Course 4: Combinatorial Game Theory

IMT Atlantique
Bretagne-Pays de la Loire
École Mines-Télécom

Summary

Last session

1. Unsupervised learning - discover structure from unlabeled data
2. Clustering
3. Decomposition - sparse dictionary learning

Today's session

- Combinatorial Game Theory

Examples

	perfect information	imperfect information
sequential		
concurrent		

Before we proceed...

- Game theory is a very rich and broad scientific domain,
- In economics, we are interested in equilibriums and payoff games,
- In mathematics, we are interested in showing existence of objects,
- And many others...
- We focus in this course on very particular games (sequential with perfect information) under the scope of combinatorial game theory.

  Following the literature, we consider two players called Eve and Adam.

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Arena

Graph

- A graph G is a pair $\langle V, E \rangle$ where V is the finite set of vertices and $E \subseteq V \times V$ is the set of edges,
- An **arena** is a triple $\langle G, V_E, V_A \rangle$ where V_E, V_A is a bipartition of V,
- We suppose each vertex in V is associated with at least one outgoing edge.
- We will also use the term "state" when referring to vertices of V.

Beware not to confuse an arena with a bipartite graph.

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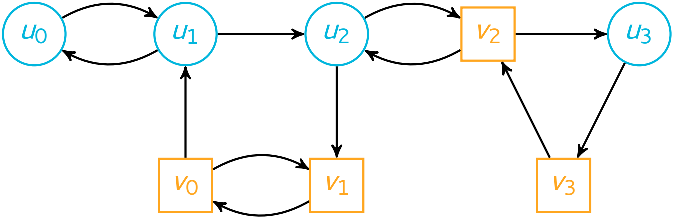

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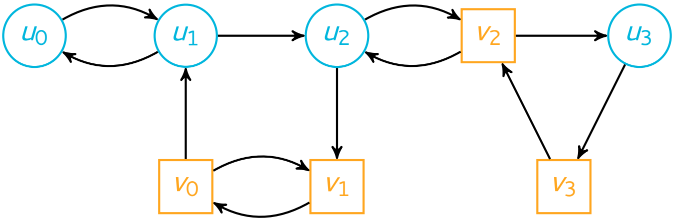

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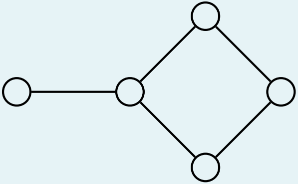
Example game

Cops and robbers

- Consider the following graph:



- First, the cop chooses a vertex to start at,
- Then, the robber chooses a vertex to start at,
- Then, alternatively each player chooses one neighbor vertex of its current vertex to go to,
- The cop wins if and only if at some turn he and the robber are at the same vertex.

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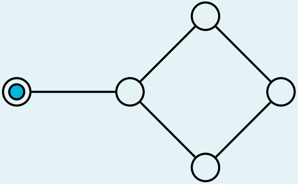

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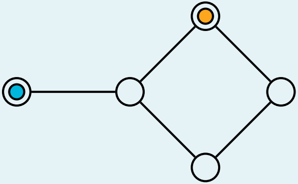

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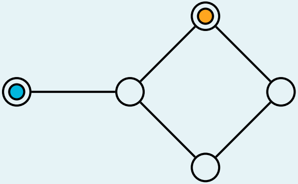

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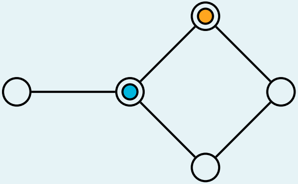

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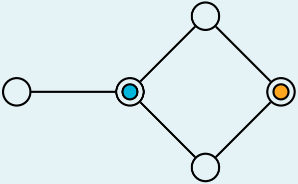

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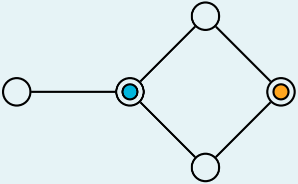

