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Self-organization and market crashes

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Abstract

This paper presents a model of stock price behavior that encompasses both short-term fluctuations and long-term exponential trends punctuated by crashes. The model represents stock market behavior as the interaction of two self-organizing processes. The first process represents business-as-usual stock price fluctuations. It builds on a model of stock price behavior introduced by Cont and Bouchaud. The second process is a risk process that determines the severity of crashes. It is a random graph process driven by macroeconomic variables. Our model is based on the assumption that stock market prices that grow at a higher rate than the real economy force structural changes that, in turn, leverage the risk of a crash. The transition from a business-as-usual regime to a crash regime is determined by trigger events. Trigger events are exponentially distributed while the size of a crash depends on the state of the underlying risk process. While in the short-term the performance of stock prices may be determined by the purely speculative behavior of agents, the long-term behavior of stock prices is essentially a function of structural and macroeconomic parameters.

JEL classification: C53: econometric modeling, forecasting and other model applications; D49: market structure and pricing, other; D79: analysis of collective decision making, other; G10: general financial markets

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1. Introduction

Financial markets might experience sudden significant downward price movements, but no comparable upward movements: crashes such as those of 1929 and 1987 have no equivalents in positive terms. There would appear to be two distinct regimes governing the stock

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markets: (1) business-as-usual nearly symmetrical price fluctuations in the short-term; and (2) smooth upward trends punctuated with crashes in the long term.

In a nutshell, we hypothesize that inflation is, more than a constant that affects the entire economy, a localized sector-dependent quantity. High- and low-inflation sectors might coexist at any given period due to the structure of the economy. A stock market whose growth rate is considerably higher than that of the nominal GDP is driven by sectoral inflationary pressure. Opinion aggregation plays a role but we postulate that, in the long run, it is the excess inflow of investment in financial markets that fuels bull markets.

Asset inflation in an otherwise low-inflation economy can be correlated with structural changes in the economy in response to the need to channel a steady flow of earnings to the market. These structural changes create underlying tensions. In the long run, sectoral asset inflation is not sustainable; dividends drop and markets experience a low or negative growth rate with increased risk of crashes. When a trigger event is activated, built-up tensions produce a sudden aggregation of sell orders and consequent (potentially large) falls in share prices.

We believe that the conceptual novelty (and importance) of this model lies in its explicit linking of economic value to structure and, in particular, in its linking inflation differentials between different sectors of the economy. Without introducing any notion of fundamentals, this model shows how markets might exhibit long-term trends.

2. Phenomenology of stock market prices

If we examine the US stock market for which coarse-grained data on stock prices are available for most of 20th century (roughly from 1918 onwards), price behavior has undergone significant changes, in particular as regards rate of growth. Taking a major index such as the Standard & Poor 500 index or the Dow Jones Industrial Average to represent the US stock market, the growth of the market underperformed the growth of the economy for most of this century. However, in the decade preceding the 1929 crash and in the 17-year period from 1982 to 1999, US stock market prices rose more rapidly than the nominal GDP. The latter period in particular has seen a price growth rate unprecedented in the history of modern stock markets, even accounting for the 1987 crash.

According to the OECD Economic Outlook no. 65, June 1999, in the 17-year period 1982–1999 both the GDP and the GDP deflator USA grew at a compound rate of about 3%. In the same period, US shares generated a compound annual return of roughly 15%. As investors, US and foreign, have been pouring money into US equities, firms have been buying back their own shares. It is estimated that in 1998 alone, US firms bought back US$ 263 billion more of their own shares than they issued; this figure is equivalent to roughly 2% of the total USA shares in the same period.

This century has also seen broad variations in the price/earnings ratio. For example, the 18-year period between 1982–1999 saw the S&P 500 Composite price index rise more than 1300% from a low of 110 in 1982 to a high of 1469 in 1999 while the P/E ratio on the same index went from a low of 7.2 in 1982 to a high of 34 in 1999 (source Datastream).

The revaluation of stock prices as a group can also vary widely depending on the index. A comparison of the behavior of three indexes—the Russell 2000, which represents the bottom two-thirds of the largest 3000 publicly traded companies domiciled in the USA, the Dow Jones Industrial Average, which includes 30 of the largest US corporations, and the wider S&P 500 reveals that the significant growth in market value has affected mainly the large firms. At the end of 1999, the ratios between the Russell 2000 and the S&P Composite and the Dow Jones Industrial Average, respectively, were roughly 65 and 58% of the corresponding ratios at the beginning of 1993. It would appear that there are asset inflation differentials even within the US stock market.

In the past two or three decades, much fine-grained data has been accumulated. The large amount of high-frequency data available in recent years has allowed establishing a number of stylized facts, among which two are of particular importance to our argument: (1) fat-tailed price/return distributions in the short-term; and (2) crashes and their precursors.

The distribution of price increments exhibits fat tails for time horizons ranging from a few minutes to a day and becomes normal at a time horizon of approximately 1 month. Empirical studies have shown that daily returns have kurtosis in the range of 30–40, a clear indication of the presence of fat tails. For comparison, Gaussian distributions have kurtosis 3 (Campbell et al., 1997).

There is no agreement, however, on how to represent these fat tails. Early studies suggested representing short-term fluctuations as Levy stable laws. Given the finite size of markets, however, the infinite-variance stable laws cannot be a faithful representation of market returns. In order to retain some of the properties of Levy distributions in a finite-mean finite-variance environment, Mantegna and Stanley proposed to represent short-term returns as truncated Levy distributions with power exponent in the range of 1.4–1.5 (Mantegna and Stanley, 1998). Other representations include stretched exponentials and multifractals. More recent studies (see, in particular, the study conducted by Plerou et al., 1999) have proposed to represent fluctuations from 5 min to 16 days through a power-law distribution with exponent $\alpha = 3$.

If the findings of this latter study on short-term market behavior are confirmed, Levy stability breaks down. While our theory of crashes and the long-term behavior of the market is not affected, the model that we use to represent short-term price fluctuations would have to be modified. In particular, more complex dynamics—as, for instance, in Stauffer and Sornette (1999) —would have to be considered.

The asymmetry in the behavior of valleys and peaks, in particular the existence of flat valleys and sharp peaks, was studied by Roehner and Sornette, 1998. As regards crashes, statistical analysis suggests that these might be outliers rather than the fat tails of business-as-usual market movements (Johansen and Sornette, 1998). Though, the fat tails of short-term fluctuations might be responsible for price movements that are definitely larger than purely Gaussian fluctuations, they cannot explain the really large movements such as those of 1929 and 1987. However, given that the time series available include only a few large crashes, firm conclusions cannot be drawn.

Recent studies seem to indicate that markets exhibit specific log-periodic movements before crashes; these might be considered precursors (Sornette et al., 1996; Sornette and Johansen, 1997 and Johansen and Sornette, 1999). However, this finding is not universally shared. Laloux et al. (1999) critically review the hypothesis that log-periodic movements can be considered precursors of crashes.

In summary, it would appear that the stock price behavior, at least as represented by the US stock market, exhibits fundamentally different behavior at different time horizons. In addition, different periods exhibit different long-term trends, both in terms of growth rate and P/E ratio. The effects of large crashes vary significantly. The 1929 crash was followed by a deep recession while the 1987 crash was a dent in a fast growth track. Because short-term behavior has been carefully analyzed only for the recent years, no long-term comparison is possible.

3. Percolation and random graphs in economic modeling

As our model is based on the theory of random graphs, we should perhaps explain why we believe that random graphs and percolation are powerful tools for analyzing the structural aspects of the economy. In the classical view of markets, agents are represented as independent optimizers coordinated solely by a price signal. In classical finance theory, agents do not exert any influence, either direct or indirect, on each other. Links between agents of various natures do however exist and might influence the behavior of economic agents, inducing correlations (Kirman, 1997 and Weisbuch et al., 1996).

Real economies are characterized by a myriad of links in areas such as production, distribution, information and financial relationships. These links result in coordination of the decision-making process of different agents. It is virtually impossible to represent the actual structure of existing links; these are either too complex or unknown. Even such a relatively simple and orderly subset of the economy as financial markets is too complex to be modeled in detail; one has to resort to statistical modeling. This is where percolation and other similar mathematical structures might come into play.

One might argue that agent aggregation represents the fundamental fact that an agent's decision-making process is influenced by that of other agents. It might seem that this would call for an understanding of how the "natural" behavior of one agent is influenced by others, giving rise to a global stochastic process. This approach would present two difficulties. First, defining the "natural" isolated behavior of an agent is somewhat arbitrary. Second, if the behavior of each agent feeds back into the environment, modifying the environment and in turn the agent's own behavior, an infinite regress is generated.

Similar problems of self-reflectivity are routinely solved in the physical sciences. For example, the motion of a charged particle in a system of other charged particles perturbs the system and is influenced, in turn, by the same perturbation it induces. In this case, and in physics in general, the problem is addressed by stating a set of conditions that the global system of coupled particles must satisfy. In the case of charged particles, the conditions are supplied by the equations of dynamics and of electrodynamics.

The approach of percolation theory, which we use, is conceptually similar. We model not the effects of an agent on other agents but the end result of aggregation. We therefore consider the probability that two agents share the same behavior rather than the probability that an agent's behavior propagates to another agent. We assume that there is a uniform random distribution of decision-making links between agents and observe the development of clusters of agents that are homogeneous as regards their decisions. This allows avoiding the problem of self-reflectivity.

