M348: Assignment 4

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Exercise 2.1.2. $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.

Proof. Show T is a linear transformation, $T(cx + y) = cT(x) + T(y)$ for $\forall x, y \in V, \forall c \in \mathbb{F}$.

Let $x = (a_1, a_2, a_3), y = (b_1, b_2, b_3), c \in \mathbb{F}$.

$$\begin{aligned}
T(ca_1 + b_1, ca_2 + b_2, ca_3 + b_3) &= (ca_1 + b_1 - ca_2 - b_2, 2(ca_3 + b_3)) \\
&= (ca_1 - ca_2, 2ca_3) + (b_1 - b_2, b_3) \\
&= c(a_1 - a_2, 2a_3) + (b_1 - b_2, b_3) \\
&= cT(a_1, a_2, a_3) + T(b_1, b_2, b_3)
\end{aligned}$$

We can see T is a linear transformation.

□

Proof. $Dim(Null(T)) = 1$.

We need to find the $Null(T)$, the set $\{(a_1, a_2, a_3) \mid T(a_1, a_2, a_3) = (0, 0)\}$.
$T(a_1, a_2, a_3) = (0, 0) \iff (a_1 - a_2, 2a_3) = (0, 0)$.
This can only be true if $a_1 = a_2$ and $a_3 = 0$.
We can see the set $\{(1, 1, 0)\}$ forms a basis for $Null(T)$, and $Dim(Null(T)) = 1$.

□

Proof. $Dim(Range(T)) = 2$.

We need to find the $Range(T)$.

$$\begin{aligned}
Range(T) &= \{T(a_1, a_2, a_3)\ \forall a_1, a_2, a_3 \in \mathbb{R}\}. \\
&= \{(a_1 - a_2, a_3)\ \forall a_1, a_2, a_3 \in \mathbb{R}\}.
\end{aligned}$$

We can see $\forall x \in \mathbb{R}, \exists a_1, a_2 \in \mathbb{R}$ such that $x = a_1 - a_2$.
Similarly, $\forall y \in \mathbb{R}, \exists a_3 \in \mathbb{R}$ such that $y = 2a_3$.
We can see the set $\{(1,0),(0,1)\}$ forms a basis for $Range(T)$, and $Dim(Range(T)) = 2$.

□

Proof. The rank nullity theorem is satisfied.

$$\begin{aligned} Dim(Null(T)) + Dim(Range(T)) &= 1 + 2 \\ &= Dim(Range(V)) \end{aligned}$$

We can see the rank nullity theorem is is satisfied.

□

Proof. T is not one-to-one.

Since $Dim(Null(T)) = 1 \neq 0$, T is not one-to-one.

□

Proof. T is onto.

Since $Dim(Range(T)) = 2 = Dim(Range(W))$, T is onto.

□

Exercise 2.1.3. $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.

Proof. Show T is a linear transformation, $T(cx + y) = cT(x) + T(y)$ for $\forall x, y \in V, \forall c \in \mathbb{F}$.

Let $x = (a_1, a_2), y = (b_1, b_2), c \in \mathbb{F}$.

$$\begin{aligned} T(ca_1 + b_1, ca_2 + b_2) &= (ca_1 + b_1 + ca_2 + b_2, 0, 2(ca_1 + b_1) - (ca_2 + b_2)) \\ &= (ca_1 + ca_2, 0, 2ca_1 - ca_2) + (b_1 + b_2, 0, 2b_1 - b_2) \\ &= c(a_1 + a_2, 0, 2a_1 - a_2) + (b_1 + b_2, 0, 2b_1 - b_2) \\ &= cT(a_1, a_2) + T(b_1, b_2) \end{aligned}$$

We can see T is a linear transformation.

□

Proof. $Dim(Null(T)) = 0$.

We need to find the $Null(T)$, the set $\{(a_1, a_2) \mid T(a_1, a_2) = (0,0,0)\}$.
$T(a_1, a_2) = (0,0,0) \iff (a_1 + a_2, 0, 2a_1 - a_2) = (0,0,0)$.

