

M346: Assignment 4

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Exercise 2.1.2. $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.

Proof. Show T is a linear transformation, $T(cx + y) = cT(x) + T(y)$ for $\forall x, y \in V, \forall c \in \mathbb{F}$.

Let $x = (a_1, a_2, a_3), y = (b_1, b_2, b_3), c \in \mathbb{F}$.

$$\begin{aligned} T(ca_1 + b_1, ca_2 + b_2, ca_3 + b_3) &= (ca_1 + b_1 - ca_2 - b_2, 2(ca_3 + b_3)) \\ &= (ca_1 - ca_2, 2ca_3) + (b_1 - b_2, b_3) \\ &= c(a_1 - a_2, 2a_3) + (b_1 - b_2, b_3) \\ &= cT(a_1, a_2, a_3) + T(b_1, b_2, b_3) \end{aligned}$$

We can see T is a linear transformation.

□

Proof. $\dim(\text{Null}(T)) = 1$.

We need to find the $\text{Null}(T)$, the set $\{(a_1, a_2, a_3) \mid T(a_1, a_2, a_3) = (0, 0)\}$.

$T(a_1, a_2, a_3) = (0, 0) \iff (a_1 - a_2, 2a_3) = (0, 0)$.

This can only be true if $a_1 = a_2$ and $a_3 = 0$.

We can see the set $\{(1, 1, 0)\}$ forms a basis for $\text{Null}(T)$, and $\dim(\text{Null}(T)) = 1$.

□

Proof. $\dim(\text{Range}(T)) = 2$.

We need to find the $\text{Range}(T)$.

$$\begin{aligned} \text{Range}(T) &= \{T(a_1, a_2, a_3) \mid \forall a_1, a_2, a_3 \in \mathbb{R}\} \\ &= \{(a_1 - a_2, a_3) \mid \forall a_1, a_2, a_3 \in \mathbb{R}\}. \end{aligned}$$

We can see $\forall x \in \mathbb{R}, \exists a_1, a_2 \in \mathbb{R}$ such that $x = a_1 - a_2$.

Similarly, $\forall y \in \mathbb{R}, \exists a_3 \in \mathbb{R}$ such that $y = 2a_3$.

We can see the set $\{(1, 0), (0, 1)\}$ forms a basis for $\text{Range}(T)$, and $\text{Dim}(\text{Range}(T)) = 2$. □

Proof. The rank nullity theorem is satisfied.

$$\begin{aligned}\text{Dim}(\text{Null}(T)) + \text{Dim}(\text{Range}(T)) &= 1 + 2 \\ &= \text{Dim}(\text{Range}(V))\end{aligned}$$

We can see the rank nullity theorem is is satisfied. □

Proof. T is not one-to-one.

Since $\text{Dim}(\text{Null}(T)) = 1 \neq 0$, T is not one-to-one. □

Proof. T is onto.

Since $\text{Dim}(\text{Range}(T)) = 2 = \text{Dim}(\text{Range}(W))$, T is onto. □

Exercise 2.1.3. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.

Proof. Show T is a linear transformation, $T(cx + y) = cT(x) + T(y)$ for $\forall x, y \in V, \forall c \in \mathbb{F}$.

Let $x = (a_1, a_2), y = (b_1, b_2), c \in \mathbb{F}$.

$$\begin{aligned}T(ca_1 + b_1, ca_2 + b_2) &= (ca_1 + b_1 + ca_2 + b_2, 0, 2(ca_1 + b_1) - (ca_2 + b_2)) \\ &= (ca_1 + ca_2, 0, 2ca_1 - ca_2) + (b_1 + b_2, 0, 2b_1 - b_2) \\ &= c(a_1 + a_2, 0, 2a_1 - a_2) + (b_1 + b_2, 0, 2b_1 - b_2) \\ &= cT(a_1, a_2) + T(b_1, b_2)\end{aligned}$$

We can see T is a linear transformation. □

Proof. $\text{Dim}(\text{Null}(T)) = 0$.

