A numerical study on electromagnetic reflection and transmission for an Ω slab

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1 Validation

The task is to use the general theory of propagation in multilayered structures in order to reproduce the transmission and reflection coefficients of Figure 4 in [2]. This text will briefly explain the numerical procedure in order to achieve this using the theory described in [1].

An electromagnetic wave is incident from the vacuum half-space ($z < 0$) towards a slab of a so-called Ω-material of thickness l. Beyond the slab ($z > l$) is again a half-space of vacuum. As described in [2], the Ω medium has the following constitutive relations in a xyz cartesian coordinate system

$$\mathbf{D} = \bar{\bar{\varepsilon}}\mathbf{E} + \bar{\bar{\xi}}\mathbf{H} \tag{1}$$

$$\mathbf{B} = \bar{\bar{\mu}}\mathbf{H} + \bar{\bar{\zeta}}\mathbf{E} \tag{2}$$

with the material parameter tensors

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}, \quad \bar{\bar{\mu}} = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \tag{3}$$

$$\bar{\bar{\xi}} = i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Omega \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\bar{\zeta}} = -i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Omega & 0 \end{bmatrix} \tag{4}$$

where the parameter values are normalized with respect to $\sqrt{\varepsilon_0\mu_0}$, so that ε_i and μ_i ($i = 1, 2, 3$) are assumed to be the dimensionless relative permittivities and permeabilities, respectively. The orientation of the Ω-elements as described in (3)-(4) are such that the straight rods are directed in y while the loop normals are in z. Note that the time convention used here $e^{-i\omega t}$ is different from [2] where $e^{+j\omega t}$ is used, which results in opposite signs in (4) compared to the article.

1.1 Method

The numerical calculations were performed in MATLAB. In preparation before the main algorithm, the parameter tensors (3)-(4) are decomposed according to

$$\bar{\bar{\varepsilon}} = \varepsilon_{\mathrm{tt}} + \varepsilon_{\mathrm{t}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\varepsilon_z + \varepsilon_{zz}\hat{\mathbf{z}}\hat{\mathbf{z}} \tag{5}$$

so that $\varepsilon_{\mathrm{tt}}$ is the 2×2 upper left block diagonal matrix containing the transversal components, ε_{t} and ε_z are both interpreted as 2×1 column vectors, and ε_{zz} is a scalar of the remaining element. Similar decomposition is done for $\bar{\bar{\mu}}, \bar{\bar{\xi}}$ and $\bar{\bar{\zeta}}$.

The following description of the algorithm is performed for each angle of incidence (θ_0, ϕ_0) and subsequently for each transversal wavevector $k_{\mathrm{t}} = k_0 \sin\theta_0[\cos\phi_0, \sin\phi_0]$. First, the eigenvectors of $\mathbf{W}_0$ in vacuum are calculated from

$$\mathbf{w}^{\pm}_{\mathrm{TM}} = \begin{bmatrix} \cos\phi_0\sqrt{\cos\theta_0} \\ \sin\phi_0\sqrt{\cos\theta_0} \\ \mp\sin\phi_0/\sqrt{\cos\theta_0} \\ \pm\sin\phi_0/\sqrt{\cos\theta_0} \end{bmatrix}, \quad \mathbf{w}^{\pm}_{\mathrm{TE}} = \begin{bmatrix} -\sin\phi_0/\sqrt{\cos\theta_0} \\ \cos\phi_0/\sqrt{\cos\theta_0} \\ \mp\cos\phi_0\sqrt{\cos\theta_0} \\ \mp\sin\phi_0\sqrt{\cos\theta_0} \end{bmatrix}. \tag{6}$$

Together, these four column vectors (6) are equivalent to the the 4×4 matrix $\mathbf{T_0}^{-1}$, see equation (35a) in [1]. The decomposed matrix tensors (5) and $\mathbf{k}_{\mathrm{t}}$ are used to construct the $\mathbf{W}$ matrix, whose elements can be found in detail in equations (11) in [1]. This matrix is a result of decomposing Maxwell's curl equations inside a layer of general bianisotropic medium into transversal and longitudinal projections, and using the constitutive relations (1)-(2) to eliminate the fields $\mathbf{D}$ and $\mathbf{B}$ from the Maxwell's equations. The longitudinal field components are finally eliminated so that one is left with the differential equation

$$\partial_z \begin{bmatrix} \mathbf{E}_{\mathrm{t}} \\ \eta_0 \mathbf{H}_{\mathrm{t}} \end{bmatrix} = ik_0 \mathbf{W} \begin{bmatrix} \mathbf{E}_{\mathrm{t}} \\ \eta_0 \mathbf{H}_{\mathrm{t}} \end{bmatrix}. \tag{7}$$

