Cuckoo Hashing

Outline for Today

- ***Towards Perfect Hashing***
  - Reducing worst-case bounds
- ***Cuckoo Hashing***
  - Hashing with worst-case O(1) lookups.
- ***Random Graph Theory***
  - Just how fast is cuckoo hashing?

Perfect Hashing

Collision Resolution

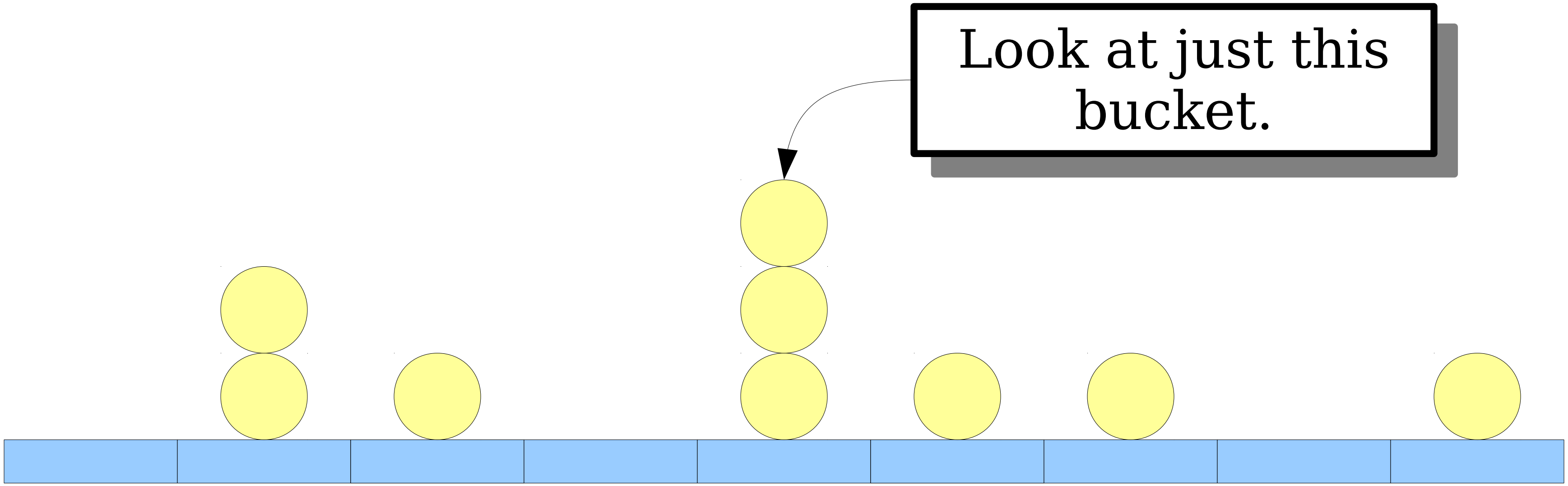
- Last time, we mentioned three general strategies for resolving hash collisions:
  - ***Closed addressing:*** Store all colliding elements in an auxiliary data structure like a linked list or BST.
  - ***Open addressing:*** Allow elements to overflow out of their target bucket and into other spaces.
  - ***Perfect hashing:*** Choose a hash function with no collisions.
- We have not spoken on this last topic yet.

Why Perfect Hashing is Hard

- For fixed, constant load factors, the expected cost of a lookup in a chained hash table or linear probing table is O(1).
- However, the expected cost of a lookup in these tables is not the same as the expected *worst-case* cost of a lookup in these tables.

Expected Worst-Case Bounds

- ***Theorem:*** Assuming truly random hash functions, the expected worst-case cost of a lookup in a chained hash table is $\Theta(\log n / \log \log n)$, assuming the number of slots is $\Theta(n)$.
- ***Theorem:*** Assuming truly random hash functions, the expected worst-case cost of a lookup in a linear probing hash table is $\Omega(\log n)$.
- ***Proofs:*** Exercises 11-1 and 11-2 from CLRS. ☺



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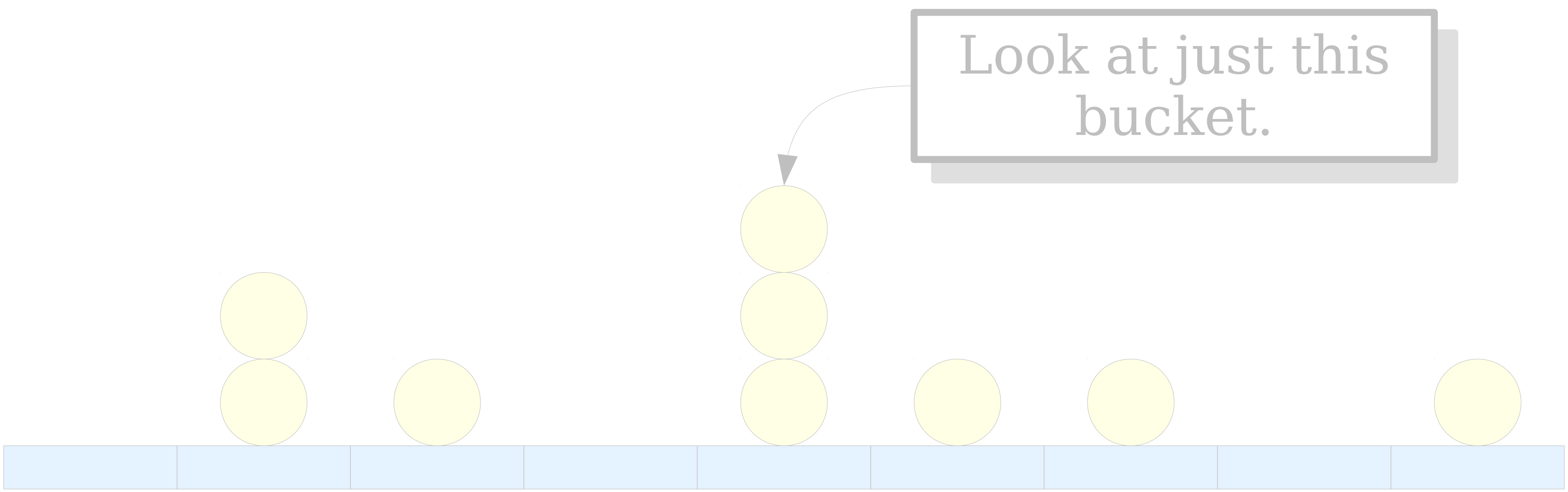
How can we get from these bounds down to O(1)?



Technique 1: Multiple-Choice Hashing

Second-Choice Hashing

- Suppose that we distribute n balls into $\Theta(n)$ bins using the following strategy:
  - For each ball, choose two bins totally at random.
  - Put the ball into the bin with fewer balls in it; tiebreak randomly.

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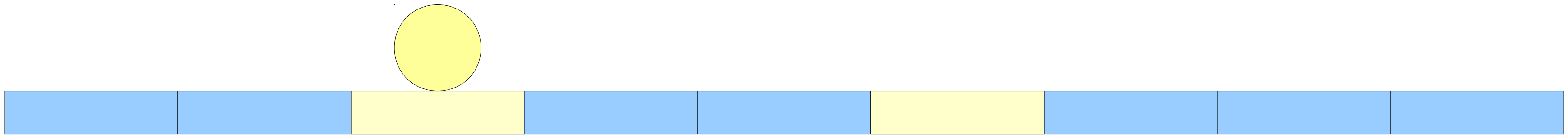

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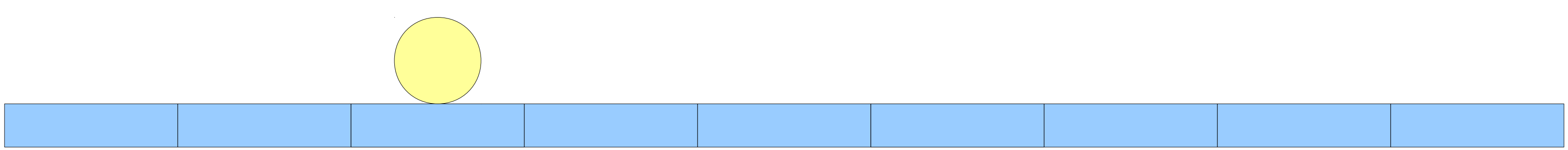

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- Suppose that we distribute *n* balls into Θ(*n*) bins using the following strategy:
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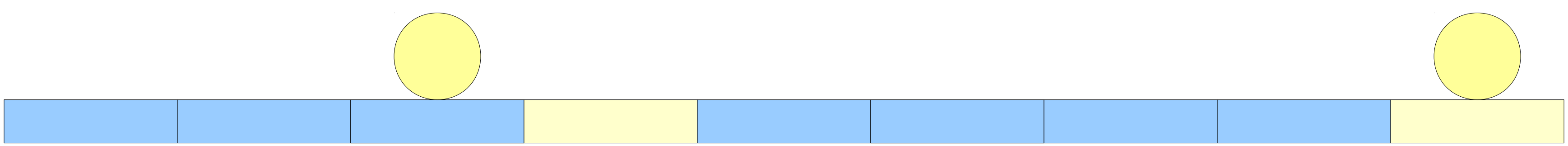

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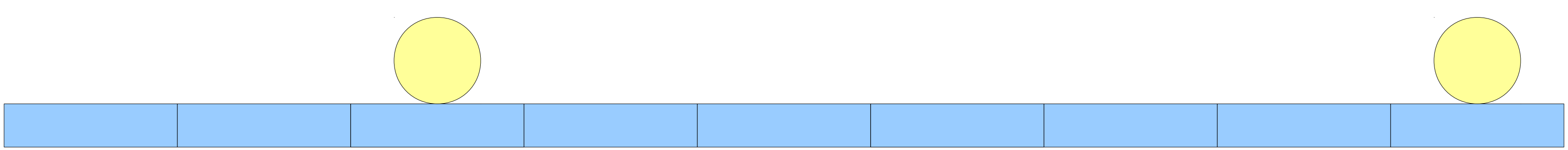

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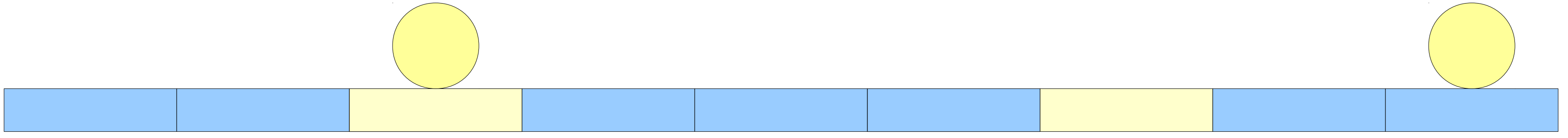

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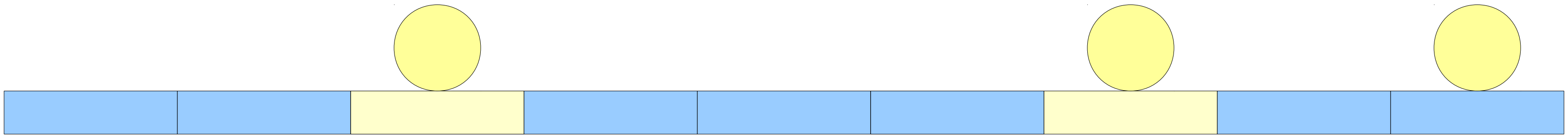

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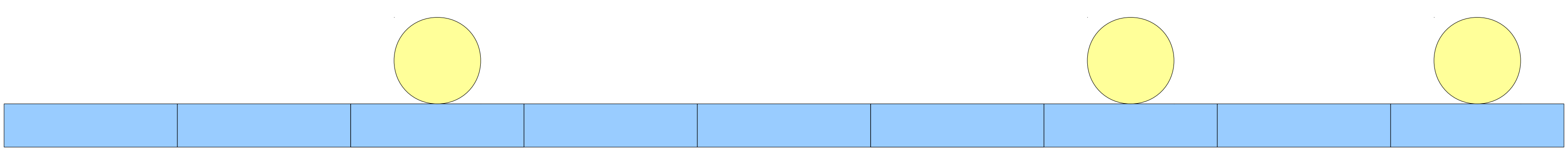

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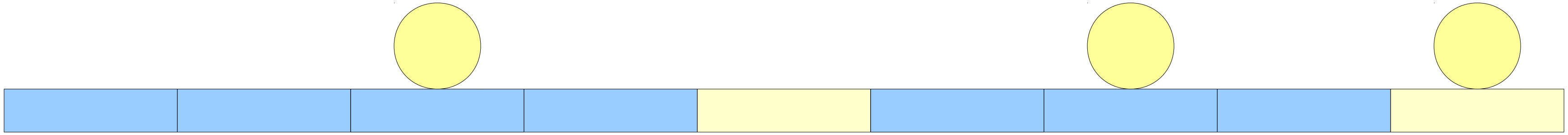

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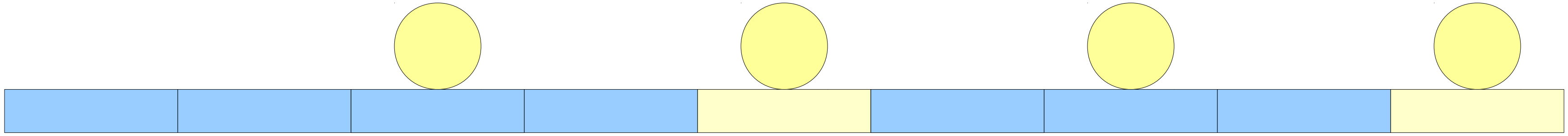

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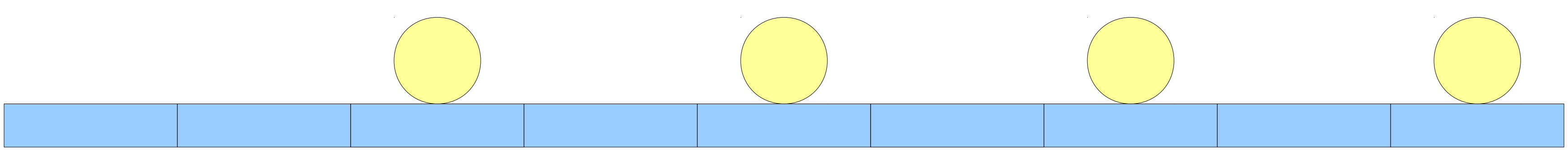

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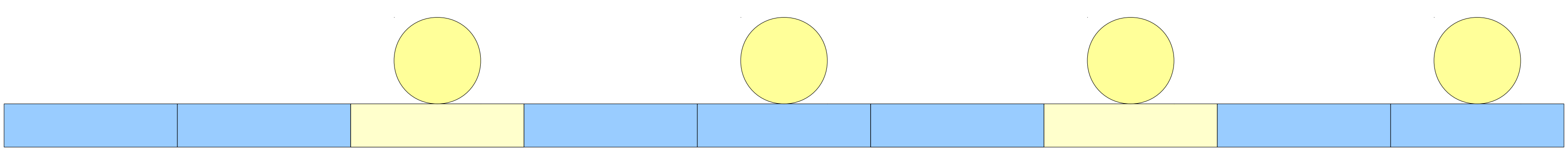

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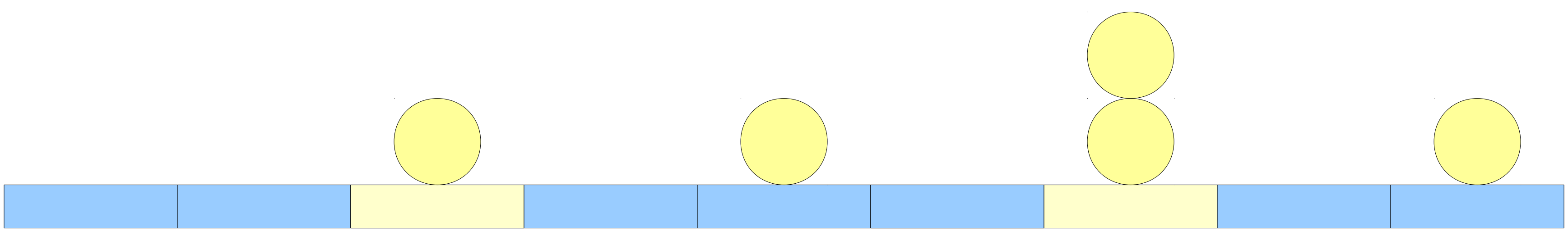

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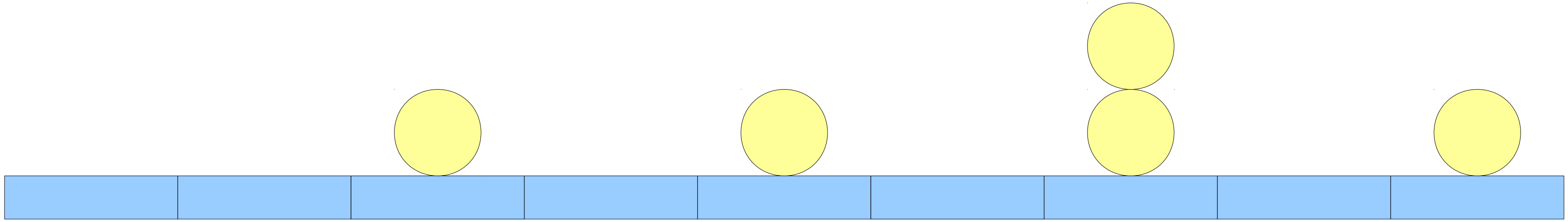

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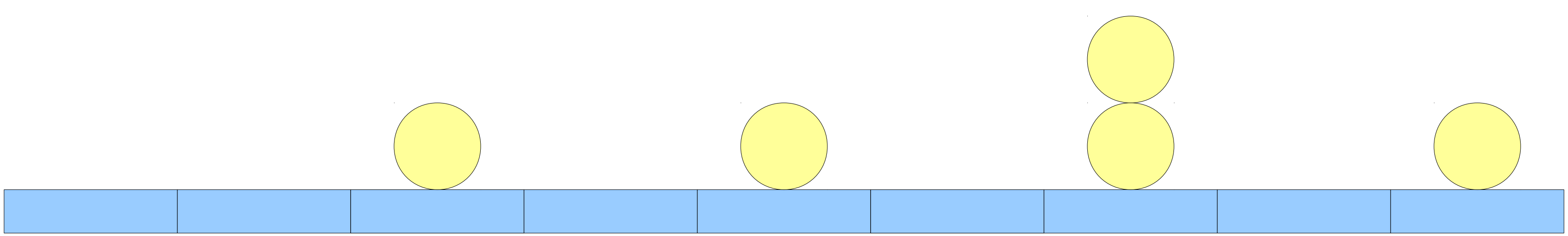
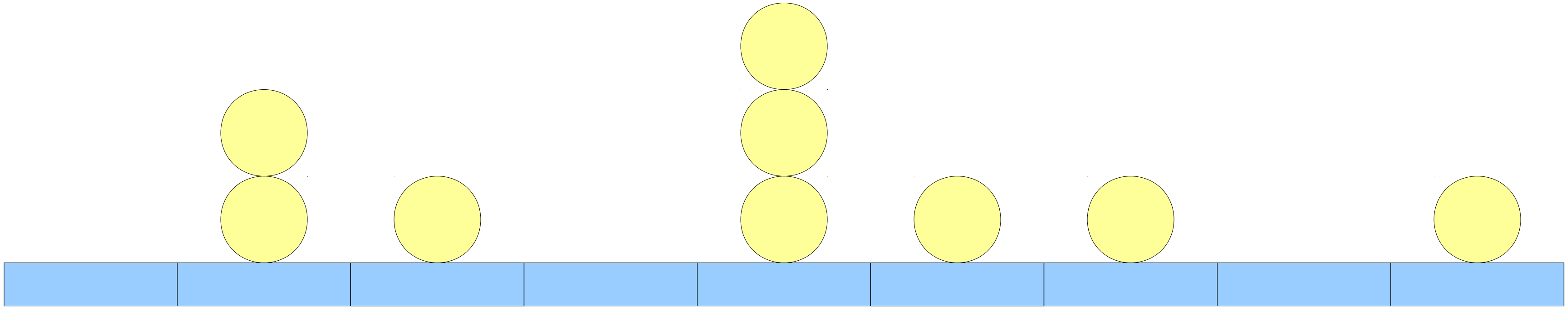
- Suppose that we distribute n balls into $\Theta(n)$ bins using the following strategy:
  - For each ball, choose two bins totally at random.
  - Put the ball into the bin with fewer balls in it; tiebreak randomly.
- ***Theorem:*** The expected value of the maximum number of balls in any urn is $\Theta(\log \log n)$.
- ***Proof:*** Nontrivial; see "Balanced Allocations" by Azar et al.



Second-Choice Hashing

- ***Idea:*** Build a chained hash table with two hash functions h_1 and h_2.
- To insert an element x, compute $h_1(x)$ and $h_2(x)$ and place x into whichever bucket is less full.
- To perform a lookup, compute $h_1(x)$ and $h_2(x)$ and search both buckets for x.



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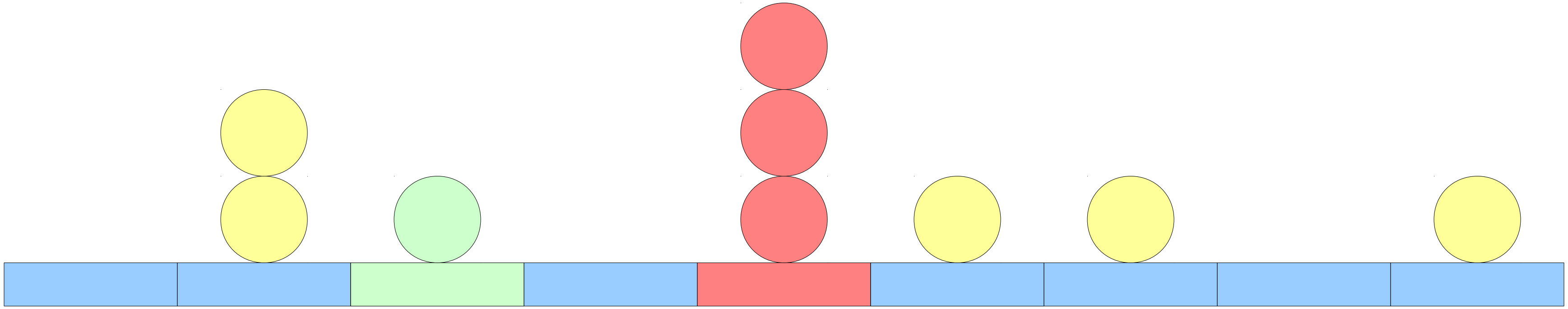

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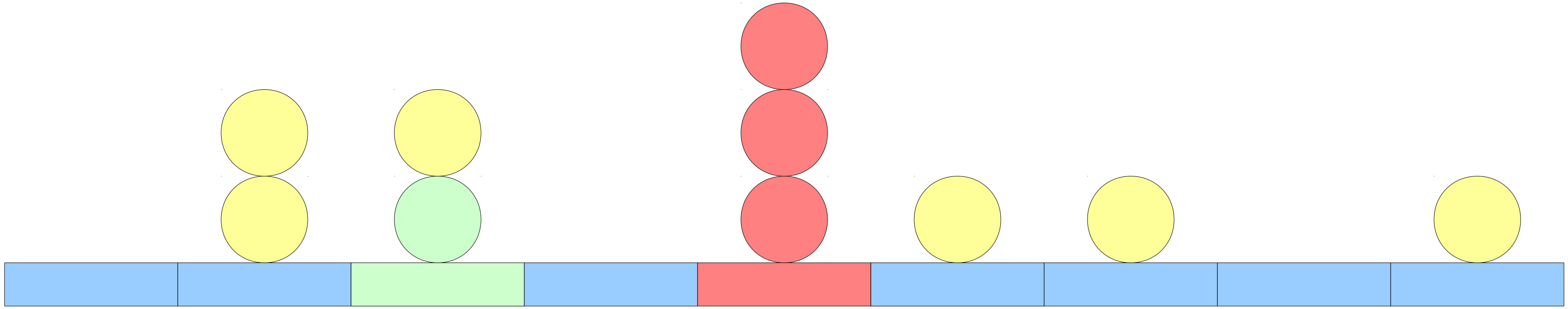

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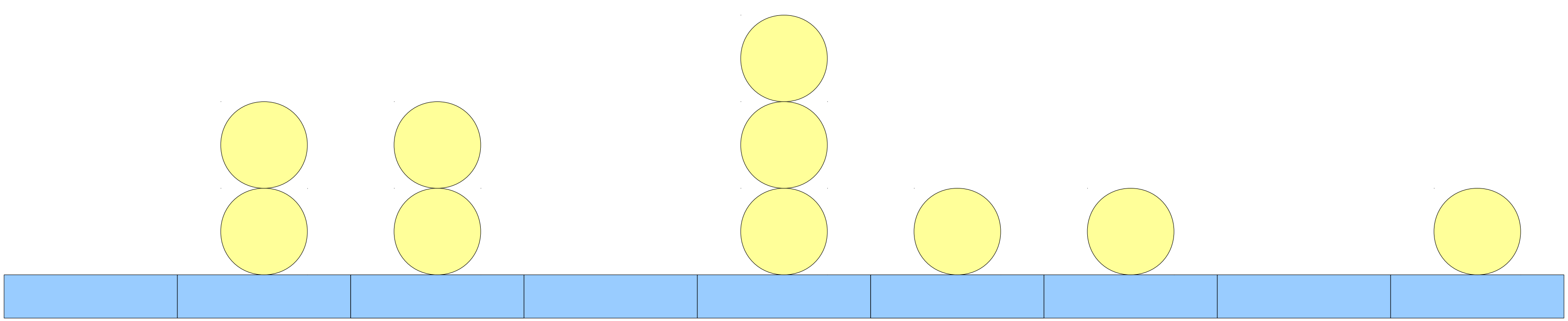
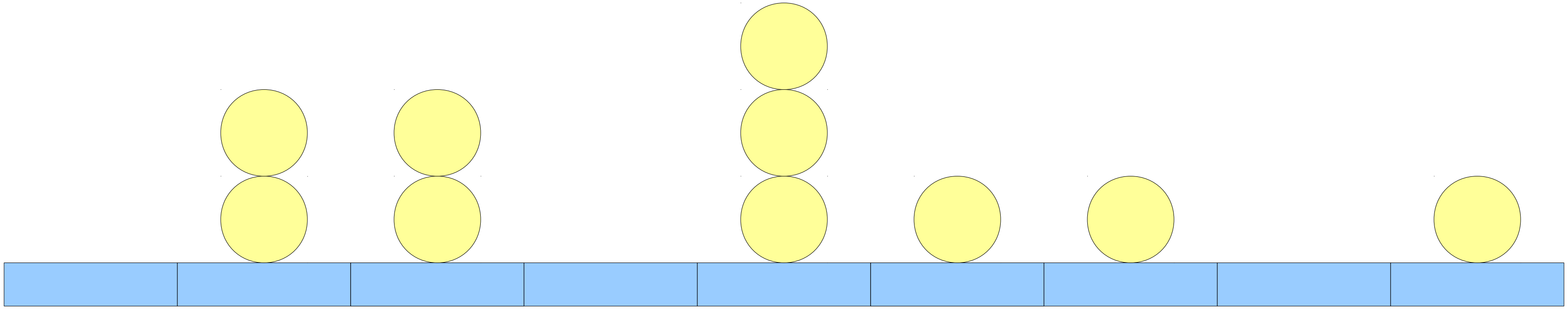


Second-Choice Hashing

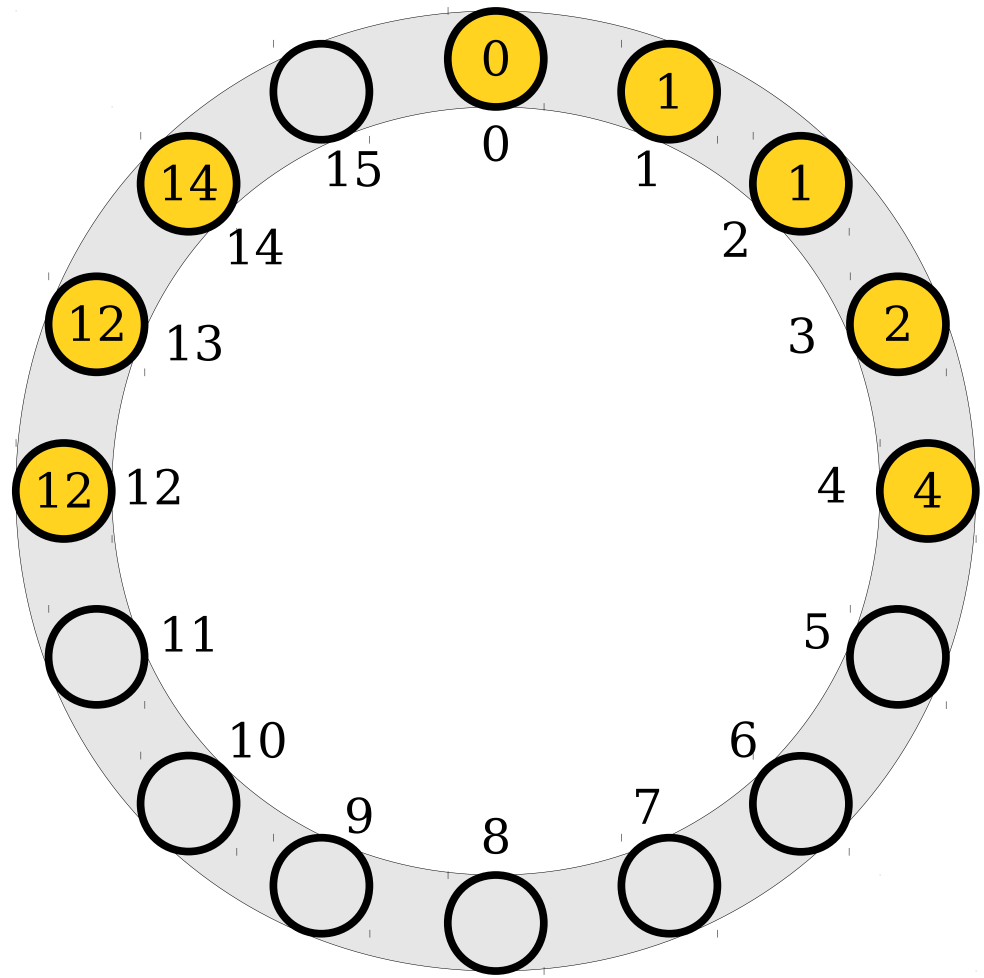
- ***Theorem:*** The expected cost of a lookup in such a hash table is $O(1 + \alpha)$. *(Why?)*
- ***Theorem:*** Assuming truly random hash functions, the expected worst-case cost of a lookup in such a hash table is $O(\log \log n)$. *(Why?)*
- ***Open problem:*** What is the smallest k for which there are k-independent hash functions that match the bounds using truly random hash functions?



Technique 2: Hashing with Relocation

Robin Hood Hashing

- ***Robin Hood hashing*** is a variation of open addressing where keys can be moved after they're placed.
- When an existing key is found during an insertion that's closer to its "home" location than the new key, it's displaced to make room for it.
- This dramatically decreases the variance in the expected number of lookups.
- It also makes it possible to terminate searches early.



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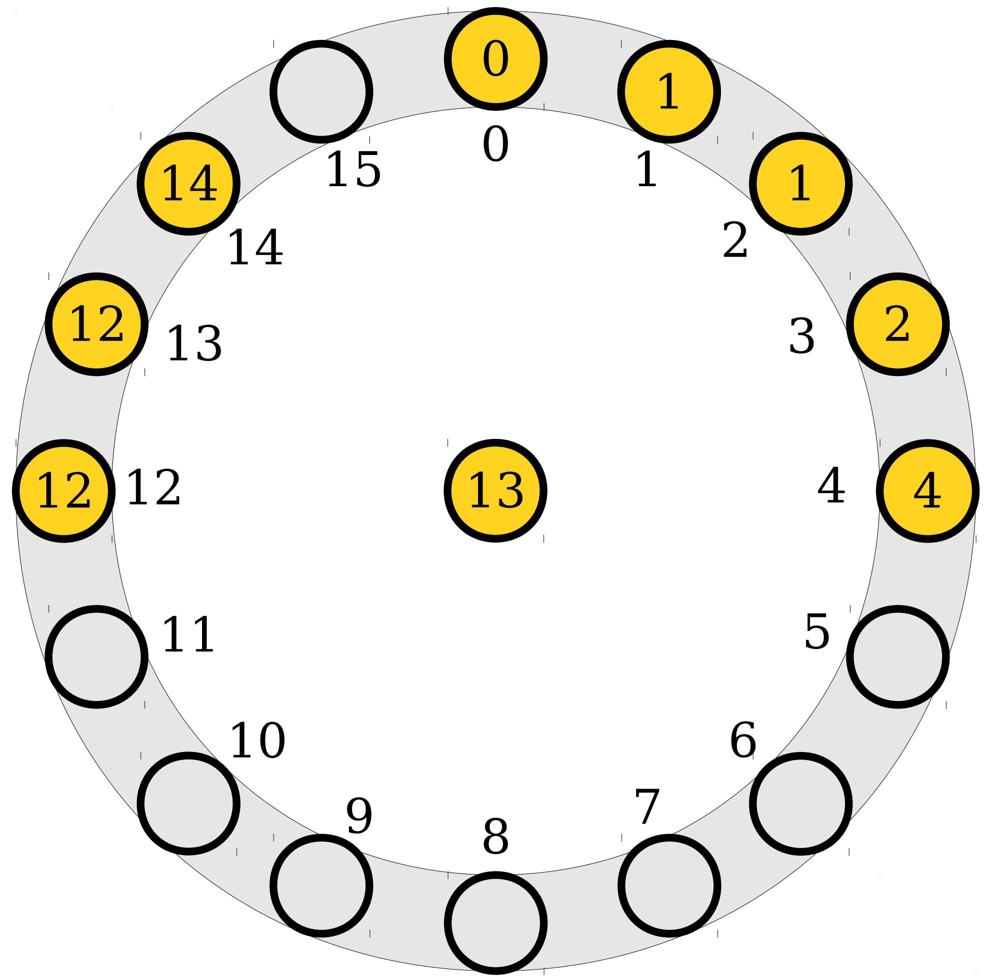

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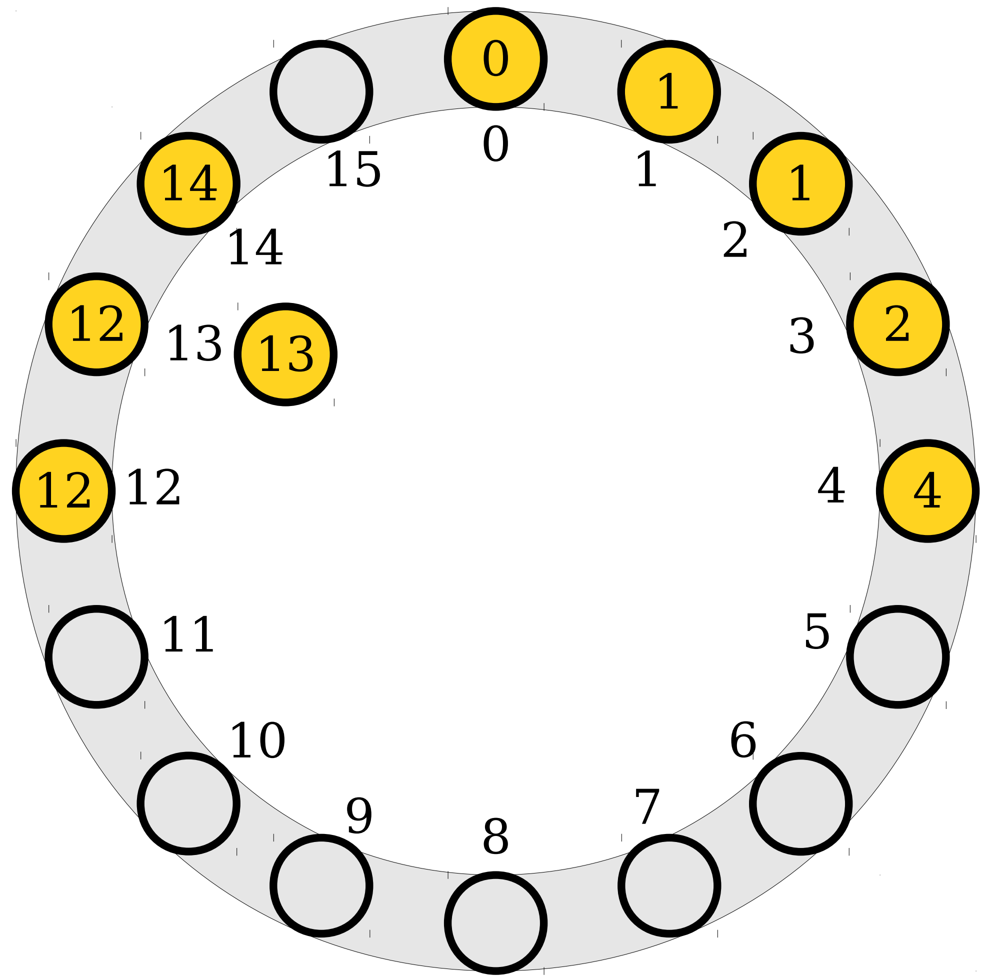

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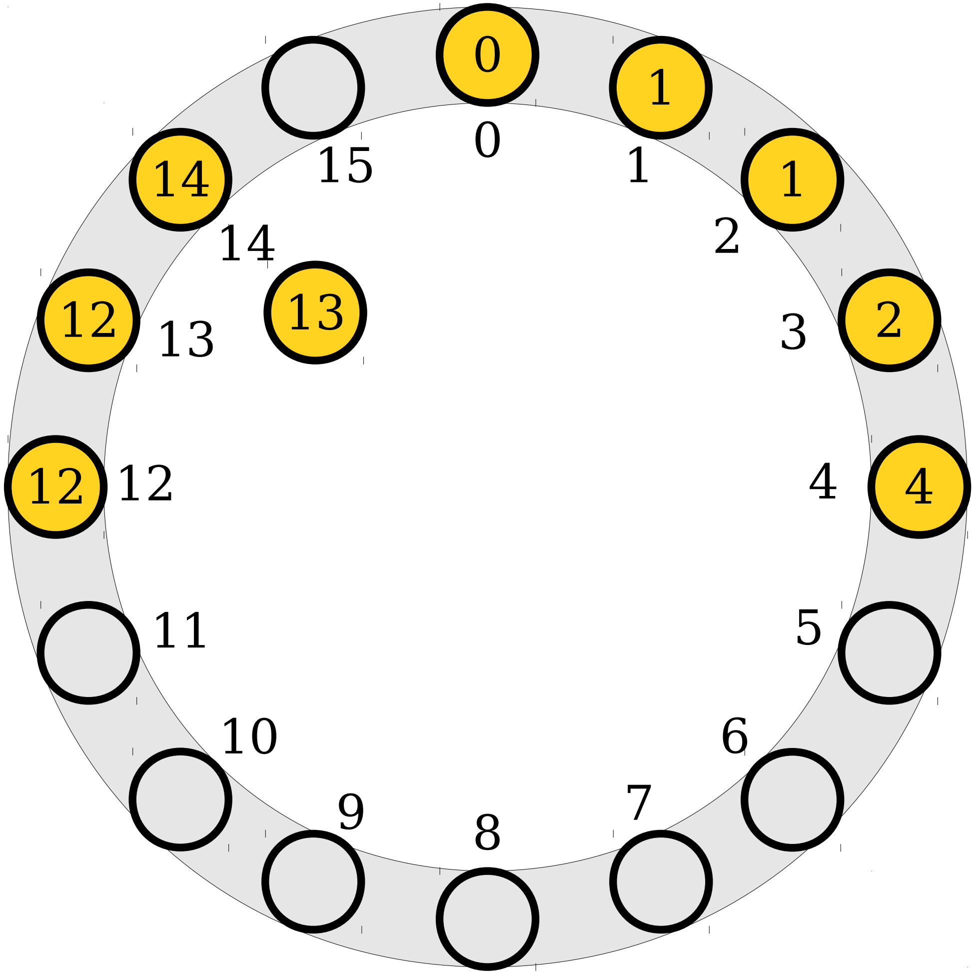

