Journal of Quality Technology

A Quarterly Journal of Methods, Applications and Related Topics

ISSN: 0022-4065 (Print) 2575-6230 (Online) Journal homepage: https://www.tandfonline.com/loi/ujqt20

Lower Confidence Limits on Process Capability Indices

Youn-Min Chou, D. B. Owen, A. Salvador & A. Borrego

To cite this article: Youn-Min Chou, D. B. Owen, A. Salvador & A. Borrego (1990) Lower Confidence Limits on Process Capability Indices, Journal of Quality Technology, 22:3, 223-229, DOI: 10.1080/00224065.1990.11979242

To link to this article: https://doi.org/10.1080/00224065.1990.11979242

Lower Confidence Limits on Process Capability Indices

YOUN-MIN CHOU

University of Texas at San Antonio, San Antonio, Texas 78285

D. B. OWEN

Southern Methodist University, Dallas, Texas 75275

SALVADOR A. BORREGO A.

SABA Consultores, Monterrey, Mexico

Lower confidence limits are derived for the common measures of process capability, usually indicated by C_p, CPU, CPL, and C_{pk}. The measures are estimated based on a random sample of observations from the process when the process is assumed to be normally distributed and has reached a state of statistical control.

Introduction

PROCESS capability indices are used to determine whether a production process is capable of producing items within a specified tolerance. See, for example, Kane (1986), Rado (1989), or Montgomery (1985). Suppose that a lower specification limit L and an upper specification limit U on the characteristic X of each item have been set. Assume that X is normally distributed with mean μ and standard deviation σ. In practice, the values of μ and σ usually are not known. The process must be stable in order to produce reliable estimates of μ and σ. Assuming that the process has reached a state of statistical control, the question often arises as to whether it can meet the tolerance.

The most commonly used measures of process capability are C_p, CPU, CPL, and C_{pk}. These indices have been utilized by a number of Japanese companies and in the U.S. automotive industry by Ford Motor Company; for example, see Kane (1986).

Dr. Chou is an Associate Professor in the Mathematics, Computer Science, and Statistics Department. She is a Member of ASQC.

Dr. Owen is a University Distinguished Professor in the Statistical Science Department. He is a Fellow of ASQC and an ASQC Certified Reliability Engineer.

Dr. Borrego is Director of SABA Consultores. He is a Member of ASQC.

Recommended minimum values for C_p, CPU, and CPL are given in Montgomery (1985, p. 279, Table 8.4). If the values are larger than or equal to the respective recommended minimum values, then we claim that the process is capable. For example, a minimum value of $C_p = 1.33$ is used for an ongoing process (Juran, Gryna, and Bingham [1979], pp. 9–22) since this value implies a fallout rate of 0.007%. Since the indices depend on the unknown parameters σ and/or μ, the true values of these indices can only be estimated. Estimators $\hat{C}_p$, $\hat{CPU}$, $\hat{CPL}$, and $\hat{C}_{pk}$, respectively, involve using $\bar{X}$ for μ and s for σ, where $\bar{X}$ and s are the sample mean and sample standard deviation, respectively, based on a random sample of n observations $X_1, \cdots, X_n$. The current practice is to compare the estimated index with the recommended minimum value. If the estimated index is larger than or equal to the minimum value, then the process is considered to be capable. However, the recommended minimum values are for the true indices, not for the estimated indices. Therefore, even if the estimated index is larger than or equal to the minimum value for the true index, one cannot be 100% sure that the true index is larger than or equal to the minimum value and 100% confident in claiming that the process is capable. Rather, we can only claim that the true index is larger than or equal to the minimum value with a certain level of confidence. Moreover, the estimated indices depend upon the sample size n. It is not appropriate to use the recommended minimum values for any sample size. In this paper, we shall consider the lower confidence limits of these indices, and the minimum values of the estimated indices in order that the process be considered capable with a high probability.

Chan, Cheng, and Spiring (1988) have introduced a further index, C_{pm}, where the measure of variability is around a target value rather than the mean. We do not consider this index in this paper. Nor do we consider the Bayesian approach of Cheng and Spiring (1989). We shall consider only the classical approach which involves confidence interval estimation.

Lower Confidence Limits on C_p and Minimum Values of $\hat{C}_p$

When the limits U and L are given, a convenient way to measure process capability is to use C_p, the process capability ratio (PCR). Recall that

$$C_p = \frac{U - L}{6\sigma} \quad \text{and} \quad \hat{C}_p = \frac{U - L}{6s}.$$

Let $c = c(X_1, X_2, \cdots, X_n)$ be a statistic satisfying $\Pr[C_p \geq c] = \gamma$, where γ does not depend on C_p. Then c is a $100\gamma\%$ lower confidence limit for C_p. Since $\dfrac{C_p}{\hat{C}_p} = \dfrac{s}{\sigma}$, we know that $(n-1)\left(\dfrac{C_p}{\hat{C}_p}\right)^2$ follows a chi-square distribution with $(n - 1)$ degrees of freedom which is denoted by χ^2_{n-1}. It follows that

$$\begin{aligned}\Pr[C_p \geq c] &= P\{(n-1)(C_p/\hat{C}_p)^2 \geq (n-1)c^2/\hat{C}_p^2\} \\ &= P\{\chi^2_{n-1} \geq (n-1)c^2/\hat{C}_p^2\}.\end{aligned}$$

