§11.8–Power Series

Tom Lewis

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Outline

Problem

For each x, consider the series

$$S(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^n = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \cdots$$

There are two natural questions to ask about this series:

1. *For which values of x, does this converge?*
2. *If this series converges, can we recognize the function?*

Definition

Let $\{c_n\}$ be a sequence of real numbers.

1. A *power series* is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x^1 + c_2 x^2 + \cdots, \quad x \in \mathbb{R}.$$

   The numbers $\{c_n\}$ are called the *coefficients* of the power series.

2. Given a number a, we define the power series centered at a by

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a)^1 + c_2 (x-a)^2 + \cdots, \quad x \in \mathbb{R}.$$

Problem

Where do the following series converge?

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$

Theorem

Given a power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$*, there are three possibilities:*

1. *The series will converge only at* a.
2. *There is a number positive number* R *such that the series converges for* $|x-a| < R$ *and diverges for* $|x-a| > R$.
3. *The series converges for all real numbers.*

The radius of convergence

The number R is called the radius of convergence. We can think of these three cases accordingly: $R = 0$, $0 < R < \infty$, or $R = \infty$.

Definition

The interval of convergence of the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is the interval consisting of all values x for which the power series converges.

Theorem

Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$.

1. *If $R=0$, then the interval of convergence is the point $\{a\}$ only.*
2. *If $R=\infty$, then the interval of convergence is $(-\infty,+\infty)$.*
3. *If $0<R<\infty$, then the interval of convergence can be any one of the following:*

$$(a-R, a+R), \quad [a-R, a+R), \quad (a-R, a+R], \quad [a-R, a+R].$$

Problem

Find the radius and interval of convergence of the power series:

1. $\displaystyle\sum_{n=0}^{\infty} 2^n(x-5)^n$
2. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n(n+1)}$
3. $\displaystyle\sum_{n=1}^{\infty} n!x^n$