This can only be true if $a_1 = -a_2$ and $a_1 = -1/2a_2$.
This is true for $a_1 = 0 = a_2$.
We can see the set $\{(0,0)\}$ forms a basis for $Null(T)$, and $Dim(Null(T)) = 0$.

$\square$

Proof. $Dim(Range(T)) = 2$.

We need to find the $Range(T)$.

$$\begin{aligned} Range(T) &= span(\{T(\beta_i)\ \forall \beta_i \in \{(1,0),(0,1)\}\}). \\ &= span(\{(1,0,2),(1,0,-1)\}). \end{aligned}$$

The two vectors are linearly independent, and we can form a basis for W.
We can see the set $\{(1,0,2),(1,0,-1)\}$ forms a basis for $Range(T)$, and $Dim(Range(T)) = 2$.

$\square$

Proof. The rank nullity theorem is satisfied.

$$\begin{aligned} Dim(Null(T)) + Dim(Range(T)) &= 0 + 2 \\ &= Dim(Range(V)) \end{aligned}$$

We can see the rank nullity theorem is is satisfied.

$\square$

Proof. T is one-to-one.

Since $Dim(Null(T)) = 0$, T is one-to-one.

$\square$

Proof. T is not onto.

Since $Dim(Range(T)) = 2 \neq Dim(Range(W))$, T is not onto.

$\square$

Exercise 2.1.4. $T : M_{2\times 3}(F) \to M_{2\times 2}(F)$ defined by

$$\mathrm{T}(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}) = \begin{bmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{bmatrix}$$

Proof. Show T is a linear transformation, $T(cx+y) = cT(x) + T(y)$ for $\forall x, y \in V, \forall c \in \mathbb{F}$.

Let $x = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $y = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$, $c \in \mathbb{F}$.

$$\begin{aligned}
T(cx+y) &= T\big(c \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\big) \\
&= T\big(\begin{bmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\big) \\
&= T\big(\begin{bmatrix} ca_{11}+b_{11} & ca_{12}+b_{12} & ca_{13}+b_{13} \\ ca_{21}+b_{21} & ca_{22}+b_{22} & ca_{23}+b_{23} \end{bmatrix}\big) \\
&= \begin{bmatrix} 2(ca_{11}+b_{11}) - (ca_{12}+b_{12}) & (ca_{13}+b_{13}) + 2(ca_{12}+b_{12}) \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2ca_{11} - ca_{12} + 2b_{11} - b_{12} & ca_{13}2ca_{12} + b_{13} + 2b_{12} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2ca_{11} - ca_{12} & ca_{13}2ca_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2b_{11} - b_{12} & b_{13} + 2b_{12} \\ 0 & 0 \end{bmatrix} \\
&= c\begin{bmatrix} ca_{11} - a_{12} & a_{13}2a_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2b_{11} - b_{12} & b_{13} + 2b_{12} \\ 0 & 0 \end{bmatrix} \\
&= cT\big(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\big) + T\big(\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\big)
\end{aligned}$$

We can see T is a linear transformation.

□

Proof. $Dim(Null(T)) = 4$.

We need to find the $Null(T)$, the set $\{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \mid T\big(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\big) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$.

$$\begin{aligned}
T\big(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\big) &= \begin{bmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

This can only be true if $2a_{11} = a_{12} = -1/2a_{13}$.
We can see the set $\{\begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\}$ forms a basis for $Null(T)$, and $Dim(Null(T)) = 4$.

□

Proof. $Dim(Range(T)) = 4$.

We need to find the $Range(T)$, we can do this by applying T to the basis of $M_{2\times 3}(F)$.
Consider $Span(\{T(\beta_i) \mid \beta_i \in \{\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\end{bmatrix}\}\})$

$$\begin{aligned}
Span(\{T(\beta)\}) = Span(\{&T(\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\end{bmatrix}), T(\begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\end{bmatrix}), T(\begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\end{bmatrix}), T(\begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\end{bmatrix}),\\
&T(\begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\end{bmatrix}), T(\begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\end{bmatrix})\})\\
= Span(\{&\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}-1 & 2\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\})\\
= Span(\{&\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix},\})
\end{aligned}$$

We can see these two matrices span $Range(T)$ and are linearly independent, so the set $\{\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix},\}$ forms a basis for $Range(T)$, and $Dim(Range(T)) = 2$.