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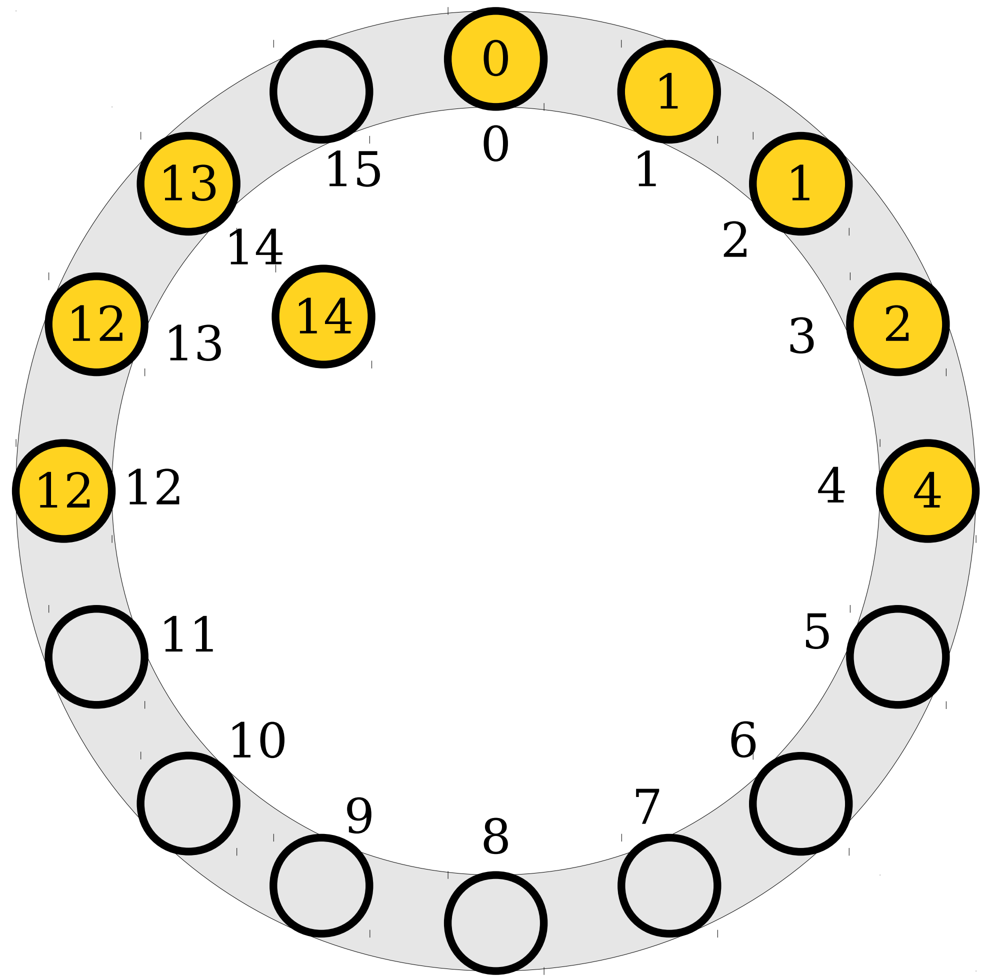

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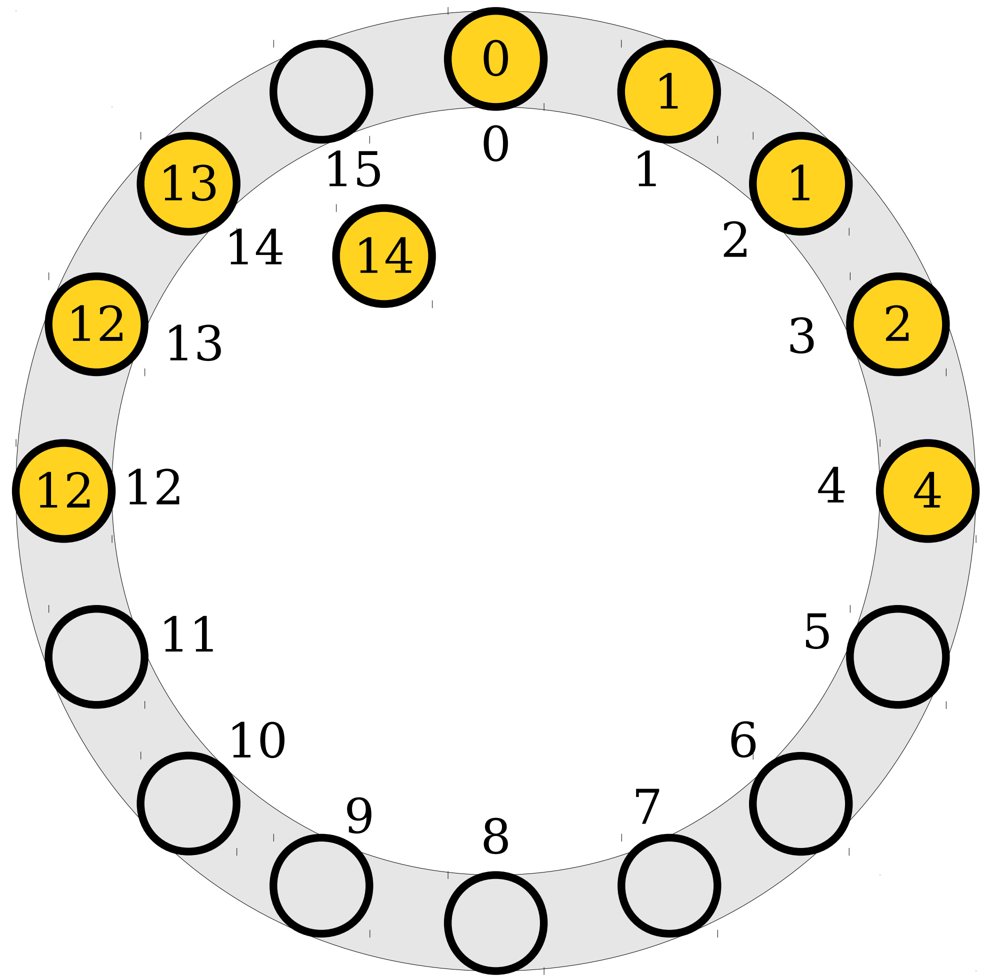

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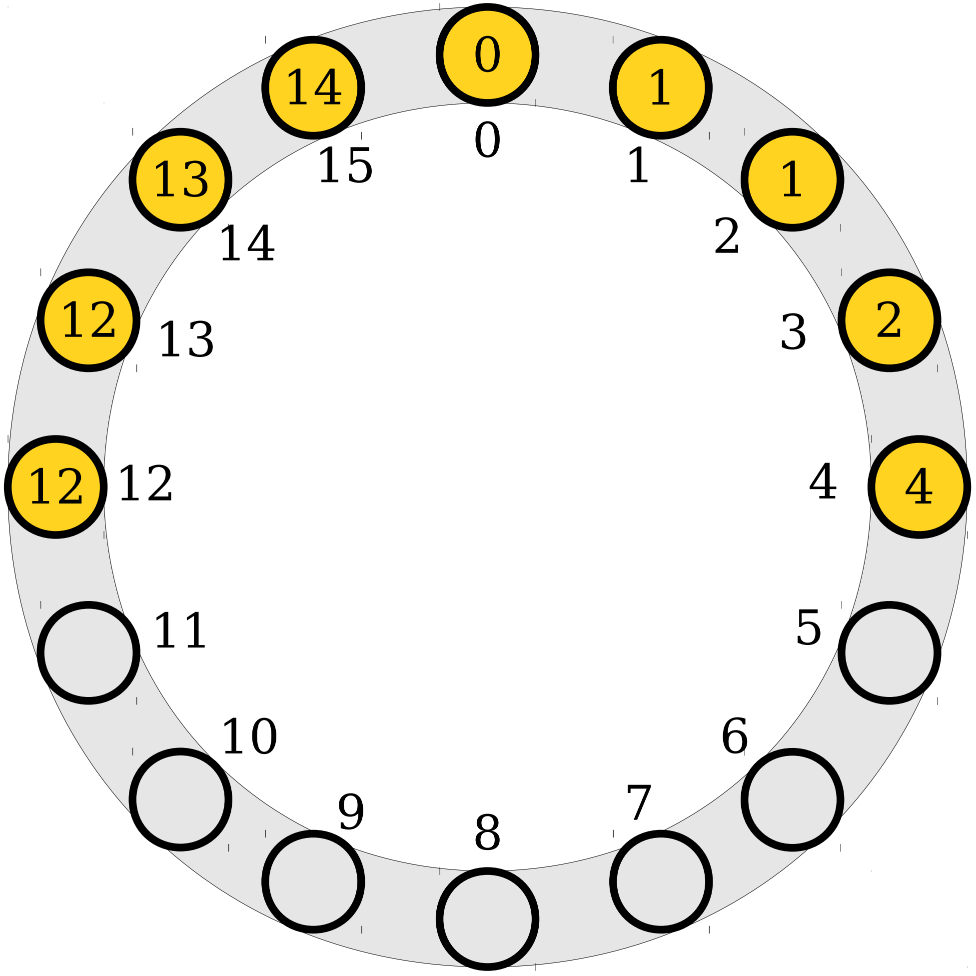

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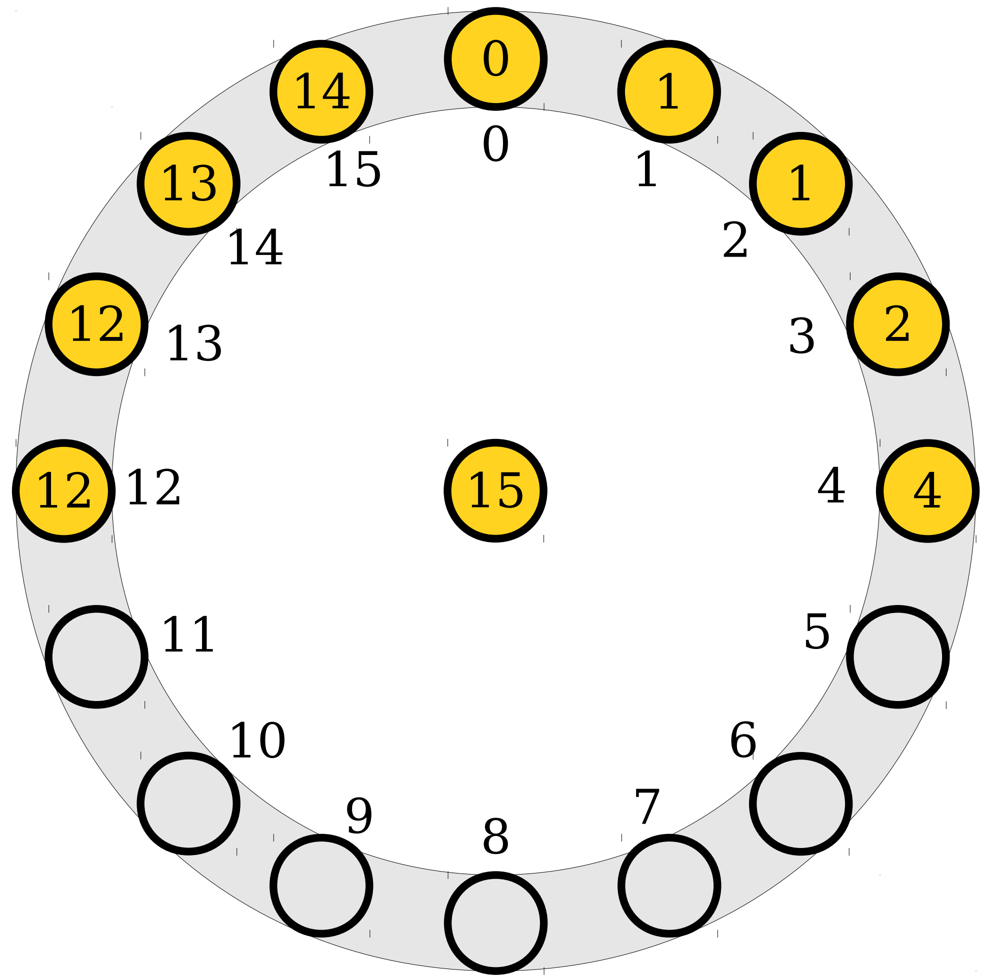

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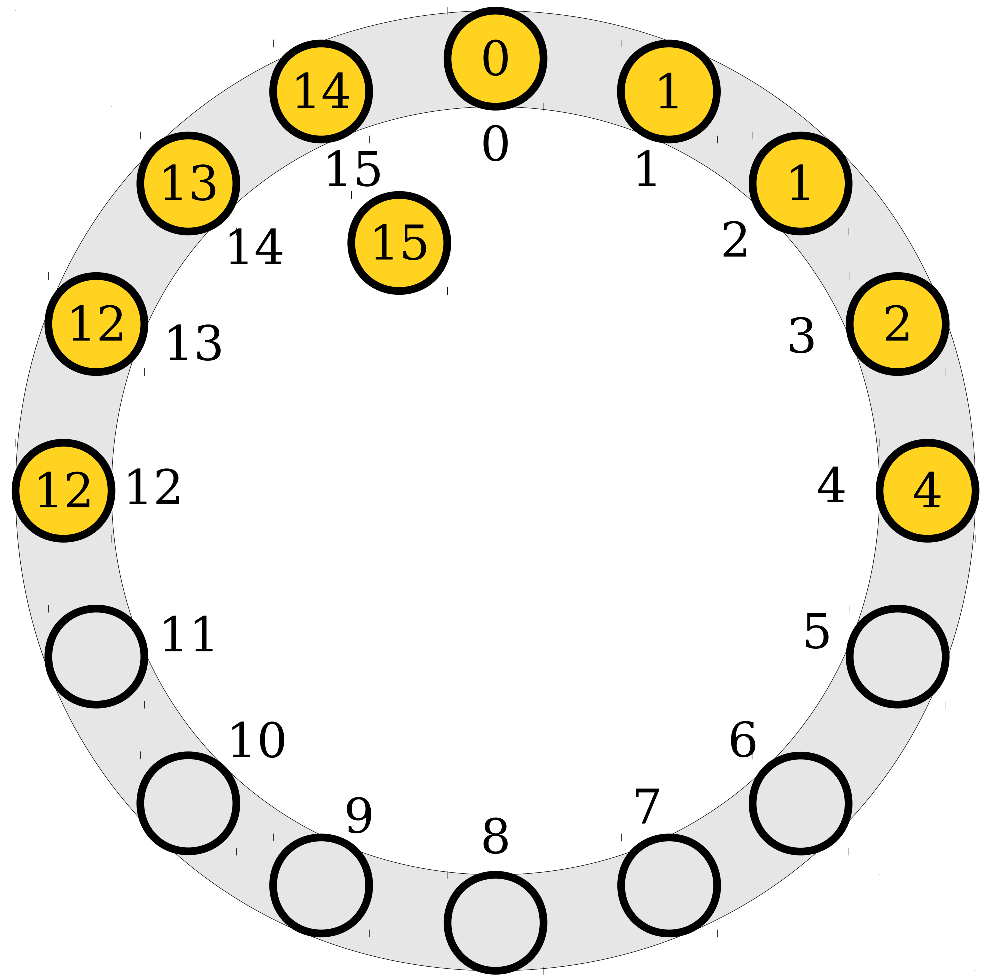

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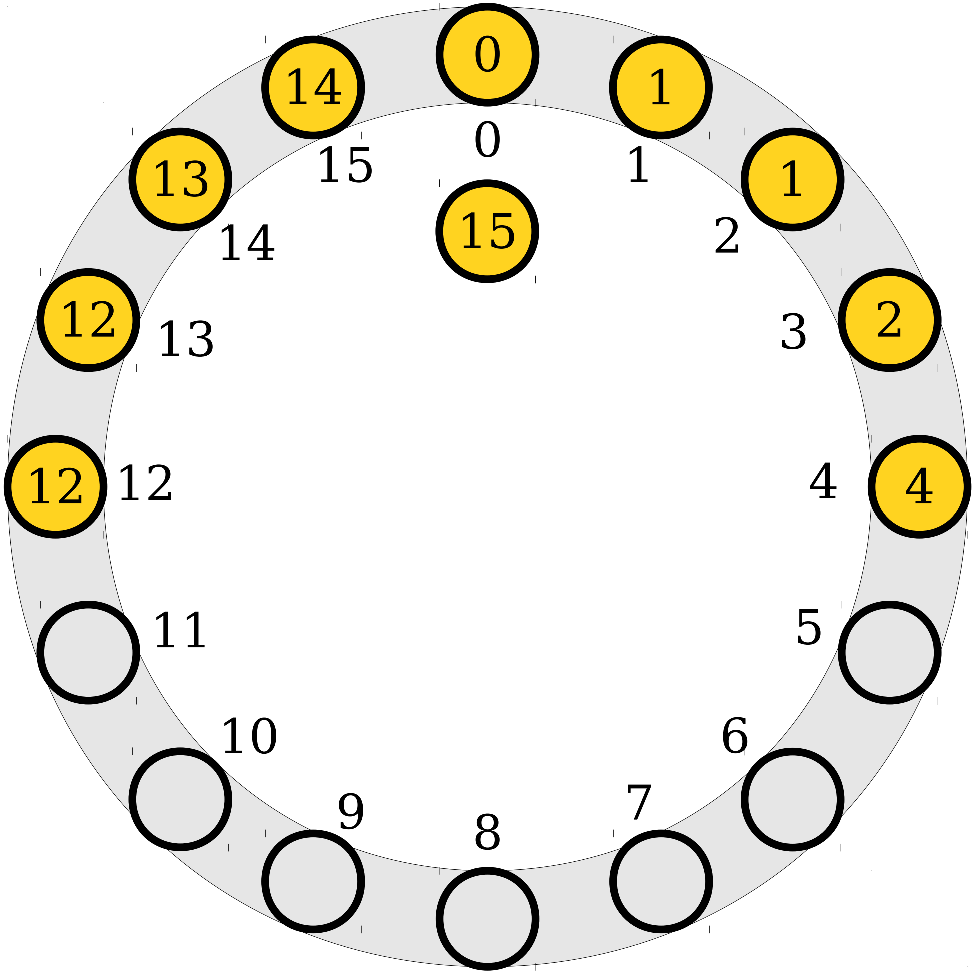

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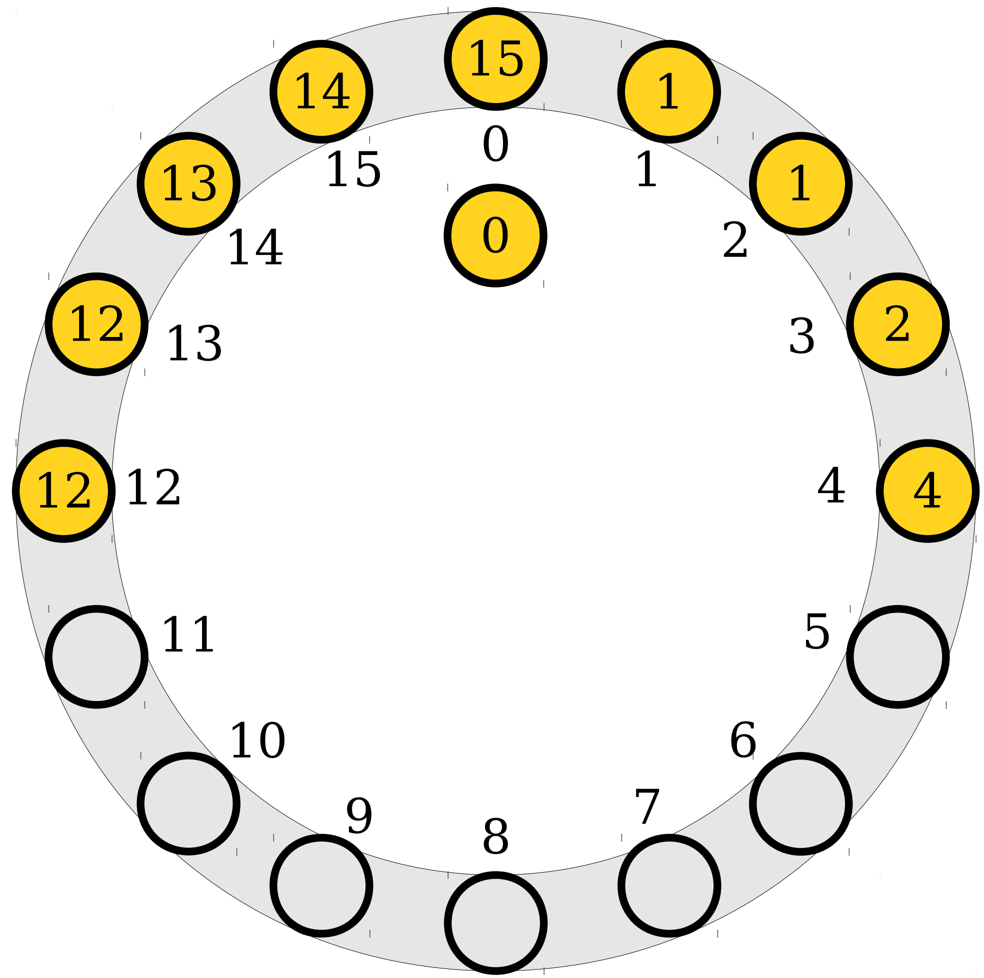

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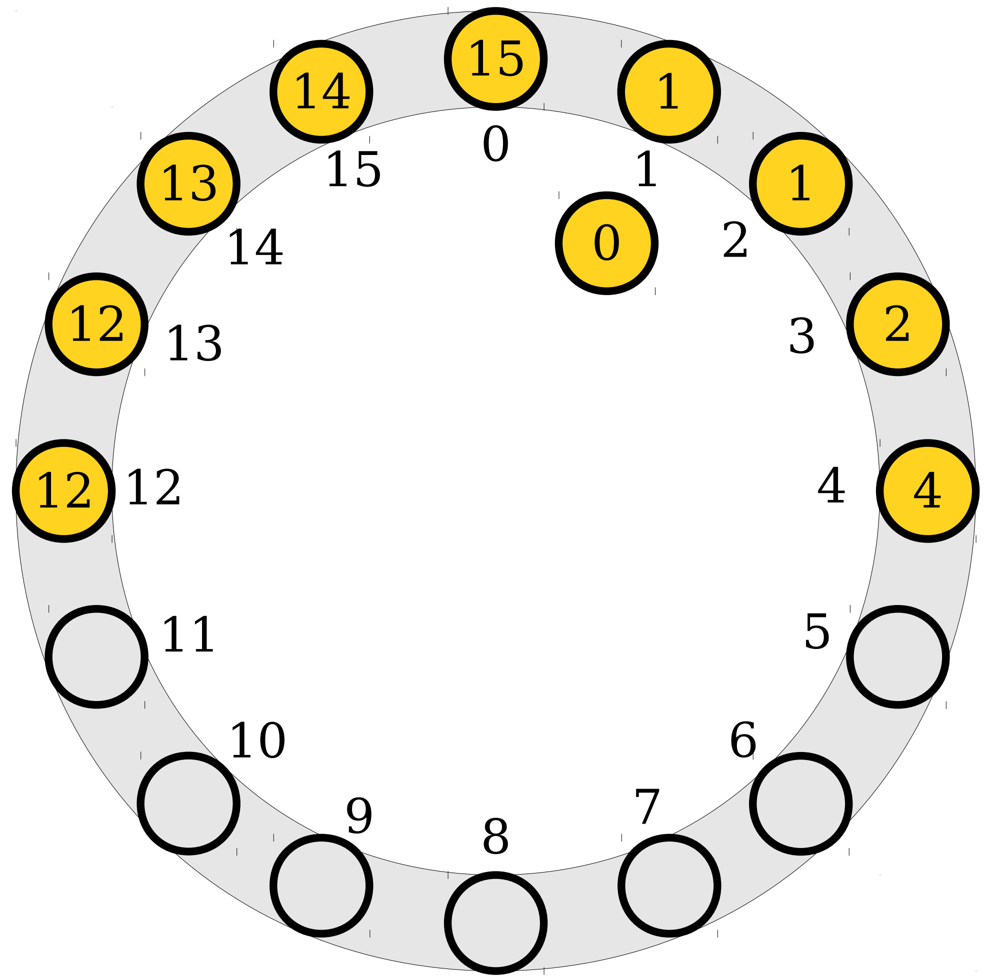

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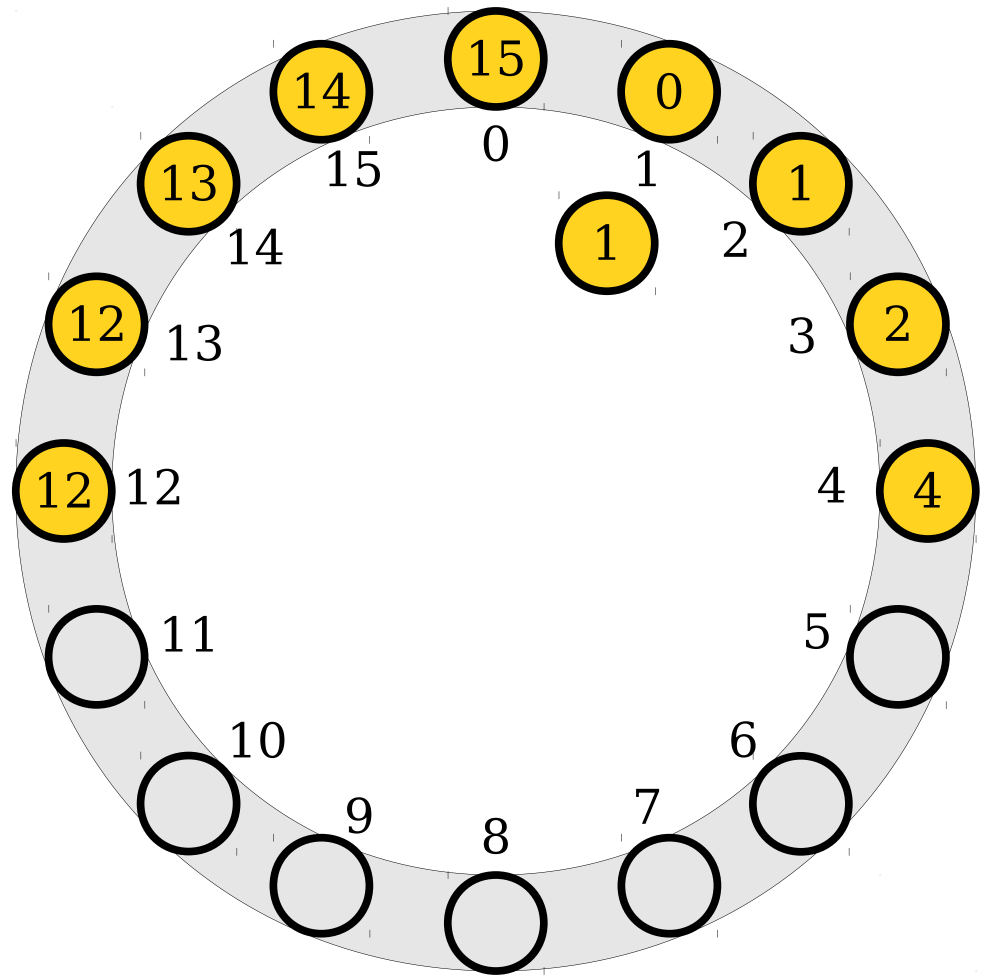

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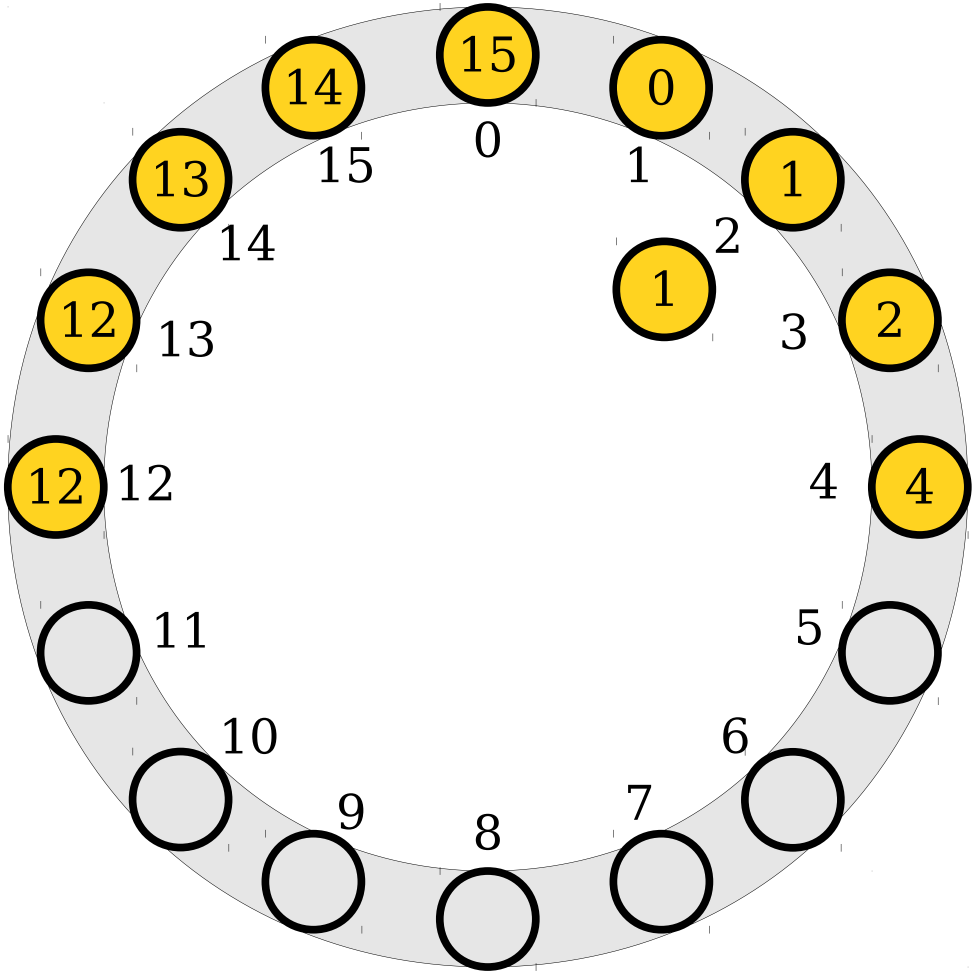

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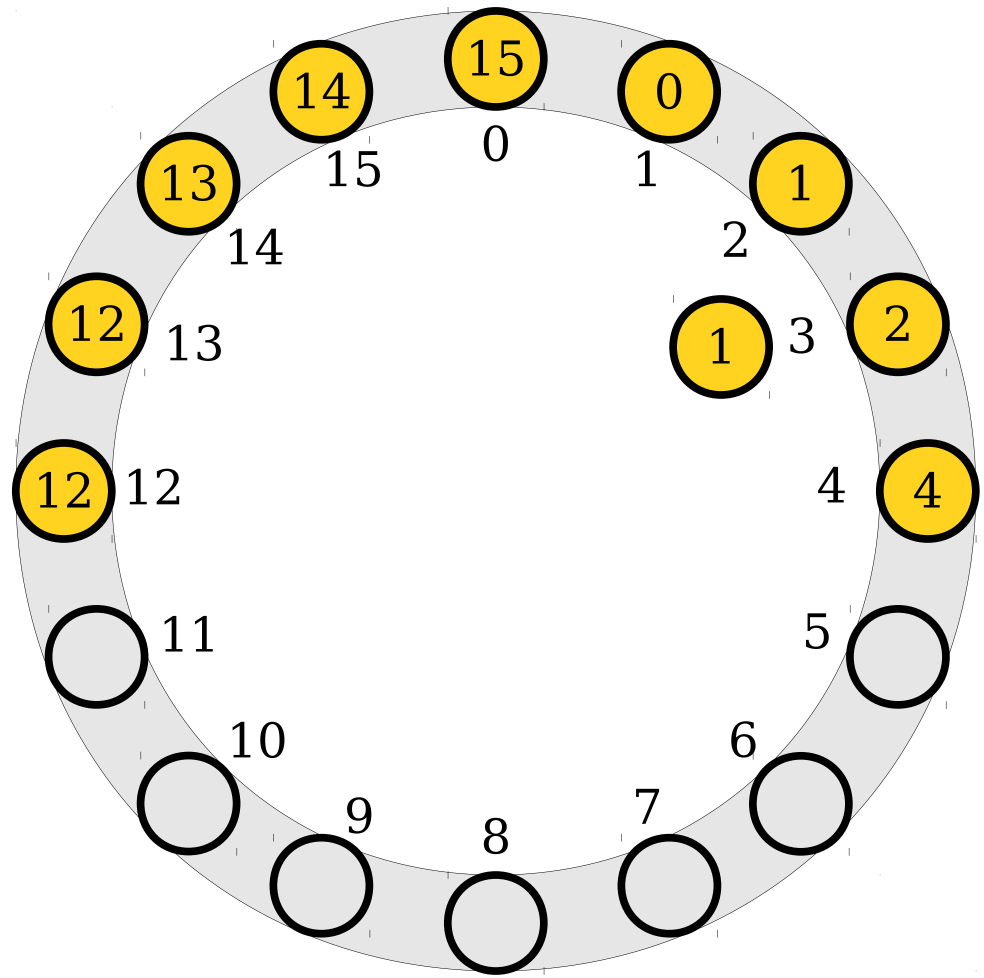

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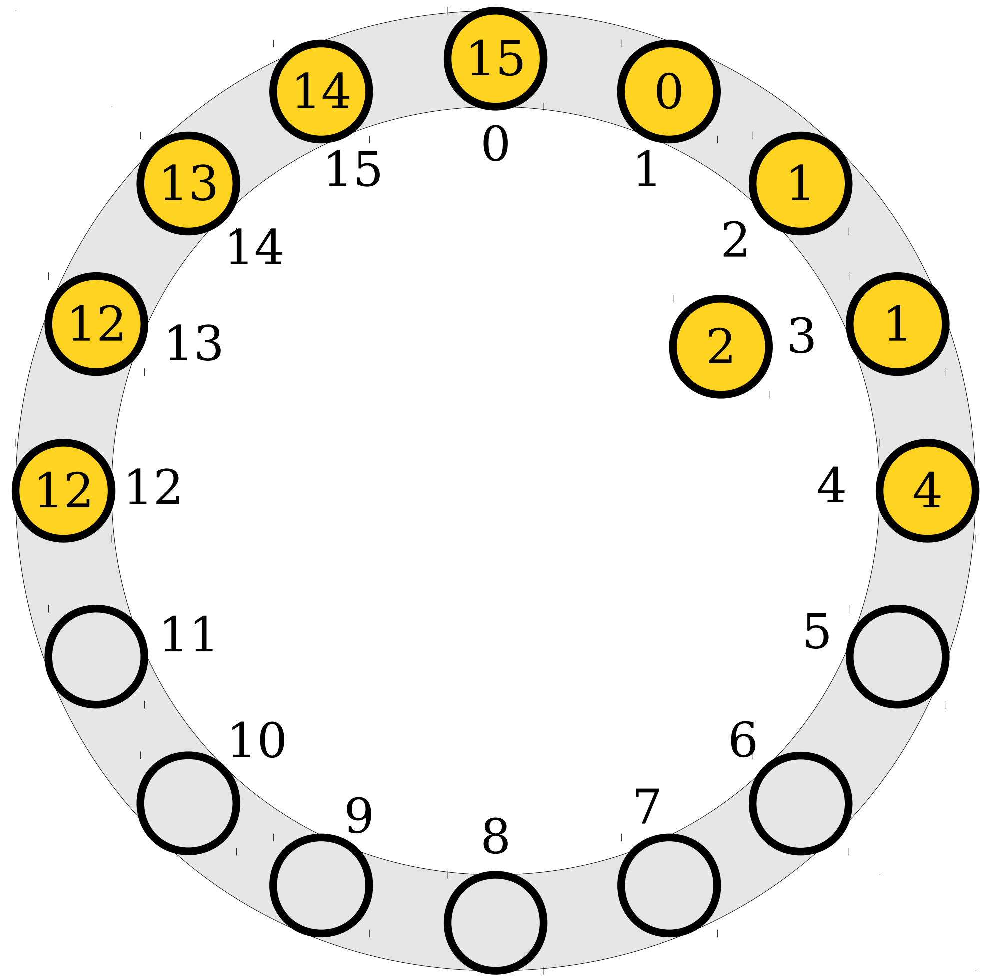

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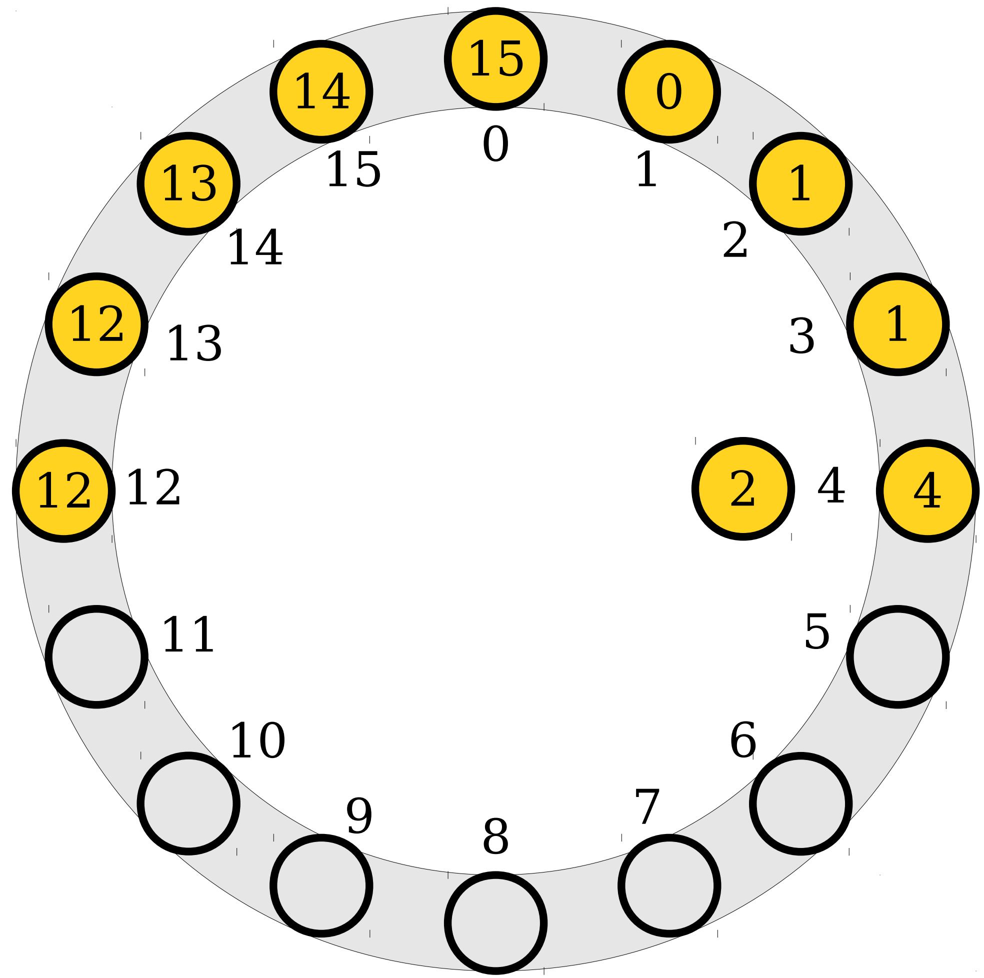

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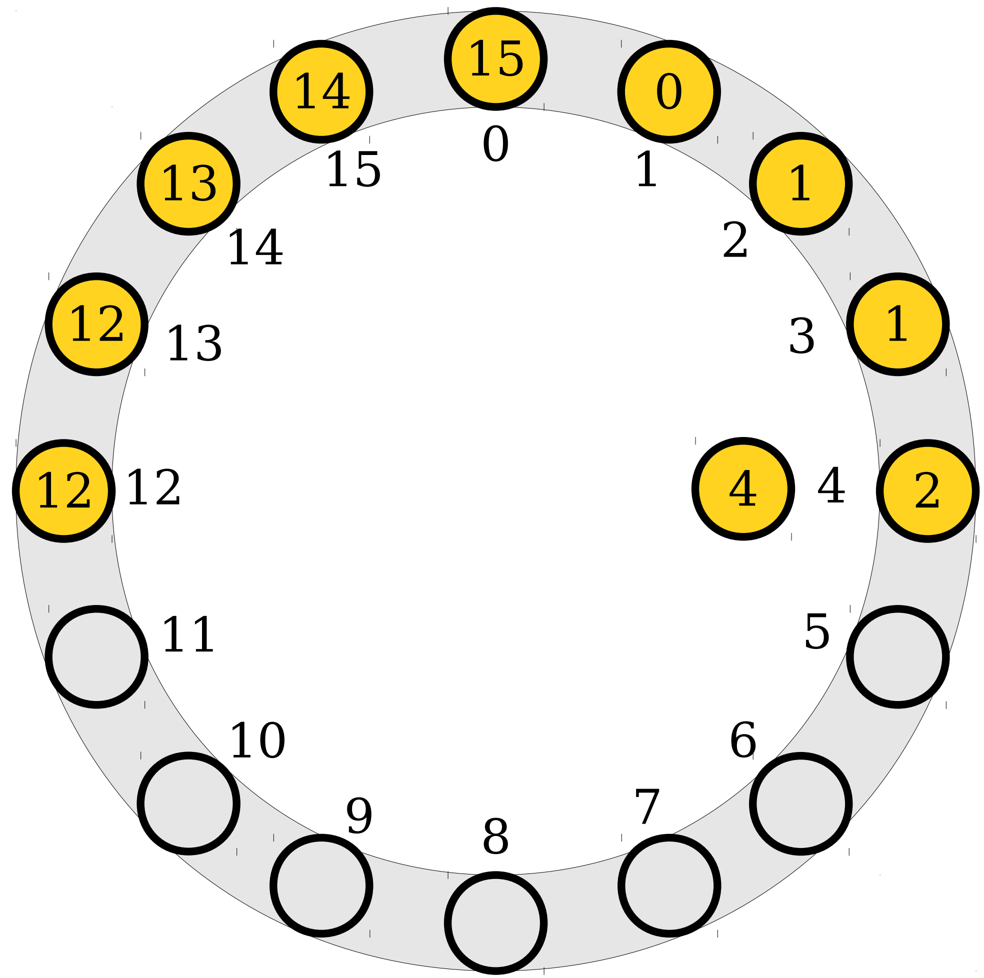

