Introduction to Locally Harmonic Manifolds

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Abstract

We introduce the definition of local harmonicity and prove its equivalence with other formulations. We show the validity of the mean value property and its converse in locally harmonic spaces. Finally, we show that two point homogeneous spaces are locally harmonic.

Contents

1 Introduction

When finding solutions to Laplace's equation $\Delta f \equiv 0$ in Euclidean space $\mathbb{R}^n$, it is natural to first look for *radially symmetric* solutions: that is, solutions of the form $f(x) = F(\|x\|)$. By introducing radial symmetry, the *partial* differential equation $\Delta f \equiv 0$ is reduced to an *ordinary* differential equation of the form

$$F''(r) + \frac{n-1}{r}F'(r) = 0. \tag{1.1}$$

Solving (1.1) yields the following solution to Laplace's equation on $\mathbb{R}^n \backslash \{0\}$:

$$f(x) = \frac{1}{\|x\|^{n-2}} \quad \text{if} \quad n > 2, \qquad f(x) = \log(\|x\|) \quad \text{if} \quad n = 2. \tag{1.2}$$

Since Laplace's equation in $\mathbb{R}^n$ is invariant under translations, it follows that for every point $p \in \mathbb{R}^n$, there exists a *non-constant* solution to $\Delta f \equiv 0$ on $\mathbb{R}^n \backslash \{p\}$ depending only on the distance to p. Given this, a natural question is the following:

Question. *Given a Riemannian manifold (M, g) and a point $p \in M$, does there exist a non-constant solution to $\Delta f \equiv 0$ on a deleted neighbourhood of p, depending only on the distance to p?*

It turns out that for a general Riemannian manifold, the answer to the question above is **no**. In 1930, H. S. Ruse attempted, on an arbitrary Riemannian manifold, to find non-constant solutions to $\Delta f \equiv 0$ that are functions only of the distance to a point $p \in M$, but failed (see [CR40]). Consequently, people defined a Riemannian manifold to be *locally harmonic* precisely when these solutions exist for every $p \in M$.

Two point homogeneous spaces are locally harmonic. The *Lichnerowicz conjecture* asserts the converse: *every (complete) locally harmonic manifold is two point homogeneous.* Lichnerowicz proved the conjecture for dimensions less than or equal to 4, and Z. I. Szabó proved it in arbitrary dimension for compact manifolds with finite fundamental group [Sza90]. However, E. Damek and F. Ricci found counterexamples in infinitely many dimensions, the smallest of them being 7 [DR92]. It remains an open question whether or not there exist locally harmonic manifolds that are not homogeneous.

In this report, we study the necessary prerequisites for understanding the definition of local harmonicity. We then use these tools to prove the equivalence of different formulations of local harmonicity; show the validity of the mean value property and its converse in these spaces; and prove that two point homogeneous spaces are locally harmonic. The main references we follow are [BGM71], [Wil93], and [Kre10]. We attempt to fill in the details of the arguments made by these previous authors.

The reader is assumed to be familiar with fundamental concepts in Riemannian geometry. In particular, vector fields along curves, geodesics, the Riemannian exponential map, and Jacobi fields are used throughout this report. For an introduction to these topics, see [Lee18].

In Section 2, we study the *Riemannian density function*. Sections 2.1 and 2.2 investigate Jacobi fields and Jacobi tensors, respectively. In Section 2.3, we introduce the definition of the density function, then prove some of its characterisations. Finally, in Section 2.4, we prove that the density function is invariant under isometries.

In Section 3, we study *radially symmetric functions*. Section 3.1 presents definitions and basic results about these functions. In Section 3.2, we prove that the density function can be characterised as a quotient of particular volume forms. In Section 3.3, we study the mean value operator. Lastly, in Section 3.4, we prove a formula for the Laplacian of a radially symmetric function.

Local harmonicity is studied in Section 4. In Section 4.1, we define local harmonicity and prove the equivalence of different formulations. In Section 4.2, we show the validity of the mean value property and its converse in locally harmonic spaces. Finally, in Section 4.3, we define two point homogeneous spaces and prove that they are locally harmonic.

For the rest of the report, we assume that (M, g) is a connected Riemannian manifold of fixed dimension n.

2 The Density Function

2.1 Jacobi Fields

In this section, we review some useful results about Jacobi fields from [Lee18, Chapter 10]. We are particularly interested in a subset of Jacobi fields, namely *normal Jacobi fields vanishing at a point*. These are used in the proofs of the characterisations of the density function in Section 2.3 and the proof of the formula for Laplacian of a radially symmetric function in Section 3.4. We skip the proofs in this section, as they can all be found in [Lee18, Chapter 10].

Definition 2.1.1 (Jacobi Field). Let $\gamma : I \to M$ be a geodesic, and let $\mathfrak{X}(\gamma)$ denote the the vector space of all smooth vector fields along γ. We say that a smooth vector field $J \in \mathfrak{X}(\gamma)$ is a *Jacobi* *field* if it satisfies the *Jacobi equation*, which is given by

$$D_t D_t J + R(J, \gamma')\gamma' \equiv 0, \tag{2.1}$$

where R is the Riemann curvature endomorphism of (M, g).

Example 2.1.2 (Jacobi Fields in Euclidean Space)**.** Suppose $M = \mathbb{R}^n$. We know that geodesics on $\mathbb{R}^n$ are precisely the constant-speed parameterisations of straight lines. Furthermore, the curvature endomorphism R vanishes, and the covariant derivative is just the standard derivative. Therefore, Jacobi fields in $\mathbb{R}^n$ are precisely the vector fields along constant-speed parameterisations of straight lines with vanishing second derivative.

A property of Jacobi fields that we use commonly throughout this report is their existence and uniqueness given initial conditions. Notice that the following proposition implies the space of Jacobi fields along γ forms a $2n$-dimensional real vector subspace of $\mathfrak{X}(\gamma)$.

Proposition 2.1.3 (Existence and Uniqueness of Jacobi Fields, [Lee18, Proposition 10.2])**.** *Let $\gamma : I \to M$ be a geodesic, and fix $t_0 \in I$. Then for any pair of vectors $v, w \in T_{\gamma(t_0)}M$, there exists a unique Jacobi field J along γ such that*

$$J(t_0) = v \qquad \textit{and} \qquad D_t J(t_0) = w.$$

Next, we introduce an object related to Jacobi fields, called a *geodesic variation*.

Definition 2.1.4 (Geodesic Variation)**.** Let $\gamma : I \to M$ be a geodesic. Then a *geodesic variation along* γ is a smooth map $\Gamma : (-\varepsilon, \varepsilon) \times I \to M$ such that $\Gamma(0, t) = \gamma(t)$ for each $t \in I$, and each curve of the form $\Gamma_s := \Gamma(s, \cdot) : I \to \mathbb{R}$ is a geodesic. The *variation field* of Γ is the vector field along γ defined by $V := \partial_s \Gamma(0, \cdot)$.

It turns out that Jacobi fields are precisely the variation fields of geodesic variations, as the next proposition shows.

Proposition 2.1.5 (Jacobi Fields as Variation Fields of Geodesic Variations, [Lee18, Propositions 10.1 and 10.4])**.** *Let $\gamma : I \to M$ be a geodesic. Then the variation field of any geodesic variation of γ is a Jacobi field. Conversely, if M is complete or I is compact, then any Jacobi field is the variation field of some geodesic variation along γ.*

In this report, we are interested in a subset of Jacobi fields, called *normal Jacobi fields*. We now define what it means for a vector field along a geodesic to be *normal*.

Definition 2.1.6 (Normal Vector Field)**.** Let $\gamma : I \to M$ be a geodesic with non-zero speed. For each $t \in I$, we denote $T^{\perp}_{\gamma(t)}\gamma \subseteq T_{\gamma(t)}M$ to be the orthogonal complement of the span of $\gamma'(t)$. We say a vector field $V \in \mathfrak{X}(\gamma)$ along γ is a *normal vector field* if $V(t) \in T^{\perp}_{\gamma(t)}\gamma$ for every $t \in I$. We denote the space of smooth normal vector fields by $\mathfrak{X}^{\perp}(\gamma)$.

In addition to being normal, the Jacobi fields that we encounter all vanish at one point. These Jacobi fields can be expressed in terms of the differential of the exponential map, as the following proposition shows.

Proposition 2.1.7 ([Lee18, Proposition 10.10])**.** *Let $\gamma : I \to M$ be a geodesic, where I is an interval containing zero. Suppose J is a Jacobi field along γ such that $J(0) = 0$. Then*

$$J(t) = d(\exp_p)_{t\gamma'(0)}(t D_t J(0)). \tag{2.2}$$

It turns out that in spaces of constant sectional curvature, normal Jacobi fields vanishing at a point have very explicit formulas relating them to *normal parallel vector fields*. We revisit the following example throughout the report.

Example 2.1.8 (Normal Jacobi Fields in Spaces of Constant Sectional Curvature)**.** Suppose M has constant sectional curvature $c \in \mathbb{R}$. We define a function $s_c : \mathbb{R} \to \mathbb{R}$ by

$$s_c(t) := \begin{cases} t & \text{if} \quad c = 0; \\ r \sin \frac{t}{r} & \text{if} \quad c = \frac{1}{r^2} = 0; \\ r \sinh \frac{t}{r} & \text{if} \quad c = -\frac{1}{r^2} = 0. \end{cases} \tag{2.3}$$

Let $\gamma : I \to M$ be a unit-speed geodesic with $0 \in I$. Then [Lee18, Proposition 10.12] tells us that for any normal parallel vector field E along γ, the vector field J along γ defined by

$$J(t) := s_c(t)E(t) \tag{2.4}$$

is a normal Jacobi field satisfying $J(0) = 0$ and $D_tJ(0) = E(0)$.

Lastly, we use the Gauss Lemma throughout Sections 2 and 3.

Proposition 2.1.9 (The Gauss Lemma, [Lee18, Theorem 6.9])**.** *Let $p \in M$. Let U be a normal neighbourhood centred at p, and define $V := \exp_p^{-1}(U)$. Fix $v, w \in V$, and define $q := \exp_p(v)$. Then*

$$\langle v, w\rangle_p = \left\langle d(\exp_p)_v(v), d(\exp_p)_v(w)\right\rangle_q.$$

2.2 Jacobi Tensors

Our goal in this section is to construct the *Jacobi tensor associated to a geodesic*, which is done in Proposition 2.2.6 and Definition 2.2.7. Moreover, Lemma 2.2.2, Lemma 2.2.4, and Lemma 2.2.5 show that Jacobi tensors can be characterised as C^∞-linear maps which send normal parallel vector fields to normal Jacobi fields vanishing at a point. We introduce Jacobi tensors because they allow us to give a basis independent definition of the density function in Section 2.3.

Definition 2.2.1 (Endomorphism Bundle Along a Geodesic)**.** Let $\gamma : I \to M$ be a geodesic. We define the *endomorphism bundle along* γ to be the set

$$\mathrm{End}(T^\perp\gamma) := \bigcup_{t\in I}\{t\} \times \mathrm{End}(T^\perp_{\gamma(t)}\gamma).$$

We say a map $\mathcal{S} : I \to \mathrm{End}(T^\perp\gamma)$ is a *section of* $\mathrm{End}(T^\perp\gamma)$ if for each $t \in I$, $\mathcal{S}(t)$ is a linear map from $T^\perp_{\gamma(t)}\gamma$ to itself. Moreover, given a normal vector field V along γ, we define $\mathcal{S}V : I \to TM$ to be the normal vector field along γ given by $\mathcal{S}V(t) := \mathcal{S}(t)V(t)$. We say that $\mathcal{S}$ is *smooth* if $\mathcal{S}V$ is smooth for any smooth normal vector field V along γ, and we denote the vector space of smooth sections by $\Gamma(\mathrm{End}(T^\perp\gamma))$. Lastly, we define the *covariant derivative of* $\mathcal{S}$ in the following way: given a smooth normal vector field V along γ, we set

$$(D_t\mathcal{S})V := D_t(\mathcal{S}V) - \mathcal{S}(D_tV).$$

Jacobi tensors are defined later to be smooth sections of endomorphism bundles satisfying additional properties, so it is important for us to understand what these smooth sections are. The following lemma gives us a useful identification for them.

Lemma 2.2.2 (Identification for Smooth Sections of Endomorphism Bundle)**.** *Let $\gamma : I \to M$ be a geodesic, and let*

$$\Phi : \Gamma(\mathrm{End}(T^\perp\gamma)) \to \left\{T : \mathfrak{X}^\perp(\gamma) \to \mathfrak{X}^\perp(\gamma) \,\middle|\, T \text{ is } C^\infty(I)\text{-linear}\right\}$$

be the map defined by the following: for each smooth section $\mathcal{S}$ of $\mathrm{End}(T^\perp\gamma)$ and $V \in \mathfrak{X}^\perp(\gamma)$, we set

$$\Phi(\mathcal{S})V := \mathcal{S}V.$$

Then Φ is a bijection.

