# EI3302 Propagation and scattering in multilayered structures

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## 1 Preliminary analysis

$\mathrm{e}^{-\mathrm{i}\omega t}$ assumed and suppressed.

Constitutive relations (CL):

$$\boldsymbol{D}(\boldsymbol{\rho}, z) = \varepsilon_0 \left[ \boldsymbol{\varepsilon}(z) \cdot \boldsymbol{E}(\boldsymbol{\rho}, z) + \eta_0 \boldsymbol{\xi}(z) \cdot \boldsymbol{H}(\boldsymbol{\rho}, z) \right], \tag{1a}$$

$$\boldsymbol{B}(\boldsymbol{\rho}, z) = c^{-1} \left[ \boldsymbol{\zeta}(z) \cdot \boldsymbol{E}(\boldsymbol{\rho}, z) + \eta_0 \boldsymbol{\mu}(z) \cdot \boldsymbol{H}(\boldsymbol{\rho}, z) \right], \tag{1b}$$

$$\boldsymbol{\rho} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}. \tag{1c}$$

Note, all constitutive dyadics are dimensionless.

Decompositions:

$$\boldsymbol{E} = \boldsymbol{E}_{\mathrm{t}} + E_z\hat{\boldsymbol{z}}, \qquad \boldsymbol{E}_{\mathrm{t}} = E_x\hat{\boldsymbol{x}} + E_y\hat{\boldsymbol{y}}. \tag{2a}$$

$$\varepsilon = \boldsymbol{\varepsilon}_{\mathrm{tt}} + \boldsymbol{\varepsilon}_{\mathrm{t}}\hat{\boldsymbol{z}} + \hat{\boldsymbol{z}}\boldsymbol{\varepsilon}_z + \varepsilon_{zz}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}, \tag{2b}$$

$$\boldsymbol{\varepsilon}_{\mathrm{tt}} = \varepsilon_{xx}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \varepsilon_{xy}\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \varepsilon_{yx}\hat{\boldsymbol{y}}\hat{\boldsymbol{x}} + \varepsilon_{yy}\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}, \tag{2c}$$

$$\boldsymbol{\varepsilon}_{\mathrm{t}} = \varepsilon_{xz}\hat{\boldsymbol{x}} + \varepsilon_{yz}\hat{\boldsymbol{y}}, \qquad \boldsymbol{\varepsilon}_z = \varepsilon_{zx}\hat{\boldsymbol{x}} + \varepsilon_{zy}\hat{\boldsymbol{y}}, \tag{2d}$$

other vectors and dyadics analogously.

Maxwell's equations (ME):

$$\nabla \times \boldsymbol{E} = \mathrm{i}\omega\boldsymbol{B}, \qquad \nabla \times \boldsymbol{H} = -\mathrm{i}\omega\boldsymbol{D}. \tag{3}$$

Decomposition into plane waves:

$$\boldsymbol{E}(\boldsymbol{k}_{\mathrm{t}}, z) = \int \boldsymbol{E}(\boldsymbol{\rho}, z)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_{\mathrm{t}}\cdot\boldsymbol{\rho}}\mathrm{d}\mathcal{S}, \tag{4a}$$

$$\boldsymbol{E}(\boldsymbol{\rho}, z) = \frac{1}{2\pi}\int \boldsymbol{E}(\boldsymbol{k}_{\mathrm{t}}, z)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\mathrm{t}}\cdot\boldsymbol{\rho}}\mathrm{d}^2k, \tag{4b}$$

$$\boldsymbol{k}_{\mathrm{t}} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}. \tag{4c}$$

In all layers, transversal wave-vector $\boldsymbol{k}_{\mathrm{t}}$ given by the plane wave impinging on the ML-structure $\Rightarrow$

$$\nabla = \mathrm{i}\boldsymbol{k}_{\mathrm{t}} + \hat{\boldsymbol{z}}\,\partial_z. \tag{5}$$