We need to find the $\text{Null}(T)$, the set $\{(a_1, a_2) \mid T(a_1, a_2) = (0, 0, 0)\}$.

$$T(a_1, a_2) = (0, 0, 0) \iff (a_1 + a_2, 0, 2a_1 - a_2) = (0, 0, 0).$$

This can only be true if $a_1 = -a_2$ and $a_1 = -1/2a_2$.

This is true for $a_1 = 0 = a_2$.

We can see the set $\{(0, 0)\}$ forms a basis for $Null(T)$, and $Dim(Null(T)) = 0$.

□

Proof. $Dim(Range(T)) = 2$.

We need to find the $Range(T)$.

$$\begin{aligned} Range(T) &= span(\{T(\beta_i) \mid \forall \beta_i \in \{(1, 0), (0, 1)\}\}). \\ &= span(\{(1, 0, 2), (1, 0, -1)\}). \end{aligned}$$

The two vectors are linearly independent, and we can form a basis for W .

We can see the set $\{(1, 0, 2), (1, 0, -1)\}$ forms a basis for $Range(T)$, and $Dim(Range(T)) = 2$.

□

Proof. The rank nullity theorem is satisfied.

$$\begin{aligned} Dim(Null(T)) + Dim(Range(T)) &= 0 + 2 \\ &= Dim(Range(V)) \end{aligned}$$

We can see the rank nullity theorem is satisfied.

□

Proof. T is one-to-one.

Since $Dim(Null(T)) = 0$, T is one-to-one.

□

Proof. T is not onto.

Since $Dim(Range(T)) = 2 \neq Dim(Range(W))$, T is not onto.

□

Exercise 2.1.4. $T : M_{2 \times 3}(F) \rightarrow M_{2 \times 2}(F)$ defined by

$$T\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) = \begin{bmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{bmatrix}$$

Proof. Show T is a linear transformation, $T(cx + y) = cT(x) + T(y)$ for $\forall x, y \in V, \forall c \in \mathbb{F}$.

Let $x = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $y = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$, $c \in \mathbb{F}$.

$$\begin{aligned}
T(cx + y) &= T\left(c \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} ca_{11} + b_{11} & ca_{12} + b_{12} & ca_{13} + b_{13} \\ ca_{21} + b_{21} & ca_{22} + b_{22} & ca_{23} + b_{23} \end{bmatrix}\right) \\
&= \begin{bmatrix} 2(ca_{11} + b_{11}) - (ca_{12} + b_{12}) & (ca_{13} + b_{13}) + 2(ca_{12} + b_{12}) \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2ca_{11} - ca_{12} + 2b_{11} - b_{12} & ca_{13} + 2ca_{12} + b_{13} + 2b_{12} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2ca_{11} - ca_{12} & ca_{13} + 2ca_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2b_{11} - b_{12} & b_{13} + 2b_{12} \\ 0 & 0 \end{bmatrix} \\
&= c \begin{bmatrix} ca_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2b_{11} - b_{12} & b_{13} + 2b_{12} \\ 0 & 0 \end{bmatrix} \\
&= cT\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) + T\left(\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\right)
\end{aligned}$$

We can see T is a linear transformation. □

Proof. $\dim(\text{Null}(T)) = 4$.

We need to find the $\text{Null}(T)$, the set $\left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \mid T\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$.

$$\begin{aligned}
T\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) &= \begin{bmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

This can only be true if $2a_{11} = a_{12} = -1/2a_{13}$.

We can see the set $\left\{ \begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$ forms a basis for $\text{Null}(T)$, and $\dim(\text{Null}(T)) = 4$. □

Proof. $\dim(\text{Range}(T)) = 4$.

We need to find the $\text{Range}(T)$, we can do this by applying T to the basis of $M_{2 \times 3}(F)$.

Consider $\text{Span}(\{T(\beta_i) \mid \beta_i \in \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}\})$

$$\begin{aligned} \text{Span}(\{T(\beta)\}) &= \text{Span}\left(\left\{T\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right), \right. \\ &\quad \left. T\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\right\}\right) \\ &= \text{Span}\left(\left\{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}\right) \\ &= \text{Span}\left(\left\{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \right\}\right) \end{aligned}$$

We can see these two matrices span $\text{Range}(T)$ and are linearly independent, so the set $\left\{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \right\}$ forms a basis for $\text{Range}(T)$, and $\dim(\text{Range}(T)) = 2$. □

Proof. The rank nullity theorem is satisfied.

$$\begin{aligned} \dim(\text{Null}(T)) + \dim(\text{Range}(T)) &= 4 + 2 \\ &= \dim(\text{Range}(V)) \end{aligned}$$

We can see the rank nullity theorem is satisfied. □

Proof. T is not one-to-one.

