Refutation of Global Regularity

A formal note on the claims of David Budden

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Abstract

This note refutes the claimed proofs of global existence, uniqueness, and smoothness for the 3D incompressible Navier-Stokes equations presented in two companion preprints by David Budden (December 2025), one on the torus $\mathbb{T}^3$ and the other in $\mathbb{R}^3$ with Schwartz-class initial data. Both proofs rely on a spectral gap framework asserting that palinstrophy dominates vortex stretching with constant $A > 1$ ($A \approx 1.067$ on $\mathbb{T}^3$ via Gaussian localization and $A = \pi^2/8$ in $\mathbb{R}^3$ via Voronoi decomposition), implying global regularity via energy depletion and contradiction with the Beale-Kato-Majda criterion.

For the $\mathbb{T}^3$ case, multiple fatal flaws are identified: (i) the conditional regularity criterion is insufficient, as $A > 1$ (but $A < 2$) provides only positive feedback in enstrophy growth, failing to rigorously bound $\int \Omega \, dt < \infty$; (ii) the 3D Faber-Krahn constant is misstated ($\pi^2/4$ instead of π^2); (iii) fixed-fraction enstrophy concentration in the diffusion-scale Gaussian lacks justification, as potential singularities may exhibit sharper profiles.

For the $\mathbb{R}^3$ case, the key coarse exclusion step relies on a purported Sobolev-Morrey embedding bounding L^∞ norms on convex domains via diameter-scaled H^1 terms with universal constants; this embedding is proved false via a counterexample of regularized logarithmic singularities on the unit ball, where H^1 norms remain bounded but L^∞ diverges, breaking the fine-component analysis and preventing universal spectral gap establishment.

These errors invalidate both proofs.

Refutation of Global Regularity on $\mathbb{T}^3$

The preprint by David Budden claims to establish global existence, uniqueness, and smoothness of solutions to the 3D incompressible Navier-Stokes equations in R^3 for divergence-free Schwartz-class initial data (Theorem 1.1). The proof relies on a spectral gap framework, where global regularity follows if the spectral gap condition holds with constant $A = \pi^2/8 > 1$ for sufficiently large maximum vorticity Ω (Theorems 1.4 and 1.5).

The proof is invalid due to reliance on a nonexistent embedding theorem. This flaw occurs in the coarse exclusion argument (Theorem 6.2), preventing establishment of the spectral gap in all cases, and thus invalidating the global regularity claim.

The Flawed Assertion

The proof invokes the following inequality (Theorem 2.8, cited as a "Sobolev-Morrey embedding"):

For any convex domain $\Omega \subset \mathbb{R}^3$ with diameter $d > 0$ and any $f \in H^1(\Omega)$,

$$\|f\|_{L^\infty(\Omega)} \leq C_1 d^{-3/2} \|f\|_{L^2(\Omega)} + C_2 d^{1/2} \|\nabla f\|_{L^2(\Omega)},$$

where $C_1, C_2 > 0$ are universal constants (independent of Ω and d).

This inequality is false. No such universal constants exist.

Counterexample

Consider a sequence of functions on the unit ball $B = B(0,1) \subset \mathbb{R}^3$, which is convex with diameter $d = 2$.

For each $k \in \mathbb{N}$, define

$$f_k(x) = \log\left(1 + \frac{k}{|x| + 1/k}\right), \quad x \in B.$$

(This regularizes the singular function $\log(1/|x|)$ while preserving essential properties.)

- Each f_k is smooth and thus in $H^1(B)$.
- As $k \to \infty$, $f_k(x) \to \log(1/|x|)$ pointwise for $x \neq 0$, and the limiting function (in the Sobolev sense) satisfies $\text{ess-sup}_B |\log(1/|x|)| = +\infty$.

Now compute the norms (up to universal factors independent of k):

- $\|\nabla f_k\|^2_{L^2(B)} \lesssim \int_B \frac{1}{(|x|+1/k)^2}\, dx \leq \int_B \frac{1}{|x|^2}\, dx < \infty$, with the integral bounded uniformly in k (converges to the finite value for $1/|x|^2$).
- $\|f_k\|^2_{L^2(B)} \lesssim \int_B (\log(1 + k/|x|))^2\, dx < \infty$, similarly bounded uniformly in k (converges to the finite integral for $(\log(1/|x|))^2$).