Proof. Fix a smooth section $\mathcal{S}$ of $\mathrm{End}(T^\perp\gamma)$. Let us check that $\Phi(\mathcal{S})$ is indeed a $C^\infty(I)$-linear map. Let $f, g \in C^\infty(I)$, and let $V, W \in \mathfrak{X}^\perp(\gamma)$. Then for each $t \in I$, by definition, we have

$$\begin{aligned}(\Phi(\mathcal{S})(fV + gW))(t) &= \mathcal{S}(t)(f(t)V(t) + g(t)W(t))\\ &= f(t)\mathcal{S}(t)(V(t)) + g(t)\mathcal{S}(t)(W(t))\\ &= (f\Phi(\mathcal{S})V + g\Phi(\mathcal{S})W)(t),\end{aligned}$$

so $\Phi(\mathcal{S})$ is $C^\infty(I)$-linear.

Next, let us show that Φ is injective. To this end, suppose $\mathcal{S}$ and $\mathcal{T}$ are smooth sections of $\mathrm{End}(T^\perp\gamma)$, and $\Phi(\mathcal{S}) = \Phi(\mathcal{T})$. We want to show that $\mathcal{S} = \mathcal{T}$. Fix $t \in I$, and let $v \in T^\perp_{\gamma(t)}\gamma$. Let V be the unique parallel vector field along γ such that $V(t) = v$. We find

$$\mathcal{S}(t)v = (\mathcal{S}V)(t) = (\Phi(\mathcal{S})V)(t) = (\Phi(\mathcal{T})V)(t) = (\mathcal{T}V)(t) = \mathcal{T}(t)v.$$

This shows that $\mathcal{S} = \mathcal{T}$, so Φ is injective.

Finally, let us show that Φ is surjective. Fix any $T : \mathfrak{X}^\perp(\gamma) \to \mathfrak{X}^\perp(\gamma)$ which is $C^\infty(I)$-linear. For each $t \in I$ and $v \in T^\perp_{\gamma(t)}\gamma$, let V be the unique parallel normal vector field along γ such that $V(t) = v$. Then set

$$\mathcal{S}(t)v := (TV)(t).$$

Let us show that $\mathcal{S}$ is indeed a smooth section of $\mathrm{End}(T^\perp\gamma)$. Fix $t \in I$, and let us first show that $\mathcal{S}(t) : T^\perp_{\gamma(t)}\gamma \to T^\perp_{\gamma(t)}\gamma$ is $\mathbb{R}$-linear. Let $\lambda, \mu \in \mathbb{R}$, and let $v, w \in T^\perp_{\gamma(t)}\gamma$. Let V and W be the unique parallel vector fields along γ such that $V(t) = v$ and $W(t) = w$. Then $\lambda V + \mu W$ is the unique parallel vector field such that $(\lambda V + \mu W)(t) = \lambda v + \mu w$. Therefore,

$$\begin{aligned}
\mathcal{S}(t)(\lambda v + \mu w) &= \mathcal{S}(t)(\lambda V(t) + \mu W(t)) \\
&= T(\lambda V + \mu W)(t) \\
&= \lambda TV(t) + \mu TW(t) = \lambda \mathcal{S}(t)v + \mu \mathcal{S}(t)w,
\end{aligned}$$

so $\mathcal{S}(t)$ is $\mathbb{R}$-linear. Thus, $\mathcal{S}$ is a section of $\mathrm{End}(T^\perp\gamma)$. Next, let us show that $\mathcal{S}$ is smooth. Let $E_2, \dots, E_n$ be normal parallel vector fields along γ which form a basis for $T^\perp_{\gamma(t)}\gamma$ at each $t \in I$. Let $V \in \mathfrak{X}^\perp(\gamma)$. Then there exist smooth functions $v^2, \dots, v^n : I \to \mathbb{R}$ such that

$$V = \sum_{i=2}^n v^i E_i.$$

Therefore, for each $t \in I$,

$$(\mathcal{S}V)(t) = \mathcal{S}(t)V(t) = \sum_{i=2}^n v^i(t)\mathcal{S}(t)E_i(t) = \sum_{i=2}^n v^i(t)(TE_i)(t) = TV(t),$$

where the last equality follows because T is $C^\infty(I)$-linear. Hence, we find $\mathcal{S}V = TV$, which is smooth. Thus, $\mathcal{S}$ is smooth, by definition. Furthermore, this computation shows that $\Phi(\mathcal{S}) = T$, which implies that Φ is surjective, as we wished. □

We are now ready to give the definition of a Jacobi tensor.

Definition 2.2.3 (Jacobi tensor)**.** Let $\gamma : I \to M$ be a geodesic. Then we say that a smooth section $\mathcal{J}$ of the endomorphism bundle $\mathrm{End}(T^\perp\gamma)$ is a *Jacobi tensor* along γ if for any normal *parallel* vector field V along γ, we have

$$(D_t D_t \mathcal{J})V + R(\mathcal{J}V, \gamma')\gamma' \equiv 0. \tag{2.5}$$

The next lemma tells us that Jacobi tensors applied to smooth normal parallel vector fields give us Jacobi fields, as expected.

Lemma 2.2.4. *Let $\gamma : I \to M$ be a geodesic. Let $\mathcal{J}$ be a Jacobi tensor along γ, and let V be a smooth normal parallel vector field along γ. Then $\mathcal{J}V$ is a Jacobi field.*

Proof. Since V is parallel, we find

$$D_t D_t(\mathcal{J}V) = (D_t D_t \mathcal{J})V = -R(\mathcal{J}V, \gamma')\gamma',$$

so rearranging gives us the result. □

The following lemma tells us that given normal Jacobi fields and a normal parallel frame, we can construct a unique Jacobi tensor.

Lemma 2.2.5. *Let $\gamma : I \to M$ be a geodesic. and let $E_2, \ldots, E_n$ be normal parallel vector fields along γ such that they form a basis at $T^{\perp}_{\gamma(t)}\gamma$ for each $t \in I$. Let $J_2, \ldots, J_n$ be normal Jacobi fields along γ. Then there exists a unique Jacobi tensor $\mathcal{J}$ along γ such that $\mathcal{J}E_i = J_i$ for each $i = 2, \ldots, n$.*

Proof. First, let us show existence. Given a normal vector field $V \in \mathfrak{X}^{\perp}(\gamma)$, there exist unique smooth functions $v^2, \ldots, v^n : I \to \mathbb{R}$ such that

$$V = \sum_{i=2}^{n} v^i E_i.$$

Define a map $\mathcal{J} : \mathfrak{X}^{\perp}(\gamma) \to \mathfrak{X}^{\perp}(\gamma)$ by

$$\mathcal{J}V := \sum_{i=2}^{n} v^i J_i.$$

By Lemma 2.2.2, to show that $\mathcal{J}$ is a smooth section of $\operatorname{End}(T^{\perp}\gamma)$, it suffices to show that $\mathcal{J}$ is $C^{\infty}(I)$-linear. To this end, fix $f, g \in C^{\infty}(I)$, and $V, W \in \mathfrak{X}^{\perp}(\gamma)$. Let $v^2, \ldots, v^n, w^2, \ldots, w^n : I \to \mathbb{R}$ be smooth functions such that

$$V = \sum_{i=2}^{n} v^i E_i \qquad \text{and} \qquad W = \sum_{i=2}^{n} w^i E_i.$$

Then

$$\mathcal{J}(fV + gW) = f\sum_{i=2}^{n} v^i J_i + g\sum_{i=2}^{n} w^i J_i = f\mathcal{J}V + g\mathcal{J}W.$$

Thus, $\mathcal{J}$ is a smooth section of $\operatorname{End}(T^{\perp}\gamma)$.

Next, let us show that $\mathcal{J}$ is a Jacobi tensor. To this end, fix a normal parallel vector field V along γ. Then we know that there exist real numbers $x^2, \ldots, x^n \in \mathbb{R}$ such that

$$V = \sum_{i=2}^{n} x^i E_i.$$

Since V is parallel, we find

$$\begin{aligned}(D_t D_t \mathcal{J})V + R(\mathcal{J}V, \gamma')\gamma' &= D_t D_t(\mathcal{J}V) + R(\mathcal{J}V, \gamma')\gamma' \\ &= D_t D_t\left(\mathcal{J}\left(\sum_{i=2}^{n} x^i E_i\right)\right) + R\left(\mathcal{J}\left(\sum_{i=2}^{n} x^i E_i\right), \gamma'\right)\gamma' \\ &= \sum_{i=2}^{n} v^i\left(D_t D_t J_i + R(J_i, \gamma')\gamma'\right) = 0,\end{aligned}$$

so $\mathcal{J}$ is a Jacobi tensor. Finally, notice that for $i = 2, \ldots, n$, $\mathcal{J}E_i = J_i$, so we have shown existence.

Now, let us show uniqueness. Suppose $\mathcal{J}$ and $\widetilde{\mathcal{J}}$ are two Jacobi tensors such that $\mathcal{J}E_i = \widetilde{\mathcal{J}}E_i = J_i$ for $i = 2, \ldots, n$. We want to show that $\mathcal{J} = \widetilde{\mathcal{J}}$. By Lemma 2.2.2, it suffices to show that they agree on normal vector fields. Thus, let us fix $V \in \mathfrak{X}^{\perp}(\gamma)$. By Lemma 2.2.2, we know that $\mathcal{J}$ and $\widetilde{\mathcal{J}}$ are $C^{\infty}(I)$-linear, so

$$\mathcal{J}V = \sum_{i=2}^{n} v^i \mathcal{J}E_i = \sum_{i=2}^{n} v^i \widetilde{\mathcal{J}}E_i = \widetilde{\mathcal{J}}V.$$

Therefore, $\mathcal{J} = \widetilde{\mathcal{J}}$, and we have proven the result. $\square$

This section has been leading to the following proposition, which allows us to define the *Jacobi tensor associated with* γ.

Proposition 2.2.6 (Existence and Uniqueness of Jacobi Tensor Associated to a Geodesic)**.** *Let* $\gamma : I \to M$ *be a geodesic with* $0 \in I$*. Then there exists a unique Jacobi tensor* $\mathcal{J}$ *along* γ *such that* $\mathcal{J}(0) = 0$ *and* $D_t\mathcal{J}(0) = \mathrm{Id}$.

Proof. Let $E_2, \ldots, E_n$ be normal parallel vector fields along γ such that they form a basis for $T^{\perp}_{\gamma(t)}\gamma$ at each $t \in I$. For each $i = 2, \ldots, n$, let J_i be the Jacobi field satisfying $J_i(0) = 0$ and $D_tJ_i(0) = E_i(0)$. Then Lemma 2.2.5 tells us that there exists a unique Jacobi tensor $\mathcal{J}$ such that $\mathcal{J}E_i = J_i$ for each $i = 2, \ldots, n$. Fix $v \in T^{\perp}_{\gamma(0)}\gamma$. Then there exist real numbers $v^2, \ldots, v^n$ such that

$$v = \sum_{i=2}^{n} v^i E_i(0).$$

Computing, we find

$$\mathcal{J}(0)v = \sum_{i=2}^{n} v^i \mathcal{J}(0)E_i(0) = \sum_{i=2}^{n} v^i J_i(0) = 0,$$

and

$$D_t\mathcal{J}(0)v = \sum_{i=2}^{n} v^i (D_t\mathcal{J}(0))E_i(0) = \sum_{i=2}^{n} v^i D_t(\mathcal{J}(0)E_i(0)) = \sum_{i=2}^{n} v^i D_tJ_i(0) = \sum_{i=2}^{n} v^i E_i(0) = v.$$

The result follows. □

The following definition is given in [Kre10, Definition 2.1.2].

Definition 2.2.7 (Jacobi tensor Associated With a Geodesic)**.** Let $\gamma : I \to M$ be a geodesic with $0 \in I$. We define *Jacobi tensor associated with* γ to be the unique Jacobi tensor $\mathcal{J}$ along γ such that $\mathcal{J}(0) = 0$ and $D_t\mathcal{J}(0) = \mathrm{Id}$.

By the existence and uniqueness theorem for Jacobi fields, 2.1.3, we know that the vector subspace of normal Jacobi fields with $J(0) = 0$ is $(n-1)$-dimensional. Observe that the vector subspace of normal *parallel* vector fields is also $(n-1)$-dimensional. Therefore, a Jacobi tensor associated with a geodesic encodes *all* the normal Jacobi fields with $J(0) = 0$ along γ.

Jacobi tensors associated with geodesics can be explicitly computed in spaces of constant sectional curvature, which the the following example demonstrates.

