Stable marriage problem

In mathematics and computer science, the **stable marriage problem** (**SMP**) is the problem of finding a **stable matching** between two sets of elements given a set of preferences for each element. A matching is a mapping from the elements of one set to the elements of the other set. A matching is stable whenever it is *not* the case that both:

a. some given element *A* of the first matched set prefers some given element *B* of the second matched set over the element to which A is already matched, and
b. *B* also prefers *A* over the element to which *B* is already matched

In other words, a matching is stable when there does not exist any alternative pairing (A, B) in which both A and B are individually better off than they would be with the element to which they are currently matched.

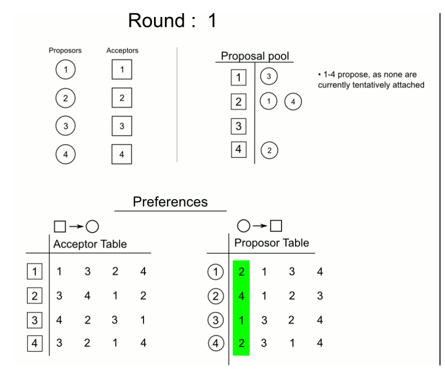
The stable marriage problem is commonly stated as:

> Given *n* men and *n* women, where each person has ranked all members of the opposite sex with a unique number between 1 and *n* in order of preference, marry the men and women together such that there are no two people of opposite sex who would both rather have each other than their current partners. If there are no such people, all the marriages are "stable".

Algorithms for finding solutions to the stable marriage problem have applications in a variety of real-world situations, perhaps the best known of these being in the assignment of graduating medical students to their first hospital appointments.[1]

Solution

In 1962, David Gale and Lloyd Shapley proved that, for any equal number of men and women, it is always possible to solve the SMP and make all marriages stable. They presented an algorithm to do so.[2][3]



Animation showing an example of the Gale-Shapley algorithm

The **Gale-Shapley algorithm** involves a number of "rounds" (or "iterations") where each unengaged man "proposes" to the most-preferred woman to whom he has not yet proposed. Each woman then considers all her suitors and tells the one she most prefers "Maybe" and all the rest of them "No". She is then provisionally "engaged" to the suitor she most prefers so far, and that suitor is likewise provisionally engaged to her. In the first round, first *a*) each unengaged man proposes to the woman he prefers most, and then *b*) each woman replies "maybe" to her suitor she most prefers and "no" to all other suitors. In each subsequent round, first *a*) each unengaged man proposes to the most-preferred woman to whom he has not yet proposed (regardless of whether the woman is already engaged), and then *b*) each woman replies "maybe" to her suitor she most prefers (whether her existing provisional partner or someone else) and rejects the rest (again, perhaps including her current provisional partner). The provisional nature of engagements preserves the right of an already-engaged woman to "trade up" (and, in the process, to "jilt" her until-then partner).

This algorithm guarantees that:

Everyone gets married

> Once a woman becomes engaged, she is always engaged to someone. So, at the end, there cannot be a man and a woman both unengaged, as he must have proposed to her at some point (since a man will eventually propose to everyone, if necessary) and, being unengaged, she would have had to have said yes.

The marriages are stable

Let Alice be a woman and Bob be a man who are both engaged, but not to each other. Upon completion of the algorithm, it is not possible for both Alice and Bob to prefer each other over their current partners. If Bob prefers Alice to his current partner, he must have proposed to Alice before he proposed to his current partner. If Alice accepted his proposal, yet is not married to him at the end, she must have dumped him for someone she likes more, and therefore doesn't like Bob more than her current partner. If Alice rejected his proposal, she was already with someone she liked more than Bob.

