Assignment 4

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1 Slab waveguide

Wave travelling in $\hat{z}$-direction, confined within a slab of height b and refractive index n_1. Confinement direction is $\hat{x}$, where the cladding regions are $x > b$ and $x < 0$ with refractive index n_2. To encourage total internal reflection, $n_1 > n_2$.

1.1 TE mode

Re-iterating the mode dispersion relation for the TE mode in a slab waveguide:

$$\frac{2\pi b}{\lambda}\sqrt{n_1^2 - N^2} = 2\arctan\sqrt{\frac{N^2 - n_2^2}{n_1^2 - N^2}} + m\pi \tag{1}$$

where λ is the working wavelength, b is the height of the waveguide core region and n_1 its refractive index, n_2 is the index of the cladding regions, N is the effective refractive index and m represents the wave modes and take values $m = \pm 0, 1, 2, ...$

Figure 1 plots equation 1, where the blue lines are the right-hand side of the equation for four different modes $m = 0, 1, 2, 3$, and the red lines are the left-hand side of the equation for seven arbitrarily chosen values of slab height $b = 0.3875, 0.7750, 1.1625, 1.5500, 1.9375, 2.3250,$ 2.7125 μm. A red line's intersections with blue lines tells us which modes are allowed for a given slab height.

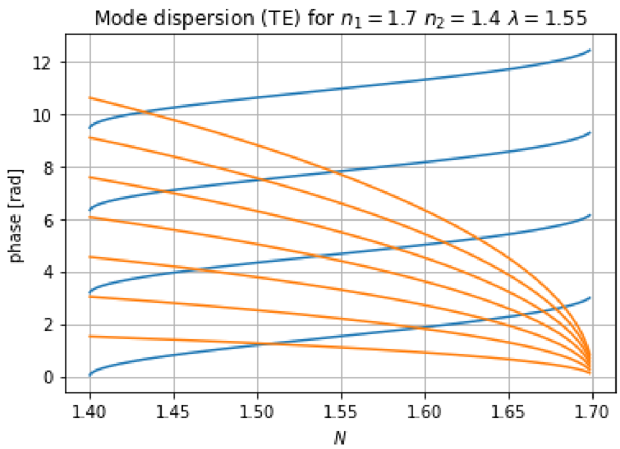

Figure 1: TE phase dispersion with respect to effective refractive index $N \epsilon [n_2, n_1]$.

Figure 2 plots the transversal electric field component E_y for each allowed mode for every b from figure 1. Slab height and effective indices are indicated in each subfigure. As both figures show, the first two slab heights $b = 0.3875, 0.7750$ allow for only the 0th mode. Thicker slabs allow for several modes, for example, slab thickness $b = 2.7125$ has four modes.