Thus, there exists $M < \infty$ (independent of k) such that

$$\|f_k\|_{L^2(B)} + \|\nabla f_k\|_{L^2(B)} \leq M \quad \forall k.$$

The right-hand side of the claimed inequality is at most $(C_1 \cdot 2^{-3/2} + C_2 \cdot 2^{1/2})M$, bounded independently of k.

However,

$$\|f_k\|_{L^\infty(B)} \geq f_k(0) = \log(1 + k) \to +\infty \quad \text{as } k \to \infty.$$

This contradicts the existence of universal C_1, C_2 bounding the left-hand side.

Remark. The actual Sobolev embeddings in $\mathbb{R}^3$ yield $H^1 \hookrightarrow L^p$ for all $p \leq 6$ (on bounded domains), but **not** L^∞. Morrey's inequality requires $W^{1,p}$ with $p > 3$ for boundedness/Holder continuity. No combination of L^2 and gradient L^2 terms yields L^∞ control in 3D.

Impact on the Proof

The invalid inequality is applied in **Case B (fine component carries the L^∞ norm)** of Theorem 6.2, on a single Voronoi cell V^* (convex, diameter $\leq 2R$) to bound the oscillation $|f(x_0)| > \Omega/2$, where $f = \omega - \bar{\omega}$ (zero mean on V^*, though the claimed theorem does not even require this).

- **Subcase B1** (L^2 term dominates): Falls back to bounded Ω (conclusion A), which is valid independently.
- **Subcase B2** (gradient term dominates): Relies on the flawed bound to derive $P \gtrsim \Omega^{5/2}/\sqrt{\nu}$ (conclusion B).

Without a valid L^∞ bound on $f|_{V^*}$, Subcase B2 fails. It is possible to construct vorticity fields where global enstrophy E is moderate, the configuration is "coarse" (Ecoarse $\geq E/2$), yet high Ω arises from fine-scale oscillations in one cell without forcing superlinear global palinstrophy P.

Consequently: - Coarse exclusion (Theorem 6.2) is incomplete for arbitrarily large Ω. - The spectral gap $P \geq (\pi^2/8)(\Omega/\nu)E$ does not hold universally for large Ω (Theorem 1.5 fails). - The energy depletion/contradiction argument (Theorem 1.4) cannot rule out finite-time blowup.

Conclusion

The preprint's proof contains a critical error in harmonic analysis: reliance on a nonexistent embedding from H^1 to L^∞ in 3D. This breaks the key technical step, leaving the spectral gap unestablished in the problematic regime. The claimed resolution of global regularity for the 3D Navier-Stokes equations in $\mathbb{R}^3$ is therefore invalid.

The spectral gap framework and Voronoi decomposition require alternative local controls (e.g., higher-order norms or different inequalities).

Refutation of Global Regularity in $\mathbb{R}^3$

The preprint by David Budden (December 2025) claims to establish global existence, uniqueness, and smoothness of smooth solutions to the 3D incompressible Navier-Stokes equations on the torus $\mathbb{T}^3$ with smooth, divergence-free, mean-zero initial data. The proof relies on a "spectral gap framework" in the vorticity formulation, asserting that a spectral gap condition with constant $A \approx 1.067 > 1$ implies global regularity, and then claiming to prove this condition holds on $\mathbb{T}^3$ via Gaussian localization and the Faber-Krahn inequality.

The proof contains fatal errors in logic and mathematics. These invalidate both the conditional regularity theorem (claimed for $A > 1$) and the establishment of the spectral gap on $\mathbb{T}^3$. Below, the primary flaws are detailed.

1. Fundamental Logical Error in the Spectral Gap Framework (Theorems 1.3 and 4.2)

The core of the conditional regularity result (Theorem 1.3) is that if a smooth solution satisfies the spectral gap

$$P(t) \geq A \cdot \frac{\Omega(t)}{\nu} \cdot E(t)$$

with $A > 1$, then no finite-time blowup occurs.

From the enstrophy identity (correctly derived as $\frac{dE}{dt} = S - \nu P$) and the stretching bound $|S| \leq 2\Omega E$ (also correct, via Holder and the L^2-isometry $\|\nabla u\|_{L^2} = \|\omega\|_{L^2}$), one obtains

$$\frac{dE}{dt} \leq 2\Omega E - \nu P.$$

Assuming the spectral gap,

$$\frac{dE}{dt} \leq 2\Omega E - A\Omega E = (2 - A)\Omega E.$$

To prevent blowup, this feedback must be sufficiently negative to bound $\int_0^{T^*} \Omega(t)\, dt < \infty$ (contradicting the Beale-Kato-Majda criterion when assuming $T^* < \infty$). However, if $1 < A < 2$, then $2 - A > 0$, and

$$\frac{dE}{dt} \leq (2 - A)\Omega E,$$

which is a **positive** feedback term. This merely slows potential enstrophy growth but does not force decay or bound the integral of Ω.

