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Research

Generalized Confidence Intervals for Process Capability Indices

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The concept of generalized confidence intervals is used to derive lower confidence limits for some of the commonly used process capability indices. For the cases where approximate lower confidence limits are already available, numerical comparisons are made among the available approximations and the generalized lower confidence limit. The numerical results indicate that the generalized confidence interval does provide coverage probabilities very close to the nominal confidence level. Two examples are given to illustrate the results. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Process capability indices (PCIs) are statistical devices used to measure the extent to which the process characteristic X under consideration meets specifications. A capability index is generally a function of the process parameters such as the mean μ, the standard deviation σ, the target value T, the lower specification limit L and the upper specification limit U of X. Of the numerous indices proposed to date, a few commonly used PCIs are

$$\begin{aligned} C_{pk} &= \frac{\min\{U-\mu,\ \mu-L\}}{3\sigma} = \frac{d-|\mu-M|}{3\sigma} \\ C_{pmk} &= \frac{\min\{U-\mu,\ \mu-L\}}{3\sqrt{\sigma^2+(\mu-T)^2}} = \frac{d-|\mu-M|}{3\sqrt{\sigma^2+(\mu-T)^2}} \\ C''_{pk} &= \frac{d^*-A^*}{3\sigma} \end{aligned} \tag{1}$$

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where

$$d^* = \min\{U - T,\, T - L\}, \qquad A^* = \max\left[\frac{d^*(\mu - T)}{U - T},\, \frac{d^*(T - \mu)}{T - L}\right]$$

with $d = (U - L)/2$, $M = (U + L)/2$. It may be noted that $C_{pmk} = C''_{pk} = C_{pk}$ when $\mu = T$. Usually $T = M$ and if $T \neq M$, the situation is described as having 'asymmetric tolerances'. Of the above three indices, the first is 'second-generation' and the last two are 'third-generation' PCIs. The index C_{pk}, having its origin in Japan, is considered as the time-tested capability index and C_{pmk} was introduced by Pearn *et al.*[1]. C''_{pk} is a generalization of C_{pk} for processes with asymmetric tolerances and was introduced by Pearn and Chen[2]. Of the three indices, the last two assume a unique role as they are quite sensitive to deviations from the target value while measuring the capability of the process. For a review of the work on PCIs during the period 1992–2000, see the article by Kotz and Johnson[3]. For a review of even earlier work, see the books by Kotz and Johnson[4] and Kotz and Lovelace[5]; see also Chapter 7 in Montgomery[6]. An extensive bibliography of papers on PCIs during the period 1990–2002 is available in Spiring *et al.*[7].

Kotz and Lovelace[5] (p. 22) point out that 'results of calculations [of PCIs] should always be qualified via confidence intervals, with a discussion of the impact of the sample size and sampling scheme on the index estimation'. Therefore, the interval estimation of PCIs is attempted in this paper. It appears difficult to obtain exact confidence intervals for PCIs based on conventional methods, and only approximate or asymptotic confidence intervals are available in the literature for most PCIs. With respect to process performance, we usually require an assurance regarding the minimum value of the PCI and, hence, lower confidence limits are of interest. Some work in this direction is available in Heavlin[8], Bissell[9], Choi and Owen[10], Chou *et al.*[11], Kushler and Hurley[12], Nagata and Nagahata[13], Chen and Hsu[14], Perakis and Xekalaki[15] and Pearn and Liao[16]. All of these methods give approximate confidence intervals. Bayesian approaches to obtaining a lower bound for C_{pm} have been investigated recently by Wu and Pearn[17] and Lin *et al.*[18].

The present work aims at deriving easily computable alternative methods for the interval estimation of commonly used PCIs. The method we have adopted is that of generalized confidence intervals, introduced by Weerahandi[19]. A detailed discussion is given by Weerahandi[20,21]. A brief introduction is given in the next section, followed by the derivation of the generalized confidence intervals. Numerical results are then reported, followed by two examples. Some concluding remarks appear in the final section.

2. GENERALIZED CONFIDENCE INTERVALS

The concept of generalized confidence intervals assumes significance in those situations where 'standard' pivotal quantities are difficult to obtain. To introduce this idea, let X be a random variable with probability distribution $f(x;\, \theta,\, \delta)$, where θ is the parameter of interest and δ is a nuisance parameter. Suppose that we want to obtain a confidence interval for θ. Let $T(X;\, x,\, \theta,\, \delta)$ be a *generalized pivotal quantity* which depends on the random variable X, its observed value x and the parameters, and suppose that $T(X;\, x,\, \theta,\, \delta)$ satisfies the following conditions.

(A1) The observed value of $T(X;\, x,\, \theta,\, \delta)$, namely $T(x;\, x,\, \theta,\, \delta)$, is free of the nuisance parameter δ.
(A2) The probability distribution of $T(X;\, x,\, \theta,\, \delta)$ is free of any unknown parameters.

A $100(1 - \alpha)\%$ generalized confidence region for θ is then given by $[\theta : T(x;\, x,\, \theta,\, \delta) \in C_{1-\alpha}]$, where $C_{1-\alpha}$ satisfies $P[T(X;\, x,\, \theta,\, \delta) \in C_{1-\alpha}] = 1 - \alpha$. Unlike the usual pivot statistics and confidence intervals, a generalized confidence interval may not satisfy the usual repeated sampling property, and the coverage probability may depend on the nuisance parameter δ.