The important finding of random graphs and percolation theory is that while uniform and independent random decisions result in average behavior with virtually no macroscopic fluctuation, uniform and independent random links between agent decisions might result in long-range correlations and macroscopic fluctuations. Random graphs and percolation therefore offer a simple but powerful tool for understanding how local influences propagate through an entire economy. In particular, these models provide the key insight that macroscopic variables can fluctuate, even assuming a random and uniform distribution of links.

We chose random graph models (Bollobas, 1985 and Palmer, 1985) for modeling agent aggregation. The exact nature of the percolation or random graph models is relatively unimportant as all share some fundamental characteristics (Stauffer, 1998 and Grimmett, 1989).

4. Presentation of the model

As mentioned, two parallel processes drive our market models: (1) a business-as-usual stochastic price behavior process that is both stable and bounded; and (2) a process that represents crashes. When a trigger event occurs, control shifts from a business-as-usual to a crash regime. The severity of the crash depends on the random state of the underlying process.

We build on the Cont and Bouchaud (2000) model in which demand in a business-as-usual regime is originated by clusters, not individuals. We use a random graph process to model the building up of links in the background. The key idea is that in the background the web of links becomes progressively more connected, without any appreciable consequences for the business-as-usual regime.

The driving parameter of the background cluster is the difference between a compound riskfree interest rate and actual compound market returns. We assume that the background will evolve as a random graph process driven by the difference between compound interest rates and actual compound market returns. In time, this random graph process becomes more connected than the random graph process related to a business-as-usual regime.

4.1. The business-as-usual component

The fundamental component of our model is the representation of business-as-usual stock price behavior for which, it should be said, high-frequency data have become available only recently. To model this behavior, we use a modified version of the herding model presented in Cont and Bouchaud and extended in Chowdhury and Stauffer (1999) and in Stauffer and Sornette (1999).

This herding model is very simple. It considers only one fundamental aggregation behavior of elementary agents. As mentioned, this model does not agree quantitatively with the recent findings that the distribution of returns at short time horizons can be fit with an inverse power law with exponent $\alpha = 3$. Stauffer and Sornette propose a modification of the model, rendering it more empirically faithful.

This model assumes that there are N agents in the market, each occupying a node of a random graph. It is assumed that each agent has an independent probability p of being linked to another agent. Linked agents make identical buy/sell decisions.

Given that agents are linked with probability p, at each moment there is a distribution of clusters of linked agents. The size distribution of such clusters (i.e. the number of agents in each cluster) can be theoretically determined by the theory of random graphs (see Appendix A for a brief presentation of the theory). Theory predicts that the cluster size distribution follows a truncated power-law distribution with exponent $-(\alpha + 1) = -5/2$. More precisely, if X is the random variable that represents cluster size and if the probability that an edge is open or closed is $p = c/N,\ c \leq 1$, we can write:

$$P(X=x)=p(x)=Ax^{-\alpha+1}\exp[(1-c+\log c)x]=Ax^{-5/2}\exp[(1-c+\log c)x] \quad (1)$$

To capture the essentials of the process, we make simplifying assumptions imposing a division of the market between potential buyers and sellers. Agents exchange a fixed amount of the stock at each trading moment. For simplicity, we assume that this amount is exactly one stock. We further impose that every seller becomes a buyer after selling its stock and that every buyer becomes a seller after buying its stock. In other words, we bar agents buying or selling for two or more time-contiguous periods. This condition precludes agents' building up unbounded stocks of financial assets.

At the start, the market is divided in two equal sets: buyers and sellers. Every seller holds at least one stock for sale. We assume that the pool of stocks is fixed. At each transaction, agents trade one stock (or the same fixed amount of stocks) and receive in exchange a variable amount of cash. Therefore, a pool of stocks rotate between agents and market-makers.

Assume that time is continuous and that trading moments are not fixed but randomly distributed. In particular, we assume that trading moments are a homogeneous Poisson process with parameter λ. At each trading moment only one cluster from the set of buying clusters and one from the set of selling clusters are chosen for trading. In this way we assume that the random nature of trading is completely captured by the variable size of the trading clusters and the distribution of trading moments.

Assuming that only one cluster is selected in each trading time from buyers and sellers, the total demand at each trading time is the difference between two identical and independent truncated power-law distributions. Therefore, the price distribution for each time step follows a truncated power law and can be approximated by a symmetrical truncated Levy distribution.

Before proceeding, a difficulty must be solved. Our model assumes a small positive average excess demand. This excess demand is supplied by the willingness of agents to reinvest earnings. To model average excess demand, we need to make a distinction between demand and actual transactions. Price movements must readjust supply and demand in order to reach a statistical equilibrium of transactions. Modeling this process would require specific assumptions as regards the fine trading mechanism. The objective of our model is to capture a more coarse-grained picture of trading. Therefore, we assume that market-makers will force supply and demand to adjust to the trend by progressively raising prices. Market-makers experience only residual zero-average fluctuations whose price change is determined by the imbalance between supply and demand.

We assume that returns, written as $\Delta P/P$, are formed by two terms: (1) a fixed amount responsible for the trend; and (2) a stochastic zero-mean amount. We can thus write:

$$\frac{\Delta P}{P} = \mu + \delta(W - X) \tag{2}$$

where W is the size of the cluster of buyers, X the size of the cluster of sellers, δ the proportionality constant or market depth, and μ the average excess demand.

We introduce the self-organization of the process in the following way. At each period, a number of sellers become buyers and buyers sellers. We assume that the clusters to which each belongs are destroyed in the corresponding trading period. We assume that at each period a (small) number of new links are randomly added. This slow growth in connectivity makes up for the loss of links after trading. In the long run, the average cluster size must be identical to the number of links added during each time step.

This latter fact can be established in the following way. First note that the number of links self organizes around some long run distributions. In fact, the evolution of the number of links $j(t)$ in the market is described by a finite Markov chain embedded in the market process, as the market is finite and there is, at each trading moment, a well defined transition probability $j \Rightarrow k$ given by the size distribution of the clusters that are selected for trading. Therefore, the system exhibits a long run equilibrium distribution of the number of links. Given that there is a long run distribution of links and thus a constant average number of links, the average size of clusters selected for trading must be identical to the number of links added at each time step.

A priori there is no reason while the equilibrium distribution of links must be identical with the binomial probability distribution of the number of links that characterizes independent links. This in turn might affect the cluster size distribution. It is reasonable to assume, however, that the difference between the two distributions will not substantially affect the behavior of the market.

The process is globally self-organizing. The average number of links added during each time step remains an exogenous parameter; this is reasonable as it represents a genuine characteristic of the economy. A global electronic market, for instance, will not have the same connectivity parameter as traditional exchanges.

Let us now look at a critical point: the cash balance, and thus the wealth, of each agent. At each transaction, the cash holding of each agent changes by an amount equal to the market price of one stock. Transactions shift cash from buyers to sellers. As we assume that agents buy and sell in sequence, the cash holding of each agent changes by a sequence of random draws from the price distribution. We assume that the market has or can create sufficient liquidity, eventually as credit lines collateralized by purchased stock—to continue operating indefinitely.

The wealth of each individual agent fluctuates with time. Individual agents might go bankrupt due to trading losses, but the cash losses of bankrupt agents are the cash gains of other agents. We therefore assume that bankrupt agents are replaced by new agents created with endowments supplied by other agents, i.e. we assume that there is some mechanism of agent replacement that holds constant the number of fully functional agents. We do not model the wealth and the cash balance of agents; agent aggregation is the only relevant parameter.

4.2. Short- and medium-term behavior of the business-as-usual model

Call $P(t)$ the random variable that represents price at moment t. At every trading moment, prices make a jump according to the equation:

$$\frac{\Delta P}{P} = \mu + \delta(W - X) = \mu + \delta\,\Delta D \tag{3}$$

where ΔD approximately follows a truncated Levy distribution. The number of trading intervals in the interval $(0, t)$ is a random variable $N(t)$ distributed as a Poisson variable. We can therefore write:

$$P(t) = P(0)(1 + \mu + \delta\,\Delta D)^{N(t)} \tag{4}$$

If we use the lower case p to indicate the logarithm of price P, i.e. if we write: $p(t) = \log(P(t))$, we can then rewrite the previous equation as:

$$p(t) = p(0) + \sum_{i=1}^{N(t)} \log(1 + \mu + \delta\,\Delta D) \tag{5}$$

With the above assumptions, the log(price) process can be represented as a Continuous Time Random Walk (CTRW) (Montroll and Schlesinger, 1984). A CTRW is a random walk with a random pausing time between successive steps. Call l the random variable that represents the step size. In the most general case, the step size and the pausing time will have a joint distribution $\varphi\ (l,t)$. In our case, however, the step size distribution $f(l)$ and the pausing time distribution $\psi(t)$ are independent so that $\varphi\,(l, t) = f\,(l)\,\psi\,(t)$. More specifically, the log(price) process p is represented by a CTRW such that the pausing time between steps is exponentially distributed with pdf $\psi(t) = N\lambda \exp(-N\lambda t)$ while $f(l)$ is the distribution of $\log(1 + \mu + \lambda\Delta D)$.