Example 2.2.8 (Jacobi Tensor Associated With a Geodesic in Space of Constant Sectional Curvature)**.** We build on Example 2.1.8. Suppose M has constant sectional curvature $c \in \mathbb{R}$, and let $\gamma : I \to M$ be a unit-speed geodesic with $0 \in I$. Let $\mathcal{J}$ be the Jacobi associated with γ. Let $E_2, \ldots, E_n$ be normal parallel vector fields such that they form a basis at $T^{\perp}_{\gamma(t)}\gamma$ for each $t \in I$.

Fix $i = 2, \ldots, n$, and define $J_i := \mathcal{J}E_i$, which we know is a Jacobi field satisfying $J_i(0) = 0$ and $D_tJ_i(0) = E_i(0)$. By Example 2.1.8, we know that the Jacobi fields $\widetilde{J}_i := s_cE_i$ also satisfy $\widetilde{J}_i(0) = 0$ and $D_t\widetilde{J}_i(0) = E_i(0)$. Therefore, by uniqueness of Jacobi fields, we must have

$$\mathcal{J}E_i = J_i = \widetilde{J}_i = s_cE_i.$$

Fix $t \in I$. Then $\mathcal{J}(t)E_i(t) = s_c(t)E_i(t)$, which means that the matrix representation of $\mathcal{J}(t)$ with respect to the basis $\{E_2(t), \ldots, E_n(t)\}$ is given by

$$[\mathcal{J}(t)]^{\{E_2(t),\ldots,E_n(t)\}}_{\{E_2(t),\ldots,E_n(t)\}} = \begin{pmatrix} s_c(t) & 0 & \cdots & 0 \\ 0 & s_c(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_c(t) \end{pmatrix}.$$

2.3 Definition of the Density Function and Characterisations

In this section, we define the density function in Definition 2.3.1, then prove two different characterisations of the *density function at a point* in Proposition 2.3.5 and Proposition 2.3.7. We do so, because these characterisations are easier to work with than Definition 2.3.1. The density function is part of the definition of local harmonicity, so it is natural that we studying it in detail.

The following definition is taken from [Kre10, Definition 2.2.1].

Definition 2.3.1 (Density function)**.** Let $\mathcal{E} \subseteq TM$ denote the domain of the exponential map. For each $(p, v) \in \mathcal{E}$ with $v \neq 0$, denote $\mathcal{J}_v$ to be the Jacobi tensor associated with the normalised geodesic $t \mapsto \exp_p(tv/\|v\|)$. We then define the *density function* $\omega : \mathcal{E} \to \mathbb{R}$ by

$$\omega(p, v) := \begin{cases} \|v\|^{1-n} \det \mathcal{J}_v(\|v\|) & \text{if} \quad v \neq 0, \\ 1 & \text{if} \quad v = 0. \end{cases} \tag{2.6}$$

Fix $p \in M$, and let U be a normal neighbourhood around p. Then we define the *density function at* p to be the map $\omega_p : U \to \mathbb{R}$ given by $\omega_p(q) := \omega(p, \exp_p^{-1}(q))$.

The advantage of this definition instead of formulating the density function using the characterisations given in Proposition 2.3.5 and Proposition 2.3.7 is that, with Definition 2.3.1, we know precisely where the density function is defined, and we are not restricted to normal neighbourhoods. In particular, if M is complete, then the density function is defined everywhere.

The density function is continuous. This implies that each ω_p is continuous, since it is a composition of continuous functions. Later, we show that ω_p is in fact smooth on its domain.

We can immediately use Example 2.2.8 to compute the density function in spaces of constant sectional curvature.

Example 2.3.2 (Density Function in Space of Constant Sectional Curvature)**.** Suppose M has constant sectional curvature $c \in \mathbb{R}$. Fix $(p, v) \in \mathcal{E}$, and suppose that $v \neq 0$. Then by Example 2.2.8, we find

$$\omega(p, v) := \frac{1}{\|v\|^{n-1}} \det \mathcal{J}_v(\|v\|) = \|v\|^{1-n} \det \begin{pmatrix} s_c(\|v\|) & 0 & \cdots & 0 \\ 0 & s_c(\|v\|) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_c(\|v\|) \end{pmatrix} = \frac{s_c(\|v\|)^{n-1}}{\|v\|^{n-1}}.$$

In particular, by the definition of s_c from Example 2.1.8, we find that $\omega(p, v)$ is equal to

$$1 \quad \text{if} \quad M = \mathbb{R}^n, \qquad \frac{\sin^{n-1}(\|v\|)}{\|v\|^{n-1}} \quad \text{if} \quad M = \mathbb{S}^n, \qquad \text{and} \qquad \frac{\sinh^{n-1}(\|v\|)}{\|v\|^{n-1}} \quad \text{if} \quad M = \mathbb{H}^n.$$

The following lemma shows that the density function is zero precisely where the differential of the exponential map fails to be a linear isomorphism.

Lemma 2.3.3 (Zero Points of Density Function)**.** *Let $(p, v) \in \mathcal{E}$ with $v \neq 0$. Write $q := \exp_p(v)$. Then $\omega(p, v) = 0$ if and only if $d(\exp_p)_v : T_pM \to T_qM$ is not injective.*

Proof. Let γ be the unit-speed geodesic given by $t \mapsto \exp_p(tv/\|v\|)$. Let $\{e_2, \ldots, e_n\}$ be an orthonormal basis at $T_p^{\perp}\gamma$, and let $E_2, \ldots, E_n$ be the unique parallel vector fields along γ such that $E_i(0) = e_i$ for each $i = 2, \ldots, n$. We know that $\{E_2(\|v\|), \ldots, E_n(\|v\|)\}$ forms an orthonormal basis for $T_q^{\perp}\gamma$. For each $i = 2, \ldots, n$, define $J_i := \mathcal{J}_v E_i$. By the proof of Proposition 2.2.6, we know that each J_i is a Jacobi field with $J_i(0) = 0$ and $D_t J_i(0) = e_i$. By Proposition 2.1.7, we can express each J_i in terms of the differential of the exponential map:

$$J_i(\|v\|) = d(\exp_p)_{\|v\| v/\|v\|}(\|v\| D_t J_i(0)) = \|v\| d(\exp_p)_v(e_i). \tag{2.7}$$

Now, suppose $\omega(p, v) = 0$. Then by definition of ω, we must have that $\det \mathcal{J}_v(\|v\|) = 0$, which means that the linear map $\mathcal{J}_v(\|v\|) : T_q^{\perp}\gamma \to T_q^{\perp}\gamma$ is not injective. This implies that the set

$$\{\mathcal{J}_v(\|v\|)E_2(\|v\|), \ldots, \mathcal{J}_v(\|v\|)E_n(\|v\|)\} = \{J_2(\|v\|), \ldots, J_n(\|v\|)\} \tag{2.8}$$

is linearly dependent in $T_q^{\perp}\gamma$. Thus, equation (2.7) shows that $d(\exp_p)_v$ is not injective, since the image of the basis $\{e_1, \ldots, e_n\}$ under it is not linearly independent.

Conversely, suppose $d(\exp_p)_v$ is not injective. Then Equation (2.7) shows that the set

$$\{J_2(\|v\|), \ldots, J_n(\|v\|)\}$$

is linearly dependent. Then Equation (2.8) implies $\mathcal{J}_v(\|v\|)$ is not injective. Therefore, its determinant $\det \mathcal{J}_v(\|v\|)$ is zero, so $\omega(p, v) = 0$, as we wished. $\square$

The following lemma shows that the density function at p is positive on any normal neighbourhood of p.

Lemma 2.3.4 (Density Function at a Point is Positive)**.** *Fix $p \in M$, and let U be a normal neighbourhood around p. Fix any $q \in U$, and write $q = \exp_p(v)$. Then $\omega_p(q) > 0$.*

Proof. For the sake of contradiction, suppose that $\omega_p(q) \leq 0$. Then either $\omega_p(q) = 0$ or $\omega_p(q) < 0$. Let us consider both cases separately.

Suppose $\omega_p(q) = 0$. Since U is a normal neighbourhood of p containing q, we know that the map $d(\exp_p)_v : T_pM \to T_qM$ is a linear isomorphism. However, Lemma 2.3.3 implies that this map is not injective, so we have reached a contradiction.

Next, suppose $\omega_p(q) < 0$. Define a curve $\gamma : [0, 1] \to M$ by $\gamma(t) := \exp_p(tv)$. Then notice that the map

$$f : [0, 1] \to \mathbb{R}, \qquad t \mapsto \omega_p(\gamma(t))$$

is continuous, with $f(0) = 1$ and $f(1) < 0$. Therefore, by the intermediate value theorem, there exists some $t_0 \in [0, 1]$ such that $0 = f(t_0) = \omega_p(\gamma(t_0))$. The argument in the first case shows that this is a contradiction. $\square$

The following proposition gives an expression for the density function at a point in normal coordinates. Observe that this proposition implies that each ω_p is smooth.

Proposition 2.3.5 (Density Function in Normal Coordinates)**.** *Fix $p \in M$, and let $(U, (x^i))$ be a normal coordinate chart centred at p. Let $q \in U$. Then*

$$\omega_p(q) = \sqrt{\det(g_{ij}(q))}. \tag{2.9}$$

Proof. If $q = p$, then we are done, since

$$\omega_p(p) := 1 = \sqrt{\det(g_{ij}(p))}.$$

Suppose $q \neq p$. Define $v := \exp_p^{-1}(q)$, and $e_1 := v/\|v\|$. Choose $e_2, \ldots, e_n \in T_pM$ such that $\{e_1, \ldots, e_n\}$ forms an orthonormal basis at T_pM. For indices $i, j = 1, \ldots, n$, define

$$A^i_j := \Big\langle d(\exp_p)_v(e_i), d(\exp_p)_v(e_j) \Big\rangle_q.$$

By the Gauss Lemma, we find $A^1_1 = \langle e_1, e_1 \rangle = 1$, and $A^1_i = A^i_1 = \langle e_1, e_i \rangle = 0$ for $i = 2, \ldots, n$. Therefore, the determinant of the matrix (A^i_j) does not change if we remove the first row and column:

$$\det(A^i_j)_{i,j=2,\ldots,n} = \det(A^i_j)_{i,j=1,\ldots,n}.$$

Next, let us show that the determinants of (A^i_j) and $(g_{ij}(q))$ are equal. Let $\{\partial_1|_p, \ldots, \partial_n|_p\}$ be the orthonormal basis at T_pM induced by the normal coordinate chart. Then for each $r = 1, \ldots, n$, we can write

$$e_r = T^i_r \partial_i|_p,$$

where (T^i_j) is an orthogonal matrix. Fix indices $r, s = 1, \ldots, n$. Then

$$\begin{aligned}
A^r_s &= \Big\langle d(\exp_p)_v(T^i_r \partial_i|_p), d(\exp_p)_v(T^j_s \partial_j|_p) \Big\rangle_q \\
&= T^i_r T^j_s \langle \partial_i|_q, \partial_j|_q \rangle_q \\
&= T^i_r g_{ij}(q) T^j_s \\
&= [(T^i_j)^\intercal \cdot (g_{ij}(q)) \cdot (T^i_j)]^r_s,
\end{aligned}$$

where $\cdot$ denotes matrix multiplication. Taking the determinant gives us

$$\begin{aligned}
\det(A^i_j)_{i,j=2,\ldots,n} &= \det(A^i_j)_{i,j=1,\ldots,n} \\
&= \cancel{\det(T^i_j)^\intercal \det(T^i_j)} \det(g_{ij}(q)) \\
&= \det(g_{ij}(q)).
\end{aligned}$$

We now want to relate the determinant of (A^i_j) with the density function at p. Let γ be the unit-speed geodesic given by $t \mapsto \exp_p(te_1)$, and let $E_2, \ldots, E_n$ be the parallel vector fields along γ such that $E_i(0) = e_i$. For each $i = 2, \ldots, n$, define $J_i := \mathcal{J}_v E_i$. By the proof of Proposition 2.2.6, we know that each J_i is a Jacobi field along γ with $J_i(0) = 0$ and $D_t J_i(0) = e_i$.