The generalized confidence interval idea was first introduced in Weerahandi[19]. Since then numerous applications have appeared in the literature. Earlier, Tsui and Weerahandi[22] had introduced the concept of a *generalized p-value*, and the generalized confidence interval idea is in the same spirit. For a detailed discussion and numerous applications, we refer the reader to Weerahandi[20,21].

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3. CONFIDENCE INTERVALS FOR C_{pk}, C_{pmk} AND C''_{pk}

In this paper we develop confidence intervals for the above three indices using the concept of a generalized confidence interval. We assume that the characteristic of interest X follows a normal distribution with mean μ and standard deviation σ. Let the upper and lower specification limits on X be U and L, respectively, and let the target value be T, where $L < T < U$. Let us start with the case of C_{pk}. Thus, we assume that the process monitoring and corrective actions are based on the values of C_{pk} as defined in Equation (1). Consider the sufficient statistics

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \quad \text{and} \quad S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$$

where $(X_1, X_2, \ldots, X_n)$ is a random sample from the production process. Let $\bar{x}$ and s^2 be the observed values of $\bar{X}$ and S^2, respectively. To derive a generalized confidence interval for C_{pk}, we shall now define a generalized pivotal quantity T_1 that is a function of $\bar{X}$, S^2, $\bar{x}$, s^2, and possibly other parameters also, and satisfying the conditions (A1) and (A2) in Section 2. Towards this, let

$$T_\mu = \bar{x} - \sqrt{\frac{n-1}{n}} \frac{((\bar{X} - \mu)/(\sigma/\sqrt{n}))}{\sqrt{(n-1)S^2/\sigma^2}} s = \bar{x} - \sqrt{\frac{n-1}{n}} \frac{Z}{U} s$$

and

$$T_{\sigma^2} = \frac{s^2\sigma^2}{S^2} = \frac{(n-1)s^2}{U^2}$$

where $Z = (\bar{X} - \mu)/(\sigma/\sqrt{n}) \sim N(0, 1)$ and $U^2 = (n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$ are statistically independent. Here, χ^2_m denotes the central chi-square distribution with m degrees of freedom. It is easily verified that the observed values of T_μ and T_{σ^2} (obtained by replacing $\bar{X}$ and S^2 with their observed values $\bar{x}$ and s^2) are μ and σ^2, respectively. Furthermore, the distribution of (T_μ, T_{σ^2}) is free from any unknown parameters. Now define

$$T_1 = \frac{d - |T_\mu - M|}{3\sqrt{T_{\sigma^2}}}$$

It is clear that the observed value of T_1 is C_{pk}, the parameter of interest, and that the distribution of T_1 is free from any unknown parameters. In other words, T_1 satisfies the conditions (A1) and (A2) in Section 2, and a $100(1-\alpha)\%$ generalized lower confidence bound for C_{pk} is given by $T_1(\alpha)$, where $T_1(\alpha)$ satisfies $P[T_1 < T_1(\alpha)] = \alpha$.

The lower confidence limit $T_1(\alpha)$ obviously can be used for testing hypotheses concerning C_{pk}. Thus, suppose that we want to test the claim that C_{pk} exceeds a specified value c_0. We have to test

$$H_0 : C_{pk} \le c_0, \quad H_1 : C_{pk} > c_0$$

Based on the lower confidence limit $T_1(\alpha)$, we reject H_0 when $T_1(\alpha) > c_0$.

The derivation of generalized lower confidence limits for the other two indices C_{pmk} and C''_{pk} can be carried out along the same lines as that for C_{pk}. A generalized pivot statistic can be derived by substituting T_μ and T_{σ^2} in the place of μ and σ^2 in the expression for the indices.

4. SIMULATION RESULTS

Using a simulation study, we now examine the performance of the lower confidence limits for the different PCIs. The interval estimates will be accurate if the estimated coverage probabilities are close to the nominal value.

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Table I. Coverage probabilities and expected values of the lower confidence limits of C_{pk} for 90% and 95% confidence levels