□

Proof. The rank nullity theorem is satisfied.

$$\begin{aligned}
Dim(Null(T)) + Dim(Range(T)) &= 4 + 2\\
&= Dim(Range(V))
\end{aligned}$$

We can see the rank nullity theorem is is satisfied.

□

Proof. T is not one-to-one.

Since $Dim(Null(T)) = 4 \neq= 0$, T is not one-to-one.

□

Proof. T is not onto.

Since $Dim(Range(T)) = 2 \neq Dim(Range(W))$, T is not onto.

□

Exercise 2.1.5. $T : P_2(R) \to P_3(R)$ defined by $T(f(x)) = xf(x) + f'(x)$.

Proof. Show T is a linear transformation, $T(cx + y) = cT(x) + T(y)$ for $\forall x, y \in V, \forall c \in \mathbb{F}$.

Let $f, g \in P_2(R), c \in \mathbb{F}$.

$$
\begin{aligned}
T(cf(x)+g(x)) &= T((cf+g)(x)) \\
&= x(cf+g)(x) + (cf'+g')(x) \\
&= xcf(x) + xg(x) + cf'(x) + g'(x) \\
&= c(xf(x) + f'(x)) + (xg(x) + g'(x)) \\
&= cT(f(x)) + T(g(x))
\end{aligned}
$$

We can see T is a linear transformation.

$\square$

Proof. $Dim(Null(T)) = 0$.
We need to find $Null(T)$, the set of polynomials $\{f \mid f \in P_2(R);\ T(f(x)) = 0\}$

Consider $T(f(x)) = xf(x) + f'(x)$.
$xf(x) + f'(x) = 0$
$\iff xf(x) = -f'(x)$
$\iff f(x) = \frac{-cf'(x)}{x}$.
If $f \in P_2(R)$ f is either a degree 0, 1, or 2 polynomial. We'll call it n.
$\implies f'$ is of degree n-1.
$\implies \frac{-cf'(x)}{x}$ is of degree n-2.
So this equation is only satisfied when f is the zero function.
We can see that set of polynomials $\{0\}$ forms a basis for $Null(T)$, so $Dim(Null(T)) = 0$.

$\square$

Proof. $Dim(Range(T)) = 3$.
We need to find $Range(T)$, the set of polynomials spanned by T.

We can do this by applying T to the basis of $P_2(F)$.

$$
\begin{aligned}
Range(T) &= Span(\{T(f(x)) \mid f(x) \in \{1, x, x^2\}\}) \\
&= Span(\{x, x^2+1, x^3+2x\})
\end{aligned}
$$

These three polynomials are linearly independent, and form a basis for $P_3(F)$. We can see $Dim(Range(t)) = Dim(\{x, x^2+1, x^3+2x\}) = 3$.

$\square$

Proof. The rank nullity theorem is satisfied.

$$
\begin{aligned}
Dim(Null(T)) + Dim(Range(T)) &= 0 + 3 \\
&= Dim(Range(V))
\end{aligned}
$$

We can see the rank nullity theorem is is satisfied.

∎

Proof. T is one-to-one.

Since $Dim(Null(T)) = 0$, T is one-to-one.

∎

Proof. T is not onto.

Since $Dim(Range(T)) = 3 \neq Dim(Range(W))$, T is not onto.

∎

Exercise 2.1.7. Prove the following properties about linear transformations.

Proof. If T is linear, then $T(0) = 0$.

Take any $x \in V$.

$$\begin{aligned} T(x) &= T(x+0) \text{ By additive zero in F.} \\ &= T(x) + T(0) \text{ By definition linear transformation.} \end{aligned}$$

By equality, we see $T(0)$ must equal 0.

∎

Proof. T is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x, y \in V,\ c \in F$.