Algorithm

```
function stableMatching {
    Initialize all m ∈ M and w ∈ W to free
    while ∃ free man m who still has a woman w to propose to {
       w = m's highest ranked such woman to whom he has not yet proposed
       if w is free
         (m, w) become engaged
       else some pair (m', w) already exists
         if w prefers m to m'
           (m, w) become engaged
           m' becomes free
         else
           (m', w) remain engaged
    }
}
```

Optimality of the solution

While the solution is stable, it is not necessarily optimal from all individuals' points of view. The traditional form of the algorithm is optimal for the initiator of the proposals and the stable, suitor-optimal solution may or may not be optimal for the reviewer of the proposals. An example is as follows:

There are three suitors (A,B,C) and three reviewers (X,Y,Z) which have preferences of:

```
A: YXZ  B: ZYX  C: XZY  X: BAC  Y: CBA  Z: ACB
```

There are 3 stable solutions to this matching arrangement:

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suitors get their first choice and reviewers their third (AY, BZ, CX)
all participants get their second choice (AX, BY, CZ)
reviewers get their first choice and suitors their third (AZ, BX, CY)
```

All three are stable because instability requires both participants to be happier with an alternative match. Giving one group their first choices ensures that the matches are stable because they would be unhappy with any other proposed match. Giving everyone their second choice ensures that any other match would be disliked by one of the parties. The algorithm converges in a single round on the suitor-optimal solution because each reviewer receives exactly one proposal, and therefore selects that proposal as its best choice, ensuring that each suitor has an accepted offer, ending the match. This asymmetry of optimality is driven by the fact that the suitors have the entire set to choose from, but reviewers choose between a limited subset of the suitors at any one time.

Similar problems

The **weighted matching problem** seeks to find a matching in a weighted bipartite graph that has maximum weight. Maximum weighted matchings do not have to be stable, but in some applications a maximum weighted matching is better than a stable one.

The **stable roommates problem** is similar to the stable marriage problem, but differs in that all participants belong to a single pool (instead of being divided into equal numbers of "men" and "women").

The **hospitals/residents problem — also known as the college admissions problem —** differs from the stable marriage problem in that the "women" can accept "proposals" from more than one "man" (e.g., a hospital can take multiple residents, or a college can take an incoming class of more than one student). Algorithms to solve the hospitals/residents problem can be *hospital-oriented* (female-optimal) or *resident-oriented* (male-optimal).

The **hospitals/residents problem with couples** allows the set of residents to include couples who must be assigned together, either to the same hospital or to a specific pair of hospitals chosen by the couple (e.g., a married couple want to ensure that they will stay together and not be stuck in programs that are far away from each other). The addition of couples to the hospitals/residents problem renders the problem NP-complete.[4]

References

[1] Stable Matching Algorithms (http://www.dcs.gla.ac.uk/research/algorithms/stable/)
[2] D. Gale and L. S. Shapley: "College Admissions and the Stability of Marriage", *American Mathematical Monthly* 69, 9-14, 1962.
[3] Harry Mairson: "The Stable Marriage Problem", *The Brandeis Review* 12, 1992 (online (http://www1.cs.columbia.edu/~evs/intro/stable/writeup.html)).
[4] D. Gusfield and R. W. Irving, *The Stable Marriage Problem: Structure and Algorithms,* p. 54; MIT Press, 1989.

Textbooks and other important references not cited in the text

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- Kleinberg, J., and Tardos, E. (2005) Algorithm Design, Chapter 1, pp 1–12. See companion website for the Text (http://www.aw-bc.com/info/kleinberg/).

External links

- Interactive Flash Demonstration of SMP (http://mathsite.math.berkeley.edu/smp/smp.html)
- http://kuznets.fas.harvard.edu/~aroth/alroth.html#NRMP
- http://www.dcs.gla.ac.uk/research/algorithms/stable/EGSapplet/EGS.html
- Gale-Shapley JavaScript Demonstration (http://www.sephlietz.com/gale-shapley/)
- SMP Lecture Notes (http://www.csee.wvu.edu/~ksmani/courses/fa01/random/lecnotes/lecture5.pdf)

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