Journal of the Operational Research Society (2006) 57, 450–459 0160-5682/06 $30.00

www.palgrave-journals.com/jors

Variables sampling plans with PPM fraction of defectives and process loss consideration

WL Pearn[1]* and Chien-Wei Wu[2]

[1]National Chiao Tung University, Hsinchu, Taiwan; and [2]Feng Chia University, Taichung, Taiwan

Acceptance sampling plans provide the vendor and the buyer decision rules for lot sentencing to meet their product quality needs. A problem the quality practitioners have to deal with is the determination of the critical acceptance values and inspection sample sizes that provide the desired levels of protection to both vendors and buyers. As today's modern quality improvement philosophy, reduction of variation from the target value is the guiding principle as well as reducing the fraction of defectives. The C_{pm} index adopts the concept of product loss, which distinguishes the product quality by setting increased penalty to products deviating from the target. In this paper, a variables sampling plan based on C_{pm} index is proposed to handle processes requiring very low parts per million (PPM) fraction of defectives with process loss consideration. We develop an effective method for obtaining the required sample sizes n and the critical acceptance value C_0 by solving simultaneously two nonlinear equations. Based on the designed sampling plan, the practitioners can determine the number of production items to be sampled for inspection and the corresponding critical acceptance value for lot sentencing.

Journal of the Operational Research Society (2006) **57,** 450–459. doi:10.1057/palgrave.jors.2602013
Published online 23 November 2005

Keywords: acceptance sampling plans; decision making; fraction of defectives; probability density function; process capability indices; quality control

Introduction

Acceptance sampling plans have been one of the most practical tools used in classical quality control applications, which involves quality contract on production orders between the factories and customers. Acceptance sampling plans provide the vendor and the buyer decision rules for lot sentencing while meeting their contract requirements for product quality. A well-designed sampling plan can significantly reduce the difference between the expected and the actual supplied product quantity. Acceptance sampling plans, however, cannot avoid the risk of accepting undesired poor product lots, nor can it avoid the risk of rejecting good product lots unless 100% inspection is implemented. Acceptance sampling plans set the required sample size for product inspection and the associated acceptance or rejection criteria for sentencing each individual product lot. The criteria used for measuring the performance of an acceptance sampling plan is based on the operating characteristic (OC) curve which quantifies the risk for vendors and buyers. The OC curve plots the probability of accepting the lot against actual product fraction defective, which displays the discriminatory power of the sampling plan. That is, the OC curve shows the probability of accepting a product lot in terms of the product fraction defective (nonconforming), which provides the producer and the buyer with a common ground for judging whether the sampling plan is appropriate.

For production quality protection and company's profit, both the vendor and the buyer would focus on certain points on the OC curve to reflect their benchmarking risk. The vendor (supplier) usually would focus on a specific production quality level, called average quality level (AQL), with a high probability of accepting a lot. The AQL also represents the poorest level of quality for the vendor's process that the consumer would consider acceptable as a process average. Therefore, a preferred sampling procedure would be one, which gives a high probability of acceptance at AQL that is normally specified in the contract. On the other hand, the buyer (consumer) would focus on a point at the other end of the OC curve, called lot tolerance per cent defective (LTPD). The LTPD is the poorest level of quality that the consumer is willing to accept. Note that the LTPD is a level of quality specified by the consumer, setting a specified low probability of accepting a lot for product with defect level as high as LTPD. There are a number of different ways of classifying the acceptance sampling plans. One major classification is by attributes and variables. The primary advantage of variables sampling plans is that the same operating characteristic curve can be obtained with a smaller sample size than would be required by an attributes sampling plan. That is, a variables acceptance sampling plan that has the same

Correspondence: WL Pearn, Department of Industrial Engineering and Management, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 30050, Taiwan.
E-mail: wlpearn@mail.nctu.edu.tw

protection as an attributes acceptance sampling plan would require less sampling. The precise measurements required by a variables plan would probably cost more than the simple classification of items required by an attributes plan, but the reduction in sample size, may more than offset this exact expense. Such saving may be especially marked if inspection is destructive and the item is expensive (see Schilling, 1982; Duncan, 1986; Montgomery, 2001).

The basic concepts and models of statistically-based variables sampling plans were introduced by Jennett and Welch (1939). Lieberman and Resnikoff (1955) developed extensive tables and OC curves for various AQLs for MIL-STD-414 sampling plan. Owen (1967) considered variables sampling plans based on the normal distribution, and developed sampling plans for various levels of probabilities of Type I error when the standard deviation is unknown. Das and Mitra (1964) have investigated the effect of non-normality on the performance of the sampling plans. Guenther (1969) developed a systematic search procedure, which can be used with published tables of binomial, hypergeometric, and Poisson distributions to obtain the desired acceptance sampling plans. Stephens (1978) provided a closed form solution for single sample acceptance sampling plans using a normal approximation to the binomial distribution. Hailey (1980) presented a computer program to obtain single sampling plans with minimum sample size based on either the Poisson or binomial distribution. Hald (1981) gave a systematic exposition of the existing statistical theory of lot-by-lot sampling inspection by attributes and provided some tables for the sampling plans. Comparisons between variables sampling plans and attributes sampling plans were investigated by Kao (1971) and Hamaker (1979), who concluded that the expected sample size required by variable sampling is smaller than those for comparable attributes sampling plans. Govindaraju and Soundararajan (1986) developed variables sampling plans that match the OC curves of MIL-STD-105D. Suresh and Ramanathan (1997) developed a sampling plan based on a more general symmetric family of distributions.

Owing to the fact that the sampling cannot guarantee that every defective item in a lot will be inspected, the sampling involves risks of not adequately reflecting the quality conditions of the lot. Such risk is even more significant as the rapid advancement of the manufacturing technology and stringent customer demand is enforced. Particularly, when the required product fraction of defectives is very low, often measured in parts per million (PPM). The required number of inspection items must be enormously large in order to adequately reflect the actual product quality. As today's modern quality improvement philosophy, reduction of the deviation from target is just as important as reducing the fraction of defectives. The capability index C_{pm} adopts the concept of the product loss, which incorporates the variation of production items with respect to the target value and the manufacturing specifications. In this paper, we introduce a variables sampling plan based on C_{pm} index to deal with lot sentencing problem for processes requiring very low fraction of defectives. The proposed sampling plan is based on exact sampling distribution. Hence the decisions made are more accurate and reliable.

Process capability indices approach

Numerous process capability indices, including C_p, C_{pk}, C_{pm}, and C_{pmk}, have been proposed to the manufacturing industry, to provide numerical measures on process performance. Those indices establish the relationships between the actual process performance and the manufacturing specifications that have been the focus of recent research in statistical and quality assurance literatures. The C_p index was developed to measure process precision (product consistency) (see Juran, 1974; Sullivan, 1984, 1985; Kane, 1986). Owing to simplicity of the design, C_p cannot reflect the tendency of process centering.

$$C_p = \frac{USL - LSL}{6\sigma}$$

In order to reflect the deviations of process mean from the target value, several indices similar in nature to C_p have been proposed. Those indices attempt to take the magnitude of process variance as well as process location into consideration. One of those indices is C_{pk} defined as:

$$C_{pk} = \min\left\{\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\}$$

where USL is the upper specification limit, LSL is the lower specification limit, μ is the process mean and σ is the process standard deviation. The C_p and C_{pk} indices are appropriate measures of progress for quality improvement paradigms in which reduction of variability is the guiding principle and process yield is the primary measure of success. However, they are not related to the cost of failing to meet customers' requirement. Taguchi, on the other hand, emphasizes the loss in a product's worth when one of its characteristics departs from the T.