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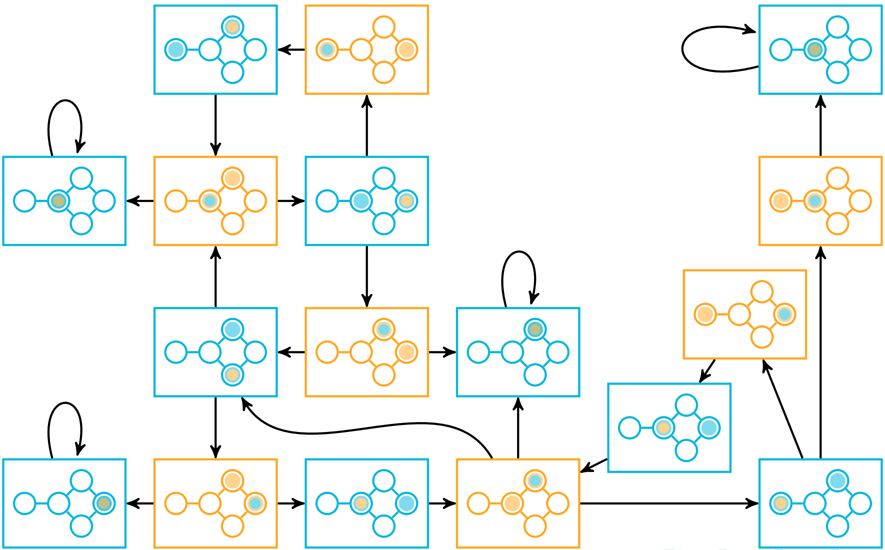
Example game

Partial arena (symmetric configurations are not represented)



Playout and winning condition

Playout

- A **playout** λ is an infinite walk on G,
- The initial vertex of a playout is called the **starting position**.

Winning conditions

Denote $F \subseteq V$ a set of final states. Eve wins a playout *if and only if*:

- **Reachability:** λ goes through at least one final state (e.g. Go),
- **Co-Reachability:** λ never goes through a final state (e.g. model-checking),

Other reachability conditions exist (Büchi, co-Büchi) , in which we consider final states finitely / infinitely often.
Note that in most practical cases in AI, reachability is considered.

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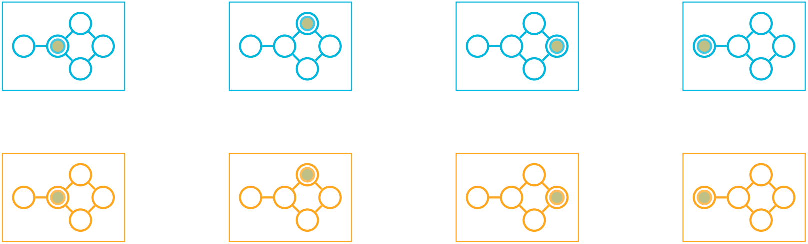
The case of cops and robbers

Here the winning condition is of type reachability for the cop and co-reachability for the robber.

Final states are:



Strategy

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- A **strategy** for Eve is a partial function $\phi_E : V^* \to V$,
- A **randomized strategy** is a partial function $\phi_E : V^* \to P(V)$, where $P(V)$ is the set of probability distributions over V.

Induced playout

- An **induced playout** associated with ϕ_E and ϕ_A is a playout λ such that:

$$\forall i > 0, \lambda_i = \begin{cases} \phi_E(\lambda_0 \ldots \lambda_{i-1}) & \text{if } \lambda_{i-1} \in V_E \\ \phi_A(\lambda_0 \ldots \lambda_{i-1}) & \text{if } \lambda_{i-1} \in V_A \end{cases},$$

  we write it $\lambda(\phi_E, \phi_A, V_0)$, where V_0 is the starting position.
- For randomized strategies, one can make use of the *Carathéodory's extension theorem*.

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Winning strategies and memory

Winning strategy

- A strategy ϕ_E for Eve is said to be winning from $V_0 \in V$ if:

$$\forall \phi_A, \lambda(\phi_E, \phi_A, V_0) \text{ is winning for Eve}$$

- For randomized strategies, we are typically interested in almost-surely winning strategies.

Positional strategy

- A strategy ϕ_E for Eve is said positional if $\exists \phi_E^P : V \to V$ such that

$$\forall i \in \mathbb{N}, \forall V_0 V_1 \dots V_{i-1} \in V^i, \forall V_i \in V_E,$$
$$\phi_E(V_0 V_1 \dots V_{i-1} V_i) = \phi_E^P(V_i).$$

A positional strategy is sometimes termed "without memory".