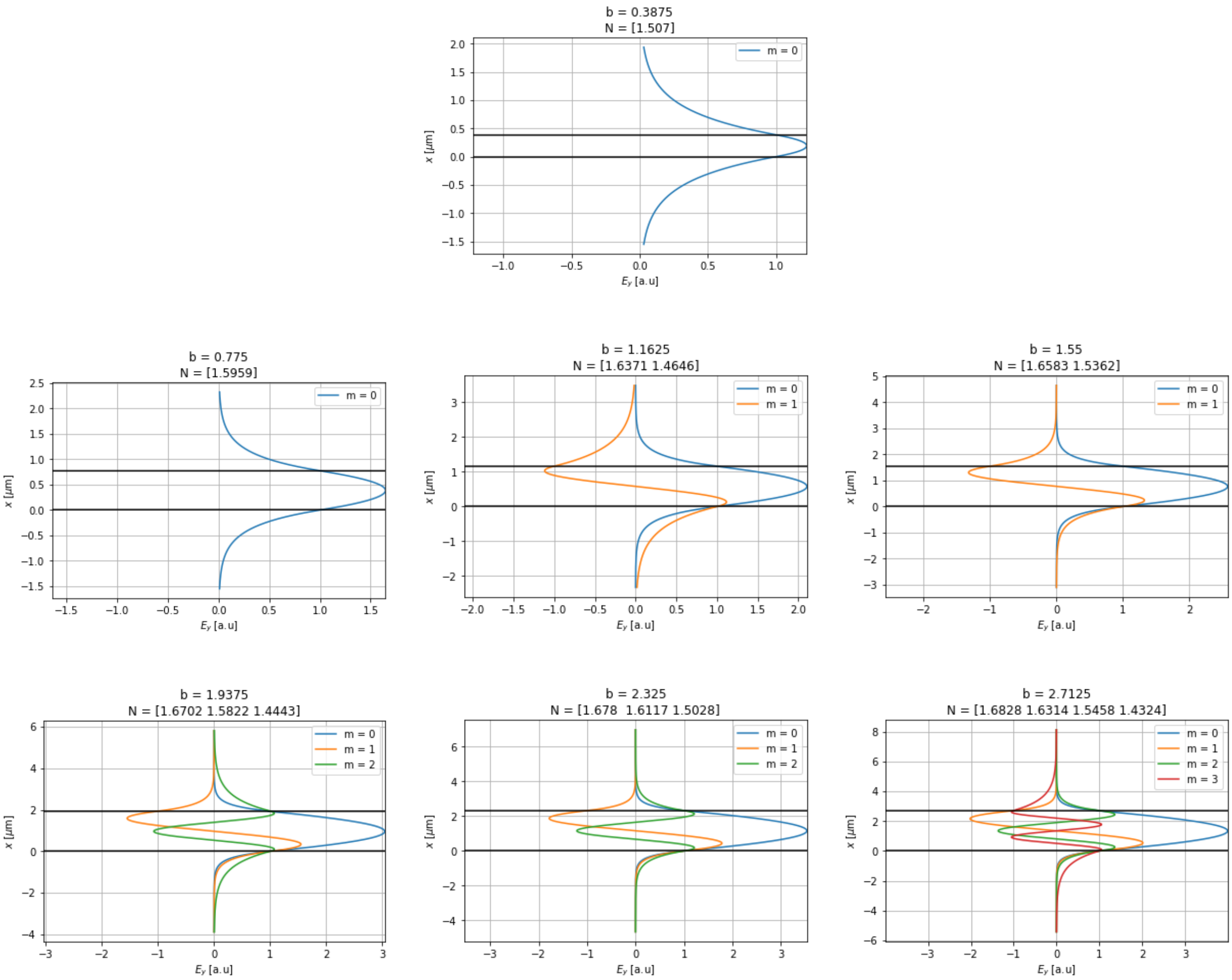

Figure 2: Electric field component E_y plotted for each slab thickness b for a TE wave propagating into the page $\hat{z}$. Black horizontal lines represent the waveguide core.

1.2 TM mode

1.2.1 Derivation of mode dispersion relation

Starting from the (second) set of coupled equations derived in lecture notes T8C,

$$i\beta E_x - \frac{dE_z}{dx} = i\omega\mu_0 H_y \tag{2a}$$

$$-i\beta H_y = -i\omega\varepsilon_0 n^2 E_x \tag{2b}$$

$$\frac{dH_y}{dx} = -i\omega\varepsilon_0 n^2 E_z \tag{2c}$$

we can solve equations 2b and 2c with respect to E_x and E_z, respectively, and insert those expressions into 2a, giving

$$i\frac{\beta^2}{\omega\varepsilon_0 n^2}H_y + \frac{1}{i\omega\varepsilon_0 n^2}\frac{d^2H_y}{dx^2} = i\omega\mu_0 H_y,$$

which, after cleaning up, results in

$$\frac{d^2H_y}{dx^2} = -(\beta^2 - k_{\text{vac}}^2 n^2)H_y$$

This differential equation has general solutions

$$\begin{aligned}
H_y &= C_1\cos Kx + C_2\sin Kx \qquad &&; K^2 \equiv \beta^2 - k_{\text{vac}}^2 n^2 \\
H_y &= C_3 e^{\gamma x} + C_4 e^{-\gamma x} \qquad &&; \gamma^2 \equiv k_{\text{vac}}^2 n^2 - \beta^2
\end{aligned}$$

For each of the three regions A, B, C as defined in the lecture notes, H_y becomes

$$H_{yA} = Ae^{-\gamma x} \qquad ; x < 0 \tag{3a}$$
$$H_{yB} = B\cos Kx + C\sin Kx \qquad ; 0 < x < b \tag{3b}$$
$$H_{yC} = De^{-\gamma(x-b)} \qquad ; x > b \tag{3c}$$

From these we can find expressions for E_z using equation 2c,

$$E_{zA} = -\frac{\gamma}{i\omega\varepsilon_0 n_2^2}Ae^{-\gamma x} \qquad ; x < 0 \tag{4a}$$
$$E_{zB} = \frac{1}{i\omega\varepsilon_0 n_1^2}\left(BK\sin Kx - CK\cos Kx\right) \qquad ; 0 < x < b \tag{4b}$$
$$E_{zC} = \frac{\gamma}{i\omega\varepsilon_0 n_2^2}De^{-\gamma(x-b)} \qquad ; x > b \tag{4c}$$

Here it is important to note that the propagators γ and K contain refractive indices depending on the medium.

