A repetitive group sampling plan by variables inspection for product acceptance determination

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Abstract: Acceptance sampling plans are practical tools for quality assurance applications and provide the producer and the consumer a general rule for lot sentencing to meet their requirements of product quality. A well-designed acceptance sampling plan not only reduces the cost and time of inspection but also provides the desired protection to the producer and the consumer. Thus, a sampling plan having smaller sample size required for inspection would be more desirable and useful especially when inspection is costly and destructive. This paper proposes an efficient and economic variables repetitive group sampling (RGS) plan based on the capability index C_{pk} for lot sentencing. The advantages of the proposed variables RGS plan over existing variables single sampling plan are discussed. Finally, to illustrate the applicability of the proposed variables RGS plan, an example is also provided. [Received 20 October 2012; Revised 21 February 2014; Accepted 25 February 2014]

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1 Introduction

Acceptance sampling has been one of the most practical tools used in classical quality control applications and is concerned with inspection and decision-making regarding acceptance or rejection of products delivered in lots. A typical application of acceptance sampling is as follows: a company receives a shipment of product, i.e., a lot from a supplier. This product is often a component or raw materials used in the company's manufacturing process. A sample is taken from the lot and some quality characteristic of the units in the sample is inspected. On the basis of the information contained in this sample, a decision is made regarding lot disposition. Usually, this decision is either to accept or reject the lot. Sometimes we refer to this decision as lot sentencing. Accepted lots are put into production; rejected lots may be returned to the supplier or may be subjected to some other lot disposition action (see Montgomery, 2009).

There are a number of different ways to classify acceptance sampling plans. One major classification is by data type, i.e., variables and attributes. When a quality characteristic is measurable on a continuous scale and is known to have a distribution of a specified type, a variables acceptance sampling plan may be used as a substitute for an attributes sampling plan, based on sample measurements such as the mean and the standard deviation of the sample. The primary advantage of the variables sampling plan is that the same operating characteristic (OC) curve can be obtained with a smaller sample size than what would be required by an attributes sampling plan. Thus, a variables acceptance sampling plan would require less sampling. The measurement data required by a variables sampling plan would probably cost more per observation than attributes data required by an attributes sampling plan. However, the reduction in sample size obtained may more than offset this increased per-observation cost. Such saving may be especially marked if inspection is destructive and the product is expensive (see e.g., Pearn and Wu, 2007; Montgomery; 2009). A second advantage is that measurement data usually provide more information about the manufacturing process or lot than do attributes data. Generally, numerical measurements of quality characteristics are more useful than simple classification of the item as conforming or non-conforming. It should also be pointed out that when acceptable quality levels (AQLs) are very small, the sample size required by the attributes sampling plans is very large. Under these circumstances, there may be significant advantages in switching to variables sampling plans. Thus, as many manufacturers begin to emphasise allowable numbers of non-conforming in terms of parts per million (PPM) variables sampling becomes very attractive.

The basic concepts and models of variables sampling plans were introduced by Jennett and Welch (1939). Lieberman and Resnikoff (1955) developed extensive tables and OC curves for various AQLs for MIL-STD-414 sampling plan. In 1980, the American National Standards Institute and the American Society for Quality Control released an updated civilian version of MIL-STD-414 known as ASNI/ASQC Z1.9. Owen (1967) considered variables sampling plans based on the normal distribution and developed sampling plans for various levels of probabilities of type 1 error when the standard deviation is unknown. Das and Mitra (1964) investigated the effect of non-normality on the performance of the sampling plans. Bender (1975) considered sampling plans by variables for assuring the percent defective in the case of the product quality characteristics obeying a normal distribution with unknown standard deviation and presented a procedure using iterative computer programme to calculate the non-central *t*-distribution. Suresh and Ramanathan (1997) developed a sampling plan by variables based on a more general symmetric family of distributions.

Sherman (1965) proposed a new type of sampling plan for the inspection of attributes' quality characteristics called the repetitive group sampling (RGS) plan. The operation of the plan is similar to that of the sequential sampling plan. Balamurali and Jun (2006) further extended the concept of RGS to variables inspection for a normally distributed quality characteristic. According to the results obtained in Balamurali and Jun (2006), the variables RGS plan will give the desired protection with minimum average sample number (ASN) compared with the variables single sampling plan. Recently, the RGS plans have been studied by many authors. Examples include Park et al. (2004), Balamurali et al. (2005), Aslam and Jun (2009), Aslam et al. (2011) and Wu (2012a).

Numerous process capability indices (PCIs) such as C_p, C_{pk}, C_{pm} and C_{pmk} have been proposed to provide numerical measures on process performance. Those indices establish the relationships between the actual process performance and the manufacturing specifications and have received much attention in statistical and quality assurance literatures (see Kotz and Johnson, 2002; Wu et al., 2009, 2012; Wu, 2012b). Several variables single acceptance sampling plans have been developed and investigated by considering PCIs recently, examples include Pearn and Wu (2006a, 2006b, 2007), Wu and Pearn (2008), Yen and Chang (2009) and Negrin et al. (2009, 2011).

Therefore, the main goal of this paper is to develop a variables RGS plan based on the most commonly used PCI C_{pk} for product acceptance determination. The OC function of the proposed sampling plan is derived based on the exact sampling distribution and hence the decisions made are more accurate and reliable. The remainder of this paper is organised as follows. In Section 2, we first introduce the relationship between the capability index C_{pk} and product quality and then an estimator of index C_{pk} and its statistical properties are also introduced. In Section 3, the concept and the operating procedure of the proposed variables RGS plan based on the index C_{pk} are presented. The OC function and the optimisation problem for solving parameters of the proposed variables RGS plan are also provided there. Furthermore, a detailed discussion and analysis is made in Section 4. The efficiency of the proposed variables RGS is also compared with the existing variables single sampling plan in terms of ASN required for inspection. For illustrative purpose, an example is presented to demonstrate the use of the proposed plan for lot acceptance. In the final section, some conclusions are given. A list of abbreviations and notations used in the manuscript are summarised in Appendix.