To help account for this, Hsiang and Taguchi (1985) introduced the index C_{pm}, which was also proposed independently by Chan *et al* (1988). The index is related to the idea of squared error loss $loss(X) = (X-T)^2$. This loss-based process capability index C_{pm} has also been called the Taguchi capability index. The index emphasizes on measuring the ability of the process to cluster around the target, which reflects the degrees of process targeting (centering). The index C_{pm} incorporates with the product variation with respect to the target value and the manufacturing specifications preset in the factory. The index C_{pm} is defined as

$$C_{pm} = \frac{USL - LSL}{6\tau} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}} \qquad (1)$$

where $USL-LSL$ is the process tolerance range, d is the half-interval length, and τ is the measure of the average product deviation from the target value T. The term $\tau^2=\sigma^2+(\mu-T)^2=E[(X-T)^2]$ incorporates two variation components: (i) variation to the process mean and (ii) deviation of the process mean from the target. Note that the definition (1) can be rewritten as

$$C_{\mathrm{pm}}=\frac{USL-LSL}{6\sqrt{\sigma^2+(\mu-T)^2}}=\frac{C_{\mathrm{p}}}{\sqrt{1+\xi^2}}$$

where

$$\xi=\frac{\mu-T}{\sigma}$$

Chan *et al* (1988) discussed this ratio and their sampling properties. Boyles (1991) has provided a definitive analysis of C_{pm} and its usefulness in measuring process centering. He notes that both C_{pk} and C_{pm} reduce to C_{p} when $\mu=T$ and decrease as μ moves away from T. However, $C_{\mathrm{pk}}<0$ for $\mu>USL$ or $\mu<LSL$, whereas C_{pm} of process with $|\mu-T|=\Delta>0$ is strictly bounded above by the C_{p} value of a process with $\sigma=\Delta$. That is,

$$C_{\mathrm{pm}}=\frac{USL-LSL}{6|\mu-T|}\tag{2}$$

C_{pm} approaches 0 asymptotically as $|\mu-T|\rightarrow\infty$. While $C_{\mathrm{pk}}=(d-|\mu-T|)/3\sigma$ increases without bound for fixed μ as $\sigma\rightarrow 0$, C_{pm} is bounded above $C_{\mathrm{pm}}<d/(3|\mu-T|)$. The right-hand side of above equation is the limiting value of C_{pm} as $\sigma\rightarrow 0$, and is equal to C_{p} value of a process with $\sigma=|\mu-T|$. It follows from (2) that a necessary condition for $C_{\mathrm{pm}}\geqslant 1$ is $|\mu-T|<d/3$. In addition, based on the C_{pm} index, Ruczinski (1996) obtained a lower bound on the process yield as *Yield* $\geqslant 2\Phi(3C_{\mathrm{pm}})-1$, or equivalently, an upper bound on the fraction of defectives as $\%NC\leqslant 2\Phi(-3C_{\mathrm{pm}})$ for $C_{\mathrm{pm}}>\sqrt{3}/3$.

Pearn *et al* (1992) proposed a third-generation capability index C_{pmk}, which is constructed by combining the merits of the three indices C_{p}, C_{pk} and C_{pm}. The index C_{pmk} alerts the user if the process variance increases and/or the process mean deviates from its target value. It is defined as

$$C_{\mathrm{pmk}}=\min\left\{\frac{USL-\mu}{3\sqrt{\sigma^2+(\mu-T)^2}},\frac{\mu-LSL}{3\sqrt{\sigma^2+(\mu-T)^2}}\right\}$$

We remark that the indices presented above, are designed to monitor the performance for stable normal or near-normal processes with symmetric tolerances. In practice, the process mean μ and the process variance σ^2 are unknown. In order to calculate the index value, sample data must be collected, and a great degree of uncertainty may be introduced into capability assessments due to sampling errors. The approach by simply looking at the calculated values of the estimated indices and then making a conclusion on whether the given process is capable, is highly unreliable since the sampling errors have been ignored. As the use of the capability indices grows more widespread, users are becoming educated and sensitive to the impact of the estimators and their sampling distributions, learning that capability measures must be reported in confidence intervals or via capability testing. Statistical properties of the estimators of those indices under various process conditions have been investigated extensively, including Chan *et al* (1988), Pearn *et al* (1992), Vännman and Kotz (1995), Vännman (1997), Kotz and Lovelace (1998), Pearn *et al* (1998), Borges and Ho (2001), Hoffman (2001), and Zimmer *et al* (2001). Kotz and Johnson (2002) presented a comprehensive review for the development of process capability indices with interpretations and comments on some 170 publications that had appeared during 1992–2000. Spiring *et al* (2003) consolidated the research findings in the field of process capability analysis for the period 1990–2002.

Distribution of the estimated C_{pm}

Owing to the index, C_{pm} involves the unknown parameters μ and σ, which must be estimated from sample. Chan *et al* (1988) and Boyles (1991) proposed two different estimators of C_{pm}, respectively defined as the following:

$$\hat{C}_{\mathrm{pm(CCS)}}=\frac{d}{3\hat{\tau}_{\mathrm{CCS}}}=\frac{d}{3\sqrt{\sum_{i=1}^{n}(x_i-T)^2/(n-1)}}$$

$$=\frac{d}{3\sqrt{s^2+\frac{n}{(n-1)}(\bar{x}-T)^2}}$$

$$\hat{C}_{\mathrm{pm(B)}}=\frac{d}{3\hat{\tau}_{\mathrm{B}}}=\frac{d}{3\sqrt{\sum_{i=1}^{n}(x_i-T)^2/n}}=\frac{d}{3\sqrt{s_n^2+(\bar{x}-T)^2}}$$

where $\bar{x}=\sum_{i=1}^{n}x_i/n$, $s^2=\sum_{i=1}^{n}(x_i-\bar{x})^2/(n-1)$ and $s_n^2=\sum_{i=1}^{n}(x_i-\bar{x})^2/n$. In fact, the two estimators, $\hat{C}_{\mathrm{pm(CCS)}}$ and $\hat{C}_{\mathrm{pm(B)}}$, are asymptotical equivalent. Detailed descriptions and proofs of the properties of $\hat{C}_{\mathrm{pm(CCS)}}$ are given in Chan *et al* (1988). Boyles (1991) considered that it would be more appropriate to replace the factor $n-1$ by n in the denominator since the term $\hat{\tau}_{\mathrm{B}}^2=\sum_{i=1}^{n}(x_i-T)^2/n=s_n^2+(\bar{x}-T)^2$ in the denominator of $\hat{C}_{\mathrm{pm(B)}}$ is the uniformly minimum variance unbiased estimator (UMVUE) of the term $\sigma^2+(\mu-T)^2$. We note that $\bar{x}$ and s_n^2 are the maximum likelihood estimators (MLEs) of μ and σ^2, respectively. Hence, the estimated $\hat{C}_{\mathrm{pm(B)}}$ is also the MLE of C_{pm}. Therefore, for this reason we adapt the estimated index $\hat{C}_{\mathrm{pm}}=\hat{C}_{\mathrm{pm(B)}}$ to evaluate the process performance in this paper.