Since $\dim(\text{Null}(T)) = 4 \neq 0$, T is not one-to-one. □

Proof. T is not onto.

Since $\dim(\text{Range}(T)) = 2 \neq \dim(\text{Range}(W))$, T is not onto. □

Exercise 2.1.5. $T : P_2(R) \rightarrow P_3(R)$ defined by $T(f(x)) = xf(x) + f'(x)$.

Proof. Show T is a linear transformation, $T(cx + y) = cT(x) + T(y)$ for $\forall x, y \in V, \forall c \in \mathbb{F}$.

Let $f, g \in P_2(R), c \in \mathbb{F}$.

$$\begin{aligned}
T(cf(x) + g(x)) &= T((cf + g)(x)) \\
&= x(cf + g)(x) + (cf' + g')(x) \\
&= xcf(x) + xg(x) + cf'(x) + g'(x) \\
&= c(xf(x) + f'(x)) + (xg(x) + g'(x)) \\
&= cT(f(x)) + T(g(x))
\end{aligned}$$

We can see T is a linear transformation. □

Proof. $\dim(\text{Null}(T)) = 0$.

We need to find $\text{Null}(T)$, the set of polynomials $\{f \mid f \in P_2(R); T(f(x)) = 0\}$

Consider $T(f(x)) = xf(x) + f'(x)$.

$$xf(x) + f'(x) = 0$$

$$\iff xf(x) = -f'(x)$$

$$\iff f(x) = \frac{-cf'(x)}{x}.$$

If $f \in P_2(R)$ f is either a degree 0, 1, or 2 polynomial. We'll call it n .

$$\implies f' \text{ is of degree } n-1.$$

$$\implies \frac{-cf'(x)}{x} \text{ is of degree } n-2.$$

So this equation is only satisfied when f is the zero function.

We can see that set of polynomials $\{0\}$ forms a basis for $\text{Null}(T)$, so $\dim(\text{Null}(T)) = 0$. □

Proof. $\dim(\text{Range}(T)) = 3$.

We need to find $\text{Range}(T)$, the set of polynomials spanned by T .

We can do this by applying T to the basis of $P_2(F)$.

$$\begin{aligned}
\text{Range}(T) &= \text{Span}(\{T(f(x)) \mid f(x) \in \{1, x, x^2\}\}) \\
&= \text{Span}(\{x, x^2 + 1, x^3 + 2x\})
\end{aligned}$$

These three polynomials are linearly independent, and form a basis for $P_3(F)$. We can see $\dim(\text{Range}(T)) = \dim(\{x, x^2 + 1, x^3 + 2x\}) = 3$. □

Proof. The rank nullity theorem is satisfied.

$$\begin{aligned}
\dim(\text{Null}(T)) + \dim(\text{Range}(T)) &= 0 + 3 \\
&= \dim(\text{Range}(V))
\end{aligned}$$

We can see the rank nullity theorem is satisfied.

□

Proof. T is one-to-one.

Since $\dim(\text{Null}(T)) = 0$, T is one-to-one.

□

Proof. T is not onto.

Since $\dim(\text{Range}(T)) = 3 \neq \dim(\text{Range}(W))$, T is not onto.

□

Exercise 2.1.7. Prove the following properties about linear transformations.

Proof. If T is linear, then $T(0) = 0$.

Take any $x \in V$.

$$\begin{aligned} T(x) &= T(x + 0) \text{ By additive zero in } F. \\ &= T(x) + T(0) \text{ By definition linear transformation.} \end{aligned}$$

By equality, we see $T(0)$ must equal 0.

□

Proof. T is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x, y \in V$, $c \in F$.