Transversal and longitudinal projections of ME $\Rightarrow$

$$\mathrm{i}\,(\boldsymbol{k}_{\mathrm{t}}\times\hat{\boldsymbol{z}})\,E_z + \hat{\boldsymbol{z}}\times\,\partial_z\boldsymbol{E}_{\mathrm{t}} = \mathrm{i}\omega\boldsymbol{B}_{\mathrm{t}}, \tag{6a}$$

$$\mathrm{i}\,(\boldsymbol{k}_{\mathrm{t}}\times\hat{\boldsymbol{z}})\,H_z + \hat{\boldsymbol{z}}\times\,\partial_z\boldsymbol{H}_{\mathrm{t}} = -\mathrm{i}\omega\boldsymbol{D}_{\mathrm{t}}, \tag{6b}$$

$$\hat{\boldsymbol{z}}\cdot(\boldsymbol{k}_{\mathrm{t}}\times\boldsymbol{E}_{\mathrm{t}}) = \omega B_z, \tag{6c}$$

$$\hat{\boldsymbol{z}}\cdot(\boldsymbol{k}_{\mathrm{t}}\times\boldsymbol{H}_{\mathrm{t}}) = -\omega D_z. \tag{6d}$$

$\hat{\boldsymbol{z}}\times \Leftrightarrow \mathbf{J}\cdot = (-\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}})\,\cdot$ and $\mathbf{J}\cdot\mathbf{J} = -\mathbf{I} \Rightarrow$

$$\partial_z\boldsymbol{E}_{\mathrm{t}} = \mathrm{i}\boldsymbol{k}_{\mathrm{t}}E_z - \mathrm{i}\omega\mathbf{J}\cdot\boldsymbol{B}_{\mathrm{t}}, \tag{7a}$$

$$\partial_z\boldsymbol{H}_{\mathrm{t}} = \mathrm{i}\boldsymbol{k}_{\mathrm{t}}H_z + \mathrm{i}\omega\mathbf{J}\cdot\boldsymbol{D}_{\mathrm{t}}, \tag{7b}$$

$$(\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}})\cdot\boldsymbol{E}_{\mathrm{t}} = \omega B_z, \tag{7c}$$

$$(\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}})\cdot\boldsymbol{H}_{\mathrm{t}} = -\omega D_z. \tag{7d}$$

Using CL:

$$\begin{aligned}\partial_z \boldsymbol{E}_{\mathrm{t}} =& -\mathrm{i}k_0\left[\left(\mathbf{J}\cdot\boldsymbol{\zeta}_{\mathrm{tt}}\right)\cdot\boldsymbol{E}_{\mathrm{t}}+\eta_0\left(\mathbf{J}\cdot\boldsymbol{\mu}_{\mathrm{tt}}\right)\cdot\boldsymbol{H}_{\mathrm{t}}\right]\\ &-\mathrm{i}\left(k_0\mathbf{J}\cdot\boldsymbol{\zeta}_{\mathrm{t}}-\boldsymbol{k}_{\mathrm{t}}\right)E_z-\mathrm{i}k_0\left(\mathbf{J}\cdot\boldsymbol{\mu}_{\mathrm{t}}\right)\eta_0 H_z,\end{aligned} \tag{8a}$$

$$\begin{aligned}\eta_0\,\partial_z \boldsymbol{H}_{\mathrm{t}} =& \mathrm{i}k_0\left[\left(\mathbf{J}\cdot\boldsymbol{\varepsilon}_{\mathrm{tt}}\right)\cdot\boldsymbol{E}_{\mathrm{t}}+\eta_0\left(\mathbf{J}\cdot\boldsymbol{\xi}_{\mathrm{tt}}\right)\cdot\boldsymbol{H}_{\mathrm{t}}\right]\\ &+\mathrm{i}k_0\left(\mathbf{J}\cdot\boldsymbol{\varepsilon}_{\mathrm{t}}\right)E_z+\mathrm{i}\left(k_0\mathbf{J}\cdot\boldsymbol{\xi}_{\mathrm{t}}+\boldsymbol{k}_{\mathrm{t}}\right)\eta_0 H_z,\end{aligned} \tag{8b}$$

$$k_0^{-1}\left(\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}}\right)\cdot\boldsymbol{E}_{\mathrm{t}} = \boldsymbol{\zeta}_z\cdot\boldsymbol{E}_{\mathrm{t}}+\zeta_{zz}E_z+\eta_0\left(\boldsymbol{\mu}_z\cdot\boldsymbol{H}_{\mathrm{t}}+\mu_{zz}H_z\right), \tag{8c}$$

$$-\eta_0 k_0^{-1}\left(\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}}\right)\cdot\boldsymbol{H}_{\mathrm{t}} = \boldsymbol{\varepsilon}_z\cdot\boldsymbol{E}_{\mathrm{t}}+\varepsilon_{zz}E_z+\eta_0\left(\boldsymbol{\xi}_z\cdot\boldsymbol{H}_{\mathrm{t}}+\xi_{zz}H_z\right). \tag{8d}$$