As demonstrated in Montroll and Schlesinger, for large values of t such a process approximates a Gaussian diffusion with growth rate $\mu = \langle l/t\rangle = \langle \log(1 + \mu + \Delta D)/t\rangle$ and variance $D = \langle l^2\rangle/2\langle t\rangle$.

To estimate $\langle \log(1 + \mu + \delta\Delta D)\rangle$ we expand the logarithm in a Taylor series. Consider that μ is a small quantity and we can therefore neglect terms in μ of order 2 and higher. As the distribution of ΔD is symmetric, all moments of odd orders are 0. We assume that we can neglect all even moments of orders higher than 4 and we retain only variance and kurtosis. We can therefore write:

$$\log(1+\mu+\delta\,\Delta D) = \mu+\delta\,\Delta D+\tfrac{1}{2}(\mu+\delta\,\Delta D)^2+\tfrac{1}{3}(\mu+\delta\,\Delta D)^3+\tfrac{1}{4}(\mu+\delta\,\Delta D)^4 \tag{6}$$

$$\langle \log(1 + \mu + \delta\,\Delta D)\rangle \approx \langle \mu + \tfrac{1}{2}\delta^2\,\Delta D^2 + \tfrac{1}{4}\delta^4\,\Delta D^4\rangle = \mu + \tfrac{1}{2}\left(\sigma^2 + \tfrac{1}{2}\sigma^4\right) \tag{7}$$

This shows that both variance and kurtosis of returns are exponentially amplified in the price process. In the Gaussian case only variance counts.

We can estimate the average number N_G of time steps needed to cross over to a lognormal regime. In fact, Mantegna and Stanley (1995) showed that if a process is represented by the sum of iid truncated Levy flights whose tail distribution $T(x)$ can be represented, after appropriate scaling, as:

$$T(x) = \frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)\exp(-\lambda x)}{x^{1+\alpha}} \tag{8}$$

then the number of steps N_G needed to cross over to a normal regime is of the order $N_G = \cos(\pi\alpha/2)/\alpha(1-\alpha)\lambda^{-\alpha}$. In the above case, $\alpha = 3/2$ and the number N_G becomes

$$N_G = \frac{\cos(\pi\alpha/2)}{\alpha(1-\alpha)}(-(1-c+\log c))^{-\alpha} \approx (-(1-c+\log c))^{-3/2} \tag{9}$$

From this expression it is clear that if $c \Rightarrow 0$ then $N_G \Rightarrow 0$, as it should be as $c = 0$ means that distributions do not have any power-law region in the tails. If on the other hand $c \Rightarrow 1$, $N_G \Rightarrow \infty$, as it should because there is no exponential truncation to the power-law region of the tails. We can conclude that if relative increments are independent and follow a truncated Levy distribution, the price behavior at long time horizons is lognormal.

4.3. The crash component

The next step is to model the transition to a crash regime and the crash regime itself. In a nutshell, the mechanics of a crash is the following. Markets rise exponentially under the pressure of demand created by the reinvestment of gains. When returns are negative and below some given threshold, the built-up tensions result in an aggregation of sell decisions.

The key assumptions of our model are the following:

- *In parallel with the business-as-usual behavior of stocks, a web of links is building up in the background*. These links are activated in case of a transition to a crash regime. They can represent a variety of conditions, such as underlying opinion links, highly leveraged positions that might require unwinding, programmed automatic trade controls which are activated in the case of price deterioration, etc.
- *The size distribution of this underlying network can be represented as a random graph process*. The speed at which links are added is a function of macroeconomic quantities. We assume that there is a reference rate of riskfree interest and choose as the leading quantity the logarithm of the difference between compound interest rates and compound stock gains. The rationale behind this choice is that the distance between compound interest rates and compound stock gains is a measure of the tensions building up in the economy. The larger this distance, the higher the probability that earnings will not be able to grow sufficiently to sustain the rise in share prices. As sensitivity to this situation grows, protection strategies are put in place.
- *We assume that there is a price change threshold for activating the transition to a crash regime*. We put a threshold on price movements and switch to a crash regime when this threshold is crossed. The threshold can be interpreted as corresponding to the maximum amount of imbalance between supply and demand that market-makers can absorb.

The next step is to represent the dynamics of the crash regime. We assume the following differences between business-as-usual and crash regimes:

- *During a crash regime, market-makers cannot absorb the excess offer*. Therefore—and because buy/sell transactions must be in equilibrium—we make the assumption that a sufficient number of agents in the set of buyers will engage in a transaction.
- *The fall in prices is quadratic with excess demand*, i.e. $\Delta P/P = \delta_{\mathrm{C}}(W - X)^2$. The rationale behind this choice is that during crashes sellers have "compulsory" reasons for selling and therefore sell at prices lower that what normal circumstances would command. On the other hand, buyers require large price drops.
- The probability distribution of price changes during a crash depends therefore on the actual level of prices. Given a price level, the excess demand $W - X$ is distributed as a truncated Levy distribution. The distribution of price drops is then amplified by the quadratic term. Put $Q_i(i - j) = \text{prob}(\text{price drop} = i - j/\text{price} = i)$.

When the crash regime is activated, we make the simple assumption that the current cluster of sellers is expanded to the cluster in the underlying web to which it belongs. Thus if *S* is the current cluster of sellers, we expand it by taking all sellers that belong to clusters in the underlying graph whose intersection with *S* is not empty. The rationale behind this choice is to capture the randomness of activated crashes. We assume that the underlying cluster involved in the crash is destroyed and that, after a crash, the market reverts to normal trading.

4.4. *The master equation of the compound model*

To understand the behavior of the compound business-as-usual/crash model in the medium and long term, it is convenient to make some simplification. Let us represent the log(price) process $p(t)$ as a discrete process, i.e. we assume that $p(t)$ can only take integer values. This is a realistic approximation: integer steps can be arbitrarily small; any real-life price process is inherently discrete. For example, prices of exchange-traded assets might change only by multiples of, say, 1/8th of a dollar. In addition, as crashes are rare, we represent the price behavior between crashes as if the business-as-usual model were in the Gaussian regime.

To reach a unified mathematical treatment of the price process, we adopt the formalism of master equations and represent both the business-as-usual and crash regimes as jump Markov processes (see Appendix B). The pausing time between steps is exponentially distributed with pdf $\psi(t) = (N + 1)\lambda \exp(-(N + 1)\lambda t)$ where λ is the crash rate and N the large integer. We assume that, at each step, the business-as-usual process p can make a positive jump of constant size υ_+ with probability $N/2(N+1)$ or a negative jump of constant size υ- with the same probability $N/2(N + 1)$ independent of the pausing time. In addition, at each step the crash process can make an independent jump of size $i - j$, $i - j > \nu_-$ with probability $q_i(i-j)/N+1$ where $q_i(i-j) = \text{prob}(\log(\text{crash} = i-j)/\text{state}(i))$, $i-j > \nu_-$.

Let us assume that the riskfree rate is zero. This assumption implies that there is no real growth in the economy. However, it will be clear from the structure of the master equation that a non-zero riskfree rate simply acts as an additive term. If we write $w_{i,j}$ for the transition rates of the global log(price) process, we can represent it as a jump Markov process with rate $(N + 1)\lambda$ and with transition rates:

$$w_{i,i} = -(N + 1)\lambda,\ w_{i,i+\nu_+} = \tfrac{1}{2}(N\lambda),\ w_{i,i-\nu_-}\ \tfrac{1}{2}(N\lambda),\ w_{i,j} \\ = q_i(i - j)\lambda,\ i - j > \nu_-,\ w_{i,j} = 0, \text{ for any other } j \tag{10}$$

We can therefore write the rate transition matrix $\boldsymbol{w} = [w_{i,j}]$ for the joint process as follows:

	.	.	$i+1$	i	$i-1$	$i-2$	.	.
.	.	.	.	.	.	.	.	.
$i-\nu_{-}-2$	.	.	q_{i+1} $(\nu_{-}+1)\lambda$	q_{i+1} $(\nu_{-}+2)\lambda$	q_{i+1} $(\nu_{-}+2)\lambda$	q_{i+2} $(\nu_{-}+4)\lambda$	.	.
$i-\nu_{-}-1$	.	.	$N\lambda/2$	$-q_{i+1}$ $(\nu_{-}+1)\lambda$	q_{i+1} $(\nu_{-}+2)\lambda$	q_{i+1} $(\nu_{-}+3)\lambda$	.	.
$i-\nu_{-}$	.	.	0	$N\lambda/2$	q_{i+1} $(\nu_{-}+1)\lambda$	q_{i+2} $(\nu_{-}+2)\lambda$	.	.
$i-\nu_{-}+1$	.	.	0	0	$N\lambda/2$	q_{i+1} $(\nu_{-}+1)\lambda$	.	.
$i-\nu_{-+2}$	.	.	0	0	0	$N\lambda/2$	.	.
.	.	..	.	.	.	.	.	.
$i-1$	.	.	$-(N+1)\lambda$	0	0	0	.	.
I	.	.	0	$-(N+1)\lambda$	0	0	.	.
$i+1$	.	.	0	0	$-(N+1)\lambda$	0	.	.
$i+2$	.	.	0	0	0	$-(N+1)\lambda$	.	.
	.	.	.	.	.	.	.	.
$j+\nu_{+}-1$	.	.	$N\lambda/2$	0	0	0	.	.
$j+\nu_{+}$	.	.	0	$N\lambda/2$	0	0	.	.
$j+\nu_{+}+1$	.	.	0	0	$N\lambda/2$	0	.	.
$j+\nu_{+}+2$	.	.	0	0	0	$N\lambda/2$	.	.