2 PCIs and product quality

PCIs provide common numerical measures for determine whether a process of reproducing items meeting the manufacturing specifications. The first index appearing in the literature was the process precision index C_p and defined as (Kane, 1986):

$$C_p = \frac{USL - LSL}{6\sigma}, \tag{1}$$

where *USL* and *LSL* are the upper and lower specification limits, σ is the standard deviation of the quality characteristic. The index C_p was designed to measure the magnitude of the overall process variation relative to the manufacturing tolerance. Due to the simplicity of its design, C_p cannot reflect the tendency of process centring and thus gives no indication of the actual process performance. For this reason, the C_{pk} index was developed by taking the magnitude of process variation as well as process location into consideration. It is defined as (Kane, 1986):

$$C_{pk} = \text{Min}\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}\right\} = \frac{d-|\mu-M|}{3\sigma}, \tag{2}$$

where μ is the process mean of the quality characteristic, $d = (USL - LSL)/2$ and $M = (USL + LSL)/2$ are the half-length and the midpoint of the specification interval, respectively.

2.1 Yield assurance based on C_{pk}

The index C_{pk} has been regarded as a yield-based index since it provides bounds on the process yield, $100[2\Phi(3C_{pk}) - 1]\% \leq Yield\% < 100\Phi(3C_{pk})\%$, for a normally distributed process (Boyles, 1991) where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution. For a C_{pk} at level 1, statistically, one would expect that the product's fraction of defectives is at most 2,700 PPM, falling outside the specification limits. At a C_{pk} level of 1.33, the defect rate drops to 66 PPM. To achieve less than 0.544 PPM defect rate, a C_{pk} level of 1.67 is needed. At a C_{pk} level of 2.0, the likelihood of a defective part drops to two parts per billion (PPB). Table 1 displays some values of C_{pk} index with the upper and lower bounds of non-conformities in PPM for a normally distributed process [see also from Pearn and Wu (2007) and Wu et al. (2009)].

Table 1 C_{pk} index values and the corresponding bounds on non-conformities in PPM for a normally distributed process

Index	*Lower bound*	*Upper bound*	*Index*	*Lower bound*	*Upper bound*
0.60	35,930	71,861	1.33	33	66
0.70	17,864	35,729	1.40	13	27
0.80	8,198	16,395	1.45	6.807	13.614
0.90	3,467	6,934	1.50	3.398	6.795
1.00	1,350	2,700	1.60	0.793	1.587
1.10	483	967	1.67	0.272	0.544
1.20	159	318	1.70	0.170	0.340
1.24	100	200	1.80	0.033	0.067
1.25	88	177	1.90	0.006	0.012
1.30	48	96	2.00	0.001	0.002

In a purchasing contract, a minimum C_{pk} value is usually specified. If the prescribed minimum C_{pk} fails to be met, the process is determined to be incapable. Otherwise, the process is considered capable. Montgomery (2009) recommended some minimum capability requirements for processes runs under some designated quality conditions. In particular, $C_{pk} \geq 1.33$ for existing processes and $C_{pk} \geq 1.50$ for new processes; $C_{pk} \geq 1.50$ also for existing processes on safety, strength or critical parameter and $C_{pk} \geq 1.67$ for new processes on safety, strength, or critical parameter. Finley (1992) also found that required C_{pk} values on all critical supplier processes are 1.33 or higher and C_{pk} values of 1.67 or higher are preferred. Many companies have recently adopted criteria for evaluating their processes that include process capability objectives more stringent than before. Motorola's 'Six Sigma' programme essentially requires the process capability at least 2.0 to accommodate the possible 1.5 σ process shift (see Harry, 1988) and no more than 3.4 PPM of non-conformities.

2.2 *Estimation of C_{pk} and its sampling distribution*

The natural estimator $\hat{C}_{pk}$ defined below is obtained by replacing the process mean μ and the process standard deviation σ by their sample estimators $\bar{x} = \sum_{i=1}^{n} x_i / n$ and $s = \left[\sum_{i=1}^{n} (x_i - \bar{x})^2 / (n-1)\right]^{1/2}$. We note that the process must be demonstrably stable (under statistical control) in order to produce a reliable estimate of process capability.

$$\hat{C}_{pk} = \text{Min}\left\{\frac{USL - \bar{x}}{3s}, \frac{\bar{x} - LSL}{3s}\right\} = \frac{d - |\bar{x} - M|}{3s}. \tag{3}$$

Under the assumption of normality, Kotz et al. (1993) obtained the r^{th} moment of $\hat{C}_{pk}$ as well as the expected value and variance. Numerous methods for constructing approximate confidence intervals of C_{pk} have been proposed in the literature. See Franklin and Wasserman (1992), Kushler and Hurley (1992), Nagata and Nagahata (1994), Tang et al. (1997), Hoffman (2001), Pearn and Shu (2003), Pearn and Wu (2005), Mathew et al. (2007) and references therein. Further, using the integration technique similar to that presented in Vännman (1997), an exact and explicit form of the CDF of the natural estimator $\hat{C}_{pk}$ can be obtained under the assumption of normality. The CDF of $\hat{C}_{pk}$ is expressed in terms of a mixture of the chi-square distribution and the normal distribution (Pearn and Shu, 2003; Lin and Pearn, 2004):

$$F_{\hat{C}_{pk}}(y) = 1 - \int_0^{b\sqrt{n}} G\left(\frac{(n-1)\left(b\sqrt{n} - t\right)^2}{9ny^2}\right)\left[\phi\left(t + \xi\sqrt{n}\right) + \phi\left(t - \xi\sqrt{n}\right)\right] dt, \tag{4}$$