The eigenvectors and eigenvalues of $\mathbf{W}$ are then assembled into the square matrix $\mathbf{T}^{-1}$ and the diagonal matrix $\mathbf{D}$, respectively. The latter is used for the exponential matrix

$$\mathbf{M} = \exp\{ik_0 \mathbf{D} l\} \tag{8}$$

where l is the thickness of the Ω slab. Equation (8) is further used to calculate the propagator matrix inside the layer

$$\mathbf{P} = \mathbf{T}^{-1}\mathbf{M}\mathbf{T}. \tag{9}$$

In this problem the multilayer system consist of one layer surrounded by vacuum, there is therefore only need for a single propagator matrix.

We now have the main matrices needed to calculate the scattering matrices. The product $\mathbf{T}_0\mathbf{P}\mathbf{T}_0^{-1}$ is decomposed into

$$\mathbf{T}_0\mathbf{P}\mathbf{T}_0^{-1} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \tag{10}$$

such that α, β, γ and δ are 2×2 matrices. The scattering matrices are then finally calculated from

$$\mathbf{S}_{11} = -\delta^{-1}\gamma \tag{11}$$
$$\mathbf{S}_{12} = \delta^{-1} \tag{12}$$
$$\mathbf{S}_{21} = \alpha - \beta\delta^{-1}\gamma \tag{13}$$
$$\mathbf{S}_{22} = \beta\delta^{-1}. \tag{14}$$

These scattering matrices provide us with the transmission and reflection coefficients for both co-polarization and cross-polarization via the relations

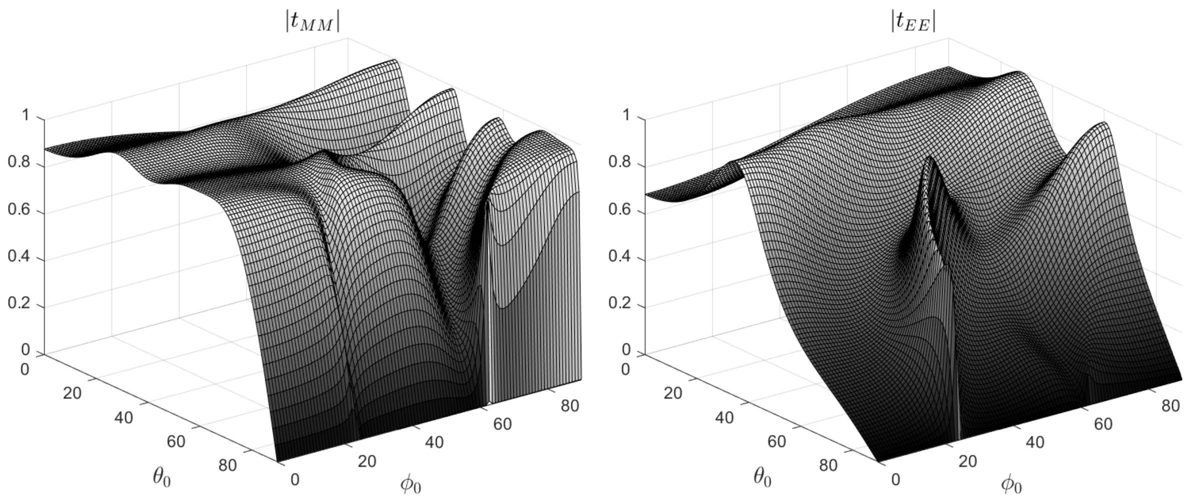
$$r_{MM} = \mathbf{S}_{11}^{(11)} \tag{15}$$
$$r_{EE} = \mathbf{S}_{11}^{(22)} \tag{16}$$
$$r_{EM} = \mathbf{S}_{11}^{(21)} \tag{17}$$
$$r_{ME} = \mathbf{S}_{11}^{(12)} \tag{18}$$
$$t_{MM} = \mathbf{S}_{21}^{(11)} \tag{19}$$
$$t_{EE} = \mathbf{S}_{21}^{(22)} \tag{20}$$
$$t_{EM} = \mathbf{S}_{21}^{(21)} \tag{21}$$
$$t_{ME} = \mathbf{S}_{21}^{(12)} \tag{22}$$

where e.g. $\mathbf{S}_{11}^{(ij)}$ is the (ij)-element of matrix $\mathbf{S}_{11}$. This procedure is iterated over all angles of incidence.