Write $\boldsymbol{Q}(t)$ for the matrix $\boldsymbol{Q}(t) = [Q_{i,j}(t)]$ where $Q_{i,j}(t)$ is the probability that the system will be in state j at time t starting from state i. The master equation can then be written:

$$\frac{\mathrm{d}Q_{i,k}(t)}{\mathrm{d}t} = \sum_{j} Q_{i,j} w_{j,k} \tag{11}$$

4.5. The time evolution of average and variance

We follow van Kampen (1992) to derive deterministic differential equations for the average and the variance. Multiply both sides of the master equation by k and sum over k:

$$\sum_{k} k \frac{\mathrm{d}Q_{i,k}}{\mathrm{d}t} = \sum_{k} k \sum_{j} Q_{i,j} w_{j,k} \tag{12}$$

Reversing the order of summation and placing k under the derivative sign, we can now write:

$$\frac{\mathrm{d}\left(\sum_{k} kQ_{i,k}\right)}{\mathrm{d}t} = \sum_{j} \left(\sum_{j} kQ_{i,j} w_{j,k} \right) \tag{13}$$

If we add and subtract j to the term multiplying $Q_{i,j}w_{j,k}$ in the inner sum we obtain:

$$\frac{\mathrm{d}\left(\sum_k kQ_{i,k}\right)}{\mathrm{d}t} = \sum_j \left(\sum_k (k-j)Q_{i,j}w_{j,k} + jQ_{i,j}\sum_k w_{j,k}\right) \tag{14}$$

To simplify the notation, assume that the market starts in state i, i.e. the market starts at a price corresponding to state i. The second term in parenthesis is zero and we can write the following differential equation for the time evolution of the average price:

$$\frac{\mathrm{d}\langle p(t)\rangle}{\mathrm{d}t} = \langle a_1(p)\rangle \tag{15}$$

where $\langle p(t)\rangle = \langle i(t)\rangle$ is the average value of the process (i.e. the log(price) p) at time t starting from state i and the $a_1(j) = \sum_k (k-j)w_{j,k}$are the (first-order) jump moments in the terminology of van Kampen.

Now consider that all $Q_{i,j}$ are positive numbers with $\sum_j Q_j = 1$. Because the average size of crashes grows with the level of states, the $a_1(i)$ are a decreasing sequence of numbers. These numbers can be positive or negative depending on whether the average size of a crash weighted by its relative rate is larger or smaller than the average size of the business-as-usual step weighted by its relative rate. As the $a_1(i)$ are a decreasing sequence of numbers, either there is a crossover index e such that $a_1(i) \geq 0, i \leq e, a_1(i) \leq 0, i > e$ or the $a_1(i)$ are all negative or all positive.

Because there is no crash for small values of the price process, the $a_1(i)$ cannot all be negative. If the $a_1(i)$ are all positive, there can be no long-term equilibrium and prices escape to infinity. If, however, there are $a_1(i)$ of opposite signs, then the process exhibits a long-term equilibrium.

Following van Kampen and reasoning as above, we can also compute the evolution of the variance of the process, arriving at:

$$\frac{\mathrm{d}\sigma^2}{\mathrm{d}t} = \langle a_2(p)\rangle + 2\langle p - \langle p\rangle\rangle a_1(p) \tag{16}$$

4.6. *Long-term equilibrium*

Suppose that there is a unique stationary probability distribution. In this case, the long-term probability distribution of states will be the same regardless of the initial state. The matrix $\boldsymbol{Q}$ is thus replaced by a vector $\boldsymbol{Q}(t) = [Q_j(t)] = [Q_i(t)\ Q_{i+1}(t)]$. The condition for a long-term equilibrium is that the time derivative of the vector $\boldsymbol{Q}$ be identically zero. The condition for a stationary long-term solution therefore becomes: $0 = Q\boldsymbol{w}$, where $\boldsymbol{Q}$ is a time-independent vector. In this case we can write $\sum_j Q_j a_j = 0$. As the Q_j are all positive, this equation tells us that a long-term equilibrium is possible only if there are a_j of opposite sign.

Given the structure of the transition matrix $\boldsymbol{w}$, in the case of a long-term equilibrium it is possible to estimate the tails of the equilibrium probabilities Q_i. Consider that the crash size distribution is truncated at some truncation level that depends on the price level, i.e. on the state i. This implies that in the transition matrix $\boldsymbol{w}$ only a finite number of entries in each row are non-zero. Given the discrete nature of states and the ultimately finite size of the market, this implies that above and below some state index the number of non-zero entries per row of the transition matrix is constant. In addition, the distribution itself becomes state-invariant.

The above implies, in turn, that we can write the following approximate recursive equations:

$$\begin{aligned} Q_i &= \textstyle\sum_{k=1}^{k} \alpha_k Q_{i-k} \\ Q_i &= \textstyle\sum_{k=1}^{M} \beta_k Q_{i-k} \end{aligned} \tag{17}$$

respectively, for large positive and negative values of *i*. Set $Q_i = \exp(ai), \quad Q_i = \exp(bi)$, respectively, for large positive and negative values of *i*. The previous equations can then be rewritten as:

$$\begin{aligned} Q_i &= \textstyle\sum_{k=1}^{k} \alpha_k Q_{i-k} = \sum_{k=1}^{k} \alpha_k \exp(a(i-k)) \exp(ai) \sum_{k=1}^{k} \alpha_k \exp(-ak) = \exp(ai) A = Q_i A \\ Q_i &= \textstyle\sum_{k=1}^{M} \beta_k Q_{i-k} = \sum_{k=1}^{M} \beta_k \exp(b(i-k)) = \exp(bi) \sum_{k=1}^{M} \beta_k \exp(-bk) = \exp(bi) B = Q_i B \end{aligned} \tag{18}$$

It is clear from the above equations that one must set $A = 1$ and $B = 1$, conditions that determine the values of *a* and *b*. It is also clear from looking at the transition matrix that $\sum_{k=1}^{k} \alpha_k > 1$, $\sum_{k=1}^{M} \beta_k > 1$, which implies that both the *a* and *b* are negative.

If, therefore, there is an equilibrium long-term distribution, its tails are exponentially distributed.

4.7. The growth regime

To understand the long-term behavior of the process when there is no equilibrium distribution, i.e. when the a_j are all positive, some approximation is required. In our process, the state transition probabilities are state-dependent. However, for large values of time *t* and for large values of the log price, the state transition probabilities become state-independent. This is because the system is finite and the size of crashes is truncated and exhibits upper bounds.

As a consequence, we can write the master equation as follows:

$$\frac{\mathrm{d}Q_{i,j}(t)}{\mathrm{d}t} = (N+1)\lambda \left[-W_{i,i} + \sum_{k \neq i} W_{i-k} Q_{k,j} \right] \tag{19}$$

where we resort to the use of transition probabilities as opposed to transition rates. Given suitable initial conditions $Q_{i,0}$ we can now Laplace-transform both members of the previous equation to find:

$$uQ_j(u) - Q_{j,0} = (N+1)\lambda \left[-Q_j(u) + \sum_{k} W_{j-k} Q(u) \right]. \tag{20}$$

Montroll and Schlesinger show that the latter master equation represents a CTRW with a Gaussian wave front. The growth rate of the log(price) process becomes $(N+1)\lambda(\nu - w)$ where w is the average size of crashes for large values of prices.

Consider the equation for the average: $\mathrm{d}\langle p(t)\rangle/\mathrm{d}t = \langle a_1(p)\rangle$. As we assume that the state transition probabilities are constant in this range of states, the jump moments are also constant. The earlier equation becomes $\mathrm{d}\langle p(t)\rangle/\mathrm{d}t = \text{const}$. Therefore, the average log(price) grows linearly with time. In the same way, consider the equation for the variance:

$d\sigma^2/dt = \langle a_2(p)\rangle + 2\langle(p - \langle p\rangle)a_1(p)\rangle$. As jump moments are constant, the first term on the right is a constant and the second term vanishes. Thus the variance grows linearly with time, a characteristic of normal diffusions.

The picture is quite different if we consider intermediate time horizons starting from low price levels. In this case, the assumption that transition probabilities are state independent cannot be maintained. The average size of transitions, i.e. of log(price) jumps, starts from a maximum at low price levels and decreases until it reaches a minimum. Therefore, the jump moments are therefore no longer constant and the equations for the average and variance involve moments of higher orders.

In general, the dynamics of the log(price) average and variance for small price levels is complex and cannot be represented by simple formulas. We can gain a qualitative understanding of the system behavior in the initial phase making first-order approximations for jump moments. We assume that in some intermediate range of log(prices) the jump moments of first-order decrease linearly with price levels and that the jump moments of the second-order are constant.

In this case, the equations for both the average and variance become of the type $dz/dt = -Az + B$, $A > 0$, $B > 0$. Both the average and the variance will thus have the functional form $z = B/A - C\exp(-At)$. In the range of values under consideration, this functional form can be approximated with a polynomial of the second-order. In particular, for both the average and the variance of the process in some initial time interval, the linear growth is depressed by a quadratic term. The variance will begin to behave as a normal diffusion only when the process reaches the price level were jump moments are constant.