For $y > 0$, where $b = d/\sigma$, $\xi = (\mu - M)/\sigma$, $G(\cdot)$ and $g(\cdot)$ are the CDF and probability density function (PDF) of the chi-square distribution with degrees of freedom $n - 1$, χ^2_{n-1}, and $\phi(\cdot)$ is the PDF of the standard normal distribution $N(0, 1)$. It is noted that we would obtain an identical equation if we substitute ξ by $-\xi$ into equation (4) for fixed values of y and n. Lin and Pearn (2004) developed *Maple* computer programmes to calculate critical values and p-values for testing process capability based on the C_{pk} index.

3 Designing a variables RGS plan based on C_{pk} index

Consider a variables sampling plan for controlling the lot fraction of defectives (non-conformities). As indicated before, PCIs are a function of process parameters (μ and σ) and manufacturing specifications. It measures the ability of the process to reproduce product units that meet the prescribed specifications. In particular, the C_{pk}

index remains the most prevalent in practice because it provides quantitative measures on process yield and bounds on fraction of non-conforming product items. Therefore, for this reason the C_{pk} index is appropriate and can be used as a quality benchmark for product acceptance.

Suppose that the quality characteristic of interest has a two-sided specification limits (*LSL* and *USL*) and follows a normal distribution. The operating procedure of the proposed VRGS plan based on C_{pk} index is stated as follows: firstly, both producer and consumer will set their requirements in the contract with the risks they can suffer. That is, the producer demands that not too many 'good' lots shall be rejected and the consumer demands that not too many 'bad' lots shall be accepted. In choosing a sampling plan attempts will be made to meet these somewhat opposing requirements. Thus, the first step for judging whether a given process meets the capability requirement is to determine the specified value of the capability requirements AQL and lot tolerance percent defective (LTPD) or limiting quality level (LQL) and the α-risk, β-risk. That is, if production process capability with $C_{pk} = C_{\text{AQL}}$ (in high quality), the probability of acceptance must greater than $1 - \alpha$. And if producer's capability is only with $C_{pk} = C_{\text{LTPD}}$ (in low quality), consumer accept for no more than β. If the estimated C_{pk} value is greater than the critical value K_a, then the consumer will accept the entire lot. If the estimated C_{pk} value is smaller than the critical value K_r, then the consumer will reject the entire lot. Otherwise, we do not have sufficient information to conclude that the process whether meets the present capability requirement. In this case, we should take a new sample for further judgment.

Therefore, the null hypothesis with fraction of defectives, H_0: p = AQL is equivalent to test process capability with H_0: $C_{pk} \geq C_{\text{AQL}}$ where C_{AQL} is the level of acceptable quality for C_{pk} index correspond to the lot or process fraction of non-conformities AQL (in PPM) as $\Phi^{-1}(1 - (\text{AQL}/2) \times 10^{-6})/3$. For instance, if the vender's fraction of defectives p = AQL is less than 66 PPM, then the probability of accepting the product is desired to be greater than $100(1 - \alpha)\%$. On the other hand, if the vender's fraction of defectives p = LTPD is more than 2,700 PPM, then the probability of accepting the product is set to be no more than $100\beta\%$. From the relationship between the index value and fraction of defectives, we could obtain the equivalent $C_{\text{AQL}} = 1.33$ and the $C_{\text{LTPD}} = 1.00$ based on C_{pk} index [see also from Pearn and Wu (2007)].

From the definition of the C_{pk} index, it may be rewritten as $C_{pk} = (d/\sigma - |\,\xi\,|)/3$, where $\xi = (\mu - M)/\sigma$. Thus, given $C_{pk} = C$, $b = d/\sigma$ can be expressed as $b = 3C + |\,\xi\,|$. The probability of accepting a lot when the lot quality with $C_{pk} = C$, $P_a(C)$ is given as:

$$\begin{aligned} P_a(C) &= P\left(\hat{C}_{pk} \geq k_a\right) = 1 - P\left(\hat{C}_{pk} < k_a\right) = 1 - F_{\hat{C}_{pk}}(k_a) \\ &= \int_0^{b\sqrt{n}} G\left(\frac{(n-1)\left(b\sqrt{n}-t\right)^2}{9n\left(k_a\right)^2}\right)\left(\phi\left(t+\xi\sqrt{n}\right)+\phi\left(t-\xi\sqrt{n}\right)\right)\mathrm{d}t. \end{aligned} \tag{5}$$

Similarly, the probability of rejecting a lot when the lot quality with $C_{pk} = C$, $P_r(C)$ can be expressed as:

$$\begin{aligned} P_r(C) &= P\left(\hat{C}_{pk} < k_r\right) = F_{\hat{C}_{pk}}(k_r) \\ &= 1 - \int_0^{b\sqrt{n}} G\left(\frac{(n-1)\left(b\sqrt{n}-t\right)^2}{9n\left(k_r\right)^2}\right)\left(\phi\left(t+\xi\sqrt{n}\right)+\phi\left(t-\xi\sqrt{n}\right)\right)\mathrm{d}t. \end{aligned} \tag{6}$$