			Coverage probability					Expected value				
C_{pk}	n	$1-\alpha$	G_{pk}	B_{pk}	KH_{pk}	H_{pk}	N_{pk}	$E(G_{pk})$	$E(B_{pk})$	$E(KH_{pk})$	$E(H_{pk})$	$E(N_{pk})$
1	10	0.90	0.9120	0.8898	0.8768	0.9702	0.9075	0.6981	0.7322	0.7604	0.5824	0.7077
		0.95	0.9588	0.9496	0.9423	0.9949	0.9598	0.6081	0.6297	0.6660	0.4379	0.6053
	20	0.90	0.9045	0.8898	0.8748	0.9342	0.9017	0.7909	0.8047	0.8253	0.7600	0.7937
		0.95	0.9563	0.9524	0.9414	0.9783	0.9582	0.7264	0.7351	0.7617	0.6779	0.7241
	30	0.90	0.9042	0.8922	0.8734	0.9224	0.9025	0.8285	0.8363	0.8534	0.8133	0.8293
		0.95	0.9504	0.9476	0.9345	0.9667	0.9525	0.7791	0.7840	0.8059	0.7543	0.7769
	40	0.90	0.9022	0.8939	0.8752	0.9146	0.9021	0.8497	0.8551	0.8700	0.8405	0.8499
		0.95	0.9487	0.9462	0.9349	0.9614	0.9503	0.8084	0.8116	0.8307	0.7928	0.8064
	50	0.90	0.9020	0.8947	0.8760	0.9120	0.9022	0.8668	0.8710	0.8843	0.8606	0.8668
		0.95	0.9517	0.9500	0.9350	0.9622	0.9541	0.8286	0.8311	0.8482	0.8178	0.8269
1.33	10	0.90	0.9031	0.8860	0.8786	0.9702	0.9041	0.9587	0.9943	1.0159	0.7928	0.9616
		0.95	0.9539	0.9478	0.9435	0.9948	0.9597	0.8431	0.8619	0.8896	0.6037	0.8293
	20	0.90	0.9001	0.8897	0.8795	0.9395	0.9006	1.0683	1.0827	1.0985	1.0225	1.0680
		0.95	0.9534	0.9507	0.9444	0.9790	0.9574	0.9860	0.9934	1.0138	0.9164	0.9788
	30	0.90	0.9014	0.8905	0.8778	0.9214	0.9022	1.1135	1.1224	1.1355	1.0912	1.1130
		0.95	0.9497	0.9478	0.9411	0.9679	0.9525	1.0503	1.0548	1.0716	1.0148	1.0454
	40	0.90	0.9020	0.8948	0.8833	0.9160	0.9026	1.1393	1.1458	1.1572	1.1260	1.1388
		0.95	0.9483	0.9470	0.9408	0.9620	0.9503	1.0887	1.0901	1.1048	1.0648	1.0832
	50	0.90	0.9005	0.8947	0.8842	0.9110	0.9010	1.1608	1.1659	1.1761	1.1520	1.1604
		0.95	0.9523	0.9500	0.9419	0.9616	0.9539	1.1122	1.1149	1.1281	1.0970	1.1094
1.5	10	0.90	0.9004	0.8856	0.8797	0.9699	0.9031	1.0907	1.1273	1.1465	0.899 30	1.0904
		0.95	0.9526	0.9479	0.9451	0.9949	0.9593	0.9619	0.9793	1.0039	0.6872	0.9424
	20	0.90	0.8998	0.8892	0.8820	0.9335	0.9008	1.2096	1.2249	1.2390	1.1568	1.2084
		0.95	0.9526	0.9509	0.9443	0.9789	0.9572	1.1180	1.1253	1.1435	1.0381	1.1088
	30	0.90	0.9002	0.8904	0.8806	0.9213	0.9011	1.2593	1.2691	1.2808	1.2338	1.2584
		0.95	0.9496	0.9476	0.9423	0.9680	0.9525	1.1888	1.1935	1.2085	1.1482	1.1829
	40	0.90	0.9011	0.8946	0.8862	0.9161	0.9020	1.2878	1.2949	1.3051	1.2726	1.2871
		0.95	0.9479	0.9467	0.9421	0.9624	0.9504	1.2293	1.2329	1.2460	1.2041	1.2250
	50	0.90	0.9008	0.8937	0.8863	0.9117	0.9011	1.3116	1.3174	1.3265	1.3016	1.3112
		0.95	0.9528	0.9505	0.9435	0.9619	0.9546	1.2575	1.2605	1.2722	1.2402	1.2543
2	10	0.90	0.9000	0.8837	0.8813	0.9705	0.9035	1.4734	1.5150	1.5295	1.2096	1.4657
		0.95	0.9507	0.9483	0.9462	0.9951	0.9590	1.3055	1.3206	1.3392	0.929 40	1.2714
	20	0.90	0.8988	0.8887	0.8844	0.9355	0.9008	1.6223	1.6415	1.6521	1.5501	1.6194
		0.95	0.9514	0.9487	0.9458	0.9791	0.9564	1.5031	1.5110	1.5248	1.3940	1.4890
	30	0.90	0.8988	0.8895	0.8853	0.9204	0.9001	1.6864	1.6992	1.7080	1.6518	1.6849
		0.95	0.9512	0.9484	0.9448	0.9678	0.9544	1.5940	1.5997	1.6111	1.5389	1.5855
	40	0.90	0.9026	0.8959	0.8906	0.9151	0.9032	1.7231	1.7325	1.7402	1.7024	1.7220
		0.95	0.9485	0.9468	0.9440	0.9633	0.9516	1.6468	1.6513	1.6612	1.6126	1.6408
	50	0.90	0.9018	0.8941	0.8886	0.9123	0.9032	1.7543	1.7618	1.7687	1.7406	1.7535
		0.95	0.9527	0.9501	0.9468	0.9623	0.9548	1.6836	1.6874	1.6963	1.6601	1.6791
2.5	10	0.90	0.9000	0.8845	0.8823	0.9716	0.9039	1.8514	1.9005	1.9121	1.5180	1.8389
		0.95	0.9507	0.9489	0.9473	0.9955	0.9591	1.6441	1.6591	1.6741	1.6921	1.5977
	20	0.90	0.8988	0.8884	0.8860	0.9360	0.9003	2.0331	2.0568	2.0654	1.9422	2.0292
		0.95	0.9503	0.9480	0.9466	0.9795	0.9566	1.8856	1.8950	1.9060	1.7483	1.8675
	30	0.90	0.8995	0.8899	0.8870	0.9198	0.9002	2.1123	2.1281	2.1352	2.0687	2.1104
		0.95	0.9510	0.9489	0.9470	0.9686	0.9551	1.9976	2.0046	2.0137	1.9282	1.9868
	40	0.90	0.9029	0.8952	0.8910	0.9158	0.9032	2.1574	2.1691	2.1753	2.1314	2.1560
		0.95	0.9495	0.9472	0.9452	0.9631	0.9523	2.0630	2.0685	2.0764	2.0200	2.0554
	50	0.90	0.9016	0.8942	0.8911	0.9119	0.9027	2.1960	2.2054	2.2110	2.1787	2.1950
		0.95	0.9513	0.9500	0.9475	0.9631	0.9544	2.1087	2.1132	2.1203	2.0789	2.1028

