CS 395T: Assignment 1

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1 Computing Directional Derivatives

Problem 1a. $f = \|(x_1, x_2) - (x_3, x_4)\|^2, \mathbf{q} = [-1, -1, 1, 1]^T, \mathbf{w} = [0, 0, 1, 0]^T$.

$$\begin{aligned}
D_{\mathbf{w}} f(\mathbf{q}) &= \frac{d}{ds} f(\mathbf{q} + s\mathbf{w})\big|_{s\to 0} \\
&= \frac{d}{ds} \|(q_1 + sw_1, q_1 + sw_2) - (q_3 + sw_3, q_4 + sw_4)\|^2\big|_{s\to 0} \\
&= \frac{d}{ds} \|\big((q_1 - q_3) + s(w_1 - w_3), (q_2 - q_4) + s(w_2 - w_4)\big)\|^2\big|_{s\to 0} \\
&= \big(2(q_1 - q_3)(w_1 - w_3) + 2s(w_1 - w_3)^2\big) + \big(2(q_2 - q_4)(w_2 - w_4) + 2s(w_2 - q_4)^2\big)\big|_{s\to 0} \\
&= 2\big((q_1 - q_3)(w_1 - w_3) + (q_2 - q_4)(w_2 - w_4)\big) \\
&= 4
\end{aligned}$$

Problem 1b. $f = \frac{(x_1, x_2)}{\|(x_1, x_2)\|}, \mathbf{q} = [1, 2]^T, \mathbf{w} = [-3, -6]^T$.

$$\begin{aligned}
D_{\mathbf{w}} f(\mathbf{q}) &= \frac{d}{ds} f(\mathbf{q} + s\mathbf{w})\big|_{s\to 0} \\
&= \frac{d}{ds} f(q_1 + sw_1, q_2 + sw_2)\big|_{s\to 0} \\
&= \frac{d}{ds} \frac{(q_1 + sw_1, q_2 + sw_2)}{\|(q_1 + sw_1, q_2 + sw_2)\|}\big|_{s\to 0}
\end{aligned}$$

$$\frac{d}{ds} f_1(\mathbf{q} + s\mathbf{w})\big|_{s\to 0} = \frac{(q_1^2 + q_2^2)^{-\frac{1}{2}}(w_1) - (q_1)\frac{1}{2}\big(q_1^2 + q_2^2\big)^{-\frac{1}{2}}(2q_1w_1 + 2q_2w_2)}{q_1^2 + q_2^2}$$

$$\frac{d}{ds} f_2(\mathbf{q} + s\mathbf{w})\big|_{s\to 0} = \frac{(q_1^2 + q_2^2)^{-\frac{1}{2}}(w_2) - (q_2)\frac{1}{2}\big(q_1^2 + q_2^2\big)^{-\frac{1}{2}}(2q_1w_1 + 2q_2w_2)}{q_1^2 + q_2^2}$$

$$D_{\mathbf{w}} f(\mathbf{q}) = (0, 0)$$

This answer makes sense intuitively. The function f returns the unit vector of $[x_1, x_2]^T$, and since the directional vector $\mathbf{w}$ is just a scalar multiple of the initial vector $\mathbf{q}$, there is no change in the normalized vector.

Problem 1c. $f = (0,0,1) \times (x_1, x_2, x_3), \mathbf{q} = [1,0,0]^T, \mathbf{w} = [1,-1,2]^T$.

$$\begin{aligned}
f(x_1, x_2, x_3) &= (0,0,1) \times (x_1, x_2, x_3) \\
&= (-x_2, x_1, 0) \\
D_{\mathbf{w}} f(\mathbf{q}) &= \frac{d}{ds} f(\mathbf{q} + s\mathbf{w})\big|_{s\to 0} \\
\frac{d}{ds} f_1(\mathbf{q} + s\mathbf{w})\big|_{s\to 0} &= \frac{d}{ds} - (q_2 + sw_2)\big|_{s\to 0} \\
&= -w_2 \\
\frac{d}{ds} f_2(\mathbf{q} + s\mathbf{w})\big|_{s\to 0} &= \frac{d}{ds} q_1 + sw_1\big|_{s\to 0} \\
&= w_1 \\
\frac{d}{ds} f_3(\mathbf{q} + s\mathbf{w})\big|_{s\to 0} &= 0 \\
D_{\mathbf{w}} f(\mathbf{q}) &= (1,1,0)
\end{aligned}$$

Problem 1d. $f = \frac{1}{2}\mathbf{q}^T M \mathbf{q}$, where M is a square symmetric matrix.