Proof of (\implies).

$$\begin{aligned} T(cx + y) &= T(cx) + T(y) \text{ Definition linear transformation.} \\ &= cT(x) + T(y) \text{ Definition linear transformation.} \end{aligned}$$

Proof of (\impliedby).

Take $c = 1$.

$$\begin{aligned} T(1x + y) &= T(x + y). \\ &= T(x) + T(y). \end{aligned}$$

This gives us the additive property of linear transformations.

Take $x \in V, y = 0, c \in F$.

$$\begin{aligned} T(cx + 0) &= T(cx). \\ &= cT(x). \end{aligned}$$

This gives us property of scalar multiplication of linear transformations.

By proof of (\Leftarrow) and (\Rightarrow), we are done.

□

Proof. If T is linear, then $T(x - y) = T(x) - T(y)$.

Take any $x \in V, c = -1, y \in V$.

$$\begin{aligned} T(x - y) &= T(x + cy) \text{ By choice of } c. \\ &= T(x) + T(cy) \text{ By addition over linear transformations.} \\ &= T(x) + cT(y) \text{ By scalar multiplication over linear transformations.} \\ &= T(x) - T(y) \text{ By choice of } c. \end{aligned}$$

We see that $T(x - y) = T(x) - T(y)$.

□

Proof. T is linear if and only if, for $x_1, x_2, \dots, x_n \in V$ and $a_1, a_2, \dots, a_n \in F$, we have $T(\sum_{i=1}^n a_i x_i) = \sum_{i=1}^n a_i T(x_i)$.

Proof of (\Rightarrow).

Let $n = 2$.

$$\begin{aligned} T\left(\sum_{i=1}^2 a_i x_i\right) &= T(a_1 x_1 + a_2 x_2) \\ &= T(a_1 x_1) + T(a_2 x_2) \text{ Definition linear transformation.} \\ &= a_1 T(x_1) + a_2 T(x_2) \text{ Scalar multiplication over linear transformations.} \\ &= \sum_{i=1}^2 a_i T(x_i) \end{aligned}$$

Assume for induction, that for some $n = k$, we have $T(\sum_{i=1}^k a_i x_i) = \sum_{i=1}^k a_i T(x_i)$.

Let $n = k+1$.

$$\begin{aligned}
T\left(\sum_{i=1}^{k+1} a_i x_i\right) &= T\left(\sum_{i=1}^k a_i x_i + a_{k+1} x_{k+1}\right) \\
&= T\left(\sum_{i=1}^k a_i x_i\right) + T(a_{k+1} x_{k+1}) \text{ By addition over linear transformations.} \\
&= \sum_{i=1}^k a_i T(x_i) + T(a_{k+1} x_{k+1}) \text{ By induction hypothesis.} \\
&= \sum_{i=1}^k a_i T(x_i) + a_{k+1} T(x_{k+1}) \text{ By scalar multiplication over linear transformations.} \\
&= \sum_{i=1}^{k+1} a_i T(x_i)
\end{aligned}$$

Proof of (\Leftarrow).

Let $n = 2$, $a_1 = a_2 = 1$, $x_1, x_2 \in V$.

We see that

$$\begin{aligned}
T\left(\sum_{i=1}^2 a_i x_i\right) &= T(a_1 x_1 + a_2 x_2) \\
&= T(x_1 + x_2) \text{ By choice of } a_1, a_2 \\
&= T(x_1) + T(x_2)
\end{aligned}$$

We see that we have addition over linear transformations.

Let $n = 1$, $a_1 \in F$, $x_1 \in V$.

We see that

$$\begin{aligned}
T\left(\sum_{i=1}^1 a_i x_i\right) &= T(a_1 x_1) \\
&= a_1 T(x_1)
\end{aligned}$$

We see that we have scalar multiplication over linear transformations.

By proof of (\Leftarrow) and (\Rightarrow), we are done.

□

Exercise 2.1.10. Suppose that $T : R^2 \rightarrow R^2$, $T(1, 0) = (1, 4)$, $T(1, 1) = (2, 5)$.