The relations (15)-(22) can be seen from the system equation for the scattering matrices

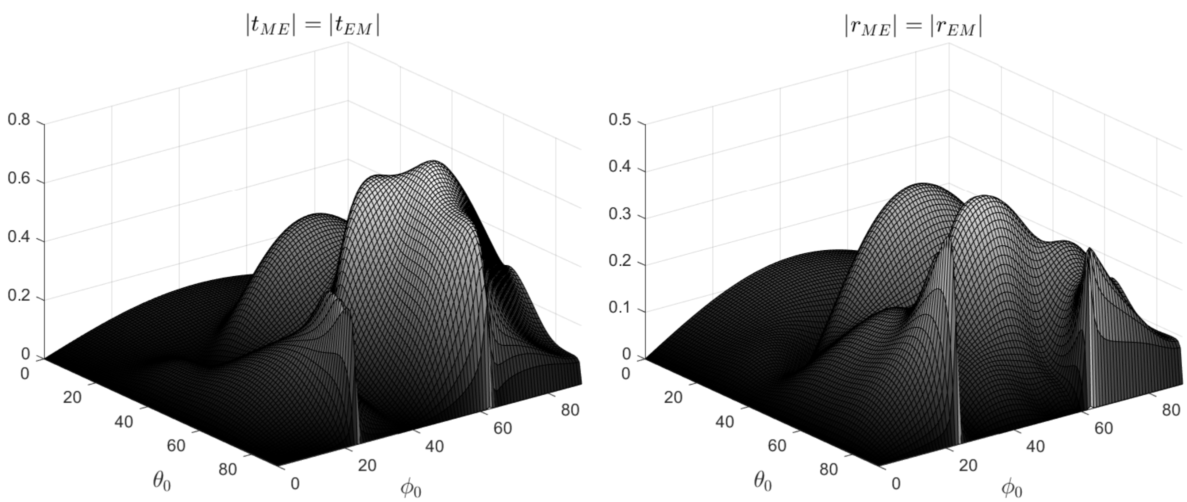
$$\begin{bmatrix} \mathbf{a}^-(0^-) \\ \mathbf{a}^+(l^+) \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{a}^+(0^-) \\ \mathbf{a}^-(l^+) \end{bmatrix} \tag{23}$$

where the incoming modes are $\mathbf{a}^+(0^-)$ and $\mathbf{a}^-(l^+) = 0$, and the scattered modes are $\mathbf{a}^-(0^-)$ and $\mathbf{a}^+(l^+)$, for the mode coefficients $\mathbf{a}^\pm(z) = [a^\pm_{\mathrm{TM}}(z) \quad a^\pm_{\mathrm{TE}}(z)]^{\mathrm{T}}$.



(a) (b)

Figure 1: Task A, transmission co-polarization.



(a) (b)

Figure 2: Task A, cross-polarization.

1.2 Results

The numerical results are calculated for a slab thickness of $l = 5.2\lambda/\sqrt{\varepsilon_1\mu_1}$. The material parameters of the metamaterial were chosen to be the same as in [2], with $\Omega = 0.9$, relative permeabilities $\mu_1 = \mu_2 = 1$, $\mu_3 = 1.12$, and relative permittivities $\varepsilon_1 = \varepsilon_3 = 3$, $\varepsilon_2 = 10$. The co-polarized transmission coefficients can be found in Figure 1, and the cross-polarized transmission and reflection coefficients can be found in Figure 2. It is noted that in [2] the figure for $|t_{ME}| = |t_{EM}|$ is incorrectly labeled and is actually the plot for $|r_{ME}| = |r_{EM}|$, and vice versa.

2 Varying the orientation of the Ω-elements

In this section, the analysis is repeated after re-orienting the principal axes of the Ω-elements. Here, the straight rods are directed in the y-direction and the loop normal is directed in x. As such, the cross-coupling tensors $\overline{\overline{\xi}}$ and $\overline{\overline{\zeta}}$ must be modified with respect to this new orientation. When the unit cells are oriented like this, the y-component of an applied electric field is oriented parallel to the straight rods and will induce a magnetic field, due to the circular Ω shape, in a direction normal to the loop, i.e. in $\pm\hat{x}$. By inspecting the constitutive relation (2) one finds that the following tensor will satisfy this behaviour

$$\bar{\bar{\zeta}} = -i \begin{bmatrix} 0 & \Omega & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