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Determined games

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- A **game** $\mathbb{G}$ is a tuple $\langle G, V_E, V_A, F, W\rangle$, where
  - $\langle G = \langle V, E\rangle, V_E, V_A\rangle$ is an arena,
  - $F \subseteq V$ is the set of final states,
  - W is a winning condition.

Determined games

- A game is said **determined** if for each starting position, either Eve or Adam admits a winning strategy.

Theorem 1

- All games considered in this course are determined,
- Moreover, winning strategies can always be chosen positional.

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T: Trap

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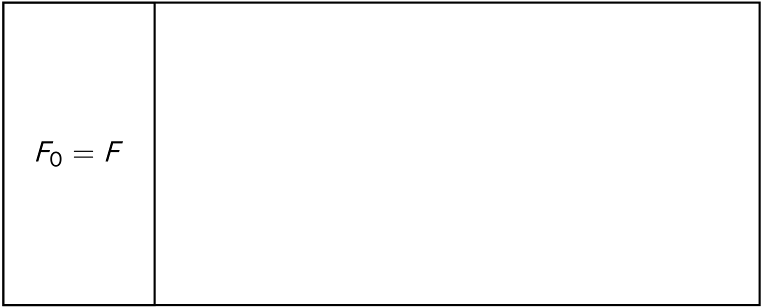

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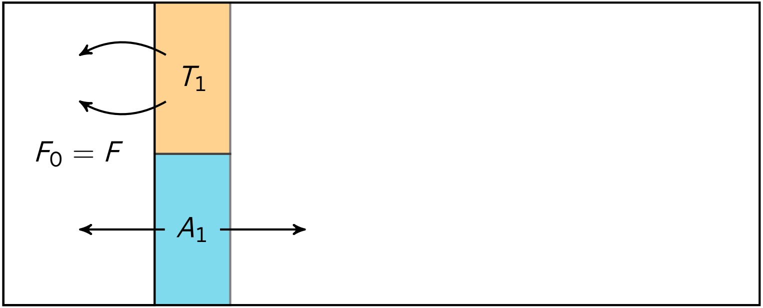

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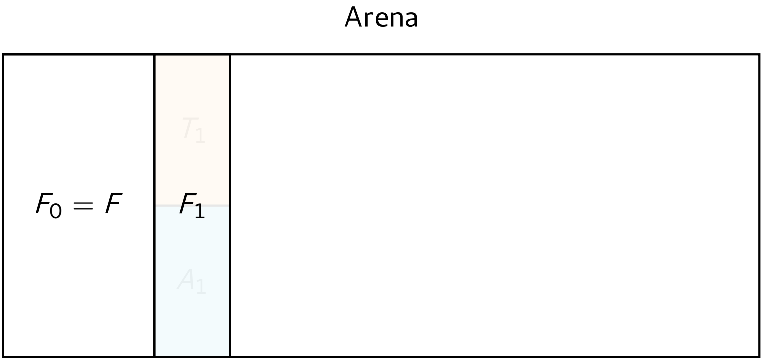

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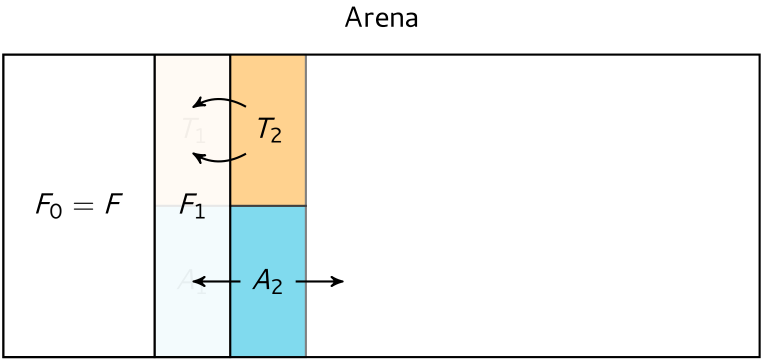

