1 Physical Simulation

1.1 Rigid Body Time Integrator

$$L(c^i, \theta^i, c^{i+1}, \theta^{i+1}) = \sum_{bodies} \Big[\frac{\rho V}{2}\Big\|\frac{c^{i+1}-c^i}{h}\Big\|^2 + \frac{\rho}{2}\omega(\theta^i, \theta^{i+1})^T R_{-\theta^i}^T M_I R_{-\theta^i}\omega(\theta^i, \theta^{i+1})\Big] - V(c^{i+1}, \theta^{i+1})$$

$$\frac{\partial}{\partial c^i} L(c^i, \theta^i, c^{i+1}, \theta^{i+1}) = \sum_{bodies} \Big[-\rho V \frac{c^{i+1}-c^i}{h^2}\Big]^T$$

$$\begin{aligned}
\frac{\partial}{\partial \theta^i} L(c^i, \theta^i, c^{i+1}, \theta^{i+1}) &= \sum_{bodies} \frac{\partial}{\partial \theta^i}\Big[\frac{\rho}{2}\omega(\theta^i, \theta^{i+1})^T R_{-\theta^i}^T M_I R_{-\theta^i}\omega(\theta^i, \theta^{i+1})\Big] \\
&= \sum_{bodies} \Big[\rho\omega(\theta^i, \theta^{i+1})^T R_{-\theta^i}^T M_I \Big(\frac{\partial}{\partial \theta^i}\big[R_{-\theta^i}\omega(\theta^i, \theta^{i+1})\big]\Big)\Big] \\
&= \sum_{bodies} \Big[\rho\omega(\theta^i, \theta^{i+1})^T R_{-\theta^i}^T M_I \Big(\frac{\partial}{\partial \theta^i} R_{-\theta^i}\omega(\theta^i, \theta^{i+1}) + R_{-\theta^i}\frac{\partial}{\partial \theta^i}\omega(\theta^i, \theta^{i+1})\Big)\Big] \\
&= \sum_{bodies} \Big[\rho\omega(\theta^i, \theta^{i+1})^T R_{-\theta^i}^T M_I \Big(-R_{-\theta^i}[\omega(\theta^i, \theta^{i+1})]_\times T(-\theta_i) + R_{-\theta^i}\frac{-1}{h}T(h\omega(\theta^i, \theta^{i+1}))^{-1}T(-\theta^i)\Big)\Big] \\
&= \sum_{bodies} \Big[\rho\omega(\theta^i, \theta^{i+1})^T R_{-\theta^i}^T M_I R_{-\theta^i}\Big(-[\omega(\theta^i, \theta^{i+1})]_\times T(-\theta_i) + \frac{-1}{h}T(h\omega(\theta^i, \theta^{i+1}))^{-1}\Big)T(-\theta^i)\Big]
\end{aligned}$$

$$\frac{\partial}{\partial c^{i+1}} L(c^i, \theta^i, c^{i+1}, \theta^{i+1}) = \sum_{bodies} \Big[\rho V \frac{c^{i+1}-c^i}{h^2}\Big]^T - \frac{\partial}{\partial c^{i+1}} V(c^{i+1}, \theta^{i+1})$$

$$\begin{aligned}
\frac{\partial}{\partial \theta^{i+1}} L(c^i, \theta^i, c^{i+1}, \theta^{i+1}) &= \sum_{bodies} \frac{\partial}{\partial \theta^{i+1}}\Big[\frac{\rho}{2}\omega(\theta^i, \theta^{i+1})^T R_{-\theta^i}^T M_I R_{-\theta^i}\omega(\theta^i, \theta^{i+1})\Big] - \frac{\partial}{\partial \theta^{i+1}} V(\theta^i, \theta^{i+1}) \\
&= \sum_{bodies} \Big[\rho\omega(\theta^i, \theta^{i+1})^T R_{-\theta^i}^T M_I R_{-\theta^i}\big(\frac{\partial}{\partial \theta^{i+1}}\omega(\theta^i, \theta^{i+1})\big)\Big] - \frac{\partial}{\partial \theta^{i+1}} V(\theta^i, \theta^{i+1}) \\
&= \sum_{bodies} \Big[\rho\omega(\theta^i, \theta^{i+1})^T R_{-\theta^i}^T M_I R_{-\theta^i}\big(\frac{1}{h}T(-h\omega(\theta^i, \theta^{i+1}))^{-1}T(-\theta^{i+1})\big)\Big] - \frac{\partial}{\partial \theta^{i+1}} V(\theta^i, \theta^{i+1})
\end{aligned}$$

1.2 Euler-Lagrange Equations

$$d_2L(q^{i-1}, q^i) + d_1L(q^i, q^{i+1}) = 0$$

For the tangential velocity, this is simply particle motion, and follows the force = mass velocity form.

$$\begin{aligned}
\sum_{bodies} \Big[\rho V \frac{c^i - c^{i-1}}{h^2}\Big]^T - \frac{\partial}{\partial c^i} V(c^i, \theta^i) + \sum_{bodies} \Big[- \rho V \frac{c^{i+1} - c^i}{h^2}\Big]^T &= 0 \\
\sum_{bodies} \Big[\rho V \frac{c^{i+1} - c^i}{h^2}\Big]^T &= \sum_{bodies} \Big[\rho V \frac{c^i - c^{i-1}}{h^2}\Big]^T - \frac{\partial}{\partial c^i} V(c^i, \theta^i) \\
\sum_{bodies} \Big[\rho V \dot{c}_{i+1}\Big] &= \sum_{bodies} \Big[\rho V \dot{c}_i\Big] - h \frac{\partial}{\partial c^i} V(c^i, \theta^i)^T \\
c_{i+1} &= c^i + h\dot{c}^i \\
\dot{c}_{i+1} &= \dot{c}_i - h \frac{1}{\rho V} dV(c^i, \theta^i)^T \quad \forall i \in bodies
\end{aligned}$$