Therefore

$$(n-1)c^2/\hat{C}_p^2 = \chi^2_{1-\gamma;n-1}$$

where $\chi^2_{1-\gamma;n-1}$ is the $(1-\gamma)^{\text{th}}$ quantile of the distribution of χ^2_{n-1}. The $100\gamma\%$ lower confidence limit on C_p can be written in terms of $\hat{C}_p$ as

$$c = \hat{C}_p \sqrt{\frac{\chi^2_{1-\gamma;n-1}}{n-1}}.$$

Table 1 gives the 95% lower confidence limit for C_p when n and $\hat{C}_p$ are given. On the other hand, $\hat{C}_p$ depends on c

$$\hat{C}_p = c\sqrt{\frac{n-1}{\chi^2_{1-\gamma;n-1}}}.$$

In the special case when c equals the recommended minimum value for C_p, the probability that $C_p \geq c$ would be either 1 or 0 if C_p were known. In practice, since C_p is unknown, we take a random sample of size n and calculate $\hat{C}_p$. Suppose that a process is capable

TABLE 1. The 95% Lower Confidence Limit c for C_p When n and $\hat{C}_p$ are Given

$\hat{C}_p$ \ n	10	20	30	40	50	75	100	125	150	200	250	300	350	400
0.7	0.43	0.51	0.55	0.57	0.58	0.60	0.62	0.63	0.63	0.64	0.65	0.65	0.66	0.66
0.8	0.49	0.58	0.63	0.65	0.67	0.69	0.71	0.72	0.72	0.73	0.74	0.75	0.75	0.75
0.9	0.55	0.66	0.70	0.73	0.75	0.78	0.79	0.81	0.81	0.83	0.83	0.84	0.84	0.85
1.0	0.61	0.73	0.78	0.81	0.83	0.86	0.88	0.89	0.90	0.92	0.93	0.93	0.94	0.94
1.1	0.67	0.80	0.86	0.89	0.92	0.95	0.97	0.98	0.99	1.01	1.02	1.03	1.03	1.04
1.2	0.73	0.88	0.94	0.97	1.00	1.04	1.06	1.07	1.08	1.10	1.11	1.12	1.12	1.13
1.3	0.79	0.95	1.02	1.06	1.08	1.12	1.15	1.16	1.18	1.19	1.20	1.21	1.22	1.22
1.4	0.85	1.02	1.09	1.14	1.16	1.21	1.24	1.25	1.27	1.28	1.30	1.31	1.31	1.32
1.5	0.91	1.09	1.17	1.22	1.25	1.30	1.32	1.34	1.36	1.38	1.39	1.40	1.41	1.41
1.6	0.97	1.17	1.25	1.30	1.33	1.38	1.41	1.43	1.45	1.47	1.48	1.49	1.50	1.51
1.7	1.03	1.24	1.33	1.38	1.41	1.47	1.50	1.52	1.54	1.56	1.57	1.59	1.59	1.60
1.8	1.09	1.31	1.41	1.46	1.50	1.55	1.59	1.61	1.63	1.65	1.67	1.68	1.69	1.69
1.9	1.15	1.39	1.48	1.54	1.58	1.64	1.68	1.70	1.72	1.74	1.76	1.77	1.78	1.79
2.0	1.22	1.46	1.56	1.62	1.66	1.73	1.76	1.79	1.81	1.83	1.85	1.86	1.87	1.88
2.1	1.28	1.53	1.64	1.70	1.75	1.81	1.85	1.88	1.90	1.93	1.94	1.96	1.97	1.98
2.2	1.34	1.61	1.72	1.79	1.83	1.90	1.94	1.97	1.99	2.02	2.04	2.05	2.06	2.07
2.3	1.40	1.68	1.80	1.87	1.91	1.99	2.03	2.06	2.08	2.11	2.13	2.14	2.16	2.17
2.4	1.46	1.75	1.88	1.95	2.00	2.07	2.12	2.15	2.17	2.20	2.22	2.24	2.25	2.26
2.5	1.52	1.82	1.95	2.03	2.08	2.16	2.21	2.24	2.26	2.29	2.31	2.33	2.34	2.35
2.6	1.58	1.90	2.03	2.11	2.16	2.25	2.29	2.33	2.35	2.38	2.41	2.42	2.44	2.45
2.7	1.64	1.97	2.11	2.19	2.25	2.33	2.38	2.42	2.44	2.48	2.50	2.52	2.53	2.54
2.8	1.70	2.04	2.19	2.27	2.33	2.42	2.47	2.51	2.53	2.57	2.59	2.61	2.62	2.64
2.9	1.76	2.12	2.27	2.35	2.41	2.50	2.56	2.59	2.62	2.66	2.69	2.70	2.72	2.73
3.0	1.82	2.19	2.34	2.43	2.50	2.59	2.65	2.68	2.71	2.75	2.78	2.80	2.81	2.82