Proof. $T(2, 3) = T(5, 11)$

Consider $T(0, 1) = T(1, 1) - T(1, 0)$.

$$\begin{aligned} T(0, 1) &= T(1, 1) - T(1, 0) \\ &= (2, 5) - (1, 4) \\ &= (1, 1) \end{aligned}$$

Now we can represent $T(2, 3) = 2T(1, 0) + 3T(0, 1)$.

$$\begin{aligned} T(2, 3) &= 2T(1, 0) + 3T(0, 1) \\ &= 2(1, 4) + 3(1, 1) \\ &= (5, 11) \end{aligned}$$

□

Proof. T is one-to-one.

We have to find $\dim(\text{Null}(T))$.

We have found $T(1, 0)$, $T(0, 1)$.

By inspection, they are linearly independent and form a basis for $\text{Range}(T)$.

We can see that $\dim(\text{Range}(T)) = 2$.

By the rank nullity theorem, we know $\dim(\text{Null}(T)) + \dim(\text{Range}(T)) = 2$.

$\implies \dim(\text{Null}(T)) = 0$, which means T is one-to-one.

□

Exercise 2.1.13. Let $T : V \rightarrow W$ be linear, $\{w_1, w_2, \dots, w_k\}$ be a linearly independent subset of $R(T)$.

Show that if $S = \{v_1, v_2, \dots, v_n\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \dots, k$, then S is linearly independent.

Proof. S is linearly independent.

Assume the contrary, that S is linearly dependent.

That is, $\exists a_i \in F$, $\sum_{i=1}^k |a_i| \neq 0$ such that $\sum_{i=1}^k a_i v_i = 0$ (\star).

Consider $\sum_{i=1}^k a_i w_i$.

$$\begin{aligned}
\sum_{i=1}^k a_i w_i &= \sum_{i=1}^k a_i T(v_i) \\
&= T\left(\sum_{i=1}^k a_i v_i\right) \text{ By definition linear transform.} \\
&= T(0) \text{ By } (\star). \\
&= 0 \text{ By zero property of linear transformation. Contradiction}
\end{aligned}$$

We can see that S is linearly independent. □

Exercise 2.1.14a. Let $T : V \rightarrow W$ be linear. Show that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W .

Proof. T one-to-one $\implies T$ preserves linearly independent subsets.

T one-to-one $\implies \dim(\text{Null}(T)) = 0$.

$\dim(\text{Null}(T)) = 0 \implies \forall a_i \in F$ such that $\sum_{i=1}^k |a_i| \neq 0$, we have $\sum_{i=1}^k a_i v_i \neq 0$.

Since $\sum_{i=1}^j a_i w_i = \sum_{i=1}^k a_i T(v_i) = T(\sum_{i=1}^k a_i v_i)$.

We know $\sum_{i=1}^k a_i v_i \neq 0$, so $T(\sum_{i=1}^k a_i v_i) \neq 0$.

We can see T preserves linearly independent subsets. □

Proof. T preserves linearly independent subsets $\implies T$ one-to-one.

Let $\beta = \{v_1, v_2, \dots, v_n\}$ be a basis for V , by definition basis, β is linearly independent.

If T preserves linearly independent subsets, then $T(\beta)$ is linearly independent.

Show that if $T(\sum_{i=1}^n a_i \beta_i) = T(\sum_{i=1}^n b_i \beta_i)$ for $a_i, b_i \in F$, then $a_i = b_i$.

Consider $T(\sum_{i=1}^n a_i \beta_i) = T(\sum_{i=1}^n b_i \beta_i)$

$\implies T(\sum_{i=1}^n a_i \beta_i) - T(\sum_{i=1}^n b_i \beta_i) = 0$

$\implies T(\sum_{i=1}^n a_i \beta_i - \sum_{i=1}^n b_i \beta_i) = 0$ By addition over linear transformation.

$\implies T(\sum_{i=1}^n (a_i - b_i) \beta_i) = 0$ By addition over F .

Since $T(\sum_{i=1}^n (a_i - b_i) \beta_i) = 0 \iff \sum_{i=1}^n (a_i - b_i) \beta_i = 0$,

and $\sum_{i=1}^n (a_i - b_i) \beta_i = 0 \iff (a_i - b_i) = 0$, since $\{\beta_i\}$ forms a basis for V ,

we have $a_i = b_i$, and we see can T is one-to-one. □

Proof. T one-to-one $\iff T$ preserves linearly independent subsets.