Eliminating longitudinal components:

$$\begin{aligned}E_z = -\frac{1}{\varepsilon_{zz}\mu_{zz}-\xi_{zz}\zeta_{zz}}\Big[&\left(\mu_{zz}\boldsymbol{\varepsilon}_z-\xi_{zz}\boldsymbol{\zeta}_z+k_0^{-1}\xi_{zz}\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}}\right)\cdot\boldsymbol{E}_{\mathrm{t}}\\ &+\eta_0\left(\mu_{zz}\boldsymbol{\xi}_z-\xi_{zz}\boldsymbol{\mu}_z+k_0^{-1}\mu_{zz}\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}}\right)\cdot\boldsymbol{H}_{\mathrm{t}}\Big],\end{aligned} \tag{9a}$$

$$\begin{aligned}\eta_0 H_z = -\frac{1}{\varepsilon_{zz}\mu_{zz}-\xi_{zz}\zeta_{zz}}\Big[&\left(\varepsilon_{zz}\boldsymbol{\zeta}_z-\zeta_{zz}\boldsymbol{\varepsilon}_z-k_0^{-1}\varepsilon_{zz}\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}}\right)\cdot\boldsymbol{E}_{\mathrm{t}}\\ &+\eta_0\left(\varepsilon_{zz}\boldsymbol{\mu}_z-\zeta_{zz}\boldsymbol{\xi}_z-k_0^{-1}\zeta_{zz}\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}}\right)\cdot\boldsymbol{H}_{\mathrm{t}}\Big].\end{aligned} \tag{9b}$$

ODE:

$$\partial_z\begin{bmatrix}\boldsymbol{E}_{\mathrm{t}}\\ \eta_0\boldsymbol{H}_{\mathrm{t}}\end{bmatrix} = \mathrm{i}k_0\begin{bmatrix}\mathbf{W}_{11} & \mathbf{W}_{12}\\ \mathbf{W}_{21} & \mathbf{W}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{E}_{\mathrm{t}}\\ \eta_0\boldsymbol{H}_{\mathrm{t}}\end{bmatrix}, \tag{10}$$

where

$$\begin{aligned}\mathbf{W}_{11} =& -\mathbf{J}\cdot\boldsymbol{\zeta}_{\mathrm{tt}} + \frac{\mathbf{J}\cdot\boldsymbol{\zeta}_{\mathrm{t}} - k_0^{-1}\boldsymbol{k}_{\mathrm{t}}}{\varepsilon_{zz}\mu_{zz} - \xi_{zz}\zeta_{zz}}\left(\mu_{zz}\boldsymbol{\varepsilon}_z - \xi_{zz}\boldsymbol{\zeta}_z + k_0^{-1}\xi_{zz}\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}}\right)\\ &+ \frac{\mathbf{J}\cdot\boldsymbol{\mu}_{\mathrm{t}}}{\varepsilon_{zz}\mu_{zz} - \xi_{zz}\zeta_{zz}}\left(\varepsilon_{zz}\boldsymbol{\zeta}_z - \zeta_{zz}\boldsymbol{\varepsilon}_z - k_0^{-1}\varepsilon_{zz}\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}}\right), \end{aligned} \tag{11a}$$