if $C_p \geq c_0$, the recommended minimum value. See Montgomery (1985), p. 279, Table 8-4. Juran and Gryna (1980) give the following limits on the reciprocal of C_p; for existing processes and two-sided limits 0.75; for existing processes and one-sided limits 0.88; for new processes and two-sided limits 0.67, and for existing processes and one-sided limits 0.83. These translate into limits on C_p of 1.33, 1.14, 1.5, and 1.2, respectively. A process is called capable according to Juran if the C_p exceeds its limit. In our formulation, if $\hat{C}_p \geq c_0 \sqrt{\frac{n-1}{\chi^2_{1-\gamma;n-1}}}$, then we claim that the process is capable at least $100\gamma\%$ of the time. Therefore, the factor $c_0 \sqrt{\frac{n-1}{\chi^2_{1-\gamma;n-1}}}$ is the minimum value of the estimated index $\hat{C}_p$ in order that the process is considered capable at least $100\gamma\%$ of the time. Table 2 gives the minimum value of $\hat{C}_p$ in order for the process to be considered capable (i.e., $C_p \geq c_0$) 95% of the time.

For a large n and $\gamma > 0.5$, it can be shown that the factor is greater than 1. This implies that the minimum values for $\hat{C}_p$ should be larger than the recommended minimum value for C_p in order to claim that the process is capable with a probability of at least γ. But one cannot be 100% sure that the process is capable even if $\hat{C}_p$ is greater than or equal to the recommended minimum value of C_p.

Example

Suppose that the requirement for a process to be capable is that $C_p \geq 1.2$ and that the specification limits U and L are given. We take a random sample of size n and calculate $\hat{C}_p = \frac{U-L}{6s}$. Using Table 2, we get the following. Based on a random sample of $n = 20$, if $\hat{C}_p \geq 1.64$, then we claim that a process is capable at least 95% of the time. For a random sample of $n = 30$ if $\hat{C}_p \geq 1.54$, then we claim that the process is capable at least 95% of the time.

Lower Confidence Limits on *CPU* (or *CPL*) and Minimum Values of $\hat{CPU}$ (or $\hat{CPL}$)

When only a single specification limit is given, one uses either *CPU* or *CPL* to measure the process capability. We define them as

$$CPU = \frac{U-\mu}{3\sigma}$$

and

$$CPL = \frac{\mu-L}{3\sigma}.$$

Their estimates are $\hat{CPU} = \frac{U-\bar{X}}{3s}$ and $\hat{CPL} = \frac{\bar{X}-L}{3s}$,

TABLE 2. The Minimum Value of $\hat{C}_p$ for Which the Process is Capable (i.e., $C_p \geqq c_0$) 95% of the Time

c_0 \ n	10	20	30	40	50	75	100	125	150	200	250	300	350	400
0.7	1.15	0.96	0.90	0.86	0.84	0.81	0.79	0.78	0.77	0.76	0.76	0.75	0.75	0.74
0.8	1.32	1.10	1.02	0.99	0.96	0.93	0.91	0.89	0.88	0.87	0.86	0.86	0.85	0.85
0.9	1.48	1.23	1.15	1.11	1.08	1.04	1.02	1.01	1.00	0.98	0.97	0.97	0.96	0.96
1.0	1.65	1.37	1.28	1.23	1.20	1.16	1.13	1.12	1.11	1.09	1.08	1.07	1.07	1.06
1.1	1.81	1.51	1.41	1.36	1.32	1.27	1.25	1.13	1.22	1.20	1.19	1.18	1.17	1.17
1.2	1.97	1.64	1.54	1.48	1.44	1.39	1.36	1.34	1.33	1.31	1.30	1.29	1.28	1.27
1.3	2.14	1.78	1.66	1.60	1.56	1.51	1.47	1.45	1.44	1.42	1.40	1.39	1.39	1.38
1.4	2.30	1.92	1.79	1.72	1.68	1.62	1.59	1.56	1.55	1.53	1.51	1.50	1.49	1.49
1.5	2.47	2.06	1.92	1.85	1.80	1.74	1.70	1.68	1.66	1.64	1.62	1.61	1.60	1.59
1.6	2.63	2.19	2.05	1.97	1.92	1.85	1.81	1.79	1.77	1.74	1.73	1.72	1.71	1.70
1.7	2.80	2.33	2.18	2.09	2.04	1.97	1.93	1.90	1.88	1.85	1.84	1.82	1.81	1.81
1.8	2.96	2.47	2.30	2.22	2.16	2.08	2.04	2.01	1.99	1.96	1.94	1.93	1.92	1.91
1.9	3.13	2.60	2.43	2.34	2.28	2.20	2.15	2.12	2.10	2.07	2.05	2.04	2.03	2.02
2.0	3.29	2.74	2.56	2.46	2.40	2.32	2.27	2.24	2.21	2.18	2.16	2.14	2.13	2.12
2.1	3.46	2.88	2.69	2.59	2.52	2.43	2.38	2.35	2.32	2.29	2.27	2.25	2.24	2.23
2.2	3.62	3.02	2.82	2.71	2.64	2.55	2.49	2.46	2.43	2.40	2.38	2.36	2.35	2.34
2.3	3.78	3.15	2.94	2.83	2.76	2.66	2.61	2.57	2.54	2.51	2.48	2.47	2.45	2.44
2.4	3.95	3.29	3.07	2.96	2.88	2.78	2.72	2.68	2.65	2.62	2.59	2.57	2.56	2.55
2.5	4.11	3.43	3.20	3.08	3.00	2.89	2.83	2.79	2.77	2.73	2.70	2.68	2.67	2.66
2.6	4.28	3.56	3.33	3.20	3.12	3.01	2.95	2.91	2.88	2.84	2.81	2.79	2.77	2.76
2.7	4.44	3.70	3.46	3.33	3.24	3.13	3.06	3.02	2.99	2.94	2.92	2.90	2.88	2.87
2.8	4.61	3.84	3.58	3.45	3.36	3.24	3.17	3.13	3.10	3.05	3.02	3.00	2.99	2.97
2.9	4.77	3.97	3.71	3.57	3.49	3.36	3.29	3.24	3.21	3.16	3.13	3.11	3.09	3.08
3.0	4.94	4.11	3.84	3.70	3.61	3.47	3.40	3.35	3.32	3.27	3.24	3.22	3.20	3.19