Since M is symmetric, $u^T M v = v^T M u$ for all u, v.

$$\begin{aligned}
D_{\mathbf{w}} f(\mathbf{q}) &= \frac{d}{ds}\frac{1}{2}(\mathbf{q} + s\mathbf{w})^T M (\mathbf{q} + s\mathbf{w})\Big|_{s\to 0} \\
&= \frac{d}{ds}\frac{1}{2}\big(\mathbf{q}^T M \mathbf{q} + 2s\mathbf{q}^T M \mathbf{w} + s^2 w^T M w\big)\Big|_{s\to 0} \\
&= \mathbf{q}^T M \mathbf{w}
\end{aligned}$$

Problem 1e. $f = (\cos x_1, \sin x_1), \mathbf{q} = [\frac{\pi}{2}], \mathbf{w} = [2]$.

$$\begin{aligned}
D_{\mathbf{w}} f(\mathbf{q}) &= \frac{d}{ds}\big(cos(q_1 + sw_1), sin(q_1 + sw_1)\big)\Big|_{s\to 0} \\
\frac{d}{ds} f_1(\mathbf{q} + s\mathbf{w})\Big|_{s\to 0} &= \frac{d}{ds} cos(q_1 + sw_1)\Big|_{s\to 0} \\
&= -sin(q_1) w_1 \\
\frac{d}{ds} f_2(\mathbf{q} + s\mathbf{w})\Big|_{s\to 0} &= \frac{d}{ds} sin(q_1 + sw_1)\Big|_{s\to 0} \\
&= cos(q_1) w_1 \\
D_{\mathbf{w}} f(\mathbf{q}) &= (-2, 0)
\end{aligned}$$

2 Computing the Symbolic Differential

Problem 2a. Least squares energy $f(\mathbf{q}) = \|A\mathbf{q} - \mathbf{b}\|^2$.

Let $\mathbf{q}, \mathbf{w} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$.

$$\begin{aligned}
[df(\mathbf{q})](\mathbf{w}) &= D_{\mathbf{w}} f(\mathbf{q}) \\
&= \frac{d}{ds} \|A(\mathbf{q} + s\mathbf{w}) - \mathbf{b}\|^2 \Big|_{s \to 0} \\
&= \frac{d}{ds} \|A(\mathbf{q} + s\mathbf{w})\|^2 - 2\mathbf{b}^T (A(\mathbf{q} + s\mathbf{w})) + \|\mathbf{b}\|^2 \Big|_{s \to 0} \\
&= \frac{d}{ds} \sum_{i=1}^{n} (A(\mathbf{q} + s\mathbf{w}))_i^2 - 2 \sum_{i=1}^{n} \mathbf{b}_i (A(\mathbf{q} + s\mathbf{w}))_i + \|\mathbf{b}\|^2 \Big|_{s \to 0} \\
&= \frac{d}{ds} \sum_{i=1}^{n} (A\mathbf{q} + sA\mathbf{w})_i^2 - 2 \sum_{i=1}^{n} \mathbf{b}_i (A\mathbf{q} + sA\mathbf{w})_i + \|\mathbf{b}\|^2 \Big|_{s \to 0} \\
&= 2 \sum_{i=1}^{n} (A\mathbf{q})_i (A\mathbf{w})_i - 2 \sum_{i=1}^{n} \mathbf{b}_i (A\mathbf{w})_i \\
&= 2(A\mathbf{q} - \mathbf{b})^T (A\mathbf{w})
\end{aligned}$$

Problem 2b. Matrix trace $f(M) = \operatorname{tr}(M)$.

Let $M, \delta M \in \mathbb{R}^{n \times n}$.

$$\begin{aligned}
[df(M)](\delta M) &= D_{\delta M} f(M) \\
&= \frac{d}{ds} \operatorname{tr}(M + s(\delta M)) \Big|_{s \to 0} \\
&= \frac{d}{ds} \operatorname{tr}(M) + (s)\operatorname{tr}(\delta M) \Big|_{s \to 0} \\
&= \operatorname{tr}(\delta M)
\end{aligned}$$

Problem 2c. The angle sine $f(\mathbf{q}) = \|\mathbf{u}(\mathbf{q}) \times \mathbf{v}(\mathbf{q})\|$, where $\mathbf{u}, \mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$, and $\mathbf{q} \in \mathbb{R}^n$.

$$\begin{aligned}
[df(q)](\delta q) &= D_{\delta \mathbf{q}} f(\mathbf{q}) \\
&= \frac{\Big(\mathbf{u}(\mathbf{q}) \times \mathbf{v}(\mathbf{q})\Big)^T \Big(([d\mathbf{u}]\delta\mathbf{q}) \times \mathbf{v}(\mathbf{q}) + \mathbf{u}(\mathbf{q}) \times \big([d\mathbf{v}]\delta\mathbf{q}\big)\Big)}{\|\mathbf{u}(\mathbf{q}) \times \mathbf{v}(\mathbf{q})\|}
\end{aligned}$$

The result follows from using chain rule on the differential of the norm and cross product.