Fix indices $i, j = 1, \ldots, n$. Since the set $\{E_2(\|v\|), \ldots, E_n(\|v\|)\}$ forms an orthonormal basis for $T^\perp_q \gamma$, we can write

$$J_j(\|v\|) = \sum_{k=2}^n \Big\langle J_j(\|v\|), E_k(\|v\|) \Big\rangle E_k(\|v\|).$$

Thus, by Equation (2.7), we find

$$\begin{aligned}
A^i_j &= \Big\langle d(\exp_p)_v(e_i), d(\exp_p)_v(e_j) \Big\rangle_q \\
&= \frac{1}{\|v\|^2} \Big\langle J_i(\|v\|), J_j(\|v\|) \Big\rangle_q \\
&= \frac{1}{\|v\|^2} \sum_{k=2}^n \Big\langle J_i(\|v\|), E_k(\|v\|) \Big\rangle_q \Big\langle J_j(\|v\|), E_k(\|v\|) \Big\rangle_q .
\end{aligned}$$

Taking the determinant gives us

$$\begin{aligned}
\det(g_{ij}(q)) &= \det(A^i_j)_{i,j=2,\ldots,n} \\
&= \frac{1}{\|v\|^{2n-2}} \left(\det \Big\langle J_i(\|v\|), E_j(\|v\|) \Big\rangle_{i,j=2,\ldots,n} \right)^2 \\
&= \frac{1}{\|v\|^{2n-2}} \det \mathcal{J}_v(\|v\|)^2 = \omega_p(q)^2.
\end{aligned}$$

Since ω_p is positive on U, taking the square root gives us the result. □

The following definition allows us to define the absolute value of the determinant of a linear map between inner product spaces.

Definition 2.3.6 (Determinant between Inner Product Spaces)**.** Let $T : V \to W$ be a linear map between inner product spaces of dimension k. We define $|\det T|$ to denote the absolute value of the determinant of the matrix representation with respect to any pair of orthonormal bases from V and W. Note that this is well-defined (that is, independent of the choice of orthonormal bases), because the change of basis matrix between any two orthonormal bases has determinant ± 1.

Observe that $|\det T|$ gives the k-volume in W of the image of a unit hypercube in V under T. Since T is linear, this means that $|\det T|$ tells us the volume scaling factor of the map T. Thus, the following proposition tells us that the density function gives the volume scaling factor of the differential of the exponential map.

Proposition 2.3.7 (Density as Volume Scaling Factor)**.** *Fix $p \in M$, and let U be a normal neighbourhood around p. Let $q \in U$, and define $v := \exp_p^{-1}(q)$. Then*

$$\omega_p(q) = |\det d(\exp_p)_v|.$$

Proof. To simplify notation, let us denote

$$T := d(\exp_p)_v : T_pM \to T_qM.$$

Let $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ be orthonormal bases at T_pM and T_qM, respectively. Let (x^i) be the normal coordinate functions defined on U, induced by $\{e_i\}_{i=1}^n$. Then for $j = 1, \dots, n$, we can write

$$\partial_j|_q = Te_j = T_j^i f_i.$$

Therefore, for indices $\alpha, \beta = 1, \dots, n$, we find

$$g_{\alpha\beta}(q) := \langle \partial_\alpha|_q, \partial_\beta|_q \rangle = \langle T_\alpha^i f_i, T_\beta^j f_j \rangle = T_\alpha^i T_\beta^j \delta_{ij} = [(T_j^i)^\intercal \cdot (T_j^i)]_\beta^\alpha.$$

By taking the determinant and using Proposition 2.3.5, we find

$$\omega_p(q)^2 = \det(g_{ij}(q)) = \det(T_j^i)^2 = |\det d(\exp_p)_v|^2.$$

The result follows since $\omega_p(q)$ is positive. $\square$

2.4 Invariance of the Density Function under Isometries

The following proposition is crucial for the proof that two point homogeneous spaces are locally harmonic, in Section 4.3.

Proposition 2.4.1 (Invariance of the Density Function Under Isometries)**.** *Let $\varphi : M \to M$ be an isometry. Let $p \in M$, and let U be a normal neighbourhood around p containing q. Then $\omega_p(q) = \omega_{\varphi(p)}(\varphi(q))$.*

Proof. Since φ is an isometry, we know that $\varphi(U)$ is a normal neighbourhood around $\varphi(p)$ containing $\varphi(q)$. [Lee18, Proposition 5.20] tells us that the following diagram commutes:

$$\begin{tikzcd}
T_pM \arrow[r, "d\varphi_p"] \arrow[d, "\exp_p"'] & T_{\varphi(p)}M \arrow[d, "\exp_{\varphi(p)}"] \\
M \arrow[r, "\varphi"] & M
\end{tikzcd}$$

Therefore, we find

$$\exp_{\varphi(p)} = \varphi \circ \exp_p \circ\, d(\varphi^{-1})_{\varphi(p)}.$$

Taking the differential of both sides at $d\varphi_p(v)$, we find

$$d(\exp_{\varphi(p)})_{d\varphi_p(v)} = d\varphi_q \circ d(\exp_p)_v \circ d(\varphi^{-1})_{\varphi(p)}.$$

Since φ is a Riemannian isometry, we know that the maps $d\varphi_q : T_qM \to T_{\varphi(q)}M$ and $d(\varphi^{-1})_{\varphi(p)} : T_{\varphi(p)}M \to T_pM$ are linear isometries. In particular, the determinant of their matrix representations with respect to orthonormal bases is equal to ± 1. Therefore, by Proposition 2.3.7, we find

$$\begin{aligned}
\omega_{\varphi(p)}(\varphi(q)) &= |\det d(\exp_{\varphi(p)})_{d\varphi_p(v)}| \\
&= \cancel{|\det d\varphi_q|}\,|\det d(\exp_p)_v|\,\cancel{|\det d(\varphi^{-1})_{\varphi(p)}|} \\
&= |\det d(\exp_p)_v| = \omega_p(q),
\end{aligned}$$

as we wished. $\square$

3 Radially Symmetric Functions

3.1 The Radial Distance Function and the Radial Vector Field

In this section, we define what it means for a function to be *radially symmetric* in Definition 3.1.2. We prove that the radial derivative of a radially symmetric function is also radially symmetric in Lemma 3.1.5.

Fix $p \in M$. We define the *injectivity radius at* p, denoted $\mathrm{Inj}(p)$, to be the supremum of all $\varepsilon > 0$ such that $\exp_p$ is a diffeomorphism from $B_\varepsilon(0) \subseteq T_pM$ onto its image.

Given $0 < \varepsilon < \mathrm{Inj}(p)$, we define the *geodesic ball of radius* ε *at* p, denoted $B_\varepsilon(p)$, to be the image of $B_\varepsilon(0) \subseteq T_pM$ under the exponential map. By [Lee18, Corollary 6.13], we know that the normal neighbourhood $B_\varepsilon(p)$ is precisely the ball induced by the Riemannian distance function d:

$$B_\varepsilon(p) = \{q \in M \mid d(p,q) < \varepsilon\}.$$

We define the *pointed ball of radius* ε *at* p to be the set $\widehat{B}_\varepsilon(p) := B_\varepsilon(p) \backslash \{p\}$. Similarly, we define $\widehat{B}_\varepsilon(0_p) := B_\varepsilon(0_p) \backslash \{0_p\}$.

We are now ready to define the *radial distance function*, the *radial vector field*, and *radially symmetric functions.*

Definition 3.1.1 (Radial Distance Function and Radial Vector Field)**.** Let $p \in M$, and let $0 < \varepsilon < \mathrm{Inj}(p)$. We define the *radial distance function at* p to be the function $r_p : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ given by $r_p(q) := d(p,q)$. We define the *radial vector field at* p to be the vector field $\partial_r : \widehat{B}_\varepsilon(p) \to TM$ given by $\partial_r|_q := d(\exp_p)_v(v/\|v\|)$, where $v := \exp_p^{-1}(q)$.

Definition 3.1.2 (Radially Symmetric Function)**.** Let $p \in M$, and let $0 < \varepsilon < \mathrm{Inj}(p)$. Let $f : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ be a function. We say that f is *radially symmetric* if there exists $F : (0,\varepsilon) \to \mathbb{R}$ such that $f = F \circ r_p$.

Lemma 3.1.3. *Let $p \in M$, and let $0 < \varepsilon < \mathrm{Inj}(p)$. Let $f : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ and $F : (0,\varepsilon) \to \mathbb{R}$ be functions such that $f = F \circ r_p$. Then f is smooth if and only if F is smooth.*

Proof. Suppose f is smooth. Let $\gamma : (-\varepsilon,\varepsilon) \to M$ be any unit-speed geodesic such that $\gamma(0) = p$. Fix $r \in (0,\varepsilon)$. Then notice that $d(p,\gamma(r)) = r$, so

$$(f \circ \gamma)(r) = F(r_p(\gamma(r))) = F(r).$$

Therefore, F is a composition of smooth functions, and so is smooth.

Conversely, suppose F is smooth. By [Lee18, lemma 6.8], we know that r_p is smooth on its domain. Thus, f is a composition of smooth functions, and so is smooth. □

The following lemma shows that unit-speed geodesics passing through p are integral curves of the radial vector field.

Lemma 3.1.4 (Integral Curves of the Radial Vector Field)**.** *Let $p \in M$, and let $0 < \varepsilon < \mathrm{Inj}(p)$. Let $v \in S_1(0_p)$, and let $\gamma_v : (-\varepsilon,\varepsilon) \to M$ be the unit-speed geodesic given by $r \mapsto \exp_p(v)$. Then for any $r \in (0,\varepsilon)$, we have*

$$\partial_r|_{\gamma(r)} = \gamma'(r).$$

Proof. Taking the derivative gives the result. □

The following lemma shows that the radial derivative of a radially symmetric function gives another radially symmetric function.

Lemma 3.1.5 (Radial Derivative of a Radially Symmetric Function)**.** *Let $p \in M$, and let $0 < \varepsilon < \mathrm{Inj}(p)$. Let $f : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ and $F : (0,\varepsilon) \to \mathbb{R}$ be smooth functions such that $f = F \circ r_p$. Then*

$$\partial_r f = F' \circ r_p.$$

Proof. Fix $q \in \widehat{B}_\varepsilon(p)$, and let $\gamma : (-\varepsilon, \varepsilon) \to M$ be the unit-speed geodesic joining p to q. Notice that $r_p \circ \gamma : (0, \varepsilon) \to \mathbb{R}$ is nothing more than the map $t \mapsto t$. Therefore, by Lemma 3.1.4, we find

$$\partial_r f(q) = \gamma'(r_p(q))f = \frac{d}{dt}\bigg|_{r_p(q)} (F \circ (r_p \circ \gamma)) = F'(r_p(q)) \cancel{\frac{d}{dt}\bigg|_{r_p(q)} (t \mapsto t)} = F'(r_p(q)),$$

as desired. $\square$

3.2 The Density Function as the Quotient of Volume Forms

In this section, we show that the density function ω_p at a point $p \in M$ can be characterised as the quotient of the pullback of the volume form on M over the volume form on T_pM (see Proposition 3.2.1). This gives us a change of variables formula when dealing with integrals in Section 4.2.

Fix $p \in M$, and let $0 < \varepsilon < \mathrm{Inj}(p)$. For each $r \in (0, \varepsilon)$, define the *geodesic sphere of radius* r *centred at* p, denoted $S_r(p)$, to be the diffeomorphic image of $S_r(0_p) \subseteq T_pM$ under the exponential map $\exp_p$. Thus, each geodesic sphere is an embedded submanifold of dimension $n - 1$.

Observe that as a consequence of the Gauss lemma, the radial vector field ∂_r on $\widehat{B}_\varepsilon(p)$ is precisely the outward unit normal vector field on each geodesic sphere. Next, define $\widetilde{\partial}_r := \exp_p^* \partial_r$, and observe that $\widetilde{\partial}_r$ is the outward unit normal field on each sphere $S_r(0_p) \subseteq T_pM$.

To discuss volume forms, we need to choose an orientation. To accomplish this, let us fix an orthonormal basis $\{e_i\}_{i=1}^n$ at T_pM, and declare it to be positively-oriented. We know that this basis induces a normal coordinate chart $(B_\varepsilon(p), (x^i))$ at p, which in turn determines an orientation on $B_\varepsilon(p)$. For the rest of the report, we assume that we have done this process every time we use volume forms. With respect to this orientation, let dV be the Riemannian volume form on $B_\varepsilon(p)$, and let $d\widetilde{V}$ be the volume form on T_pM.

Proposition 3.2.1 (Density Function as Quotient of Volume Forms)**.** *Fix* $p \in M$*, and let* $0 < \varepsilon < \mathrm{Inj}(p)$*. Fix* $v \in B_\varepsilon(0_p)$*. Then*

$$\exp_p^* dV|_v = \omega(p, v) d\widetilde{V}|_v.$$

Proof. First, for any $q \in B_\varepsilon(p)$, we find

$$dV|_q = \sqrt{\det(g_{ij}(q))} dx^1 \wedge \cdots \wedge dx^n = \omega_p(q) dx^1 \wedge \cdots \wedge dx^n.$$

The first equality follows from the formula for the volume form in coordinates, and the second equality follows from Proposition 2.4.1. Next, let $\{\varepsilon^i\}_{i=1}^n$ denote the dual basis of $\{e_i\}_{i=1}^n$. Then since $\{e_i\}_{i=1}^n$ is orthonormal and positively-oriented, we know that

$$d\widetilde{V} = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n.$$

Therefore, by [Lee13, Proposition 14.20], we find that for any $v \in B_\varepsilon(0_p)$, we have

$$\exp_p^* dV|_v = \exp_p^*(\omega_p dx^1 \wedge \cdots \wedge dx^n) = \underbrace{\omega_p(\exp_p(v))}_{=\omega(p,v)} \underbrace{\det[d(\exp_p)_v]_{\{e_i\}}^{\{\partial_i|_{\exp_p(v)}\}}}_{=1} \underbrace{\varepsilon^1 \wedge \cdots \wedge \varepsilon^n|_v}_{=d\widetilde{V}|_v}.$$

The determinant of the differential of the exponential map above is equal to 1, because $d(\exp_p)_v(e_i) = \partial_i|_{\exp_p(v)}$ for $i = 1, \ldots, n$. The result follows. $\square$

For each $r \in (0, \varepsilon)$, let dS_r denote the volume form on $S_r(p)$, and let $d\widetilde{S}_r$ denote the volume form on $S_r(0_p) \subseteq T_pM$.