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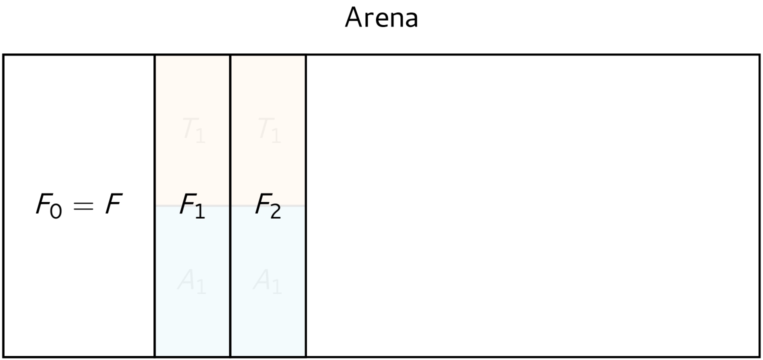

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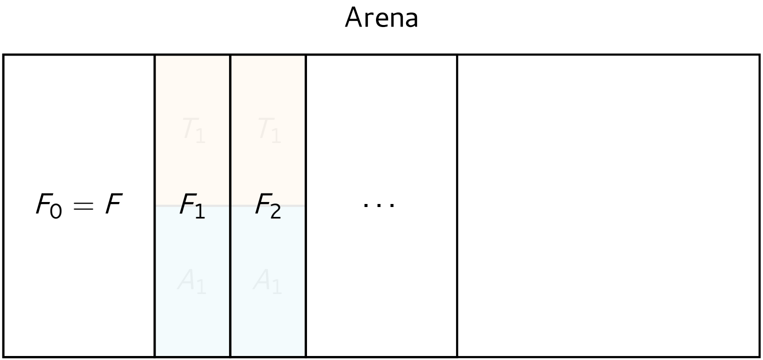

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$$\bigcup_{i=0}^{+\infty} F_i$$

$$V - \bigcup_{i=0}^{+\infty} F_i$$

Sketch of the proof for reachability games

A: Attractor
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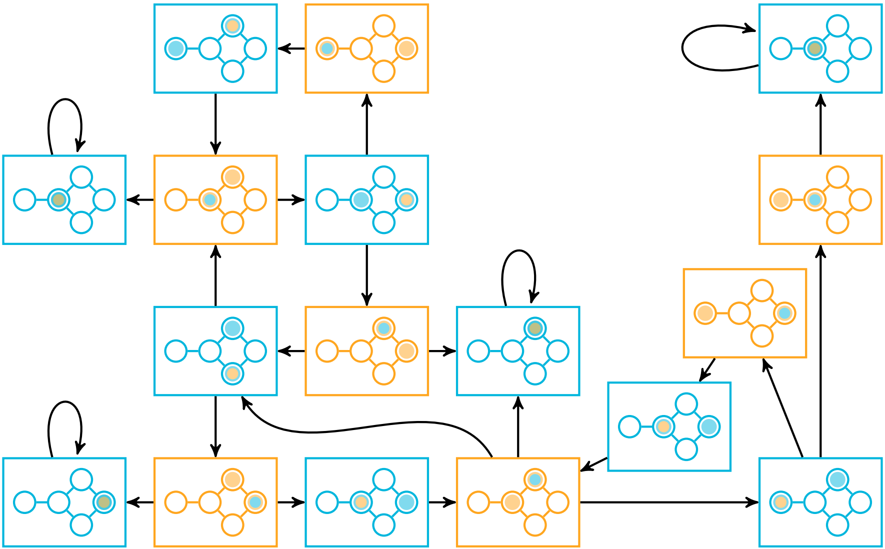
$\bigcup_{i=0}^{+\infty} F_i$ Winning region for Eve	$V - \bigcup_{i=0}^{+\infty} F_i$ Winning region for Adam