Problem 2d. Tetrahedron volume $f(\mathbf{q}) = \frac{1}{6}((\mathbf{u}-\mathbf{q}) \times (\mathbf{v}-\mathbf{q})) \cdot (\mathbf{w}-\mathbf{q})$ as a function of one of the tetrahedron's corners $\mathbf{q}$.

Let $\mathbf{q}, \delta\mathbf{q}, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, where $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{q}$, are positions of the tetrahedron's vertices.

$$\begin{aligned}
[df(\mathbf{q})](\delta\mathbf{q}) &= D_{\delta\mathbf{q}} f(\mathbf{q}) \\
&= \frac{d}{ds}\frac{1}{6}((\mathbf{u}-(\mathbf{q}+s\delta\mathbf{q})) \times (\mathbf{v}-(\mathbf{q}+s\delta\mathbf{q}))) \cdot (\mathbf{w}-(\mathbf{q}+s\delta\mathbf{q}))\Big|_{s\to 0} \\
&= \frac{d}{ds}\frac{1}{6}((\mathbf{u}-\mathbf{q}-s\delta\mathbf{q}) \times (\mathbf{v}-\mathbf{q}-s\delta\mathbf{q})) \cdot (\mathbf{w}-(\mathbf{q}+s\delta\mathbf{q}))\Big|_{s\to 0} \\
&= -\frac{1}{6}\big(((\mathbf{u}-\mathbf{q}) \times (\mathbf{v}-\mathbf{q})) \cdot (\mathbf{w}-\delta\mathbf{q}) + \\
&\qquad ((\mathbf{u}-\mathbf{q}) \times \delta\mathbf{q}) \cdot (\mathbf{w}-\mathbf{q}) + \\
&\qquad (\delta\mathbf{q} \times (\mathbf{v}-\mathbf{q})) \cdot (\mathbf{w}-\mathbf{q})\big)
\end{aligned}$$

Problem 2e. (Challenging): smallest eigenvalue $\lambda_1(M)$ of a real symmetric matrix M (with nondegenerate eigenvalues).

$$\begin{aligned}
\|v_1(M)\| &= 1 \\
v_1(M)^T\Big(\big[dv_1(M)\big]\big(\delta M\big)\Big) &= 0 \\
\lambda_1(M) &= \|Mv_1(M)\| \\
[d\lambda_1(M)](\delta M) &= \frac{1}{2}\Big((Mv_1(M))^T Mv_1(M)\Big)^{-\frac{1}{2}}\Big[d\Big((Mv_1(M))^T Mv_1(M)\Big)\Big](\delta M) \\
&= \frac{1}{2}\frac{\Big[d\Big((Mv_1(M))^T Mv_1(M)\Big)\Big](\delta M)}{\|Mv_1(M))\|} \\
&= \frac{\Big(Mv_1(M)\Big)^T\Big[d\big(Mv_1(M)\big)\Big]\Big(\delta M\Big)}{\|Mv_1(M))\|} \\
&= \frac{\Big(Mv_1(M)\Big)^T\Big(M\big([dv_1(M)](\delta M)\big) + \delta M\big(v_1(M)\big)\Big)}{\|Mv_1(M))\|} \\
&= \frac{\Big(Mv_1(M)\Big)^T\Big(M\big([dv_1(M)](\delta M)\big) + \delta M\big(v_1(M)\big)\Big)}{\lambda_1(M)} \\
&= \Big(v_1(M)\Big)^T\Big(M\big([dv_1(M)](\delta M)\big) + \delta M\big(v_1(M)\big)\Big) \\
&= v_1(M)^T\Big(\delta M\big(v_1(M)\big)\Big)
\end{aligned}$$

Note to Grader: All problems except for 2e were coded up in Matlab and verified through comparison of numerical and analytical solutions.

Also, there are a lot of answers using the $\frac{d}{ds} f(\mathbf{q} + s\mathbf{w})\Big|_{s\to 0}$ method because that's the only one I understood before 1/24's lecture and I was too lazy to go back and edit them.