Proof of ($\Longrightarrow$).

$$\begin{aligned} T(cx+y) &= T(cx) + T(y) \text{ Definition linear transformation.} \\ &= cT(x) + T(y) \text{ Definition linear transformation.} \end{aligned}$$

Proof of ($\Longleftarrow$).

Take $c = 1$.

$$\begin{aligned} T(1x+y) &= T(x+y). \\ &= T(x) + T(y). \end{aligned}$$

This gives us the additive property of linear transformations.

Take $x \in V, y = 0, c \in F$.

$$\begin{aligned} T(cx + 0) &= T(cx). \\ &= cT(x). \end{aligned}$$

This gives us property of scalar multiplication of linear transformations.

By proof of ($\Longleftarrow$) and ($\Longrightarrow$), we are done.

$\square$

Proof. If T is linear, then $T(x - y) = T(x) - T(y)$.

Take any $x \in V, c = -1, y \in V$.

$$\begin{aligned} T(x - y) &= T(x + cy) \text{ By choice of c.} \\ &= T(x) + T(cy) \text{ By addition over linear transformations.} \\ &= T(x) + cT(y) \text{ By scalar multiplication over linear transformations.} \\ &= T(x) - T(y) \text{ By choice of c.} \end{aligned}$$

We see that $T(x - y) = T(x) - T(y)$.

$\square$

Proof. T is linear if and only if, for $x_1, x_2, \ldots, x_n \in V$ and $a_1, a_2, \ldots, a_n \in F$, we have $T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i T(x_i)$.

Proof of ($\Longrightarrow$).
Let n = 2.

$$\begin{aligned} T(\sum_{i=1}^{2} a_i x_i) &= T(a_1 x_1 + a_2 x_2) \\ &= T(a_1 x_1) + T(a_2 x_2) \text{ Definition linear transformation.} \\ &= a_1 T(x_1) + a_2 T(x_2) \text{ Scalar multiplication over linear transformations.} \\ &= \sum_{i=1}^{2} a_i T(x_i) \end{aligned}$$

Assume for induction, that for some n = k, we have $T(\sum_{i=1}^{k} a_i x_i) = \sum_{i=1}^{k} a_i T(x_i)$.

Let n = k+1.

$$
\begin{aligned}
T(\sum_{i=1}^{k+1} a_i x_i) &= T(\sum_{i=1}^{k} a_i x_i + a_{k+1}x_{k+1}) \\
&= T(\sum_{i=1}^{k} a_i x_i) + T(a_{k+1}x_{k+1}) \text{ By addition over linear transformations.} \\
&= \sum_{i=1}^{k} a_i T(x_i) + T(a_{k+1}x_{k+1}) \text{ By induction hypothesis.} \\
&= \sum_{i=1}^{k} a_i T(x_i) + a_{k+1}T(x_{k+1}) \text{ By scalar multiplication over linear transformations.} \\
&= \sum_{i=1}^{k+1} a_i T(x_i)
\end{aligned}
$$

Proof of ($\Longleftarrow$).

Let n = 2, $a_1 = a_2 = 1,\ x_1,\ x_1 \in V$.
We see that

$$
\begin{aligned}
T(\sum_{i=1}^{2} a_i x_i) &= T(a_1 x_1 + a_2 x_2) \\
&= T(x_1 + x_2) \text{ By choice of } a_1, a_2 \\
&= T(x_1) + T(x_2)
\end{aligned}
$$

We see that we have addition over linear transformations.

Let n = 1, $a_1 \in F,\ x_1, \in V$.
We see that

$$
\begin{aligned}
T(\sum_{i=1}^{i=1} a_i x_i) &= T(a_1 x_1) \\
&= a_1 T(x_1)
\end{aligned}
$$

We see that we have scalar multiplication over linear transformations.

By proof of ($\Longleftarrow$) and ($\Longrightarrow$), we are done.

$\square$

Exercise 2.1.10. Suppose that $T : R^2 \to R^2$, $T(1,0) = (1,4)$, $T(1,1) = (2,5)$.