Corollary 3.2.2. *Fix* $p \in M$*, and let* $0 < \varepsilon < \mathrm{Inj}(p)$*. Fix* $r \in (0, \varepsilon)$*. Then for each* $v \in S_r(0_p)$*, we have*

$$\exp_p^* dS_r|_v = \omega(p, v) d\widetilde{S}_r|_v.$$

Proof. Since ∂_r and $\widetilde{\partial}_r$ are the outward unit normal fields on $S_r(p)$ and $S_r(0_p)$ respectively, by [Lee13, Proposition 15.32], we know that

$$dS_r = \iota_{\partial_r} dV \qquad \text{and} \qquad d\widetilde{S}_r = \iota_{\widetilde{\partial}_r} d\widetilde{V},$$

where ι is the interior product. Therefore, by Proposition 3.2.1, we find that for any $v \in S_r(0_p)$,

$$\exp_p^* dS_r|_v = \exp_p^*(\iota_{\partial_r} dV)|_v = \iota_{\exp_p^* \partial_r} \exp_p^* dV|v = \omega(p,v)\iota_{\widetilde{\partial}_r} d\widetilde{V}|_v = \omega(p,v) d\widetilde{S}_r|_v,$$

as we wished. □

For each $r \in (0, \varepsilon)$, define the map $T_r : S_1(0_p) \to S_r(0_p)$ by $v \mapsto rv$. It can be shown by a procedure analogous to Proposition 3.2.1 and Corollary 3.2.2 that

$$T_r^* d\widetilde{S}_r = r^{n-1} d\widetilde{S}_1. \tag{3.1}$$

3.3 The Mean Value Operator

In this section, we introduce the *mean value operator*. The main result in this section is Proposition 3.3.4, which tells us that the mean value operator is smooth. The properties of the mean value operator are important for understanding the mean value property in Section 4.2.

Definition 3.3.1 (Mean Value Operator)**.** Fix $p \in M$ and let $0 < \varepsilon < \operatorname{Inj}(p)$. We define the *mean value operator* MV_p by the following: for each smooth function $f : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ and $r \in (0, \varepsilon)$, we define

$$\mathrm{MV}_p(f, r) := \frac{1}{\operatorname{Vol}(S_r(p))} \int_{S_r(p)} f \, dS_r.$$

Notice that $\mathrm{MV}_p(\cdot, r) : C^\infty(\widehat{B}_\varepsilon(p)) \to \mathbb{R}$ is linear, since integrals are linear.

The following lemma tells us that the mean value operator and composition with the distance function "cancel".

Lemma 3.3.2 (Mean Value Operator and Composition with Distance Function)**.** *Fix $p \in M$ and let $0 < \varepsilon < \operatorname{Inj}(p)$. Let $F : (0, \varepsilon) \to \mathbb{R}$ and $u : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ be smooth functions. Then for each $r \in (0, \varepsilon)$,*

$$\mathrm{MV}_p((F \circ r_p) \cdot u, r) = F(r)\mathrm{MV}_p(u, r),$$

where $\cdot$ denotes product of functions. In particular, if $u \equiv 1$, then $\mathrm{MV}_p(F \circ r_p, r) = F(r)$.

Proof. Fix $r \in (0, \varepsilon)$. Then notice that $F \circ r_p \equiv F(r)$ on $S_r(p)$. Therefore,

$$\begin{aligned}
\mathrm{MV}_p((F \circ r_p) \cdot u, r) &= \frac{1}{\operatorname{Vol}(S_r(p))} \int_{S_r(p)} F(r) u \, dS_r \\
&= F(r) \frac{1}{\operatorname{Vol}(S_r(p))} \int_{S_r(p)} u \, dS_r \\
&= F(r)\mathrm{MV}_p(u, r),
\end{aligned}$$

as desired. □

The following lemma does all of the heavy lifting to show that the mean value operator is smooth.

Lemma 3.3.3. *Fix $p \in M$ and let $0 < \varepsilon < \operatorname{Inj}(p)$. Let $f : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ be a smooth function. Then the map $\Phi : (0, \varepsilon) \to \mathbb{R}$ given by*

$$\Phi(r) := \int_{S_r(p)} f \, dS_r$$

is smooth.

Proof. Since the diffeomorphisms $\exp_p : S_r(0_p) \to S_r(p)$ and $T : S_1(0_p) \to S_r(0_p)$ are orientation preserving, by [Lee13, Proposition 16.6], we find

$$\begin{aligned}
\Phi(r) &= \int_{S_r(p)} f \, dS_r \\
&= \int_{S_r(0_p)} f(\exp_p(v))\omega(p,v) \, d\widetilde{S}_r(v) && \text{by Corollary 3.2.2} \\
&= \int_{S_1(0_p)} f(\exp_p(rv)) r^{n-1} \omega(p, rv) \, d\widetilde{S}_1(v),
\end{aligned}$$

where we have used (3.1) in the last step. Define $G : (0, \varepsilon) \times S_1(0_p) \to \mathbb{R}$ by

$$G(r, v) := f(\exp_p(rv)) r^{n-1} \omega(p, rv).$$

Then notice that for any $r \in (0, \varepsilon)$,

$$\Phi(r) = \int_{S_1(0_p)} G(r, v) \, d\widetilde{S}_1(v).$$

By the product rule and Lemma 3.1.4, we find that for any $r \in (0, \varepsilon)$ and $v \in S_1(0_p)$,

$$\begin{aligned}
\frac{\partial}{\partial r} G(r, v) &= \partial_r f(\exp_p(rv)) r^{n-1} \omega_p(\exp_p(rv)) \\
&\quad + f(\exp_p(rv)) \frac{n-1}{r} r^{n-1} \omega_p(\exp_p(rv)) \\
&\quad + f(\exp_p(rv)) r^{n-1} \partial_r \omega_p(\exp_p(rv)) \\
&= \left(\partial_r f(\exp_p(rv)) + \left(\frac{n-1}{r_p(\exp_p(rv))} + \frac{\partial_r \omega_p(\exp_p(rv))}{\omega_p(\exp_p(rv))} \right) f(\exp_p(rv)) \right) r^{n-1} \omega_p(\exp_p(rv)).
\end{aligned}$$

By the Leibniz integral rule [Che06, Theorem 3.2], we know that Φ is differentiable, and for any $r \in (0, \varepsilon)$,

$$\begin{aligned}
\Phi'(r) &= \int_{S_r(0_p)} \frac{\partial}{\partial r} G(r, v) \, d\widetilde{S}_1(v) \\
&= \int_{S_r(p)} \partial_r f + \left(\frac{n-1}{r_p} + \frac{\partial_r \omega_p}{\omega_p} \right) f \, dS_r.
\end{aligned}$$

Since the integrand is another smooth function, it follows by induction that Φ is smooth. □

Proposition 3.3.4 (Smoothness of the Mean Value Operator)**.** *Fix $p \in M$, and let $0 < \varepsilon < \mathrm{Inj}(p)$. Let $f : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ be smooth. Then the map $\mathrm{MV}_p(f, \cdot) : (0, \varepsilon) \to \mathbb{R}$ is smooth.*

Proof. Lemma 3.3.3 tells us that $\mathrm{MV}_p(f, \cdot)$ is equal to a product of smooth functions, and so is smooth. □

3.4 The Laplacian of a Radially Symmetric Function

The goal of this section is to prove Proposition 3.4.4, which is the main tool used in the proof of equivalences in Section 4.1. By the Laplacian Δ, we mean the Laplace-Beltrami operator.

The following lemma gives formulas fo the Laplacian in coordinates.

Lemma 3.4.1 (Laplacian in Coordinates)**.** *Let $f : M \to \mathbb{R}$ be a smooth function. Let $(U, (x^i))$ be a coordinate chart. Then*

$$\begin{aligned}
\Delta f &= \frac{1}{\sqrt{\det(g_{il})}} \partial_j \left(g^{jk} \sqrt{\det(g_{il})} \partial_k f \right) && (3.2) \\
&= g^{jk} \partial_j \partial_k f + \partial_j g^{jk} \partial_k f + \frac{1}{2} g^{jk} g^{il} \partial_j g_{il} \partial_k f && (3.3) \\
&= g^{jk} \partial_j \partial_k f - g^{jk} \Gamma^l_{jk} \partial_l f. && (3.4)
\end{aligned}$$

Proof. The fact that Δf is equal to (3.2) is presented in [Lee18, Proposition 2.46], so we omit the proof here.

Let us now show that (3.2) is equal to (3.3). First of all, by Jacobi's formula [MN99], we find that for any index $j = 1, \dots, n$, we have

$$\partial_j \det(g_{il}) = (\partial_j g_{il}) g^{il} \det(g_{il}).$$

Therefore, by the chain rule, we find

$$\partial_j \sqrt{\det(g_{il})} = \frac{1}{2}(\partial_j g_{il}) g^{il} \sqrt{\det(g_{il})}.$$

Expanding (3.2) using the product rule gives us

$$\begin{aligned}
\Delta f &= \frac{1}{\sqrt{\det(g_{il})}} \left(\partial_j g^{jk} \sqrt{\det(g_{il})} \partial_k f + g^{jk} \partial_j \sqrt{\det(g_{il})} \partial_k f + g^{jk} \sqrt{\det(g_{il})} \partial_j \partial_k f \right) \\
&= \partial_j g^{jk} \partial_k f + \frac{1}{2} g^{jk} g^{il} \partial_j g_{il} \partial_k f + g^{jk} \partial_j \partial_k f,
\end{aligned}$$

which is (3.3).

Next, let us show that (3.3) is equal to (3.4). Recall that the Christoffel symbols are given by

$$\Gamma^l_{jk} = \frac{1}{2} g^{il} (\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}), \tag{3.5}$$

and the inverse components of the metric are given by $g_{ij} g^{jk} = \delta_{ik}$. By taking the partial derivative ∂_l of both sides and using the product rule, we find $\partial_l g_{ij} g^{jk} + \partial_l g^{jk} g_{ij} = 0$. Rearranging gives us $\partial_j g^{jk} = -g^{jk} g^{il} (\partial_l g_{ij})$.

Summing over all indices, we find

$$\begin{aligned}
\partial_j g^{jk} + \frac{1}{2} g^{jk} g^{ij} \partial_j g_{il} &= -g^{jk} g^{il} \partial_l g_{ij} + \frac{1}{2} g^{jk} g^{ij} \partial_j g_{il} \\
&= -\frac{1}{2} g^{jk} g^{il} \partial_i g_{jl} - \frac{1}{2} g^{jk} g^{il} \partial_l g_{ij} + \frac{1}{2} g^{jk} g^{ij} \partial_j g_{il} \\
&= -\frac{1}{2} g^{jk} g^{il} \left(\partial_i g_{jl} + \partial_l g_{ij} - \partial_j g_{il} \right) \\
&= -\frac{1}{2} g^{jk} g^{il} \left(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk} \right) \\
&= -g^{jk} \Gamma^l_{jk}.
\end{aligned}$$

Therefore,

$$\Delta f = \partial_j g^{jk} \partial_k f + \frac{1}{2} g^{jk} g^{il} \partial_j g_{il} \partial_k f + g^{jk} \partial_j \partial_k f = g^{jk} \partial_j \partial_k f - g^{jk} \Gamma^l_{jk} \partial_l f,$$

which shows that Δf is equal to (3.4). □

The next lemma gives a characterisation of the Laplacian using geodesics.