By proof of (\implies) and (\impliedby) , we are done. □

Exercise 2.1.14b. Let $T : V \rightarrow W$ be linear. Suppose that T is one-to-one and S is a subset of V . Prove that $S = \{v_1, v_2, \dots, v_n\}$ is linearly independent if and only if $T(S)$ is linearly independent.

Proof. S linearly independent $\implies T(S)$ linearly independent.

Assume for contradiction that $T(S)$ linearly dependent.

S linearly independent means $\sum_{i=1}^n a_i v_i = 0 \iff a_i = 0$.

$\implies \forall v \in V, v = \sum_{i=1}^k a_i v_i = 0 \iff a_i = 0$.

Consider $T(S)$. By assumption, $\sum_{i=1}^k a_i T(v_i) = 0$ and $\sum_{i=1}^k |a_i| > 0$.

$\implies \exists a_i$ such that $T(\sum_{i=1}^k a_i v_i) = 0$, contradiction.

We can see S linearly independent $\implies T(S)$ linearly independent. □

Proof. $T(S)$ linearly independent $\implies S$ linearly independent.

$T(S)$ linearly independent $\implies \sum_{i=1}^k a_i T(v_i) = 0 \iff \sum_{i=1}^k |a_i| = 0$.

$\implies T(\sum_{i=1}^k a_i v_i) \iff \sum_{i=1}^k |a_i| = 0$.

$\implies \sum_{i=1}^k a_i v_i = 0 \iff \sum_{i=1}^k |a_i| = 0$.

By definition, we can see S is linearly independent. □

Proof. S linearly independent $\iff T(S)$ linearly independent.

By proof of (\implies) and (\impliedby), we are done. □

Exercise 2.1.15. Define $T : P(R) \rightarrow P(R)$ by $T(f(x)) = \int_0^x f(t)dt$.

Show T is linear, one-to-one, but not onto.

Proof. T is linear.

We need to show that $T(cf(x) + g(x)) = cT(f(x)) + T(g(x))$ for $f, g \in P(R)$, $c \in R$.

$$\begin{aligned}
 T(cf(x) + g(x)) &= T((cf + g)(x)) \\
 &= \int_0^x (cf + g)(t)dt \\
 &= \int_0^x cf(t)dt + \int_0^x g(t)dt \\
 &= c \int_0^x f(t)dt + \int_0^x g(t)dt \\
 &= cT(f(x)) + T(g(x))
 \end{aligned}$$

We can see that T is linear.

□

Proof. T is one-to-one.

We need to show that $\dim(\text{Null}(T)) = 0$. Consider the basis for $P(R)$, the set $\beta = \{1, x^1, x^2, \dots, x^n\}$.

We find $\text{Range}(T)$ by applying T to β .

$$\begin{aligned}\text{Range}(T) &= \text{Span}(\{T(\beta_i) \mid \beta_i \in \beta\}) \\ &= \text{Span}(\{\int_0^x x^i dt \mid 0 \leq i \leq n\}) \\ &= \text{Span}(\{\frac{1}{i+1}x^{i+1} \mid 0 \leq i \leq n\})\end{aligned}$$

We can see that $\{\frac{1}{i+1}x^{i+1} \mid 0 \leq i \leq n\}$ is linearly independent and forms a basis for $P(R)$. This means $\dim(\text{Range}(T)) = n$ and by the rank nullity theorem, $\dim(\text{Null}(T)) = 0$. We can see that T is one-to-one.

□

Proof. T is not onto.

We need to show that $\text{Range}(T) \neq \text{Range}(P(R))$.

In the previous problem we showed that $\text{Range}(T) = \text{Span}(\{\frac{1}{i+1}x^{i+1} \mid 0 \leq i \leq n\})$.

But we can see that $1 \notin \text{Span}(\{\frac{1}{i+1}x^{i+1} \mid 0 \leq i \leq n\})$.

This implies T is not onto.

□