Further, Boyles (1991) has shown that $\hat{\tau}_{(\mathrm{B})}^2/\tau^2$ is approximately distributed as central χ_v^2/v, where

$v = [2(1+\xi^2)^2]/(1+2\xi^2)$, $\xi = (\mu - T)/\sigma$. An approximate 100 $(1-\alpha)$% lower confidence bound for $C_{\rm pm}$ can be constructed as

$$\hat{C}_{\rm pm(B)} \Big/ \sqrt{\chi^2_{1-\alpha,\hat{v}}/\hat{v}} \tag{3}$$

where $v = [2(1 + \hat{\xi}^2)^2]/(1 + 2\hat{\xi}^2)$ and $\hat{\xi} = (\bar{x} - T)/s_n$. Other approximation formulae have been developed for this confidence interval, such as $\log(\hat{C}_{\rm pm})$ or $\sqrt{\hat{C}_{\rm pm}}$, etc. Subbaiah and Taam (1993) conducted a comparative Monte Carlo study on many of these functions. Under the assumption of normality, Kotz and Johnson (1993) derived the r-th moment of $\hat{C}_{\rm pm}$, and calculated the first two moments, the mean, and the variance of $\hat{C}_{\rm pm}$ for selected values of n, d/σ, $|\mu - T|/\sigma$ and their corresponding values of $C_{\rm pm}$. Zimmer and Hubele (1997) provided tables of exact percentiles for the sampling distribution of the estimator $\hat{C}_{\rm pm}$. Zimmer *et al* (2001) proposed a graphical procedure to obtain exact confidence intervals for $C_{\rm pm}$, where the parameter $\xi = (\mu - T)/\sigma$ is assumed to be a known constant.

In fact, the estimator of $C_{\rm pm}$ can be expressed as $\hat{C}_{\rm pm} = D/(3\sqrt{K + H})$, where $D = n^{1/2}d/\sigma$, $K = ns_n^2/\sigma^2 \sim \chi^2_{n-1}$, $H = n(\bar{x} - T)^2/\sigma^2$, $\delta = n^{1/2}|\mu - T|/\sigma$. Using variable transformation and the integration technique similar to that presented in Vännman (1997), an exactly explicit form of the cumulative distribution function of $\hat{C}_{\rm pm}$ can be derived as

$$F_{\hat{C}_{\rm pm}}(y) = 1 - \int_0^{b\sqrt{n}/(3y)} G\left(\frac{b^2 n}{9y^2} - t^2\right) \times [\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})]\,{\rm d}t \tag{4}$$

for $y > 0$, where $b = d/\sigma$, $\xi = (\mu - T)/\sigma$, $G(\cdot)$ is the cumulative distribution function of the χ^2 distribution with degree of freedom $n-1$, χ^2_{n-1}, and $\phi(\cdot)$ is the probability density function of the standard normal distribution $N(0, 1)$. It is noted that we would obtain an identical equation if we substitute ξ by $-\xi$, into Equation (4) for fixed values of y and n.

Designing $C_{\rm pm}$ variables sampling plans

Consider a variables sampling plan for controlling the lot fraction of defectives (nonconformities (NC)). Since the quality characteristic is a variable, there will exist either an *LSL* or a *USL*, or both, that defined the acceptable values of this parameter. A well-designed sampling plan must provide a probability of at least $1-\alpha$ of accepting a lot if the lot fraction of defectives is at the contracted AQL. The sampling plan must also provide a probability of acceptance no more than β if the lot fraction of defectives is at the LTPD level, the designated undesired level preset by the buyer. Thus, the acceptance sampling plan must have its OC curve passing through those two designated points (AQL, $1-\alpha$) and (LTPD, β). To determine whether a given process is capable, we can first consider the following testing hypothesis

H_0: $p = $ AQL (process is capable),
H_1: $p = $ LTPD (process is not capable).

According to today's modern quality improvement philosophy, customers do notice unit-to-unit differences in these characteristics, especially if the variance is large and/or the mean is offset from the target. With the increasing importance of clustering around the target rather than conforming to specification limits, the understanding of loss functions is the guiding principle to assess the process capability. Therefore, for this reason the $C_{\rm pm}$ index can be used as a quality benchmark for acceptance of a production lot. That is, the null hypothesis with proportion defective, H_0: $p = $ AQL is equivalent to test process capability index with H_0: $C_{\rm pm} \geqslant C_{\rm AQL}$, where $C_{\rm AQL}$ is the level of acceptable quality for $C_{\rm pm}$ index correspond to the lot or process fraction of defectives AQL as $\Phi^{-1}(1-({\rm AQL}/2)\times 10^{-6})/3$. For instance, if the fraction of defectives $p = $ AQL of vendor's product is less than 66 PPM, then the probability of the consumer accepting the lots will be larger than $100(1-\alpha)$%. On the other hand, if the fraction of defectives of vendor's product, $p = $ LTPD, is more than 2700 PPM, then the probability of the consumer accepting would be no more than 100β%. Then, from the relationship between the index value and fraction of defectives, we could obtain the equivalent $C_{\rm AQL} = 1.33$ and $C_{\rm LPTD} = 1.00$ based on the capability index $C_{\rm pm}$. Therefore, the required inspection sample size n and critical acceptance value C_0 for the sampling plans are the solution to the following two nonlinear simultaneous equations.

$$\Pr\{\text{Accepting the lot} \mid \quad p = {\rm AQL}\} \geqslant 1 - \alpha \tag{5}$$

$$\Pr\{\text{Accepting the lot} \mid \quad p = {\rm LTPD}\} \leqslant \beta \tag{6}$$

As described earlier, the sampling distribution of $\hat{C}_{\rm pm}$ is expressed in terms of a mixture of the χ^2 and the normal distributions. Thus, for processes with target value setting to the mid-point of the specification limits (ie $T = M$), the index may be rewritten as: $C_{\rm pm} = d/[3\sigma \times (1 + \xi^2)^{1/2}]$, where $\xi = (\mu - T)/\sigma$. Further, given $C_{\rm pm} = C$, $b = d/\sigma$ can be expressed as $b = 3C(1 + \xi^2)^{1/2}$. The probability of accepting the lot can be expressed as:

$$\begin{aligned}\pi_A(C_{\rm pm}) &= P(\hat{C}_{\rm pm} \geqslant C_0 | C_{\rm pm} = C) \\ &= \int_0^{b\sqrt{n}/(3C_0)} G\left(\frac{b^2 n}{9C_0^2} - t^2\right) \times [\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})]\,{\rm d}t\end{aligned} \tag{7}$$

Therefore, the required inspection sample size n and critical acceptance value C_0 of $\hat{C}_{\rm pm}$ for the sampling plans can be obtained by solving the following two nonlinear simultaneous equations (8) and (9).

$$1-\alpha \leqslant \int_0^{b_1\sqrt{n}/(3C_0)} G\left(\frac{b_1^2 n}{9C_0^2}-t^2\right) \times[\phi(t+\xi\sqrt{n})+\phi(t-\xi\sqrt{n})]\mathrm{d}t \tag{8}$$

$$\beta \geqslant \int_0^{b_2\sqrt{n}/(3C_0)} G\left(\frac{b_2^2 n}{9C_0^2}-t^2\right) \times[\phi(t+\xi\sqrt{n})+\phi(t-\xi\sqrt{n})]\mathrm{d}t \tag{9}$$

where $b_1=3C_{\mathrm{AQL}}(1+\xi^2)^{1/2}$ and $b_2=3C_{\mathrm{LTPD}}(1+\xi^2)^{1/2}$, $C_{\mathrm{AQL}}>C_{\mathrm{LTPD}}$. We note that the required sample size n is the smallest possible value of n satisfying equations (8) and (9), and determining the $\lceil n \rceil$ as sample size, where $\lceil n \rceil$ means the least integer greater than or equal to n.