$$\begin{aligned}\mathbf{W}_{12} =& -\mathbf{J}\cdot\boldsymbol{\mu}_{\mathrm{tt}} + \frac{\mathbf{J}\cdot\boldsymbol{\zeta}_{\mathrm{t}} - k_0^{-1}\boldsymbol{k}_{\mathrm{t}}}{\varepsilon_{zz}\mu_{zz} - \xi_{zz}\zeta_{zz}}\left(\mu_{zz}\boldsymbol{\xi}_z - \xi_{zz}\boldsymbol{\mu}_z + k_0^{-1}\mu_{zz}\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}}\right)\\ &+ \frac{\mathbf{J}\cdot\boldsymbol{\mu}_{\mathrm{t}}}{\varepsilon_{zz}\mu_{zz} - \xi_{zz}\zeta_{zz}}\left(\varepsilon_{zz}\boldsymbol{\mu}_z - \zeta_{zz}\boldsymbol{\xi}_z - k_0^{-1}\zeta_{zz}\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}}\right), \end{aligned} \tag{11b}$$

$$\begin{aligned}\mathbf{W}_{21} =& \mathbf{J}\cdot\boldsymbol{\varepsilon}_{\mathrm{tt}} - \frac{\mathbf{J}\cdot\boldsymbol{\xi}_{\mathrm{t}} + k_0^{-1}\boldsymbol{k}_{\mathrm{t}}}{\varepsilon_{zz}\mu_{zz} - \xi_{zz}\zeta_{zz}}\left(\varepsilon_{zz}\boldsymbol{\zeta}_z - \zeta_{zz}\boldsymbol{\varepsilon}_z - k_0^{-1}\varepsilon_{zz}\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}}\right)\\ &- \frac{\mathbf{J}\cdot\boldsymbol{\varepsilon}_{\mathrm{t}}}{\varepsilon_{zz}\mu_{zz} - \xi_{zz}\zeta_{zz}}\left(\mu_{zz}\boldsymbol{\varepsilon}_z - \xi_{zz}\boldsymbol{\zeta}_z + k_0^{-1}\xi_{zz}\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}}\right), \end{aligned} \tag{11c}$$

$$\begin{aligned}\mathbf{W}_{22} =& \mathbf{J}\cdot\boldsymbol{\xi}_{\mathrm{tt}} - \frac{\mathbf{J}\cdot\boldsymbol{\xi}_{\mathrm{t}} + k_0^{-1}\boldsymbol{k}_{\mathrm{t}}}{\varepsilon_{zz}\mu_{zz} - \xi_{zz}\zeta_{zz}}\left(\varepsilon_{zz}\boldsymbol{\mu}_z - \zeta_{zz}\boldsymbol{\xi}_z - k_0^{-1}\zeta_{zz}\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}}\right)\\ &- \frac{\mathbf{J}\cdot\boldsymbol{\varepsilon}_{\mathrm{t}}}{\varepsilon_{zz}\mu_{zz} - \xi_{zz}\zeta_{zz}}\left(\mu_{zz}\boldsymbol{\xi}_z - \xi_{zz}\boldsymbol{\mu}_z + k_0^{-1}\mu_{zz}\mathbf{J}\cdot\boldsymbol{k}_{\mathrm{t}}\right). \end{aligned} \tag{11d}$$

In the compact notation

$$\boldsymbol{F}(z) = \begin{bmatrix}\boldsymbol{E}_{\mathrm{t}}(z)\\ \eta_0\boldsymbol{H}_{\mathrm{t}}(z)\end{bmatrix}, \qquad \mathbf{W}(z) = \begin{bmatrix}\mathbf{W}_{11}(z) & \mathbf{W}_{12}(z)\\ \mathbf{W}_{21}(z) & \mathbf{W}_{22}(z)\end{bmatrix}, \tag{12}$$

(10) becomes

$$\partial_z\boldsymbol{F}(z) = \mathrm{i}k_0\mathbf{W}(z)\cdot\boldsymbol{F}(z)\,. \tag{13}$$

Note that (13) is valid for arbitrary $z$-dependence in the constitutive dyadics.