respectively. Let $L = \bar{X} - k_1 s$ and $U = \bar{X} + k_2 s$. A $100\gamma\%$ lower confidence limit c_U for CPU satisfies

$$\Pr[CPU \geq c_U] = \gamma. \quad (1)$$

It can be shown that (1) can be written as

$$\Pr[T_{n-1}(\delta = 3\sqrt{n}\, c_U) \leq k_2\sqrt{n}] = \gamma \quad (2)$$

where $k_2 = 3 \times \hat{CPU}$. Similarly, a $100\gamma\%$ lower confidence limit c_L for CPL satisfies

$$\Pr[T_{n-1}(\delta = 3\sqrt{n}\, c_L) \leq k_1\sqrt{n}] = \gamma \quad (3)$$

where $k_1 = 3 \times \hat{CPL}$. Since (2) and (3) are of the same form, the relationship between c_U and $\hat{CPU}$ holds for c_L and $\hat{CPL}$. Hence, the minimum values of $\hat{CPU}$ for which the process is considered capable with a probability of at least γ can be also applied to $\hat{CPL}$. In the following discussion we will consider the situation where only U is given. Since we cannot establish the relationship between c_U and $\hat{CPU}$ explicitly, we will find it numerically. Given $\hat{CPU}$, we can solve for c_U from (2). Given the criterion that the process is capable if $CPU \geq c_1$, the recommended minimum value for CPU, we can solve for the minimum value for $\hat{CPU}$ such that the process is capable with a high probability. Table 3 gives the minimum values of $\hat{CPU}$ (or $\hat{CPL}$) for which the process is considered capable [i.e., $CPU \geq c_1$ (or $CPL \geq c_1$)] with a probability of at least 0.95. Table 4 gives the 95% lower confidence limit c_U (or c_L) for CPU (or CPL) when n and $\hat{CPU}$ (or $\hat{CPL}$) are given.

Example

Consider the situation where only U is given. Suppose that a process is considered capable if $CPU \geq 1.2$. Using Table 3, we have the following results. Since CPU is unknown, we take a random sample of size n and calculate $\hat{CPU} = \dfrac{U - \bar{X}}{3s}$. For a random sample of size $n = 20$, if $\hat{CPU} \geq 1.67$, then we claim that the process is capable at least 95% of the time. For a random sample of size $n = 100$, if $\hat{CPU} \geq 1.37$, then we claim that the process is capable at least 95% of the time.

Lower Confidence Limits on C_{pk}

When both specification limits are given, the C_p and C_{pk} indices can be used, where

$$C_{pk} = \min\{CPL, CPU\}.$$

Unlike C_p, C_{pk} depends on both μ and σ and its minimum values have not been tabulated anywhere in the literature. The C_{pk} index can also be written as