According to Balamurali and Jun (2006), the OC function in the case of variables RGS plan with $C_{pk} = C$, $\pi_A(C)$ can be expressed as follows:

$$\begin{aligned}\pi_A(C) &= P_a(C) + [1 - P_a(C) - P_r(C)]P_a(C) + [1 - P_a(C) - P_r(C)]^2 P_a(C) \\ &\quad + [1 - P_a(C) - P_r(C)]^3 P_a(C) + \dots \\ &= \frac{P_a(C)}{P_a(C) + P_r(C)} = \frac{1 - F_{\hat{C}_{pk}}(k_a)}{1 - F_{\hat{C}_{pk}}(k_a) + F_{\hat{C}_{pk}}(k_r)}.\end{aligned} \tag{7}$$

Thus, the parameters of the proposed variables RGS plan can be obtained by solving the following two non-linear simultaneous equations.

$$\pi_A(C_{\text{AQL}}) = \frac{P_a(C_{\text{AQL}})}{P_a(C_{\text{AQL}}) + P_r(C_{\text{AQL}})} \geq 1 - \alpha \tag{8}$$

and

$$\pi_A(C_{\text{LTPD}}) = \frac{P_a(C_{\text{LTPD}})}{P_a(C_{\text{LTPD}}) + P_r(C_{\text{LTPD}})} \leq \beta. \tag{9}$$

But, please note that there are three parameters (n, k_a, k_r) for the variables RGS plan based on the C_{pk} index should be determined, there might be several combinations of the design parameters for the proposed variables RGS plan that satisfy the given producer and the consumer risks simultaneously. So, we find the plan parameters by minimising ASN required for inspection at LTPD. The ASN for the variables RGS plan evaluated at $C_{pk} = C_{\text{LTPD}}$ can be obtained as:

$$\text{ASN}(C_{\text{LTPD}}) = \frac{n}{P_a(C_{\text{LTPD}}) + P_r(C_{\text{LTPD}})}. \tag{10}$$

That is, the parameters (n, k_a, k_r) of the proposed VRGS plan based on the C_{pk} index can be determined by solving the following optimisation problem:

$$\text{Min ASN}(C_{\text{LTPD}}) = \frac{n}{\left\{\begin{aligned}&1 + \int_0^{b_L\sqrt{n}} G\left(\frac{(n-1)(b_L\sqrt{n} - t)^2}{9n(k_a)^2}\right)\left(\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})\right)\mathrm{d}t \\ &- \int_0^{b_L\sqrt{n}} G\left(\frac{(n-1)(b_L\sqrt{n} - t)^2}{9n(k_r)^2}\right)\left(\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})\right)\mathrm{d}t\end{aligned}\right\}}$$

subject to

$$\pi_A\left(C_{\mathrm{AQL}}\right)=\frac{\int_0^{b_A\sqrt{n}} G\left(\frac{(n-1)\left(b_A\sqrt{n}-t\right)^2}{9n\left(k_a\right)^2}\right)\left(\phi\left(t+\xi\sqrt{n}\right)+\phi\left(t-\xi\sqrt{n}\right)\right)\mathrm{d}t}{\left\{1+\int_0^{b_A\sqrt{n}} G\left(\frac{(n-1)\left(b_A\sqrt{n}-t\right)^2}{9n(k_a)^2}\right)\left(\phi\left(t+\xi\sqrt{n}\right)+\phi\left(t-\xi\sqrt{n}\right)\right)\mathrm{d}t -\int_0^{b_A\sqrt{n}} G\left(\frac{(n-1)\left(b_A\sqrt{n}-t\right)^2}{9n\left(k_r\right)^2}\right)\left(\phi\left(t+\xi\sqrt{n}\right)+\phi\left(t-\xi\sqrt{n}\right)\right)\mathrm{d}t\right\}}\geq 1-\alpha,$$

$$\pi_A\left(C_{\mathrm{LTPD}}\right)=\frac{\int_0^{b_L\sqrt{n}} G\left(\frac{(n-1)\left(b_L\sqrt{n}-t\right)^2}{9n\left(k_a\right)^2}\right)\left(\phi\left(t+\xi\sqrt{n}\right)+\phi\left(t-\xi\sqrt{n}\right)\right)\mathrm{d}t}{\left\{1+\int_0^{b_L\sqrt{n}} G\left(\frac{(n-1)\left(b_L\sqrt{n}-t\right)^2}{9n(k_a)^2}\right)\left(\phi\left(t+\xi\sqrt{n}\right)+\phi\left(t-\xi\sqrt{n}\right)\right)\mathrm{d}t -\int_0^{b_L\sqrt{n}} G\left(\frac{(n-1)\left(b_L\sqrt{n}-t\right)^2}{9n\left(k_r\right)^2}\right)\left(\phi\left(t+\xi\sqrt{n}\right)+\phi\left(t-\xi\sqrt{n}\right)\right)\mathrm{d}t\right\}}\leq \beta,$$

$$C_{\mathrm{AQL}} > C_{\mathrm{LTPD}}, k_a \geq k_r \geq 0,$$

where $b_A = 3C_{\mathrm{AQL}} + |\,\xi\,|$ and $b_L = 3C_{\mathrm{LTPD}} + |\,\xi\,|$.

4 Analysis and discussion

For the purpose of practical applications, we calculate and tabulate the parameters, n, k_a, k_r and ASN of the proposed variables RGS plan based on the C_{pk} index with commonly used values of α, β, C_{AQL} and C_{LTPD}. As noted by Pearn and Wu (2007) that the process parameters μ and σ are unknown, then the distribution characteristic parameter, ξ is also unknown which has to be estimated in real applications. Such approach introduces additional sampling errors from estimating ξ in finding the critical acceptance values and the required sample sizes. To eliminate the need for estimating the distribution characteristic parameter ξ, they performed extensive calculations to investigate the behaviour of the critical acceptance value and sample size for various parameters and found that the required sample size and the critical acceptance value will be conservative by setting ξ = 1.00. Thus, we calculate the parameters (n, k_a, k_r) with the condition ξ = 1.00 to ensure that the decisions made are reliable.