5. Simulation results

We can summarize the above discussion as follows. Over short time horizons, crashes are negligible and the compound business-as-usual/crash model behaves like the business-as-usual model, i.e. log(prices) exhibit a truncated Levy distribution. Over intermediate time horizons, the interaction of the Gaussian diffusion process generated by the business-as-usual model and the large jumps (crashes) can be approximated as an anomalous diffusion. Over long time horizons, the compound model either exhibits an equilibrium stationary distribution or grows as a geometric Gaussian diffusion.

The numerical simulation proceeds as follows. The system is initialized creating $N = 1000$ agents. Half of the agents are marked as buyers $\boldsymbol{B}$, the other half as sellers $\boldsymbol{S}$. There are $500 \times 499/2$ possible links between agents in each set. Links are selected from both sets with probability p_a. Clusters are formed using a fast cluster-formation algorithm. A random number of clusters is selected from both sets $\boldsymbol{B}$, $\boldsymbol{S}$ with probability a. The total number of buyers N_{B} and of sellers N_{S} are computed. The return is computed as:

$$\frac{\Delta P}{P} = [\mu + \delta(N_{\mathrm{B}} - N_{\mathrm{S}})] \tag{21}$$

The clusters of buyers and sellers are destroyed. Each buyer becomes a seller and is randomly aggregated to existing sellers and each seller becomes a buyer and is randomly aggregated to existing buyers. New clusters are then formed using the probability p_a. New prices are computed and the process repeated until negative returns exceed the threshold S_c.

When $(P_{k-1} - P)/P_k > S_c$, the crash regime is activated. Agents are aggregated with probability of links $p_S = c_s/N$ where $c_s = \arctan(k_S \log(P_K/C_f))/k_A$. To ensure that c_s is only slightly larger than 1, the coefficient k_A is slightly less than $\pi/2$. Clusters are then selected with probability a. All clusters made up of sellers are successively expanded to the underlying clusters to which they belong. The new number of sellers $N_S{}'$ is computed. The new price is computed as, $\Delta P = -\delta(N_B - N_S')^2$. Clusters of sellers and buyers are then destroyed and the process continues.

Different numerical simulations were performed on an artificial market made up of 1000 agents. The model requires the exogenous setting of the following parameters:

tr	the risk free interest rate
P_0	the initial price of the stock
c_s	aggregation factor of new nodes at each simulation step
S_c	threshold of negative returns that trigger crashes
k_A, k_S	coefficients for computing the aggregation probability of underlying clusters
a	probability of choosing a cluster in the normal regime
δ	market depth
δ_C	market depth in a crash regime.

The parameters were set as follows: $t_r = 0.0002$; $P_0 = 10$; $c_s = 0.035$; $k_A = 1.5$; $k_S = 10$; $a = 0.05$; $\delta = 0.005$; $\delta_C = 0.000015$.

Three sets of simulations were performed with the crash threshold set at 2, 12 and 20%, respectively. By changing the crash threshold, we let the crash rate vary; the average jump size of the process thereby also changes. The average, variance, and price distributions were analyzed. Their behavior is described in the following sections.

Figs. 1 and 2 show the results of simulations under different choices of the parameters. Although, the system is still highly idealized, simulations show a rich dynamics. A number

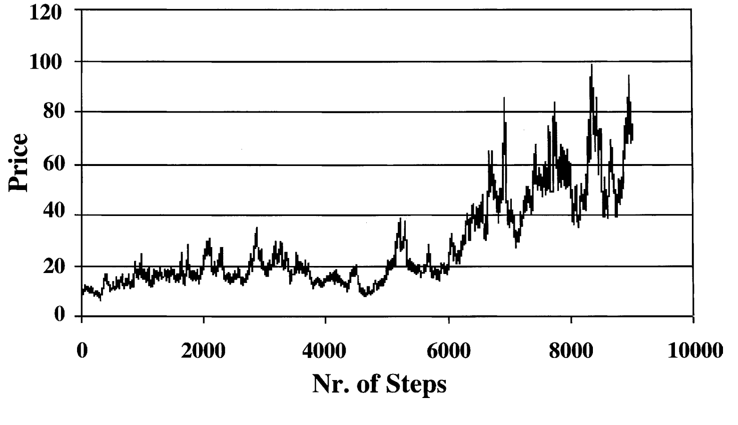

Fig. 1. Price trend with crash threshold set at 0.02.

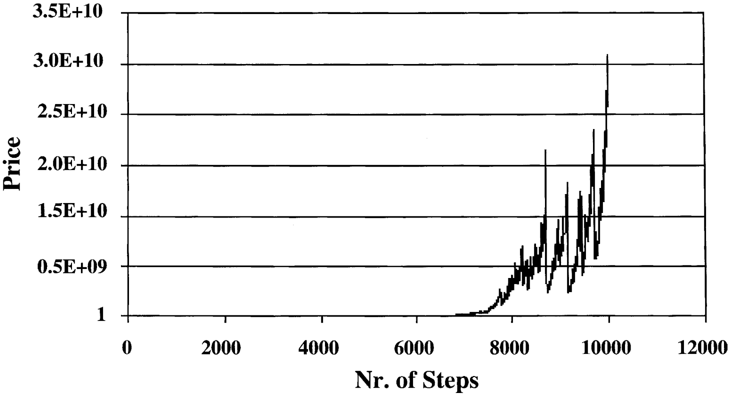

Fig. 2. Price trend with crash threshold set at 0.12.

of crashes occur in each simulation. Some drops in price are essentially without effect on the upward trend, others produce large price falls. In Fig. 1 the crash threshold is very low (2%), and high number of crashes keeps the average price growth at a low level. In fact, the average, after a transient period, sets to a growth rate essentially determined by the risk-free rate.

Fig. 2, that refers to a simulation with the crash threshold set at 12%, exhibit a small number of crashes and, thereby, an exponential growth of the price average.

6. Average

First the behavior of the average was checked. Simulations were performed with the crash parameter set at 2, 12, and 20%, respectively. The system ran for 12,800 steps and averages were computed at regular intervals starting from 50 steps. If we assume that one time step corresponds to one day of trading, 12,800 time steps correspond to roughly 60 years of trading. In interpreting results, recall that the only market feedback in this model is the presence of crashes. Both the riskfree rate and the growth rate are assumed to be constant throughout the entire simulation period.

The simulation results confirm the theoretical prediction. With the crash threshold set at 20%, the price average grows exponentially and crashes are very rare. The transient period is barely noticeable, the log(price) plot growth shows a slight convexity in the first portion. The behavior of the average is shown in Fig. 3.

Fig. 4 shows that when the crash threshold is set at 12%, there is still a slight convexity in the first part of the plot, but the two different regimes are difficult to identify visually.

When the crash threshold is set at 2%, crashes are frequent. After a short transient period, the system is in the steady regime and exhibits only the growth relative to the growth of the riskfree rate. Two sets of simulation runs were performed with the market depth parameter set at 0.000015 and 0.00015, respectively. In the steady regime, the steepness of the plot is the same in both cases as it is determined by the riskfree rate. The average, however, is lower in the case 0.00015. The behavior obtained is represented in Fig. 5a and b.

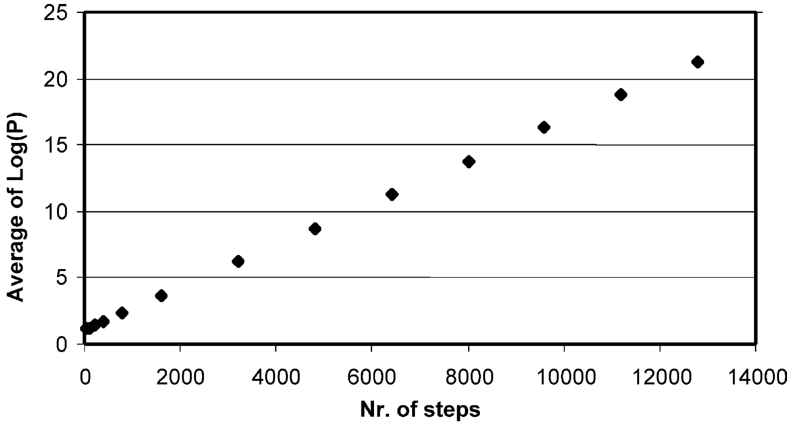

Fig. 3. Average of log(P) with crash threshold set at 0.2.

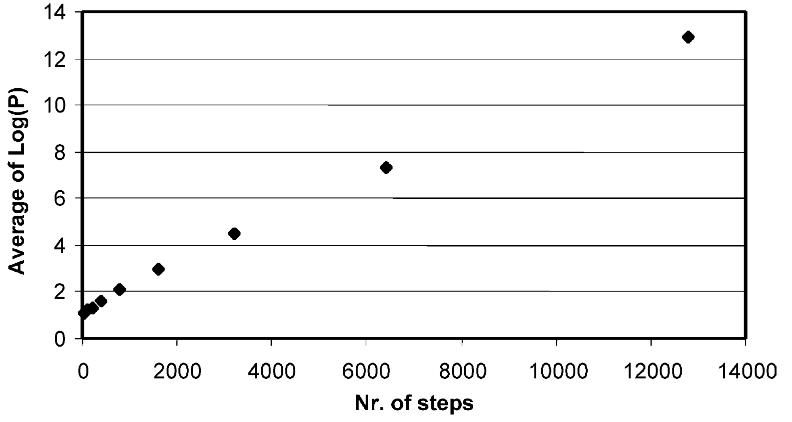

Fig. 4. Average of log(P) with crash threshold set at 0.12.