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Table I. *Continued*

			Coverage probability					Expected value				
C_{pk}	n	$1-\alpha$	G_{pk}	B_{pk}	KH_{pk}	H_{pk}	N_{pk}	$E(G_{pk})$	$E(B_{pk})$	$E(KH_{pk})$	$E(H_{pk})$	$E(N_{pk})$
3	10	0.90	0.8986	0.8844	0.8832	0.9719	0.9037	2.2273	2.2850	2.2947	1.8255	2.2111
		0.95	0.9515	0.9491	0.9482	0.9954	0.9588	1.9803	1.9964	2.0090	1.4079	1.9227
	20	0.90	0.8987	0.8888	0.8868	0.9359	0.9004	2.4430	2.4714	2.4785	2.3336	2.4383
		0.95	0.9499	0.9481	0.9473	0.9798	0.9563	2.2671	2.2781	2.2873	2.1017	2.2451
	30	0.90	0.8988	0.8899	0.8875	0.9207	0.8992	2.5376	2.5566	2.5625	2.4851	2.5352
		0.95	0.9507	0.9487	0.9476	0.9691	0.9545	2.4005	2.4086	2.4163	2.3168	2.3873
	40	0.90	0.9014	0.8945	0.8924	0.9165	0.9026	2.5912	2.6052	2.6104	2.5598	2.5895
		0.95	0.9497	0.9472	0.9461	0.9638	0.9528	2.485	2.4850	2.4917	2.4267	2.4693
	50	0.90	0.9009	0.8944	0.8928	0.9113	0.9021	2.6373	2.6485	2.6532	2.6164	2.6361
		0.95	0.9509	0.9496	0.9480	0.9629	0.9536	2.5331	2.5384	2.5444	2.4972	2.5260

For the simulation we used $L = 7$, $U = 14$, $\mu = 10$ and various values of σ that provide $C_{pk} = 1, 1.33, 1.5, 2, 2.5$ and 3. We calculated the expected value of the generalized confidence limit and its coverage probability for $n = 10(10)50$. The generalized confidence limit was computed using 10 000 simulated values of (Z, U^2), keeping the observed values $\bar{x}$ and s^2 fixed. These observed values were generated 10 000 times, resulting in 10 000 values of the generalized lower confidence limit. The estimated coverage probability is, then, the proportion of times the true value of C_{pk} exceeds the generalized lower confidence limit. The expected value of the generalized lower confidence limit is simply the average of the 10 000 values of the generalized lower confidence limit. For each pair of (n, C_{pk}), the coverage probability and the expected value were computed for $1 - \alpha = 0.90$ and 0.95. The results are given in Table I, where we have used the notation G_{pk} for the generalized lower confidence limit of C_{pk}.

The index C_{pk} is considered as the time-tested capability index and it has been used extensively in production circles. The usual estimator of C_{pk} is

$$\hat{C}_{pk} = \frac{\min\{U - \bar{X}, \bar{X} - L\}}{3S} = \frac{d - |\bar{X} - M|}{3S}$$

As the estimator $\hat{C}_{pk}$ is the minimum of two non-central Student's t random variables, the actual probability distribution is complicated; moreover, only approximate confidence intervals are available in the literature. Four important approximate lower confidence limits are reported in Kotz and Lovelace[5]. For a confidence level of $1 - \alpha$, these approximate limits are:

1. Bissell's[9] approximation

$$B_{pk} = \hat{C}_{pk} - Z_{1-\alpha}\sqrt{\frac{1}{9n} + \frac{\hat{C}_{pk}^2}{2(n-1)}}$$

2. Heavlin's[8] approximation

$$H_{pk} = \hat{C}_{pk} - Z_{1-\alpha}\sqrt{\frac{n-1}{9n(n-3)} + \hat{C}_{pk}^2 . \frac{1}{2(n-3)}\left(1 + \frac{6}{n-1}\right)}$$

