-1)^n c_n 9^n = \sum_{n=0}^{\infty} c_n (-9)^n \leftarrow x=-9$
diverges $(R \leq 6)$.

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§ 11.9 #6.

$$\frac{1}{x+10} = \frac{1}{10} \frac{1}{1-(-x/10)}$$

$$= \frac{1}{10} \sum_{n=0}^{\infty} \left(\frac{-x}{10}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{10^{n+1}} x^n$$

valid for $|x| < 10$.

#10. $$\frac{x^2}{a^3-x^3} = \frac{x^2}{a^3}\left\{\frac{1}{1-\left(\frac{x}{a}\right)^3}\right\}$$

$$= \frac{x^2}{a^3}\sum_{n=0}^{\infty}\left(\frac{x}{a}\right)^{3n}$$

$$= \sum_{n=0}^{\infty}\frac{x^{3n+2}}{a^{3n+3}}$$

valid if $\left|\left(\frac{x}{a}\right)^3\right| < 1$ or $|x| < a$.

#12. $$\frac{x+2}{2x^2-x-1} = \frac{1}{x-1} - \frac{1}{1+2x}$$

$$= -\left(\frac{1}{1-x}\right) - \left(\frac{1}{1-(-2x)}\right)$$

$$= -\left(\sum_{n=0}^{\infty} x^n\right) - \left(\sum_{n=0}^{\infty}(-1)^n 2^n x^n\right)$$

$$= \sum_{n=0}^{\infty}\left(-1-(-1)^n 2^n\right)x^n$$

valid if $|x|<1$ and $|2x|<1$ & thus,
valid if $|x| < 1/2$.

#18. $f(x) = \dfrac{x^3}{(2-x)^3} = \dfrac{x^3}{8(1-\frac{x}{2})^3}$

$$\frac{1}{1-x/2} = \sum_{n=0}^{\infty} \frac{x^n}{2^n}$$

$$+1\left(1-x/2\right)^{-2}\left(+\tfrac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{n\, x^{n-1}}{2^n}$$

$$+\cancel{2}\left(1-x/2\right)^{-3}\left(\tfrac{1}{\cancel{2}}\right)\left(+\tfrac{1}{2}\right) = \sum_{n=2}^{\infty} \frac{(n)(n-1)\, x^{n-2}}{2^n}$$

$$\frac{1}{(1-x/2)^3} = 2\sum_{n=2}^{\infty} \frac{n(n-1)\, x^{n-2}}{2^n}$$

$$f(x) = \frac{\cancel{2}x^3}{\cancel{8}4} \sum_{n=2}^{\infty} \frac{n(n-1)\, x^{n-2}}{2^n}$$

$$= \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n+2}}\, x^{n+1}$$

#32. $\dfrac{x^2}{1+x^4} = x^2\left(\dfrac{1}{1-(-x^4)}\right) = x^2 \displaystyle\sum_{n=0}^{\infty} (-1)^n x^{4n}$

$$= \sum_{n=0}^{\infty} (-1)^n x^{4n+2}$$

$$\int_0^{.3} \frac{x^2}{1+x^4}\,dx = \sum_{n=0}^{\infty} (-1)^n \int_0^{.3} x^{4n+2}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(.3)^{4n+3}}{(4n+3)}$$

$$= .009 - .000312429 \,\Big|\, + (1.61 \times 10^{-7})$$

stop here.

$$\int_0^{.3} \frac{x^2}{1+x^4}\,dx \approx \frac{9}{1000} - \frac{2187}{70,000,000}$$

$$= \frac{627813}{70,000,000}$$

§11.10 #6. $f(x) = \ln(1+x)$ $\quad f(0) = 0$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$ $\quad f'(0) = 1$

$f''(x) = -1(1+x)^{-2}$ $\quad f''(0) = -1$

$f'''(x) = 2 \cdot 1 \, (1+x)^{-3}$ $\quad f'''(0) = 2!$

$f^{(4)}(x) = -3!\,(1+x)^{-4}$ $\quad f^{(4)}(0) = -3!$

$\vdots$ $\quad \vdots$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n}$ $\quad f^{(n)}(0) = (-1)^{n-1}(n-1)!$

The Maclaurin series is thus

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

#16. $f(x) = \frac{1}{x}$; $a = -3$

$f(x) = x^{-1}$ $\qquad f(a) = -1/3$

$f'(x) = -1\, x^{-2}$ $\qquad f'(a) = (-1)(-3)^{-2} = \frac{-1}{3^2}$

$f''(x) = 1 \cdot 2 \cdot x^{-3}$ $\qquad f''(a) = 2!\,(-3)^{-3} = \frac{-2!}{3^3}$

$f'''(x) = -3!\, x^{-4}$ $\qquad f'''(a) = -3!\,(-3)^{-4} = \frac{-3!}{3^4}$

$\vdots$ $\qquad \vdots$

$f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}$ $\qquad f^{(n)}(a) = (-1)^n n!\,(-3)^{-(n+1)}$

$$= \frac{-n!}{3^{n+1}}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{\left(\frac{-n!}{3^{n+1}}\right)}{n!} (x-(-3))^n = \sum_{n=0}^{\infty} \frac{-1}{3^{n+1}} (x+3)^n$$

#26. $\sqrt[3]{8+x} = 2\left(1+\frac{x}{8}\right)^{1/3}$

$$= 2\cdot\sum_{n=0}^{\infty}\binom{1/3}{n}\left(\frac{x}{8}\right)^n \qquad \left|\frac{x}{8}\right|<1$$

or

$$= \sum_{n=0}^{\infty}\frac{2}{8^n}\binom{1/3}{n}x^n \qquad |x|<8$$

$R=8$

#28. $(1-x)^{2/3} = (1+(-x))^{2/3}$

$$= \sum_{n=0}^{\infty}\binom{2/3}{n}(-x)^n$$

$$= \sum_{n=0}^{\infty}\binom{2/3}{n}(-1)^n x^n$$

$|x|<1$

$R=1$

#32. e^x+2e^{-x}

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{2(-1)^n x^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1+2(-1)^n}{n!}\right)x^n$$

#36. $\dfrac{x^2}{\sqrt{2+x}} = x^2 (2+x)^{-\frac{1}{2}}$

$$= x^2 \frac{1}{\sqrt{2}} \left(1+\frac{x}{2}\right)^{-\frac{1}{2}}$$

$$= x^2 \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{2}} \binom{-1/2}{n} \frac{x^{n+2}}{2^n}$$

#54. $\displaystyle\int_0^{.5} x^2 e^{-x^2} dx$

$$= \int_0^{.5} x^2 \left[\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\right] dx$$

$$= \int_0^{.5} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+2} \, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{.5} x^{2n+2} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{x^{2n+3}}{2n+3}\right]\Bigg|_0^{.5}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(.5)^{2n+3}}{(2n+3)}$$

$$= \frac{1}{24} - \frac{1}{160} + \frac{1}{1792} - \cdots$$

stop $\quad$ $< .001$

$$\int_0^{.5} x^2 e^{-x^2}\,dx \approx \frac{17}{480}$$

#68 $\quad$ ~~$3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!}$~~

oops!

#68. $\quad 1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \cdots$

$$= 1 + \frac{(-\ln 2)^1}{1!} + \frac{(-\ln 2)^2}{2!} + \frac{(-\ln 2)^3}{3!} + \cdots$$

$$= e^{-\ln 2} = \frac{1}{2}.$$