7. Variance

We then analyzed the variance of the model for the same values of crash parameters 2, 12 and 20%. Variances were computed over 12,800 steps as averages (over 3200 steps in the case). When the crash threshold is set at 2%, crashes are very frequent and the system is in a stationary regime. The growth of the log(price) process is due only to the riskfree rate and the variance—after a rather long transient period—sets on a constant value. Fig. 6 shows the behavior of the variance for the first 12800 steps with the crash threshold set at 2%.

With the crash parameter set at 12%, the behavior of the variance clearly exhibits a transient regime after which it stabilizes and grows linearly. The growth of the variance in the transient regime is less than linear. Fig. 7 shows the variance of the log(price) process

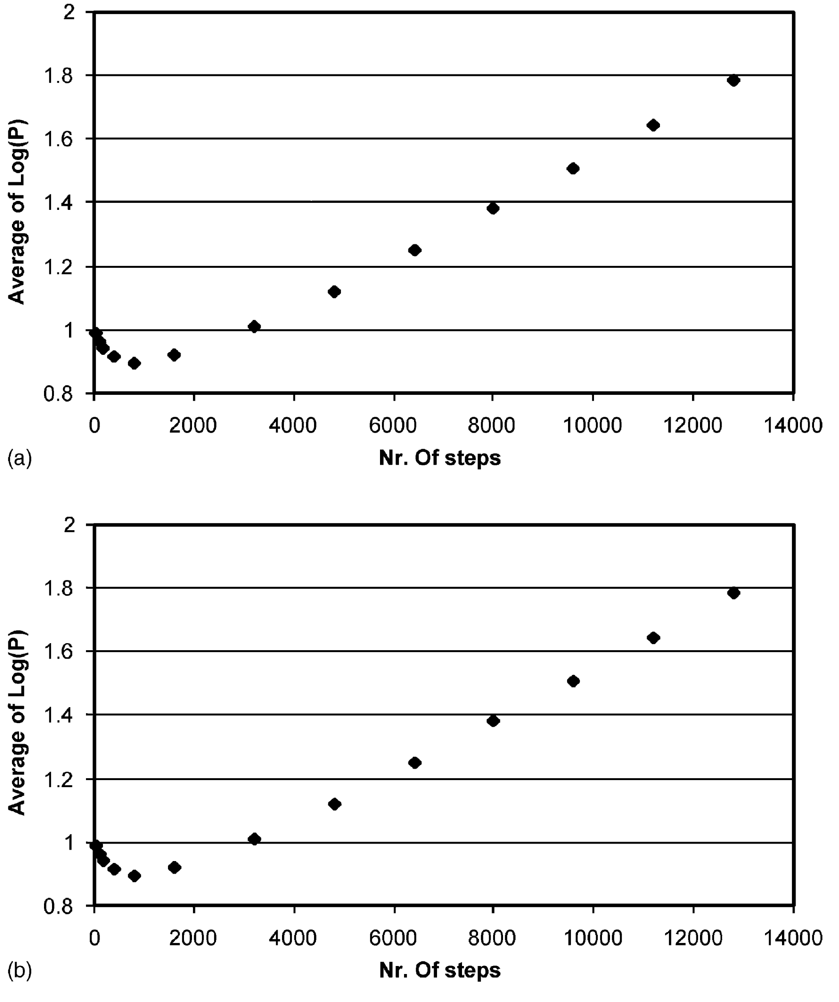

Fig. 5. (a) Average of log(P) with crash threshold set at 0.02 and $\delta = 0.000015$; (b) average of log(P) with crash threshold set at 0.02 and $\delta = 0.00015$.

at different time horizons. When the crash threshold is set at 20%, crashes are rare. Prices behave as a Gaussian diffusion as indicated by the linear growth of the variance shown in Fig. 8.

8. Distributions

Distributions of log(P) were computed letting the system run 3000 times with the crash threshold set at 2% and 5000 times with the crash threshold set at 12%. In the first case time horizon was 12,800 steps, while in the second case it was 3200 steps. Assuming that

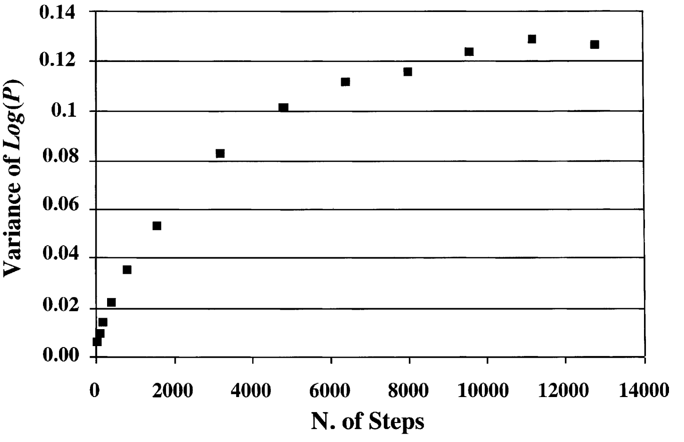

Fig. 6. Variance of log(P) with crash threshold set at 0.02 and $\delta = 0.00015$.

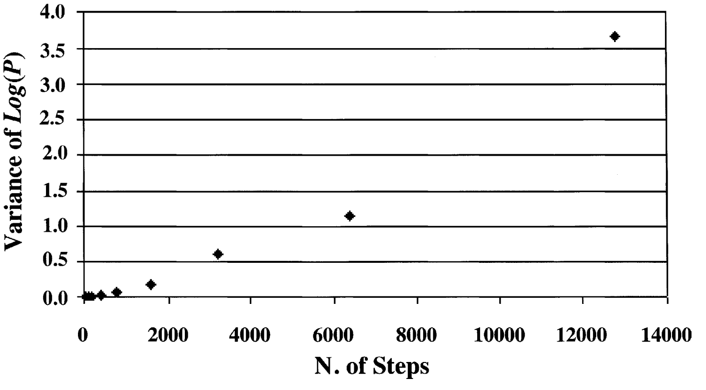

Fig. 7. Variance of log(P) with crash threshold set at 0.12.

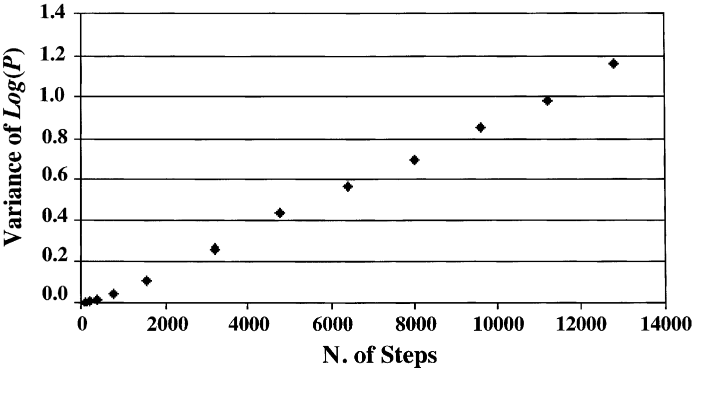

Fig. 8. Variance of log(P) with crash threshold set at 0.2.