Illustration on the example



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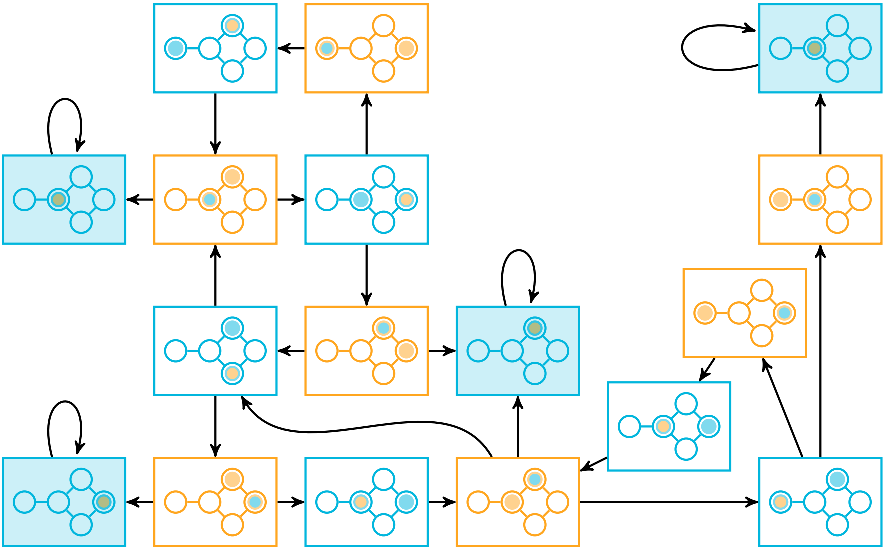

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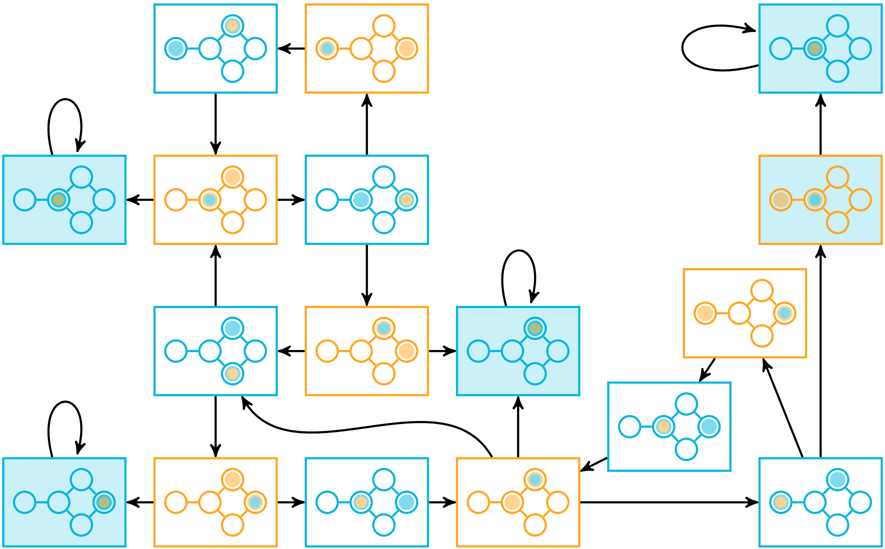

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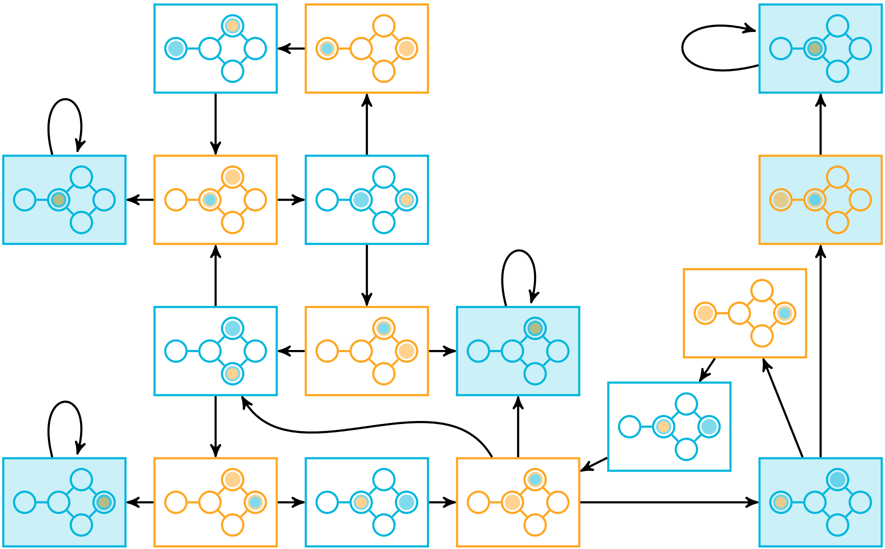