Lemma 3.4.2 (Laplacian using Geodesics)**.** *Let $f : M \to \mathbb{R}$ be a smooth function. Let $p \in M$, and let $\{e_i\}_{i=1}^n$ be an orthonormal basis at T_pM. For each index i, let $\gamma_i : (\delta, \delta) \to \mathbb{R}$ be the unit-speed geodesic starting at p with initial velocity e_i. Then*

$$\Delta f(p) = \sum_{i=1}^n \frac{d^2}{dt^2} (f \circ \gamma_i)(0).$$

Proof. Let $(U, (x^i))$ be the normal coordinates around p induced by $\{e_i\}_{i=1}^n$. By the properties of normal coordinates, we know that $g_{ij}(p) = g^{ij}(p) = \delta_{ij}$, and the Christoffel symbols vanish at p. Therefore, Equation (3.4) tells us that

$$\Delta f(p) = \sum_{i=1}^n \partial_i|_p (\partial_i f).$$

Fix $i \in 1, \dots, n$. Then for any $t \in (-\delta, \delta)$, we find

$$(\partial_i f \circ \gamma_i)(t) = \partial_i|_{\gamma_i(t)} f = \gamma_i'(t) f = \frac{d}{dt}(f \circ \gamma_i)(t).$$

Therefore,

$$\partial_i|_p(\partial_i f) = \gamma_i'(0)(\partial_i f) = \frac{d}{dt}(\partial_i f \circ \gamma_i)(0) = \frac{d}{dt}\left(\frac{d}{dt}(f \circ \gamma_i)\right)(0) = \frac{d^2}{dt^2}(f \circ \gamma_i)(0).$$

The result follows. $\square$

The following definition will be used in Proposition 3.4.4, so we include it here.

Definition 3.4.3. Let V be a real inner product space of dimension k, and let $v_1, \dots, v_k$ be vectors in V. Then we define the expression $\|v_1 \wedge \cdots \wedge v_k\|$ to be the volume of the k-dimensional parallelepiped with vertices $0, v_1, \dots, v_n, v_1 + \cdots + v_n$. If β is an orthonormal basis of V, then we can write

$$\|v_1 \wedge \cdots \wedge v_k\| = \left|\det\left([v_1]_\beta \middle| \cdots \middle| [v_k]_\beta\right)\right|, \tag{3.6}$$

where $[v_i]_\beta$ is the column vector in $\mathbb{R}^k$ of the components of v_i with respect to β.

Let $T : V \to W$ be a linear map between inner product spaces of dimension k. Let $\{e_i\}_{i=1}^k$ be an orthonormal basis for V. Then notice that a consequence of Definition 3.4.3 is

$$|\det T| = \|Te_1 \wedge \cdots \wedge Te_k\|, \tag{3.7}$$

where $|\det T|$ is given in Definition 2.3.6.

This section has been leading to the following proposition. We present the proof from [BGM71, G.V].

Proposition 3.4.4 (The Laplacian of a Radially Symmetric Function)**.** *Let $p \in M$, and let $0 < \varepsilon < \mathrm{Inj}(p)$. Let $f : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ and $F : (0, \varepsilon) \to \mathbb{R}$ be smooth functions such that $f = F \circ r_p$. Fix $q \in \widehat{B}_\varepsilon(p)$. Then*

$$\Delta f(q) = F''(r_p(q)) + F'(r_p(q))\left(\frac{n-1}{r_p(q)} + \frac{\partial_r \omega_p(q)}{\omega_p(q)}\right). \tag{3.8}$$

Proof. We know that there exists a unique unit-speed geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma(r_p(q)) = q$.

Define $y_1 := \gamma'(r_p(q))$, and choose vectors $y_2, \dots, y_n$ in T_qM such that $\{y_i\}_{i=1}^n$ forms an orthonormal basis. For each $i = 2, \dots, n$, let $\gamma_i : (-\delta, \delta) \to \mathbb{R}$ denote the geodesic satisfying $\gamma_i(0) = q$ and $\gamma_i'(0) = y_i$. Then Proposition 3.4.2 implies that

$$\Delta f(q) = \frac{d^2}{dt^2}(f \circ \gamma)(r_p(q)) + \sum_{i=2}^{n} \frac{d^2}{dt^2}(f \circ \gamma_i)(0). \tag{3.9}$$

Notice that $F \circ r = f = f \circ \gamma \circ r$, so the first term of the expression above is equal to $F''(r(q))$. It remains for us to evaluate the second term.

Fix an index $i = 2, \dots, n$. Define a geodesic variation $\Gamma : (-\delta, \delta) \times [0, r_p(q)] \to \mathbb{R}$ of γ by

$$\Gamma(s, t) := \exp_p\left(\frac{t}{r_p(q)} \exp_p^{-1}(\gamma_i(s))\right).$$

Let J_i be the variation field of Γ. By Proposition 2.1.5, we know that J_i is a Jacobi field. Moreover, we find

$$J_i(0) = 0 \qquad \text{and} \qquad J_i(r_p(q)) = \gamma_i'(0) = y_i.$$

By Proposition 2.1.7, we find that $y_i = J_i(r_p(q)) = d(\exp_p)_v(r_p(q)D_tJ_i(0))$, which implies that

$$D_tJ_i(0) = \frac{1}{r_p(q)} d(\exp_p^{-1})_q(y_i).$$

Next, notice that for each $s \in (-\delta, \delta)$, we have $\Gamma_s(r_p(q)) = \gamma_i(s)$. Hence, we find

$$(f \circ \gamma_i)(s) = F(L(\Gamma_s)),$$

where $L(\Gamma_s)$ is the length of the curve $\Gamma_s : [0, r(q)] \to M$. Thus, the standard chain and product rules tell us that

$$\frac{d^2}{dt^2}(f \circ \gamma_i)(0) = F''(r_p(q)) \left(\frac{d}{ds}L(\Gamma_s)(0)\right)^2 + F'(r_p(q))\frac{d^2}{ds^2}L(\Gamma_s)(0). \tag{3.10}$$

Since γ is unit-speed, by the First Variation Formula [Lee18, Theorem 6.3], we find that

$$\frac{d}{ds}L(\Gamma_s)(0) = -\int_0^{r_p(q)} \langle J_i, \underbrace{D_t\gamma'}_{=0}\rangle dt + \underbrace{\langle J_i(r_p(q)), \gamma'(r_p(q))\rangle}_{=0} - \langle \underbrace{J_i(0)}_{=0}, \gamma'(0)\rangle = 0. \tag{3.11}$$

The second term vanishes because $J_i(r_p(q)) = y_i$ and $\gamma'(r_p(q)) = y_1$ are orthogonal by definition. Next, notice that J_i is orthogonal to γ' at two distinct points, so J_i is a normal vector field. Thus, the Second Variation Formula [Lee18, Proposition 10.4] tells us that

$$\begin{aligned}\frac{d^2}{ds^2}L(\Gamma_s)(0) &= -\int_0^{r_p(q)} \langle \underbrace{D_tD_tJ_i + R(J_i, \gamma')\gamma'}_{=0}, J_i\rangle dt + \langle D_tJ_i(r_p(q)), J_i(r_p(q))\rangle - \langle D_tJ_i(0), \underbrace{J_i(0)}_{=0}\rangle \\ &= \Big\langle D_tJ_i(r_p(q)), J_i(r_p(q))\Big\rangle.\end{aligned}$$

Therefore, by substituting the first and second variations back into Equation (3.10), we find

$$\frac{d^2}{dt^2}(f \circ \gamma_i)(0) = \Big\langle D_tJ_i(r_p(q)), J_i(r_p(q))\Big\rangle,$$

so by substituting the above into Equation (3.9), we obtain

$$\Delta f(q) = F''(r_p(q)) + f'(r_p(q)) \sum_{i=2}^{n} \Big\langle D_tJ_i(r_p(q)), J_i(r_p(q))\Big\rangle. \tag{3.12}$$

It remains to express

$$\Big\langle D_tJ_i(r_p(q)), J_i(r_p(q))\Big\rangle$$

in terms of $\omega_p(q)$ and $\partial_r\omega_p(q)$. Define $e_1 := \gamma'(0)$, and choose $e_2, \ldots, e_n \in T_pM$ such that $\{e_i\}_{i=1}^n$ forms an orthonormal basis. Recall that $\{y_i\}_{i=1}^n$ is an orthonormal basis at T_qM. Since the reciprocal of a non-zero determinant is equal to the determinant of the inverse, we know that

$$\frac{1}{\omega_p(q)} = \Big|\det(d\exp_p^{-1})_q\Big| = \Big\|(d\exp_p^{-1})_q(y_1) \wedge \cdots \wedge (d\exp_p^{-1})_q(y_n)\Big\|,$$

where the second equality follows from (3.7). Observe that $d(\exp_p^{-1})_q(y_1) = e_i$. Moreover, by the Gauss Lemma, for each $j = 2, \ldots, n$, the component of $d(\exp_p^{-1})_q(y_j)$ in the direction of e_1 vanishes. Hence, by computing the matrix representation of $d(\exp_p^{-1})_q$ with respect to the orthonormal bases $\{y_i\}_{i=1}^n$ and $\{e_i\}_{i=1}^n$, we find that

$$\frac{1}{\omega_p(q)} = \Big\|(d\exp_p^{-1})_q(y_2) \wedge \cdots \wedge (d\exp_p^{-1})_q(y_n)\Big\|, \tag{3.13}$$

which is the $n-1$-volume of the parallelepiped with vertices $0, (d\exp_p^{-1})_q(y_2), \ldots, (d\exp_p^{-1})_q(y_n)$, and $(d\exp_p^{-1})_q(y_2) + \cdots + (d\exp_p^{-1})_q(y_n)$ in the subspace $T_p^\perp\gamma \subseteq T_pM$.

Next, fix $t \in (0, r_p(q)]$. By Proposition 2.1.7, for each $i = 2, \ldots, n$, we have

$$J_i(t) = \frac{t}{r_p(q)} d(\exp_p)_{t\gamma'(0)}(d(\exp_p^{-1})_q(y_i)). \tag{3.14}$$

Since $d(\exp_p^{-1})_q : T_qM \to T_pM$ and $d(\exp_p)_{t\gamma'(0)} : T_pM \to T_{\gamma(t)}M$ are both linear isomorphisms, it follows that the set $\{J_2(t), \ldots, J_n(t)\}$ is linearly independent in $T_{\gamma(t)}M$. Since each J_i is a normal field, we know that $\{\gamma'(t), J_2(t), \ldots, J_n(t)\}$ is a basis for $T_{\gamma(t)}M$. Let us also construct an orthonormal basis at $T_{\gamma(t)}M$: define $v_1 := \gamma'(t)$ and choose $v_2, \ldots, v_n$ in $T_{\gamma(t)}M$ such that $\{v_i\}_{i=1}^n$ forms an orthonormal basis. Then

$$\begin{aligned}
\frac{1}{\omega_p(\gamma(t))} &= \left| \det d(\exp_p)^{-1}_{\gamma(t)} \right| \\
&= \left| \det[d(\exp_p)^{-1}_{\gamma(t)}]^{\{e_1,\ldots,e_n\}}_{\{v_1,\ldots,v_n\}} \right| \\
&= \left| \det[d(\exp_p)^{-1}_{\gamma(t)}]^{\{e_1,\ldots,e_n\}}_{\{\gamma'(t),J_2(t),\ldots,J_n(t)\}} \right| \cdot \left| \det[\mathrm{Id}_{T_{\gamma(t)}M}]^{\{\gamma'(t),J_2(t),\ldots,J_n(t)\}}_{\{v_1,\ldots,v_n\}} \right| \\
&= \frac{\left\| (d\exp_p^{-1})_{\gamma(t)}(\gamma'(t)) \wedge (d\exp_p^{-1})_{\gamma(t)}(J_2(t)) \wedge \cdots \wedge (d\exp_p^{-1})_{\gamma(t)}(J_n(t)) \right\|}{\|\gamma'(t) \wedge J_2(t) \wedge \cdots \wedge J_n(t)\|},
\end{aligned} \tag{3.15}$$

where the last equality follows from (3.6). Since each J_i is a normal vector field and γ' is unit-length, we have

$$\|\gamma'(t) \wedge J_2(t) \wedge \cdots \wedge J_n(t)\| = \|J_2(t) \wedge \cdots \wedge J_n(t)\|.$$

Furthermore, by applying an argument analogous to (3.13), we find that the numerator of (3.15) is

$$\left\| (d\exp_p^{-1})_{\gamma(t)}(J_2(t)) \wedge \cdots \wedge (d\exp_p^{-1})_{\gamma(t)}(J_n(t)) \right\|.$$

Thus, we have

$$\frac{1}{\omega_p(\gamma(t))} = \frac{\left\| (d\exp_p^{-1})_{\gamma(t)}(J_2(t)) \wedge \cdots \wedge (d\exp_p^{-1})_{\gamma(t)}(J_n(t)) \right\|}{\|J_2(t) \wedge \cdots \wedge J_n(t)\|}.$$