Solving for angular velocity, we can see each rigid body is an independent system, and since the potential $V(c, \theta)$ in this case only includes gravity, $d_2V(c, \theta) = 0$.

$$\begin{aligned}
\Big[\rho\omega(\theta^{i-1}, \theta^i)^T R^T_{-\theta^{i-1}} M_I R_{-\theta^{i-1}} (\frac{1}{h} T(-h\omega(\theta^{i-1}, \theta^i))^{-1} T(-\theta^i))\Big] + \Big[\rho\omega(\theta^i, \theta^{i+1})^T R^T_{-\theta^i} M_I R_{-\theta^i} \Big([\omega(\theta^i, \theta^{i+1})]_\times T(-\theta_i) + \frac{-1}{h} T(h\omega(\theta^i, \theta^{i+1}))^{-1}\Big) T(-\theta^i)\Big] &= 0 \\
\Big[\rho\omega(\theta^{i-1}, \theta^i)^T R^T_{-\theta^{i-1}} M_I R_{-\theta^{i-1}} \frac{1}{h} T(-h\omega(\theta^{i-1}, \theta^i))^{-1}\Big] + \Big[\rho\omega(\theta^i, \theta^{i+1})^T R^T_{-\theta^i} M_I R_{-\theta^i} \Big([\omega(\theta^i, \theta^{i+1})]_\times T(-\theta_i) + \frac{-1}{h} T(h\omega(\theta^i, \theta^{i+1}))^{-1}\Big)\Big] &= 0
\end{aligned}$$

The goal is to compute the orientation at the next time step, θ^{i+1}. This can be done through solving for $\omega(\theta^i, \theta^{i+1})$, since $R_{h\omega(\theta^i, \theta^{i+1})} = R_{\theta^{i+1}} R_{-\theta^i}$.

Using properties of rotation matrices, we can see $R^T_{-\theta^i} R_{h\omega(\theta^i, \theta^{i+1}} = R_{\theta^{i+1}}$.

Let $\omega^i = \omega(\theta^{i-1}, \theta^i)$

$$\begin{aligned}
\Big[\rho(\omega^i)^T R^T_{-\theta^{i-1}} M_I R_{-\theta^{i-1}} \frac{1}{h} T(-h\omega^i)^{-1}\Big] + \Big[\rho(\omega^{i+1})^T R^T_{-\theta^i} M_I R_{-\theta^i} \Big([\omega^{i+1}]_\times T(-\theta_i) + \frac{-1}{h} T(h\omega^{i+1})^{-1}\Big)\Big] &= 0 \\
\Big[(\omega^i)^T R^T_{-\theta^{i-1}} M_I R_{-\theta^{i-1}} T(-h\omega^i)^{-1}\Big] + \Big[(\omega^{i+1})^T R^T_{-\theta^i} M_I R_{-\theta^i} \Big(h[\omega^{i+1}]_\times T(-\theta_i) - T(h\omega^{i+1})^{-1}\Big)\Big] &= 0
\end{aligned}$$

We can use Newton's method to solve for ω^{i+1}, where

$$\begin{aligned}
f(\omega^{i+1}) &= \Big[(\omega^i)^T R^T_{-\theta^{i-1}} M_I R_{-\theta^{i-1}} T(-h\omega^i)^{-1}\Big] + \Big[(\omega^{i+1})^T R^T_{-\theta^i} M_I R_{-\theta^i} \Big(h[\omega^{i+1}]_\times T(-\theta_i) - T(h\omega^{i+1})^{-1}\Big)\Big] \\
&= 0 \\
df(\omega^{i+1}) &= R^T_{-\theta^i} M_I R_{-\theta^i} \Big(h[\omega^{i+1}]_\times T(-\theta_i) - T(h\omega^{i+1})^{-1}\Big) + (\omega^i)^T R^T_{-\theta^{i-1}} M_I R_{-\theta^{i-1}} \Big(h \; d[\omega^{i+1}]_\times T(-\theta_i) - h \; dT(h\omega^{i+1})^{-1}\Big) \\
&\approx R^T_{-\theta^i} M_I R_{-\theta^i} \Big(- T(h\omega^{i+1})^{-1}\Big)
\end{aligned}$$

1.3 Integration over a Triangle

Given - arbitrary points of a triangle u, v, w, and a function f that you want to integrate over the face of the triangle.

Let's consider the basic function $f(u, v, w) = 1$ for now, so this integration should give us the area of the triangle - $\|(v - u) \times (w - u)\|$.

First step is to parameterize this into a function of two parameters, s, t, using the basis vectors $v - u$, and $w - u$.

This can be thought of sliding across the plane that is covered by the triangle.

This leads us to the double integral -

$$\begin{aligned}
\int_0^1 \int_0^{1-s} f(s,t)\ dt\ ds &= \int_0^1 t|_0^{1-s}\ ds \\
&= \int_0^1 1 - s\ ds \\
&= s - \frac{1}{2}s^2|_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

The true area of this triangle should be $\frac{\|(v-u)\times(w-u)\|}{2}$, so we can see we're off by a factor of $\|(v - u) \times (w - u)\|$, which is caused by fact that these basis vectors are no longer unit length.