Critical acceptance value C_0 and the sample size n with parameter ξ

Since the process parameters μ and σ are usually unknown, then the distribution characteristic parameter $\xi=(\mu-T)/\sigma$ is also unknown, which has to be estimated in real applications. Such an approach introduces additional sampling errors from estimating ξ in finding the critical acceptance values and the required sample sizes for inspection. To eliminate the need for estimating the distribution characteristic parameter ξ, we examine the behaviour of the critical acceptance values C_0 and the sample size n against the parameter ξ. We perform extensive calculations to obtain the critical acceptance value C_0 and the sample size n for $\xi=0(0.05)3.00$, with various parameters. Note that the parenthesis $(\cdot)$ in $\xi=0(0.05)3.00$ means ξ from 0 to 3.00 by the increment of the sequence 0.05. Figure 1(a) displays the surface plot for the required sample size n, critical acceptance value C_0, *versus* ξ value for $C_{\mathrm{AQL}}=1.33$, $C_{\mathrm{LTPD}}=1.00$ with $\alpha=0.05$, $\beta=0.05$. Figure 1(b) plots the required sample size n *versus* ξ value for $C_{\mathrm{AQL}}=1.33$, 1.50, 1.67, 2.00, $C_{\mathrm{LTPD}}=1.00$ with $a=0.05$, $\beta=0.05$. Noting that parameter values we investigated, $\xi=0(0.05)3.00$, cover a wide range of applications and it has the same result replace ξ by $-\xi$.

From our analysis, we observe that the required sample size n is decreasing in ξ, and reaches its maximum at $\xi=0$ in all cases. Further, we find that the critical acceptance value C_0 does not change a lot as the ξ increases and stays the same acceptance value with accuracy up to 10^{-2} in all cases (and for $n\geqslant 100$, $\xi=0$ with accuracy up to 10^{-3}). Hence, for practical purpose we may solve equations with $\xi=0$ to obtain the criterion of $\hat{C}_{\mathrm{pm}}$ and the required sample size n, without having to estimate the parameter ξ. This approach ensures that the decisions made based on those criterions are more reliable than all existing methods.

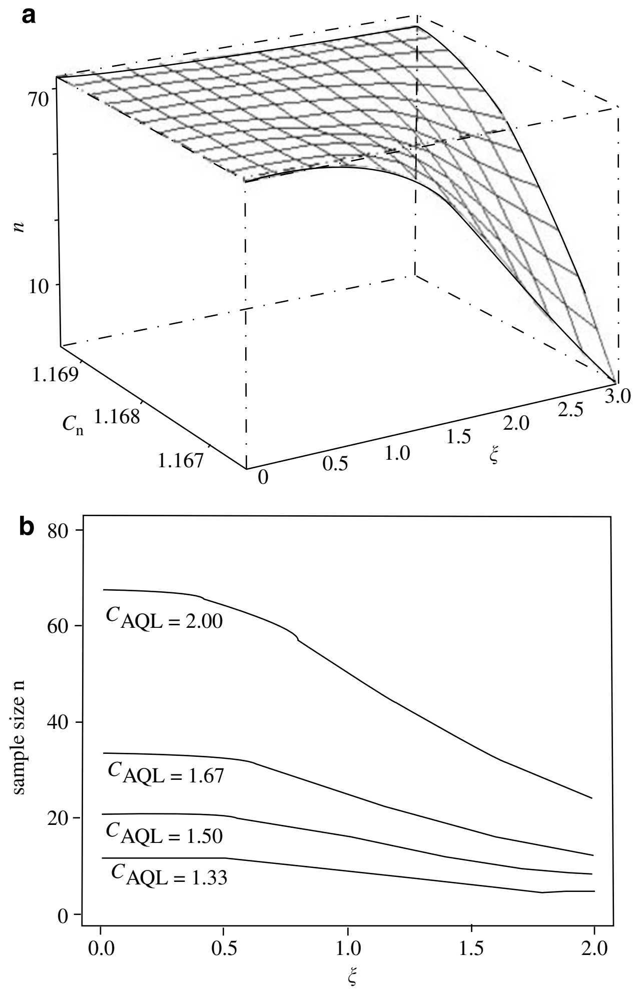

Figure 1 (a) The surface plot for the required sample size n and critical acceptance value C_0, *versus* ξ value for $C_{\mathrm{AQL}}=1.33$, $C_{\mathrm{LTPD}}=1.00$ with $\alpha=0.05$, $\beta=0.05$. (b) The plot of the sample size n *versus* ξ value for $C_{\mathrm{AQL}}=1.33$, 1.50, 1.67, 2.00, $C_{\mathrm{LTPD}}=1.00$ with $\alpha=0.05$, $\beta=0.05$ (from bottom to top in plot).

Solving the nonlinear simultaneous equations

In order to illustrate how we solve the above two nonlinear simultaneous equations (8) and (9), let

$$S_1(n,C_0)=\int_0^{b_1\sqrt{n}/(3C_0)} G\left(\frac{b_1^2 n}{9C_0^2}-t^2\right) \times[\phi(t+\xi\sqrt{n})+\phi(t-\xi\sqrt{n})]\mathrm{d}t-(1-\alpha) \tag{10}$$

$$S_2(n,C_0)=\int_0^{b_2\sqrt{n}/(3C_0)} G\left(\frac{b_2^2 n}{9C_0^2}-t^2\right) \times[\phi(t+\xi\sqrt{n})+\phi(t-\xi\sqrt{n})]\mathrm{d}t-\beta \tag{11}$$

