[TOPCODER]

Algorithm Tutorials

Assignment Problem and Hungarian Algorithm

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Introduction

Are you familiar with the following situation? You open the Div I Medium and don't know how to approach... room submitted it in less than 10 minutes. Then, after the contest, you find out in the editorial that this pro... classical one. If yes, then this tutorial will surely be useful for you.

Problem statement

In this article we'll deal with one optimization problem, which can be informally defined as:

Assume that we have N workers and N jobs that should be done. For each pair (worker, job) we know sa... for him to perform the job. Our goal is to complete all jobs minimizing total inputs, while assigning each w... versa.

Converting this problem to a formal mathematical definition we can form the following equations:

$\{c_{ij}\}_{N \times N}$ - cost matrix, where c_{ij} - cost of worker i to perform job j.

$\{x_{ij}\}_{N \times N}$ - resulting binary matrix, where x_{ij} = 1 if and only if i^{th} worker is assigned to j^{th} job.

$$\sum_{j=1}^{N} x_{ij} = 1, \qquad \forall i \epsilon \overline{1, N}$$
 - one worker to one job assignment.

$$\sum_{i=1}^{N} x_{ij} = 1, \qquad \forall j \epsilon \overline{1, N}$$
 - one job to one worker assignment.

$$\sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} x_{ij} \rightarrow min$$
 - total cost function.

We can also rephrase this problem in terms of graph theory. Let's look at the job and workers as if they w... each edge between the ith worker and j^{th} job has weight of c_{ij}. Then our task is to find minimum-weight... matching will consists of N edges, because our bipartite graph is complete).

Small example just to make things clearer:
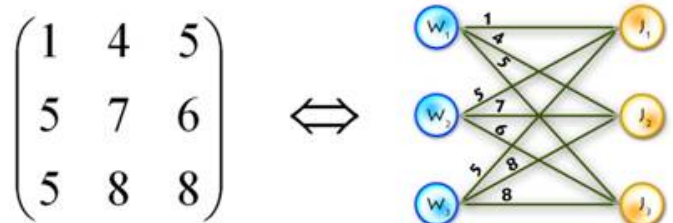
General description of the algorithm

This problem is known as the assignment problem. The assignment problem is a special case of the tran... is a special case of the [min-cost flow](#) problem, so it can be solved using algorithms that solve the more ge... a special case of binary integer linear programming problem (which is NP-hard). But, due to the specifics... efficient algorithms to solve it. We'll handle the assignment problem with the Hungarian algorithm (or Kuh...

illustrate two different implementations of this algorithm, both graph theoretic, one easy and fast to impler
the other one with $O(n^3)$ complexity, but harder to implement.

There are also implementations of Hungarian algorithm that do not use graph theory. Rather, they just op
different transformation of it (see [1] for clear explanation). We'll not touch these approaches, because it's
needs.

$O(n^4)$ algorithm explanation

As mentioned above, we are dealing with a bipartite graph. The main idea of the method is the following:
matching using only edges of weight 0 (hereinafter called "0-weight edges"). Obviously, these edges will I
problem. If we can't find perfect matching on the current step, then the Hungarian algorithm changes wei
such a way that the new 0-weight edges appear and these changes do not influence the optimal solution.

To clarify, let's look at the step-by-step overview:

Step 0)

A. For each vertex from left part (workers) find the minimal outgoing edge and subtract its weight from all
vertex. This will introduce 0-weight edges (at least one).

B. Apply the same procedure for the vertices in the right part (jobs).
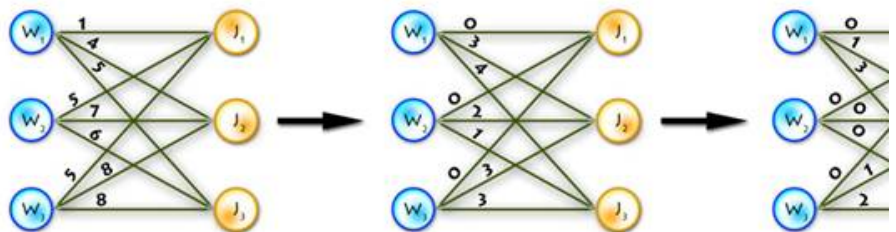
Actually, this step is not necessary, but it decreases the number of main cycle iterations.