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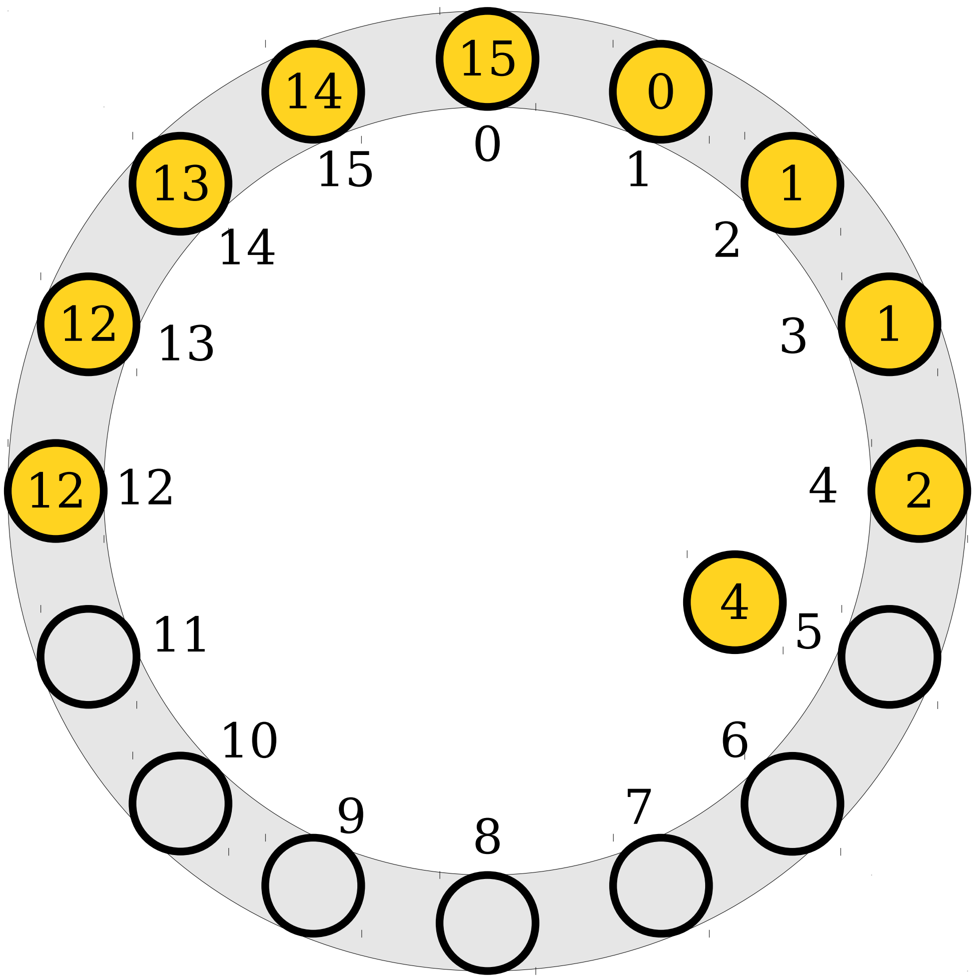

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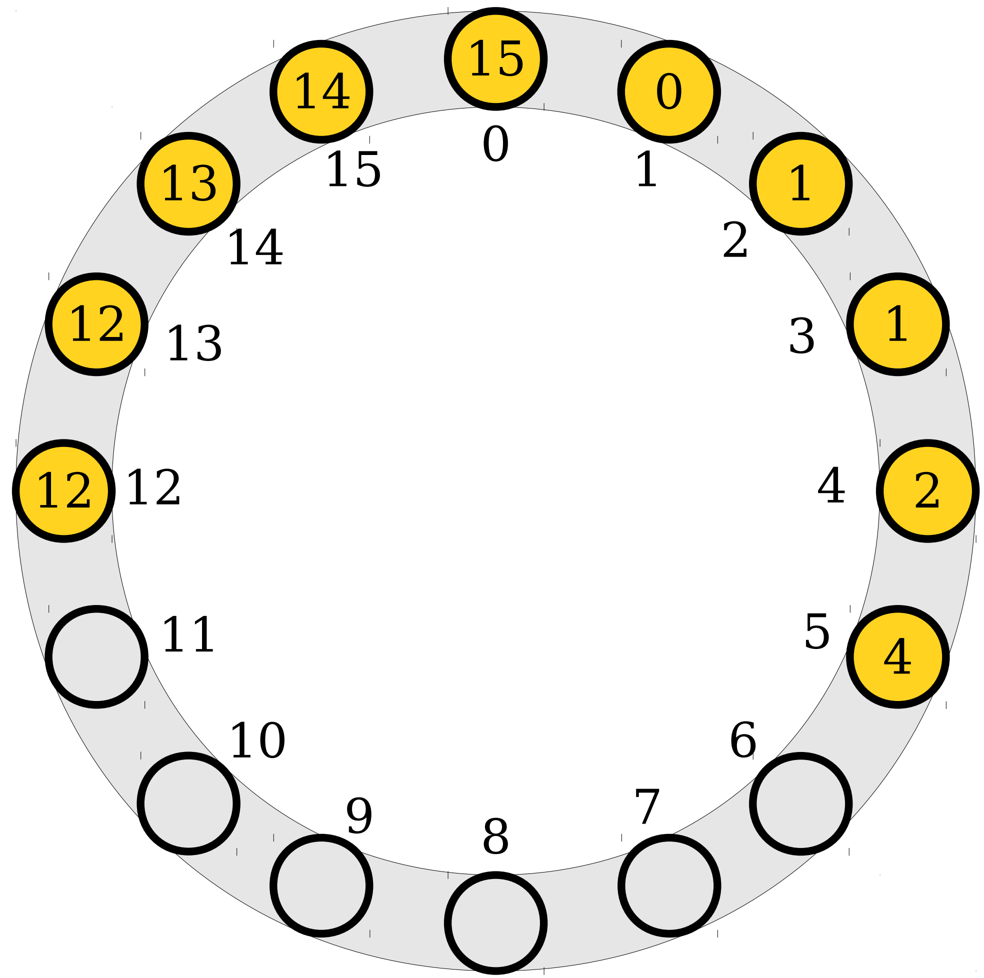
- ***Robin Hood hashing*** is a variation of open addressing where keys can be moved after they're placed.
- When an existing key is found during an insertion that's closer to its “home” location than the new key, it's displaced to make room for it.
- This dramatically decreases the variance in the expected number of lookups.
- It also makes it possible to terminate searches early.



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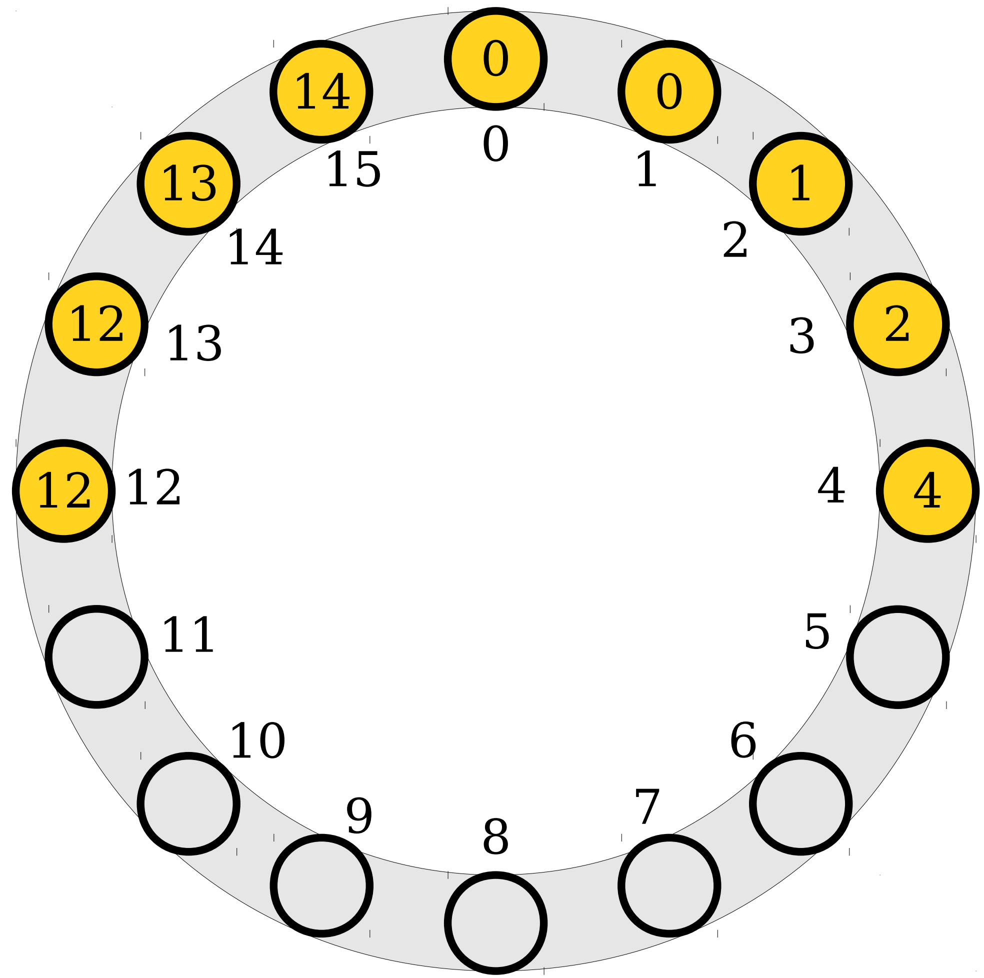

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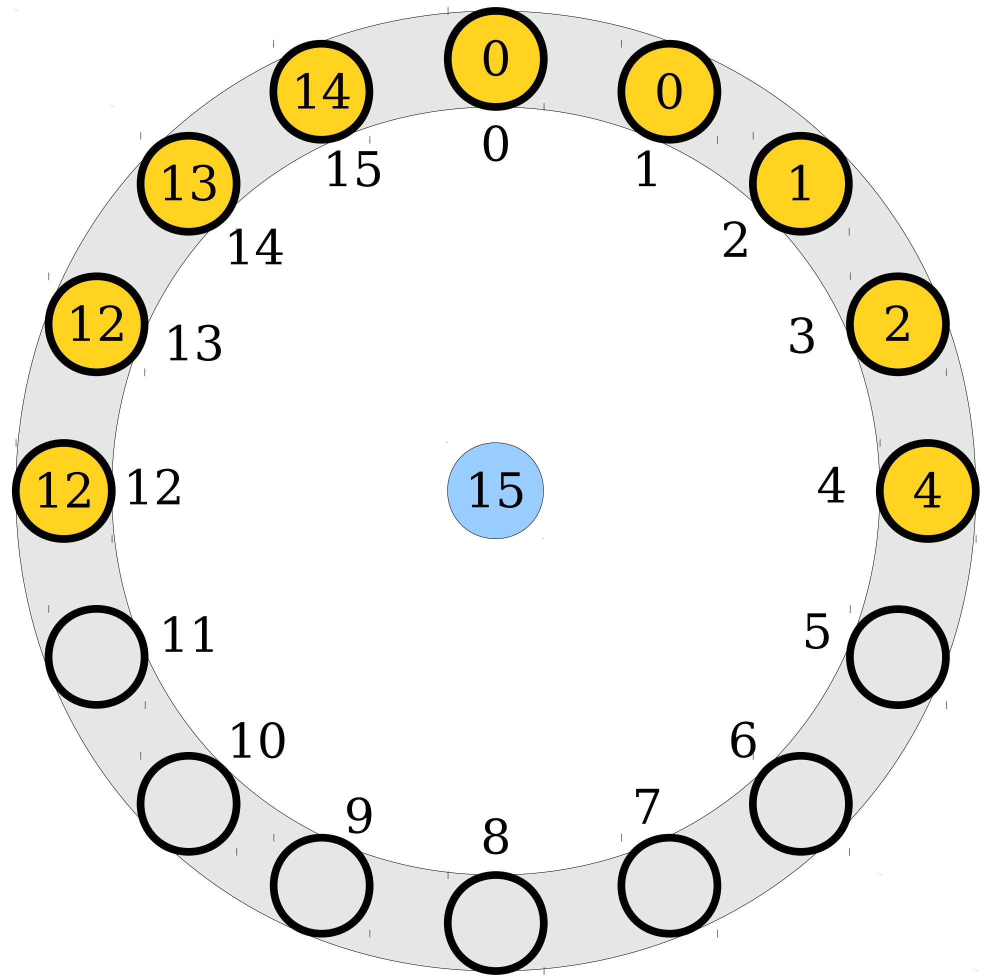

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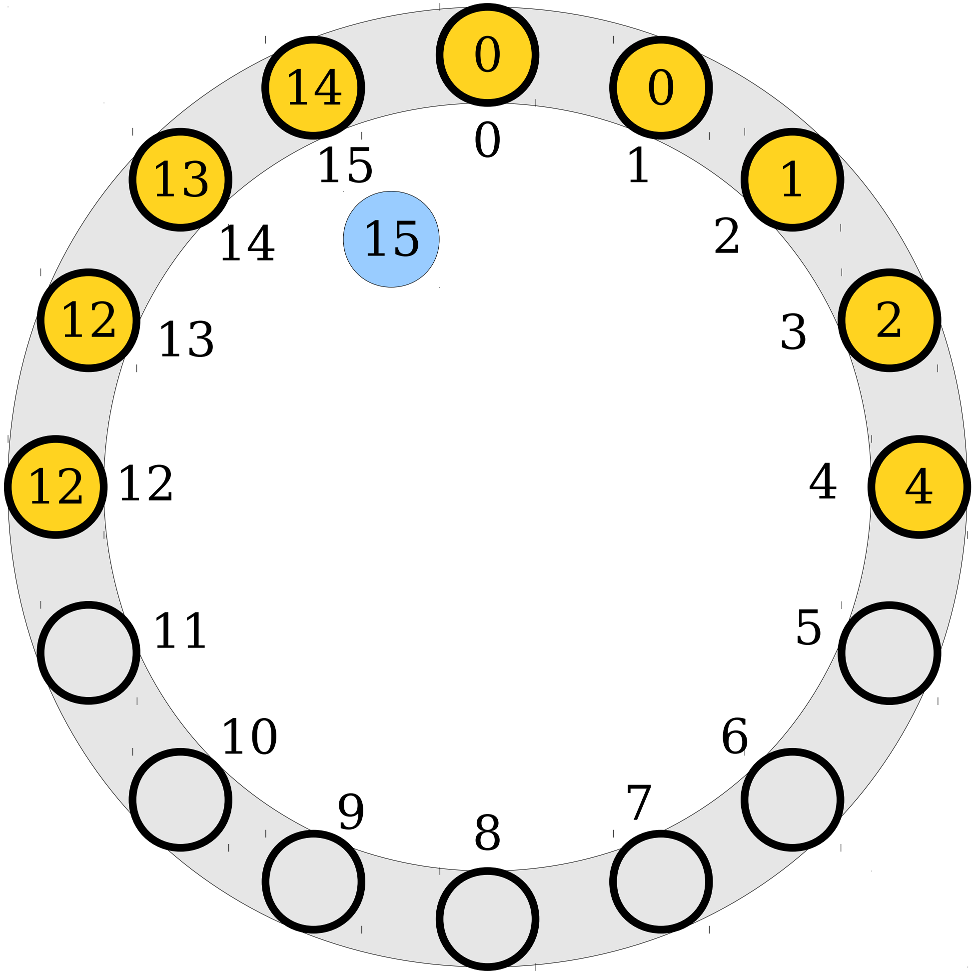

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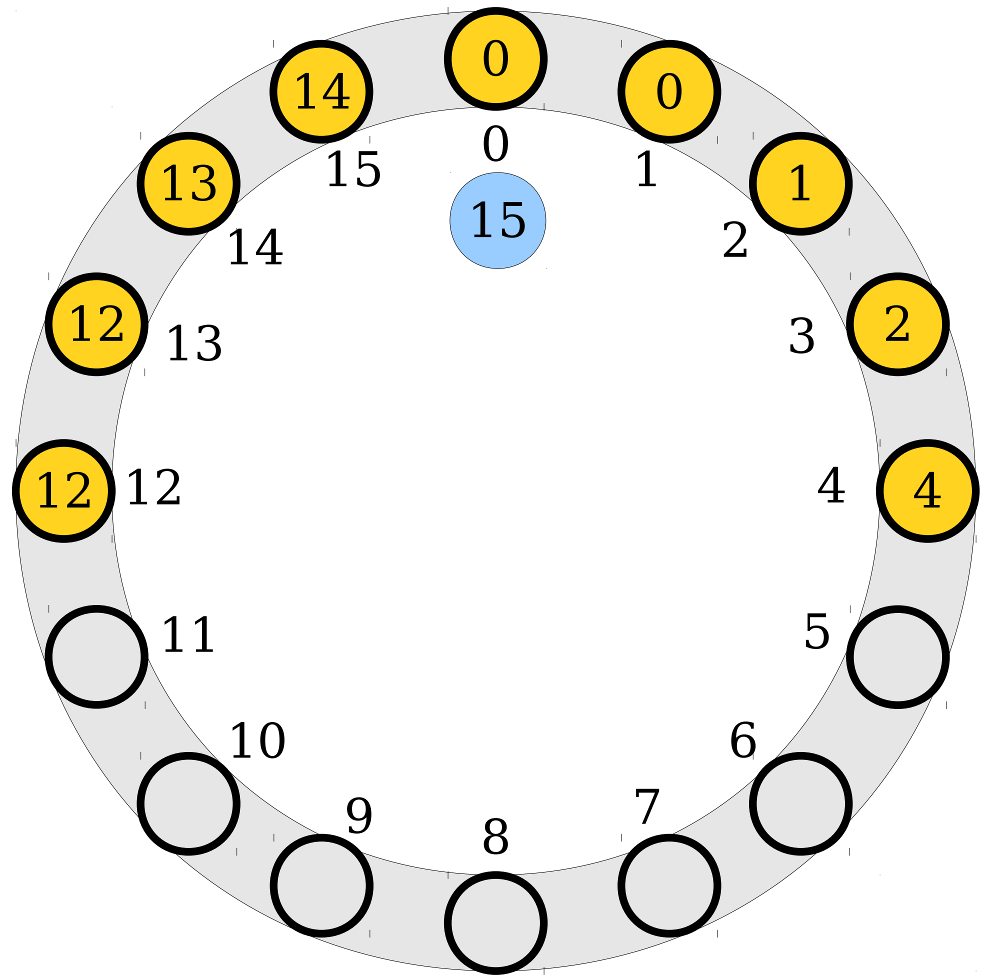

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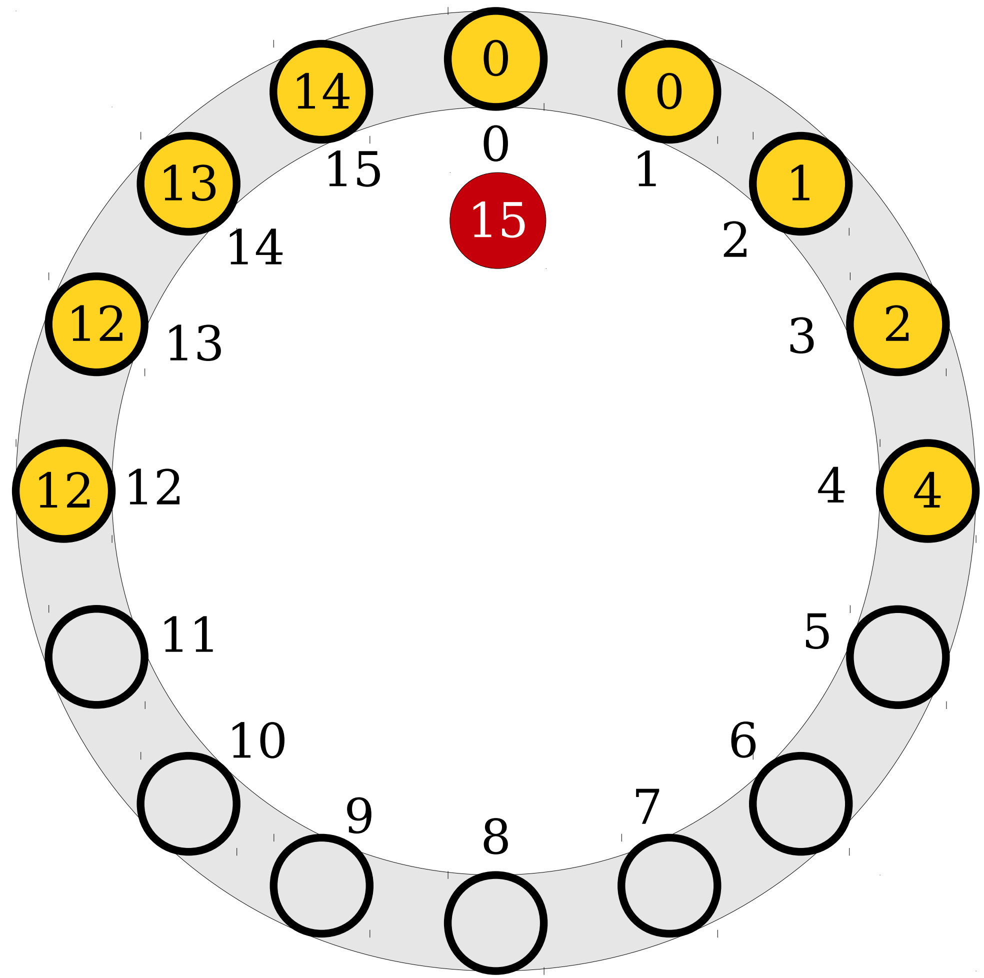
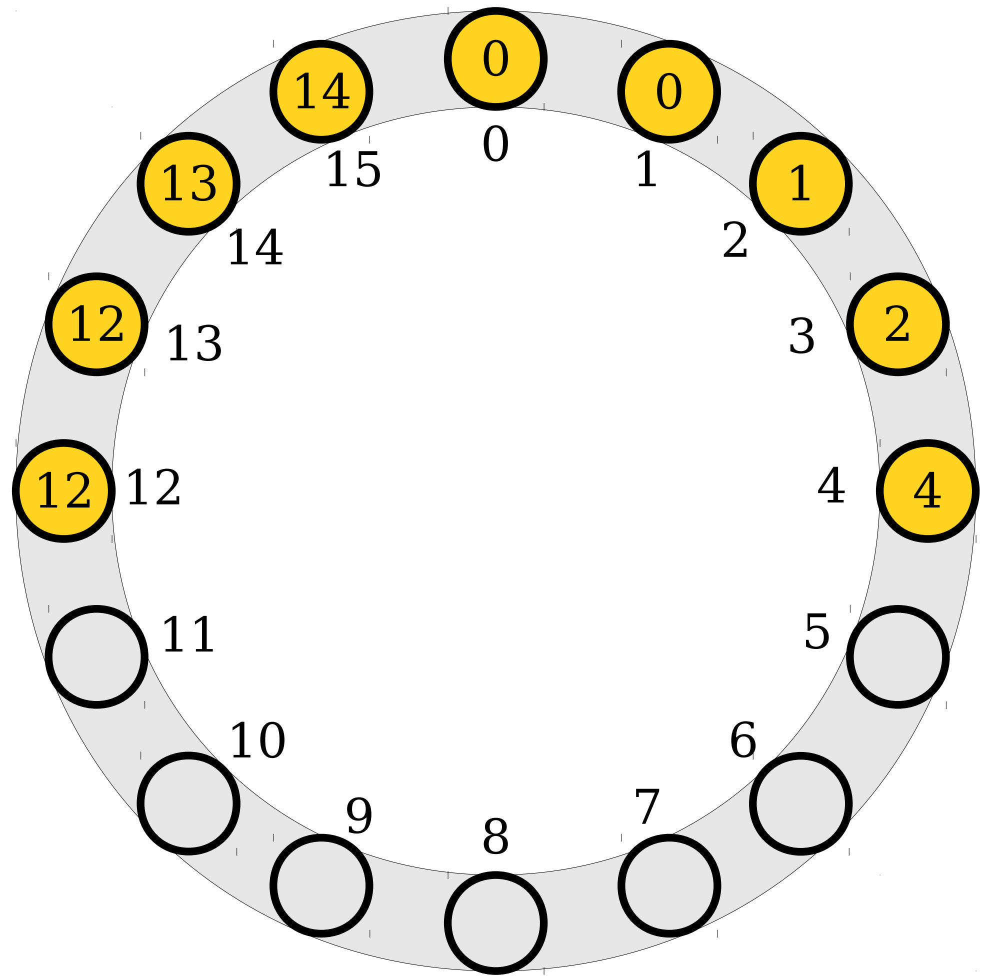


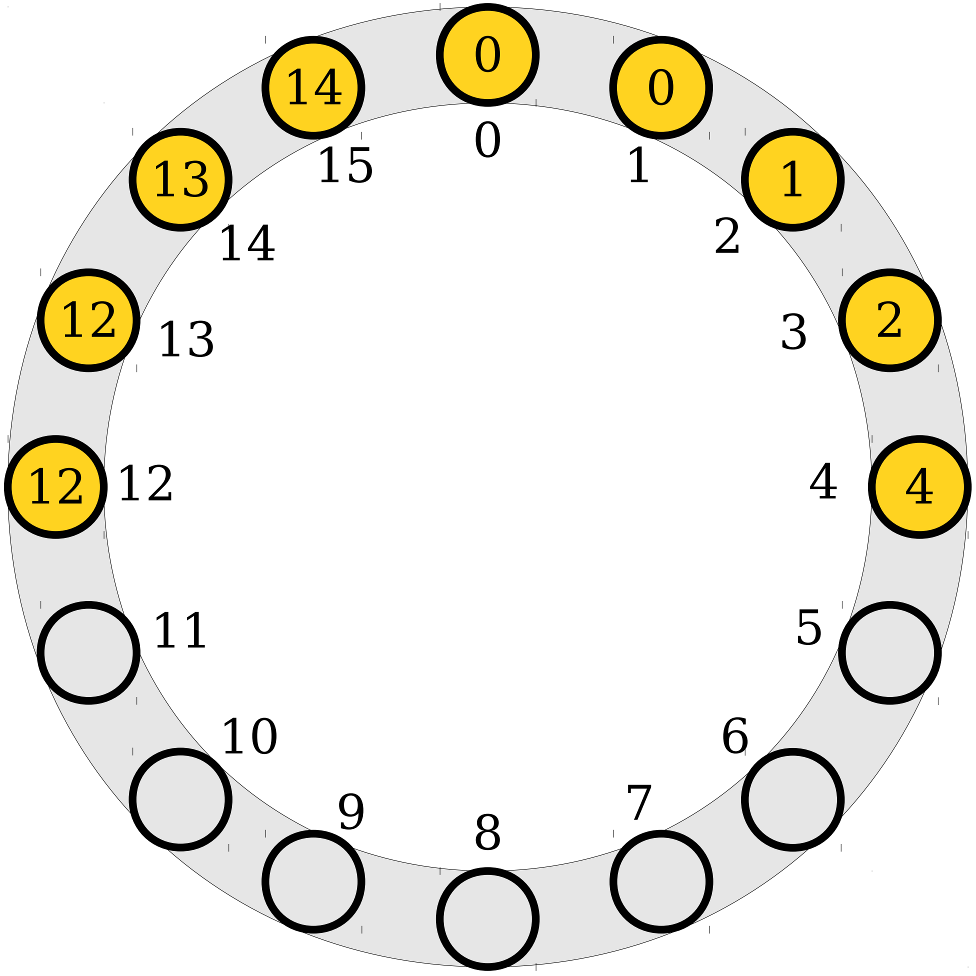
Robin Hood Hashing

- ***Theorem:*** The expected cost of a lookup in Robin Hood hashing, using 5-independent hashing, is O(1), assuming a constant load factor.
- ***Proof idea:*** Each element is hashed into the same run as it would have been hashed to in linear probing.



Robin Hood Hashing

- ***Theorem:*** Assuming truly random hash functions, the variance of the expected number of probes required in Robin Hood hashing is O(log log n).
- ***Proof:*** Tricky; see Celis' Ph.D thesis.



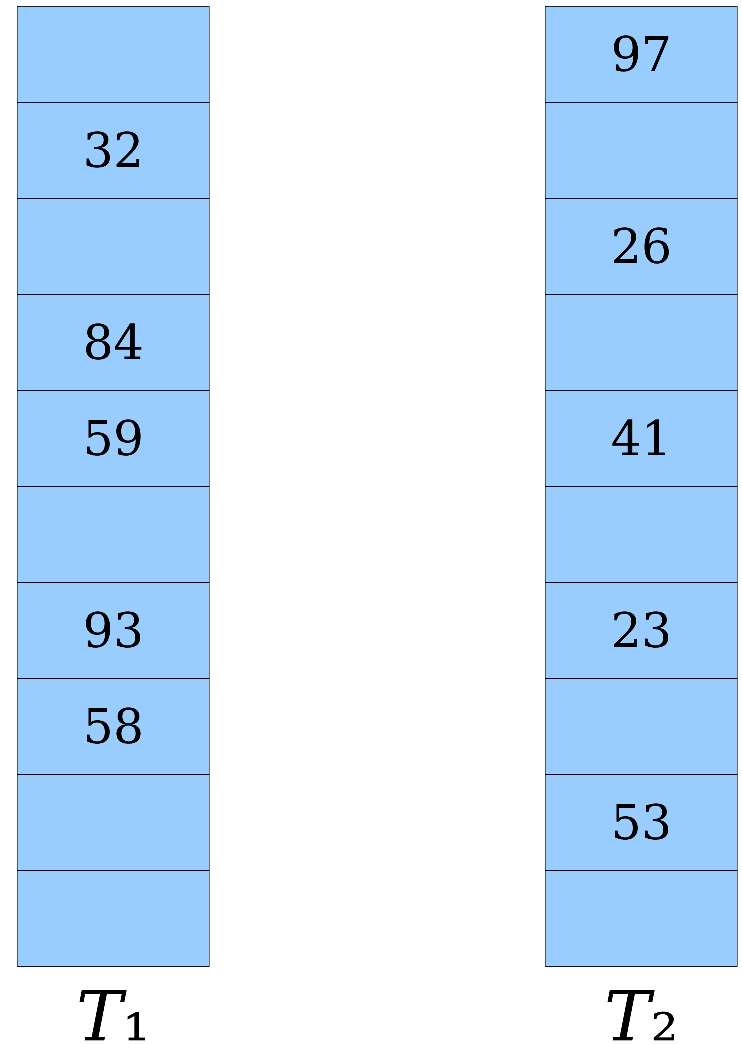
Where We Stand

- We now have two interesting ideas for improving performance:
  - ***Second-choice hashing:*** Give each element multiple homes to pick from.
  - ***Relocation hashing:*** Move elements after placing them.
- Each idea, individually, exponentially decreases the worst-case cost of a lookup by decreasing the variance in the element distribution.
- What happens if we combine these ideas together?

Cuckoo Hashing

Cuckoo Hashing

- Maintain two tables, each of which has m elements.
- We choose two hash functions h_1 and h_2 from $\mathscr{U}$ to $[m]$.
- Every element $x \in \mathscr{U}$ will either be at position $h_1(x)$ in the first table or $h_2(x)$ in the second.



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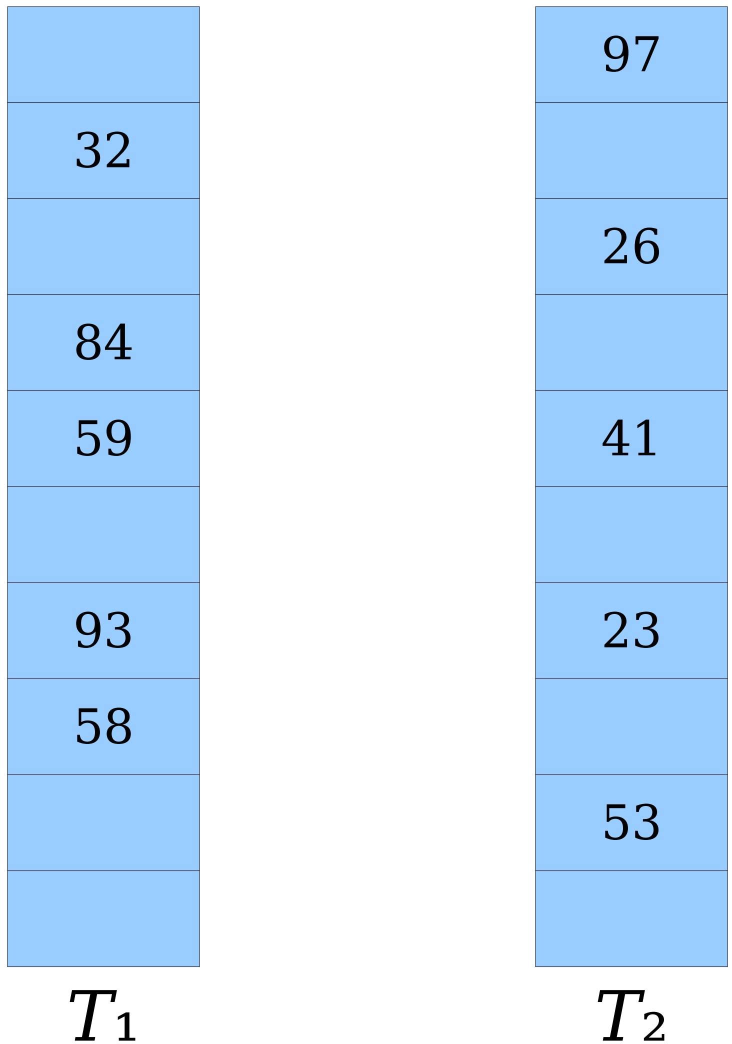

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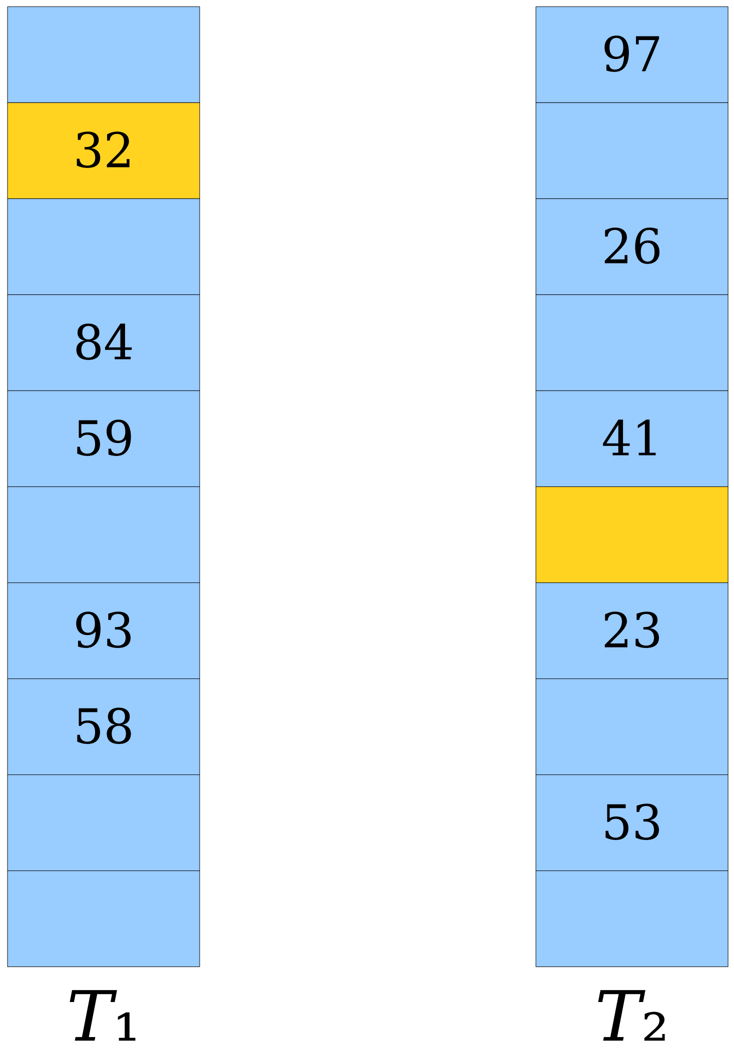

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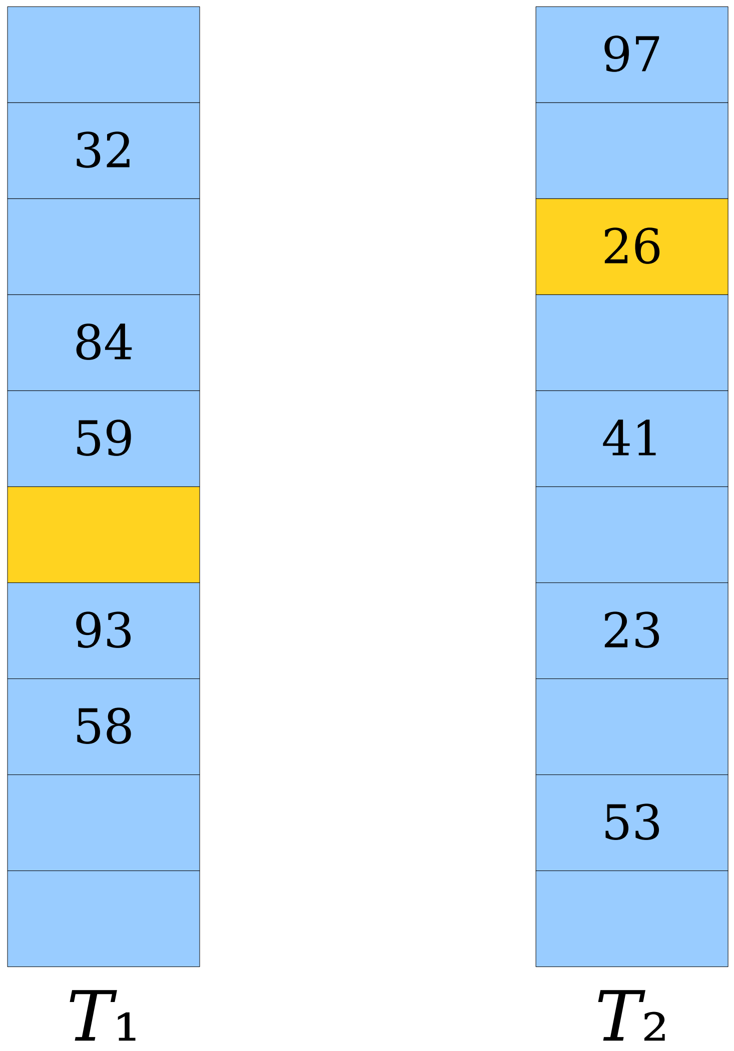

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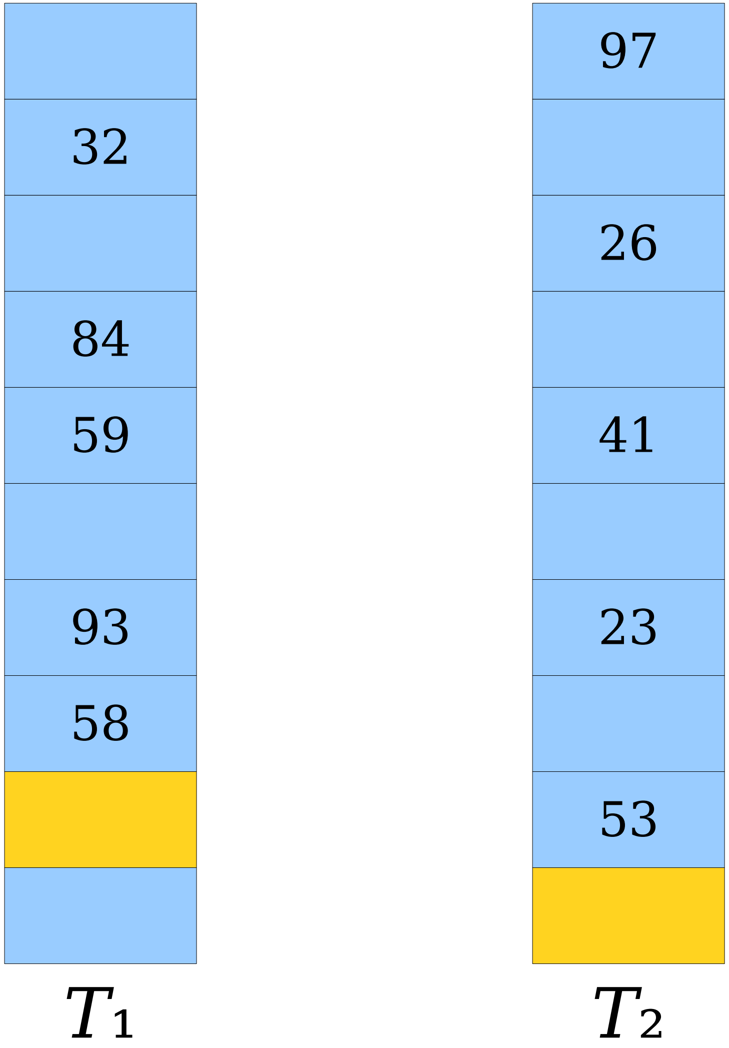