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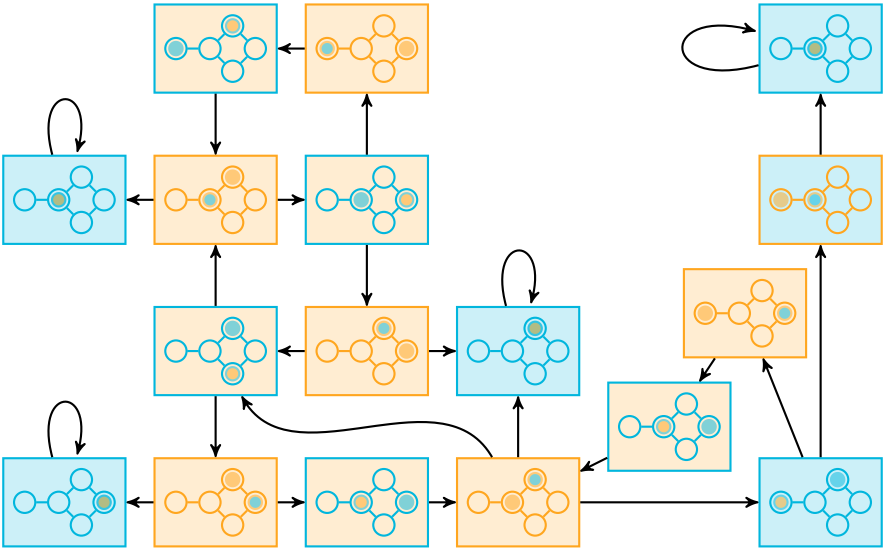

Playout tree

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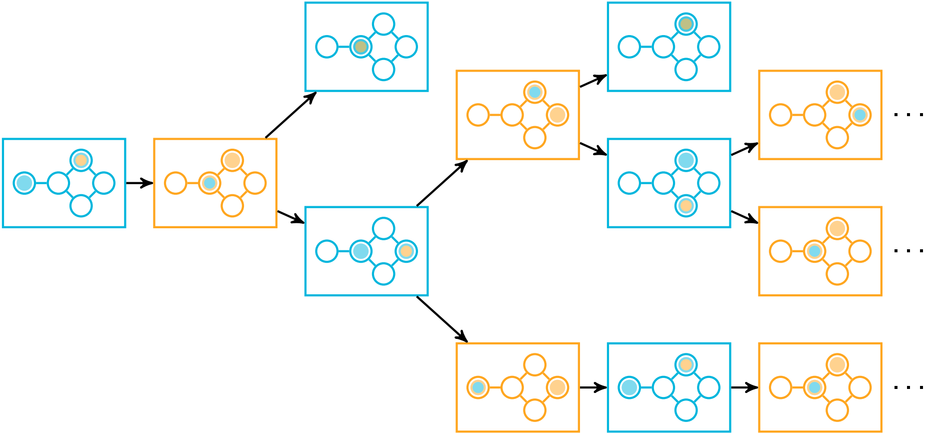

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Finding winning strategies

- In practice, it is possible to use the construction of the proof of Theorem 1 to find winning strategies,
- When V is too large, it may be better to search the playout tree.

Exploring large playout trees

- Even when playouts are finite, playouts trees can quickly become untractably large,
- Randomly explore to find interesting strategies,
- A possible such method is Monte-Carlo Tree-Search (MCTS) or to derive machine learning strategies.

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Lab Session 4

TP Combinatorial Game Theory (TP3)

- Pyrat game with the python playing using a greedy approach (closest cheese)
- Program an exhaustive playout tree search for the rat to beat the python

Challenge annoucement!

Solo pyrat game against a Greedy algorithm, followed by a tournament on December 5th.

- Complete rules and modalities (deadline, etc) are on Moodle
- Baseline with supervised learning
- Oral presentation (10+5 minutes) of your solution

More details on Moodle (section : Challenge Information).