Step 1)

A. Find the maximum matching using only 0-weight edges (for this purpose you can use max-flow algoritl
etc.).

B. If it is perfect, then the problem is solved. Otherwise find the minimum vertex cover **V** (for the subgrapl
best way to do this is to use [Köning's graph theorem](#).

Graph with 0-weight edges only   Maximum matching and minimum vertex cover
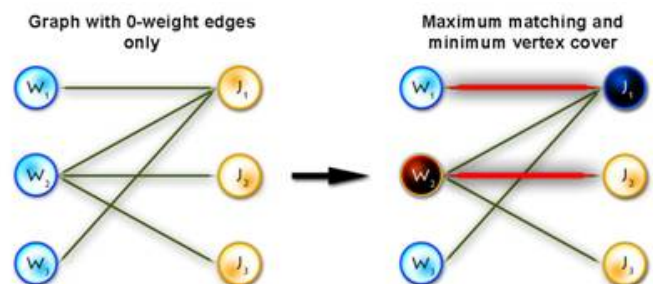
Step 2) Let $\Delta = \min_{i \notin V, j \notin V}(c_{ij})$ and adjust the weights using the following rule:

$$c_{ij} = \begin{cases} c_{ij} - \Delta, i \notin V \wedge j \notin V \\ c_{ij} \quad, i \in V \vee j \in V \\ c_{ij} + \Delta, i \in V \wedge j \in V \end{cases}$$

Graph with modified weights (delta = 1)   Maximum matching. It's perfect!

STEP 1

Step 3) Repeat Step 1 until solved.

But there is a nuance here; finding the maximum matching in step 1 on each iteration will cause the algor to avoid this, on each step we can just modify the matching from the previous step, which only takes $O(n^2$

It's easy to see that no more than n2 iterations will occur, because every time at least one edge becomes complexity is $O(n^4)$.

$O(n^3)$ algorithm explanation

Warning! In this section we will deal with the maximum-weighted matching problem. It's obviously easy to the maximum one, just by setting:

$$w(x,y) = -w(x,y), \forall (x,y) \in E$$

or

$$w(x,y) = M - w(x,y), M = \max_{(x,y) \in E} w(x,y)$$

Before discussing the algorithm, let's take a look at some of the theoretical ideas. Let's start off by consid bipartite graph **G=(V,E)** where $V = X \cup Y (X \cap Y = \varnothing)$ and $E \subseteq X \times Y$, **w(x,y)** - weight of edge *(x,*

Vertex and set neighborhood

Let $v \in V$. Then $J_G(v) = \{u \mid (v,u) \in E\}$ is **v's** neighborhood, or all vertices that share an edge with **v**

$$J_G(S) = \bigcup_{v \in S} J_G(v)$$

Let $S \subseteq V$. Then is **S's** neighborhood, or all vertices that share an edge with a ver

Vertex labeling

This is simply a function $l : V \to R$ (for each vertex we assign some number called a label). Let's call th the following condition: $l(x) + l(y) \geq w(x,y), \forall x \in X, \forall y \in Y$. In other words, the sum of the labels given edge are greater than or equal to the weight of that edge.

Equality subgraph

Let **G_l=(V,E_l)** be a spanning subgraph of **G** (in other words, it includes all vertices from **G**). If **G** only those following condition: $(x,y) \in E_l \Leftrightarrow (x,y) \in E \wedge l(x) + l(y) = w(x,y)$, then it is an equality subgraph those edges from the bipartite matching which allow the vertices to be perfectly feasible.

Now we're ready for the theorem which provides the connection between equality subgraphs and maximu

*If **M*** is a perfect matching in the equality subgraph **G_l**, then **M*** is a maximum-weighted matching in **G**.*

The proof is rather straightforward, but if you want you can do it for practice. Let's continue with a few fina

Alternating path and alternating tree

Consider we have a matching **M** ($M \subseteq E$).