## 2 Piecewise homogeneous stratification

### 2.1 Homogeneous layer

$\mathbf{W}$ is independent of $z$. A suitable approach to (13) is diagonalisation:

$$\mathbf{W} = \mathbf{T}^{-1}\mathbf{D}\mathbf{T}, \tag{14}$$

where

$$\mathbf{D} = \operatorname{diag}\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} \tag{15}$$

is a diagonal matrix, where the elements $\{\gamma_i\}$ are the eigenvalues of $\mathbf{W}$, determined from

$$\det(\mathbf{W} - \gamma\mathbf{I}) = 0. \tag{16}$$

The matrix

$$\mathbf{T}^{-1} = \begin{bmatrix} | & | & | & | \\ \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3 & \boldsymbol{w}_4 \\ | & | & | & | \end{bmatrix}, \tag{17}$$

where the columns are formed by the eigenvectors $\{\boldsymbol{w}_i\}$ of $\boldsymbol{W}$, determined from

$$(\boldsymbol{W} - \gamma_i\mathbf{I})\,\boldsymbol{w}_i = \mathbf{0}. \tag{18}$$

Now, $\partial_z\boldsymbol{F} = \mathbf{T}^{-1}\mathbf{D}\mathbf{T}\boldsymbol{F} \Rightarrow$ diagonalised system

$$\partial_z\left(\mathbf{T}\boldsymbol{F}\right) = \mathrm{i}k_0\mathbf{D}\left(\mathbf{T}\boldsymbol{F}\right). \tag{19}$$

Solving using the matrix exponential:

$$\mathbf{M}(z) = \exp(\mathrm{i}k_0\mathbf{D}z) = \operatorname{diag}\left\{\mathrm{e}^{\mathrm{i}k_0\gamma_1 z}, \mathrm{e}^{\mathrm{i}k_0\gamma_2 z}, \mathrm{e}^{\mathrm{i}k_0\gamma_3 z}, \mathrm{e}^{\mathrm{i}k_0\gamma_4 z}\right\}, \tag{20}$$

we obtain

$$\left(\mathbf{T}\boldsymbol{F}(z_2)\right) = \mathbf{M}(z_2 - z_1)\left(\mathbf{T}\boldsymbol{F}(z_1)\right) \Rightarrow \tag{21a}$$

$$\boldsymbol{F}(z_2) = \mathbf{T}^{-1}\mathbf{M}(z_2 - z_1)\,\mathbf{T}\boldsymbol{F}(z_1) = \mathbf{P}(z_2 - z_1)\cdot\boldsymbol{F}(z_1). \tag{21b}$$

The propagator:

$$\mathbf{P}(z) = \mathbf{T}^{-1}\mathbf{M}(z)\,\mathbf{T}. \tag{22}$$

Decomposing the propagator into $2 \times 2$-blocks:

$$\mathbf{P}(z) = \begin{bmatrix} \mathbf{A}(z) & \mathbf{B}(z) \\ \mathbf{C}(z) & \mathbf{D}(z) \end{bmatrix}, \tag{23}$$

we obtain more explicitly

$$\begin{bmatrix} \boldsymbol{E}_{\mathrm{t}}(z_2) \\ \eta_0 \boldsymbol{H}_{\mathrm{t}}(z_2) \end{bmatrix} = \begin{bmatrix} \mathbf{A}(z_2 - z_1) & \mathbf{B}(z_2 - z_1) \\ \mathbf{C}(z_2 - z_1) & \mathbf{D}(z_2 - z_1) \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_{\mathrm{t}}(z_1) \\ \eta_0 \boldsymbol{H}_{\mathrm{t}}(z_1) \end{bmatrix}. \tag{24}$$

Note, by definition the propagator in the homogeneous layer must satisfy

$$[\mathbf{P}(z)]^{-1} = \mathbf{P}(-z) \Leftrightarrow \mathbf{P}(z)\,\mathbf{P}(-z) = \mathbf{I}. \tag{25}$$

### 2.2 Multiple layers

$N$ layers, $1, \ldots, N$.

Let the $n$:th layer have boundaries at $z_{n-1}$ and $z_n$ ($z_{n-1} < z_n$).

Propagation from $z_{n-1}$ to $z_n$ by $\mathbf{P}_n = \mathbf{P}_n(z_n - z_{n-1})$.