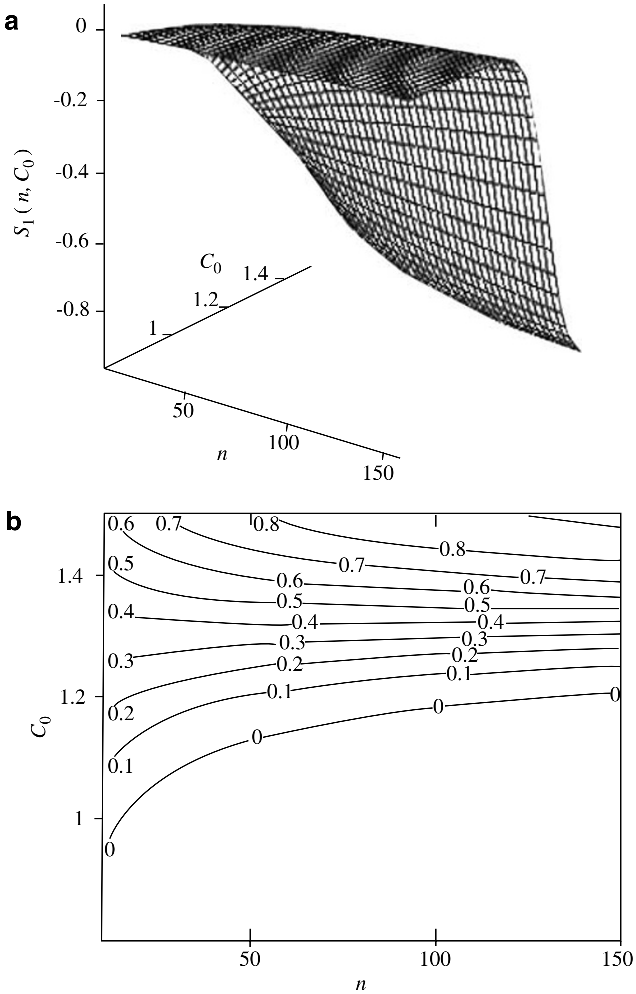

Figure 2 (a) Surface plot of $S_1(n, C_0)$. (b) Contour plot of S_1 (n, C_0).

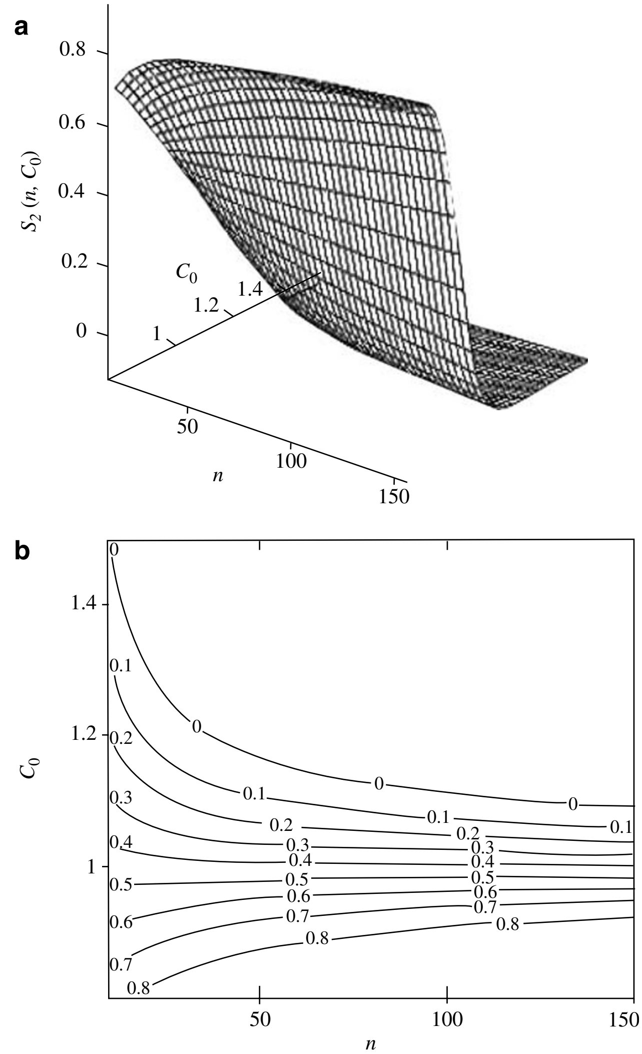

Figure 3 (a) Surface plot of $S_2(n, C_0)$. (b) Contour plot of S_2 (n, C_0).

For $C_{\mathrm{AQL}} = 1.33$ and $C_{\mathrm{LTPD}} = 1.00$, Figures 2(a)–(b) and Figures 3(a)–(b) display the surface and contour plots of Equations (10) and (11) with α-risk $= 0.05$ and β-risk $= 0.05$, respectively.

Figures 4(a) and (b) display the surface and contour plots of Equations (10) and (11) simultaneously with α-risk $= 0.05$ and β-risk $= 0.05$ under $C_{\mathrm{AQL}} = 1.33$ and $C_{\mathrm{LTPD}} = 1.00$, respectively. From the Figure 4(b), we can see that the interaction of $S_1(n,C_0)$ and $S_2(n,C_0)$ contour curves at level 0 is $(n, C_0) = (68, 1.1668)$, which is the solution to nonlinear simultaneous equations (8) and (9). That is, in this case, the minimum required sample size $n = 68$ and critical acceptance value $C_0 = 1.1668$ of the sampling plan based on the capability index C_{pm}.

To investigate the behaviour of the critical acceptance values and required sample sizes, we perform extensive calculations to obtain the solution of (8) and (9) with various parameters. Figure 5(a) displays the required sample size n as surface plot with the probabilities $\alpha = 0.01(0.01)0.10$ and $\beta = 0.01(0.01)0.10$ under $(C_{\mathrm{AQL}}, C_{\mathrm{LPTD}}) = (1.05, 1.00)$. Figure 5(b) displays the critical acceptance value C_0 as surface plot with the probabilities $\alpha = 0.01(0.01)0.10$ and $\beta = 0.01(0.01)0.10$ under $(C_{\mathrm{AQL}}, C_{\mathrm{LPTD}}) = (1.05, 1.00)$. Figures 5(c) and (d) show the required sample size n and the critical acceptance value C_0 as surface plot with $\alpha = 0.01(0.01)0.10$ and $\beta = 0.01(0.01)0.10$ under $(C_{\mathrm{AQL}}, C_{\mathrm{LPTD}}) = (1.10, 1.00)$, respectively.

From Figures 5(a) to (d), we observe that the greater of the risk (α and/or β) which producer or customer could suffer, the smaller is the required sample size n. This phenomenon can be explained intuitively, as if we hope that the chance of wrongly concluding a bad process as good or good lots as bad ones is small, then more sample information is needed to judge the lot quality. Further, for fixed α risk, C_{AQL} and C_{LTPD}, the corresponding critical acceptance values become smaller when the β risk becomes larger. On the other hand, for fixed β risk, C_{AQL} and C_{LTPD}, the corresponding critical acceptance values become larger when the α risk becomes larger. This can also be explained by the same reasoning as above. Consequently, the required sample size is smaller when the difference between C_{AQL} and C_{LTPD} is significant since the judgement will then make it relatively easier to reach the correct decision.