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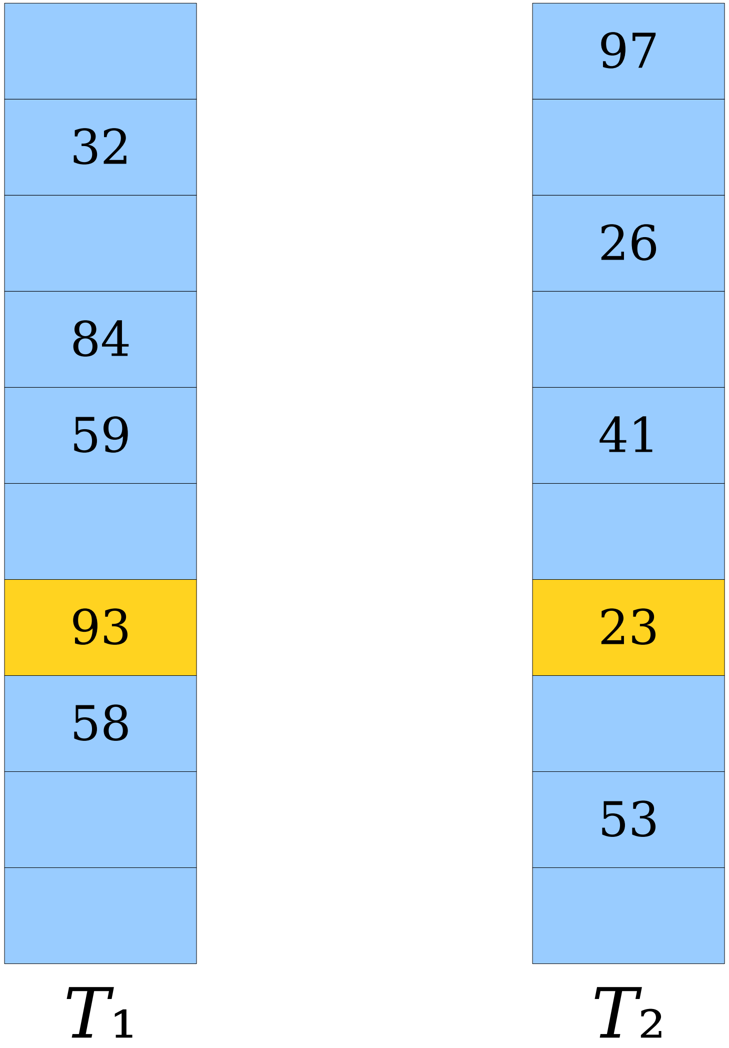

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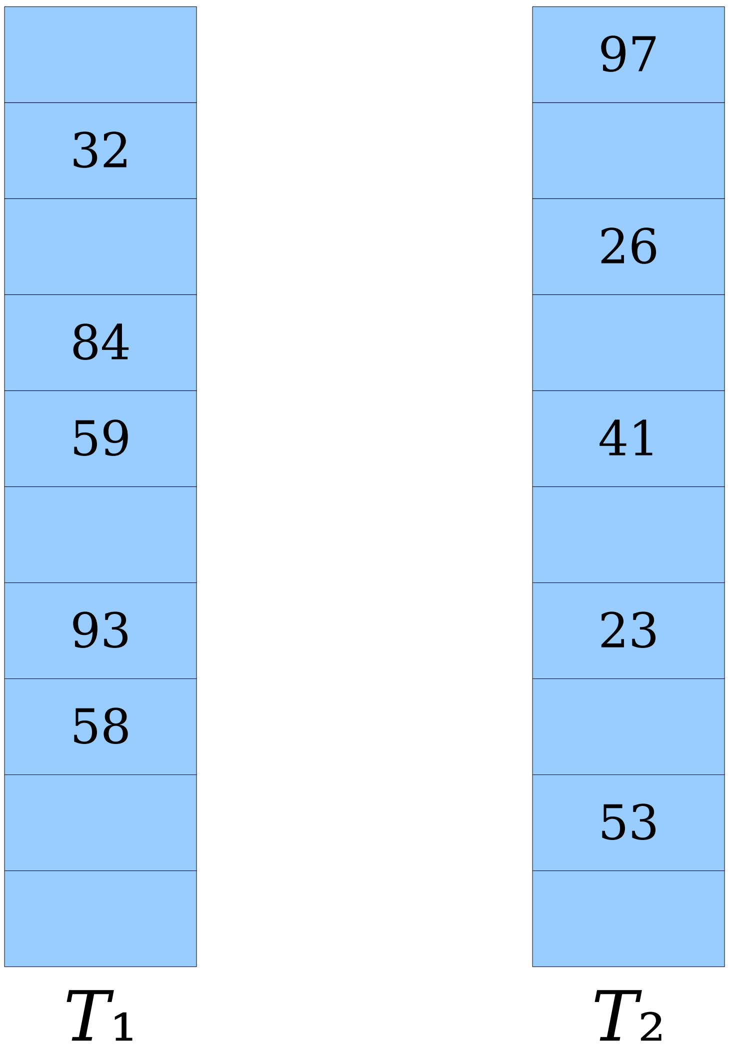

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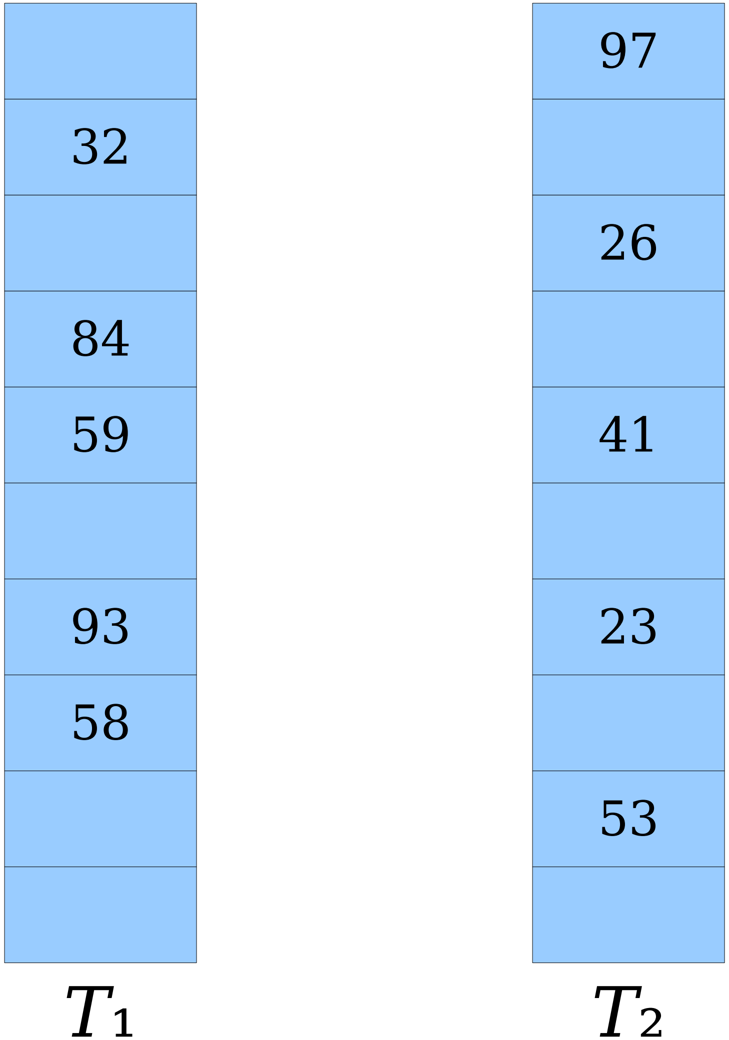

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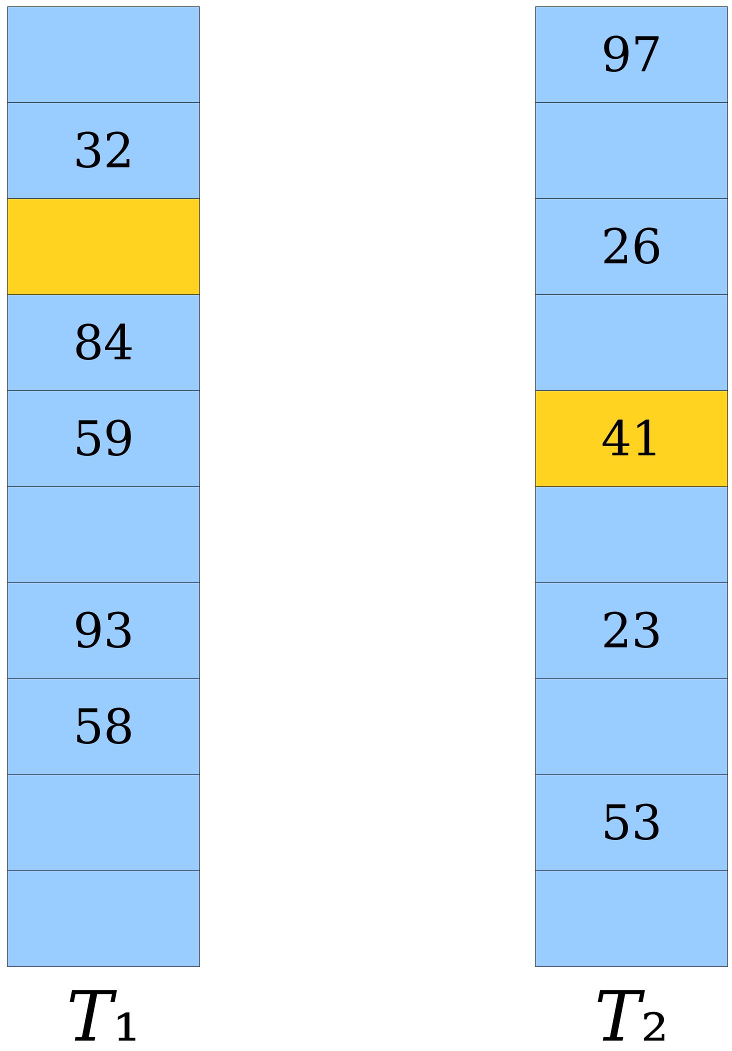

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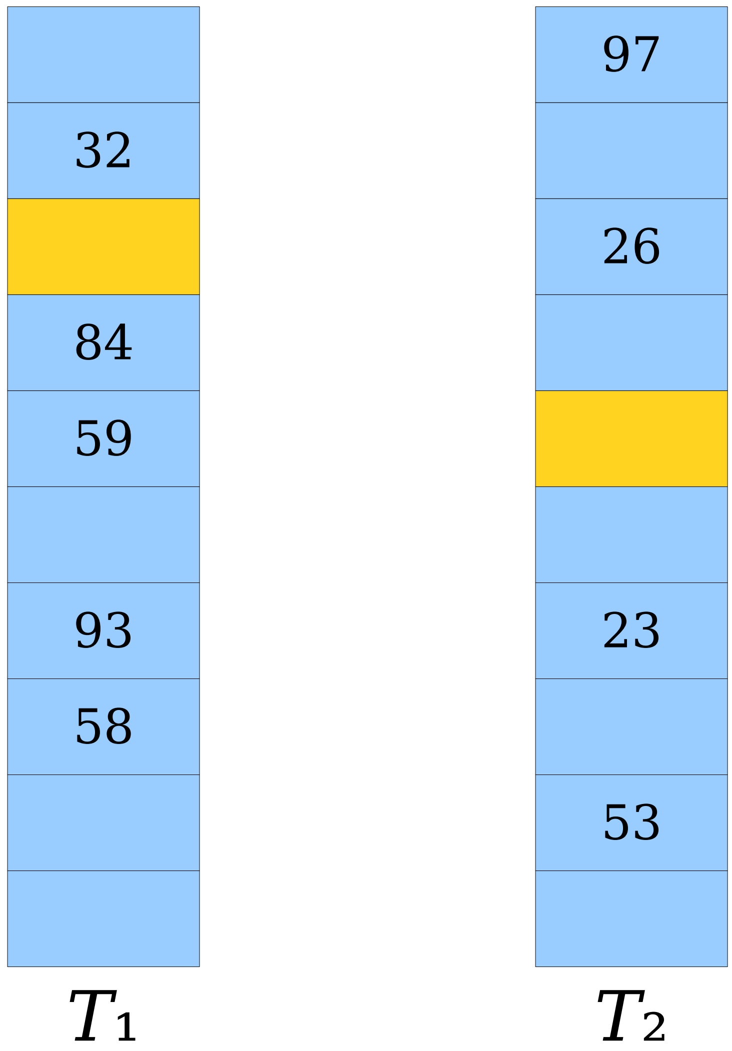

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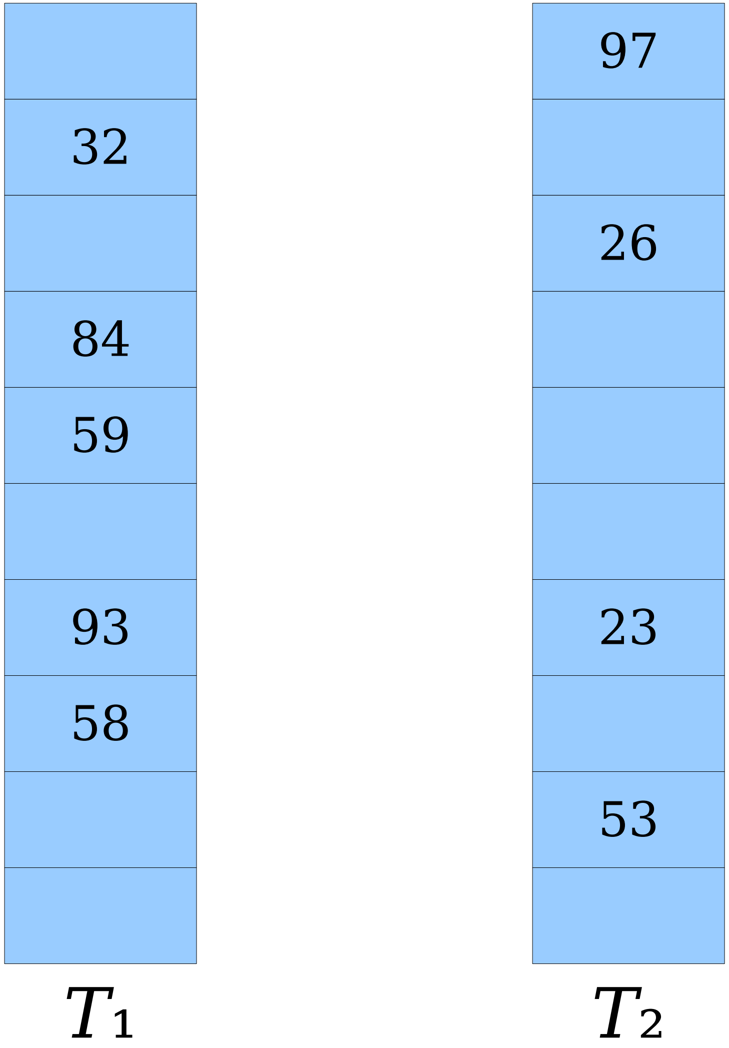

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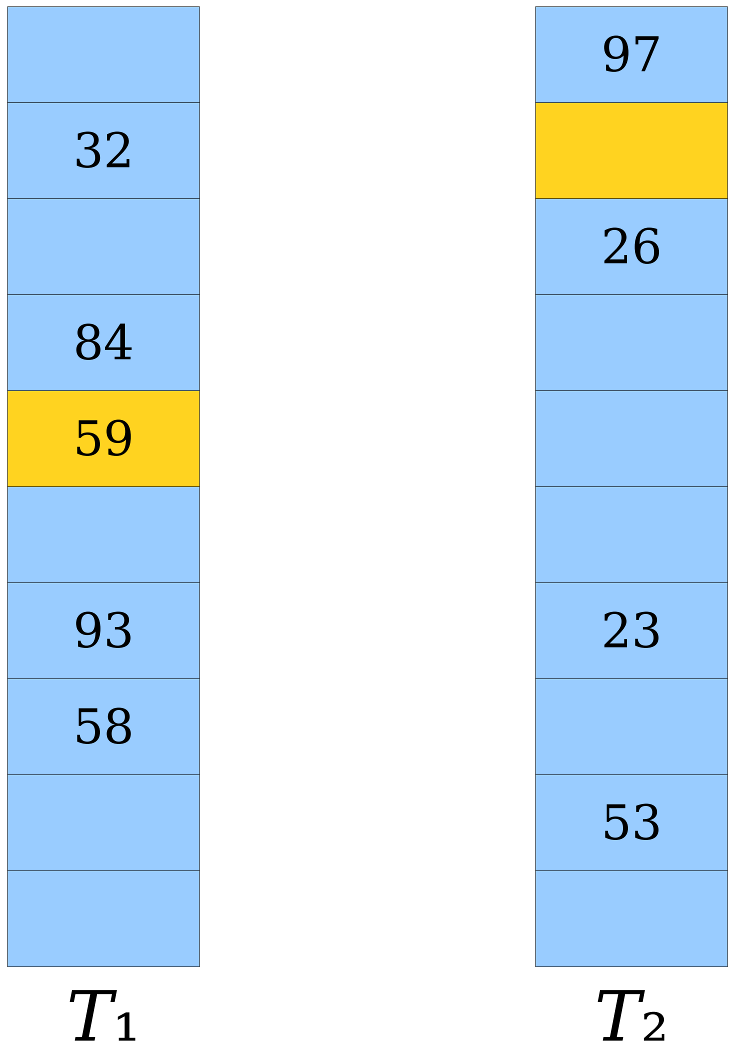

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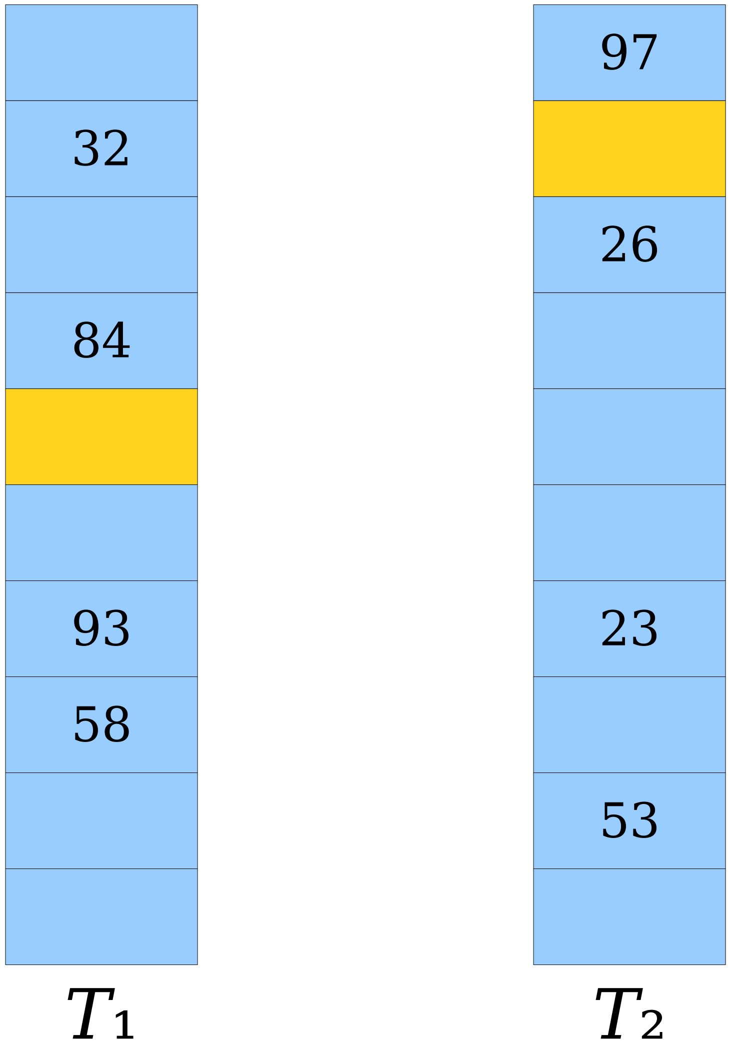

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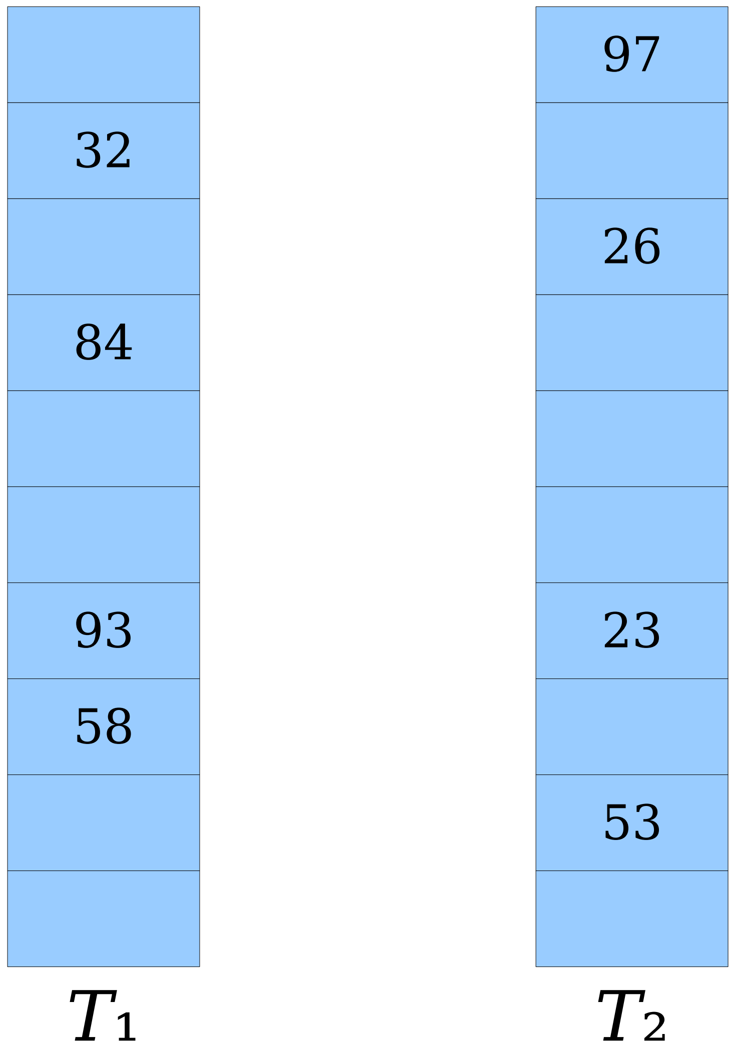

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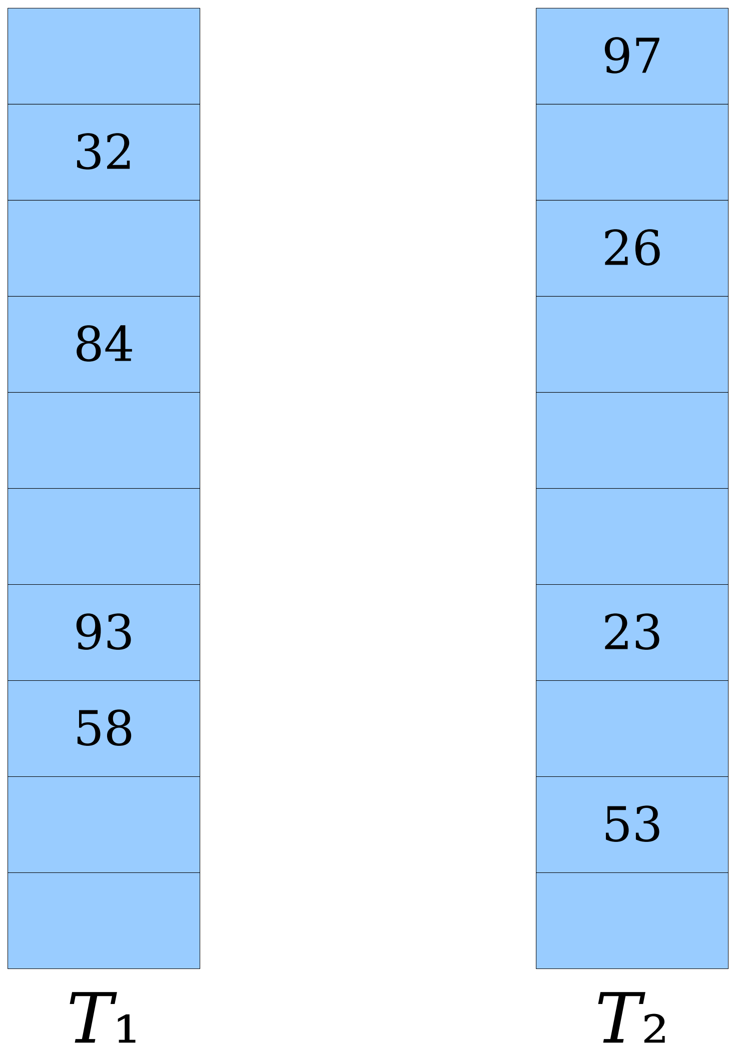

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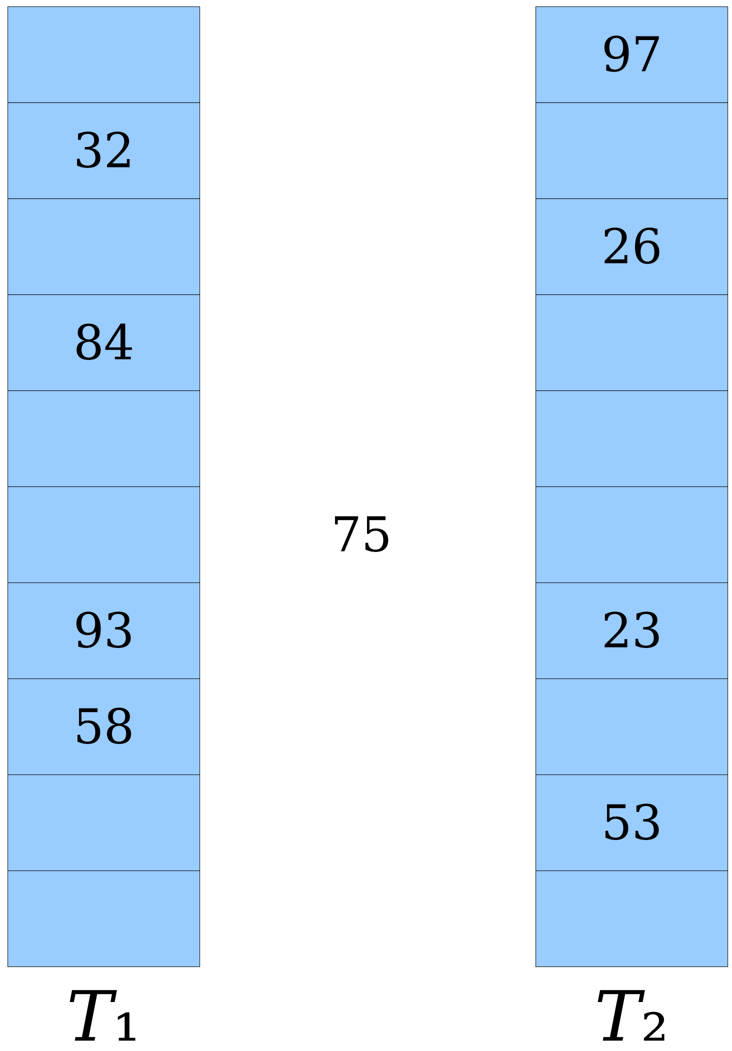

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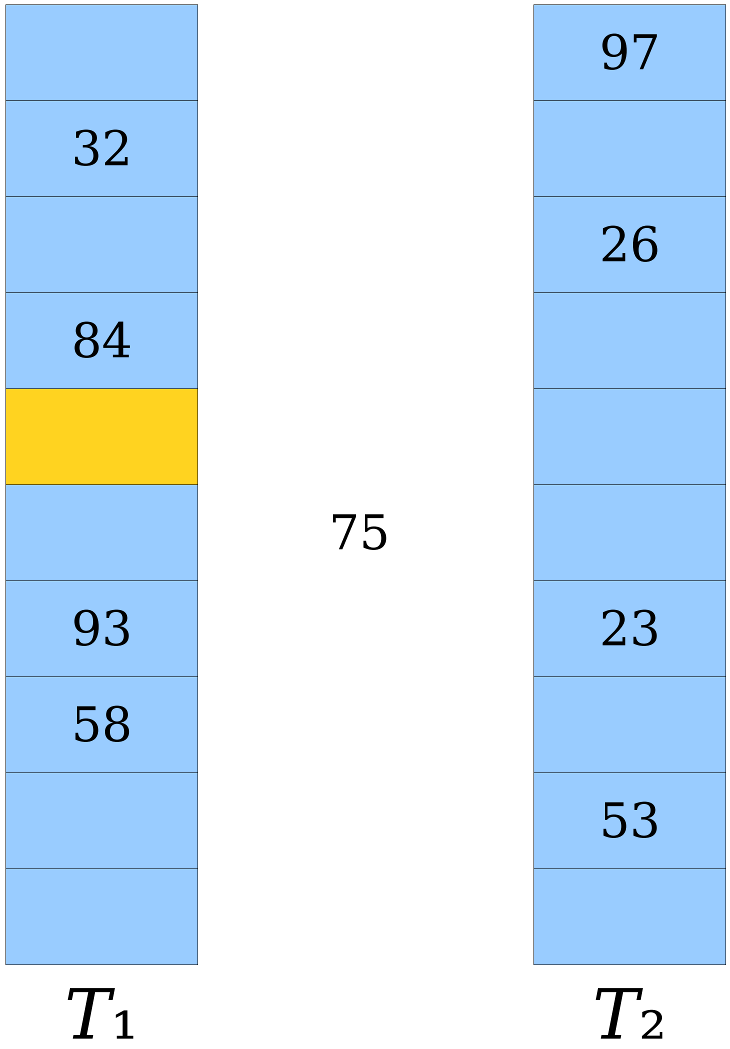

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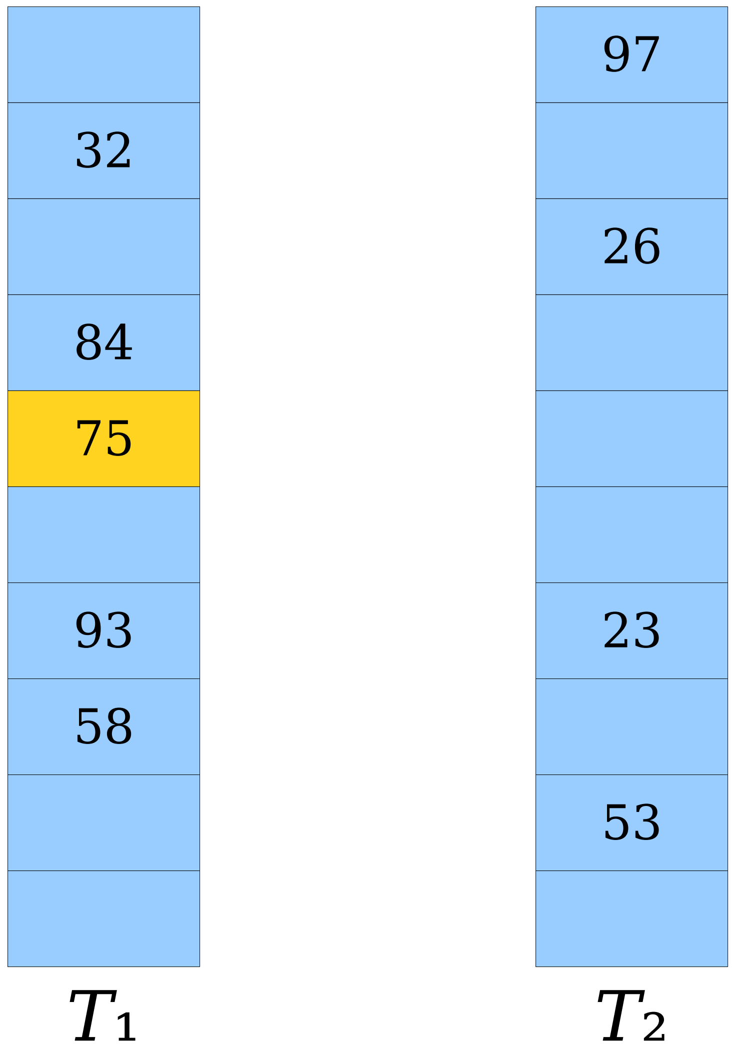

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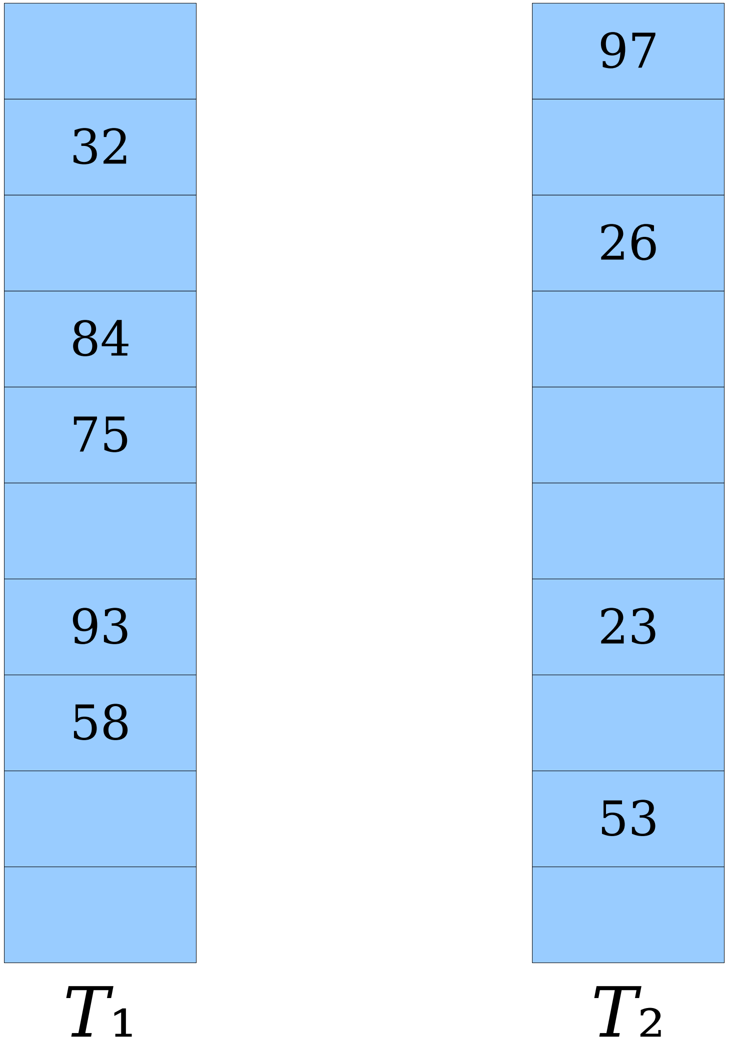

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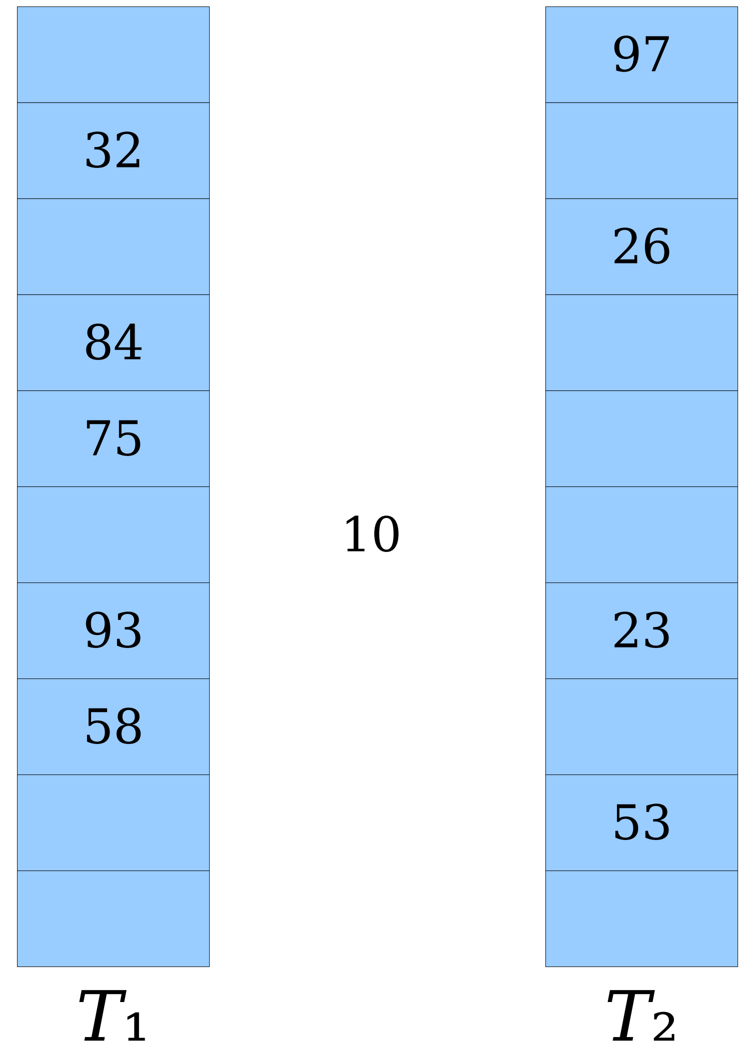

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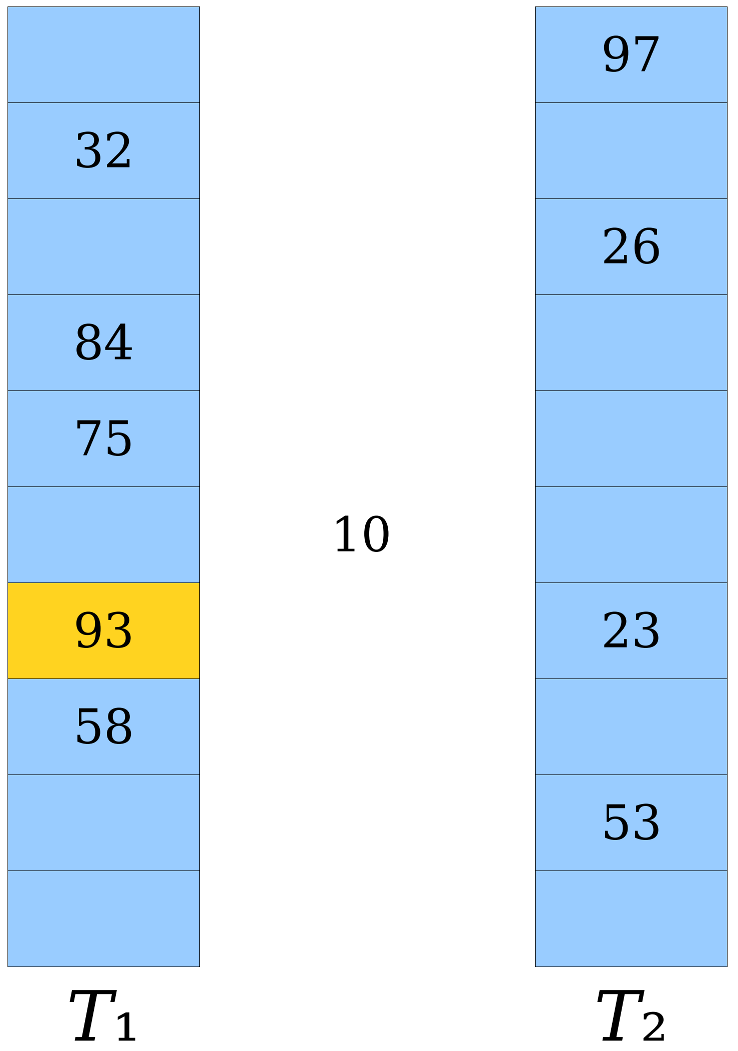