By (3.14), we find that for each $i = 2, \ldots, n$,

$$d(\exp_p^{-1})_{\gamma(t)}(J_i(t)) = \frac{t}{r_p(q)} d(\exp_p^{-1})_q(y_i).$$

Therefore, by the multilinearity of the determinant, we find that

$$\frac{1}{\omega_p(\gamma(t))} = \frac{t^{n-1}}{r_p(q)^{n-1}} \frac{\left\| (d\exp_p^{-1})_{\gamma(t)}(y_2) \wedge \cdots \wedge (d\exp_p^{-1})_{\gamma(t)}(y_n) \right\|}{\|J_2(t) \wedge \cdots \wedge J_n(t)\|},$$

so taking the reciprocal and applying (3.13) gives us

$$\omega_p(\gamma(t)) = \frac{r(q)^{n-1}}{t^{n-1}} \|J_2(t) \wedge \cdots \wedge J_n(t)\| \omega_p(q). \tag{3.16}$$

Next, let us compute the derivative of the map $t \mapsto \|J_2(t) \wedge \cdots \wedge J_n(t)\|$ at $r_p(q)$. First, let $E_2, \ldots, E_n$ be parallel normal vector fields along γ such that $E_i(r_p(q)) = y_i = J_i(r_p(q))$ for $i = 2, \ldots, n$. Define a map $A : (0, r_p(q)] \to \mathrm{M}(n-1, \mathbb{R})$ by

$$A(t) := \begin{pmatrix} \left\langle J_2(t), E_2(t) \right\rangle & \cdots & \left\langle J_n(t), E_2(t) \right\rangle \\ \vdots & \ddots & \vdots \\ \left\langle J_2(t), E_n(t) \right\rangle & \cdots & \left\langle J_n(t), E_n(t) \right\rangle \end{pmatrix}.$$

Notice that $A(r_p(q)) = I_{n-1}$, and each E_i being parallel implies

$$A'(t) = \begin{pmatrix} \Big\langle D_t J_2(t), E_2(t) \Big\rangle & \cdots & \Big\langle D_t J_n(t), E_2(t) \Big\rangle \\ \vdots & \ddots & \vdots \\ \Big\langle D_t J_2(t), E_n(t) \Big\rangle & \cdots & \Big\langle D_t J_n(t), E_n(t) \Big\rangle \end{pmatrix}.$$

Observe that $\det A(t) = \|J_2(t) \wedge \cdots \wedge J_n(t)\|$. Therefore, by Jacobi's formula [MN99], we find that

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=r_p(q)} \Big(t \mapsto \|J_2(t) \wedge \cdots \wedge J_n(t)\|\Big) &= \frac{d}{dt}\Big|_{t=r_p(q)} \det A(t) \\
&= \operatorname{trace}(A'(r_p(q)) \cdot \underbrace{A^{-1}(r_p(q))}_{=I_{n-1}}) \underbrace{\det A(r_p(q))}_{=1} \\
&= \sum_{i=2}^{n} \Big\langle D_t J_i(r_p(q)), E_i(r_p(q)) \Big\rangle \\
&= \sum_{i=2}^{n} \Big\langle D_t J_i(r_p(q)), J_i(r_p(q)) \Big\rangle. \qquad (3.17)
\end{aligned}$$

By the product rule, the derivative of (3.16) is given by

$$\begin{aligned}
\frac{d}{dt}(\omega_p \circ \gamma)(t) &= -(n-1)\frac{r_p(q)^{n-1}}{t^n}\|J_2(t) \wedge \cdots \wedge J_n(t)\|\omega_p(q) \\
&\quad + \frac{r_p(q)^{n-1}}{t^{n-1}} \frac{d}{dt}\Big|_t \Big(t \mapsto \|J_2(t) \wedge \cdots \wedge J_n(t)\|\Big)\omega_p(q).
\end{aligned}$$

Evaluating at $t = r_p(q)$ gives us

$$\frac{d}{dt}(\omega_p \circ \gamma)(r_p(q)) = -\frac{n-1}{r_p(q)}\omega_p(q) + \sum_{i=2}^{n} \Big\langle D_t J_2(r_p(q)), J_2(r_p(q)) \Big\rangle \omega_p(q),$$

which can be rearranged to become

$$\sum_{i=2}^{n} \Big\langle D_t J_2(r_p(q)), J_2(r_p(q)) \Big\rangle = \frac{n-1}{r_p(q)} + \frac{\frac{d}{dt}(\omega_p \circ \gamma)(r_p(q))}{\omega_p(q)}.$$

By Lemma 3.1.4, we know that

$$\frac{d}{dt}(\omega_p \circ \gamma)(r_p(q)) = \gamma'(r(q))\omega_p = \partial_r \omega_p(q).$$

Therefore, by substituting the above into (3.12), we finally obtain

$$\Delta f(q) = F''(r_p(q)) + F'(r_p(q)) \left(\frac{n-1}{r_p(q)} + \frac{\partial_r \omega_p(q)}{\omega_p(q)} \right),$$

as desired. □

4 Local Harmonicity

4.1 Definition of Local Harmonicity and Equivalences

In this section we define what it means for a Riemannian manifold to be *locally harmonic at a point* in Definition 4.1.1, then prove equivalent formulations in Theorem 4.1.3.

Definition 4.1.1 (Local Harmonicity)**.** We say that M is *locally harmonic at* $p \in M$ if there exists $0 < \varepsilon < \operatorname{Inj}(p)$ such that $\omega_p|_{\widehat{B}_\varepsilon(p)}$ is radially symmetric. If M is locally harmonic at every point, then we say that M is *locally harmonic*, and we call M a *LH-manifold*.

As seen in Example 2.3.2, the density function in spaces of constant sectional curvature can be computed explicitly. We now use these computations to show that spaces of constant sectional curvature are locally harmonic.

Example 4.1.2 (Spaces of Constant Sectional Curvature are Locally Harmonic)**.** Suppose M has constant sectional curvature $c \in \mathbb{R}$. Fix $p \in M$, and let $0 < \varepsilon < \mathrm{Inj}(p)$. Define $\Omega : (0, \varepsilon) \to \mathbb{R}$ by

$$\Omega(r) := \frac{s_c(r)^{n-1}}{r^{n-1}}.$$

Fix any $q \in \widehat{B}_\varepsilon(p)$, and let $v := \exp_p^{-1}(q)$. By Example 2.3.2, we know that

$$\omega_p(q) = \omega(p, v) = \frac{s_c(\|v\|)^{n-1}}{\|v\|^{n-1}} = \Omega(\|v\|) = \Omega(r_p(q)),$$

so M is locally harmonic.

In order to simplify notation, let us make the following definition. First, fix a point $p \in M$, and let $0 < \varepsilon < \mathrm{Inj}(p)$. We define the map $h_p : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ by

$$h_p(q) := \frac{n-1}{r_p(q)} + \frac{\partial_r \omega_p(q)}{\omega_p(q)}. \tag{4.1}$$

We know that h_p is a smooth function, since ω_p and r_p are smooth. By Proposition 3.4.4, if $F : (0, \varepsilon) \to \mathbb{R}$ is a smooth function, the Laplacian of the function $f := F \circ r_p : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ is given by

$$\Delta f(q) = F''(r_p(q)) + F'(r_p(q)) h_p(q), \qquad \forall q \in \widehat{B}_\varepsilon(p). \tag{4.2}$$

The following proof is adapted from [Kre10].

Theorem 4.1.3 (Equivalences for Local Harmonicity at a Point)**.** *Fix $p \in M$. Then the following are equivalent:*

(i) M is locally harmonic at p.

(ii) There exists $0 < \varepsilon < \mathrm{Inj}(p)$ such that $h_p : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ is radially symmetric.

(iii) There exists $0 < \varepsilon < \mathrm{Inj}(p)$ and a non-constant, radially symmetric smooth function $f : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ satisfying $\Delta f \equiv 0$.

Proof of (i) ⇒ (ii). Suppose M is locally harmonic at p. By definition, this means that there exists $0 < \varepsilon < \mathrm{Inj}(p)$ and a function $\Omega : (0, \varepsilon) \to \mathbb{R}$ such that $\omega_p(q) = \Omega(r_p(q))$ for every $q \in \widehat{B}_\varepsilon(p)$. By Lemma 3.1.5, this means that the radial derivative $\partial_r \omega_p|_{\widehat{B}_\varepsilon(p)}$ is also radially symmetric, and $\partial_r \omega_p(q) = \Omega'(r_p(q))$ for every $q \in \widehat{B}_\varepsilon(p)$. Let us define $H : (0, \varepsilon) \to \mathbb{R}$ by

$$H(r) := \frac{n-1}{r} + \frac{\Omega'(r)}{\Omega(r)}.$$

Then we find

$$h_p(q) := \frac{n-1}{r_p(q)} + \frac{\Omega'(r_p(q))}{\Omega(r_p(q))} = H(r_p(q)) \qquad \forall q \in \widehat{B}_\varepsilon(p),$$

so $h_p : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ is radially symmetric. □

Proof of (ii) ⇒ (i). Suppose that there exists $0 < \varepsilon < \mathrm{Inj}(p)$ and a function $H : (0, \varepsilon) \to \mathbb{R}$ such that $h_p(q) = H(r_p(q))$ for every $q \in \widehat{B}_\varepsilon(p)$. We wish to show that $\omega_p|_{\widehat{B}_\varepsilon(p)}$ is radially symmetric. First, choose $G : (0, \varepsilon) \to \mathbb{R}$ to be a *fixed* anti-derivative of the map

$$r \mapsto H(r) - \frac{n-1}{r}.$$

Fix a unit vector $v \in S_1(0_p)$, and consider the initial value problem

$$y'(r) + \left(\frac{n-1}{r} - H(r)\right) y(r) = 0, \qquad y\left(\frac{\varepsilon}{2}\right) = \omega_p\left(\exp_p\left(\frac{\varepsilon}{2}v\right)\right). \tag{4.3}$$

By the existence and uniqueness theorem for linear ODEs [Lee18, Theorem 4.31], we know that there exists a unique solution to (4.3). Solving the ODE tells us that the solution $y_v : (0, \varepsilon) \to \mathbb{R}$ is given by

$$y_v(r) = C(v)e^{G(r)},$$

where $C(v) \in \mathbb{R}$ is constant. Next, let $\gamma_v : (-\varepsilon, \varepsilon) \to M$ be the unit-speed geodesic defined by $r \mapsto \exp_p(rv)$, and define the map $\Omega_v : (0, \varepsilon) \to \mathbb{R}$ by

$$\Omega_v(r) := \omega_p(\gamma_v(r)).$$

Let us show that Ω_v is also a solution to (4.3). By Lemma 3.1.4, we find

$$\Omega_v'(r) = \frac{d}{dt}(\omega_p \circ \gamma_v)(r) = \gamma_v'(r)\omega_p = \partial_r \omega_p(\gamma_v(r)).$$

Fix $r \in (0, \varepsilon)$. Since γ_v is unit-speed, we know that $r_p(\gamma_v(r)) = r$. Therefore, by Equation (4.1), we find

$$H(r) = H(r_p(\gamma_v(r))) = h_p(\gamma_v(r)) = \frac{n-1}{r_p(\gamma_v(r))} + \frac{\partial_r \omega_p(\gamma_v(r))}{\omega_p(\gamma_v(r))} = \frac{n-1}{r} + \frac{\Omega_v'(r)}{\Omega_v(r)}.$$

Rearranging gives us

$$\Omega_v'(r) + \left(\frac{n-1}{r} - H(r)\right)\Omega_v(r) = 0,$$

so Ω_v is also a solution to (4.3), since the initial condition is satisfied. Thus, the uniqueness of the solution to (4.3) implies that

$$\omega_p(\gamma_v(r)) = \Omega_v(r) = y_v(r) = C(v)e^{G(r)} \qquad \forall r \in (0, \varepsilon).$$

Since ω_p is continuous and $\omega_p(\gamma_v(0)) = 1$, we find that

$$1 = \lim_{r \to 0} \omega_p(\gamma_v(r)) = \lim_{r \to 0} C(v)e^{G(r)} = C(v) \lim_{r \to 0} e^{G(r)}.$$

By rearranging, this shows that $C(v)$ is independent of our choice of v, since the anti-derivative G is fixed. Thus, Ω_v is also independent of v, which means we have a function $\Omega : (0, \varepsilon) \to \mathbb{R}$ such that $\Omega \equiv \Omega_v$ for any $v \in S_1(0_p)$. Finally, fix $q \in \widehat{B}_\varepsilon(p)$ and let $v := \exp_p^{-1}(q)/r_p(q)$. Then v is a unit vector in T_pM, so

$$\omega_p(q) = \omega_p(\gamma_v(r_p(q))) = \Omega(r_p(q)),$$

which implies that ω_p is radially symmetric. $\square$

Proof of (ii) $\Rightarrow$ (iii). Suppose there exists $0 < \varepsilon < \mathrm{Inj}(p)$ and a function $H : (0, \varepsilon) \to \mathbb{R}$ such that $h_p = H \circ r_p : \widehat{B}_\varepsilon(p) \to \mathbb{R}$. Consider the initial value problem

$$y''(r) + H(r)y'(r) = 0, \qquad y\left(\frac{\varepsilon}{2}\right) = 0, \quad y'\left(\frac{\varepsilon}{2}\right) = 1. \tag{4.4}$$