The proof of Theorem 4.2 attempts to derive a contradiction assuming $T^* < \infty$ (implying $\int_0^{T^*} \Omega\, dt = \infty$) by claiming $E(t) \to \infty$ as $t \to T^*$, contradicting the bounded dissipation integral $\int_0^{T^*} E\, dt \leq E(0)/(2\nu) < \infty$.

This claim relies on

$$\frac{d}{dt} \log E \leq (2 - A)\Omega,$$

so

$$\log \frac{E(T)}{E(0)} \leq (2 - A) \int_0^T \Omega\, dt.$$

With $2 - A > 0$ (as in the paper, $A \approx 1.067$ gives $2 - A \approx 0.933 > 0$), infinite integral implies $E(T) \to \infty$. However, this does **not** rigorously contradict the bounded $\int E\, dt$, because sub-exponential growth (effective rate constant < 1) can yield $E \to \infty$ while keeping $\int E\, dt < \infty$.

For example, hypothetical blowup profiles with $\Omega \sim (T^* - t)^{-\alpha}$ for suitable $\alpha > 1$ could produce super-linear Ω growth without diverging the enstrophy integral over finite time. The crude linear-in-E stretching bound permits such scenarios when the palinstrophy domination is only marginal ($A < 2$).

In contrast, negative feedback ($A > 2$) would yield $\frac{dE}{dt} \leq -c\Omega E$ for $c > 0$, forcing exponential decay of E and bounding $\int \Omega \, dt < \infty$ directly via Gronwall-type arguments.

Thus, the spectral gap with $A > 1$ (but $A < 2$) is **insufficient** for global regularity. The proof of Theorem 1.3 (and hence Theorem 4.2) fails logically for the range of A claimed.

2. Incorrect Faber-Krahn Constant and Misapplication (Section 5)

Even if the framework were repaired to require $A > 2$, the establishment of the spectral gap (Theorem 1.4) is invalid.

The paper states the 3D Faber-Krahn constant as $c_{\mathrm{FK}} = \pi^2/4 \approx 2.467$, claiming $\lambda_1(\Omega) \geq c_{\mathrm{FK}}/r^2$ for domain Ω with inradius comparable to ball of radius r.

This is erroneous. The classical 3D Faber-Krahn inequality for the Dirichlet Laplacian states

$$\lambda_1(\Omega) \geq \left(\frac{\pi}{r}\right)^2 \approx 9.869/r^2,$$

where r is the radius of the ball of equal volume (explicit ground state on the ball: radial solution vanishing at boundary, first zero at π). The factor $\pi^2/4$ appears in lower-dimensional or different normalizations (e.g., 1D interval), but not in 3D.

Moreover, applying Faber-Krahn to improve the sharp Ornstein-Uhlenbeck Poincare constant (σ^2) on a **soft Gaussian-weighted** domain is unjustified. Faber-Krahn requires hard Dirichlet boundaries and zero extension; the Gaussian weight allows tails, and the vorticity does not vanish outside the "effective support." Subtracting the local mean (as sketched) does not rigorously yield the claimed improvement without additional error terms that break the constant.

3. Unsubstantiated Energy Concentration (Theorem 6.2)

The claim that a fixed fraction ($\kappa/2 \approx 43\%$) of enstrophy concentrates in a Gaussian of width $\sigma = \sqrt{\nu/\Omega}$ around the maximum vorticity point lacks proof. While continuity ensures $|\omega| \gtrsim \Omega/2$ in some small neighborhood, there is no a priori parabolic regularity forcing this neighborhood to have radius $\gtrsim \sigma$.

In potential singularity scenarios (e.g., vortex filaments with thickness $o(\sigma)$), the L^2-mass could concentrate on sets of measure $o(\sigma^3)$, capturing only $o(1)$ fraction in the fixed Gaussian. No quantitative lower bound independent of the solution profile is provided.

Conclusion

The proof does not establish global regularity for the 3D Navier-Stokes equations on $\mathbb{T}^3$. The primary flaw is the insufficient spectral gap threshold ($A > 1$ instead of $A > 2$), rendering the conditional regularity argument invalid. Secondary errors in constants and localization further undermine the proof.
