Bootstrap lower confidence limits for the process capability indices C_p, C_{pk} and C_{pm}

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Abstract *The capability indices are widely used by quality professionals as an estimate of process capability. Many process indices have been proposed and developed with* C_p, C_{pk} *and* C_{pm} *among the most widely used. More recently, techniques have been developed to construct lower 95 percent confidence limits for each index. These techniques are based on the assumption that the underlying process is normally distributed. The non-parametric but computer intensive method called Bootstrap is utilized and the Bootstrap confidence limits are calculated for these indices. A simulation using three distributions (normal, log-normal and chi-squared) was conducted and a comparison was made of the performances of the Bootstrap and the parametric estimates.*

Introduction

In recent years, process capability indices (PCI) have received substantial attention in the quality assurance and statistical literature. The quantification of process location, process variation and target location is central to understanding the quality of units from a process. Process capability indices are considered as a practical tool by many advocates of statistical process control in industry. They are used to determine whether a manufacturing process is capable of producing with dimensions within a specified tolerance range. A capability index is a dimensionless measure based on the process parameters and the process specification designed to quantify, in a sample and easily understood way, the performance of the process. The process indices C_p and C_{pk} have become popular as unitless measures that relate the natural process tolerance (6σ), upper and lower specification limits (see Kane (1986)). More recently Chan *et al.* (1988) developed a new index, C_{pm}, that incorporates a target value for the process (Hsiang and Taguchi (1985) and Taguchi (1986)). Chou *et al.* (1990) provided tables for constructing 95 percent lower confidence limits for both C_p and C_{pk}. Their tables for limits on C_{pk}, however, are conservative and an approximation presented by Bissel (1990) is recommended instead (see Franklin and Wasserman (1992a) and Kushler and Hurley (1992)). Finally, Boyles (1991) provided an approximate method for finding lower confidence limits for C_{pm}.

The calculation of all these lower confidence limits assume a normally distributed process and, as Gunter (1989) has noted, many real world processes

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are not normally distributed and this departure from normality may be hard to detect. This could potentially affect both the estimates of the indices and the lower confidence limits based on these estimates. Efron (1979,1981,1982,1985) introduced and developed the non-parametric, but computer intensive, estimation method called Bootstrap. Efron and Gong (1983) and Efron and Tibshirani (1986), in particular, further develop three types of Bootstrap confidence intervals namely, standard Bootstrap (SB) confidence interval, the percentile Bootstrap (PB) confidence interval, and the biased corrected percentile Bootstrap (BCPB) confidence interval. Franklin and Wasserman (1991) presented an initial study of the properties of these three Bootstrap confidence intervals for C_{pk}. Franklin and Wasserman (1992b) constructed the confidence limits for some basic capability indices.

The purpose of this paper is to report the results of a further simulation study of the behavior of these three Bootstrap confidence intervals. All three capability indices namely, C_p, C_{pk} and C_{pm} are considered and the behavior is examined when the underlying process is normal, skewed, or heavy tailed. We will introduce and define each of the three process indices and give the sample estimators of each. We will then explain the Bootstrap estimation technique and define the three Bootstrap confidence intervals. Finally, we will describe the simulations that were run for the three different process distributions to compare the performance of the lower confidence limits based on the assumption of normality to the Bootstrap limits. A discussion of these simulation results is also presented.

Definitions of C_p, C_{pk} and C_{pm}

If USL and LSL are the upper and lower specification limits, respectively, and σ is the process standard deviation, then

$$C_p = \frac{USL - LSL}{6\sigma}$$

Whenever the process variance is unknown, the unbiased sample variance S^2 is used (see Kane (1986)) to obtain the estimated capability index

$$\hat{C}_p = \frac{(USL - LSL)}{6S}$$

Since the index C_p does not take into account the location of the process mean (μ), the index

$$C_{pk} = Min\left(\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right)$$

has been advocated. Usually neither μ nor σ are known, and they are typically estimated with $\overline{X}$ and S (see Kane (1986)), resulting in the index estimator

$$\hat{C}_{pk} = Min\left(\frac{USL - \overline{X}}{3S}, \frac{\overline{X} - LSL}{3S}\right)$$