TABLE 3. The Minimum Value of $\hat{CPU}$ (or $\hat{CPL}$) for Which the Process is Capable [i.e. $CPU \geqq c_1$ (or $CPL \geqq c_1$)] 95% of the Time

c_1 \ n	10	20	30	40	50	75	100	125	150	200	250	300	350	400
0.7	1.21	1.00	0.93	0.89	0.87	0.83	0.81	0.80	0.79	0.78	0.77	0.76	0.76	0.75
0.8	1.37	1.13	1.05	1.01	0.98	0.94	0.92	0.91	0.90	0.88	0.87	0.87	0.86	0.86
0.9	1.53	1.26	1.18	1.13	1.10	1.06	1.03	1.02	1.01	0.99	0.98	0.97	0.97	0.96
1.0	1.69	1.40	1.30	1.25	1.22	1.17	1.15	1.13	1.12	1.10	1.09	1.08	1.07	1.07
1.1	1.85	1.53	1.43	1.37	1.34	1.29	1.26	1.24	1.23	1.21	1.20	1.19	1.18	1.17
1.2	2.01	1.67	1.56	1.50	1.46	1.40	1.37	1.35	1.34	1.32	1.30	1.29	1.29	1.28
1.3	2.17	1.80	1.68	1.62	1.58	1.52	1.48	1.46	1.45	1.42	1.41	1.40	1.39	1.39
1.4	2.33	1.94	1.81	1.74	1.70	1.63	1.60	1.57	1.56	1.53	1.52	1.51	1.50	1.49
1.5	2.50	2.08	1.94	1.86	1.81	1.75	1.71	1.68	1.67	1.64	1.63	1.61	1.60	1.60
1.6	2.66	2.21	2.06	1.98	1.93	1.86	1.82	1.80	1.78	1.75	1.73	1.72	1.71	1.70
1.7	2.82	2.35	2.19	2.11	2.05	1.98	1.93	1.91	1.89	1.86	1.84	1.83	1.82	1.81
1.8	2.98	2.48	2.32	2.23	2.17	2.09	2.05	2.02	2.00	1.97	1.95	1.93	1.92	1.92
1.9	3.15	2.62	2.44	2.35	2.29	2.21	2.16	2.13	2.11	2.08	2.06	2.04	2.03	2.02
2.0	3.31	2.76	2.57	2.47	2.41	2.32	2.27	2.24	2.22	2.19	2.16	2.15	2.14	2.13
2.1	3.48	2.89	2.70	2.60	2.53	2.44	2.39	2.35	2.33	2.29	2.27	2.26	2.24	2.23
2.2	3.64	3.03	2.83	2.72	2.65	2.55	2.50	2.46	2.44	2.40	2.38	2.36	2.35	2.34
2.3	3.80	3.16	2.95	2.84	2.77	2.67	2.61	2.58	2.55	2.51	2.49	2.47	2.46	2.45
2.4	3.97	3.30	3.08	2.97	2.89	2.79	2.73	2.69	2.66	2.62	2.60	2.58	2.56	2.55
2.5	4.13	3.44	3.21	3.09	3.01	2.90	2.84	2.80	2.77	2.73	2.70	2.68	2.67	2.66
2.6	4.29	3.57	3.34	3.21	3.13	3.02	2.95	2.91	2.88	2.84	2.81	2.79	2.78	2.76
2.7	4.46	3.71	3.46	3.33	3.25	3.13	3.07	3.02	2.99	2.95	2.92	2.90	2.88	2.87
2.8	4.62	3.85	3.59	3.46	3.37	3.25	3.18	3.13	3.10	3.06	3.03	3.01	2.99	2.98
2.9	4.79	3.98	3.72	3.58	3.49	3.36	3.29	3.25	3.21	3.17	3.14	3.11	3.10	3.08
3.0	4.95	4.12	3.85	3.70	3.61	3.48	3.41	3.36	3.32	3.27	3.24	3.22	3.20	3.19

TABLE 4. The 95% Lower Confidence Limit c_U (or c_L) for CPU (or CPL) for Values of n and $\hat{CPU}$ (or $\hat{CPL}$)