For the purpose of practical applications, we calculate and tabulate the critical acceptance values and required sample sizes for the sampling plans, with commonly used α, β, C_{AQL} and C_{LTPD}. Table 1 displays (n, C_0) values for producer's α-risk = 0.01, 0.025(0.025)0.10 and buyer's β-risk = 0.01, 0.025(0.025)0.10, with various benchmarking quality levels,

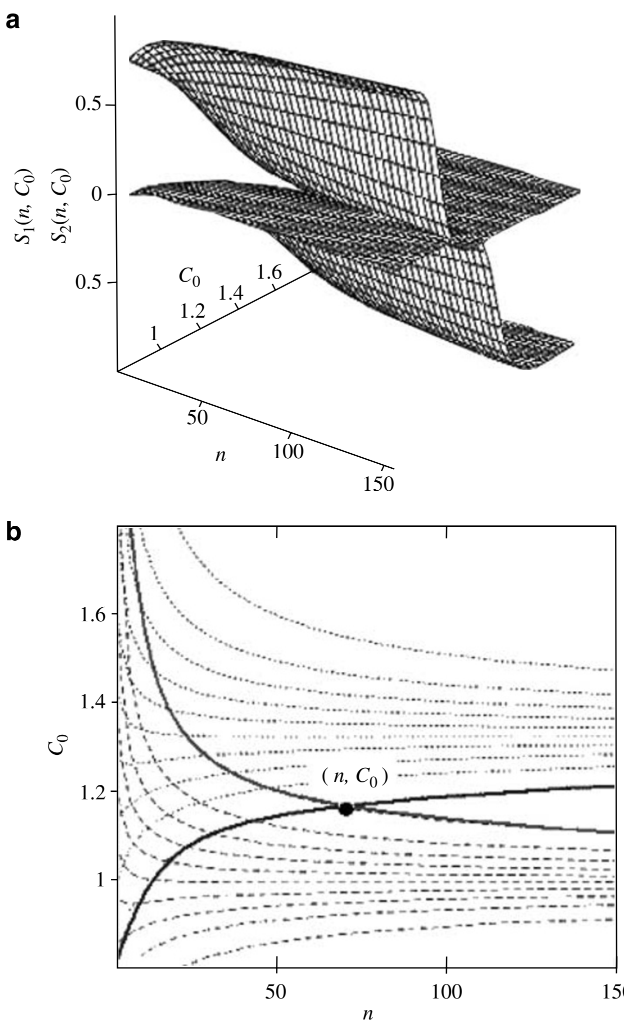

Figure 4 (a) Surface plot of S_1 and S_2. (b) Contour plot of S_1 and S_2.

Figure 5 (a) The critical acceptance value C_0 as surface plot with $\alpha = 0.01(0.01)0.10$ and $\beta = 0.01(0.01)0.10$, under (C_{AQL}, C_{LTPD}) = (1.05, 1.00). (b) The required sample size n as surface plot with $\alpha = 0.01(0.01)0.10$ and $\beta = 0.01(0.01)0.10$, under (C_{AQL}, C_{LTPD}) = (1.05, 1.00). (c) The critical acceptance value C_0 as surface plot with $\alpha = 0.01(0.01)0.10$ and $\beta = 0.01(0.01)0.10$ under (C_{AQL}, C_{LTPD}) = (1.10, 1.00). (d) The required sample size n as surface plot with $\alpha = 0.01(0.01)0.10$ and $\beta = 0.01(0.01)0.10$ under (C_{AQL}, C_{LTPD}) = (1.10, 1.00).

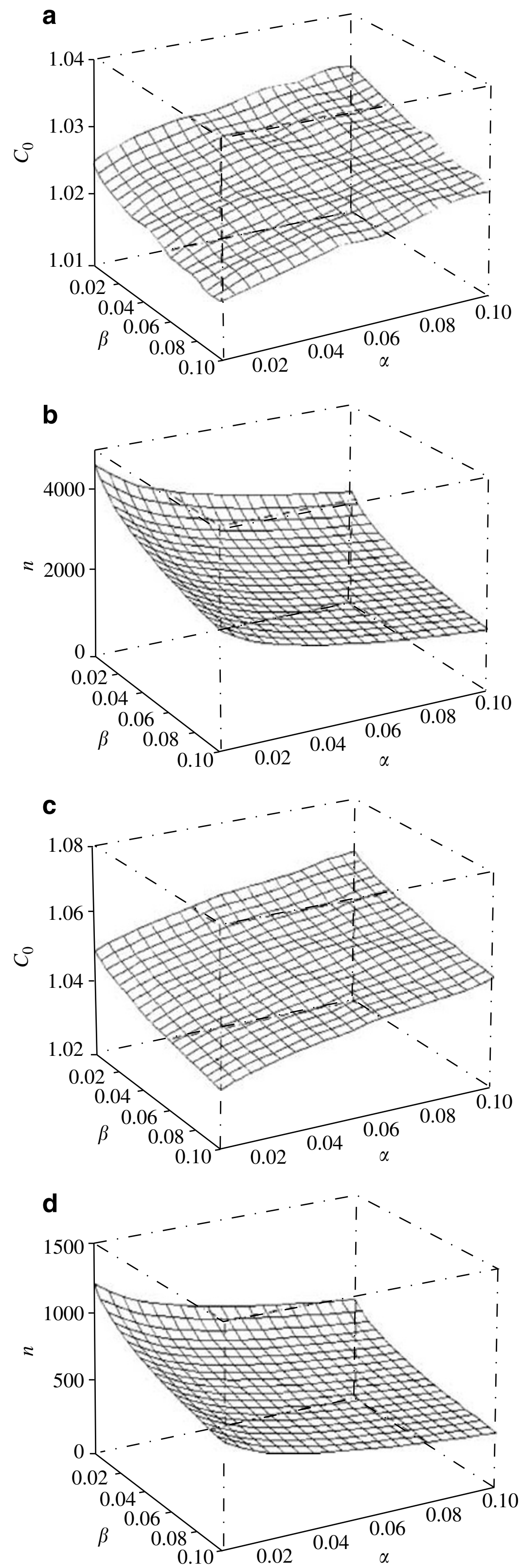

Table 1 (n, C_0) values for α-risk = 0.01, 0.025(0.025)0.10, with β-risk = 0.01, 0.025(0.025)0.10 with various (C_{AQL}, C_{LPTD})