Vertex $v \in V$ is called matched if $\exists x \in X : (x,v) \in M \vee \exists y \in Y : (v,y) \in M$, otherwise it is called *ex*

(In the diagram below, **W_1**, **W_2**, **W_3**, **J_1**, **J_3**, **J_4** are matched, **W_4**, **J_2** are exposed)
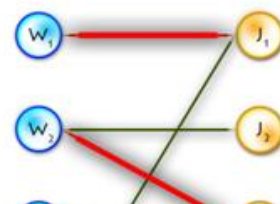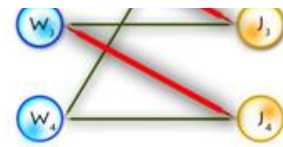
Path **P** is called alternating if its edges alternate between **M** and **E\M**. (For example, (**W4**, **J4**, **W3**, **J3**, **W2** alternating paths)

If the first and last vertices in alternating path are exposed, it is called *augmenting* (because we can incre inverting edges along this path, therefore matching unmatched edges and vice versa). ((**W4**, **J4**, **W3**, **J3**, path)

A tree which has a root in some exposed vertex, and a property that every path starting in the root is alte *tree*. (Example on the picture above, with root in **W4**)

That's all for the theory, now let's look at the algorithm:

First let's have a look on the scheme of the Hungarian algorithm:

Step 0. Find some initial feasible vertex labeling and some initial matching.

Step 1. If **M** is perfect, then it's optimal, so problem is solved. Otherwise, some exposed $x \in X$ exists; s of the alternating tree we're going to build). Go to step 2.

Step 2. If $J_{G_l}(S) \neq T$ go to step 3, else $J_{G_l}(S) = T$. Find

$$\Delta = \min_{x \in S, y \in Y \setminus T} (l(x) + l(y) - w(x, y)) \qquad (1)$$

and replace existing labeling with the next one:

$$l'(v) = \begin{cases} l(v) - \Delta, v \in S \\ l(v) + \Delta, v \in T \\ l(v), otherwise \end{cases} \qquad (2)$$

Now replace G_l with $G_{l'}$

**Step 3.** Find some vertex $y \in T \setminus J_{G_l}(S)$. If **y** is exposed then an alternating path from **x** (root of the tre along this path and go to step 1. If **y** is matched in **M** with some vertex **z** add *(z,y)* to the alternating tree a $S = S \cup \{z\}, T = T \cup \{y\}$, go to step 2.

And now let's illustrate these steps by considering an example and writing some code.

As an example we'll use the previous one, but first let's transform it to the maximum-weighted matching p method from the two described above. (See Picture 1)
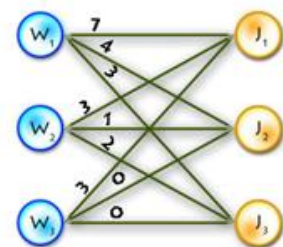
Picture 1

Here are the global variables that will be used in the code:

```
#define N 55              //max number of vertices in one part
#define INF 100000000     //just infinity

int cost[N][N];           //cost matrix
int n, max_match;         //n workers and n jobs
int lx[N], ly[N];         //labels of X and Y parts
```

```
int xy[N];                    //xy[x] - vertex that is matched with x,
int yx[N];                    //yx[y] - vertex that is matched with y
bool S[N], T[N];              //sets S and T in algorithm
int slack[N];                 //as in the algorithm description
int slackx[N];                //slackx[y] such a vertex, that
                              // l(slackx[y]) + l(y) - w(slackx[y],y) = slack[y]
int prev[N];                  //array for memorizing alternating paths
```

Step 0:

It's easy to see that next initial labeling will be feasible:

$$l(x) = \max_{(x,y)\in E} (w(x,y))$$

$$l(y) = 0, y \in Y$$

And as an initial matching we'll use an empty one. So we'll get equality subgraph as on Picture 2. The co
I'll paste it for completeness:
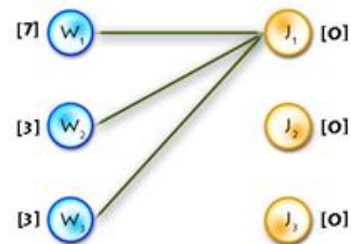

```
void init_labels()
{
    memset(lx, 0, sizeof(lx));
    memset(ly, 0, sizeof(ly));
    for (int x = 0; x < n; x++)
        for (int y = 0; y < n; y++)
            lx[x] = max(lx[x], cost[x][y]);
}
```