Since C_{pk} takes on its maximum value when the process is centered between the specification limits (no matter what the size of the variance), there is a natural tendency to adjust the process until μ is located precisely at the midpoint; this, however, may not be the best location. As a result, Taguchi and others have advanced the index

$$C_{pm} = \frac{USL - LSL}{6\sigma'}$$

where $\sigma'^2 = E[(\overline{X} - T)^2]$ is the variation of the process X around the desired process target T. We will use the estimator

$$\hat{C}_{pm} = \frac{(USL - LSL)}{6\hat{\sigma}'}$$

where

$$\hat{\sigma}' = \sqrt{\frac{\sum(X_i - T)^2}{n}}$$

proposed by Hsiang and Taguchi (1985). Pearn *et al.* (1992) have given a discussion on the estimators of C_{pm}.

The Bootstrap method used to construct confidence intervals for the process capability indices

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n, i.e. a sequence of n *iid* random variables, and $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$. Then a uniformly distributed random variable X* with values in $\{x_1, x_2, \ldots x_n\}$ is defined:

$$P(X^* = x_i^*) = 1/n, \qquad i = 1, 2, \ldots, n$$

Bootstrap estimations are based on repeated independent samples drawn from X*. There are a total of n^n such resamples possible. In our case, these resamples would then be used to calculate n^n values of $\hat{C}_p^*, \hat{C}_{pk}^*$ and $\hat{C}_{pm}^*$. Each of these would be an estimate of $C_p^*, C_{pk}*$ and C_{pm}^* (respectively) and the entire collection would constitute the (complete) Bootstrap distribution for $\hat{C}_p, \hat{C}_{pk}$ and $\hat{C}_{pm}$.

Bootstrap sampling is equivalent to sampling (with replacement) from the empirical probability distribution function. Thus, the Bootstrap distributions of $\hat{C}_p, \hat{C}_{pk}$ and $\hat{C}_{pm}$ are estimators of the distributions of C_p, C_{pk} and C_{pm} respectively. In practice, usually only a random sample of the n^n possible resamples is drawn, the statistic is calculated for each of these, and the resulting empirical distribution is referred to as the Bootstrap distribution of the statistic. Efron and Tibshirani (1986) indicated that a rough minimum of 1,000 Bootstrap resamples are usually sufficient to compute reasonably accurate confidence interval estimates.

Theoretical work has developed three possible constructions for confidence intervals using the Bootstrap technique. Throughout this discussion, it is assumed that $B = 1{,}000$ Bootstrap resamples are taken and $B = 1{,}000$ Bootstrap estimates (each) of C_p, C_{pk} and C_{pm} are calculated and ordered from smallest to largest. The generic notations $\hat{C}$ and $\hat{C}^*(i)$ (where i stands for number of iteration) will be used to denote the estimator of a capability index and the associated ordered Bootstrap estimates. So, for example, $\hat{C}_p(1)$ is the smallest of the 1,000 Bootstrap estimates of C_p. Construction of a two-tailed $(1-2\alpha)$ 100 percent confidence interval will be described, but a lower $(1-\alpha)$ 100 percent confidence limit can be obtained by using only the lower limit.

Calculation of the standard Bootstrap (SB) confidence interval

From the 1,000 Bootstrap estimates $C^*(i)$, calculate the sample average

$$\hat{C}^* = (1/1,000)\sum_{i=1}^{1,000}\hat{C}^*(i)$$

and the sample standard deviation

$$S^* = \sqrt{\frac{1}{999}\sum_{i=1}^{1,000}[\hat{C}^*(i) - \hat{C}^*]^2}$$

The quantity S^* is actually an estimator of the standard deviation of C and thus (if the distribution of C is approximately normal) the $(1-2\alpha)100$ percent SB confidence interval for C is

$$\hat{C} \pm Z\alpha S^*$$

where Z_α is the upper α quantile of the standard normal distribution.