3. Kushler and Hurley's[12] approximation

$$KH_{pk} = \hat{C}_{pk}\left[1 - \frac{Z_{1-\alpha}}{\sqrt{2(n-1)}}\right]$$

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Table II. Coverage probabilities and expected values of the lower confidence limits of C_{pmk} for 90% and 95% confidence levels

			Coverage probability		Expected value					Coverage probability		Expected value	
C_{pmk}	n	$1-\alpha$	G_{pmk}	CH_{pmk}	$E(G_{pmk})$	$E(CH_{pmk})$	C_{pmk}	n	$1-\alpha$	G_{pmk}	CH_{pmk}	$E(G_{pmk})$	$E(CH_{pmk})$
1	50	0.90	0.9189	0.8655	0.8407	0.8807	2	50	0.90	0.9156	0.8602	1.7351	1.7974
		0.95	0.9604	0.9281	0.7974	0.8444			0.95	0.9565	0.9204	1.6621	1.7342
	60	0.90	0.9192	0.8716	0.8533	0.8879		60	0.90	0.9170	0.8643	1.7555	1.8103
		0.95	0.9574	0.9260	0.8165	0.8565			0.95	0.9553	0.9190	1.6941	1.7573
	70	0.90	0.9134	0.8694	0.8649	0.8958		70	0.90	0.9102	0.8626	1.7742	1.8239
		0.95	0.9572	0.9278	0.8286	0.8644			0.95	0.9571	0.9216	1.7137	1.7710
	80	0.90	0.9139	0.8676	0.8725	0.9006		80	0.90	0.9119	0.8615	1.7863	1.8317
		0.95	0.9549	0.9246	0.8382	0.8718			0.95	0.9542	0.9177	1.7291	1.7827
	90	0.90	0.9112	0.8674	0.8812	0.9067		90	0.90	0.9084	0.8586	1.8012	1.8432
		0.95	0.9563	0.9297	0.8477	0.8782			0.95	0.9538	0.9196	1.7448	1.7942
	100	0.90	0.9142	0.8695	0.8870	0.9110		100	0.90	0.9088	0.8644	1.8103	1.8497
		0.95	0.9588	0.9301	0.8547	0.8817			0.95	0.9559	0.9206	1.7565	1.8031
1.33	50	0.90	0.9190	0.8604	1.1304	1.1812	2.5	50	0.90	0.9117	0.8630	2.2166	2.2771
		0.95	0.9598	0.9227	1.0763	1.1355			0.95	0.9545	0.9205	2.1369	2.2072
	60	0.90	0.9181	0.8678	1.1461	1.1902		60	0.90	0.9114	0.8636	2.2383	2.9221
		0.95	0.9579	0.9208	1.1001	1.1510			0.95	0.9519	0.9188	2.1719	2.2346
	70	0.90	0.9129	0.8631	1.1604	1.2000		70	0.90	0.9095	0.8633	2.2581	2.3073
		0.95	0.9570	0.9224	1.1151	1.1607			0.95	0.9562	0.9221	2.1928	2.2502
	80	0.90	0.9135	0.8636	1.1699	1.2059		80	0.90	0.9076	0.8616	2.2706	2.3160
		0.95	0.9552	0.9206	1.1270	1.1698			0.95	0.9522	0.9170	2.2089	2.2627
	90	0.90	0.9097	0.8621	1.1807	1.2136		90	0.90	0.9067	0.8582	2.2870	2.3294
		0.95	0.9564	0.9239	1.1389	1.1778			0.95	0.9534	0.9195	2.2258	2.2759
	100	0.90	0.9130	0.8663	1.1880	1.2189		100	0.95	0.9072	0.8632	2.2964	2.3362
		0.95	0.9583	0.9262	1.1476	1.1841			0.95	0.9552	0.9195	2.2385	2.2860
1.5	50	0.90	0.9174	0.8586	1.2812	1.3363	3	50	0.90	0.9089	0.8574	2.7620	2.8085
		0.95	0.9596	0.9216	1.2217	1.2858			0.95	0.9547	0.9191	2.6931	2.7489
	60	0.90	0.9182	0.8653	1.2982	1.3462		60	0.90	0.9058	0.8622	2.7804	2.8222
		0.95	0.9573	0.9194	1.2478	1.3031			0.95	0.9494	0.9149	2.7238	2.7743
	70	0.90	0.9124	0.8622	1.3139	1.3571		70	0.90	0.9058	0.8598	2.7971	2.8356
		0.95	0.9573	0.9213	1.2642	1.3139			0.95	0.9536	0.9201	2.7416	2.7881
	80	0.90	0.9129	0.8631	1.3242	1.3635		80	0.90	0.9066	0.8595	2.8072	2.8430
		0.95	0.9552	0.9197	1.2772	1.3237			0.95	0.9501	0.9138	2.7551	2.7986
	90	0.90	0.9097	0.8601	1.3361	1.3722		90	0.90	0.9036	0.8563	2.8217	2.8553
		0.95	0.9558	0.9215	1.2901	1.3327			0.95	0.9513	0.9181	2.7695	2.8104
	100	0.90	0.9123	0.8639	1.3441	1.3778		100	0.90	0.9046	0.8610	2.8291	2.8609
		0.95	0.9581	0.9236	1.2998	1.3397			0.95	0.9511	0.9158	2.7805	2.8192

4. Nagata and Nagahata's[13] approximation

$$N_{pk} = \sqrt{1 - \frac{2}{5(n-1)}}\hat{C}_{pk} - Z_{1-\alpha}\sqrt{\frac{\hat{C}_{pk}^2}{2(n-1)} + \frac{1}{9n}}$$

where $Z_{1-\alpha}$ is the $(1-\alpha)$th percentile of the standard normal distribution. To facilitate a comparison of the above approximate confidence limits and the generalized confidence limit G_{pk}, the coverage probabilities and expected values are given in Table I. The coverage probabilities and expected values of the above approximate confidence limits were also computed based on 10 000 simulations.