Tables 2–3 display the values of n, k_a, k_r and ASN for various producer's α-risk and consumer's β-risk = 0.01, 0.025(0.025)0.10 with various benchmarking quality levels (C_{AQL}, C_{LTPD}) = (1.33, 1.00), (1.50, 1.33), (1.67, 1.33) and (2.00, 1.67). Based on the designed variables RGS plan, the practitioner can determine the number of production items (n) to be sampled for inspection and the corresponding critical values (k_a, k_r) for product acceptance and rejection. For instance, if the benchmarking quality level (C_{AQL}, C_{LTPD}) is set to (1.33, 1.00) with producer's α-risk = 0.01 and buyer's β-risk = 0.05, then the corresponding sample size and critical values for product acceptance and rejection can be obtained as (n, k_a, k_r) = (45, 1.2742, 1.0296). This implies that the lot will be accepted if the 45 inspected production items yield measurements with $\hat{C}_{pk}$ > 1.2746 and the lot will be rejected if $\hat{C}_{pk}$ < 1.0296. Otherwise, a new sample is needed to take for further judgment.

Table 2 The values of n, k_a, k_r and ASN for various α and β with quality levels (C_{AQL}, C_{LTPD}) = (1.33, 1.00) and (1.50, 1.33)

α	β	C_{AQL} = 1.33, C_{LTPD} = 1.00				C_{AQL} = 1.50, C_{LTPD} = 1.33			
		n	k_a	k_r	ASN	n	k_a	k_r	ASN
0.010	0.010	56	1.3328	1.0460	85	305	1.4861	1.3535	466
	0.025	50	1.3021	1.0379	80	278	1.4711	1.3496	439
	0.050	45	1.2742	1.0296	74	255	1.4573	1.3456	413
	0.075	42	1.2552	1.0235	71	241	1.4478	1.3427	394
	0.100	40	1.2401	1.0183	67	231	1.4401	1.3403	377
0.025	0.010	47	1.3731	1.0510	72	250	1.5044	1.3556	384
	0.025	41	1.3440	1.0421	66	221	1.4904	1.3508	354
	0.050	36	1.3176	1.0325	60	198	1.4774	1.3457	326
	0.075	33	1.2995	1.0252	56	183	1.4683	1.3418	307
	0.100	31	1.2852	1.0188	53	172	1.4611	1.3385	291
0.050	0.010	40	1.4109	1.0565	62	212	1.5216	1.3579	325
	0.025	35	1.3842	1.0472	55	182	1.5088	1.3525	293
	0.050	30	1.3598	1.0370	50	159	1.4968	1.3467	265
	0.075	27	1.3432	1.0289	46	144	1.4886	1.3420	246
	0.100	25	1.3300	1.0218	43	133	1.4819	1.3379	230
0.075	0.010	37	1.4373	1.0606	56	190	1.5336	1.3597	292
	0.025	31	1.4127	1.0514	50	161	1.5218	1.3541	259
	0.050	27	1.3903	1.0410	44	138	1.5109	1.3477	231
	0.075	24	1.3750	1.0327	40	124	1.5033	1.3427	211
	0.100	22	1.3629	1.0253	37	113	1.4973	1.3381	196
0.100	0.010	34	1.4586	1.0642	52	175	1.5433	1.3613	269
	0.025	29	1.4361	1.0552	46	146	1.5325	1.3555	235
	0.050	24	1.4156	1.0449	40	124	1.5225	1.3489	207
	0.075	22	1.4017	1.0365	36	110	1.5157	1.3434	188
	0.100	20	1.3906	1.0290	33	99	1.5102	1.3385	173

Table 3 The values of n, k_a, k_r and ASN for various α and β with quality levels (C_{AQL}, C_{LTPD}) = (1.67, 1.33) and (2.00, 1.67)

α	β	$C_{AQL} = 1.67, C_{LTPD} = 1.33$				$C_{AQL} = 2.00, C_{LTPD} = 1.67$			
		n	k_a	k_r	ASN	n	k_a	k_r	ASN
0.010	0.010	83	1.6627	1.3774	127	129	1.9844	1.7159	197
	0.025	74	1.6317	1.3690	119	116	1.9549	1.7079	185
	0.050	68	1.6035	1.3606	111	106	1.9279	1.6998	173
	0.075	63	1.5842	1.3544	105	100	1.9090	1.6940	165
	0.100	60	1.5689	1.3491	101	95	1.8941	1.6891	158
0.025	0.010	69	1.7028	1.3822	106	106	2.0222	1.7202	164
	0.025	60	1.6737	1.3728	97	93	1.9942	1.7111	150
	0.050	53	1.6473	1.3625	89	83	1.9685	1.7013	138
	0.075	49	1.6285	1.3552	83	77	1.9509	1.6938	130
	0.100	46	1.6140	1.3485	79	72	1.9368	1.6873	123
0.050	0.010	59	1.7405	1.3875	91	91	2.0573	1.7252	140
	0.025	51	1.7138	1.3776	82	78	2.0318	1.7153	125
	0.050	44	1.6889	1.3668	73	68	2.0083	1.7043	113
	0.075	40	1.6721	1.3582	68	61	1.9920	1.6956	105
	0.100	37	1.6585	1.3506	63	57	1.9791	1.6881	98
0.075	0.010	54	1.7668	1.3916	82	82	2.0819	1.7290	126
	0.025	45	1.7420	1.3817	73	69	2.0585	1.7188	112
	0.050	39	1.7193	1.3704	64	59	2.0368	1.7073	99
	0.075	35	1.7039	1.3614	59	53	2.0220	1.6981	91
	0.100	32	1.6916	1.3533	55	48	2.0102	1.6899	84
0.100	0.010	50	1.7878	1.3952	76	76	2.1021	1.7321	116
	0.025	42	1.7652	1.3853	67	63	2.0803	1.7221	102
	0.050	35	1.7445	1.3739	58	54	2.0606	1.7103	89
	0.075	31	1.7305	1.3646	53	47	2.0472	1.7007	81
	0.100	28	1.7193	1.3563	49	43	2.0364	1.6921	74