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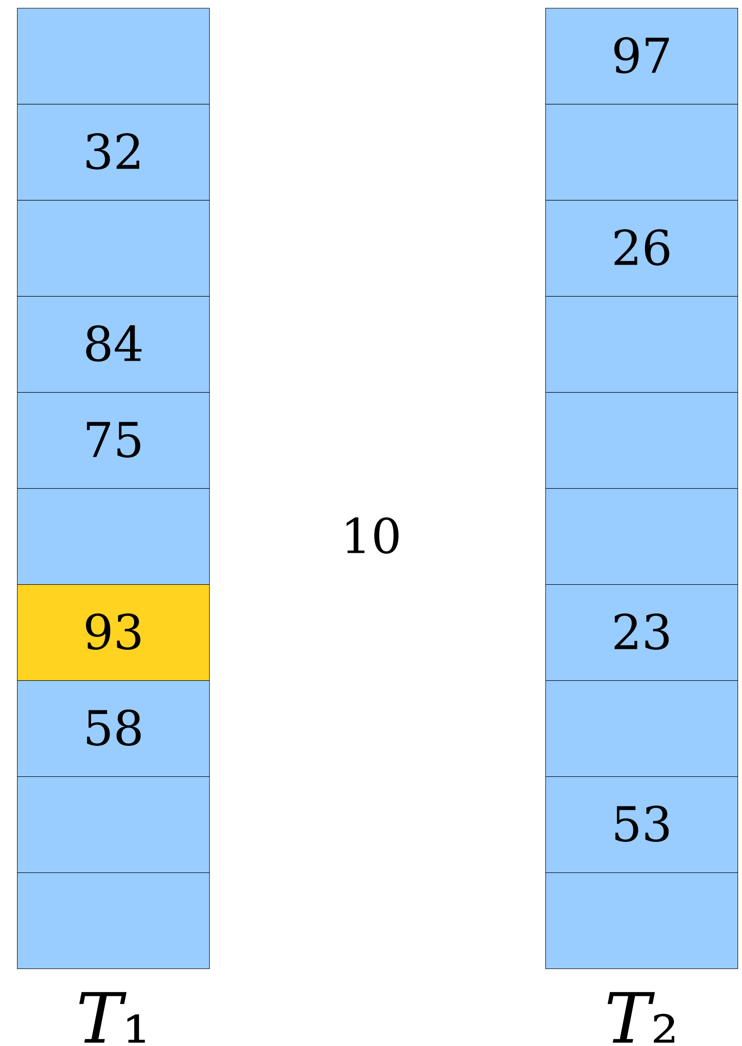

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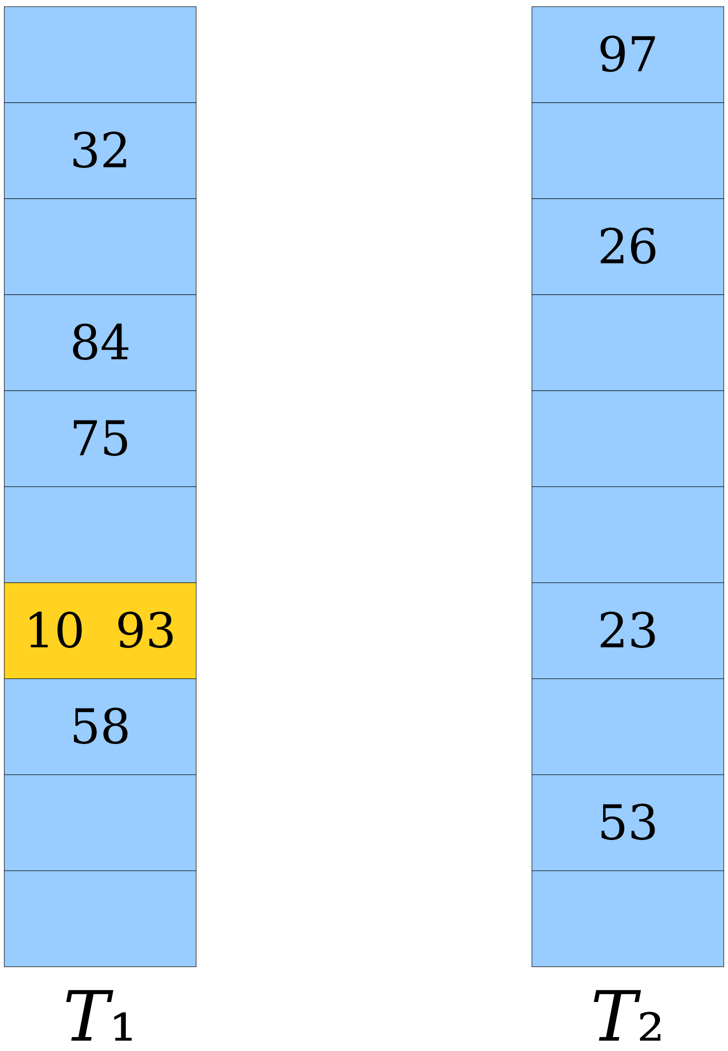

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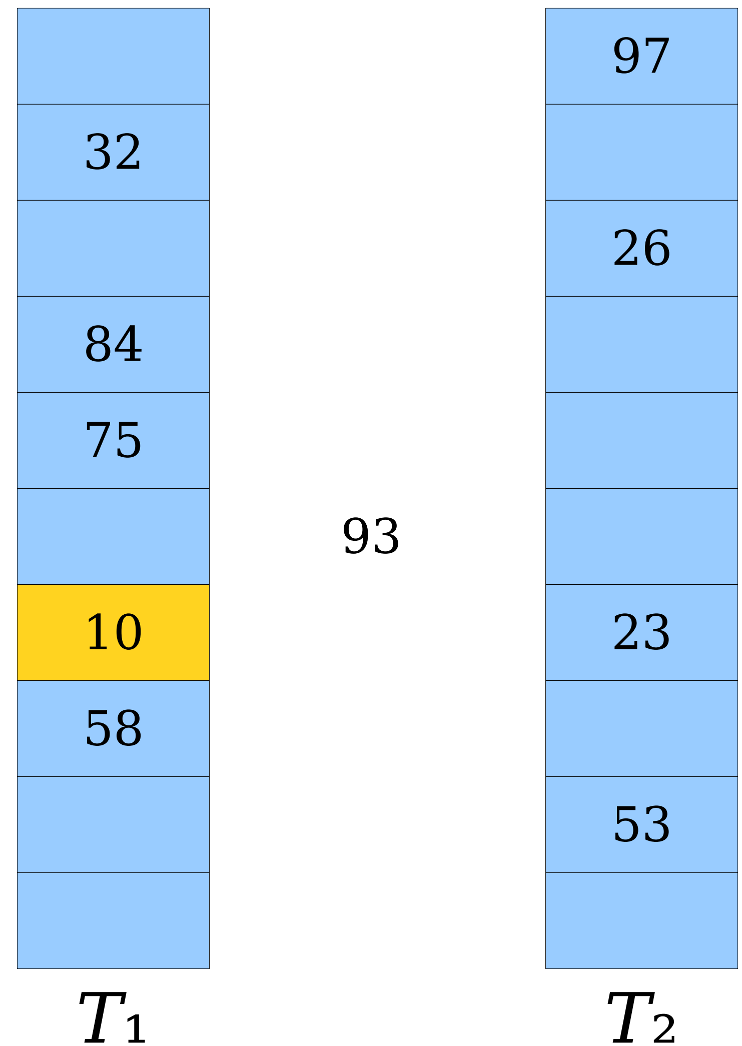

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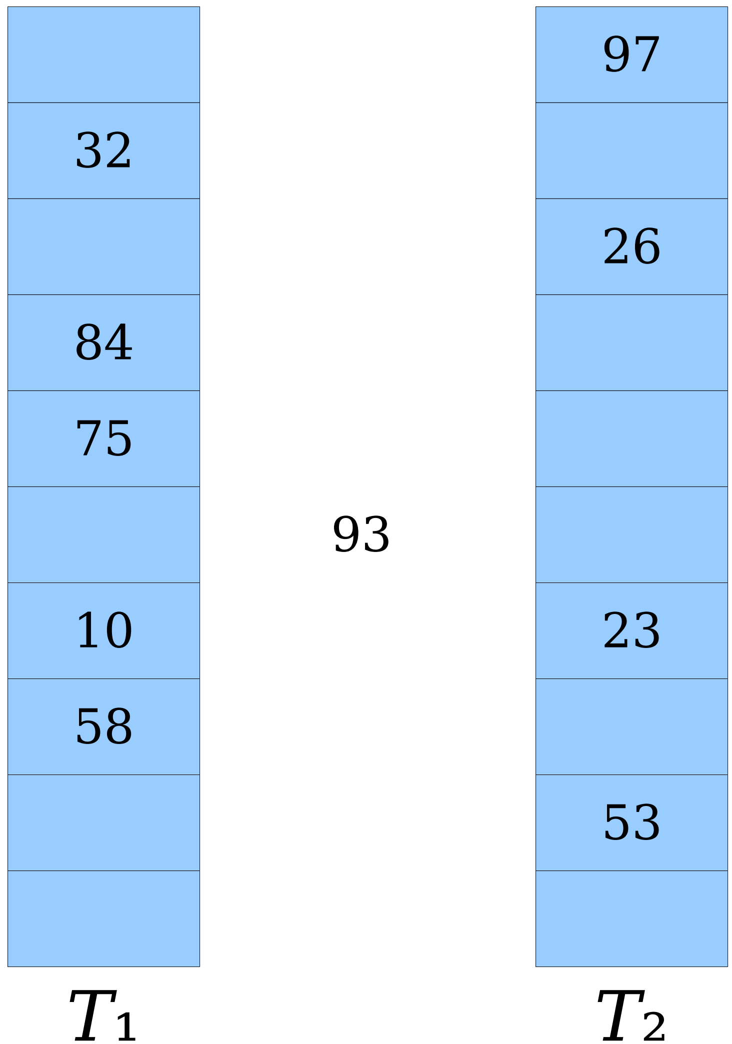

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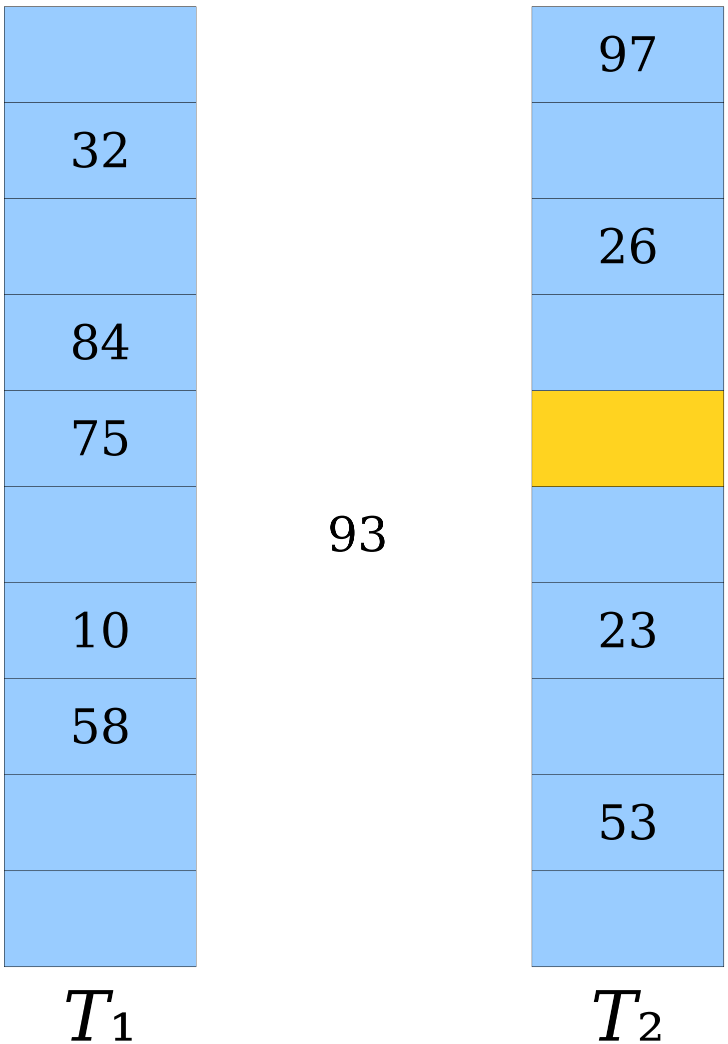

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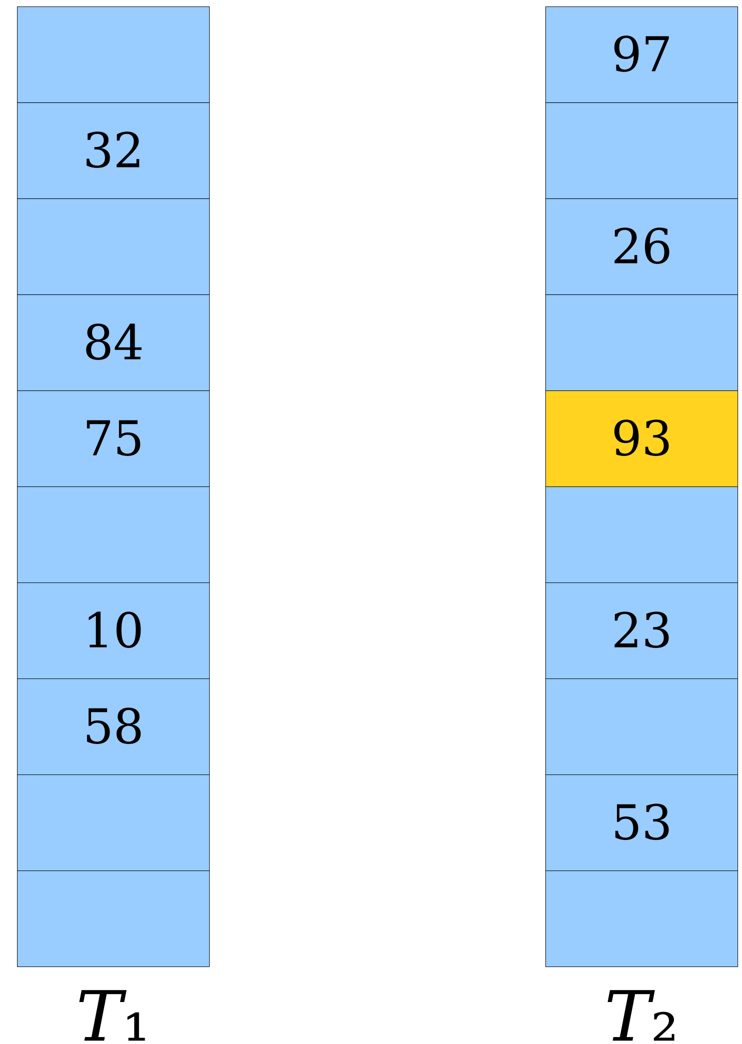

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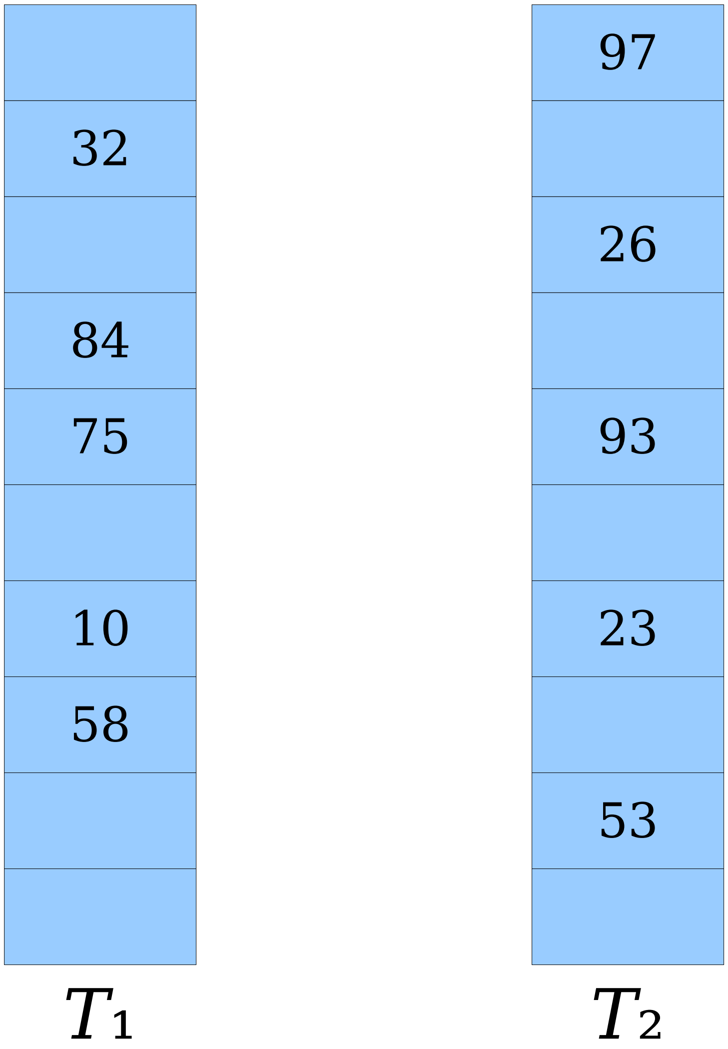

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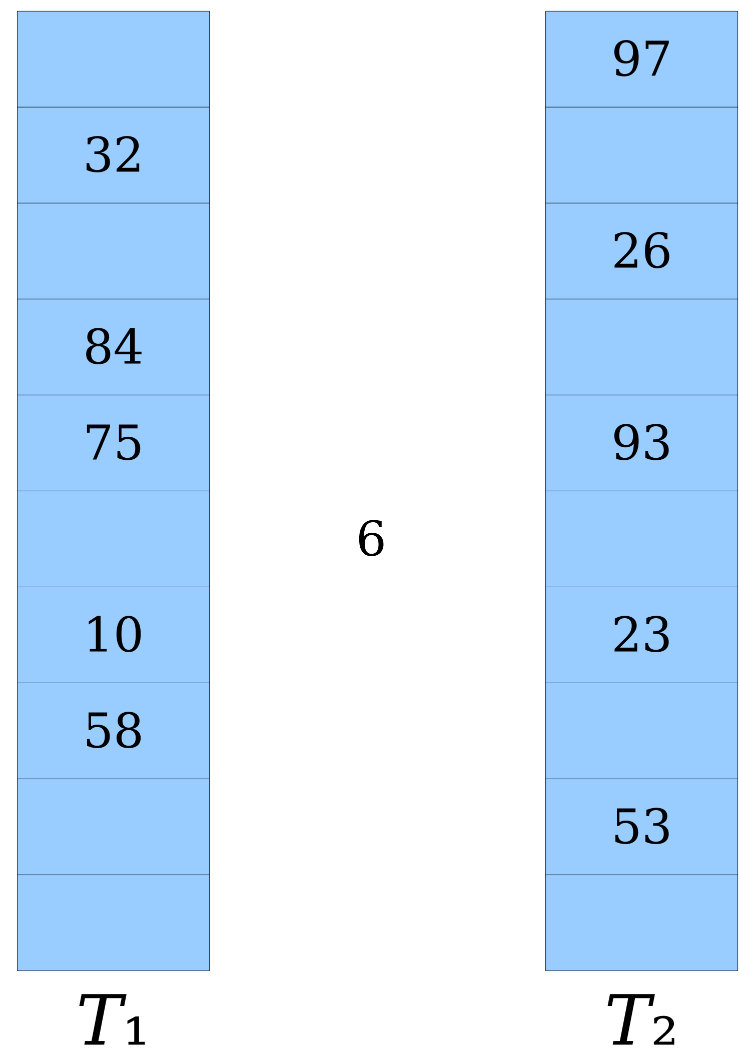

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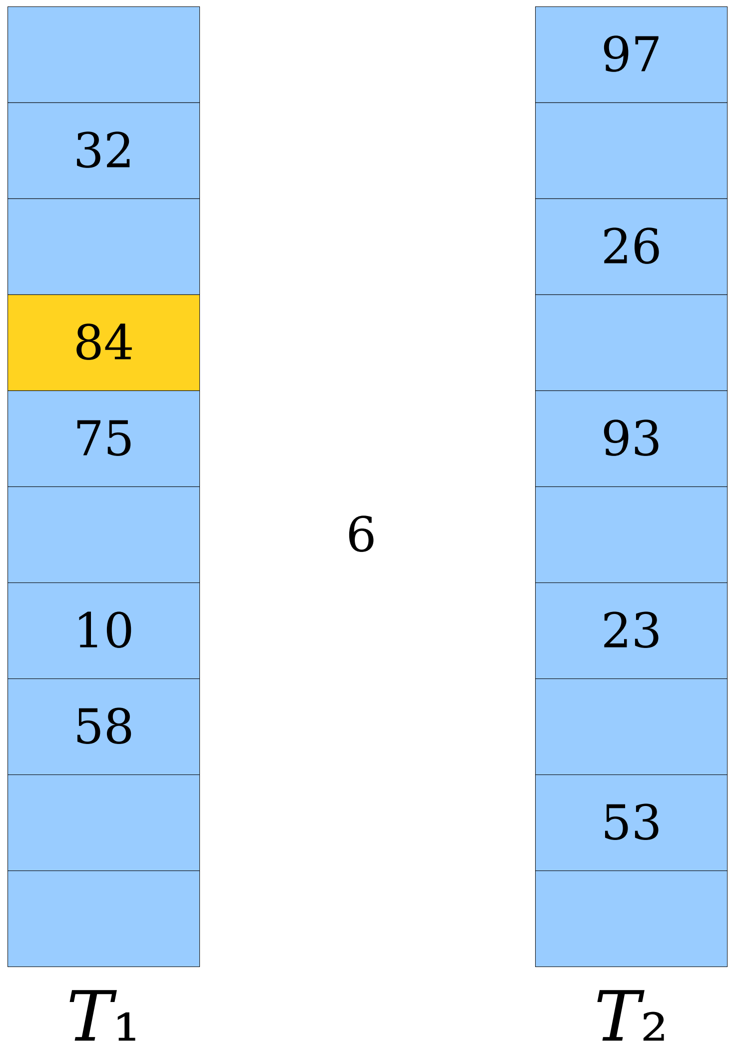

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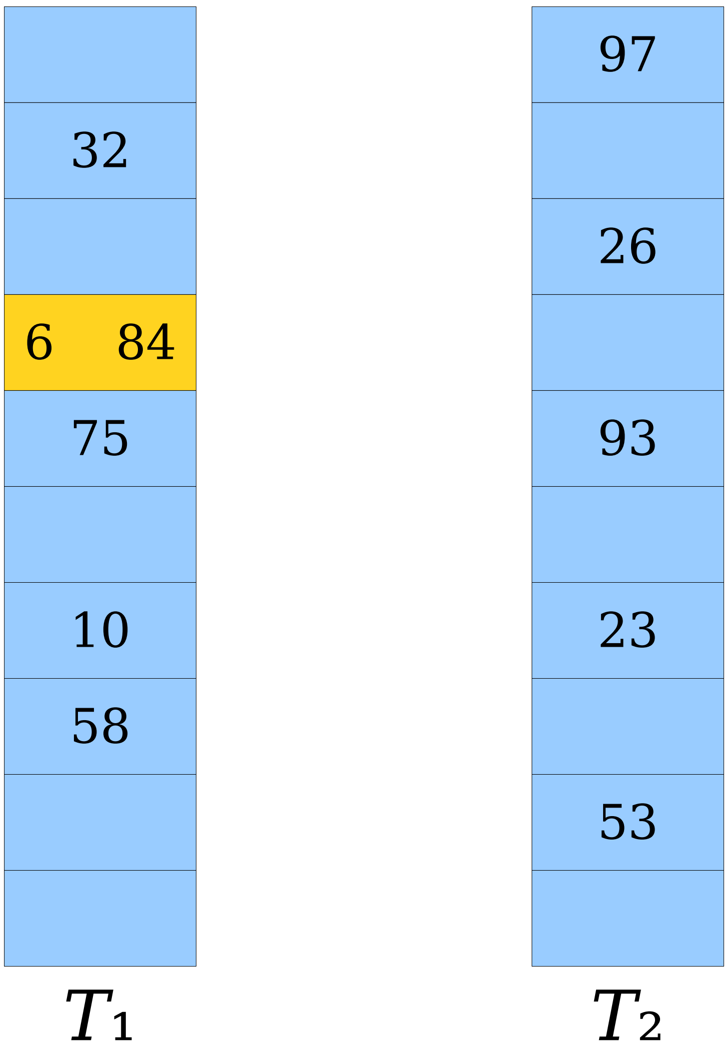

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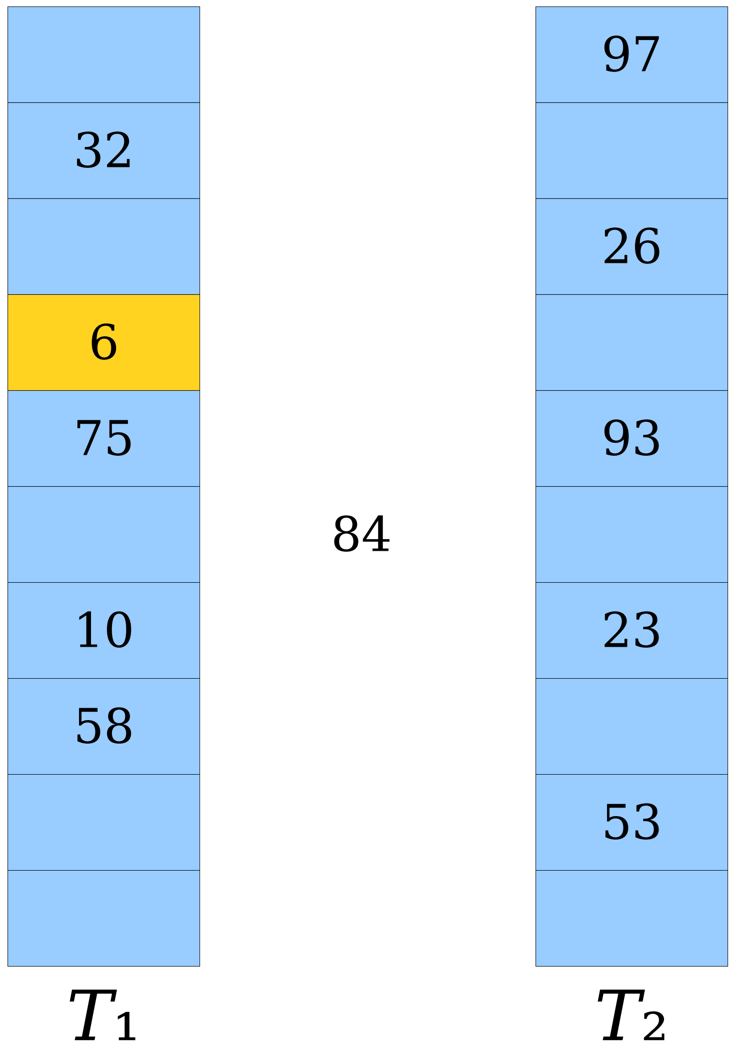

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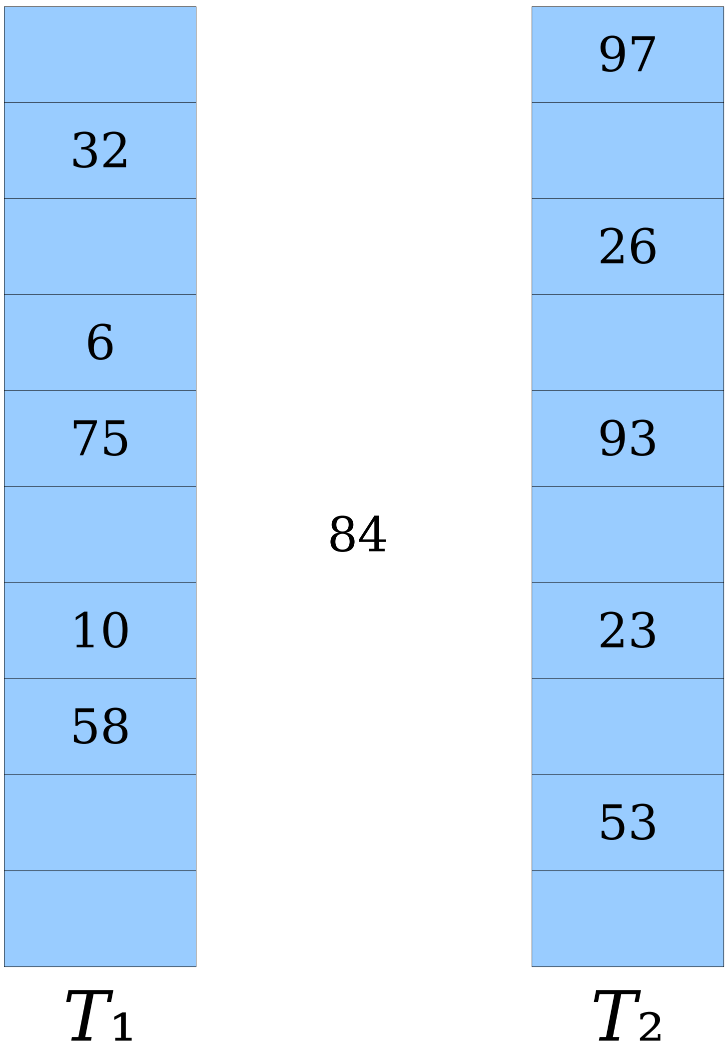

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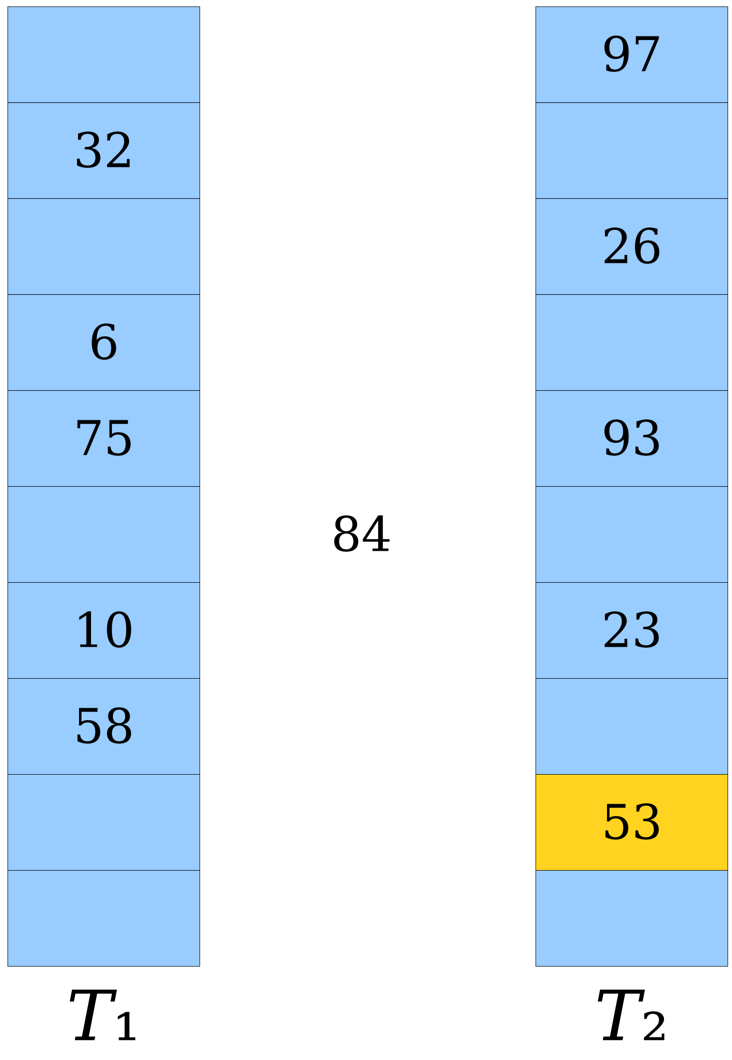

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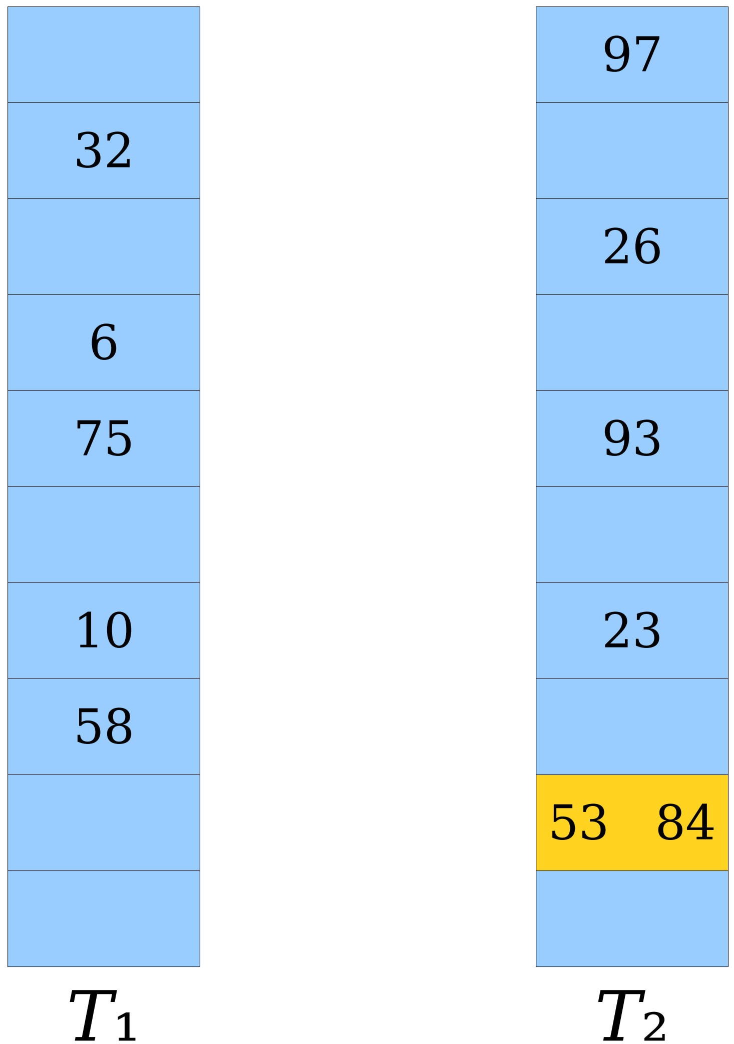

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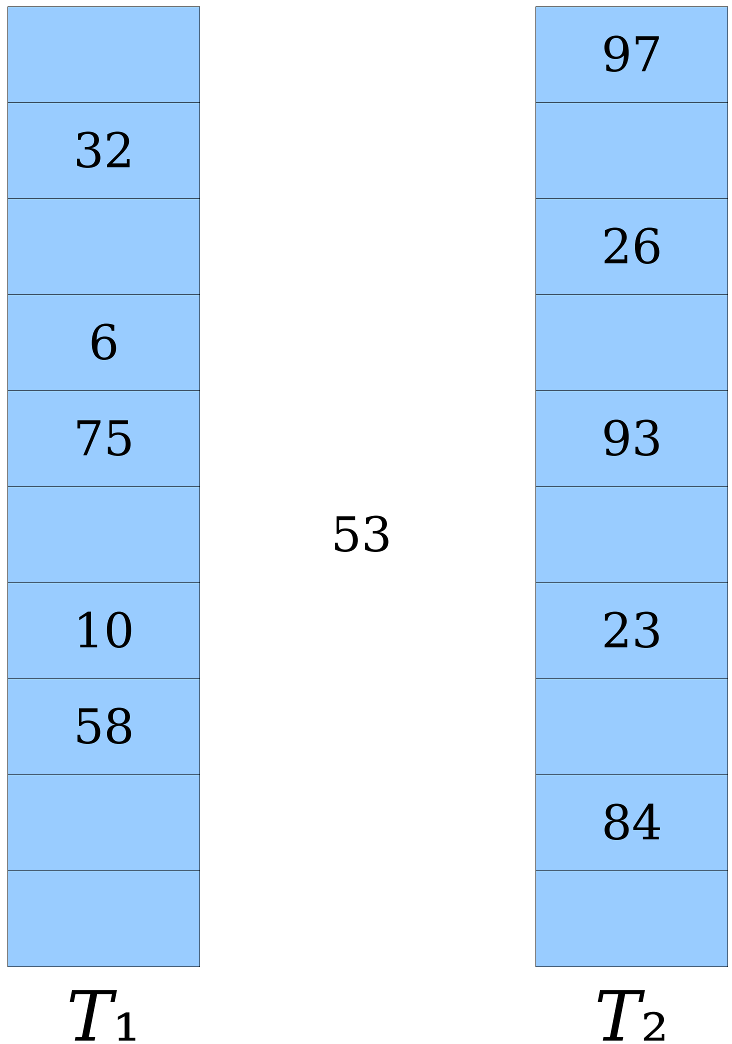

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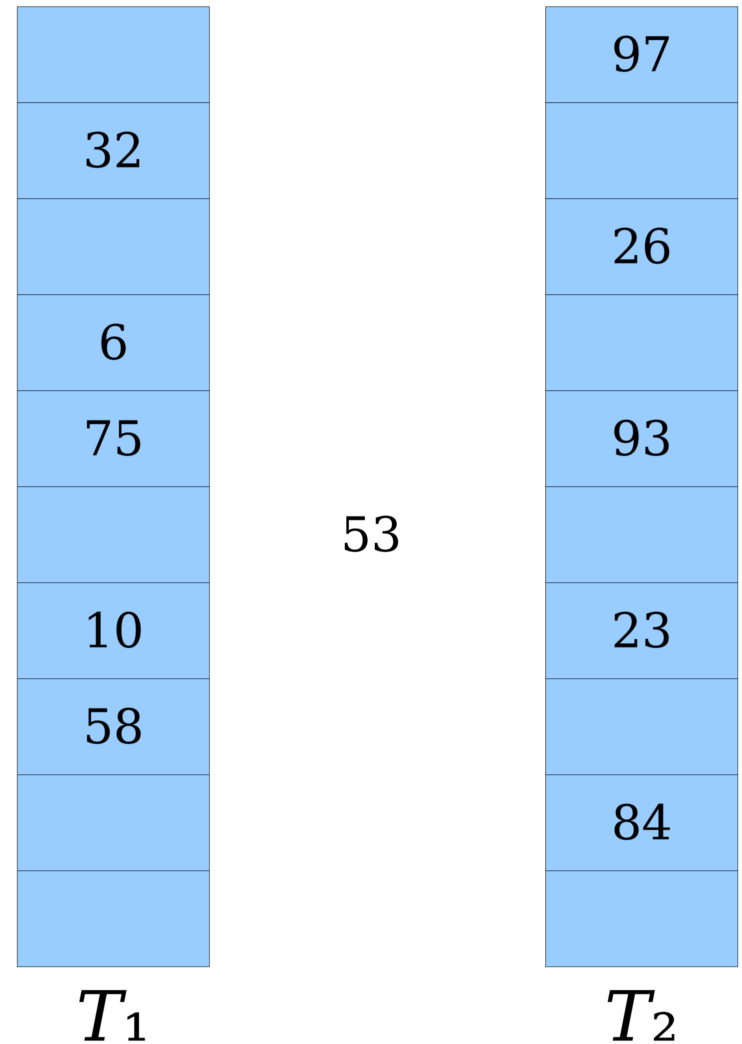

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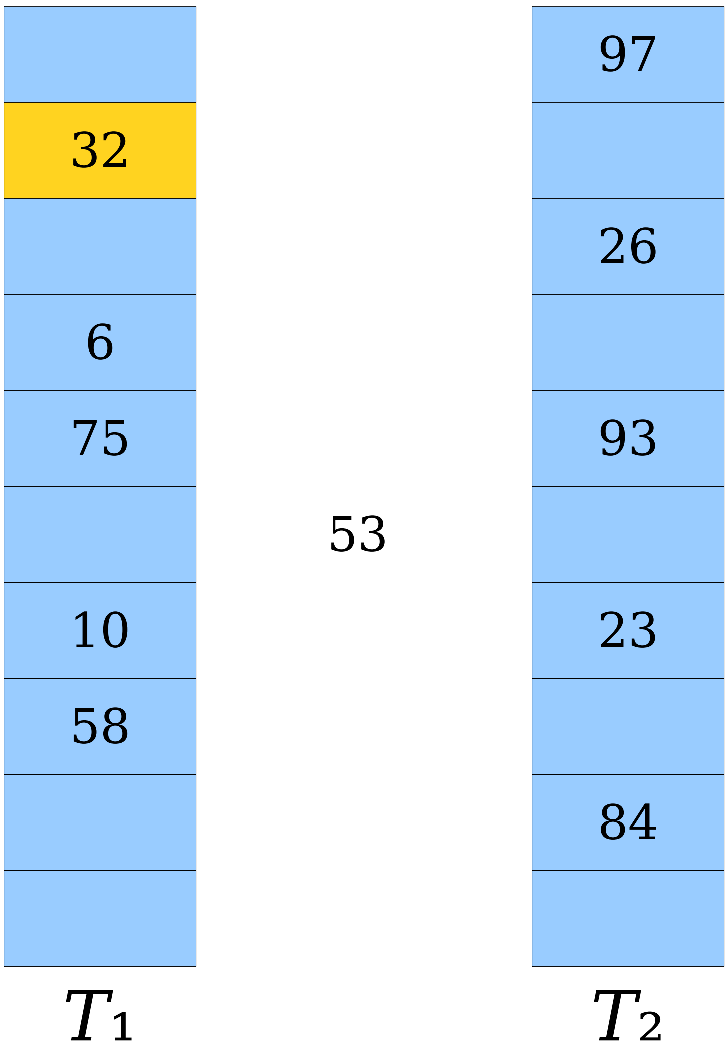