Continuity of $\boldsymbol{E}_{\mathrm{t}}$ and $\boldsymbol{H}_{\mathrm{t}}$ $\Rightarrow$ the total propagator, from $z_0$ to $z_N$:

$$\mathbf{P}(z_N, z_0) = \mathbf{P}_N \mathbf{P}_{N-1} \ldots \mathbf{P}_2 \mathbf{P}_1. \tag{26}$$

By definition,

$$[\mathbf{P}(z_N, z_0)]^{-1} = \mathbf{P}(z_0, z_N)\,. \tag{27}$$

## 3 Eigenmodes in vacuum

Assuming a propagating plane wave, the wavenumber

$$\boldsymbol{k} = k_0 \hat{\boldsymbol{k}} = k_0\,(\hat{\boldsymbol{x}} \sin\theta \cos\varphi + \hat{\boldsymbol{y}} \sin\theta \sin\varphi + \hat{\boldsymbol{z}} \cos\theta) \Rightarrow \tag{28a}$$

$$\boldsymbol{k}_{\mathrm{t}} = k_0 \sin\theta\,(\hat{\boldsymbol{x}} \cos\varphi + \hat{\boldsymbol{y}} \sin\varphi)\,. \tag{28b}$$

In the general expression (11) we have $\boldsymbol{\varepsilon} = \boldsymbol{\mu} = \mathbf{I}$, $\boldsymbol{\xi} = \boldsymbol{\zeta} = \mathbf{0}$, yielding

$$\mathbf{W}_{11} = \mathbf{W}_{22} = \mathbf{0}, \tag{29a}$$

$$\begin{aligned}\mathbf{W}_{12} = -\mathbf{W}_{21} &= -\mathbf{J} - \frac{1}{k_0^2}\boldsymbol{k}_\mathrm{t}\left(\mathbf{J}\cdot\boldsymbol{k}_\mathrm{t}\right) \\ &= \hat{\boldsymbol{x}}\hat{\boldsymbol{y}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{x}} - \sin^2\theta\left(\hat{\boldsymbol{x}}\cos\varphi + \hat{\boldsymbol{y}}\sin\varphi\right)\left(-\hat{\boldsymbol{x}}\sin\varphi + \hat{\boldsymbol{y}}\cos\varphi\right) \\ &= \sin^2\theta\sin\varphi\cos\varphi\left(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\right) \\ &\quad + \hat{\boldsymbol{x}}\hat{\boldsymbol{y}}\left(1 - \sin^2\theta\cos^2\varphi\right) - \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}\left(1 - \sin^2\theta\sin^2\varphi\right).\end{aligned} \tag{29b}$$

As matrices,

$$\mathbf{W}_{12} = -\mathbf{W}_{21} = \begin{bmatrix} \sin^2\theta\sin\varphi\cos\varphi & 1 - \sin^2\theta\cos^2\varphi \\ -1 + \sin^2\theta\sin^2\varphi & -\sin^2\theta\sin\varphi\cos\varphi \end{bmatrix}. \tag{30}$$

The eigenvalues of $\mathbf{W} = \begin{bmatrix} \mathbf{0} & \mathbf{W}_{12} \\ -\mathbf{W}_{12} & \mathbf{0} \end{bmatrix}$ become (cf Maple-script):

$$\gamma_{1,2} = \cos\theta, \qquad \gamma_{3,4} = -\cos\theta. \tag{31}$$

With $\theta < \pi/2$, we have with $k_z = k_0\gamma$ that

$$\begin{aligned}\gamma_{1,2} &\Rightarrow +z\text{-going modes}, \\ \gamma_{3,4} &\Rightarrow -z\text{-going modes}.\end{aligned}$$

The result is expected from the outset, assuming a propagating plane wave, and the two-fold degeneracy in isotropic media.

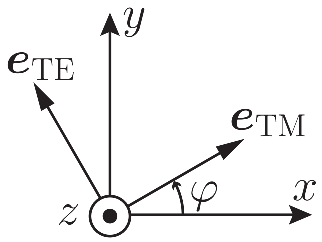
Modal transversal fields in the eigenvectors of $\mathbf{W}$:

$$\boldsymbol{w} = \begin{bmatrix}\boldsymbol{e} \\ \boldsymbol{h}\end{bmatrix}, \qquad (\mathbf{W} - \gamma\mathbf{I})\cdot\boldsymbol{w} = \begin{bmatrix} -\gamma\mathbf{I} & \mathbf{W}_{12} \\ -\mathbf{W}_{12} & -\gamma\mathbf{I}\end{bmatrix}\cdot\begin{bmatrix}\boldsymbol{e} \\ \boldsymbol{h}\end{bmatrix} = \mathbf{0} \Rightarrow \tag{32a}$$

$$\boldsymbol{h} = -\frac{1}{\gamma}\mathbf{W}_{12}\cdot\boldsymbol{e}, \qquad \left(\gamma^2\mathbf{I} + \mathbf{W}_{12}^2\right)\cdot\boldsymbol{e} = \mathbf{0}. \tag{32b}$$

