MA2.101: Linear Algebra (Spring 2022)

Mid-semester Exam

Tuesday, 17 May 2022

Attention

1. Mid-sem exam is 15% of total marks for this course (50% of the total marks is allocated to the first part). 2 marks out of 15 (assuming total of 100 marks for full course) allocated for Mid-sem exam is from the in-house Mid-sem exam that asked you to submit a question with solution by 8 PM, 14 May 2022.
2. Total points for in-class Midsem exam questions is 65: $Q1 = 23, Q2 = 11, Q3 = 11, Q4 = 20$. 65 points $\equiv$ 13 marks allocated for in-class Mid-sem exam.
3. You may opt to submit solution of either Question 1 or Question 4 by Tuesday midnight (11:59:59 PM, 17 May 2022) via moodle. However, total points allocated for given question will be reduced to half of the original.

Question 1

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation whose action is given by

$$T(x, y, z) = (x + 2y + z, z, -x + y - 5z, 3x + 2z)$$

- What are the standard ordered bases for $\mathbb{R}^3$ and $\mathbb{R}^4$. (2 points).
- Write the matrix representation of T relative to the standard ordered bases of $\mathbb{R}^3$ and $\mathbb{R}^4$. (12 points).
- If the standard ordered basis for $\mathbb{R}^4$ is $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ (in the given sequence). Then find the matrix representation of T relative to the standard ordered basis for $\mathbb{R}^3$ and the ordered basis $\{\epsilon_2, \epsilon_1, \epsilon_3, \epsilon_4\}$ (in the given sequence) for $\mathbb{R}^4$. (7 points)
- Is T invertible? Why or why not? (2 point)

Question 2

If a linear transformation $T : V \to W$ is an isomorphism, then the inverse map $T^{-1} : W \to V$ is also an isomorphism. Give a formal proof of this statement. (11 points)

Question 3

$n=7$

Let A be an $m \times n$ matrix representing a linear transformation with respect to a pair of ordered bases. Suppose that the nullspace of A is a plane in $\mathbb{R}^3$ and the range is spanned by a nonzero vector $\vec{v} \in \mathbb{R}^5$.

- Determine m and n. (2 points)
- Find the rank and nullity of A. (6 points)
- Write the expression for rank-nullity theorem for given A. (3 points)

Question 4

Consider a linear transformation $T : \mathbb{R}^3 \to P_2$ from $\mathbb{R}^3$ to the vector space P_2 of polynomials in x with real coefficients, where the degree of polynomial is at most 2:

$$T(a, b, c) = (a - b)x^2 + cx + (a + b + c). \quad (1)$$

Find the matrix representation of T with respect to the pair of ordered bases $\{(1, 0, 0), (1, 1, 0), (0, -1, 1)\}$ and $\{x + 1, x^2 - x, x^2 + x + 1\}$ for $\mathbb{R}^3$ and P_2, respectively. (20 points)