Furthermore, in order to compare the efficiency of the proposed variables RGS plan and the existing variables single sampling plan based on C_{pk} proposed by Pearn and Wu (2007), we summarises the required sample size of the single sampling plan and the ASN of the proposed variables RGS plan with various values of AQL, LTPD, α-risk and β-risk in Table 4. It can be seen from Table 4 that the variables RGS plan is economically superior to the variables single sampling plan in terms of ASN. That is, the proposed variables RGS plan requires a smaller sample size for the product lot sentencing than the single sampling plan under the same values of AQL, LTPD, α-risk and β-risk. For example, if $\alpha = 0.01$, $\beta = 0.05$ and (C_{AQL}, C_{LTPD}) = (1.33, 1.00), the sample size required by the variables single sampling plan is 112 while the sample size required by the proposed variables RGS plan is 45 and ASN is 74 under the same specified values of parameters. A similar reduction in ASN can be achieved for any combination of AQL and LTPD values under the same α and β risks. This implies that the proposed variables RGS plan will give the desired protection with minimum inspection so that the cost of inspection will be greatly reduced.

Table 4 The required sample size for inspection of variables single sampling plan and the ASN of variables RGS plan under selected values of AQL, LTPD, α-risk and β-risk

		$C_{AQL} = 1.33$, $C_{LTPD} = 1.00$		$C_{AQL} = 1.50$, $C_{LTPD} = 1.33$		$C_{AQL} = 1.67$, $C_{LTPD} = 1.33$		$C_{AQ\ L} = 2.00$, $C_{LTPD} = 1.67$	
α	β	*Variables single sampling plan*	*Variables RGS plan*	*Variables single sampling plan*	*Variables RGS plan*	*Variables single sampling plan*	*Variables RGS plan*	*Variables single sampling plan*	*Variables RGS plan*
0.01	0.01	158	85	834	466	232	127	358	197
	0.05	112	74	600	413	166	111	256	173
	0.10	91	67	491	377	135	101	209	158
0.05	0.01	119	62	616	325	174	91	266	140
	0.05	80	50	418	265	117	73	180	113
	0.10	62	43	328	230	91	63	141	98
0.10	0.01	100	52	513	269	146	76	223	116
	0.05	65	40	334	207	95	58	145	89
	0.10	49	33	254	173	72	49	110	74

Figure 1 OC curves of a variables single sampling plan and a variables RGS plan with $n = 50$ (see online version for colours)

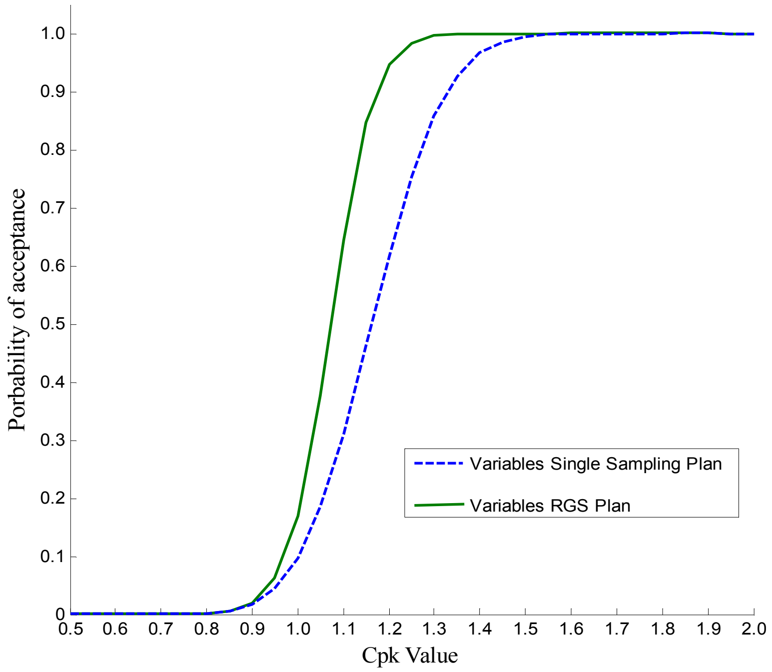
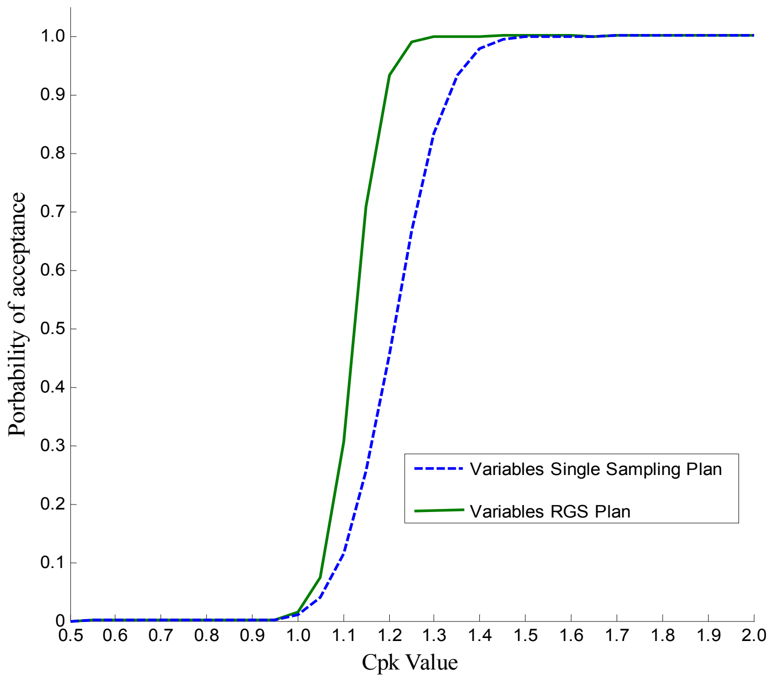