		$C_{AQL}=1.33$ $C_{LPTD}=1.00$		$C_{AQL}=1.50$ $C_{LPTD}=1.00$		$C_{AQL}=1.50$ $C_{LPTD}=1.33$		$C_{AQL}=1.67$ $C_{LPTD}=1.33$		$C_{AQL}=1.67$ $C_{LPTD}=1.50$		$C_{AQL}=2.00$ $C_{LPTD}=1.67$	
		n	C_0	n	C_0	n	C_0	n	C_0	n	C_0	n	C_0
0.010	0.010	135	1.1640	67	1.2477	750	1.4148	211	1.4992	941	1.5848	335	1.8343
	0.025	113	1.1504	56	1.2276	632	1.4076	177	1.4851	794	1.5776	281	1.8206
	0.050	95	1.1369	47	1.2076	539	1.4005	150	1.4710	677	1.5705	239	1.8068
	0.075	85	1.1269	42	1.1929	482	1.3951	133	1.4606	606	1.5651	213	1.7966
	0.100	77	1.1185	38	1.1805	441	1.3907	122	1.4518	554	1.5606	195	1.7880
0.025	0.010	116	1.1786	59	1.2702	641	1.4222	181	1.5141	803	1.5922	287	1.8488
	0.025	96	1.1651	48	1.2502	532	1.4150	150	1.5001	668	1.5850	238	1.8351
	0.050	79	1.1515	40	1.2301	447	1.4078	125	1.4859	562	1.5778	199	1.8212
	0.075	70	1.1414	35	1.2152	396	1.4023	110	1.4753	497	1.5723	176	1.8108
	0.100	63	1.1328	32	1.2026	358	1.3977	100	1.4663	450	1.5680	159	1.8019
0.050	0.010	102	1.1934	52	1.2930	554	1.4296	158	1.5293	694	1.5996	249	1.8634
	0.025	83	1.1803	42	1.2736	454	1.4226	128	1.5156	569	1.5926	203	1.8560
	0.050	68	1.1668	34	1.2539	375	1.4154	106	1.5015	471	1.5854	168	1.8362
	0.075	59	1.1567	30	1.2391	328	1.4099	92	1.4909	412	1.5799	146	1.8257
	0.100	53	1.1480	26	1.2264	294	1.4052	82	1.4818	369	1.5751	131	1.8167
0.075	0.010	93	1.2045	47	1.3101	501	1.4352	143	1.5406	627	1.6051	226	1.8743
	0.025	74	1.1918	38	1.2914	406	1.4284	115	1.5273	508	1.5983	182	1.8612
	0.050	61	1.1786	31	1.2722	332	1.4212	94	1.5135	416	1.5912	149	1.8477
	0.075	52	1.1686	26	1.2576	288	1.4158	81	1.5030	361	1.5857	129	1.8373
	0.100	46	1.1600	23	1.2451	256	1.4111	72	1.4939	321	1.5810	114	1.8283
0.100	0.010	86	1.2139	44	1.3246	462	1.4399	133	1.5502	578	1.6098	209	1.8835
	0.025	69	1.2016	35	1.3067	371	1.4333	106	1.5373	464	1.6032	167	1.8709
	0.050	55	1.1888	28	1.2881	300	1.4263	85	1.5239	376	1.5962	135	1.8576
	0.075	47	1.1790	24	1.2740	258	1.4209	73	1.5136	324	1.5908	116	1.8474
	0.100	42	1.1706	21	1.2618	228	1.4162	65	1.5046	286	1.5861	102	1.8385

(C_{AQL}, C_{LPTD}) = (1.33, 1.00), (1.50, 1.00), (1.50, 1.33), (1.67, 1.33), (1.67, 1.50), (2.00, 1.67). Based on the designed sampling plan, the practitioners can determine the number of production items to be sampled for inspection and the corresponding critical acceptance value. For example, if the benchmarking quality level (C_{AQL}, C_{LPTD}) is set to (1.33, 1.00) with the producer's α-risk = 0.01 and buyer's β-risk = 0.05, then the corresponding sample size and critical acceptance value can be obtained as (n, C_0) = (95, 1.1369). The lot will be accepted if the 95 inspected production items yield measurements with $\hat{C}_{pm} \geqslant 1.1369$.

Sampling procedures and decision making

Both producer and consumer will set their requirements in the contract: the producer demands that not too many 'good' lots shall be rejected, and the consumer demands that not too many 'bad' lots shall be accepted. In choosing a sampling plan attempts will be made to meet these somewhat opposing requirements. Thus, the first step for judging whether a given process meets the capability requirement is to determine the specified value of the capability requirement C_{AQL} and C_{LTPD}, and the α-risk, β-risk. That is, if production process capability with $C_{pm} = C_{AQL}$ (in high quality), the probability of acceptance must be greater than $1-\alpha$. And if the producer's capability is only with $C_{pm} = C_{LTPD}$ (in low quality), a consumer accepts no more than β. Then, by checking the Table 1, we would obtain the sample size n and the critical value C_0 based on given values of α-risk, β-risk, C_{AQL} and C_{LTPD}. If the estimated C_{pm} value is greater than the critical value C_0, then the consumer will accept the entire lot. Otherwise, we do not have sufficient information to conclude that the process meets the present capability requirement. In this case, the consumer will reject the lot. For the proposed sampling plan to be practical and convenient to use, a step-by-step procedure is provided below.

Step 1: Decide the process capability requirements (ie set the values of C_{AQL} and C_{LTPD}) and choose the α-risk, the chance of wrongly concluding a capable process as incapable, and the β-risk, the chance of wrongly concluding a bad lot as good one.

Step 2: Check Table 1 to find the critical value C_0 and the required sample size n for inspection based on given values of α-risk, β-risk, C_{AQL} and C_{LTPD}.
Step 3: Calculate the value of $\hat{C}_{pm}$ from these n inspected samples.
Step 4: Make decisions that accept the entire lot if the estimated $\hat{C}_{pm}$ value is greater than the critical value C_0 ($\hat{C}_{pm} > C_0$). Otherwise, we reject the entire lot.

Capability requirements

In a purchasing contract, a minimum value of the PCI is usually specified. If the prescribed minimum value of the PCI fails to be met, the process is determined to be incapable. Otherwise, the process will be determined to be capable. Montgomery (2001) recommended some minimum capability requirements for some special types of processes. In particular, it is recommended that 1.33 for existing processes, and 1.50 for new processes; 1.50 also for existing processes on safety, strength, or critical parameter, and 1.67 for new processes on safety, strength, or critical parameter. The recommended guidelines for minimum quality requirements and the corresponding PPM of NC for those processes are summarized in Table 2. In recent years, many companies have adopted criteria for evaluating their processes that include process capability objectives that are more stringent than those recommended values. For example, the 'six-sigma' programme pioneered by Motorola essentially requires that when the process mean is in control, it will not be closer than six standard deviations from the nearest specification limit. Thus, in effect, it requires that the process capability ratio be at least 2.0.

An application example

To illustrate how the sampling plan can be established and applied to the actual data collected from the factories, we present a case study on an electronic component manufacturer, which developing passive and active components for the personal computers, telecommunications, industrial controls, automotive parts, and avionics. The factory manufactures various types of the resistors. For a particular model of the resistors investigated, the target value is set to $T = 10.0$ mil, and the tolerance of thickness is 2.0 mil, that is, the lower and upper specification limit are set to, $LSL = 8.0$ mil, $USL = 12.0$ mil, respectively. If the characteristic data does not fall within the tolerance (LSL, USL), the lifetime or reliability of the resistors will be discounted. In the contract, the values of C_{AQL} and C_{LTPD} are set to 1.50 and 1.00 with the α-risk = 0.05 and β-risk = 0.10, respectively. That is, the sampling plans must provide a probability of at least 0.95 of accepting the lot if the lot proportion defective is at the $C_{AQL} = 1.50$ (which is equivalent to no more than 6.80 PPM fraction of defectives), and also provide a probability of no more than 0.10 of accepting the lot if the lot proportion defective is at the $C_{LTPD} = 1.00$ (which is equivalent to 2700 PPM fraction of defectives).