Proof. $T(2,3) = T(5,11)$

Consider $T(0,1) = T(1,1) - T(1,0)$.

$$\begin{aligned} T(0,1) &= T(1,1) - T(1,0) \\ &= (2,5) - (1,4) \\ &= (1,1) \end{aligned}$$

Now we can represent $T(2,3) = 2T(1,0) + 3T(0,1)$.

$$\begin{aligned} T(2,3) &= 2T(1,0) + 3T(0,1) \\ &= 2(1,4) + 3(1,1) \\ &= (5,11) \end{aligned}$$

□

Proof. T is one-to-one.

We have to find $Dim(Null(T))$.
We have found $T(1,0),\ T(0,1)$.
By inspection, they are linearly independent and form a basis for $Range(T)$.
We can see that $Dim(Range(T)) = 2$.
By the rank nullity theorem, we know $Dim(Null(T)) + Dim(Range(T)) = 2$.
$\implies Dim(Null(T)) = 0$, which means T is one-to-one.

□

Exercise 2.1.13. Let $T : V \to W$ be linear, $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of $R(T)$.
Show that if $S = \{v_1, v_2, \ldots, v_n\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \ldots, k$, then S is linearly independent.

Proof. S is linearly independent.

Assume the contrary, that S is linearly dependent.
That is, $\exists a_i \in F,\ \sum_{i=1}^{k} |a_i| \neq 0$ such that $\sum_{i=1}^{k} a_i v_i = 0$ $(\star)$.
Consider $\sum_{i=1}^{k} a_i w_i$.

$$\begin{aligned}
\sum_{i=1}^{k} a_i w_i &= \sum_{i=1}^{k} a_i T(v_i) \\
&= T\Big(\sum_{i=1}^{k} a_i v_i\Big) \text{ By definition linear transform.} \\
&= T(0) \text{ By } (\star). \\
&= 0 \text{ By zero property of linear transformation. Contradiction}
\end{aligned}$$

We can see that S is linearly independent.

□

Exercise 2.1.14a. Let $T : V \to W$ be linear. Show that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

Proof. T one-to-one $\implies$ T preserves linearly independent subsets.

T one-to-one $\implies$ $Dim(Null(T)) = 0$.
$Dim(Null(T)) = 0 \implies \forall a_i \in F$ such that $\sum_{i=1}^{k} |a_i| \neq 0$, we have $\sum_{i=1}^{k} a_i v_i \neq 0$.
Since $\sum_{i=1}^{j} a_i w_i = \sum_{i=1}^{k} a_i T(v_i) = T(\sum_{i=1}^{k} a_i v_i)$.
We know $\sum_{i=1}^{k} a_i v_i \neq 0$, so $T(\sum_{i=1}^{k} a_i v_i) \neq 0$.
We can see T preserves linearly independent subsets.

□

Proof. T preserves linearly independent subsets $\implies$ T one-to-one.

Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a basis for V, by definition basis, β is linearly independent.
If T preserves linearly independent subsets, then $T(\beta)$ is linearly independent.
Show that if $T(\sum_{i=1}^{n} a_i \beta_i) = T(\sum_{i=1}^{n} b_i \beta_i)$ for $a_i, b_i \in F$, then $a_i = b_i$.
Consider $T(\sum_{i=1}^{n} a_i \beta_i) = T(\sum_{i=1}^{n} b_i \beta_i)$
$\implies T(\sum_{i=1}^{n} a_i \beta_i) - T(\sum_{i=1}^{n} b_i \beta_i) = 0$
$\implies T(\sum_{i=1}^{n} a_i \beta_i \sum_{i=1}^{n} b_i \beta_i) = 0$ By addition over linear transformation.
$\implies T(\sum_{i=1}^{n} (a_i - b_i) \beta_i) = 0$ By addition over F.
Since $T(\sum_{i=1}^{n} (a_i - b_i) \beta_i) = 0 \iff \sum_{i=1}^{n} (a_i - b_i) \beta_i = 0$,
and $\sum_{i=1}^{n} (a_i - b_i) \beta_i = 0 \iff (a_i - b_i) = 0$, since $\{\beta_i\}$ forms a basis for V,
we have $a_i = b_i$, and we see can T is one-to-one.