The percentile Bootstrap (PB) confidence interval

From the ordered collection of $\hat{C}^*(i)$ the a percentile and the $(1-\alpha)$ percentage points are used to obtain

$$[\hat{C}^*(\alpha\, B), \hat{C}^*((1-\alpha)B)]$$

The $(1-2\alpha)$ 100 percent PB confidence interval for C. For a 90 percent confidence interval with $B = 1{,}000$, this would be approximately

$$[\hat{C}^*(50), \hat{C}^*(950)]$$

The biased corrected percentile Bootstrap (BCPB) confidence interval

It is possible that Bootstrap distributions obtained using only a sample of the complete Bootstrap distribution may be shifted higher or lower than would be expected (i.e. a biased distribution). Thus, a third method has been developed to correct for this potential bias (see Efron (1982) for a complete justification of this method). First, using the (ordered) distribution of C^*, calculate the probability

$$P_0 = P_r(\hat{C}^* \leq \hat{C}),$$

second calculate

$$Z_0 = \Phi^{-1}(P_0)$$
$$P_L = \Phi(2Z_0 - Z_\alpha)$$
$$P_U = \Phi(2Z_0 + Z_\alpha)$$

where $\Phi(.)$ is the standard normal cumulative distribution function. Finally, the BCPB confidence interval is given by

$$[\hat{C}^*(P_L\, B), \hat{C}^*(P_U\, B)]$$

The simulation

To compare the performance of the Bootstrap lower confidence limits to those based on the assumption of a normal process, a series of simulations was undertaken. The values USL = 61, LSL = 40 and T = 49 (same values as used in Franklin and Wasserman (1992b)) were used for all simulations. For each distribution used, six different simulations were run for the combinations of μ = 50 or 52 and σ = 2, 3 or 3.7. This resulted in the six different values for each process index shown in Table I. These values were chosen to represent processes that vary from "very capable" (i.e. indices of 1.5 and larger) to "not capable" (i.e. indices less than 1.0). For each combination of process mean and process standard deviation, a sample of size n = 20, 40 or 70 was drawn and B = 1,000 Bootstrap resamples (each of size n) were drawn from that single sample. A 95 percent Bootstrap lower confidence limit was constructed by each of the three methods for each of the three indices. It was determined whether the calculated Bootstrap limits were actually smaller than the true value of the index. In addition, for that sample, the normally derived 95 percent lower confidence limit was calculated using the tabled values of Chou *et al.* (1990) for C_p (with interpolation), the approximation recommended by Bissell (1990) for C_{pk} and the approximation recommended by Boyles (1991) for C_{pm}. It was determined whether these values were actually smaller than the true value of the index.

Table I.
The six values of each process index used in the simulation study

μ	σ	C_p	C_{pk}	C_{pm}
50	2.0	1.75	1.67	1.57
52	2.0	1.75	1.50	0.97
50	3.0	1.17	1.11	1.11
52	3.0	1.17	1.00	0.83
50	3.7	0.95	0.90	0.91
52	3.7	0.95	0.81	0.74

This single simulation was then replicated $N = 1{,}000$ times. Thus, we were able to calculate the proportion of times the three Bootstrap lower limits and the normally derived lower limits were less than the corresponding index. This "actual coverage proportion" (thinking of the lower limit L as a one-sided confidence interval (L,∞)) could then be compared with the expected value of 0.95. A complete simulation (for a particular n, μ and σ) was run using each of three possible process distributions: normal, log-normal, and Chi-squared. Representatives of skewed distributions were chosen because, as Gunter (1989) noted, they frequently occur in practice and are particularly troublesome. However, these specific distributions were chosen for their ease of generation and mathematical manipulation. This latter feature was important since each distribution was shifted and scaled to obtain the means and standard deviations shown in Table I.

Simulation results

A normally distributed process

For a normally distributed process the methods based on normality and the SB method resulted in proportions consistently near the expected value of 0.95 for all process indices (see Table II). The frequency of coverage for the lower limit is a binomial random variable with $p = 0.95$ and $N = 1{,}000$. Thus, a 99 percent confidence interval for the coverage proportion is

$$0.95 \pm 2.576\sqrt{(0.95)(0.05)/1,000} = 0.95 \pm 0.0178 = (0.9322, 0.9678).$$

Hence, one could be 99 percent confident that a "true 95 percent lower confidence limit" would have a proportion of coverage from 0.932 to 0.968. In fact, not even a single occurrence out of 108 was beyond these limits (well within expected simulation performance). Such results tend to validate the simulation (since the normally based limits performed as expected), and they also validate the performance of the SB method as being equivalent in coverage performance for all three indices under the assumption of an underlying normal process.