Based on the above specified values in the contract, we could find the critical acceptance value and inspected sample size of the sampling plan $(n, C_0) = (26, 1.2264)$ from Table 1. Hence, the inspected samples are taken from the lot randomly and the observed measurements are displayed in Table 3. Based on these inspections, we obtain that

$$\bar{x} = 10.3731,\ s_n^2 = 0.3166, \quad \text{and}$$
$$\hat{C}_{pm} = \frac{USL - LSL}{6\sqrt{s_n^2 + (\bar{x} - T)^2}} = 1.1255$$

Therefore, in this case, the lot will be rejected by the consumer, since the sample estimator from the inspections, 1.1255, is smaller than the critical acceptance value 1.2264 of the sampling plan. We note that if existing sampling plans are applied here, it is almost certain that any sample of 26 resistors taken from the process will contain zero defective items. All the products therefore, will be accepted, which obviously provides no protection to the buyer at all.

Table 2 Some recommended minimum capability requirements of PCI for processes

Production process types	*Capability requirement*	*NC (PPM)*
Existing processes	1.33	66.07
New processes, or existing processes on safety, strength, or critical parameters	1.50	6.80
New processes on safety, strength, or critical parameters	1.67	0.54

Table 3 Sample data with 26 observations (unit: mil)

11.29	10.68	9.66	9.87	10.71	11.23	9.81	9.96	10.45
9.87	9.06	10.79	10.03	10.20	10.19	9.62	9.91	10.56
10.69	10.04	10.45	9.68	9.56	10.46	10.84	9.19	

Conclusions

Process capability indices have been widely used in the manufacturing industry to determine whether a process is capable of reproducing items within a specified tolerance, which provides common quantitative measures on production quality. The index C_{pm} emphasizes on the ability of the process to cluster around the target, which also provides a lower bound on process yield. In this paper, we developed a variables sampling plan based on process capability index C_{pm}, to deal with the lot sentencing problem for situations with very low fraction of defectives. The proposed sampling plan provides a feasible inspection policy, which can be applied to products requiring low fraction of defectives, where classical sampling plans cannot be applied. We developed an analytical method to obtain the critical acceptance values and the corresponding sample size required for inspection, providing the desired levels of protection to both producers and consumers. We also tabulated the required sample size n and the critical acceptance value C_0 for various α-risks, β-risks, and the fraction of defectives of process that correspond to acceptable quality levels. Practitioners can determine the number of required inspection units and the critical acceptance value, and make reliable decisions. For illustrative purpose, we demonstrated the use of the derived results by presenting a case study on resistors manufacturing process.

References

Borges WS and Ho LL (2001). A fraction defective based capability index. *Qua Relia Eng Inter* **17**(6): 447–458.

Boyles RA (1991). The Taguchi capability index. *J Qual Technol* **23**: 17–26.

Chan LK, Cheng SW and Spiring FA (1988). A new measure of process capability: C_{pm}. *J Qual Technol* **20**: 162–175.

Das NG and Mitra SK (1964). The effect of non-normality on sampling inspection. *Sankhya* **26A**: 169–176.

Duncan AJ (1986). *Quality Control and Industrial Statistics*, 5th edn. Irwin: Homewood, III.

Govindaraju K and Soundararajan V (1986). Selection of single sampling plans for variables matching the MIL-STD-105 scheme. *J Qual Technol* **18**: 234–238.

Guenther WC (1969). Use of the binomial, hypergeometric, and Poisson tables to obtain sampling plans. *J Qual Technol* **1**(2): 105–109.

Hailey WA (1980). Minimum sample size single sampling plans: a computerized approach. *J Qual Technol* **12**(4): 230–235.

Hald A (1981). *Statistical Theory of Sampling Inspection by Attributes*. Academic Press Inc.: London.

Hamaker HC (1979). Acceptance sampling for percent defective by variables and by attributes. *J Qual Technol* **11**: 139–148.

Hsiang TC and Taguchi G (1985). A tutorial on quality control and assurance—the Taguchi methods. *ASA Annual Meeting* Las Vegas, Nevada, USA.

Hoffman LL (2001). Obtaining confidence intervals for C_{pk} using percentiles of the distribution of C_p. *Qual Reliability Eng Int* **17**(2): 113–118.

Jennett WJ and Welch BL (1939). The control of proportion defective as judged by a single quality characteristic varying on a continuous scale. *J R Stat Soc (Ser B)* **6**: 80–88.

Juran JM (1974). *Quality Control Handbook*, 3rd edn. McGraw-Hill: New York.

Kane VE (1986). Process capability indices. *J Qual Technol* **18**(1): 41–52.

Kao JHK (1971). MIL-STD-414: Sampling procedures and tables for inspection by variables for percent defective. *J Qual Technol* **3**: 28–37.

Kotz S and Johnson NL (1993). *Process Capability Indices*. Chapman & Hall: London.

Kotz S and Johnson NL (2002). Process capability indices—a review, 1992–2000. *J Qual Technol* **34**(1): 1–19.

Kotz S and Lovelace C (1998). *Process Capability Indices in Theory and Practice*. Arnold: London, UK.

Lieberman GJ and Resnikoff GJ (1955). Sampling plans for inspection by variables. *J Am Stat Assoc* **50**: 72–75.

Montgomery DC (2001). *Introduction to Statistical Quality Control*, 4th edn. Wiley: New York.

Owen DB (1967). Variables sampling plans based on the normal distribution. *Technometrics* **9**: 417–423.

Pearn WL, Kotz S and Johnson NL (1992). Distributional and inferential properties of process capability indices. *J Qual Technol* **24**(4): 216–233.

Pearn WL, Lin GH and Chen KS (1998). Distributional and inferential properties of the process accuracy and process precision indices. *Communications Stat: Theory Methods* **27**(4): 985–1000.

Ruczinski I (1996). *The relation between C_{pm} and the degree of includence. Doctoral Dissertation*. University of Würzberg, Würzberg, Germany.

Schilling EG (1982). *Acceptance Sampling in Quality Control*. Marcel Dekker Inc.: New York.

Spiring FA, Leung B, Cheng SW and Yeung A (2003). A bibliography of process capability papers. *Qual Reliab Eng Int* **19**(5): 445–460.

Stephens LJ (1978). A closed form solution for single sample acceptance sampling plans. *J Qual Technol* **10**(4): 159–163.

Subbaiah P and Taam W (1993). Inference on the capability index: C_{pm}. *Communications Stat: Theory Methods* **22**(2): 537–560.

Sullivan LP (1984). Targeting variability—a new approach to quality. *Qual Prog* **17**(7): 15–21.

Sullivan LP (1985). Letters. *Qual Prog* **18**(4): 7–8.

Suresh RP and Ramanathan TV (1997). Acceptance sampling plans by variables for a class of symmetric distributions. *Commun in Statistics: Simulation and Computation* **26**(4): 1379–1391.

Vännman K and Kotz S (1995). A superstructure of capability indices distributional properties and implications. *Scand J Stat* **22**: 477–491.

Vännman K (1997). Distribution and moments in simplified form for a general class of capability indices. *Commun Stat: Theory Methods* **26**: 159–179.

Zimmer LS, Hubele NF and Zimmer WJ (2001). Confidence intervals and sample size determination for C_{pm}. *Qual Reliab Eng Int* **17**: 51–68.

Zimmer LS and Hubele NF (1997). Quantiles of the sampling distribution of C_{pm}. *Qual Eng* **10**: 309–329.

Received May 2004;
accepted April 2005 after two revisions