The next three steps will be implemented in one function, which will correspond to a single iteration of the
halts, we will have a perfect matching, that's why we'll have n iterations of the algorithm and therefore **(n**

Step 1

According to this step we need to check whether the matching is already perfect, if the answer is positive
we need to clear **S**, **T** and alternating tree and then find some exposed vertex from the **X** part. Also, in thi
array, I'll describe it on the next step.

```
void augment()                            //main function of the algorithm
{
    if (max_match == n) return;           //check wether matching is already perfect
    int x, y, root;                       //just counters and root vertex
    int q[N], wr = 0, rd = 0;             //q - queue for bfs, wr,rd - write and read
                                          //pos in queue
    memset(S, false, sizeof(S));          //init set S
    memset(T, false, sizeof(T));          //init set T
    memset(prev, -1, sizeof(prev));       //init set prev - for the alternating tree
    for (x = 0; x < n; x++)               //finding root of the tree
        if (xy[x] == -1)
        {
            q[wr++] = root = x;
            prev[x] = -2;
            S[x] = true;
            break;
        }

    for (y = 0; y < n; y++)               //initializing slack array
    {
        slack[y] = lx[root] + ly[y] - cost[root][y];
        slackx[y] = root;
```

```
    }
```

Step 2

On this step, the alternating tree is completely built for the current labeling, but the augmenting path hasr
improve the labeling. It will add new edges to the equality subgraph, giving an opportunity to expand the
idea of the method; *we are improving the labeling until we find an augmenting path in the equality graph*
labeling. Let's turn back to step 2. There we just change labels using formulas **(1)** and **(2)**, but using them
the algorithm to have **O(n^4)** time. So, in order to avoid this we use a *slack* array initialized in **O(n)** time be
array created in step 1:

$$slack[y] = \min_{x \in S}(l(x) + l(y) - w(x, y))$$

Then we just need O(n) to calculate a delta Δ (see (1)):

$$\Delta = \min_{y \in Y \setminus T} slack[y]$$

Updating slack:
1) On **step 3**, when vertex *x* moves from *X\S* to *S*, this takes **O(n)**.
2) On **step 2**, when updating labeling, it's also takes **O(n)**, because:

$$\Delta = \min_{y \in Y \setminus T} slack[y]$$

So we get **O(n)** instead of **O(n^2)** as in the straightforward approach.
Here's code for the label updating function:

```
void update_labels()
{
    int x, y, delta = INF;              //init delta as infinity
    for (y = 0; y < n; y++)             //calculate delta using slack
        if (!T[y])
            delta = min(delta, slack[y]);
    for (x = 0; x < n; x++)             //update X labels

        if (S[x]) lx[x] -= delta;
    for (y = 0; y < n; y++)             //update Y labels
        if (T[y]) ly[y] += delta;
    for (y = 0; y < n; y++)             //update slack array
        if (!T[y])
            slack[y] -= delta;
}
```

Step 3

In step 3, first we build an alternating tree starting from some exposed vertex, chosen at the beginning of
using breadth-first search algorithm. If on some step we meet an exposed vertex from the *Y* part, then fin
finishing up with a call to the main function of the algorithm. So the code will be the following:

1) Here's the function that adds new edges to the alternating tree:

```
void add_to_tree(int x, int prevx)
//x - current vertex,prevx - vertex from X before x in the alternating path,
//so we add edges (prevx, xy[x]), (xy[x], x)
{
    S[x] = true;                    //add x to S
    prev[x] = prevx;                //we need this when augmenting
    for (int y = 0; y < n; y++)     //update slacks, because we add new vertex to S
        if (lx[x] + ly[y] - cost[x][y] < slack[y])
        {
            slack[y] = lx[x] + ly[y] - cost[x][y];
            slackx[y] = x;
        }
}
```