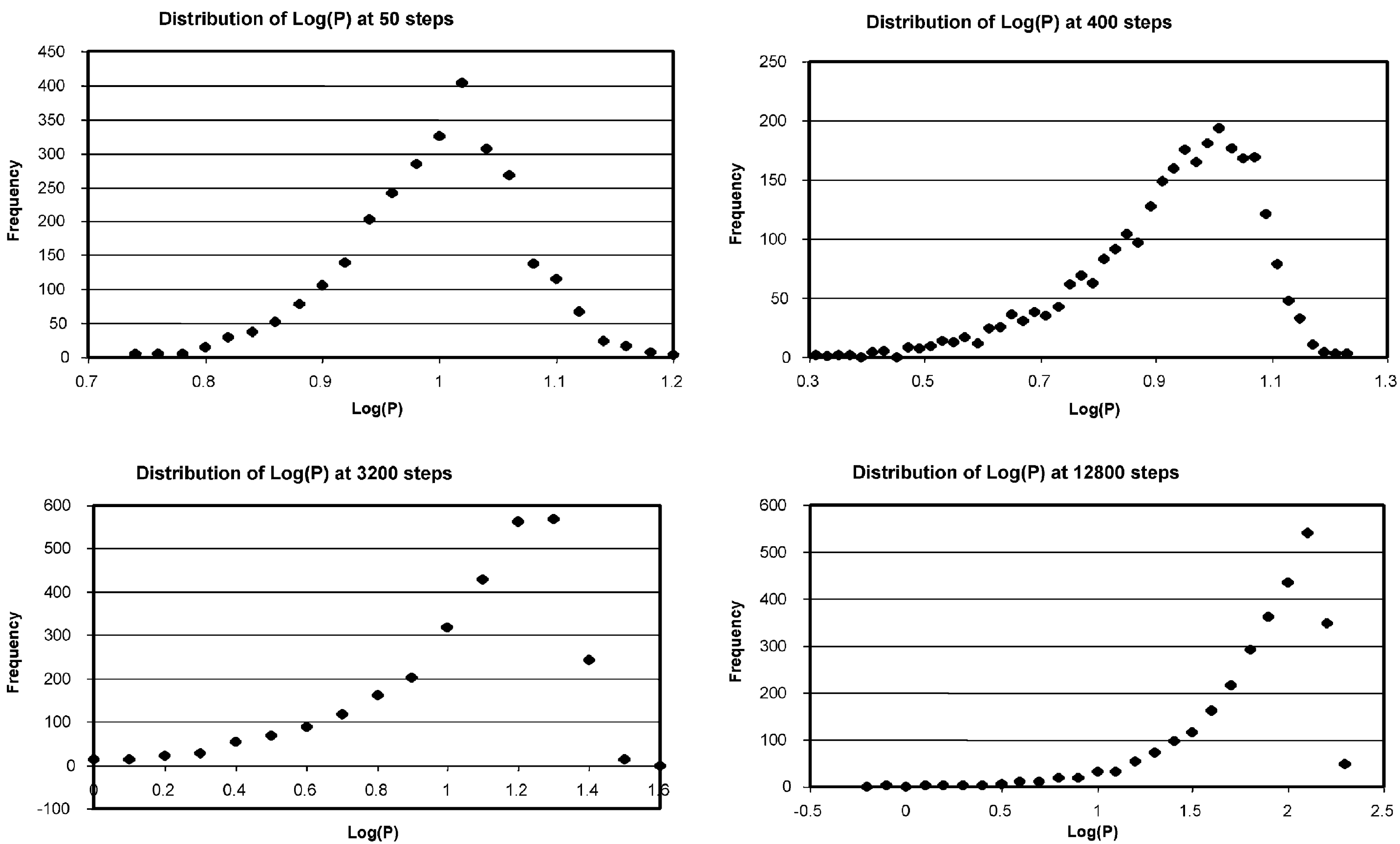

Fig. 9. Distributions of log(P) at different steps with crash threshold set at 0.02.

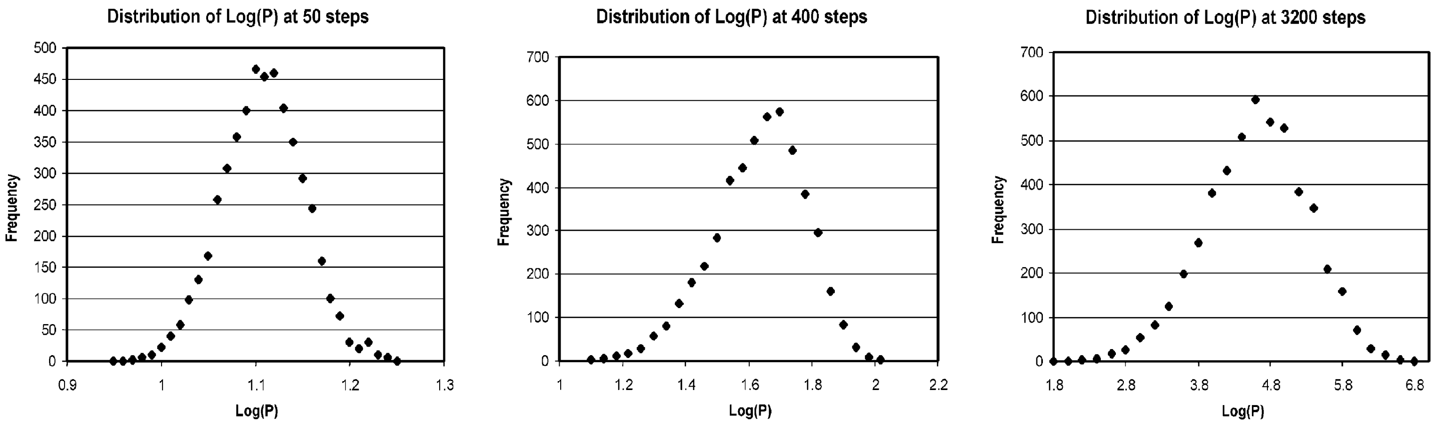

Fig. 10. Distributions of $\log(P)$ at different steps with crash threshold set at 0.12.

the average time step corresponds to one day of trading, 3200 steps correspond to roughly 15 years of price history.

When the crash threshold is set at 2%, crashes are very frequent and the system sets on the equilibrium distribution. The equilibrium distribution is highly asymmetrical with exponential tails as predicted by our model. Fig. 9 shows the behavior of distributions at 50, 400, 3200 and 12,800 steps when the crash threshold is set at 2% and the market depth d at 0.00015.

When the crash threshold is set at 12% the transition through the different regimes can be clearly seen. At the time horizon of 50 steps, the log(price) distribution is well approximated by a Gaussian distribution. As in the previous case, this is the Gaussian regime of the business-as-usual model.

At 400 steps, however, the distribution is clearly not Gaussian. It exhibits fat tails and a significant skewness. At 3200 steps the distribution is again Gaussian. Fig. 10 shows the shape of the distribution at 50, 400 and 3200 steps.

9. Conclusions

Though still highly idealized, we believe that our model exhibits a number of essential features of stock price behavior, i.e. fat-tailed fluctuations around an exponential trend punctuated with rare crashes. The market grows at a higher rate than the economy under inflationary pressure created by investment inflows but crashes pull it back to reduced growth level or eventually to equilibrium.

The price behavior is determined by random steps and feedback that we assume are originated by structural changes in the economy. Feedback is represented mathematically through the formalism of a master equation. Feedback generates a rich dynamics with different regimes at different time horizons.

The model assumes that the price growth rate in the business-as-usual regime is exogenously given. Future work will take into account additional feedback, exploring how this rate itself is pulled back by growing stock prices. The representation of this essential feature would, however, require the development of a growth theory that assumes that the real economy exhibits multiple inflation differentials.

Appendix A. Random graphs

Random graphs can be loosely considered as a model of percolation in infinite dimensions. Random graphs and finite-dimensional percolation models are characterized by critical points (i.e. critical values of some parameters) and threshold functions that mark an abrupt and qualitative change of behavior. The relevant parameter for both random graphs and finite-dimensional percolation is the probability p that an edge is open or closed.

We are interested in the limit behavior when the number of nodes $N \Rightarrow \infty$. It is possible to imagine a sequence of graphs of growing N and the limit distribution as the limit of distributions for $N \Rightarrow \infty$ if this limit exists. The probability threshold is replaced by different threshold functions that prescribe how the probability p grows with the number of vertices.

Suppose that a finite set of N vertices V_i, $i = 1, N$, is given. A link (or edge) is defined as an unordered pair of vertices (i, j). A graph is defined by a set of vertices V and by a set of edges E. Each of the N vertices can be connected with an edge to $N - 1$ other nodes. Therefore, there are a total of $N/2 = N(N - 1)/2$ possible edges.

Suppose that each pair of vertices has a probability p of being connected by an open edge and a probability $1 - p$ of not being connected and that these probabilities are independent. As each edge is randomly open or closed, the corresponding graph is called a random graph. The sample space is the set of all possible configurations. There are $(n/2)$ possible edges and thus $2^{(n/2)}$ configurations. On average, each vertex will be connected to $p(N - 1)$ vertices. It is convenient to represent p as, $p = c/N$.

We are interested in the distribution of the size of clusters (i.e. connected components). It can be demonstrated (Bollobas, 1985, Chapters IV and V) that for $0 < c < 1$ most clusters will be trees or will contain at most one cycle. It is possible to infer that the probability distribution of the number of nodes in a cluster, S, decays as an exponentially truncated power law for large values of S:

$$P(S) = AS^{-(\alpha+1)}\exp[(1 - c + \log c)] = AS^{-5/2}\exp[(1 - c + \log c)S] \tag{22}$$

For $c = 1$, the distribution becomes:

$$P(S) = AS^{-(\alpha+1)} = AS^{-(3/2+1)} = AS \tag{23}$$

If $c = 1$, the size distribution has an infinite variance in the limit of an infinite cluster while for $c < 1$, the size distribution has a finite variance. For $c > 1$, a giant cluster is formed in the limit of infinite N.

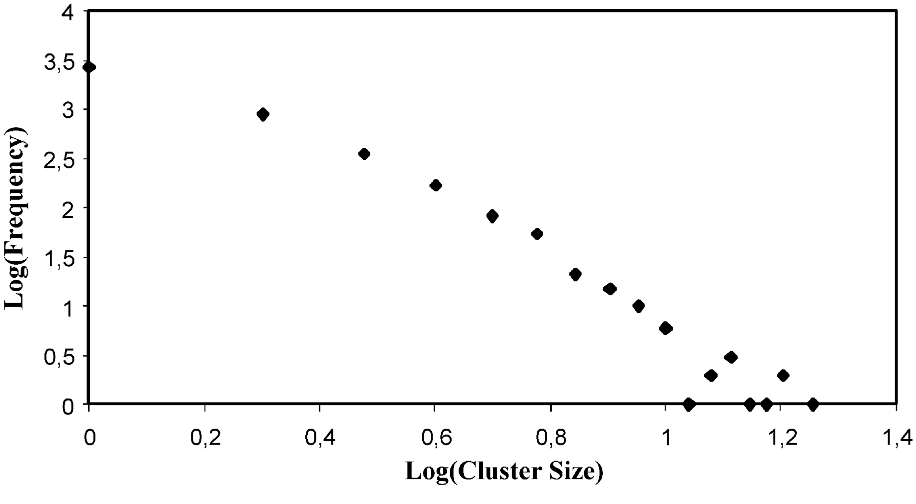

Fig. 11. Cluster size distribution in the case $c = 1$. The power-law distribution with exponent 5/2 of the cluster size is evident in the log–log plot.