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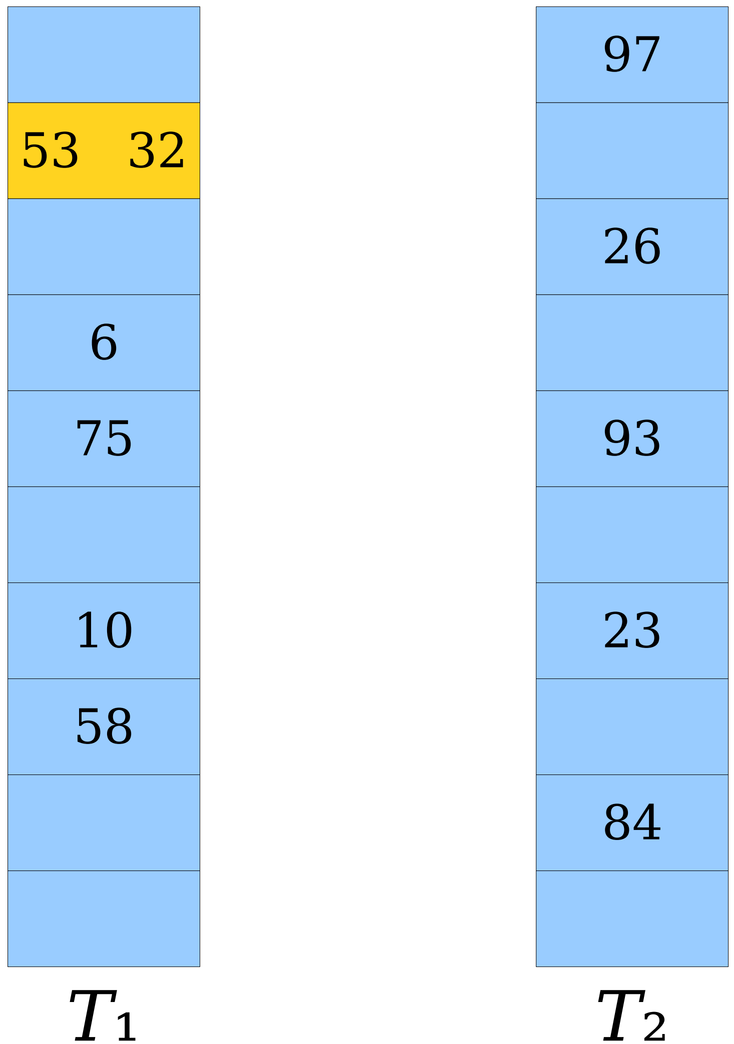

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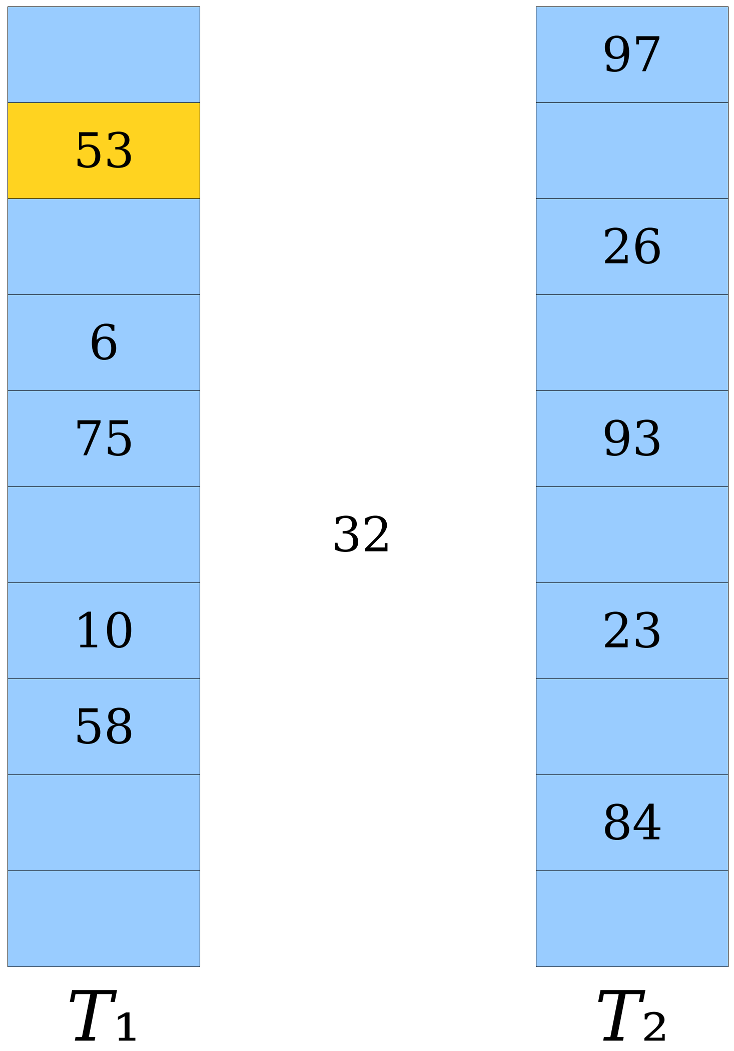

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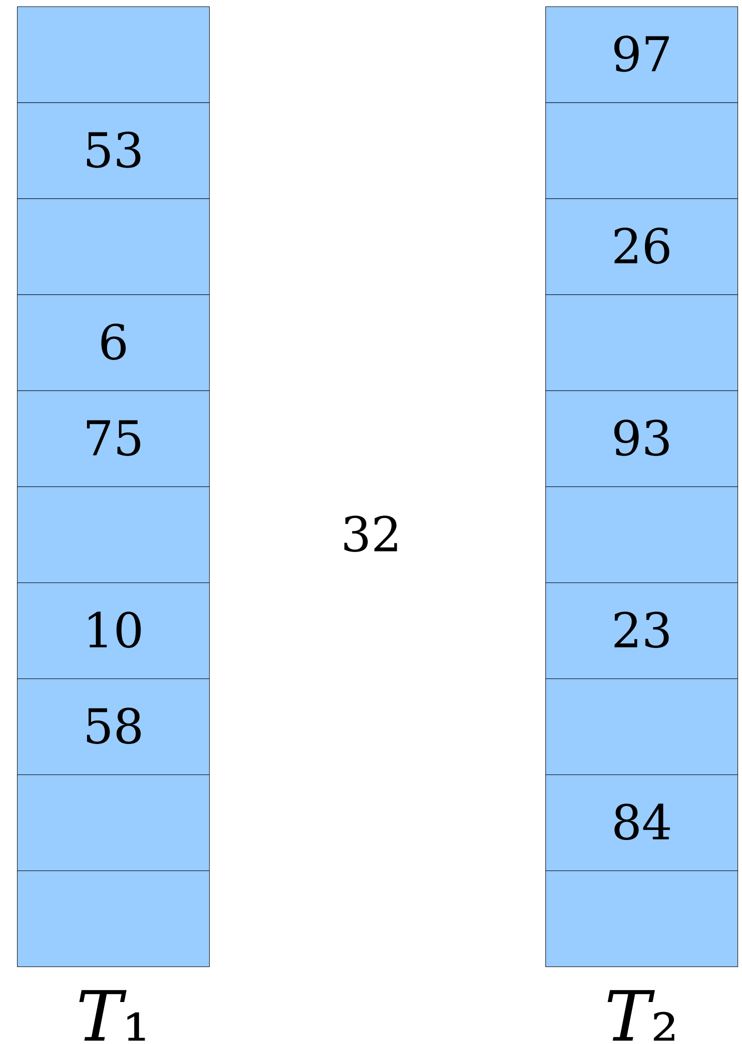

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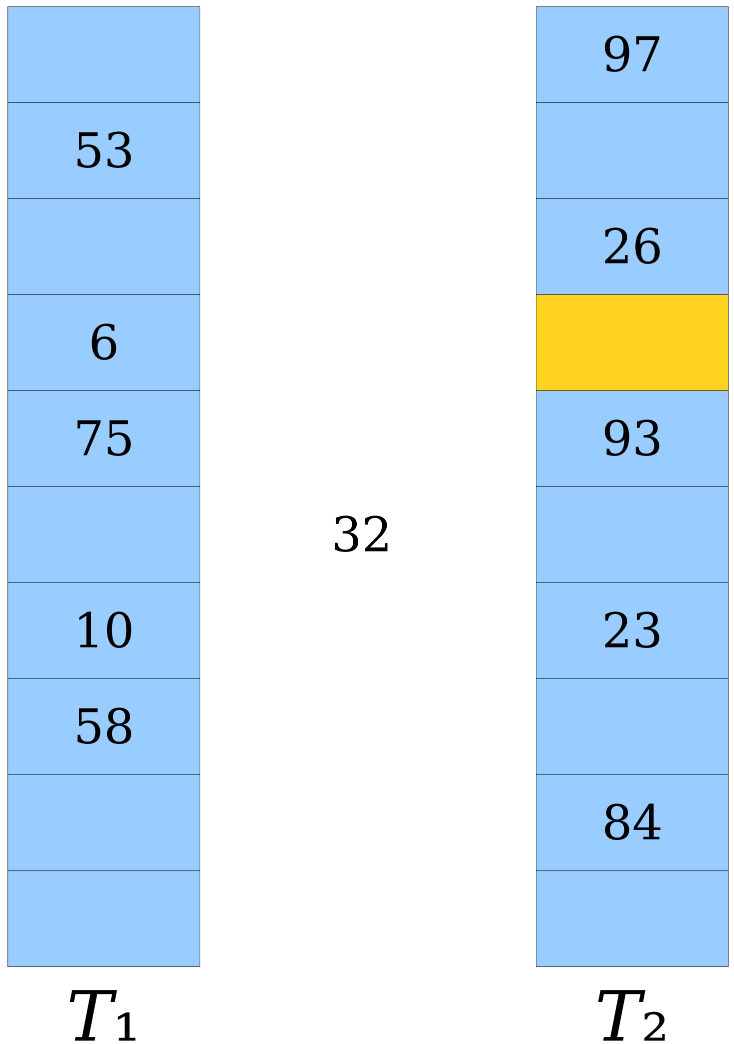

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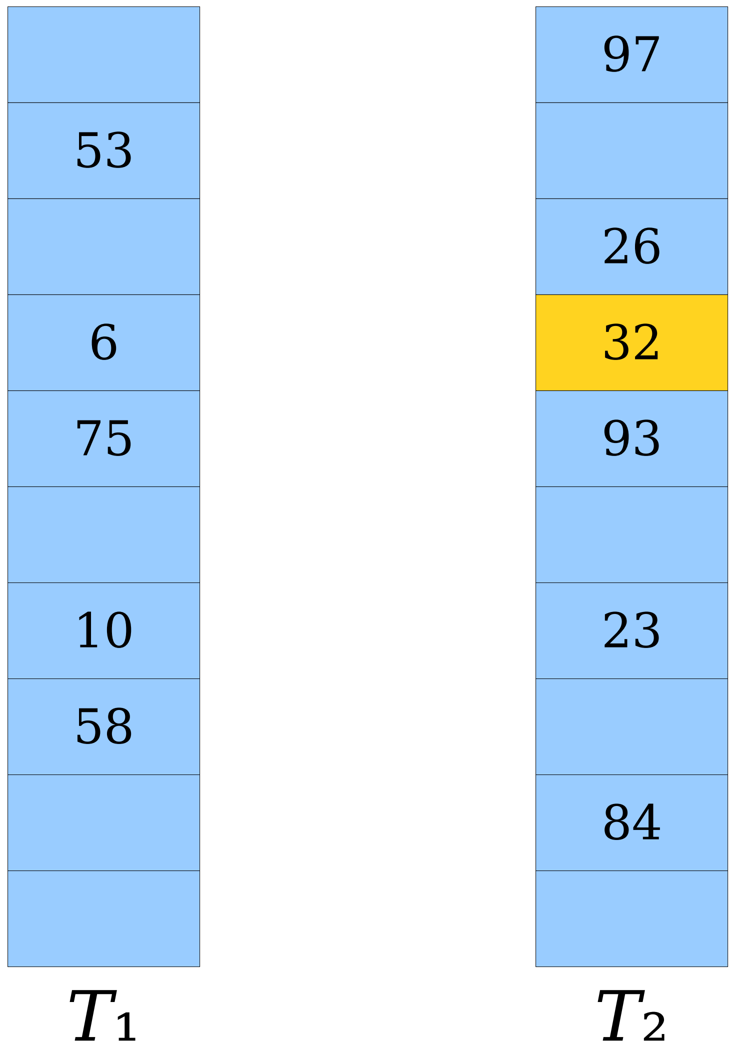

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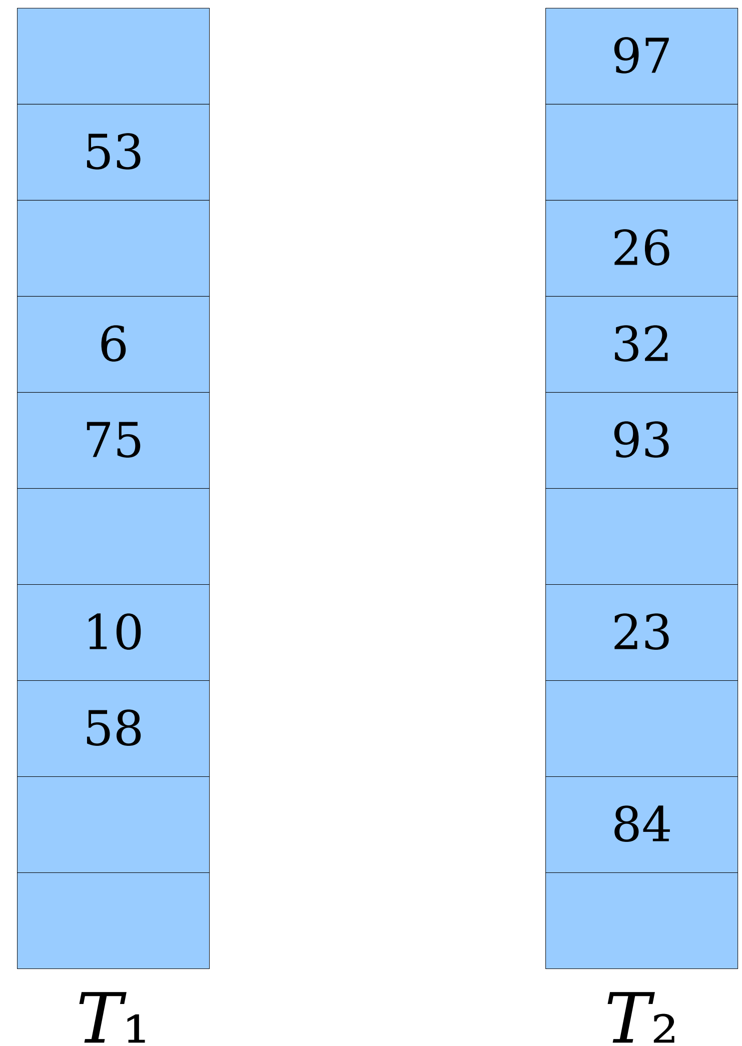

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Cuckoo Hashing

- An insertion ***fails*** if the displacements form an infinite cycle.

T_1	T_2
	97
53	
	26
6	32
75	93
10	23
58	
	84

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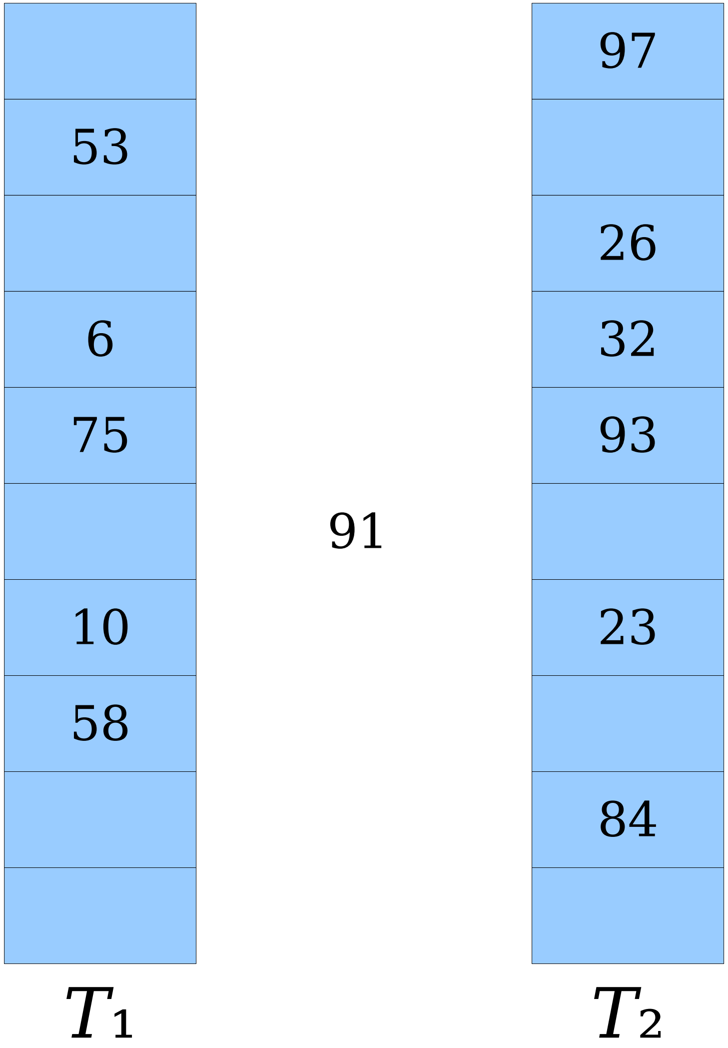

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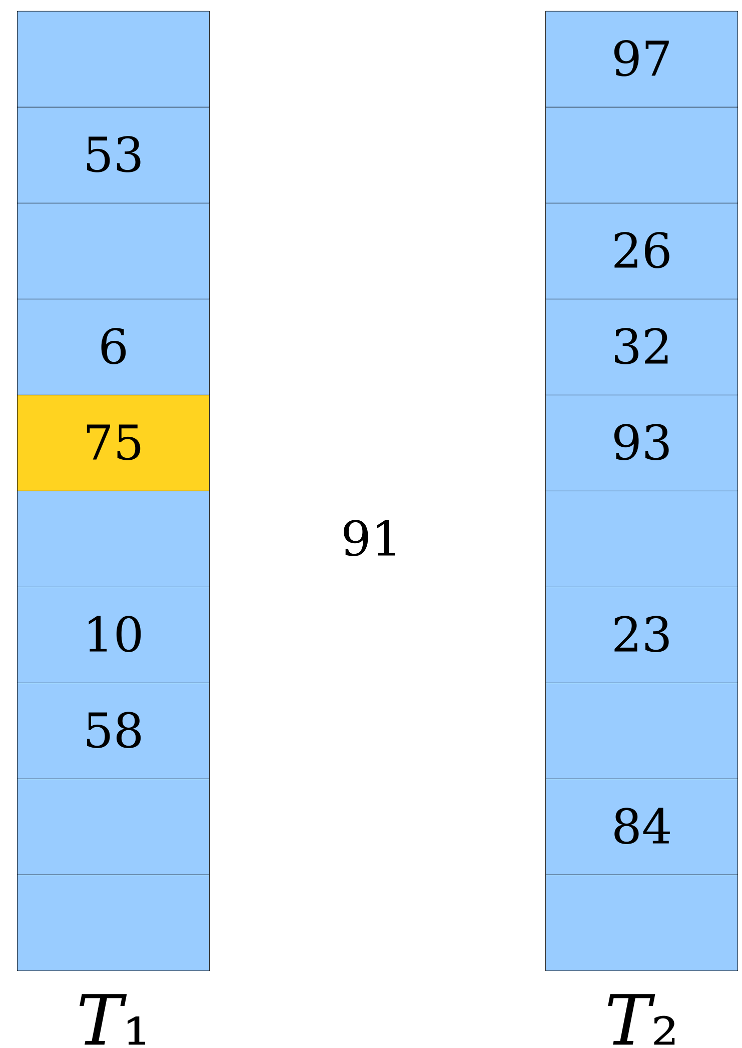

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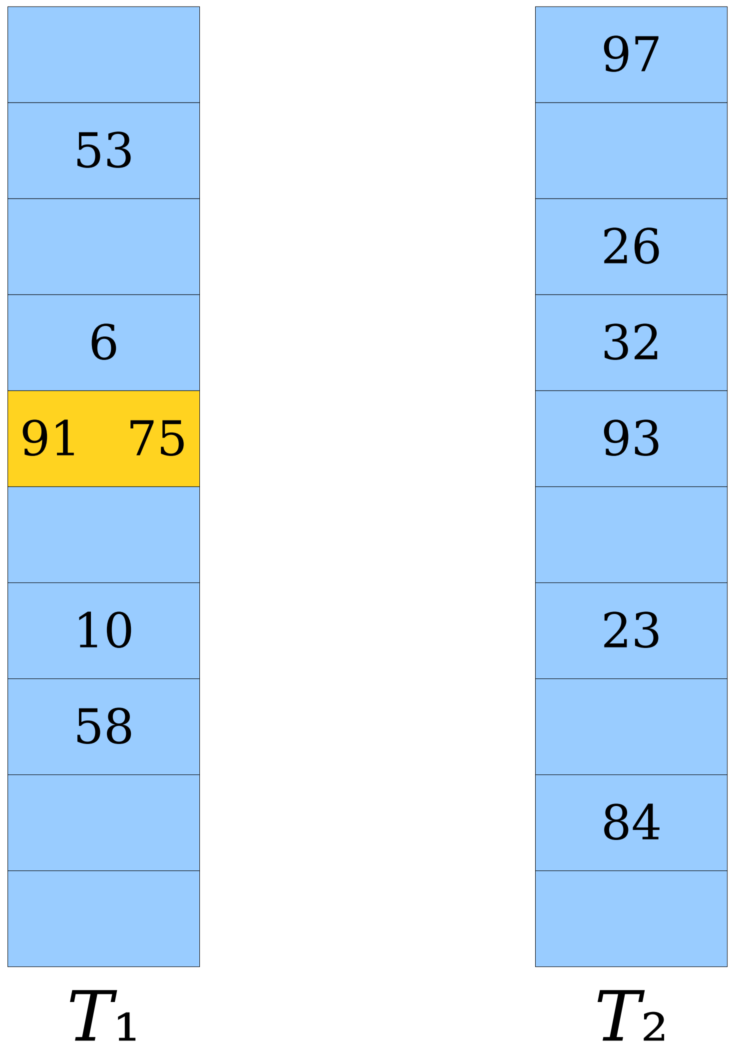

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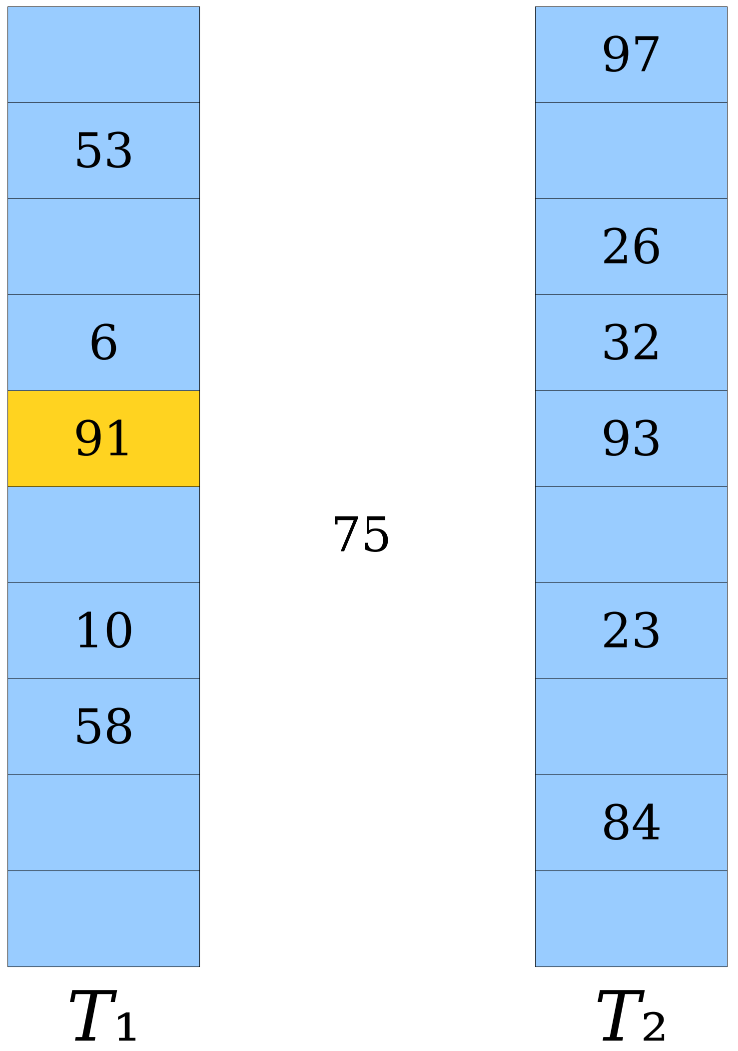

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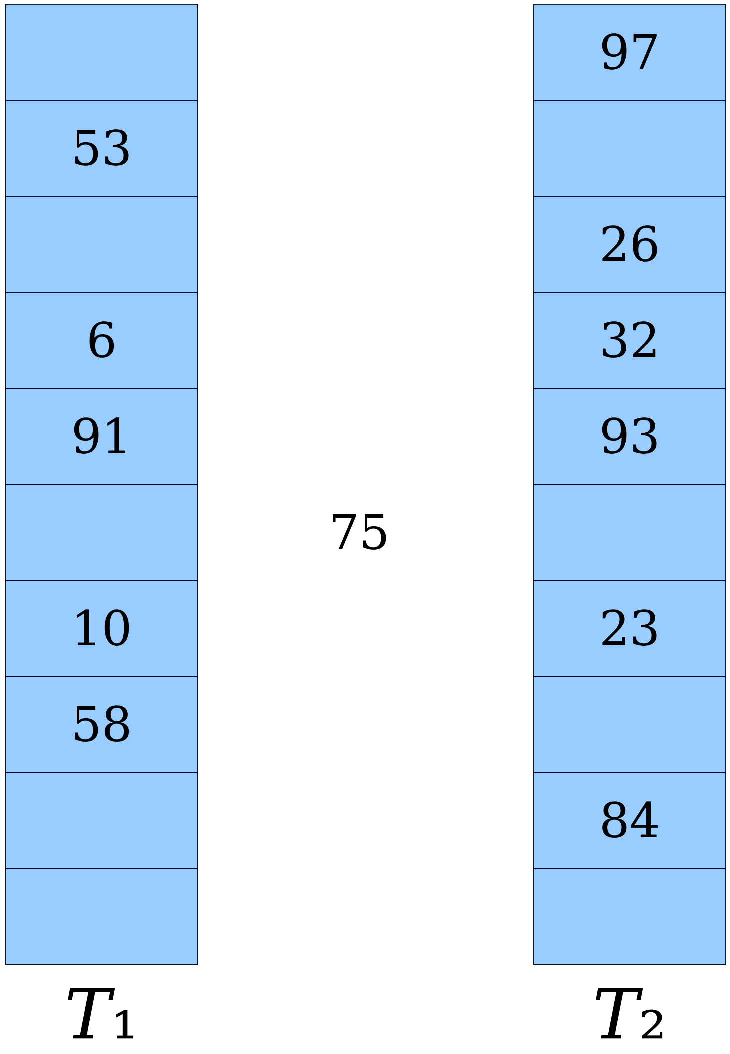

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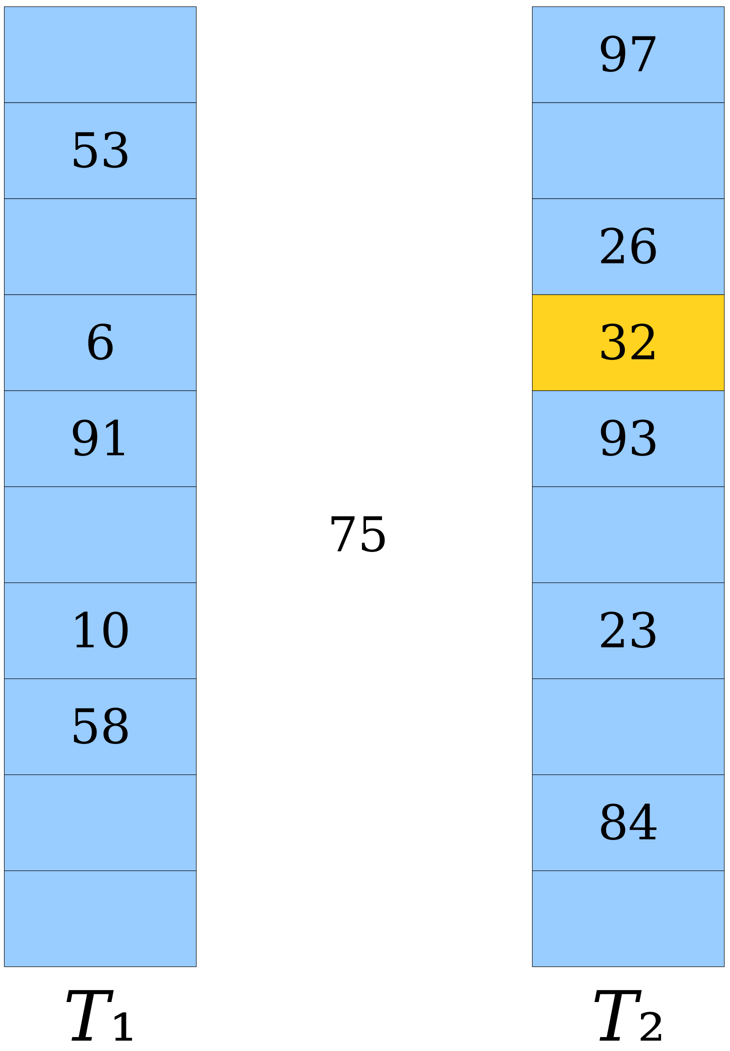

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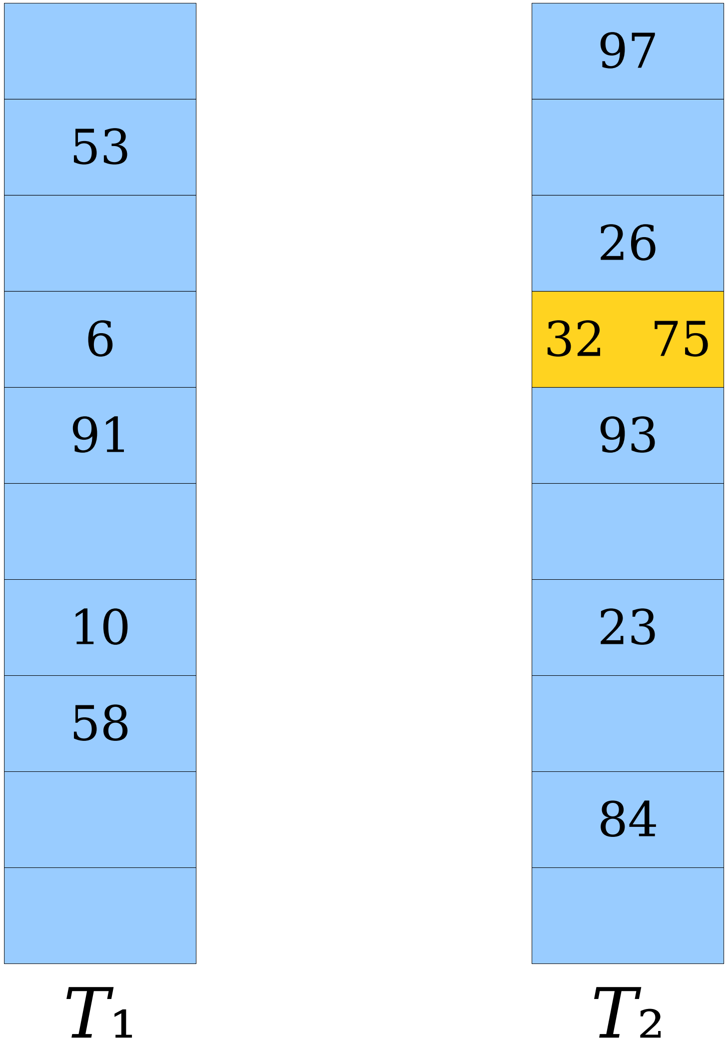

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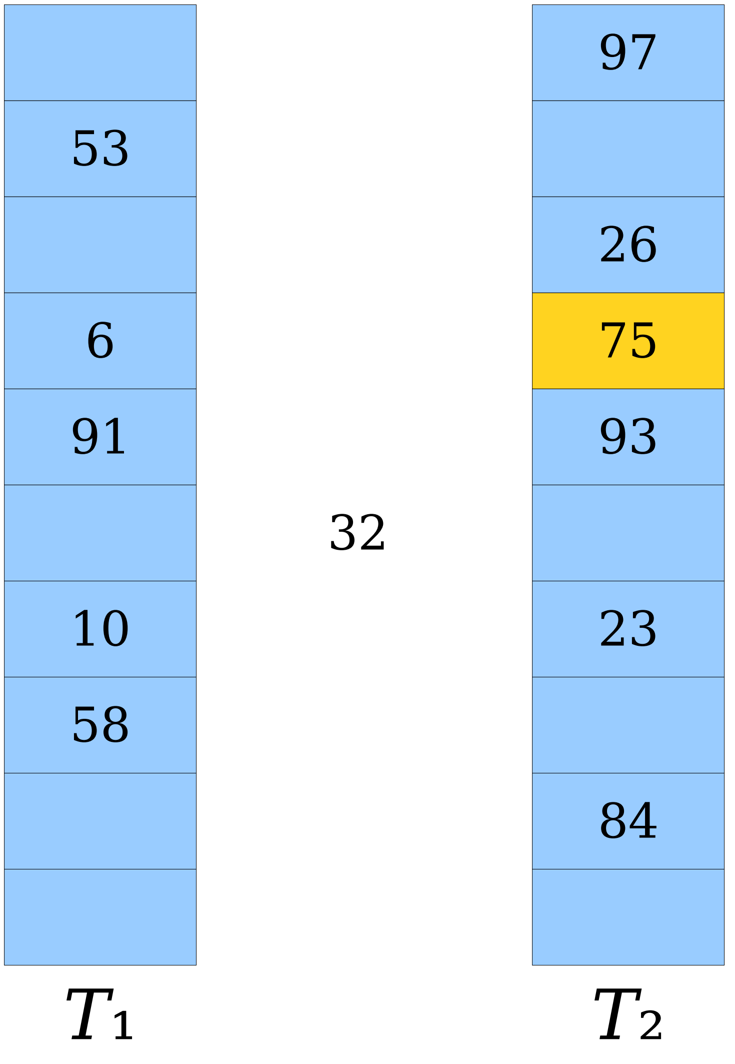

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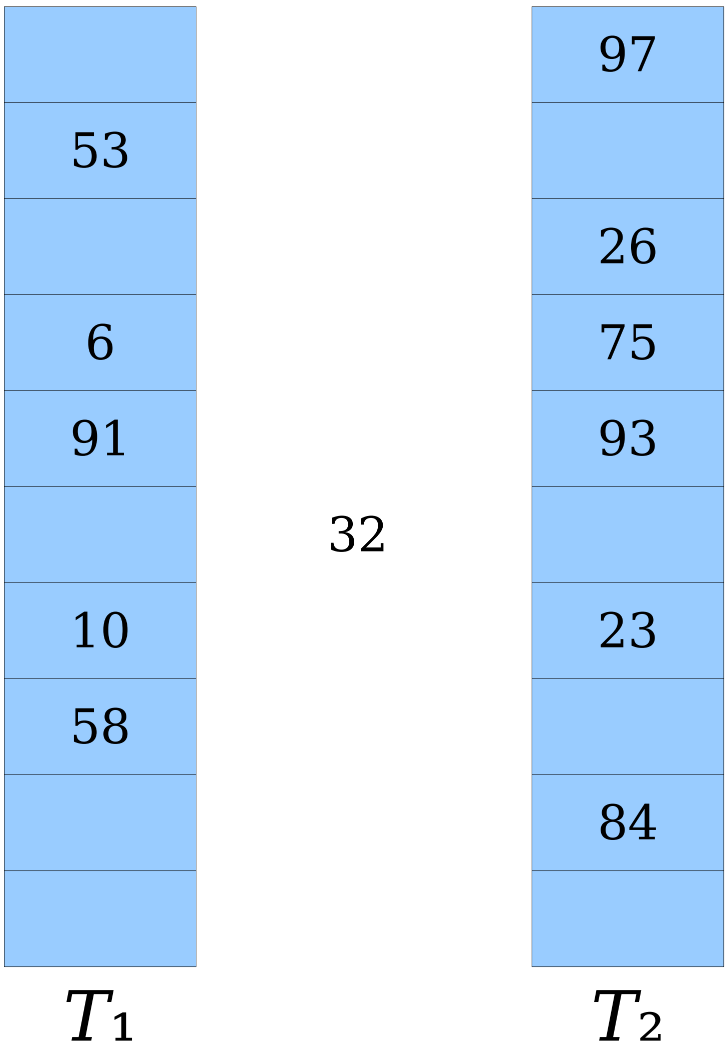

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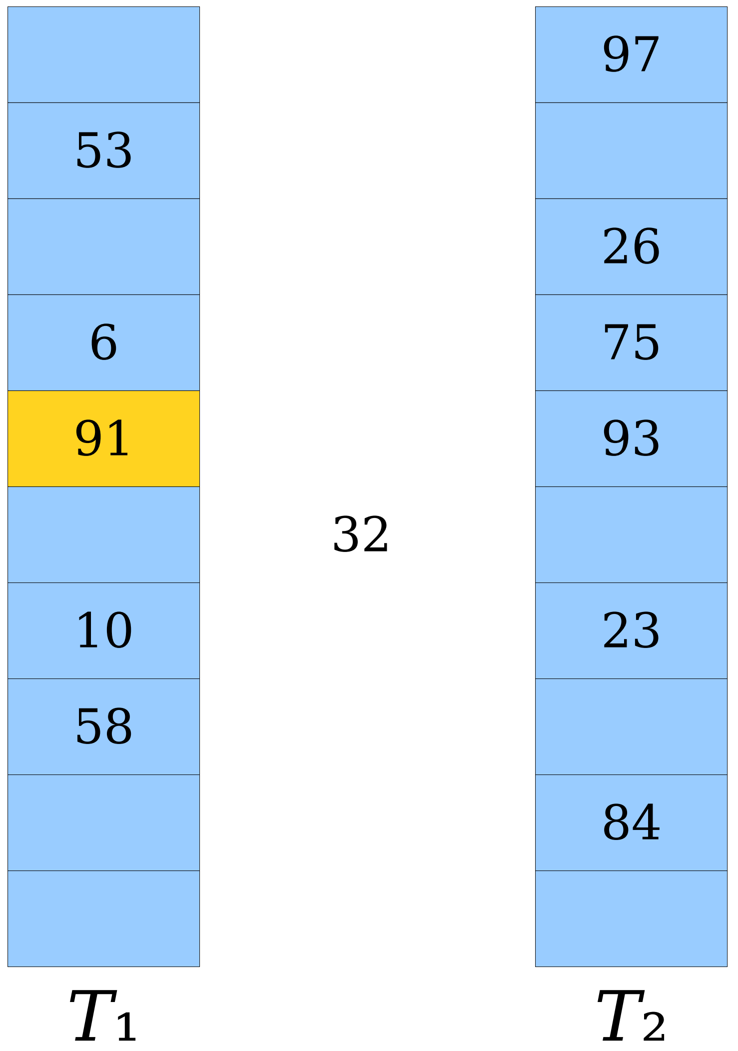