Figure 2 OC curves of a variables single sampling plan and a variable RGS plan with $n = 100$ (see online version for colours)



Besides, the OC curves of these two sampling plans, variables single sampling plan and the variables RGS plan, are considered. Figure 1 shows the OC curves of the variables single sampling plan with parameters (sample size $n = 50$, critical value $c_0 = 1.17$) and the variables RGS plan with parameters (sample size $n = 50$, critical value for acceptance and rejection $(k_a, k_r) = (1.17, 1.00)$. The OC curves of the variables single sampling plan with parameters $(n, c_0) = (100, 1.2140)$ and the variables RGS plan with parameters $(n, k_a, k_r) = (100, 1.2140, 1.0500)$ is displayed in Figure 2. From Figures 1–2, the probability of acceptance will become larger as the value of C_{pk} increases for both variables single sampling plan and RGS plan. In addition, these figures apparently show that the proposed variables RGS plan has a better OC curve than the variables single sampling plan at good quality levels and also ensures protection against the consumer point of view at poor quality levels. This is also an important feature of the variables RGS plan.

5 An application example

To illustrate how the sampling plan can be established and applied to the actual data collected from factories, we consider the following example taken from a company which manufacturing printed circuit boards (PCBs). The thickness of the PCB is the critical characteristic the company focuses on, which has significant impact to product quality. For a typical type of PCB investigated, the lower and upper specification limits are $LSL = 1.36$ mm and $USL = 1.64$ mm, respectively. In the contract, the values of C_{AQL} and C_{LTPD} are set to 1.33 and 1.00 with the α-risk = 0.01 and β-risk = 0.05, respectively. This implies that the sampling plans must provide a probability of at least 0.99 of accepting the lot if the lot proportion defective is at the $C_{\text{AQL}} = 1.33$ (which is equivalent to no more than 66 PPM fraction of defectives) and also provide a probability of no more than 0.05 of accepting the lot if the lot proportion defective is at the $C_{\text{LTPD}} = 1.00$ (which is equivalent to 2,700 PPM fraction of defectives).

Based on the proposed variables RGS plan based on the C_{pk} index, we can obtain the sample size required for inspection and the critical value for acceptance and rejection are $(n, k_a, k_r) = (45, 1.2742, 1.0296)$ from Table 2. Hence, 45 samples are taken from the lot randomly and the observed measurements are showed in Table 5. These data are also shown to be fairly close to the normal distribution using Anderson-Darling normality test. The normal probability plot of the collected data with p-value for testing the normality is displayed in Figure 3. The histogram of the collected data with lower and upper specification limits is displayed in Figure 4.

Table 5 The collected data sampling from the lot (45 measurements)

1.5444	1.5516	1.5515	1.5517	1.5054	1.5377	1.4148	1.4654	1.5761
1.4891	1.5353	1.5904	1.4785	1.4752	1.4970	1.5406	1.5104	1.4743
1.4465	1.5027	1.5097	1.4950	1.5078	1.4688	1.5456	1.4184	1.5268
1.5512	1.4804	1.5173	1.5336	1.4913	1.5194	1.5392	1.4762	1.5590
1.5711	1.5350	1.5207	1.4656	1.5431	1.4632	1.5943	1.5446	1.5324

Figure 3 The normal probability plot of the collected data (see online version for colours)