□

Proof. T one-to-one $\iff$ T preserves linearly independent subsets.

By proof of ($\implies$) and ($\impliedby$), we are done.

□

Exercise 2.1.14b. Let $T : V \to W$ be linear. Suppose that T is one-to-one and S is a subset of V. Prove that $S = \{v_1, v_2, \dots, v_n\}$ is linearly independent if and only if $T(S)$ is linearly independent.

Proof. S linearly independent $\implies T(S)$ linearly independent.

Assume for contradiction that $T(S)$ linearly dependent.
S linearly independent means $\sum_{i=1}^{n} a_i v_i = 0 \iff a_i = 0$.
$\implies \forall v \in V, v = \sum_{i=1}^{k} a_i v_i = 0 \iff a_i = 0$.
Consider $T(S)$. By assumption, $\sum_{i=1}^{k} a_i T(v_i) = 0$ and $\sum_{i=1}^{k} |a_i| > 0$.
$\implies \exists a_i$ such that $T(\sum_{i=1}^{k} a_i v_i) = 0$, contradiction.
We can see S linearly independent $\implies T(S)$ linearly independent.

□

Proof. $T(S)$ linearly independent $\implies S$ linearly independent.

$T(S)$ linearly independent $\implies \sum_{i=1}^{k} a_i T(v_i) = 0 \iff \sum_{i=1}^{k} |a_i| = 0$.
$\implies T(\sum_{i=1}^{k} a_i v_i) \iff \sum_{i=1}^{k} |a_i| = 0$.
$\implies \sum_{i=1}^{k} a_i v_i = 0 \iff \sum_{i=1}^{k} |a_i| = 0$.
By definition, we can see S is linearly independent.

□

Proof. S linearly independent $\iff T(S)$ linearly independent.

By proof of ($\implies$) and ($\impliedby$), we are done.

□

Exercise 2.1.15. Define $T : P(R) \to P(R)$ by $T(f(x)) = \int_0^x f(t)dt$.
Show T is linear, one-to-one, but not onto.

Proof. T is linear.

We need to show that $T(cf(x) + g(x)) = cT(f(x)) + T(g(x))$ for $f, g \in P(R),\ c \in R$.

$$\begin{aligned} T(cf(x) + g(x)) &= T((cf + g)(x)) \\ &= \int_0^x (cf + g)(t)dt \\ &= \int_0^x cf(t)dt + \int_0^x g(t)dt \\ &= c\int_0^x f(t)dt + \int_0^x g(t)dt \\ &= cT(f(x)) + T(g(x)) \end{aligned}$$

We can see that T is linear.

$\square$

Proof. T is one-to-one.

We need to show that $Dim(Null(T)) = 0$. Consider the basis for $P(R)$, the set $\beta = \{1, x^1, x^2, \ldots, x^n\}$.
We find $Range(T)$ by applying T to β.

$$\begin{aligned}
Range(T) &= Span(\{T(\beta_i) \mid \beta_i \in \beta\}) \\
&= Span(\{\int_0^x x^i dt \mid 0 <= i <= n\}) \\
&= Span(\{\frac{1}{i+1}x^{i+1} \mid 0 <= i <= n\})
\end{aligned}$$

We can see that $\{\frac{1}{i+1}x^{i+1} \mid 0 <= i <= n\}$ is linearly independent and forms a basis for $P(R)$.
This means $Dim(Range(T)) = n$ and by the rank nullity theorem, $Dim(Null(T)) = 0$.
We can see that T is one-to-one.

$\square$

Proof. T is not onto.

We need to show that $Range(T) \neq Range(P(R))$.
In the previous problem we showed that $Range(T) = Span(\{\frac{1}{i+1}x^{i+1} \mid 0 <= i <= n\})$.
But we can see that $1 \notin Span(\{\frac{1}{i+1}x^{i+1} \mid 0 <= i <= n\})$.
This implies T is not onto.

$\square$