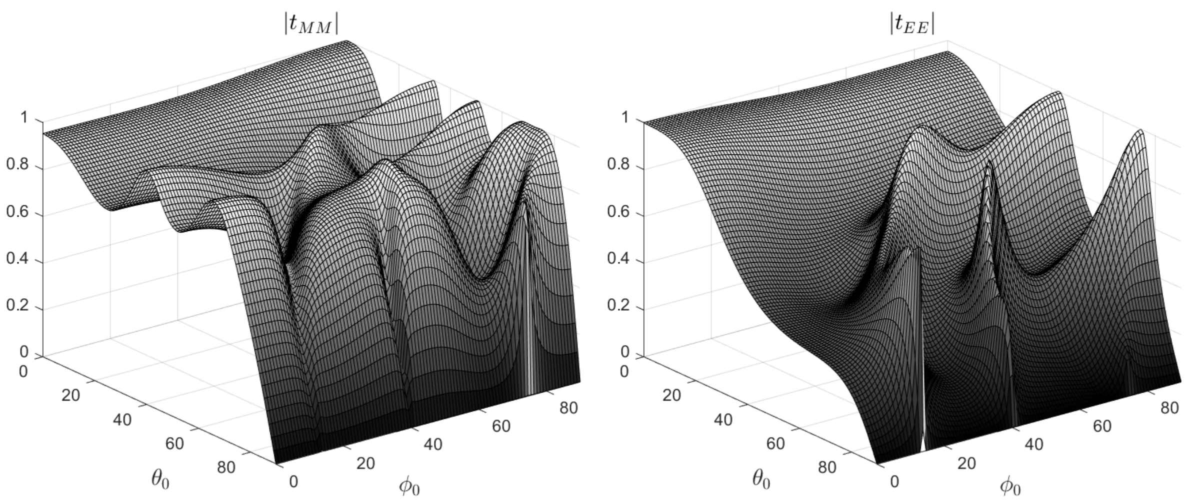
Similarly, an applied magnetic field will have its x-component couple with the loop and induce an electric field directed along the rods in $\pm\hat{y}$. This is satisfied by

$$\bar{\bar{\xi}} = i \begin{bmatrix} 0 & 0 & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{25}$$

as found by inspection of (1).

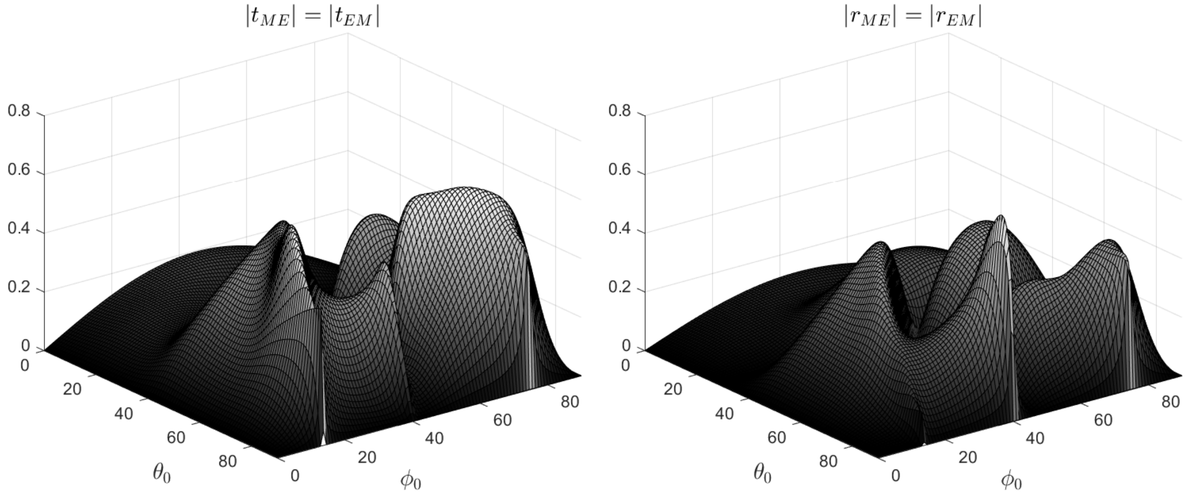
2.1 Results

The same parameter values as described in section 1.2 were used to calculate the response for the Ω slab with the new orientation of the elements. The transmission and reflection coefficients can be found in Figures 3 and 4.



(a) (b)

Figure 3: Task B, co-polarization.



(a) (b)

Figure 4: task B, cross-polarization.

References

[1] Norgren M. Ei3302 propagation and scattering in multilayered structures. *Lecture notes for the KTH course EI3302 Electromagnetic Waves in Complex Media*, 2022.

[2] Norgren M. and He S. Electromagnetic reflection and transmission for a dielectric-ω interface and an ω slab. *International Journal of Infrared and Millimeter Waves*, 15(9):1537–1554, 1994.