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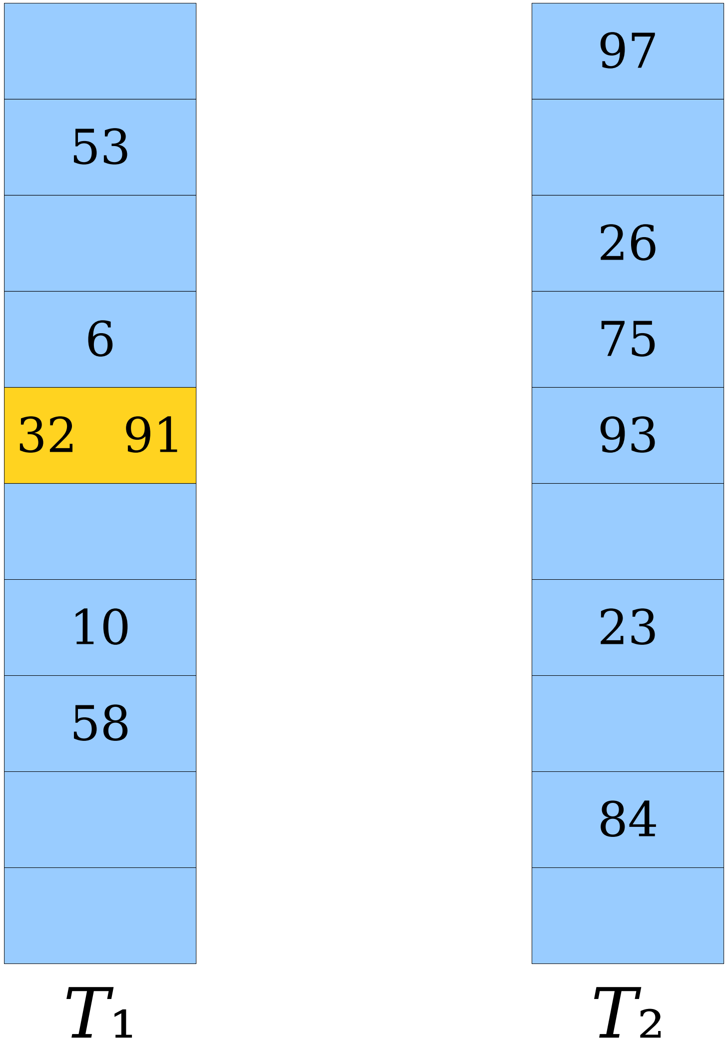

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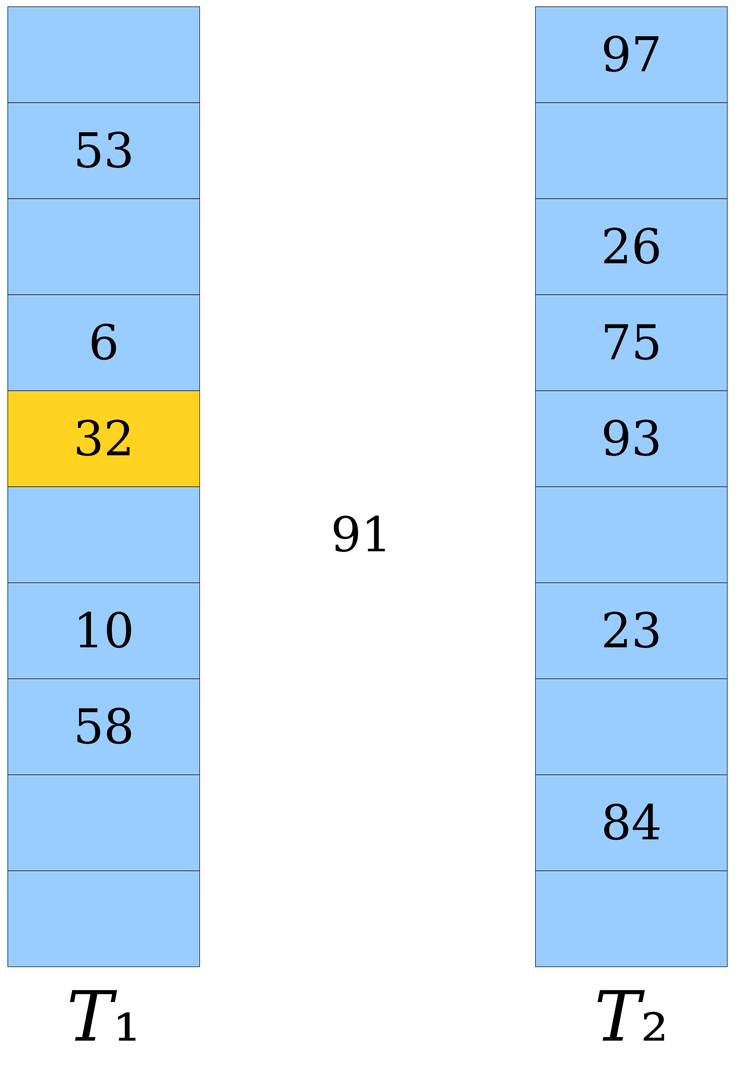

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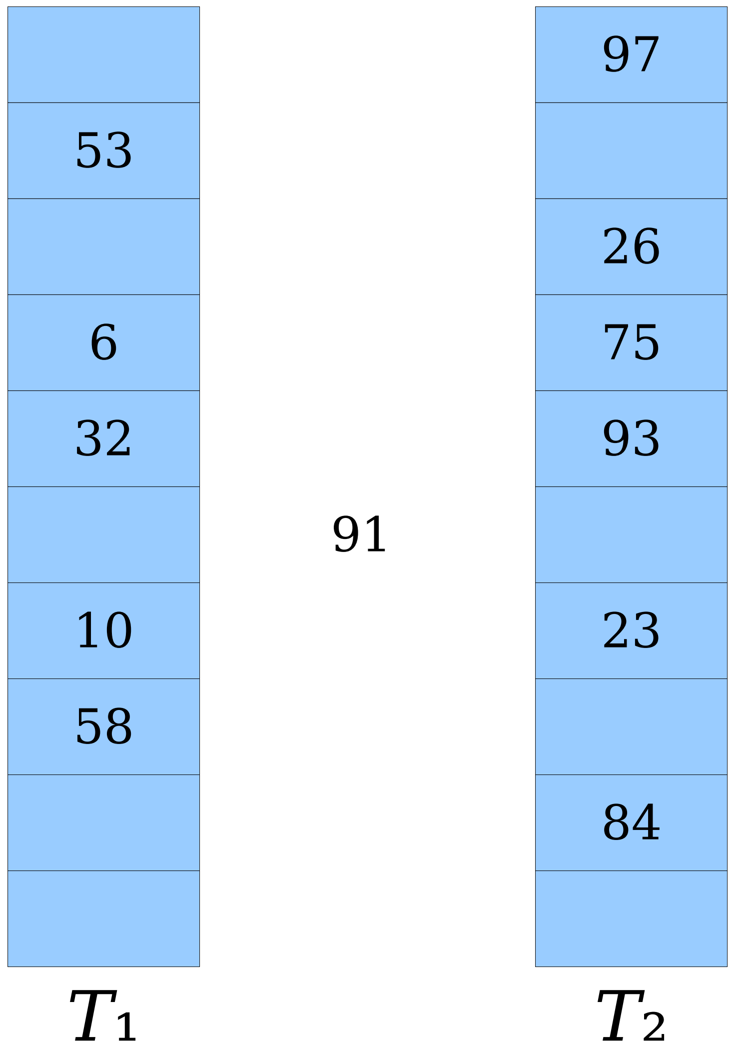

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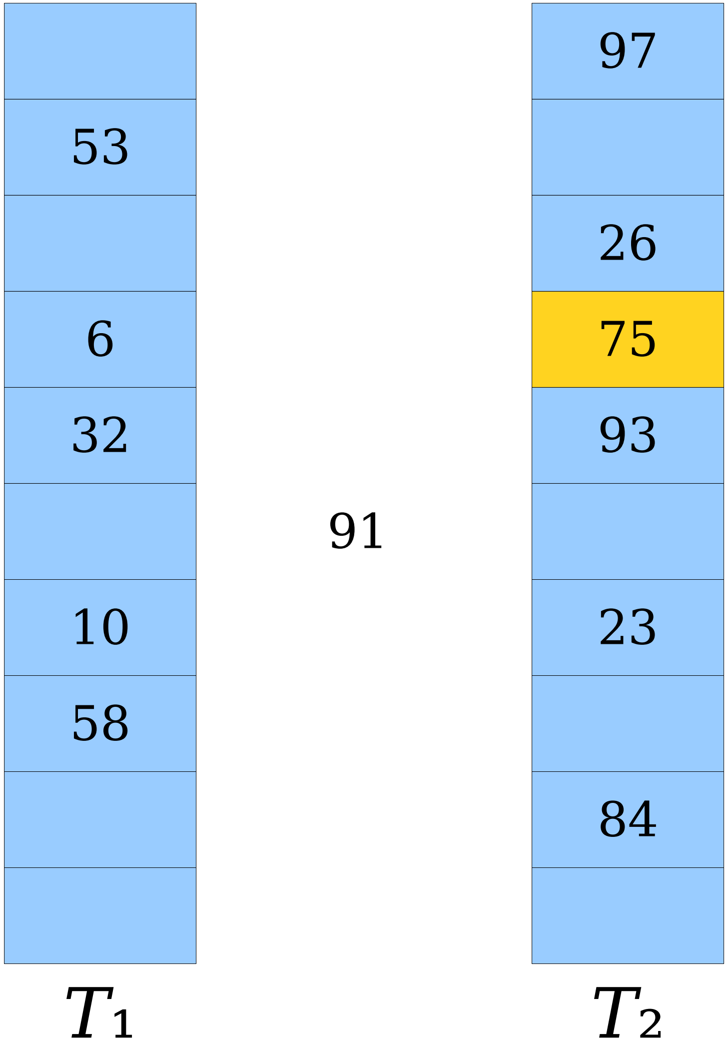

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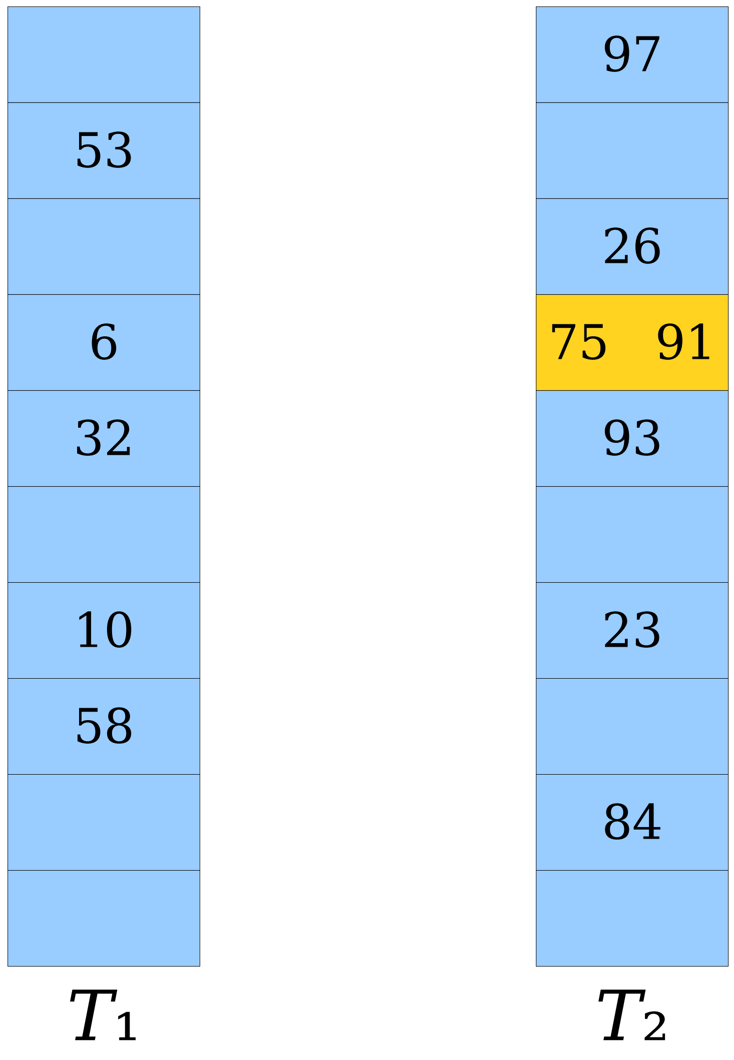

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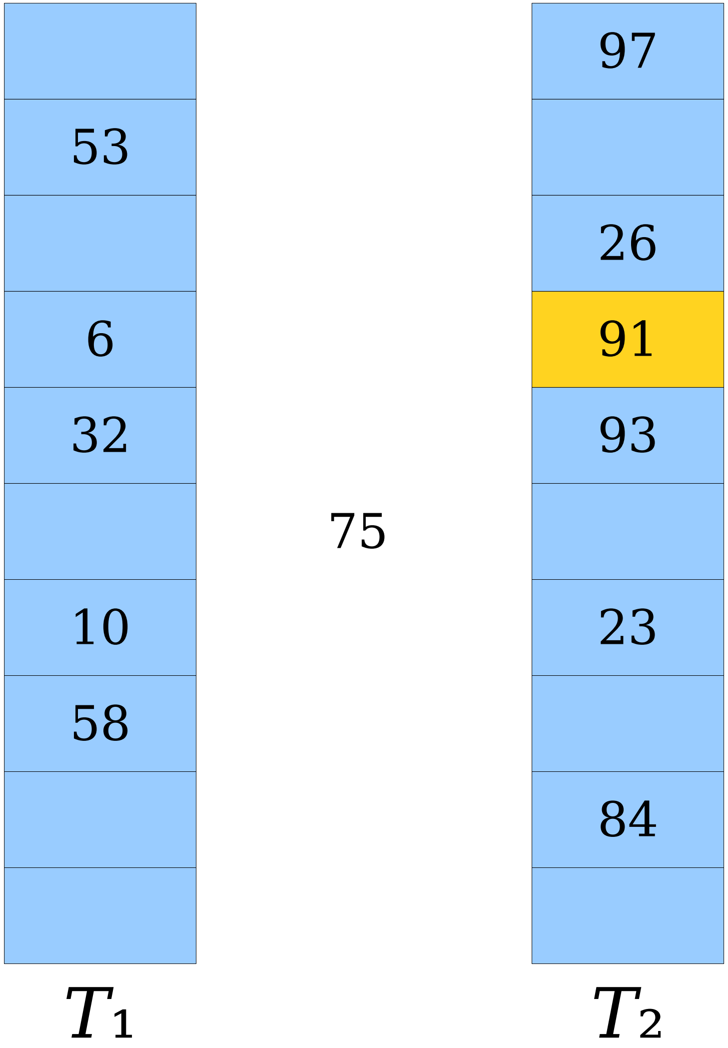

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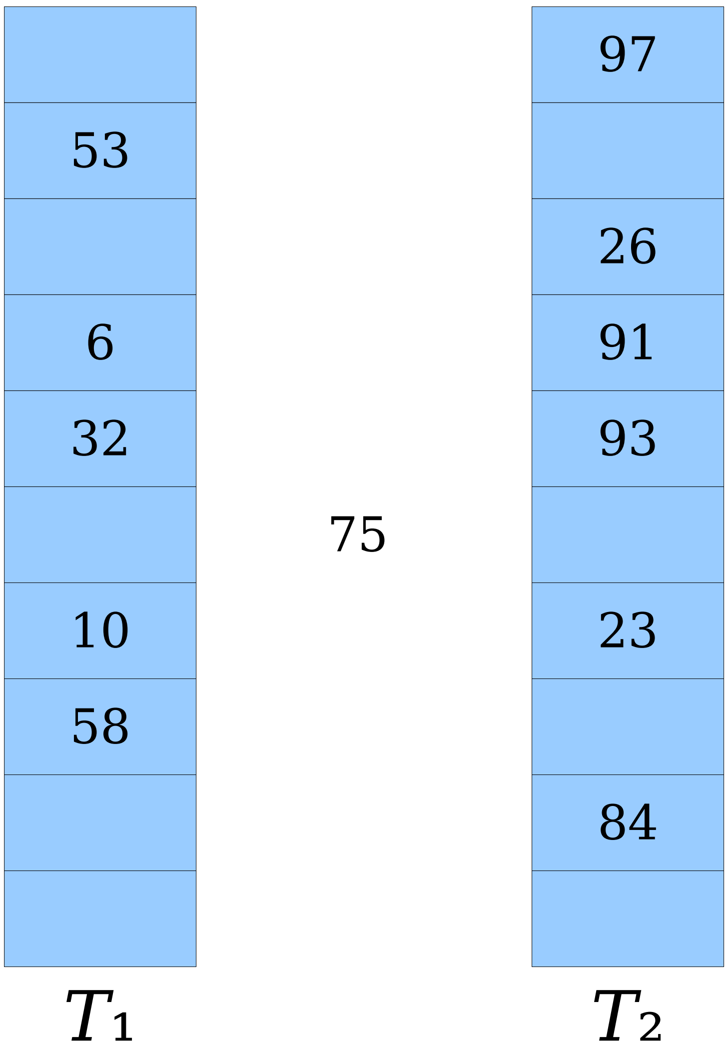

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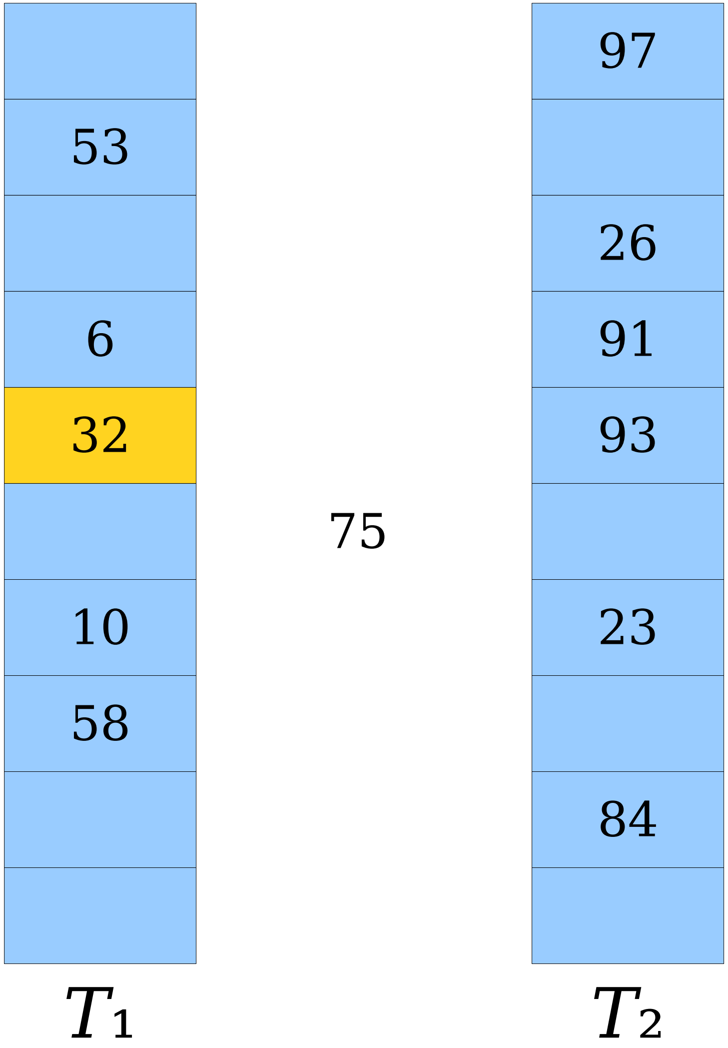

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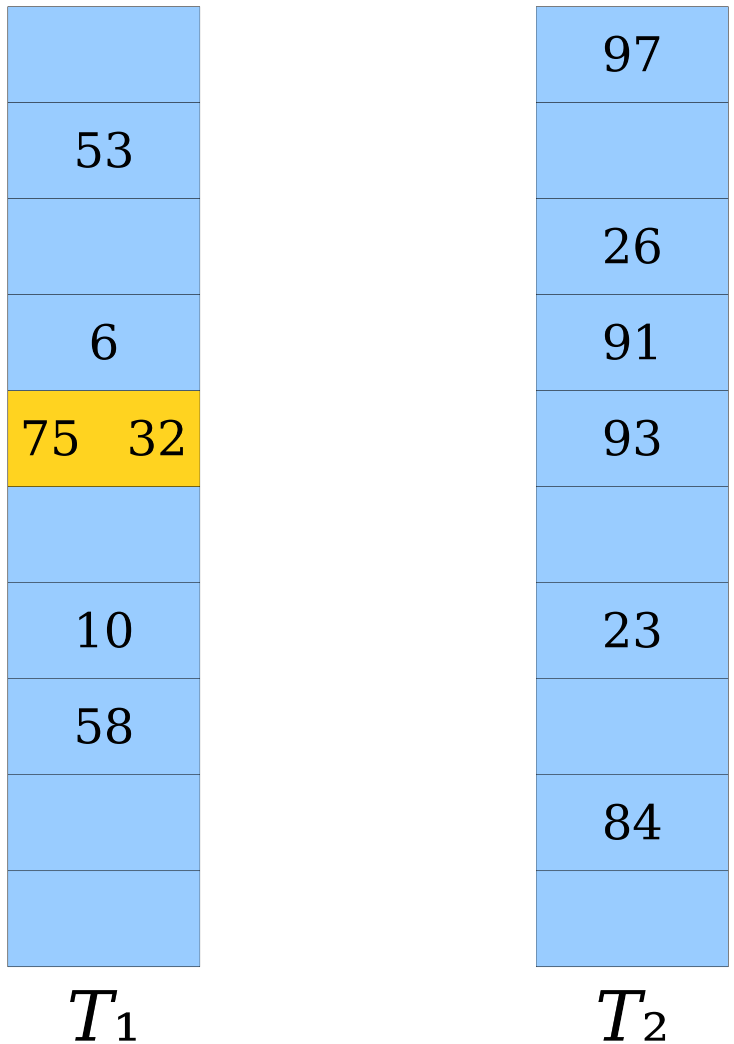

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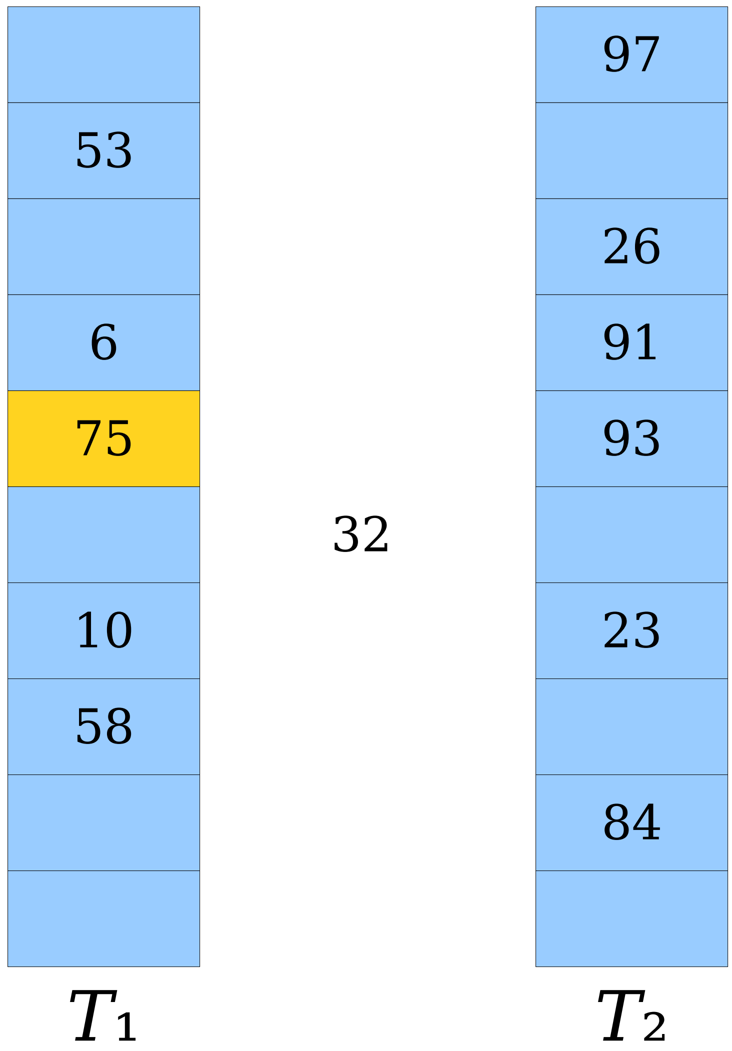

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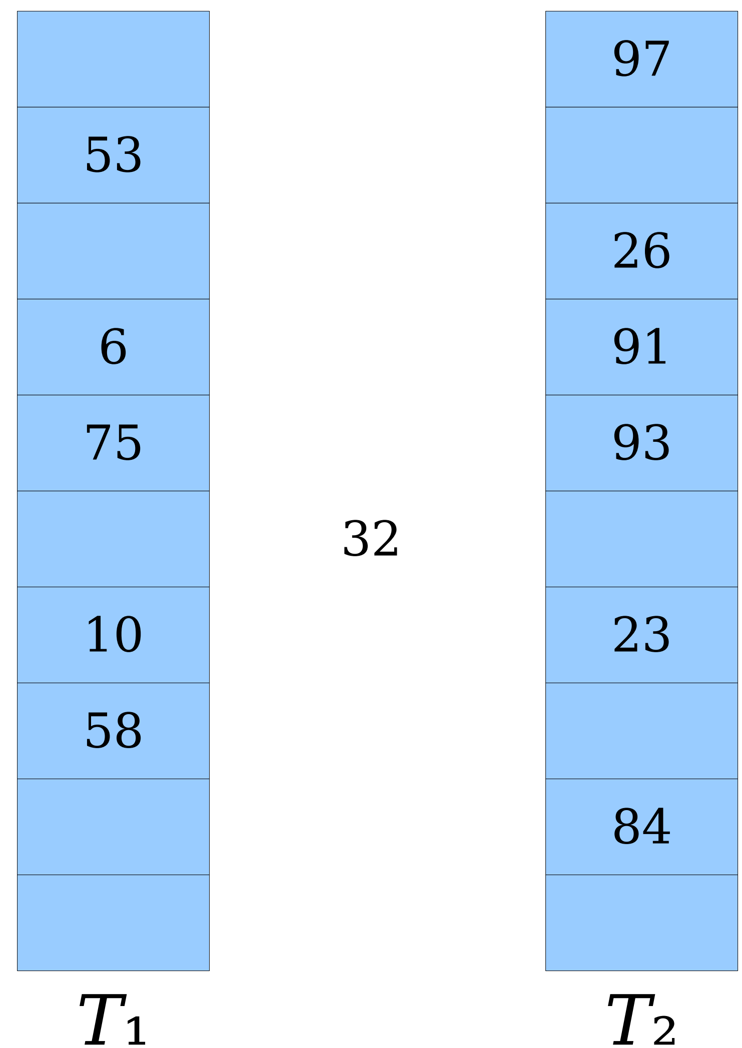

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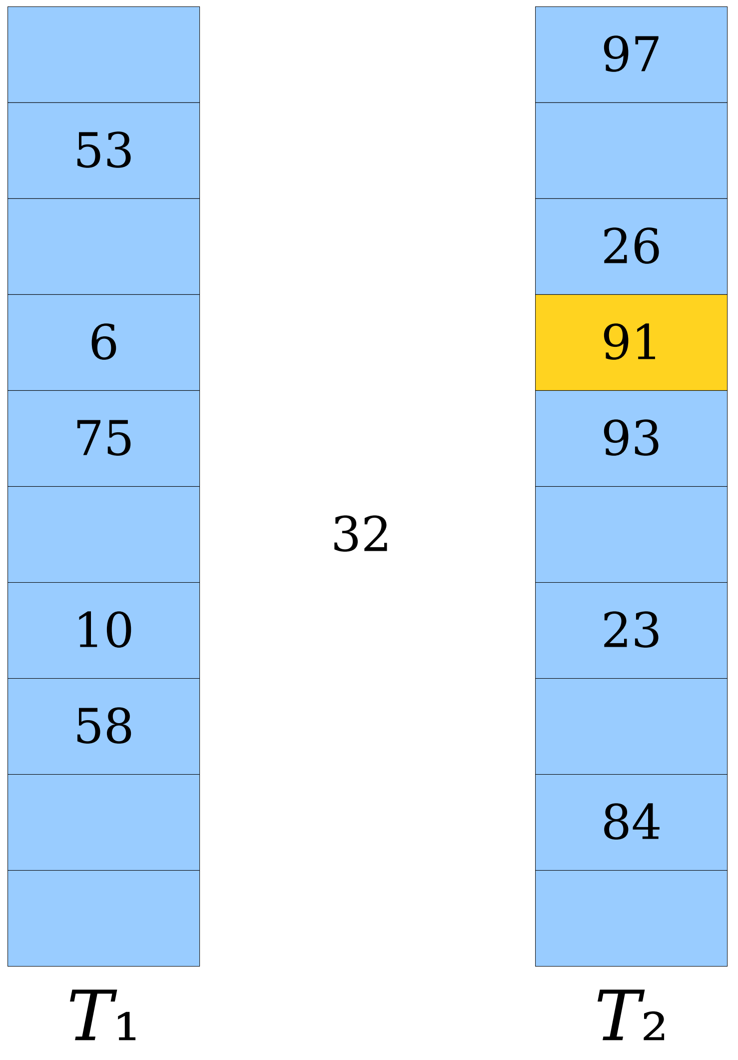

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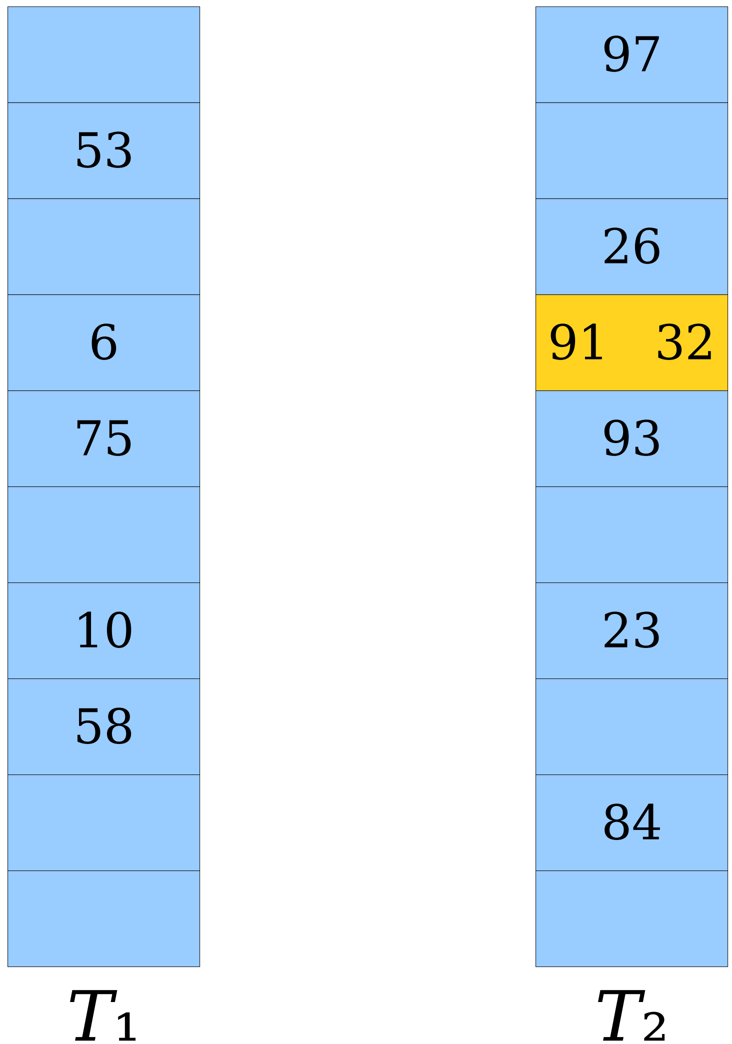

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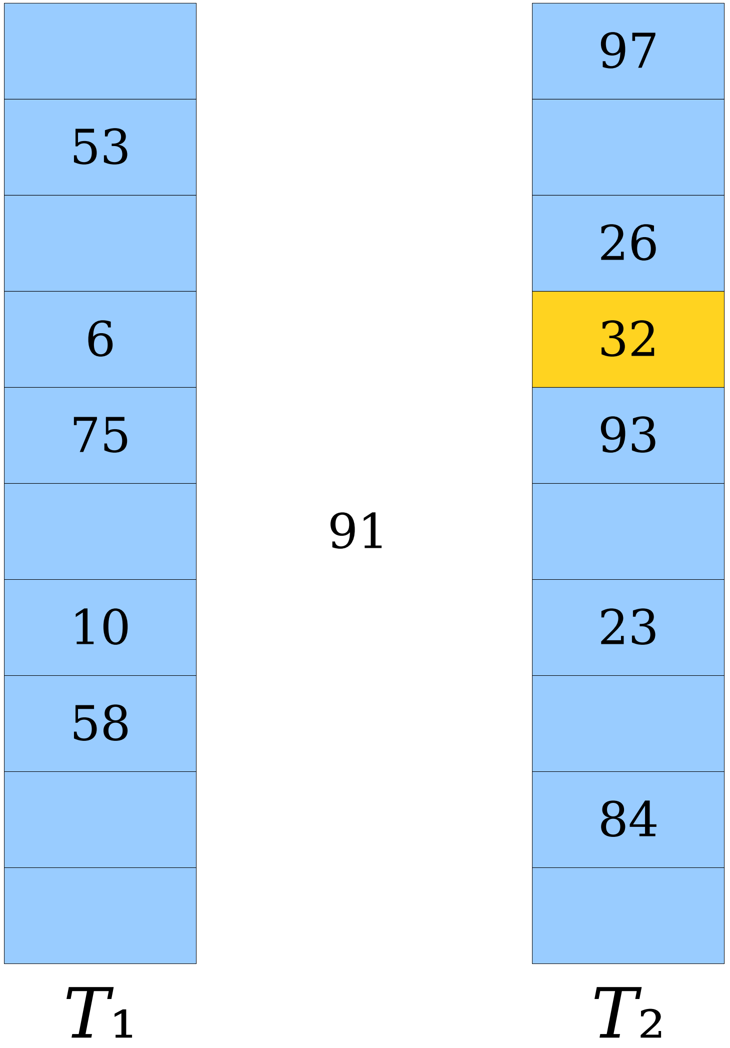

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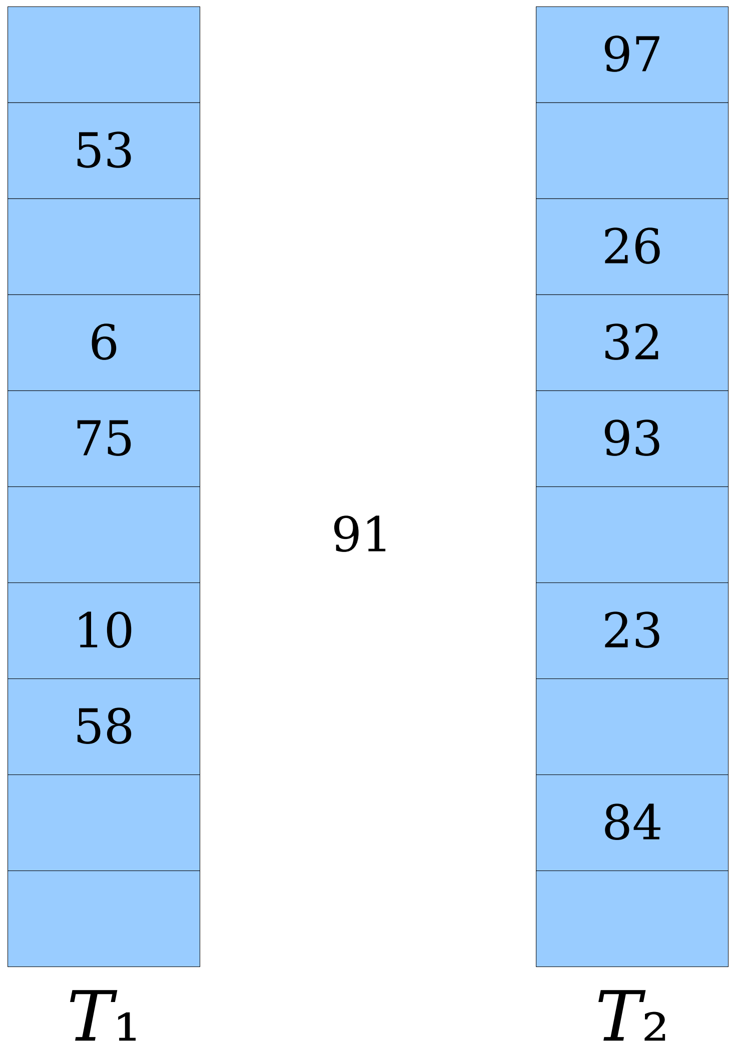

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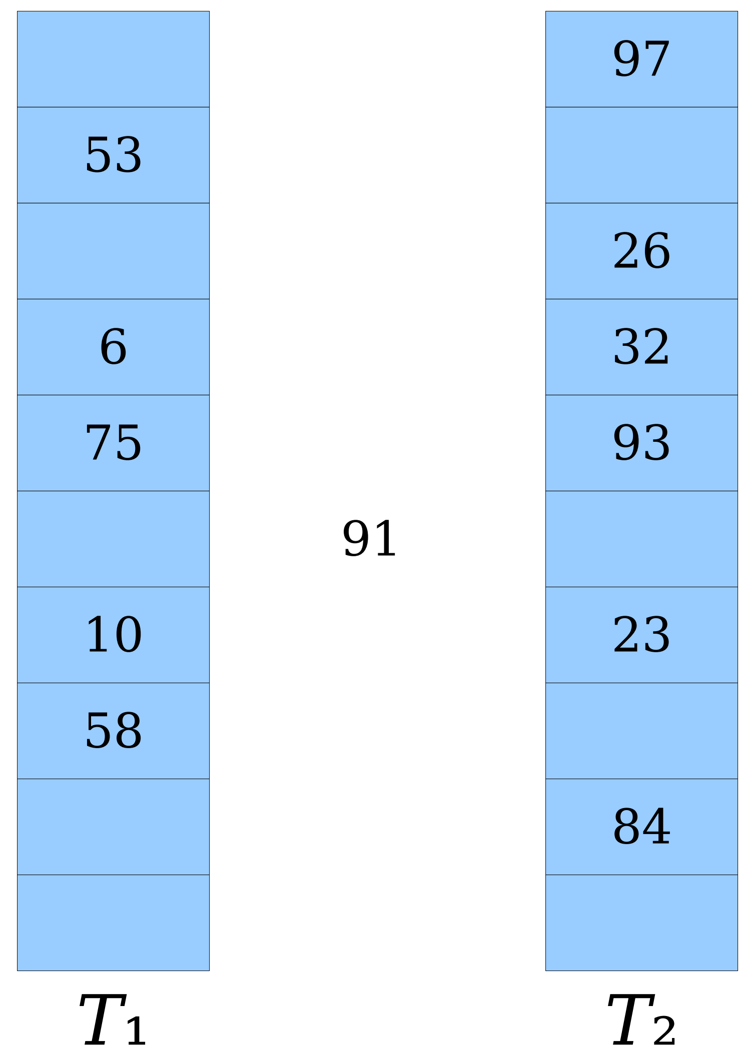

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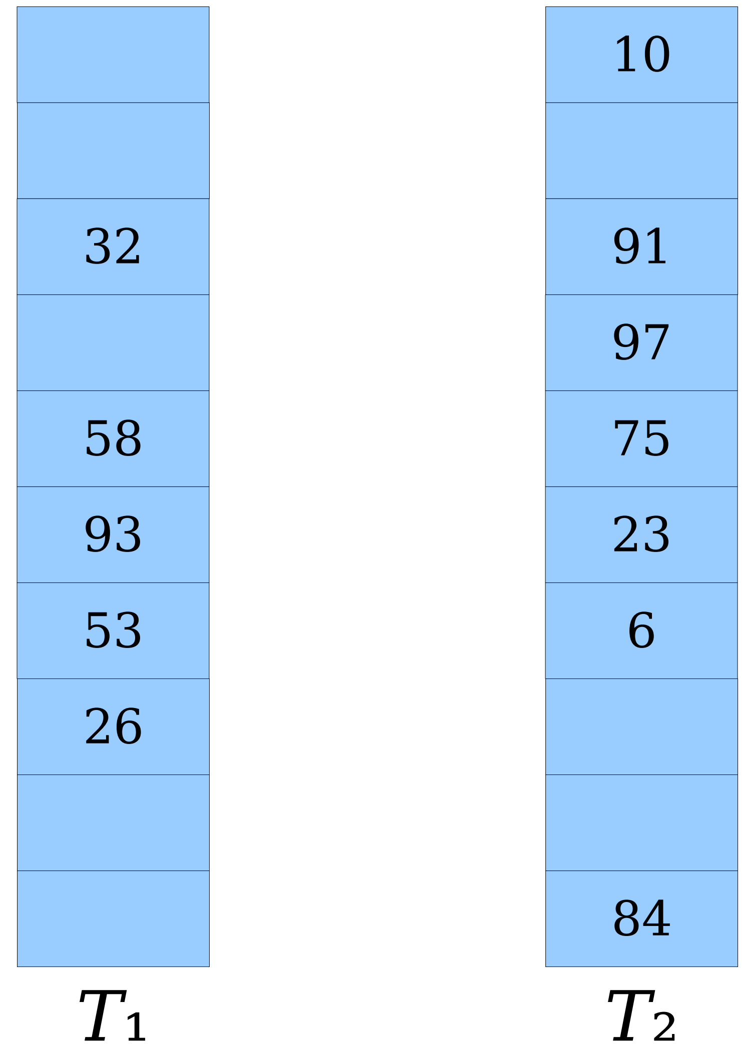