Table III. Coverage probabilities and expected values of the lower confidence limits of C''_{pk} for 90% and 95% confidence levels

C''_{pk}	n	$1-\alpha$	G''_{pk}	$\mathrm{E}(G''_{pk})$	C''_{pk}	n	$1-\alpha$	G''_{pk}	$\mathrm{E}(G''_{pk})$
1	10	0.90	0.9249	0.6820	2	10	0.90	0.9036	1.4625
		0.95	0.9643	0.5938			0.95	0.9533	1.2967
	20	0.90	0.9164	0.7809		20	0.90	0.9004	1.6197
		0.95	0.9626	0.7174			0.95	0.9521	1.5003
	30	0.90	0.9141	0.8226		30	0.90	0.8991	1.6858
		0.95	0.9571	0.7734			0.95	0.9515	1.5933
	40	0.90	0.9109	0.8458		40	0.90	0.9026	1.7229
		0.95	0.9527	0.8050			0.95	0.9486	1.6466
	50	0.90	0.9069	0.8644		50	0.90	0.9018	1.7542
		0.95	0.9571	0.8264			0.95	0.9527	1.6836
1.33	10	0.90	0.9125	0.9431	2.5	10	0.90	0.9011	1.8443
		0.95	0.9581	0.8288			0.95	0.9522	1.6365
	20	0.90	0.9064	1.0610		20	0.90	0.8989	2.0321
		0.95	0.9567	0.9792			0.95	0.9506	1.8845
	30	0.90	0.9051	1.1102		30	0.90	0.8995	2.1122
		0.95	0.9528	1.0470			0.95	0.9510	1.9964
	40	0.90	0.9044	1.1376		40	0.90	0.9029	2.1573
		0.95	0.9499	1.0852			0.95	0.9495	2.0630
	50	0.90	0.9021	1.1599		50	0.90	0.9016	2.1960
		0.95	0.9536	1.1113			0.95	0.9513	2.1086
1.5	10	0.90	0.9084	1.0760	3	10	0.90	0.8995	2.2230
		0.95	0.9565	0.9481			0.95	0.9523	1.9753
	20	0.90	0.9036	1.2038		20	0.90	0.8989	2.4427
		0.95	0.9552	1.1125			0.95	0.9499	2.2667
	30	0.90	0.9018	1.2570		30	0.90	0.8988	2.5376
		0.95	0.9514	1.1864			0.95	0.9507	2.4005
	40	0.90	0.9024	1.2867		40	0.90	0.9014	2.5912
		0.95	0.9488	1.2283			0.95	0.9497	2.4785
	50	0.90	0.9018	1.3111		50	0.90	0.9009	2.6373
		0.95	0.9531	1.2571			0.95	0.9509	2.5331

From Table I it is clear that the coverage probabilities of the generalized confidence limits are consistently close to the nominal value. With respect to the coverage probabilities, the only approximation that shows consistently good performance is that of Nagata and Nagahata[13]. We also note that for small values of C_{pk}, the coverage probabilities are slightly smaller for G_{pk} compared with N_{pk}, and the converse is true for large values of C_{pk}. In addition, we note a corresponding ordering among the expected values of these confidence limits. However, the differences are not significant and we conclude that both G_{pk} and N_{pk} are satisfactory lower confidence limits. We also note that, with the exception of Heavlin's[8] approximate limit H_{pk}, all of the limits considered in Table I perform reasonably well. The numerical results indicate that H_{pk} can be quite conservative.

We also carried out simulations to study the behavior of the generalized lower confidence limit G_{pmk} of C_{pmk}, in the same setup as that used for C_{pk}, with the additional choice $T = 10.3$. In the case of C_{pmk}, an asymptotic lower confidence limit was obtained by Chen and Hsu[14]. They found that, under certain regularity conditions, the distribution of the natural estimator $\hat{C}_{pmk}$, namely, $\hat{C}_{pmk} = (d - |\bar{X}_n - M|)/[3\sqrt{S_n^2 + (\bar{X}_n - T)^2}]$, is normal. They obtained the asymptotic $100(1-\alpha)\%$ lower confidence limit, say CH_{pmk}, as

$$CH_{pmk} = \hat{C}_{pmk} - Z_{1-\alpha}\frac{\hat{\sigma}_{pmk}}{\sqrt{n}}$$

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Table IV. 90% and 95% lower confidence limits of C_{pk} for the piston rings data