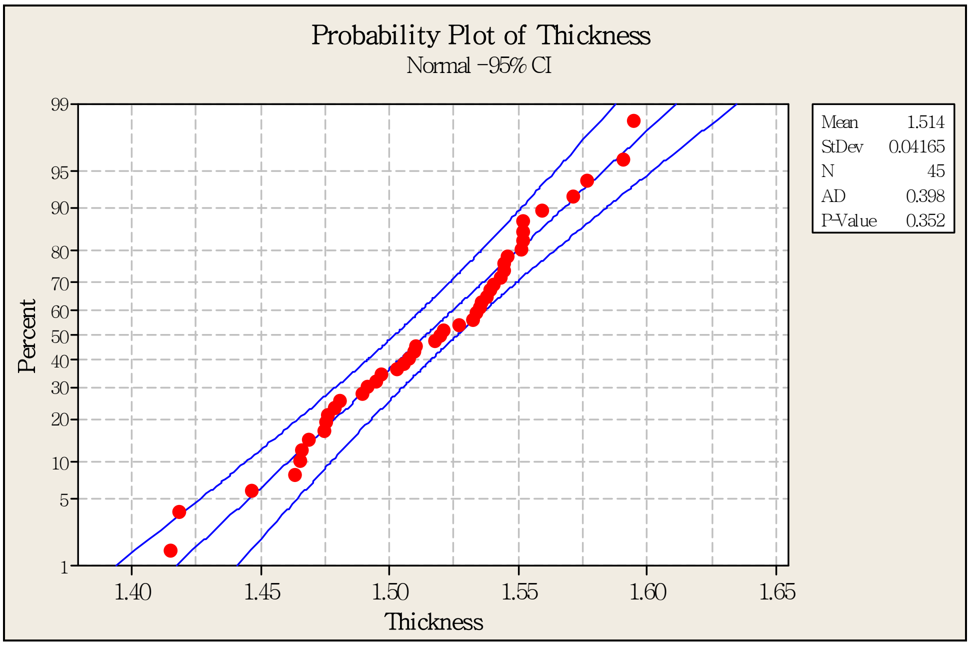
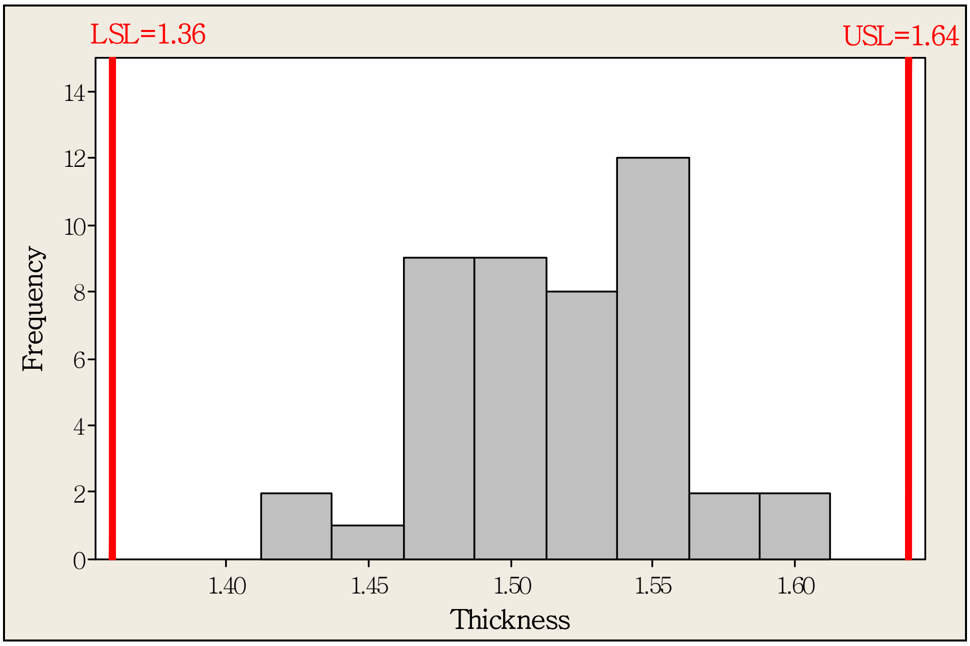


Figure 4 The histogram of the collected data with specification limits (see online version for colours)



The sample mean and sample standard deviation from the collected samples are calculated as $\bar{x} = 1.5140$ mm, s = 0.04165 mm. And the estimation of C_{pk} is $\hat{C}_{pk} = 1.0051$. Thus, in this case, the consumer would reject the entire lot since the sample estimator from these 45 measurements, 1.0051 is smaller than the critical value for rejection 1.0296. We note that if the classical attributes sampling plans are applied here, it is almost certain that any sample of 45 PCBs taken from the lot will contain no defective items. All the lots therefore will be accepted, which obviously provides no protection to the consumer at all. Besides, if the traditional variable single sampling plan based on the C_{pk} index is applied to this case, the sample size required for inspection is 112 with the same condition. Therefore, the proposed variables RGS plan can provide a more efficient and economic scheme for lot sentencing.

6 Conclusions

Acceptance sampling plans provide the producer and the consumer a general rule for lot sentencing to satisfy the desired quality requirement and protection. Since the sampling cannot guarantee that every defective item in a lot will be inspected, then the sampling involves risks of not adequately reflecting the quality conditions of the lot. Pearn and Wu (2007) developed a variables single sampling plan based on the C_{pk} index, to deal with product acceptance decision-making problem for situations with very low fraction of non-conformities. In this paper, we develop a more efficient sampling plan, variables RGS plan based on the C_{pk} index for lot sentencing. The proposed variables RGS plan is developed based on the exact sampling distribution rather than approximation. The sample size required for inspection and the corresponding acceptance and rejection criteria are determined by minimising ASN with two constraints which provide the desired protection to both producers and consumers. The efficiency of the proposed variables RGS plan and the existing variables single sampling plan is also compared in terms of ASN required for inspection. The results indicate that the proposed variables RGS plan requires a smaller sample size for the product lot sentencing than the single sampling plan under the same conditions. This implies that the proposed variables RGS plan will give the desired protection with smaller ASN so that the cost of inspection will be greatly reduced. For practical purpose, an example is presented for illustration and tables for the required sample size and the corresponding critical values for lot acceptance and rejection are also provided.

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Appendix

Acronyms

AQL	Acceptable quality levels
ASN	Average sample number
CDF	Cumulative distribution function
LTPD	Lot tolerance percent defective
LQL	Limiting quality level
LSL	Lower specification limit
OC	Operating characteristic
PCBs	Printed circuit boards
PCIs	Process capability indices
PDF	Probability density function
PPB	Parts per billion
PPM	Parts per million
RGS	Repetitive group sampling
USL	Upper specification limit

Notations

C_p	Process potential index
C_{pk}	Process performance index
d	Half-length of the specification interval
M	Midpoint of the specification interval
C_{AQL}	C_{pk} index at AQL level
C_{LTPD}	C_{pk} index at LTPD level
α	Producer's risk
β	Consumer's risk
p	Fraction of non-conformities in PPM
P_r	Probability of rejecting the lot
P_a	Probability of accepting the lot
π_A	OC function of the variables RGS plan
n	Sample size required for inspection
k_a	Critical value for acceptance
k_r	Critical value for rejection