$\gamma^2 = \cos^2\theta$, $\mathbf{W}_{12}^2 = -\cos^2\theta\mathbf{I}$ (cf. Maple-script) $\Rightarrow \gamma^2\mathbf{I} + \mathbf{W}_{12}^2 = \mathbf{0} \Rightarrow$ $\boldsymbol{e} \in$ any two linearly independent vectors (two-fold degeneracy).

### 3.1 TM- and TE-modes




Convenient choice of linearly independent eigenmodes are the TM- and TE-modes.

$$\boldsymbol{e}_{\mathrm{TM}} \sim \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \Rightarrow \boldsymbol{h}_{\mathrm{TM}} \sim \frac{1}{\gamma}\begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}, \tag{33a}$$

$$\boldsymbol{e}_{\mathrm{TE}} \sim \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix} \Rightarrow \boldsymbol{h}_{\mathrm{TE}} \sim -\frac{\cos^2\theta}{\gamma}\begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \tag{33b}$$

$\pm z$-going modes $\Rightarrow \gamma = \pm\cos\theta$. Relabel 1=TM, 2=TE, and power normalise such that $\boldsymbol{e}_{\mathrm{TM}} \times \boldsymbol{h}_{\mathrm{TM}} = \boldsymbol{e}_{\mathrm{TE}} \times \boldsymbol{h}_{\mathrm{TE}} = \pm\hat{\boldsymbol{z}} \Rightarrow$

$$\boldsymbol{w}_1^{\pm} = \begin{bmatrix} \cos\varphi\sqrt{\cos\theta} \\ \sin\varphi\sqrt{\cos\theta} \\ \mp\sin\varphi/\sqrt{\cos\theta} \\ \pm\cos\varphi/\sqrt{\cos\theta} \end{bmatrix}, \qquad \boldsymbol{w}_2^{\pm} = \begin{bmatrix} -\sin\varphi/\sqrt{\cos\theta} \\ \cos\varphi/\sqrt{\cos\theta} \\ \mp\cos\varphi\sqrt{\cos\theta} \\ \mp\sin\varphi\sqrt{\cos\theta} \end{bmatrix}. \tag{34}$$

Total transversal fields in vacuum region:

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{E}_{\mathrm{t}} \\ \eta_0\boldsymbol{H}_{\mathrm{t}} \end{bmatrix} = \begin{bmatrix} | & | & | & | \\ \boldsymbol{w}_1^+ & \boldsymbol{w}_2^+ & \boldsymbol{w}_1^- & \boldsymbol{w}_2^- \\ | & | & | & | \end{bmatrix} \begin{bmatrix} a_1^+ \\ a_2^+ \\ a_1^- \\ a_2^- \end{bmatrix} = \mathbf{T}_0^{-1}\begin{bmatrix} \boldsymbol{a}^+ \\ \boldsymbol{a}^- \end{bmatrix}, \tag{35a}$$

$$\boldsymbol{a}^{\pm}(z) = \begin{bmatrix} a_1^{\pm} \\ a_2^{\pm} \end{bmatrix}(z) = \text{ mode coefficients}, \tag{35b}$$

$$\begin{bmatrix} \boldsymbol{a}^+ \\ \boldsymbol{a}^- \end{bmatrix} = \mathbf{T}_0\boldsymbol{F}. \tag{35c}$$