We will now continue by looking at boundary conditions for equations 3-4:

- $x = 0$:
  - Equation 3a = 3b $\longrightarrow A = B$
  - Equation 4a = 4b $\longrightarrow n_2^2 KC = n_1^2\gamma A$
- $x = b$:
  - Equation 3b = 3c $\longrightarrow D = B\cos Kb + C\sin Kb$
  - Equation 4b = 4c $\longrightarrow \gamma D = K\frac{n_2^2}{n_1^2}\left(B\sin Kb - C\cos Kb\right)$

These relations we can summarize into a matrix equation form as

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & \cos Kb & \sin Kb & -1 \\ n_1^2\gamma & 0 & -n_2^2 K & 0 \\ 0 & -n_2^2 K\sin Kb & n_2^2 K\cos Kb & n_1^2\gamma \end{pmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{5}$$

which has valid solutions only if the matrix determinant is zero. Finding the determinant and setting it to zero, the resulting equation yields

$$\tan Kb = \frac{\frac{n_1^2}{n_2^2}\frac{\gamma}{K} + \frac{n_1^2}{n_2^2}\frac{\gamma}{K}}{1 - \left(\frac{n_1^2}{n_2^2}\frac{\gamma}{K}\right)\left(\frac{n_1^2}{n_2^2}\frac{\gamma}{K}\right)} \tag{6}$$

where we recognize that the form of equation 6 has the same form as the trigonometric relation[1]

$$\tan(r+q) = \frac{\tan r + \tan q}{1 - r\tan q}.$$

with $r = q = \arctan\left(\frac{n_1^2}{n_2^2}\frac{\gamma}{K}\right)$. We can exploit this relation by taking the inverse tangent of equation 6 to get

$$\begin{aligned} Kb &= r + q + m\pi \\ &= 2\arctan\left(\frac{n_1^2}{n_2^2}\frac{\gamma}{K}\right) + m\pi \end{aligned} \tag{7}$$

where $m = \pm 0, 1, 2, ...$

From before we have the propagators

$$\begin{aligned} \gamma &= \sqrt{N^2k^2 - n_2^2k^2} = \frac{2\pi}{\lambda}\sqrt{N^2 - n_2^2} \\ K &= \sqrt{n_1^2k^2 - N^2k^2} = \frac{2\pi}{\lambda}\sqrt{n_1^2 - N^2} \end{aligned}$$

where we have defined the effective refractive index from $\beta = Nk$. Finally, inserting these into equation equation 7, we arrive at the mode dispersion relation for TM,

$$\frac{2\pi b}{\lambda}\sqrt{n_1^2 - N^2} = 2\arctan\left[\frac{{n_1}^2}{n_2}\sqrt{\frac{N^2 - n_2^2}{n_1^2 - N^2}}\right] + m\pi. \tag{8}$$

1.2.2 Mode plots

Similar as for TE, figure 3 displays the TM mode dispersion relation, equation 8, in terms of its left-hand side of the equation (red lines) and its right-hand side (blue lines) for the same slab thicknesses $b = 0.3875, 0.7750, 1.1625, 1.5500, 1.9375, 2.3250, 2.7125$ μm.

The transversal field components H_y are calculated from equation 3, where the differential equation constants A, B, C, D are given in equation 5. Figure 4 plots H_y with respect to x-position for all seven values of slab thickness b. Comparing with figure 2 for TE, we see that the effective refractive indices N are not the same, albeit close.

[1] I don't see how $r\tan q = \arctan\left(\frac{n_1^2}{n_2^2}\frac{\gamma}{K}\right)\frac{n_1^2}{n_2^2}\frac{\gamma}{K} = ...???... = \left(\frac{n_1^2}{n_2^2}\frac{\gamma}{K}\right)\left(\frac{n_1^2}{n_2^2}\frac{\gamma}{K}\right)$

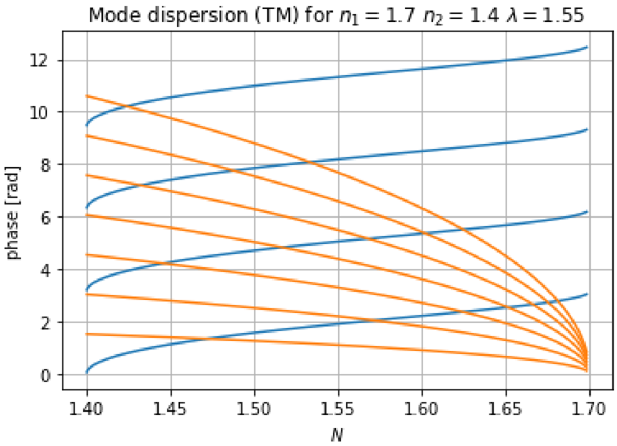

Figure 3: TM phase dispersion with respect to effective refractive index $N \epsilon [n_2, n_1]$.