$\hat{CPU}$ \ n	10	20	30	40	50	75	100	125	150	200	250	300	350	400
0.7	0.37	0.47	0.52	0.54	0.56	0.58	0.60	0.61	0.62	0.63	0.64	0.64	0.65	0.65
0.8	0.44	0.55	0.60	0.63	0.64	0.67	0.69	0.70	0.71	0.72	0.73	0.74	0.74	0.75
0.9	0.50	0.63	0.68	0.71	0.73	0.76	0.78	0.79	0.80	0.82	0.82	0.83	0.84	0.84
1.0	0.57	0.70	0.76	0.79	0.81	0.85	0.87	0.88	0.89	0.91	0.92	0.93	0.93	0.94
1.1	0.63	0.78	0.84	0.87	0.90	0.94	0.96	0.97	0.99	1.00	1.01	1.02	1.02	1.03
1.2	0.70	0.85	0.92	0.96	0.98	1.02	1.05	1.06	1.08	1.09	1.10	1.11	1.12	1.12
1.3	0.76	0.93	1.00	1.04	1.07	1.11	1.14	1.15	1.17	1.19	1.20	1.21	1.21	1.22
1.4	0.82	1.00	1.08	1.12	1.15	1.20	1.23	1.24	1.26	1.28	1.29	1.30	1.31	1.31
1.5	0.88	1.08	1.16	1.20	1.24	1.29	1.31	1.33	1.35	1.37	1.38	1.39	1.40	1.41
1.6	0.95	1.15	1.24	1.29	1.32	1.37	1.40	1.42	1.44	1.46	1.48	1.49	1.50	1.50
1.7	1.01	1.22	1.31	1.37	1.40	1.46	1.49	1.51	1.53	1.55	1.57	1.58	1.59	1.60
1.8	1.07	1.30	1.39	1.45	1.49	1.55	1.58	1.60	1.62	1.65	1.66	1.67	1.68	1.69
1.9	1.13	1.37	1.47	1.53	1.57	1.63	1.67	1.69	1.71	1.74	1.75	1.77	1.78	1.79
2.0	1.19	1.44	1.55	1.61	1.65	1.72	1.76	1.78	1.80	1.83	1.85	1.86	1.87	1.88
2.1	1.26	1.52	1.63	1.69	1.74	1.81	1.85	1.87	1.89	1.92	1.94	1.95	1.96	1.97
2.2	1.32	1.59	1.71	1.78	1.82	1.89	1.93	1.96	1.98	2.01	2.03	2.05	2.06	2.07
2.3	1.38	1.67	1.79	1.86	1.91	1.98	2.02	2.05	2.07	2.11	2.13	2.14	2.15	2.16
2.4	1.44	1.74	1.87	1.94	1.99	2.07	2.11	2.14	2.17	2.20	2.22	2.23	2.25	2.26
2.5	1.50	1.81	1.94	2.02	2.07	2.15	2.20	2.23	2.26	2.29	2.31	2.33	2.34	2.35
2.6	1.56	1.89	2.02	2.10	2.16	2.24	2.29	2.32	2.35	2.38	2.40	2.42	2.43	2.45
2.7	1.63	1.96	2.10	2.18	2.24	2.33	2.38	2.41	2.44	2.47	2.50	2.51	2.53	2.54
2.8	1.69	2.03	2.18	2.27	2.32	2.41	2.47	2.50	2.53	2.56	2.59	2.61	2.62	2.63
2.9	1.75	2.11	2.26	2.35	2.41	2.50	2.55	2.59	2.62	2.66	2.68	2.70	2.72	2.73
3.0	1.81	2.18	2.34	2.43	2.49	2.59	2.64	2.68	2.71	2.75	2.77	2.79	2.81	2.82

TABLE 5. The 95% Lower Confidence Limit c_k for C_{pk} for Values of n and $\hat{C}_{pk}$

$\hat{C}_{pk}$ \ n	10	20	30	40	50	75	100	125	150	200	250	300	350	400
0.7	0.32	0.43	0.48	0.51	0.53	0.57	0.58	0.60	0.61	0.62	0.63	0.63	0.64	0.64
0.8	0.38	0.51	0.56	0.60	0.62	0.65	0.67	0.69	0.70	0.71	0.72	0.73	0.73	0.74
0.9	0.44	0.58	0.64	0.68	0.70	0.74	0.76	0.78	0.79	0.80	0.81	0.82	0.83	0.83
1.0	0.51	0.66	0.72	0.76	0.79	0.83	0.85	0.87	0.88	0.89	0.91	0.91	0.92	0.93
1.1	0.57	0.73	0.80	0.84	0.87	0.91	0.94	0.96	0.97	0.99	1.00	1.01	1.01	1.02
1.2	0.63	0.81	0.88	0.93	0.95	1.00	1.03	1.05	1.06	1.08	1.09	1.10	1.11	1.11
1.3	0.69	0.88	0.96	1.01	1.04	1.09	1.12	1.14	1.15	1.17	1.18	1.19	1.20	1.21
1.4	0.76	0.95	1.04	1.09	1.12	1.17	1.20	1.23	1.24	1.26	1.28	1.29	1.30	1.30
1.5	0.82	1.03	1.12	1.17	1.21	1.26	1.29	1.32	1.33	1.35	1.37	1.38	1.39	1.40
1.6	0.88	1.10	1.20	1.25	1.29	1.35	1.38	1.41	1.42	1.45	1.46	1.47	1.48	1.49
1.7	0.94	1.17	1.27	1.33	1.37	1.43	1.47	1.49	1.51	1.54	1.56	1.57	1.58	1.59
1.8	1.00	1.25	1.35	1.41	1.46	1.52	1.56	1.58	1.60	1.63	1.65	1.66	1.67	1.68
1.9	1.06	1.32	1.43	1.50	1.54	1.61	1.65	1.67	1.69	1.72	1.74	1.75	1.77	1.77
2.0	1.12	1.39	1.51	1.58	1.62	1.69	1.74	1.76	1.78	1.81	1.83	1.85	1.86	1.87
2.1	1.18	1.47	1.59	1.66	1.71	1.78	1.82	1.85	1.87	1.91	1.93	1.94	1.95	1.96
2.2	1.25	1.54	1.67	1.74	1.79	1.87	1.91	1.94	1.97	2.00	2.02	2.03	2.05	2.06
2.3	1.31	1.61	1.74	1.82	1.87	1.95	2.00	2.03	2.06	2.09	2.11	2.13	2.14	2.15
2.4	1.37	1.69	1.82	1.90	1.96	2.04	2.09	2.12	2.15	2.18	2.20	2.22	2.23	2.25
2.5	1.43	1.76	1.90	1.98	2.04	2.13	2.18	2.21	2.24	2.27	2.30	2.31	2.33	2.34
2.6	1.49	1.83	1.98	2.07	2.12	2.21	2.26	2.30	2.33	2.36	2.39	2.41	2.42	2.43
2.7	1.55	1.91	2.06	2.15	2.21	2.30	2.35	2.39	2.42	2.46	2.48	2.50	2.52	2.53
2.8	1.61	1.98	2.14	2.23	2.29	2.39	2.44	2.48	2.51	2.55	2.57	2.59	2.61	2.62
2.9	1.67	2.05	2.21	2.31	2.37	2.47	2.53	2.57	2.60	2.64	2.67	2.69	2.70	2.72
3.0	1.73	2.13	2.29	2.39	2.46	2.56	2.62	2.66	2.69	2.73	2.76	2.78	2.80	2.81