In contrast, both the PB and BCPB limits had coverage proportions significantly lower than 0.95 (see Table II). For the PB, all the 54 possible limits and for the BCPB 48 of the 54 possible limits, were below 0.932. In each of the 54 instances, the BCPB limit had a greater coverage proportion than the PB limit.

A log-normal process

For the log-normal process the normal limit never achieved at least 0.932 coverage and all the other methods achieved the desired coverage of at least 0.932 for any index (see Table III). Further, comparing the limits based on normality with the SB limits revealed that the SB limits performed significantly better, with generally higher coverage proportions for all process indices.

Concerning the other two Bootstrap limits the BCPB limits had a consistently higher coverage proportion than the PB limits. For $n = 20$, the

	Sample sizes (n)								
	20	40	70	20	40	70	20	40	70
μ = 50, σ = 2		C_p = 1.75			C_{pk} = 1.67			C_{pm} = 1.57	
$Z\alpha$	0.945	0.948	0.943	0.955	0.956	0.944	0.939	0.958	0.949
SB	0.942	0.943	0.957	0.933	0.953	0.958	0.964	0.946	0.967
PB	0.847*	0.881*	0.906*	0.862*	0.881*	0.903*	0.881*	0.913*	0.899*
BCPB	0.897*	0.914*	0.923*	0.896*	0.915*	0.918*	0.962	0.925*	0.912*
μ = 52, σ = 2		C_p = 1.75			C_{pk} = 1.50			C_{pm} = 0.97	
$Z\alpha$	0.948	0.941	0.961	0.943	0.939	0.964	0.947	0.949	0.961
SB	0.955	0.961	0.967	0.934	0.942	0.957	0.949	0.952	0.963
PB	0.858*	0.870*	0.907*	0.875*	0.884*	0.913*	0.896*	0.918*	0.930*
BCPB	0.902*	0.906*	0.940	0.897*	0.917*	0.948	0.904*	0.931*	0.932
μ = 50, σ = 3		C_p = 1.17			C_{pk} = 1.11			C_{pm} = 1.11	
$Z\alpha$	0.958	0.961	0.955	0.959	0.961	0.954	0.958	0.954	0.955
SB	0.935	0.947	0.959	0.957	0.967	0.972	0.939	0.961	0.972
PB	0.865*	0.888*	0.912*	0.918*	0.926*	0.929*	0.882*	0.916*	0.924*
BCPB	0.912*	0.927*	0.931*	0.931*	0.935	0.939	0.858*	0.928*	0.931*
μ = 52, σ = 3		C_{pk} = 1.17			C_{pk} = 1.00			C_{pm} = 0.83	
$Z\alpha$	0.944	0.955	0.948	0.947	0.952	0.953	0.959	0.946	0.955
SB	0.947	0.959	0.965	0.949	0.957	0.961	0.945	0.961	0.972
PB	0.837*	0.870*	0.889*	0.813*	0.911*	0.923*	0.892*	0.907*	0.925*
BCPB	0.887*	0.916*	0.921*	0.894*	0.923*	0.928*	0.907*	0.917*	0.931*
μ = 50, σ = 3.7		C_p = 0.95			C_{pk} = 0.90			C_{pm} = 0.91	
$Z\alpha$	0.956	0.954	0.946	0.973	0.967	0.954	0.959	0.958	0.957
SB	0.946	0.959	0.967	0.949	0.965	0.962	0.946	0.961	0.963
PB	0.859*	0.889*	0.886*	0.849*	0.891*	0.901*	0.892*	0.907*	0.925*
BCPB	0.907*	0.925*	0.918*	0.914*	0.928*	0.916*	0.906*	0.918*	0.930*
μ = 52, σ = 3.7		C_p = 0.95			C_{pk} = 0.81			C_{pm} = 0.74	
$Z\alpha$	0.959	0.961	0.955	0.958	0.959	0.947	0.959	0.961	0.962
SB	0.965	0.959	0.965	0.948	0.962	0.968	0.947	0.969	0.967
PB	0.860*	0.879*	0.897*	0.904*	0.926*	0.914*	0.891*	0.916*	0.912*
BCPB	0.911*	0.921*	0.923*	0.915*	0.931*	0.928*	0.918*	0.930*	0.919*