n	C_{pk}	$1-\alpha$	B_{pk}	H_{pk}	N_{pk}	KH_{pk}	G_{pk}
10	1.22	0.90	0.8301	0.6613	0.8026	0.8541	0.7929
		0.95	0.7186	0.5019	0.6911	0.7493	0.7032
20	1.27	0.90	0.9906	0.9355	0.9771	1.0073	0.9700
		0.95	0.9109	0.8403	0.8974	0.9323	0.9070
30	1.34	0.90	1.1023	1.0717	1.0930	1.1154	1.0860
		0.95	1.0346	0.9954	1.0253	1.0514	1.0277
40	1.43	0.90	1.2145	1.1936	1.2071	1.2252	1.2016
		0.95	1.1525	1.1256	1.1452	1.6625	1.1452
50	1.55	0.90	1.3429	1.3267	1.3365	1.3518	1.3326
		0.95	1.2834	1.2627	1.2770	1.2948	1.2786
60	1.67	0.90	1.4644	1.4513	1.4587	1.4720	1.4502
		0.95	1.4065	1.3896	1.4008	1.4162	1.3999
70	1.58	0.90	1.4043	1.3945	1.3997	1.4117	1.3789
		0.95	1.3532	1.3406	1.3486	1.3627	1.3336
80	1.62	0.90	1.4473	1.4390	1.4432	1.4540	1.4260
		0.95	1.3985	1.3880	1.3944	1.4072	1.3846
90	1.62	0.90	1.4586	1.4518	1.4550	1.4650	1.4470
		0.95	1.4127	1.4039	1.4090	1.4209	1.4057
100	1.62	0.90	1.4660	1.4602	1.4627	1.4721	1.4560
		0.95	1.4225	1.4150	1.4192	1.4303	1.4173
110	1.67	0.90	1.5193	1.5140	1.5162	1.5249	1.5122
		0.95	1.4766	1.4699	1.4735	1.4838	1.4718
120	1.66	0.90	1.5185	1.5139	1.5157	1.5239	1.5146
		0.95	1.4778	1.4720	1.4750	1.4848	1.4756
125	1.62	0.90	1.4866	1.4824	1.4840	1.4920	1.4821
		0.95	1.4476	1.4422	1.4449	1.4545	1.4445

Table V. The transformed amplifier gain data

−1.1	−0.9	0.2	2.6	0.4	−0.6	−1.6	0.9	−0.2	−1.1	−0.4	−0.3	1.4	0.0	−0.6
−1.3	0.8	0.2	−0.4	1.5	0.9	1.1	−0.9	0.3	−0.1	1.1	0.7	−1.6	−0.2	1.1
0.3	−0.3	−0.4	0.7	0.1	−0.4	0.8	0.5	2.0	−0.7	−0.9	−2.0	−0.2	0.5	0.0
−1.1	1.3	−0.1	0.1	−0.7	−0.4	1.6	−0.7	−2.0	0.7	−1.3	0.4	0.8	−0.6	−0.3
−0.7	1.2	0.7	−2.0	0.3	−1.6	−0.2	0.0	−1.1	1.2	−0.1	0.2	−1.3	0.6	0.1
0.9	−1.6	−0.4	1.0	−0.3	−0.1	−1.6	−0.9	−0.6	0.9	−0.7	1.6	2.9	0.6	−0.2
0.5	0.6	−1.3	0.9	1.4	−1.1	0.1	0.0	−0.3	−1.3	1.8	−2.0	0.1	0.5	0.3
−0.7	0.8	0.6	0.0	0.2	1.0	0.4	−2.0	1.4	0.3	−0.1	−0.6	0.1	0.2	−0.2

where

$$\hat{\sigma}^2_{pmk} = \frac{1}{9(1+\lambda^2)} + \frac{2\lambda}{3(1+\lambda^2)^{3/2}}\hat{C}_{pmk} + \frac{144\lambda^2 + (U-L)(m_4/S^4 - 1)}{144(1+\lambda^2)^2}\hat{C}^2_{pmk}$$

$$m_4 = \frac{\Sigma(X_i - \bar{X})^4}{n}, \qquad \lambda = \frac{\bar{X} - T}{S}$$

and $Z_{1-\alpha}$ is the $(1-\alpha)$th percentile of the standard normal distribution. We computed the coverage probabilities and expected values associated with G_{pmk} and CH_{pmk}; these are given in Table II. It is clear that the coverages associated with CH_{pmk} are below the nominal level. However, the performance of G_{pmk} is very satisfactory.

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Table VI. 90% and 95% lower confidence limits of C_{pmk} for the amplifier gain data

n	$1-\alpha$	C_{pmk}	G_{pmk}	CH_{pmk}	n	$1-\alpha$	C_{pmk}	G_{pmk}	CH_{pmk}
10	0.90	0.4301	0.2378	0.3526	70	0.90	0.5286	0.4509	0.4983
	0.95	0.4301	0.1976	0.3306		0.95	0.5286	0.4298	0.4897
20	0.90	0.5169	0.3702	0.4535	80	0.90	0.5586	0.4802	0.5226
	0.95	0.5169	0.3326	0.4355		0.95	0.5586	0.4589	0.5124
30	0.90	0.5741	0.4436	0.5188	90	0.90	0.5453	0.4728	0.5129
	0.95	0.5741	0.4108	0.5031		0.95	0.5453	0.4533	0.5037
40	0.90	0.6191	0.4997	0.5634	100	0.90	0.5573	0.4876	0.5255
	0.95	0.6191	0.4697	0.5475		0.95	0.5573	0.4675	0.5165
50	0.90	0.5627	0.4651	0.5192	110	0.90	0.5589	0.4923	0.5288
	0.95	0.5627	0.4378	0.5069		0.95	0.5589	0.4751	0.5202
60	0.90	0.5467	0.4607	0.5104	120	0.90	0.5491	0.4870	0.5215
	0.95	0.5467	0.4370	0.5001		0.95	0.5491	0.4691	0.5137