TABLE 6. The Minimum Value of $\hat{C}_{pk}$ for Which the Process is Considered Capable (i.e. $C_{pk} \geqq c_k$) 95% of the Time

c_k \ n	10	20	30	40	50	75	100	125	150	200	250	300	350	400
0.7	1.31	1.06	0.97	0.93	0.90	0.85	0.83	0.82	0.80	0.79	0.78	0.77	0.77	0.76
0.8	1.47	1.19	1.10	1.05	1.02	0.97	0.94	0.93	0.91	0.90	0.89	0.88	0.87	0.87
0.9	1.64	1.33	1.22	1.17	1.13	1.08	1.06	1.04	1.02	1.01	0.99	0.98	0.98	0.97
1.0	1.80	1.46	1.35	1.29	1.25	1.20	1.17	1.15	1.13	1.11	1.10	1.09	1.08	1.08
1.1	1.96	1.60	1.48	1.41	1.37	1.31	1.28	1.26	1.24	1.22	1.21	1.20	1.19	1.18
1.2	2.12	1.73	1.61	1.54	1.49	1.43	1.39	1.37	1.35	1.33	1.32	1.31	1.30	1.29
1.3	2.29	1.87	1.73	1.66	1.61	1.55	1.51	1.48	1.47	1.44	1.42	1.41	1.40	1.49
1.4	2.45	2.01	1.86	1.78	1.73	1.66	1.62	1.59	1.58	1.55	1.53	1.52	1.51	1.50
1.5	2.62	2.14	1.99	1.90	1.85	1.78	1.73	1.71	1.69	1.66	1.64	1.63	1.62	1.61
1.6	2.78	2.28	2.12	2.03	1.97	1.89	1.85	1.82	1.80	1.77	1.75	1.73	1.72	1.72
1.7	2.94	2.42	2.24	2.15	2.09	2.01	1.96	1.93	1.91	1.88	1.86	1.84	1.83	1.82
1.8	3.11	2.55	2.37	2.27	2.21	2.12	2.07	2.04	2.02	1.99	1.96	1.95	1.94	1.93
1.9	3.27	2.69	2.50	2.40	2.33	2.24	2.19	2.15	2.13	2.09	2.07	2.06	2.04	2.03
2.0	3.44	2.83	2.63	2.52	2.45	2.35	2.30	2.26	2.24	2.20	2.18	2.16	2.15	2.14
2.1	3.60	2.96	2.75	2.64	2.57	2.47	2.41	2.38	2.35	2.31	2.29	2.27	2.26	2.25
2.2	3.76	3.10	2.88	2.77	2.69	2.59	2.53	2.49	2.46	2.42	2.40	2.38	2.36	2.35
2.3	3.93	3.24	3.01	2.89	2.81	2.70	2.64	2.60	2.57	2.53	2.50	2.48	2.47	2.46
2.4	4.09	3.38	3.14	3.01	2.93	2.82	2.75	2.71	2.68	2.64	2.61	2.59	2.58	2.56
2.5	4.26	3.51	3.27	3.14	3.05	2.93	2.87	2.82	2.79	2.75	2.72	2.70	2.68	2.67
2.6	4.42	3.65	3.39	3.26	3.17	3.05	2.98	2.93	2.90	2.86	2.83	2.81	2.79	2.78
2.7	4.59	3.79	3.52	3.38	3.29	3.16	3.09	3.05	3.01	2.97	2.94	2.91	2.90	2.88
2.8	4.75	3.92	3.65	3.50	3.41	3.28	3.21	3.16	3.12	3.08	3.04	3.02	3.00	2.99
2.9	4.92	4.06	3.78	3.63	3.53	3.40	3.32	3.27	3.23	3.18	3.15	3.13	3.11	3.10
3.0	5.08	4.20	3.90	3.75	3.65	3.51	3.43	3.38	3.34	3.29	3.26	3.23	3.22	3.20

$$C_{pk} = C_p(1 - k) \quad \text{where} \quad k = \frac{|U + L - 2\mu|}{U - L}.$$

If it is assumed that $L \le \mu \le U$, then $0 \le k \le 1$ and $C_{pk} \le C_p$. In this case, it is reasonable to assume that the minimum value of C_{pk} should be generally less than that of C_p.