3) And now, the end of the ***augment()*** function:

```
//second part of augment() function
    while (true)                                                //main cycle
    {
        while (rd < wr)                                         //building tree with bf
        {
```

```cpp
            x = q[rd++];                                    //current vertex from )
            for (y = 0; y < n; y++)                         //iterate through all e
                if (cost[x][y] == lx[x] + ly[y] &&  !T[y])
                {
                    if (yx[y] == -1) break;                 //an exposed vertex in
                                                            //augmenting path exist
                    T[y] = true;                            //else just add y to T,
                    q[wr++] = yx[y];                        //add vertex yx[y], whi
                                                            //with y, to the queue
                    add_to_tree(yx[y], x);                  //add edges (x,y) and (
                }
            if (y < n) break;                               //augmenting path founc
        }
        if (y < n) break;                                   //augmenting path founc

        update_labels();                                    //augmenting path not f
        wr = rd = 0;
        for (y = 0; y < n; y++)
        //in this cycle we add edges that were added to the equality graph as a
        //result of improving the labeling, we add edge (slackx[y], y) to the tree if
        //and only if !T[y] &&  slack[y] == 0, also with this edge we add another one
        //(y, yx[y]) or augment the matching, if y was exposed
            if (!T[y] &&  slack[y] == 0)
            {
                if (yx[y] == -1)                            //exposed vertex in Y f
                {
                    x = slackx[y];
                    break;
                }
                else
                {
                    T[y] = true;                            //else just add y to T,
                    if (!S[yx[y]])
                    {
                        q[wr++] = yx[y];                    //add vertex yx[y], whi
                                                            //y, to the queue
                        add_to_tree(yx[y], slackx[y]);      //and add edges (x,y) a
                                                            //yx[y]) to the tree
                    }
                }
            }
        if (y < n) break;                                   //augmenting path founc
    }

    if (y < n)                                              //we found augmenting p
    {
        max_match++;                                        //increment matching
        //in this cycle we inverse edges along augmenting path
        for (int cx = x, cy = y, ty; cx != -2; cx = prev[cx], cy = ty)
        {
            ty = xy[cx];
            yx[cy] = cx;
            xy[cx] = cy;
        }
        augment();                                          //recall function, go t
    }
}//end of augment() function
```
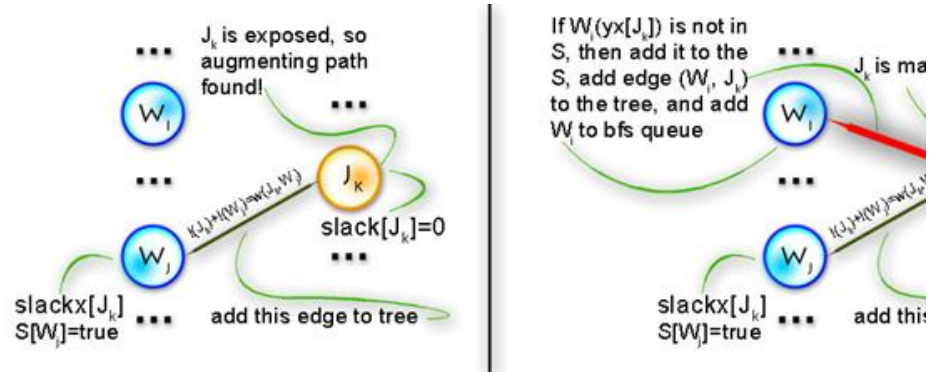
The only thing in code that hasn't been explained yet is the procedure that goes after labels are updated.

now we need to complete our alternating tree; to do this and to keep algorithm in $O(n^3)$ time (it's only pos
more than one time per iteration) we need to know what edges should be added without iterating through
this question is to use BFS to add edges only from those vertices in **Y**, that are not in **T** and for which **sla**

in such way we'll add all edges and keep algorithm to be $O(n^3)$). See picture below for explanation:

J$_k$ is exposed, so augmenting path found!