## 4 Scattering parameters

Assume an $N$-layered structure sandwiched between vacuum half-spaces at $z_0$ and $z_N$. Using the earlier derived propagator (26), we obtain

$$\begin{aligned}\begin{bmatrix}\boldsymbol{a}^+\\ \boldsymbol{a}^-\end{bmatrix}\left(z_N^+\right) &= \mathbf{T}_0\boldsymbol{F}(z_N) = \mathbf{T}_0\mathbf{P}(z_N,z_0)\,\boldsymbol{F}(z_0)\\ &= \mathbf{T}_0\mathbf{P}(z_N,z_0)\,\mathbf{T}_0^{-1}\begin{bmatrix}\boldsymbol{a}^+\\ \boldsymbol{a}^-\end{bmatrix}\left(z_0^-\right) = \begin{bmatrix}\boldsymbol{\alpha} & \boldsymbol{\beta}\\ \boldsymbol{\gamma} & \boldsymbol{\delta}\end{bmatrix}\begin{bmatrix}\boldsymbol{a}^+\\ \boldsymbol{a}^-\end{bmatrix}\left(z_0^-\right).\end{aligned} \tag{36}$$

Incoming modes: $\left\{\boldsymbol{a}^+\left(z_0^-\right),\ \boldsymbol{a}^-(z_N^+)\right\}$.

Scattered modes: $\left\{\boldsymbol{a}^-\left(z_0^-\right),\ \boldsymbol{a}^+(z_N^+)\right\}$.

Scattering matrices:

$$\begin{bmatrix}\boldsymbol{a}^-\left(z_0^-\right)\\ \boldsymbol{a}^+(z_N^+)\end{bmatrix} = \begin{bmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\\ \mathbf{S}_{21} & \mathbf{S}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{a}^+\left(z_0^-\right)\\ \boldsymbol{a}^-(z_N^+)\end{bmatrix} \tag{37}$$

where

$$\mathbf{S}_{11} = -\boldsymbol{\delta}^{-1}\boldsymbol{\gamma}, \tag{38a}$$

$$\mathbf{S}_{12} = \boldsymbol{\delta}^{-1}, \tag{38b}$$

$$\mathbf{S}_{21} = \boldsymbol{\alpha} - \boldsymbol{\beta}\boldsymbol{\delta}^{-1}\boldsymbol{\gamma}, \tag{38c}$$

$$\mathbf{S}_{22} = \boldsymbol{\beta}\boldsymbol{\delta}^{-1}. \tag{38d}$$

(note, the labels 1,2 are just the "standard" notation, not referring to the TM- and TE-modes which are in the components of $\boldsymbol{a}^\pm$)

$with(LinearAlgebra):$

$W12 := Matrix([[kx \cdot ky, 1 - kx^2], [-1 + ky^2, -kx \cdot ky]]); G := Matrix([[-g, 0], [0, -g]]);$

$$W12 := \begin{bmatrix} kx\, ky & -kx^2 + 1 \\ ky^2 - 1 & -kx\, ky \end{bmatrix}$$

$$G := \begin{bmatrix} -g & 0 \\ 0 & -g \end{bmatrix} \tag{1}$$

$W := Matrix([[G, W12], [-W12, G]]);$

$$W := \begin{bmatrix} -g & 0 & kx\, ky & -kx^2 + 1 \\ 0 & -g & ky^2 - 1 & -kx\, ky \\ -kx\, ky & kx^2 - 1 & -g & 0 \\ -ky^2 + 1 & kx\, ky & 0 & -g \end{bmatrix} \tag{2}$$

$CP := Determinant(W);$

$$CP := g^4 + 2\, g^2\, kx^2 + 2\, g^2\, ky^2 + kx^4 + 2\, kx^2\, ky^2 + ky^4 - 2\, g^2 - 2\, kx^2 - 2\, ky^2 + 1 \tag{3}$$

$solve(KP, g);$

$$\sqrt{-kx^2 - ky^2 + 1}, -\sqrt{-kx^2 - ky^2 + 1}, \sqrt{-kx^2 - ky^2 + 1}, -\sqrt{-kx^2 - ky^2 + 1} \tag{4}$$

$W12 \cdot W12;$

$$\begin{bmatrix} kx^2\, ky^2 + (-kx^2 + 1)\,(ky^2 - 1) & 0 \\ 0 & kx^2\, ky^2 + (-kx^2 + 1)\,(ky^2 - 1) \end{bmatrix} \tag{5}$$