By the existence and uniqueness theorem for linear ODEs [Lee18, Theorem 4.31], we know that there exists a unique smooth solution $F : (0, \varepsilon) \to \mathbb{R}$ to (4.4). Since F has non-vanishing derivative, it is non-constant. Define $f := F \circ r_p : \widehat{B}_\varepsilon(p) \to \mathbb{R}$. Then f is also smooth and non-constant. Furthermore, by Equation (4.2), we find that for any $q \in \widehat{B}_\varepsilon(p)$,

$$\Delta f(q) = F''(r_p(q)) + h_p(q)F'(r_p(q)) = F''(r_p(q)) + H(r_p(q))F'(r_p(q)) = 0.$$

Therefore, f is a smooth, non-constant, radially symmetric solution to Laplace's equation, as we wished. $\square$

Proof of (iii) ⇒ (ii). Suppose there exists $0 < \varepsilon < \mathrm{Inj}(p)$ and non-constant smooth function $F : (0, \varepsilon) \to \mathbb{R}$ such that $f := F \circ r_p : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ satisfies $\Delta f \equiv 0$. By Equation (4.2), we know that for any $q \in \widehat{B}_\varepsilon(p)$,

$$0 = \Delta f(q) = F''(r_p(q)) + h_p(q)F'(r_p(q)), \tag{4.5}$$

It remains to show that F' is non-vanishing. By applying the mean value operator to Equation (4.5), we find that for any $r \in (0, \varepsilon)$,

$$\begin{aligned} 0 = \mathrm{MV}_p(0, r) &= \mathrm{MV}_p(F'' \circ r_p + (F' \circ r_p)h_p, r) \\ &= \mathrm{MV}_p(F'' \circ r_p, r) + \mathrm{MV}_p((F' \circ r_p)h_p, r) \\ &= F''(r) + \mathrm{MV}_p(h_p, r)F'(r). \end{aligned}$$

Now, for the sake of contradiction, suppose that there exists some $r_0 \in (0, \varepsilon)$ such that $F'(r_0) = 0$. Consider the initial value problem

$$y''(r) + \mathrm{MV}_p(h_p, r)y'(r) = 0, \qquad y(r_0) = F(r_0), \quad y'(r_0) = 0. \tag{4.6}$$

Notice that both $F : (0, \varepsilon) \to \mathbb{R}$ and the constant function $r \mapsto F(r_0)$ are solutions to (4.6). By uniqueness of solutions to linear ODEs, we must have that $F \equiv F(r_0)$. However, we assumed that F is non-constant. Thus, we have reached a contradiction, and F' does not vanish.

Thus, by rearranging Equation (4.5), we find that

$$h_p(q) = -\frac{F''(r_p(q))}{F'(r_p(q))} \qquad \forall q \in \widehat{B}_\varepsilon(p),$$

so h_p is radially symmetric, as desired. □

4.2 The Mean Value Property

In this section, we prove that the mean value property and its converse hold in locally harmonic spaces. The proofs are adapted from [Wil93].

Theorem 4.2.1 (Mean Value Property)**.** *Suppose M is locally harmonic at $p \in M$, and U is a neighbourhood of p. Let $f : U \to \mathbb{R}$ be a smooth function satisfying $\Delta f \equiv 0$. Then there exists $\varepsilon > 0$ with $B_\varepsilon(p) \subseteq U$ such that for each $r \in (0, \varepsilon)$,*

$$f(p) = \mathrm{MV}_p(f, r).$$

Proof. Since M is locally harmonic at $p \in M$, we know that there exists $0 < \varepsilon_1 < \mathrm{Inj}(p)$ and a function $\Omega : (0, \varepsilon_1) \to \mathbb{R}$ such that $\omega_p(q) = \Omega(r_p(q))$ for each $q \in \widehat{B}_{\varepsilon_1}(p)$. Since U is open, there exists $\varepsilon_2 > 0$ such that $B_{\varepsilon_2}(p) \subseteq U$. Define $\varepsilon := \min(\varepsilon_1, \varepsilon_2)$.

Let us first show that for each $r \in (0, \varepsilon)$,

$$\frac{d}{dr}(\mathrm{MV}_p(f, \cdot))(r) = \mathrm{MV}_p(\partial_r f, r), \tag{4.7}$$

where $\frac{d}{dr}$ is the standard derivative operator on $C^\infty(\mathbb{R})$. Fix $r \in (0, \varepsilon)$. Then by Corollary 3.2.2 and (3.1), we find

$$\begin{aligned} \mathrm{MV}_p(f, r) &= \frac{1}{\int_{S_r(p)} dS_r} \int_{S_r(p)} f \, dS_r \\ &= \frac{1}{\int_{S_r(0_p)} \omega_p(\exp_p(v)) d\widetilde{S}_r(v)} \int_{S_r(0_p)} f(\exp_p(v))\omega_p(\exp_p(v)) \, d\widetilde{S}_r(v) \\ &= \frac{1}{\int_{S_1(0_p)} r^{n-1}\omega_p(\exp_p(rv)) d\widetilde{S}_1(v)} \int_{S_1(0_p)} f(\exp_p(rv)) r^{n-1}\omega_p(\exp_p(rv)) \, d\widetilde{S}_1(v) \\ &= \frac{1}{\int_{S_1(0_p)} \cancel{r^{n-1}\Omega(r)} d\widetilde{S}_1(v)} \int_{S_1(0_p)} f(\exp_p(rv)) \cancel{r^{n-1}\Omega(r)} \, d\widetilde{S}_1(v) \\ &= \frac{1}{\mathrm{Vol}(S_1(0_p))} \int_{S_1(0_p)} f(\exp_p(rv)) \, d\widetilde{S}_1(v). \end{aligned}$$

Therefore, we find that for any $r \in (0, \varepsilon)$,

$$\begin{aligned}\frac{d}{dr}(\mathrm{MV}_p(f, \cdot))(r) &= \frac{1}{\mathrm{Vol}(S_1(0_p))} \int_{S_1(0_p)} \frac{d}{dr}(r \mapsto f(\exp_p(rv)))\, d\widetilde{S}_1(v) \\ &= \frac{1}{\mathrm{Vol}(S_1(0_p))} \int_{S_1(0_p)} \partial_r f(\exp_p(rv))\, d\widetilde{S}_1(v) \\ &= \mathrm{MV}_p(\partial_r f, r),\end{aligned}$$

where we have used the Leibniz integral rule [Che06] in the first equality, and Lemma 3.1.4 in the second equality. Thus, we have shown (4.7). Note that (4.7) holds even when f does not satisfy Laplace's equation.

Next, let us show that $\frac{d}{dr}(\mathrm{MV}_p(f, \cdot)) \equiv 0$. We know that ∂_r is the outward unit normal vector field on each geodesic sphere. Therefore, by Green's first identity [Lee18, Exercise 2-23], we find that for each $r \in (0, \varepsilon)$,

$$\begin{aligned}\frac{d}{dr}(\mathrm{MV}_p(f, \cdot))(r) &= \mathrm{MV}_p(\partial_r f, r) \\ &= \frac{1}{\mathrm{Vol}(S_r(p))} \int_{S_r(p)} \partial_r f \; dS_r \\ &= \frac{1}{\mathrm{Vol}(S_r(p))} \int_{B_\varepsilon(p)} \underbrace{\Delta f}_{=0} \; dV = 0.\end{aligned}$$

Thus, there exists a constant $C(f) \in \mathbb{R}$ such that $\mathrm{MV}_p(f, \cdot) \equiv C(f)$. By Lebesgue's Differentiation Theorem [Rud87], we know that the limit of $\mathrm{MV}_p(f, t)$ as $t \to 0^+$ exists and is equal to $f(p)$. Therefore, for each $r \in (0, \varepsilon)$, we find

$$\mathrm{MV}_p(f, r) = C(f) = \lim_{t \to 0^+} C(f) = \lim_{t \to 0^+} \mathrm{MV}_p(f, t) = f(p),$$

as we wished. □

Theorem 4.2.2 (Converse of Mean Value Property). *Suppose M is locally harmonic (at every point). Let $U \subseteq M$ be open, and let $f : U \to \mathbb{R}$ be a smooth function. Suppose that for each $p \in M$, there exists $\varepsilon > 0$ with $B_\varepsilon(p) \subseteq U$ such that for every $r \in (0, \varepsilon)$, we have*

$$f(p) = \mathrm{MV}_p(f, r).$$

Then $\Delta f \equiv 0$.

Proof. For the sake of contradiction, suppose that there exists a point $p \in M$ such that $\Delta f(p) \neq 0$. Without loss of generality, suppose that $\Delta f(p) > 0$. Since $\Delta f : U \to \mathbb{R}$ is a continuous function, we know that there must be some $\varepsilon_1 > 0$ with $B_{\varepsilon_1}(p) \subseteq U$ such that Δf is positive on $B_{\varepsilon_1}(p)$.

By assumption, there exists some $\varepsilon_2 > 0$ with $B_{\varepsilon_2}(p) \subseteq U$ such that for each $r \in (0, \varepsilon_2)$, we have $f(p) = \mathrm{MV}_p(f, r)$. Define $\varepsilon := \min(\varepsilon_1, \varepsilon_2)$. Then for each $r \in (0, \varepsilon)$, we have

$$\begin{aligned}0 &= \frac{d}{dr}(\mathrm{MV}_p(f, \cdot))(r) && \text{by (4.7)} \\ &= \mathrm{MV}_p(\partial_r f, r) && \text{by Green's first identity} \\ &= \frac{1}{\mathrm{Vol}(S_r(p))} \int_{B_\varepsilon(p)} \Delta f \; dV > 0.\end{aligned}$$

Thus, we have reached a contradiction, so $\Delta f \equiv 0$. □

4.3 Two Point Homogeneous Spaces are Locally Harmonic

Definition 4.3.1. We say that M is *two point homogeneous* if for every $p_1, p_2, q_1, q_2 \in M$ satisfying $d(p_1, p_2) = d(q_1, q_2)$, there exists an isometry $\varphi : M \to M$ such that $\varphi(p_1) = q_1$ and $\varphi(p_2) = q_2$.

We say that M is *homogeneous* if for any two points $p, q \in M$, there exists an isometry on M taking p to q. By letting $p_1 = p_2$ and $q_1 = q_2$ in the definition above, we see that every two point homogeneous space is also homogeneous, as we would expect. In particular, this implies that two point homogeneous spaces are complete.

Two point homogeneous spaces are fully classified, as the following theorem from [Wol11, Corollary 8.12.9] shows.

Theorem 4.3.2 (Classification of Two Point Homogeneous Spaces)**.** *The connected Riemannian manifold M is two point homogeneous if and only if M is isometric to one of the following: a Euclidean space; a sphere; a real, complex, or quaternionic projective space; a real, complex, or quaternionic hyperbolic space; the Caley projective plane; or the Caley hyperbolic plane.*

We now show that two point homogeneous spaces are locally harmonic.

Theorem 4.3.3. *Suppose M is two point homogeneous. Then M is locally harmonic.*

Proof. Fix $p \in M$, and choose any $0 < \varepsilon < \mathrm{Inj}(p)$. We want to show that $\omega_p|_{\widehat{B}_\varepsilon(p)} : \widehat{B}_\varepsilon(p) \to \mathbb{R}$ is radially symmetric.

Fix $r \in (0, \varepsilon)$. We want to show that ω_p is constant on $S_r(p)$. To this end, fix two arbitrary points $q_1, q_2 \in S_r(p)$. Then we know that $d(p, q_1) = d(p, q_2)$, so by two point homogeneity, there exists an isometry $\varphi : M \to M$ which fixes p and sends q_1 to q_2. By Proposition 2.4.1, we know that the density function is invariant under φ:

$$\omega_p(q_1) = \omega_{\varphi(p)}(\varphi(q_1)) = \omega_p(q_2).$$

This shows that ω_p is constant on each geodesic sphere.

Next, let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}$ be any unit-speed geodesic with $\gamma(0) = p$. Define $\Omega : (0, \varepsilon) \to \mathbb{R}$ by $\Omega(r) := \omega_p(\gamma(r))$. Fix any $q \in \widehat{B}_\varepsilon(p)$. We know that $q \in S_{r_p(q)}(p)$, and since γ is unit-speed, we know that $\gamma(r_p(q)) \in S_{r_p(q)}(p)$. Therefore,

$$\Omega(r_p(q)) = \omega_p(\gamma(r_p(q))) = \omega_p(q),$$

so ω_p is radially symmetric. Since our choice of p was arbitrary, we have shown that M is locally harmonic. ☐

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