slack[J$_k$]=0

f(J$_j$)+f(W$_j$)=w(J$_k$,W$_j$)

slackx[J$_k$]
S[W$_j$]=true

add this edge to tree

If W$_i$(yx[J$_k$]) is not in S, then add it to the S, add edge (W$_i$, J$_k$) to the tree, and add W$_i$ to bfs queue

J$_k$ is ma

f(J$_j$)+f(W$_j$)=w(J$_k$,

slackx[J$_k$]
S[W$_j$]=true

add thi

At last, here's the function that implements Hungarian algorithm:

```
int hungarian()
{
    int ret = 0;                    //weight of the optimal matching
    max_match = 0;                  //number of vertices in current matching
    memset(xy, -1, sizeof(xy));
    memset(yx, -1, sizeof(yx));
    init_labels();                  //step 0
    augment();                      //steps 1-3
    for (int x = 0; x < n; x++)     //forming answer there
        ret += cost[x][xy[x]];
    return ret;
}
```

To see all this in practice let's complete the example started on step 0.

Build alternating tree → Augmenting path found →
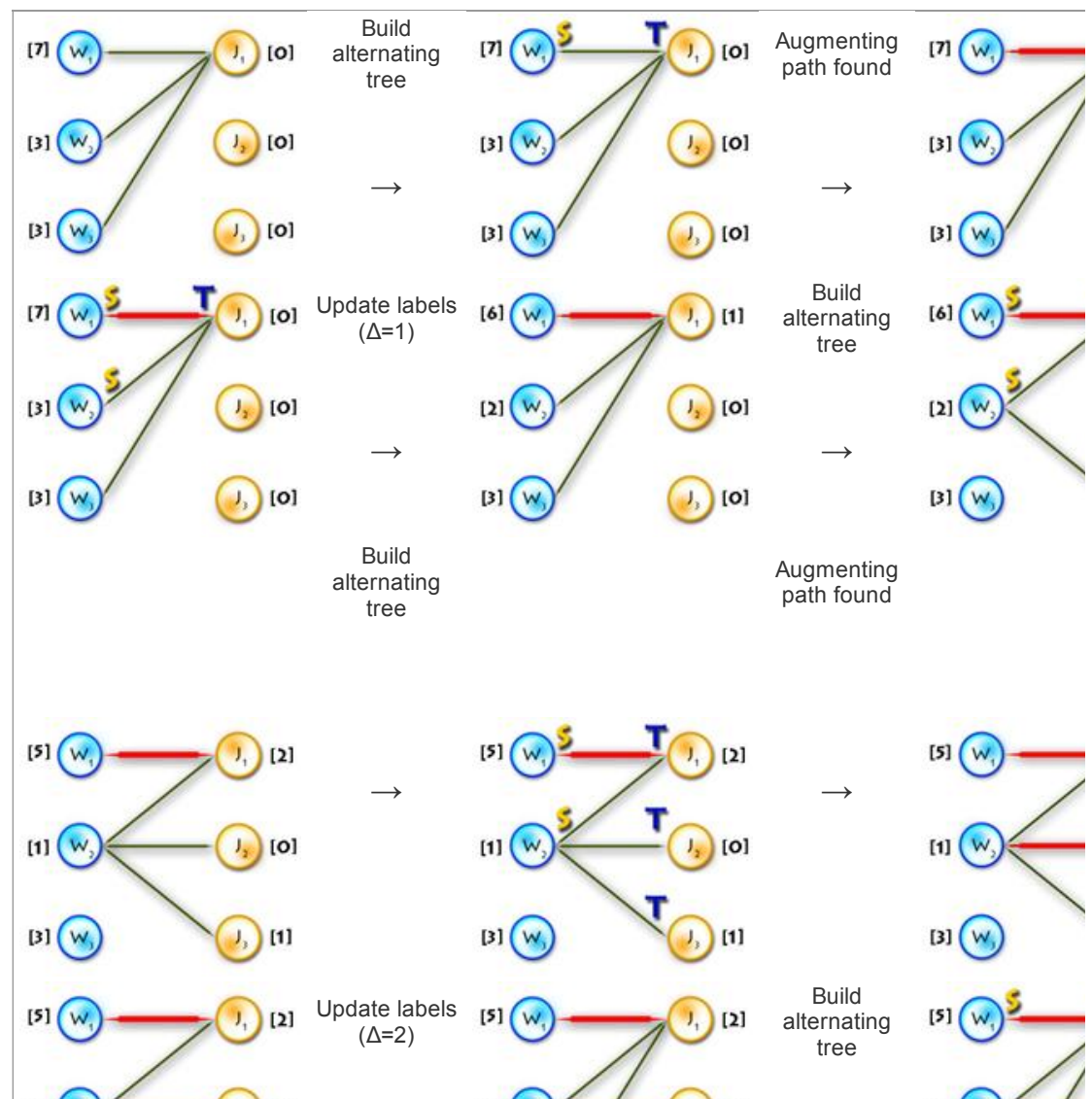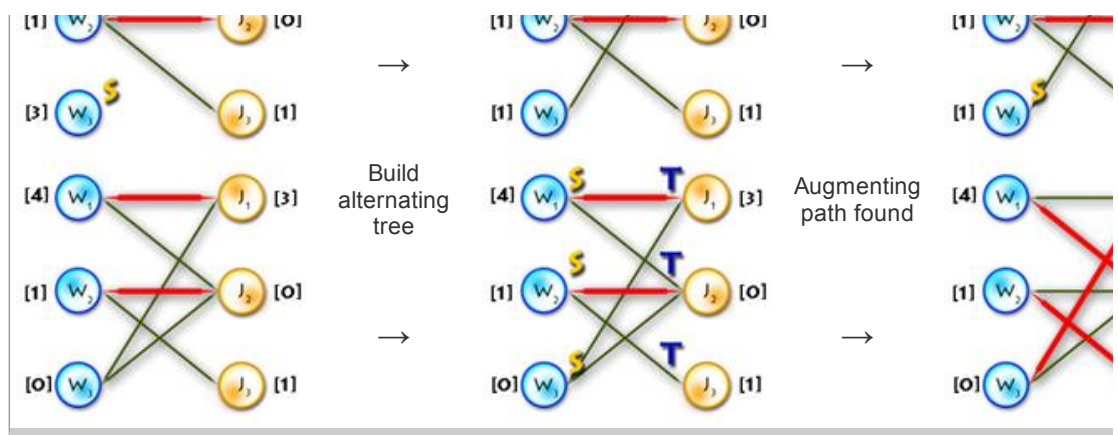
Finally, let's talk about the complexity of this algorithm. On each iteration we increment matching so we ha[ve] iterations each edge of the graph is used no more than one time when finding augmenting path, so we've [...] Concerning labeling we update *slack* array each time when we insert vertex from **X** into **S**, so this happen[s ...] iteration, updating *slack* takes **O(n)** operations, so again we've got **O(n²)**. Updating labels happens no m[ore than ...] (because we add at least one vertex from **Y** to **T** per iteration), it takes **O(n)** operations - again **O(n²)**. So [...] implementation is **O(n³)**.

Some practice

For practice let's consider the medium [problem](#) from SRM 371 (div. 1). It's obvious we need to find the m[aximum ...] graph, where the **X** part is our players, the **Y** part is the opposing club players, and the weight of each ed[ge ...]

$$w(x,y) = \begin{cases} 0, us[x] < them[y] \\ 1, us[x] = them[y] \\ 2, us[x] > them[y] \end{cases}$$

Though this problem has a much simpler solution, this one is obvious and fast coding can bring more poi[nts ...]

Try this [one](#) for more practice. I hope this article has increased the wealth of your knowledge in classical [...] fun!

References

1. Mike Dawes "[The Optimal Assignment Problem](#)"

2. Mordecaj J. Golin "[Bipartite Matching and the Hungarian Method](#)"
3. Samir Khuller "[Design and Analysis of Algorithms: Course Notes](#)"
4. Lawler E.L. "[Combinatorial Optimization: Networks and Matroids](#)"