C_{pk} is estimated by $\hat{C}_{pk} = \min(\hat{CPL}, \hat{CPU})$. Let $L = \bar{X} - k_1 s$ and $U = \bar{X} + k_2 s$ so that $\hat{CPL} = \frac{k_1}{3}$ and $\hat{CPU} = \frac{k_2}{3}$. A 100γ% lower confidence limit, c_k, for C_{pk} satisfies

$$\Pr[C_{pk} \ge c_k] = \gamma.$$

It is shown in the appendix that this equation can be written as

$$\Pr[T_{n-1}(\delta_1) \le t_1, T_{n-1}(\delta_2) \ge t_2] = \gamma \qquad (4)$$

where $t_1 = k_1\sqrt{n}$, $t_2 = -k_2\sqrt{n}$, $\delta_1 = 3c_k\sqrt{n}$, and $\delta_2 = -3c_k\sqrt{n}$.

The lower confidence limit c_k can be obtained from (4). Table 5 gives the 95% lower confidence limit c_k for C_{pk} for given values of n and $\hat{C}_{pk}$. Table 6 gives the minimum value of $\hat{C}_{pk}$ so that the process is considered capable (i.e., $C_{pk} \ge c_k$) 95% of the time. For the special case where $k_1 = k_2$, we have

$$c_k = \frac{-K_p^*}{3}$$

where p^* can be obtained from Chou and Owen (1984), K_p^* is the $p^{*\text{th}}$ quantile of the standard normal distribution, and the γ value is the same as the confidence coefficient η in their Tables 1–3. Their tables also give the upper limits of the fallout rates (or percents nonconforming). For example, if $n = 30$, $\hat{CPU} = \hat{CPL} = 1$ (i.e. $k_1 = k_2 = 3$), then when $\gamma = \eta = 0.95$, $p^* = 0.015$, and $c_k = \frac{-K_p^*}{3} = \frac{2.17}{3} = 0.723$. That is, we can be 95% confident that the true C_{pk} is greater than or equal to 0.723. Table 5 gives 0.72 for this lower confidence limit. In addition, since $p^* = 0.015$, we are 95% confident that the percent nonconforming in each tail is no more than 0.015.

Acknowledgments

The work of Youn-Min Chou was partially supported by the Texas Advanced Research Program under Grant No. 010115-006. The authors also want to express their appreciation to the referees for many helpful comments.

Appendix: Derivation of Equation (4)

The equation $\Pr[C_{pk} \ge c_k] = \gamma$ can be written as $\Pr[CPL \ge c_k, CPU \ge c_k] = \gamma$. Since $L = \bar{X} - k_1 s$ and $U = \bar{X} + k_2 s$, we have

$$\Pr\left[\frac{\bar{X} - k_1 s - \mu}{3\sigma} \leq -c_k, \frac{\bar{X} + k_2 s - \mu}{3\sigma} \geq c_k\right] = \gamma.$$

This equation is similar to the one given in Chou and Owen (1984). Therefore, it can be written as

$$\Pr[T_{n-1}(\delta_1) \leq t_1 \text{ and } T_{n-1}(\delta_2) \geq t_2] = \gamma.$$

where $t_1 = k_1\sqrt{n}$, $t_2 = -k_2\sqrt{n}$, $\delta_1 = 3c_k\sqrt{n}$, and $\delta_2 = -3c_k\sqrt{n}$.

References

CHAN, L. K.; CHENG, S. W.; and SPIRING, F. A. (1988). "A New Measure of Process Capability: C_{pm}." *Journal of Quality Technology* 20, pp. 162–175.

CHENG, S. W. and SPIRING, F. A. (1989). "Assessing Process Capability: A Bayesian Approach." *IIE Transactions* 21, pp. 97–98.

CHOU, Y. M. and OWEN, D. B. (1984). "One-Sided Confidence Regions on the Upper and Lower Tail Areas of the Normal Distribution." *Journal of Quality Technology* 16, pp. 150–158.

JURAN, J. M. and GRYNA, F. M. (1980). *Quality Planning and Analysis.* McGraw-Hill, New York, NY.

JURAN, J. M.; GRYNA, F. M.; and BINGHAM, R. S., EDS. (1979). *Quality Control Handbook.* McGraw-Hill, New York, NY.

KANE, V. E. (1986). "Process Capability Indices." *Journal of Quality Technology* 18, pp. 41–52.

MONTGOMERY, D. C. (1985). *Introduction to Statistical Quality Control.* John Wiley & Sons, New York, NY.

RADO, L. E. (1989). "Enhance Product Development by Using Capability Indexes." *Quality Progress* 22, 4, pp. 38–41.

Key Words: *Estimators of C_p, CPU, CPL, and C_{pk} Indices.*