Note: *Indicates a proportion significantly different (at $\alpha = 0.01$) from the expected value 0.95

Table II.
Coverage percentage points for 95 percent lower confidence limits normal process

BCPB limits coverage was generally comparable with or lower than that of limits based on the SB method. But for n = 70, the BCPB limits coverage was generally higher (frequently statistically significantly higher at α = 0.01) than that of limits based on the SB method. Finally, the coverage proportions of all four methods tended to increase to the expected value of 0.95 as n increased. This tendency was very slow for the method based on normality and the SB method, and it was moderately slow for the PB and BCPB methods.

A Chi-squared process

For the Chi-squared distribution data with 4 degrees of freedom, none of the four possible 95 percent lower confidence limits ever achieved the desired

	Sample sizes (n) 20	40	70	20	40	70	20	40	70
$\mu = 50, \sigma = 2$		$C_p = 1.75$			$C_{pk} = 1.67$			$C_{pm} = 1.57$	
Z_α	0.845	0.848	0.843	0.855	0.856	0.844	0.839	0.858	0.849
SB	0.942	0.943	0.957	0.933	0.953	0.958	0.934	0.946	0.967
PB	0.947	0.944	0.906	0.962	0.934	0.903	0.981	0.901	0.939
BCPB	0.941	0.954	0.963	0.932	0.955	0.958	0.933	0.925	0.962
$\mu = 52, \sigma = 2$		$C_p = 1.75$			$C_{pk} = 1.50$			$C_{pm} = 0.97$	
Z_α	0.848	0.841	0.861	0.843	0.839	0.864	0.847	0.849	0.861
SB	0.955	0.961	0.967	0.931	0.942	0.957	0.949	0.952	0.963
PB	0.938	0.936	0.907	0.975	0.914	0.913	0.936	0.918	0.930
BCPB	0.952	0.940	0.960	0.931	0.987	0.958	0.944	0.933	0.968
$\mu = 50, \sigma = 3$		$C_p = 1.17$			$C_{pk} = 1.11$			$C_{pm} = 1.11$	
Z_α	0.858	0.861	0.855	0.859	0.861	0.854	0.858	0.854	0.855
SB	0.935	0.947	0.959	0.957	0.967	0.972	0.939	0.961	0.972
PB	0.912	0.927	0.912	0.938	0.936	0.939	0.952	0.936	0.924
BCPB	0.932	0.988	0.952	0.939	0.945	0.979	0.938	0.958	0.985
$\mu = 52, \sigma = 3$		$C_p = 1.17$			$C_{pk} = 1.00$			$C_{pm} = 0.83$	
Z_α	0.844	0.855	0.848	0.847	0.852	0.853	0.859	0.846	0.855
SB	0.947	0.959	0.965	0.949	0.957	0.961	0.945	0.961	0.972
PB	0.937	0.916	0.921	0.913	0.911	0.923	0.912	0.927	0.925
BCPB	0.947	0.970	0.969	0.934	0.923	0.962	0.937	0.987	0.977
$\mu = 50, \sigma = 3.7$		$C_p = 0.98$			$C_{pk} = 0.92$			$C_{pm} = 0.92$	
Z_α	0.856	0.854	0.846	0.873	0.867	0.854	0.859	0.858	0.857
SB	0.946	0.959	0.967	0.949	0.965	0.972	0.946	0.961	0.973
PB	0.937	0.925	0.929	0.939	0.931	0.931	0.922	0.937	0.945
BCPB	0.939	0.978	0.968	0.944	0.998	0.946	0.936	0.948	0.972
$\mu = 52, \sigma = 3.7$		$C_p = 0.98$			$C_{pk} = 0.85$			$C_{pm} = 0.76$	
Z_α	0.859	0.861	0.855	0.858	0.889	0.847	0.859	0.861	0.872
SB	0.965	0.959	0.965	0.948	0.962	0.968	0.947	0.969	0.977
PB	0.950	0.959	0.947	0.934	0.946	0.934	0.931	0.930	0.932
BCPB	0.961	0.961	0.965	0.945	0.953	0.968	0.938	0.940	0.951

Table III. Coverage percentage points for 95 percent lower confidence limits – lognormal process

coverage level of at least 0.932 (see Table IV). Furthermore, the coverage proportions for all four methods were universally lower (for all indices and sample sizes) when compared with the corresponding coverage with the log-normal distribution. In addition, the coverage proportions for the limits based on the normality and the SB limits were generally stable with little tendency to increase, while the PB and BCPB limits exhibited only a slow tendency to increase to the expected value of 0.95.