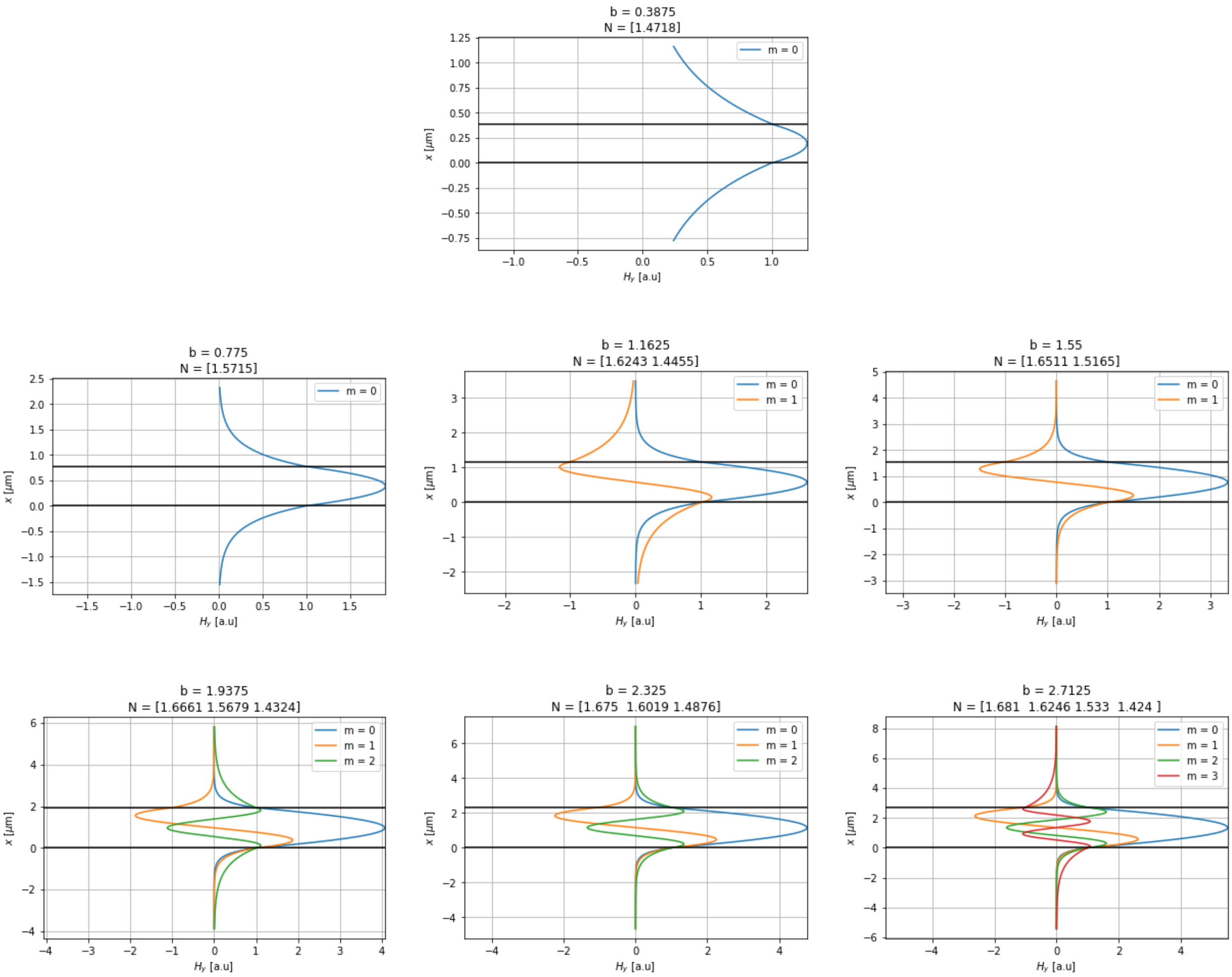

Figure 4: Magnetic field component H_y plotted for each slab thickness b for a TM wave propagating into the page $\hat{z}$. Black horizontal lines represent the waveguide core.