Probability and Random Processes

MA6.102, Monsoon-2022

Exam: End-Sem
Total Marks: 100

Date: 23 Nov 2022
Time: 03:00-06:00

Instructions:

- This is a closed book exam.
- Answering all the questions is compulsory. There are optional subsqestions in third and fourth questions.
- Clearly state the assumptions (*if any*) made that are not specified in the questions.

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1. Answer the following statements are true or false [Marks: 10 (10x1)]
   (a) If $X \sim \mathcal{N}(0, \sigma)$, then $\mathbb{P}(X = 0) = 0$.
   (b) MGF of the sum of random variables is always equal to the product of their individual MGFs.
   (c) If $\text{Cov}(X, Y) > 0$, then $\text{Var}(X - Y) \leq \sigma_X^2 + \sigma_Y^2$.
   (d) All normal random processes are stationary processes.
   (e) Strong law of large number suggests that the sample mean converges in probability to the exact mean.
   (f) If X is a positive random variable, then $\mathbb{E}[\log(1 + X)] \leq \log(1 + \mathbb{E}[X])$.
   (g) If X_1, X_2 and X_3 are independent random variables, then X_1 and X_2 are also conditionally independent given X_3.
   (h) Given ζ, $X(t; \zeta)$ is a sample function of the random process.
   (i) Two processes are orthogonal if they are zero-mean and uncorrelated processes.
   (j) Output of the linear time invariant system is a stationary process if its input is a stationary process.

2. Answer the following questions in short. [Marks: 20 (2x10)]
   (a) If $X_i \in \{0, 1\}$ follows Burnoulli distribution with parameter p and

$$Y = \sum_{i=1}^{N} X_i \quad \text{and} \quad Z = \sum_{i=1}^{N} (1 - X_i),$$

   then is the covariance of Y and Z, and the variance of $Y - Z$.
   (b) Mention any three properties of covariance matrix.
   (c) State Chebyshev and Chernoff inequalities.
   (d) State the weak law of large number and central limit theorem.
   (e) State the conditions under which the Binomial distribution can be approximated with Poisson and Normal distributions.

(f) Find the mean of $\sum_{n=1}^{N} X_n$ where $X_i \sim \text{Exp}(\mu)$ and $N \sim \text{Poisson}(\lambda)$.

(g) Consider $X = [X_1,\ X_2]$ is a bivariate Normal random variable. What is $\mathbb{E}[X_1|X_2]$ and $\text{Var}[X_1|X_2]$?

(h) Show that $\lim_{n\to\infty} \mathbb{P}([n,\infty]) = 0$.

(i) Show that the convergence in mean square implies the convergence in probability.

(j) Define the strict sense stationary and wide sense stationary processes.

3. Answer any six of the following questions. [Marks: 42 (7x6)]

(a) Lets $X = [X_1,\ X_2,\ X_3]$ be a random vector such that X_i follows $\mathcal{N}(0,\sigma)$ independently of each other. Find the distribution of $\|X\|^2$.

(b) If $Z = \sum_{i=1}^{N} X_i$ such that X_is are i.i.d. zero-mean unit variance normal random variables and N is a Poisson random variable with mean λ. Find the MGF of Z. Also, find its mean and variance.

(c) Consider independent Bernoulli trials of successes and failures. Find the p.m.f. of the number of trials required of the occurrence of n-th success.

(d) Prove the central limit theorem.

(e) Find the distribution $Z = X + Y$ where X and Y are independent. Further, find distribution of Z when $X \sim \text{Exp}(\lambda_1)$ and $Y \sim \text{Exp}(\lambda_2)$. Also, comment on the case when $\lambda_1 = \lambda_2$.

(f) Find the joint probability density function of $W = X + Y$ and $Z = X - Y$ when X and Y independently follow exponential distribution with mean $\frac{1}{\lambda}$.

(g) Consider a Poisson process $N(t)$ for counting the number of occurrences of some event. Assume $N(0) = 0$ and derive

i. probability that the time of the first occurrence of event is greater than T

ii. distribution of the time required for the n-th occurrence of event

iii. mean and variance of the number of occurrences of event in time interval $[T_1, T_2]$.

(h) If X is a zero-mean bivariate normal random variable with covariance matrix

$$K = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}.$$

i. Find $\mathbb{E}[X_1|X_2 = \frac{1}{2}]$.

ii. Find the distribution of $Y = HX$ where

$$H = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

4. Answer any two of the following questions. [Marks: 28 (14x2)]

(a) Consider that the customers are randomly arriving in a bank according to a Poisson process with parameter λ (i.e., their inter arrival times follow exponential distribution independently of each other). The bank has a large number of service counters so that each customer directly gets service without waiting in a queue. The service time required for an individual customer is exponentially distributed with parameter μ independently of others' service times. Let $N(t)$ represents counting process of the number of customers in the bank. Assume $N(0) = 0$ and answer the following questions.

i. Find the p.m.f of $N(T)$.

ii. Comment on the stationarity of $N(t)$.

(b) Consider $X = [X_1, \ldots, X_N]^T$ follows a multivariate zero-mean normal distribution with covariance matrix K. Answer the following questions

i. Derive the joint MGF of X, i.e., $M_X(s) = \mathbb{E}[e^{s^T X}]$.

$\lambda^x \; e^{\lambda}$