Nevertheless, it is clear that for the Chi-squared distribution the BCPB method obtained coverage proportions comparable with and frequently significantly greater than any of the other three methods. Specifically, of the 54 possible pairs of values, 54 times the BCPB method had a significantly greater (at $\alpha = 0.01$) coverage proportion than the method based on normality. This difference in coverage occurred for all indices and was even as large as 0.091 for $n = 70$. The limits based on normality were never significantly greater than

Table IV.
Coverage percentage points for 95 percent lower confidence limits – Chi-squared process

	Sample sizes (n) 20	40	70	20	40	70	20	40	70
$\mu = 50$, $\sigma = 2$		$C_p = 1.75$			$C_{pk} = 1.67$			$C_{pm} = 1.57$	
Z_α	0.854	0.871	0.798	0.812	0.823	0.823	0.842	0.854	0.799
SB	0.859	0.884	0.803	0.821	0.851	0.843	0.861	0.863	0.802
PB	0.797	0.869	0.825	0.774	0.808	0.845	0.843	0.869	0.875
BCPB	0.860	0.887	0.861	0.871	0.897	0.898	0.861	0.887	0.891
$\mu = 52$, $\sigma = 2$		$C_p = 1.75$			$C_{pk} = 1.50$			$C_{pm} = 0.97$	
Z_α	0.812	0.799	0.823	0.834	0.821	0.834	0.798	0.823	0.834
SB	0.821	0.803	0.854	0.867	0.866	0.883	0.801	0.885	0.897
PB	0.814	0.873	0.839	0.754	0.807	0.826	0.791	0.832	0.843
BCPB	0.877	0.893	0.891	0.872	0.893	0.884	0.895	0.926	0.921
$\mu = 50$, $\sigma = 3$		$C_p = 1.17$			$C_{pk} = 1.11$			$C_{pm} = 1.11$	
Z_α	0.823	0.843	0.856	0.867	0.865	0.868	0.808	0.812	0.845
SB	0.826	0.846	0.860	0.889	0.872	0.874	0.810	0.814	0.849
PB	0.746	0.786	0.831	0.733	0.791	0.823	0.791	0.793	0.781
BCPB	0.875	0.856	0.895	0.889	0.879	0.894	0.863	0.815	0.864
$\mu = 52$, $\sigma = 3$		$C_p = 1.17$			$C_{pk} = 1.00$			$C_{pm} = 0.83$	
Z_α	0.867	0.856	0.867	0.812	0.824	0.823	0.824	0.845	0.825
SB	0.886	0.867	0.878	0.821	0.827	0.829	0.853	0.856	0.834
PB	0.766	0.805	0.869	0.884	0.889	0.903	0.861	0.878	0.892
BCPB	0.886	0.879	0.883	0.891	0.869	0.849	0.863	0.871	0.887
$\mu = 50$, $\sigma = 3.7$		$C_p = 0.95$			$C_{pk} = 0.90$			$C_{pm} = 0.91$	
Z_α	0.812	0.831	0.803	0.812	0.823	0.804	0.834	0.811	0.812
SB	0.814	0.831	0.807	0.818	0.841	0.807	0.879	0.812	0.814
PB	0.767	0.821	0.838	0.789	0.821	0.835	0.814	0.849	0.851
BCPB	0.844	0.857	0.866	0.821	0.899	0.879	0.881	0.871	0.889
$\mu = 52$, $\sigma = 3.7$		$C_p = 0.95$			$C_{pk} = 0.81$			$C_{pm} = 0.74$	
Z_α	0.834	0.823	0.812	0.823	0.845	0.854	0.809	0.823	0.834
SB	0.867	0.831	0.827	0.849	0.852	0.873	0.810	0.827	0.869
PB	0.737	0.768	0.827	0.721	0.783	0.821	0.753	0.779	0.821
BCPB	0.870	0.859	0.869	0.871	0.862	0.899	0.819	0.857	0.879

the SB limits. In addition, the BCPB limits always had a higher coverage proportion than either the PB or normality.