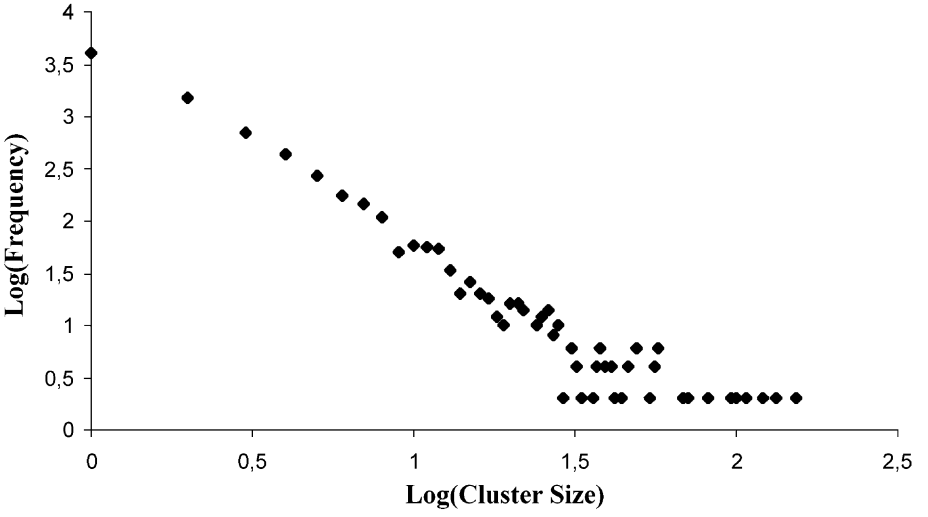

Fig. 12. Cluster size distribution in the case $c = 0.5$. The distribution follows the same power law with exponent 5/2 but is exponentially truncated.

$F(S)$ has an ensemble probability interpretation as the probability distribution of the size of a randomly chosen cluster. Figs. 11 and 12 illustrate the cluster size distribution obtained through simulation of a network of 30,000 vertices with $c = 1$ and 0.5.

Appendix B. Derivation of the master equation

Suppose that the log(price) process $p(t)$ is a discrete process, i.e. that $p(t)$ can only take integer values and that the model is in the Gaussian regime. We show that we can represent the log(price) process p by a CTRW. Suppose that prices change by discrete steps at random times. Suppose that the pausing time between steps is exponentially distributed with pdf $\psi(t) = N\lambda \exp(-N\lambda t)$, where λ is the crash rate and N a large integer that we will determine shortly. Suppose also that, at each step, the process p can make a positive jump of constant size ν_+ with probability 1/2 or a negative jump of constant size ν_- with probability 1/2 independent of the pausing time. The jump distribution $f(p)$ has therefore a mean $\langle p\rangle = \nu = \nu_+ - \nu_-$.

As demonstrated in Montroll and Schlesinger, for large values of t, such a process approximates a Gaussian diffusion with growth rate $\mu = \langle l/l\rangle = N\lambda\nu$ and variance $D = \langle l^2\rangle/2\langle t\rangle = (\nu_+^2 + \nu_-^2)N\lambda/4$. By a suitable choice of N, ν_+, ν_- it is therefore possible to approximate the log(price) process $p$for large values of t with any desired level of accuracy.

Because we model crashes as activated events of constant probability per time step, the number of crashes per time interval is a Poisson variable. Crashes are therefore distributed as a Poisson process. Thus it is not possible to represent the crash process either as a fixed-step random walk (the steps are exponentially distributed) or as a CTRW (the crash distribution itself depends on the price level). We can, however, find a suitable representation of both business-as-usual and crash processes as jump Markov processes (Aoki, 1996, for an introduction to the use of jump Markov processes in economics).

Jump Markov processes (Cox and Miller, 1994 or van Kampen, 1992) are Markov processes in continuous time with a countable number of states. Time spent in a state is called holding or sojourn time. A jump Markov process remains in the same state *i* for a lapse of time distributed according to an exponential distribution with a (possibly state-dependent) rate λ_i and then jumps to another state *j*. The exponential distribution of sojourn time is needed to preserve the Markovian character of the process.

When a jump occurs, the probability of jumps are governed by a probability transition matrix $\boldsymbol{W} = [W_{i,j}]$, that gives the probability of a transition from state *i* to state *j*. The distribution of sojourn time and the jump size distribution are assumed to be independent.

Given the probability transition matrix $\boldsymbol{W} = [W_{i,j}]$, we can compute the process transition rate matrix, $\boldsymbol{w} = [w_{i,j}]$, according to the following rules (Aoki, 1996):

$$w_{i,i} = -\lambda_i, \qquad w_{i,j} = \lambda_i W_{i,j}, \quad i \neq j. \tag{24}$$

Given that a state must jump to some other state, we can derive the relationship

$$\sum_j w_{i,j} = 0. \tag{25}$$

The crash process is clearly a jump Markov process with only one rate λ where states represent prices. The time between crashes is exponentially distributed; the distribution of crash size depends on the price level, i.e. on the current state of the process.

In general, jump Markov processes and CTRW are different processes. However, a CTRW with exponential distribution of pausing time and with pausing time and step size independently distributed is a jump Markov process. As these conditions are met by the log(price) process *p*(*t*), we can represent both the business-as-usual and the crash processes as jump Markov processes. We can thus write one transition matrix for the joint process.

Let us assume that the riskfree rate is zero. This assumption implies that there is no real growth in the economy. However, it will be clear from the structure of the master equation that a non-zero riskfree rate simply acts as an additive term. The probability transition matrix $\boldsymbol{W} = [W_{i,j}]$ can be written as

$$W_{i,i+\nu_+} = \tfrac{1}{2}, \qquad W_{i,j-\nu_-} = \tfrac{1}{2}, \qquad W_{i,j} = 0, \quad j \neq i + \nu_+, i - \nu_- \tag{26}$$

for the business-as-usual process and as:

$$\begin{aligned} W_{i,j} = 0, \quad j \geq i - \nu_-, W_{i,j} &= \mathrm{Prob}(\log(\mathrm{crash} = i - j))/\mathrm{state}(i) \\ &\equiv q_i(i-j), i - j > \nu_-, \end{aligned} \tag{27}$$

for the crash regime, where $q_i(i-j)$ is the probability that the logarithm of the crash be $j-i$. We also assume that the crash size is always larger than the normal price step size. Following Aoki, we can now combine the business-as-usual and the crash price processes and consider a single global log(price) process. If we write $w_{i,j}$ for the transition rates of the global log(price) process, we can represent this process as a jump Markov process with rate $(N+1)\lambda$ and with transition rates:

$$\begin{aligned} &w_{i,i} = -(N+1)\lambda,\ w_{i,i+\nu_+} = \tfrac{1}{2}(N\lambda),\ w_{i,i-\nu_-}\tfrac{1}{2}(N\lambda)\\ &w_{i,j} = p_i(i-j)\lambda,\ i-j>\nu_-\\ &w_{i,j} = 0 \end{aligned} \tag{28}$$

for any other j. We can therefore write the rate transition matrix $\boldsymbol{w} = [w_{i,j}]$ for the joint process as follows:

	·	·	$i+1$	I	$i-1$	$i-2$	·	·
·	·	·	·	·	·	·	·	·
$i-\nu_--2$	·	·	q_{i+1} $(\nu_-+1)\lambda$	q_{i+1} $(\nu_-+2)\lambda$	q_{i+1} $(\nu_-+2)\lambda$	q_{i+2} $(\nu_-+4)\lambda$	·	·
$i-\nu_--1$	·	·	$N\lambda/2$	$-q_{i+1}$ $(\nu_-+1)\lambda$	q_{i+1} $(\nu_-+2)\lambda$	q_{i+1} $(\nu_-+3)\lambda$	·	·
$i-\nu_-$	·	·	0	$N\lambda/2$	q_{i+1} $(\nu_-+1)\lambda$	q_{i+2} $(\nu_-+2)\lambda$	·	·
$i-\nu_-+1$	·	·	0	0	$N\lambda/2$	q_{i+1} $(\nu_-+1)\lambda$	·	·
$i-\nu_{-+2}$	·	·	0	0	0	$N\lambda/2$	·	·
·	·	··	·	·	·	·	·	·
$i-1$	·	·	$-(N+1)\lambda$	0	0	0	·	·
i	·	·	0	$-(N+1)\lambda$	0	0	·	·
$i+1$	·	·	0	0	$-(N+1)\lambda$	0	·	·
$i+2$	·	·	0	0	0	$-(N+1)\lambda$	·	·
	·	·	·	·	·	·	·	·
$j+\nu_+-1$	·	·	$N\lambda/2$	0	0	0	·	·
$j+\nu_+$	·	·	0	$N\lambda/2$	0	0	·	·
$j+\nu_++1$	·	·	0	0	$N\lambda/2$	0	·	·
$j+\nu_++2$	·	·	0	0	0	$N\lambda/2$	·	·

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