Table VII. 90% and 95% lower confidence limits of C''_{pk} for the amplifier gain data

n	$1-\alpha$	C''_{pk}	G''_{pk}	n	$1-\alpha$	C''_{pk}	G''_{pk}	n	$1-\alpha$	C''_{pk}	G''_{pk}
10	0.90	0.5849	0.3490	50	0.90	0.7899	0.6668	90	0.90	0.7615	0.6742
	0.95	0.5849	0.2893		0.95	0.7899	0.6384		0.95	0.7615	0.6506
20	0.90	0.7194	0.5335	60	0.90	0.7923	0.6802	100	0.90	0.7740	0.6888
	0.95	0.7194	0.4874		0.95	0.7923	0.6545		0.95	0.7740	0.6683
30	0.90	0.7827	0.6227	70	0.90	0.7816	0.6777	110	0.90	0.7925	0.7106
	0.95	0.7827	0.5821		0.95	0.7816	0.6536		0.95	0.7925	0.6893
40	0.90	0.8449	0.6977	80	0.90	0.7715	0.6764	120	0.90	0.7831	0.7059
	0.95	0.8449	0.6591		0.95	0.7715	0.6537		0.95	0.7831	0.6851

The performance of C''_{pk} was also evaluated under the same setup. The generalized lower confidence limit is now denoted by G''_{pk}. The coverage probability and the expected value of G''_{pk} are given in Table III. The coverage probabilities indicate that the performance of G''_{pk} is quite satisfactory. As far as we are aware, no confidence limits are available for C''_{pk}.

5. EXAMPLES

We first compute lower confidence limits for C_{pk} based on the piston ring data given in Montgomery[6] (Example 5.3). The quality characteristic of interest is the inside diameter measurements of piston rings. 125 observations in millimeters are available with $L = 73.95$ and $U = 74.05$. The data consist of 25 samples with five observations for each sample. The $\bar{x}$ and s charts for these data, given in Montgomery[6] (p. 226), show that the process is in control. Hence, we proceed under the assumption that we have a single sample of 125 observations from a normal distribution.

The generalized lower confidence limits, together with the various approximate confidence limits, were computed for confidence levels $1-\alpha = 0.90$ and 0.95, and for values of $n = 10, 20, 30, \ldots, 120$ and 125. The computations for $n = 10$ were based on the first ten observations of the piston ring data given in Montgomery[6] (Example 5.3), and similarly for the other values of n. Table IV shows the results. The true values of C_{pk} are also included in the table. We note that, with the exception of the lower confidence limit H_{pk}, the remaining limits are nearly the same. This is to be expected in view of the numerical results shown in Table I.

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To illustrate the computation of the lower confidence limits for C_{pmk} and C''_{pk}, in the context of asymmetric tolerances, we use the data on the boosting ability (gain) of an electronic amplifier, considered in Pearn *et al.*[23]. The amplifier would be considered acceptable if the gain were between 7.75 and 12.25 dB; the target value was 10 dB. Since the original data did not follow a normal distribution, they were suitably transformed to obtain the data shown in Table V, taken from Pearn *et al.*[23].

After the transformation, the specification limits and the target value are $L = -2.31$, $U = 5.06$ and $T = 1$. The 90% and 95% generalized lower confidence limits for G_{pmk} and the Chen and Hsu[14] asymptotic limit CH_{pmk}, for various values of n, are given in Table VI. Table VII shows the values of the generalized lower confidence limit G''_{pk} for the index C''_{pk}. Once again, the computations for $n = 10$ were based on the first ten observations given in Table V, and similarly for the other values of n.

As expected, the limit CH_{pmk} is significantly larger compared with G_{pmk}, since, as already noted, the coverage probability associated with CH_{pmk} falls below the nominal level. The limit CH_{pmk} clearly cannot be recommended for practical use.

6. CONCLUSIONS

The generalized confidence interval approach pursued in this paper has provided a unified methodology for computing confidence limits for various PCIs. In the context of quality control applications, the approach has already been used by Hamada and Weerahandi[24] and Burdick *et al.*[25] for gauge repeatability and reproducibility studies. Our numerical results indicate that the generalized confidence interval approach provides confidence intervals that perform well, in terms of providing coverage probabilities close to the nominal level. Numerical results also show that not all of the approximate confidence limits available in the literature perform satisfactorily. We have investigated the computation of generalized lower confidence limits for three PCIs, namely C_{pk}, C_{pmk} and C''_{pk}, but the generalized confidence interval idea can easily be implemented for computing confidence limits for other PCIs, under the normality assumption.

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