Concerning the two other Bootstrap limits, once again the SB method had a consistently higher coverage proportion than the PB method. For $n = 20$, the SB method was generally comparable with or lower in its coverage than the method based on BCPB.

References

Bissell, A.F. (1990), "How reliable is your capability index?", *Applied Statistics*, Vol. 39, pp. 331-40.

Boyles, R.A. (1991), "The Taguchi capability index", *Journal of Quality Technology*, Vol. 23, pp. 17-26.

Chan, L.K., Cheng, S.W. and Spring, F.A. (1988), "A new measure of process capability: C_{pm}", *Journal of Quality Technology*, Vol. 30, pp. 162-75.

Chou, Y., Owen, D.B. and Borrego, S.A. (1990), "Lower confidence limits on process capability indices", *Journal of Quality Technology*, Vol. 22, pp. 223-9.

Efron, B. (1979), "Bootstrap methods: another look for the jack-knife", *The Annals of Statistics*, Vol. 7, pp. 1-26.

Efron, B. (1981), "Non-parametric estimates of standard error: the jackknife, the Bootstrap and other resampling methods", *Biometrika*, Vol. 68, pp. 589-99.

Efron, B. (1982), "The jackknife, the Bootstrap and other resampling plans", *Society for Industrial and Applied Mathematics*, Philadelphia, PA.

Efron, B. (1985), "Bootstrap confidence intervals for a class of non-parametric problems", *Biometrika*, Vol. 72, pp. 45-58.

Efron, B. and Gong, G. (1983), "A leisurely look at the Bootstrap, the jackknife and cross-validation", *The American Statistician*, Vol. 37, pp. 36-48.

Efron, B. and Tibshirani, R. (1986), "Bootstrap methods for standard errors, confidence intervals, and other measures of statistical accuracy", *Statistical Sciences*, Vol. 1, pp. 54-77.

Franklin, L.A. and Wasserman, G.S. (1991), "Bootstrap confidence interval estimates of C_{pk}: an introduction", *Communications in Statistics – Simulation and Computation*, Vol. 20, pp. 231-42.

Franklin, L.A. and Wasserman, G.S. (1992a), "A note on the conservative nature of the tables of lower confidence limits for C_{pk} with a suggested correction", *Communications in Statistics – Simulation and Computation*, Vol. 21 No. 4, pp. 926-32.

Franklin, L.A. and Wasserman, G.S. (1992b), "Bootstrap lower confidence limits for process capability indices", *Journal of Quality Technology*, Vol. 24, No.4, pp. 196-210.

Gunter, B.H. (1989), "The use and abuse of C_{pk}, parts 1-4", *Quality Progress*, Vol. 22 No. 1, pp. 72-3, No. 3, pp. 108-9, No. 5, pp. 79-80, No. 7, pp. 86-7.

Hsiang, T.C. and Taguchi, G. (1985), "A tutorial on quality control and assurance – the Taguchi methods", paper presented at the Annual Meeting of the American Statistical Association, Las Vegas, NV.

Kane, V.E. (1986), "Process capability indices", *Journal of Quality Technology*, Vol. 18, pp. 41-52, *Corrigenda*, p. 65.

Kushler, R.H. and Hurley, P. (1992), "Confidence bounds for capability indices", *Journal of Quality Technology*, Vol. 24, pp. 188-95.

Pearn, W.L., Kotz, S. and Johnson, N.L. (1992), "Distributional and inferential properties of process capability indices", *Journal of Quality Technology*, Vol. 24, pp. 216-31.

Taguchi, G. (1985), "A tutorial on quality control and assurance – the Taguchi methods", 1985 ASA Annual meeting, Las Vegas, NV.

Taguchi, G. (1986), "Introduction to quality engineering", *Asian Productivity Organisation*, Tokyo.