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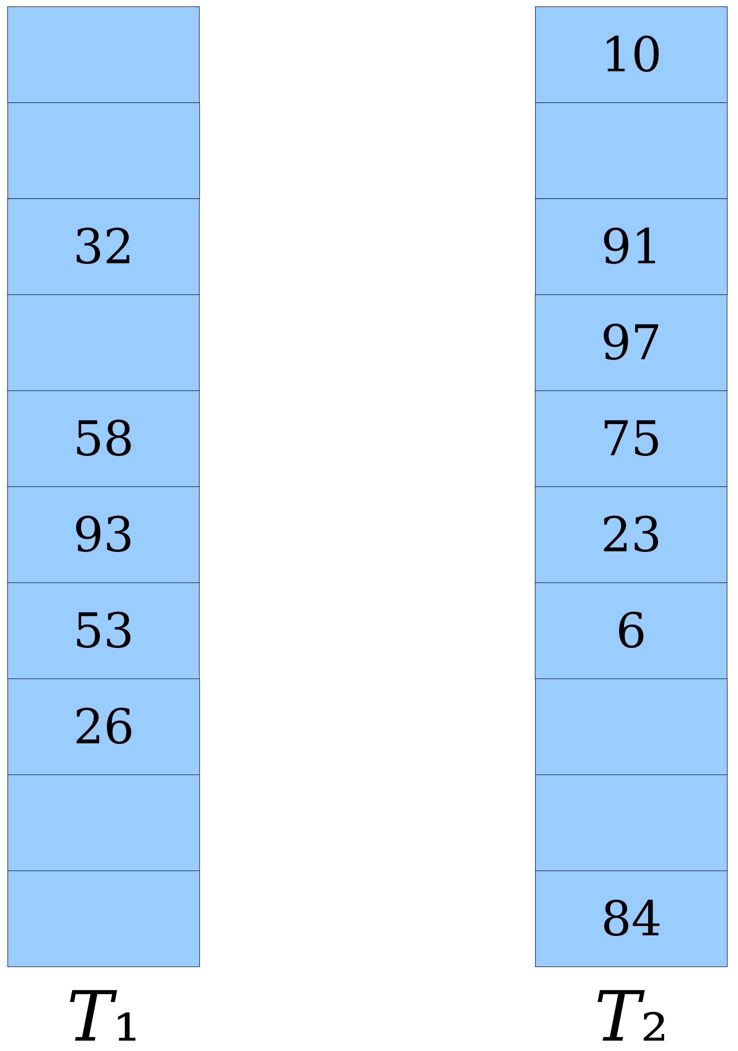
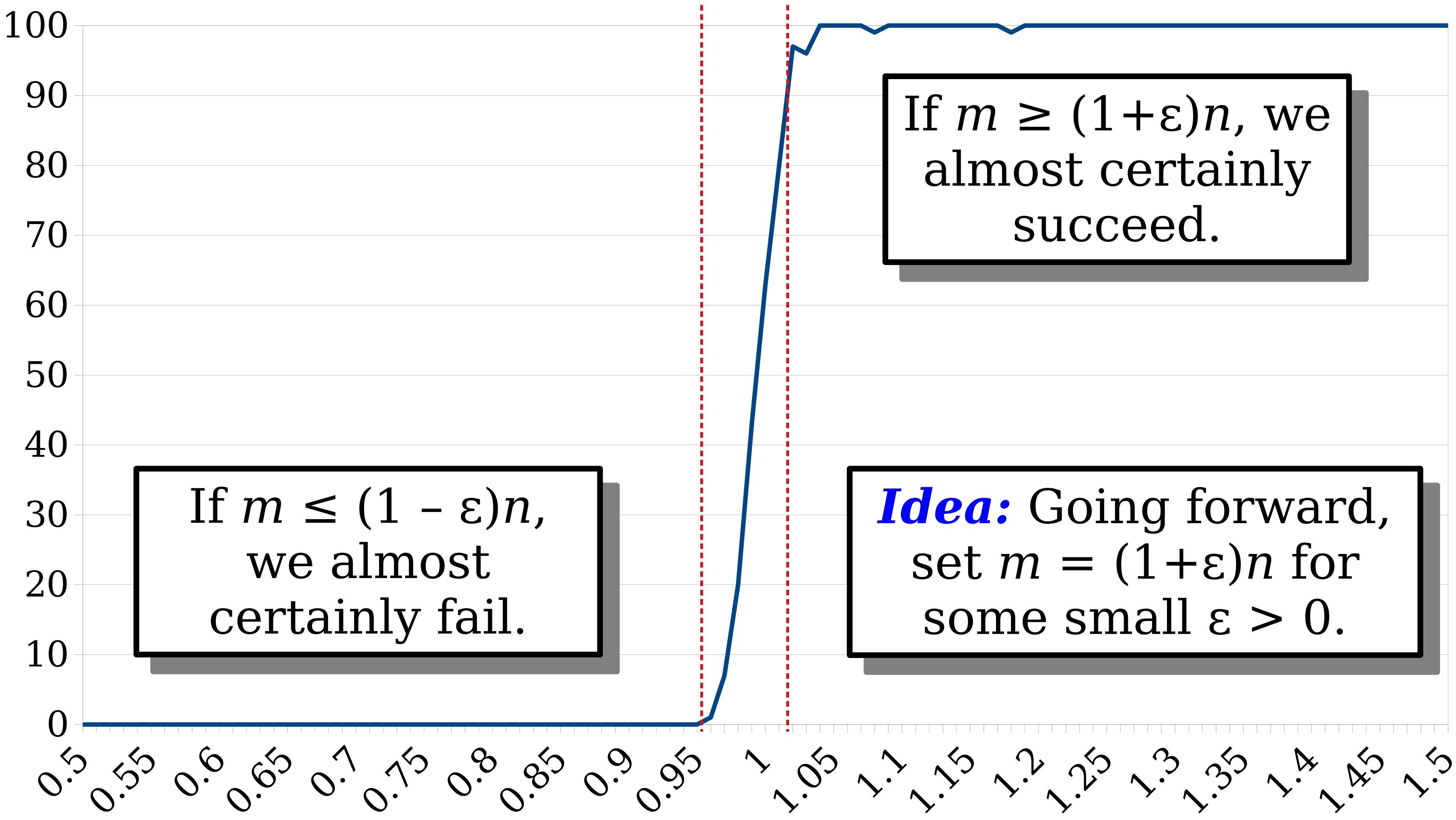
- Multiple rehashes might be necessary before this succeeds.



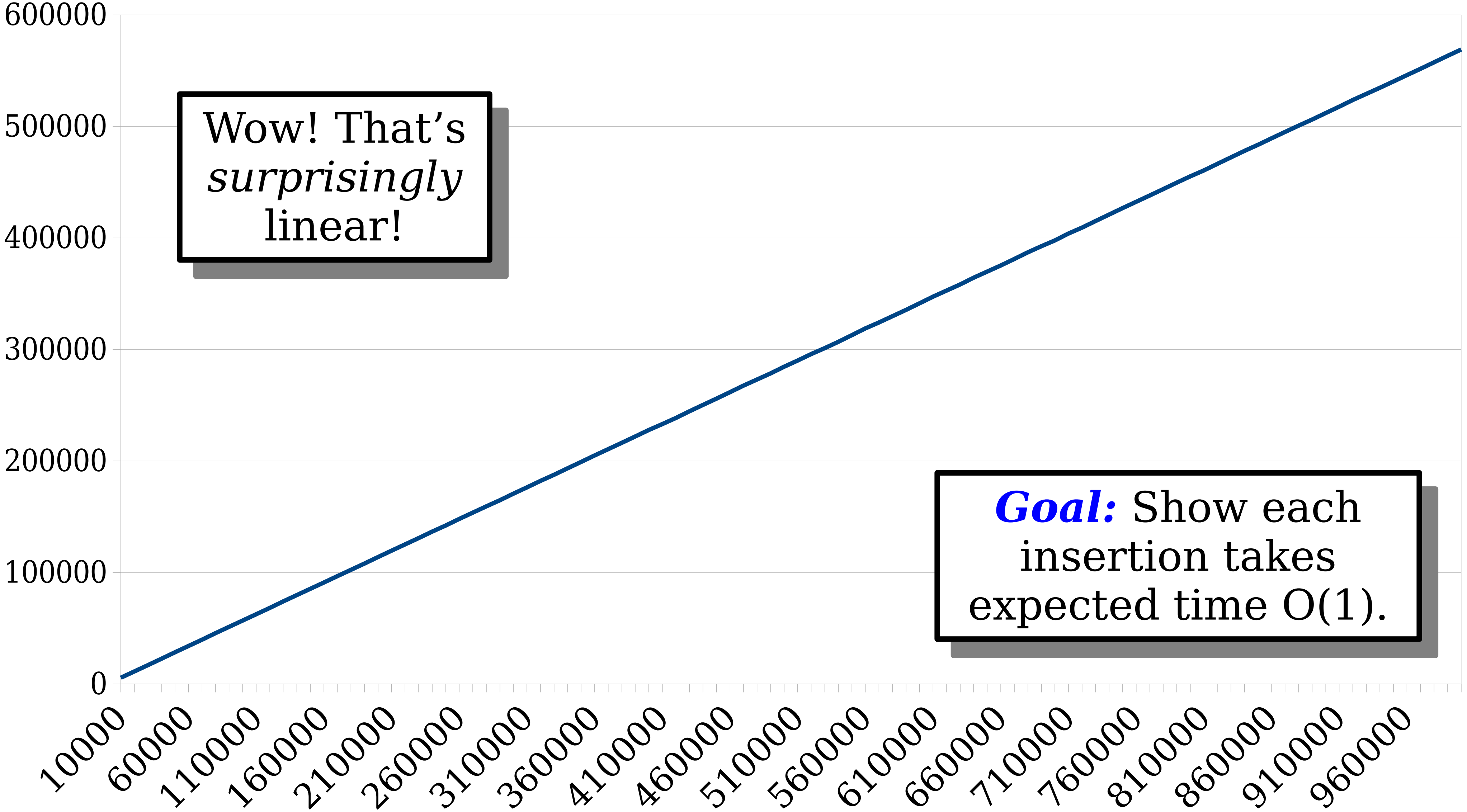
How efficient is cuckoo hashing?

Pro tip: When analyzing a data structure, it never hurts to get some empirical performance data first.



Suppose we store n total elements in two tables of m slots each.

What's probability all insertions succeed, assuming $m = \alpha n$?



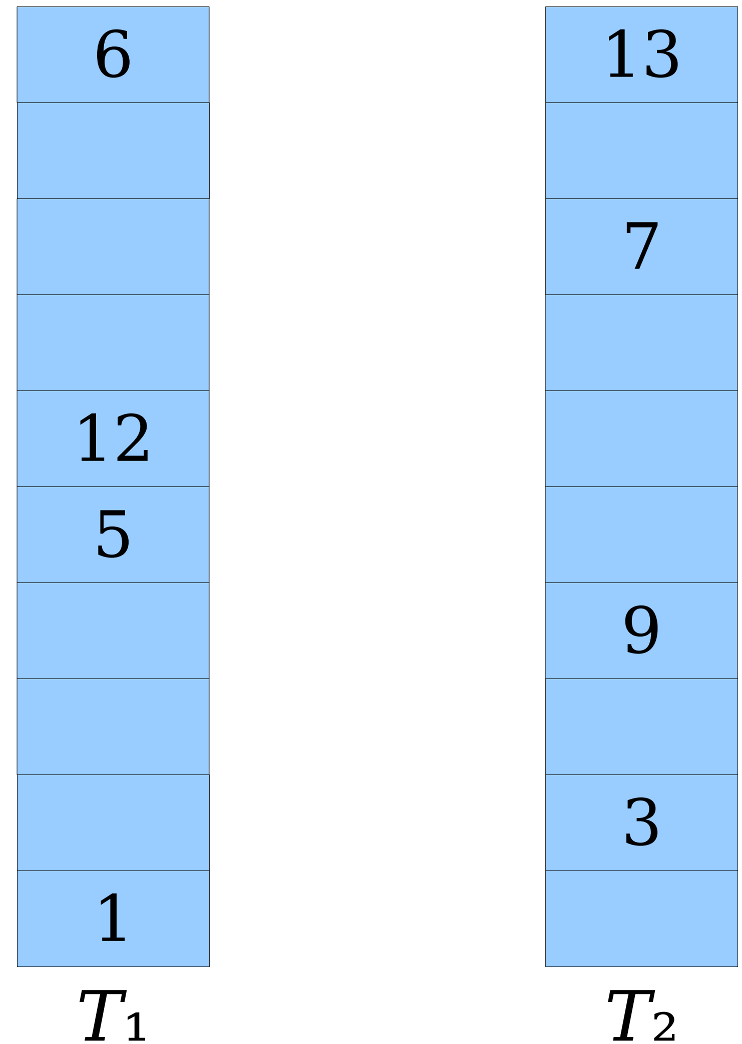
Suppose we store n total elements with $m = (1+\varepsilon)n$.

How many total displacements occur across all insertions?

Goal: Show that insertions take expected time O(1), under the assumption that $m = (1+\varepsilon)n$ for some $\varepsilon > 0$.

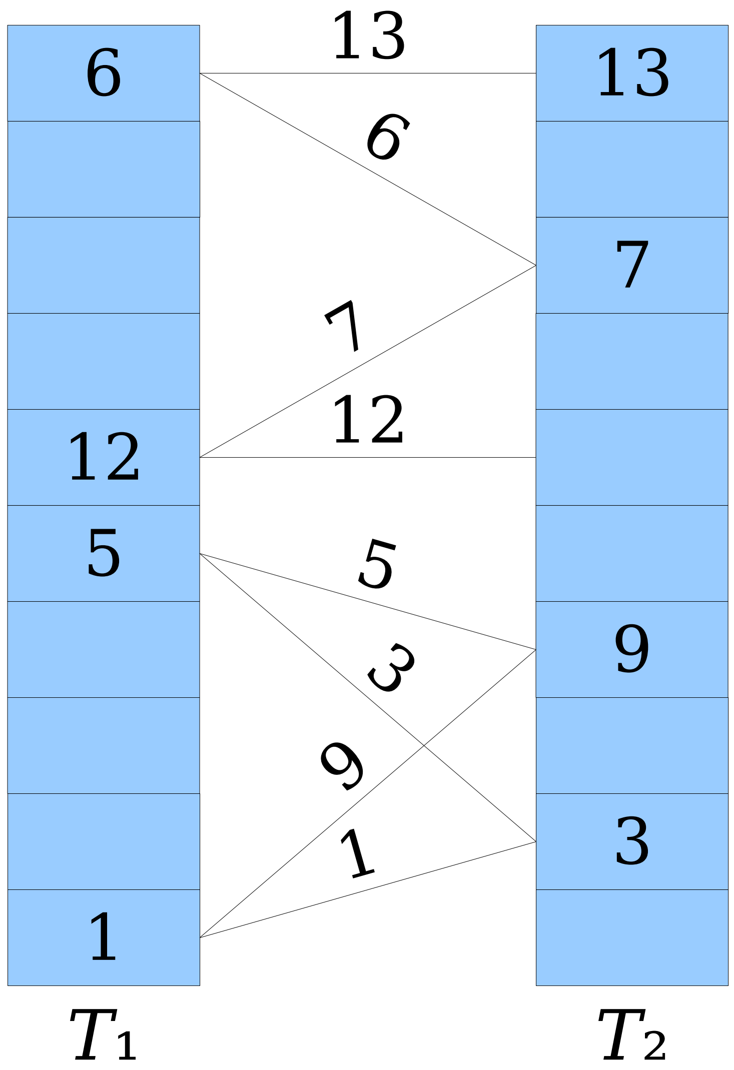
Analyzing Cuckoo Hashing

- The analysis of cuckoo hashing presents some challenges.
- ***Challenge 1:*** We may have to consider hash collisions across multiple hash functions.
- ***Challenge 2:*** We need to reason about chains of displacement, which can be fairly complicated.
- To resolve these challenges, we'll need to bring in some new techniques.



The Cuckoo Graph

- The ***cuckoo graph*** is a bipartite multigraph derived from a cuckoo hash table.
- Each table slot is a node.
- Each element is an edge.
- Edges link slots where each element can be.
- Each insertion introduces a new edge into the graph.



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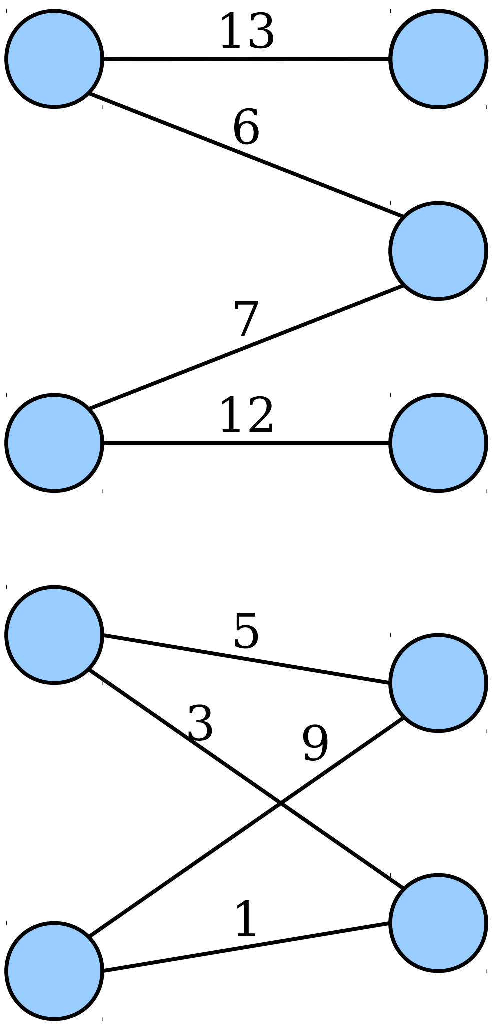

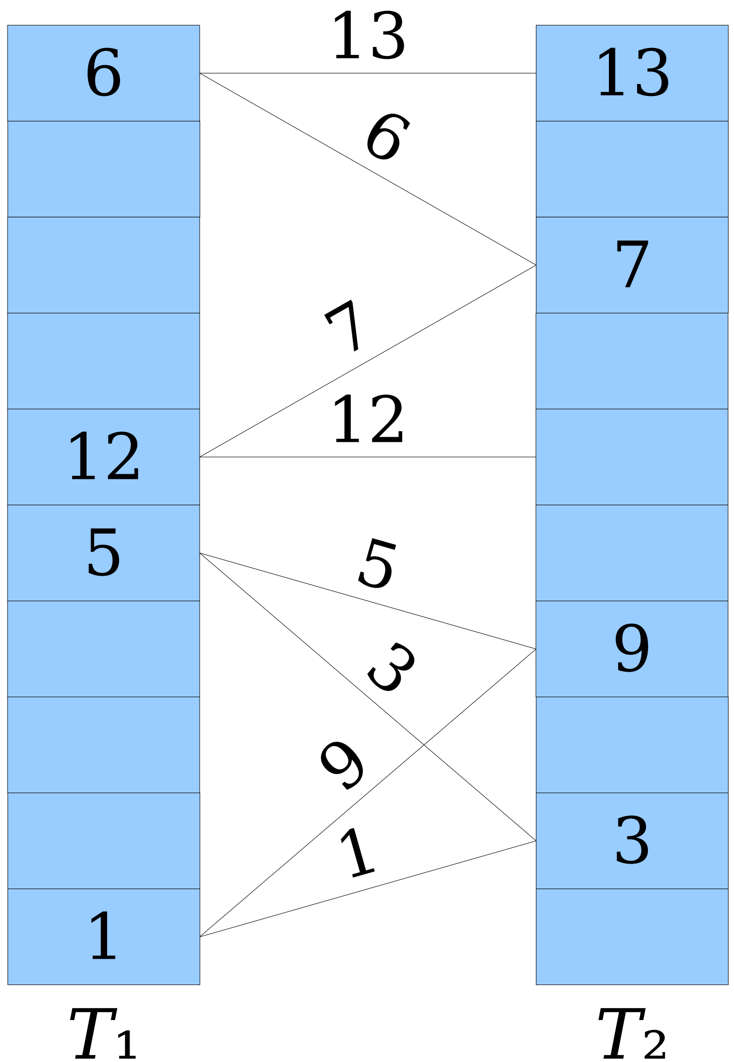

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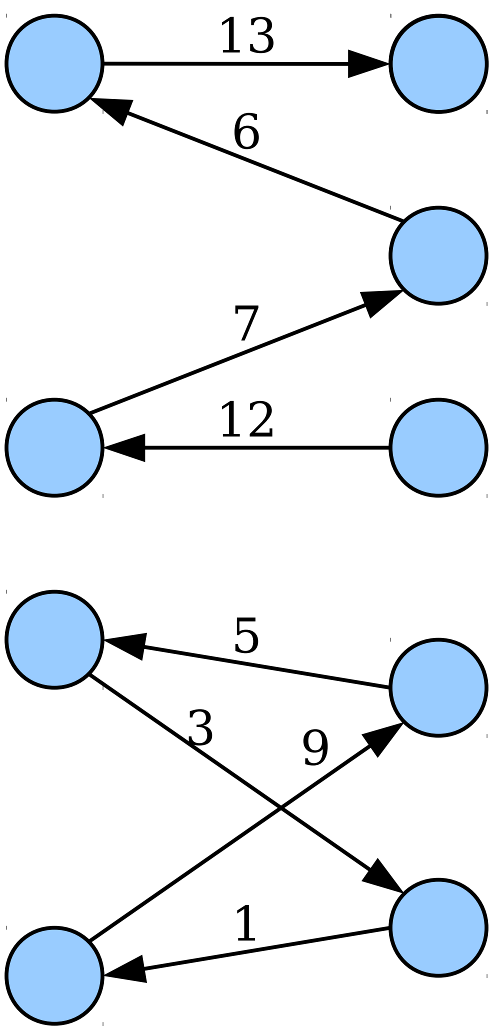
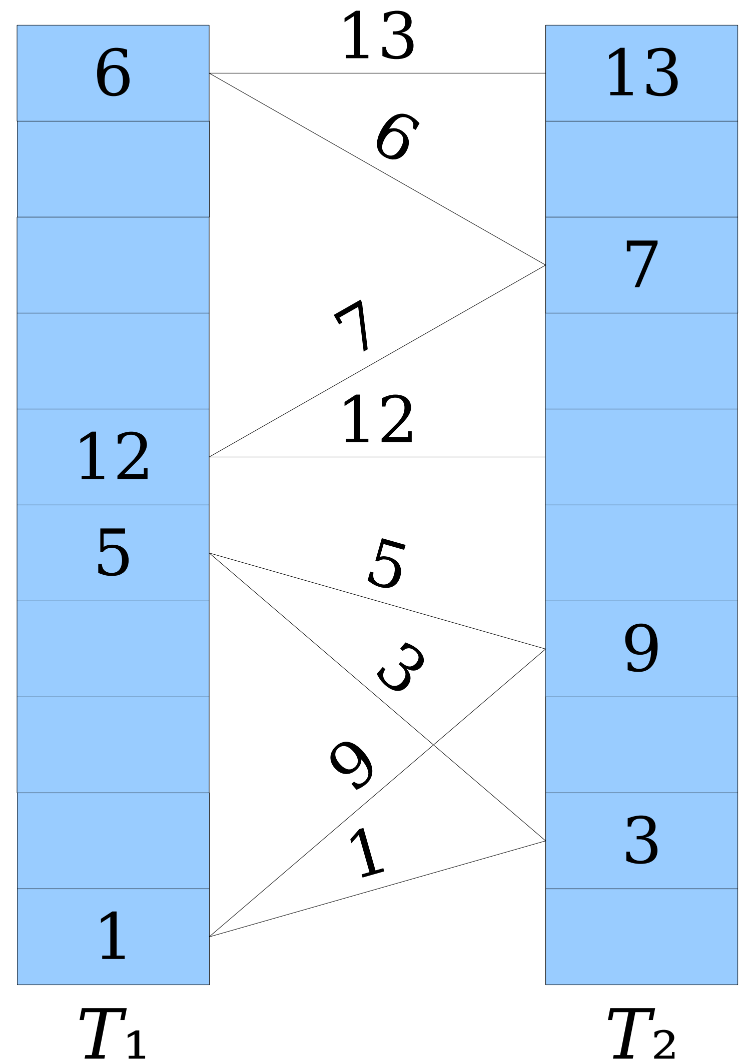


Arrowheads indicate which slots elements is stored in.

Each node has at most one incoming edge.



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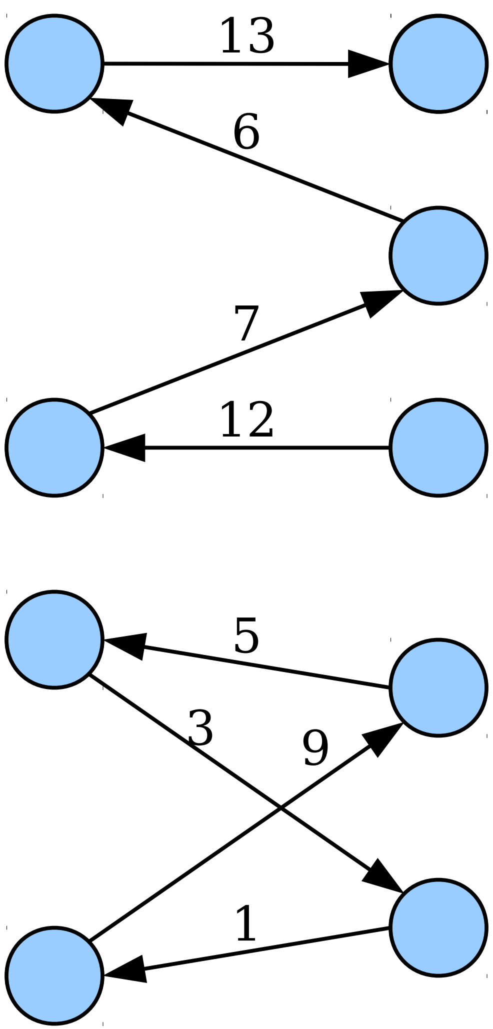
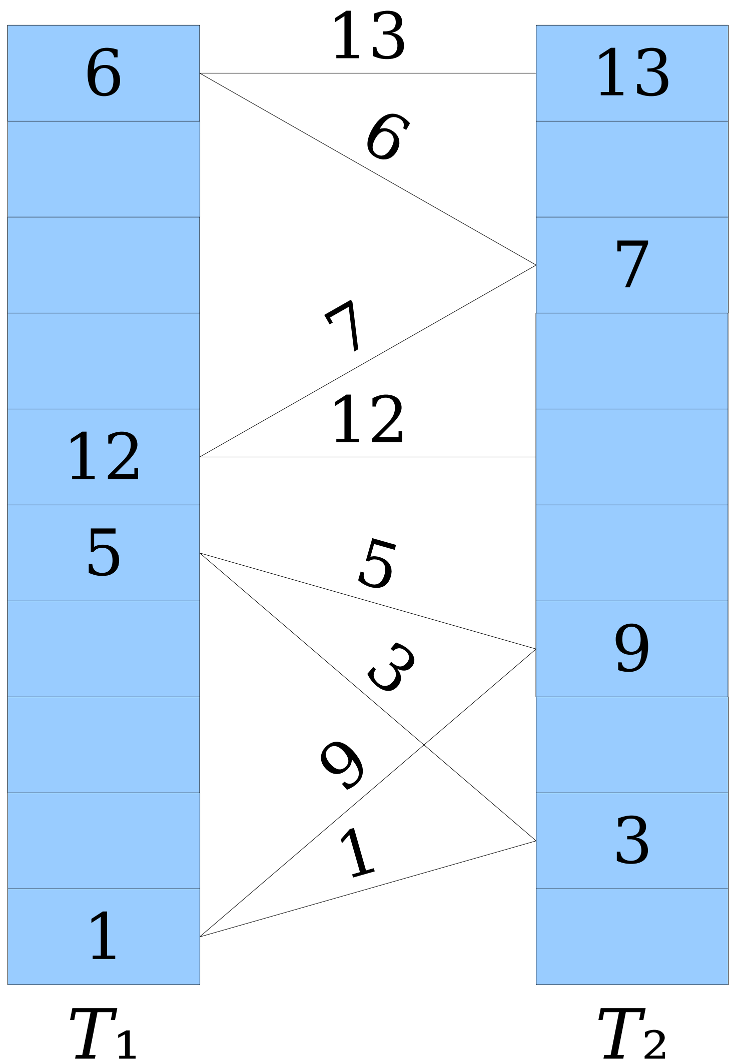


Insertions correspond to sequences of flipping arrowheads.



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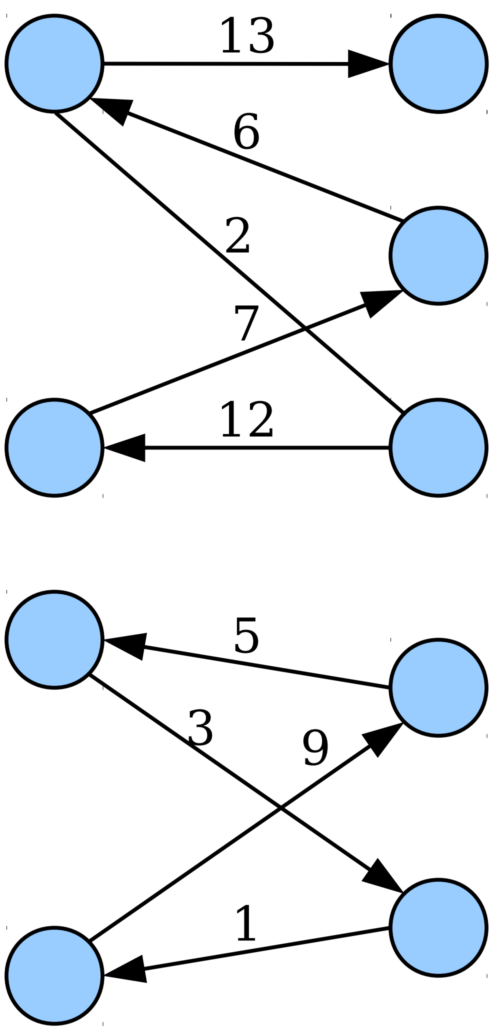

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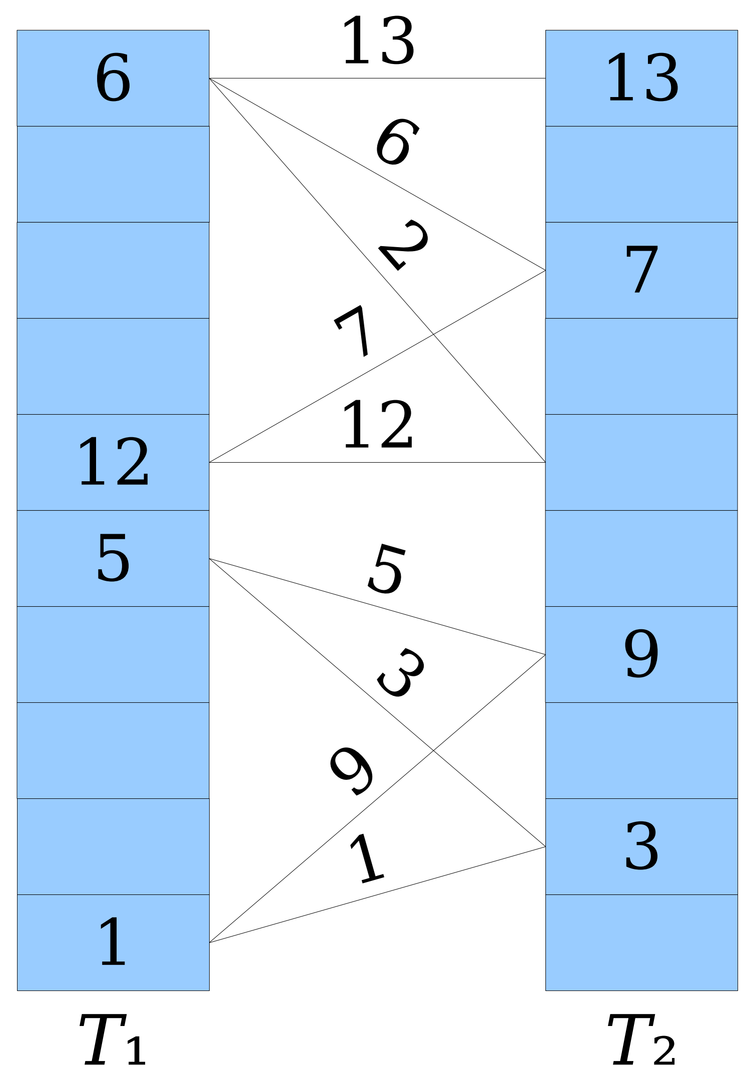

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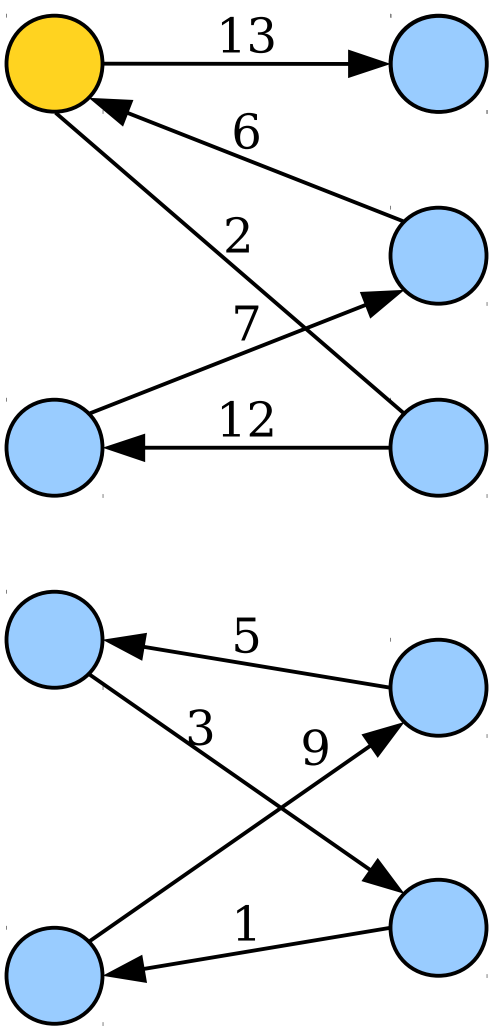

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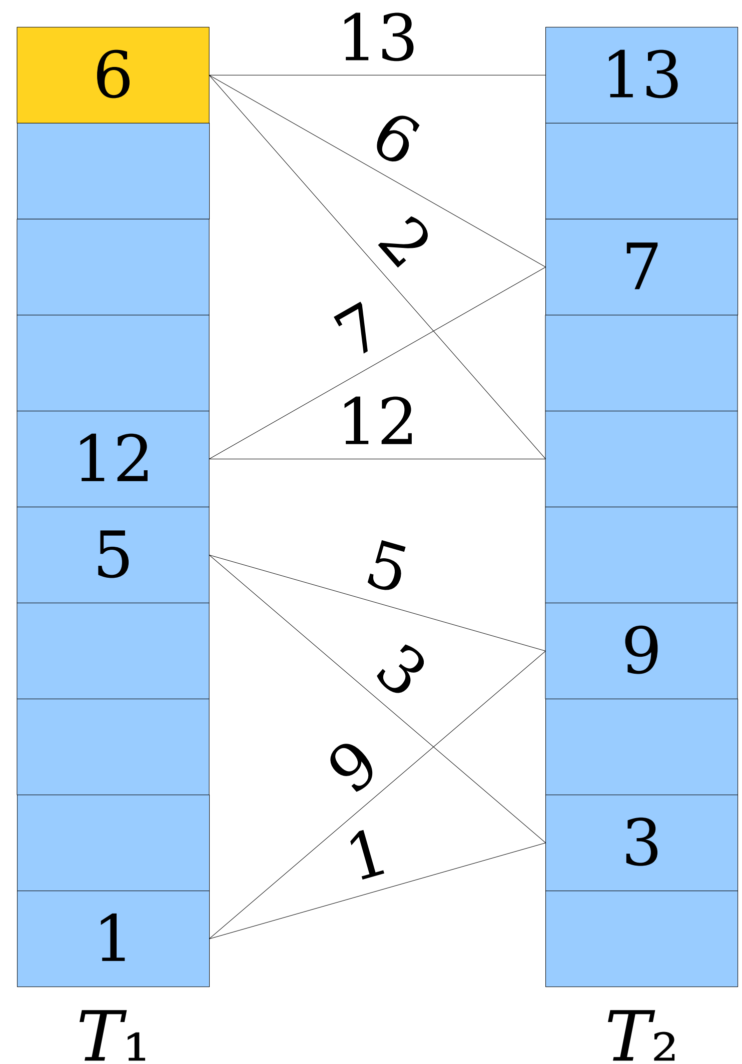

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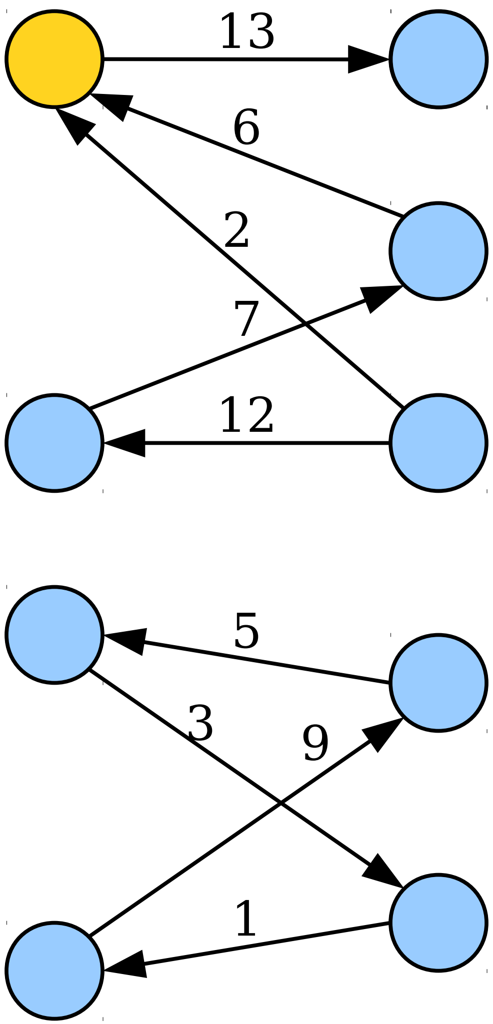

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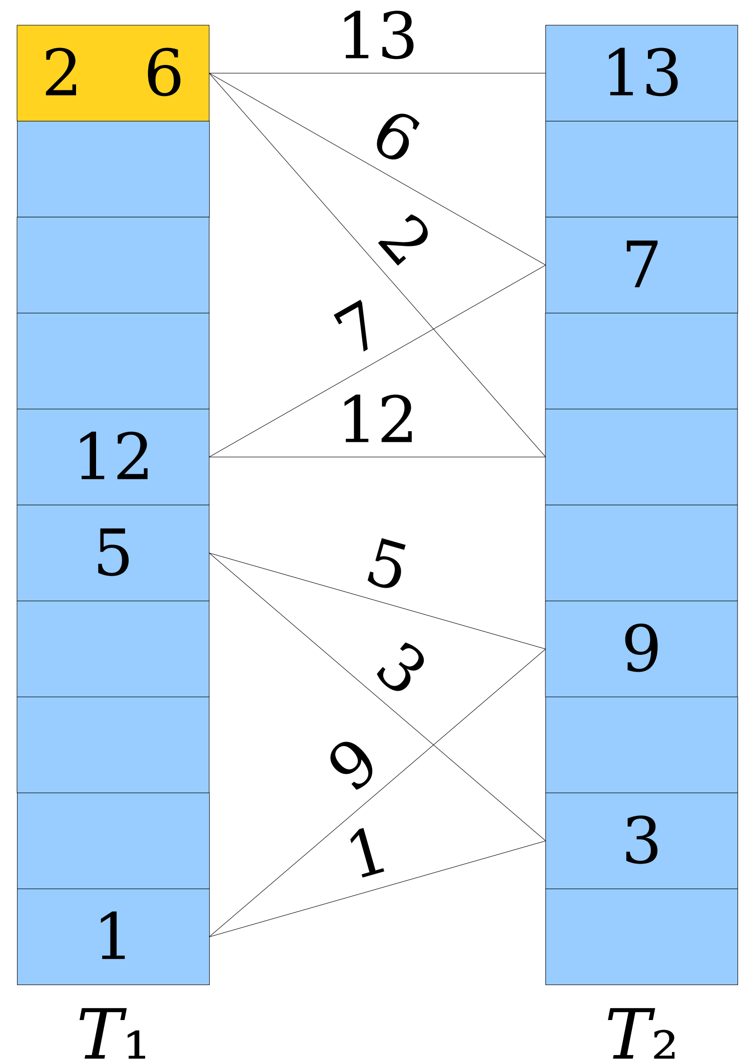

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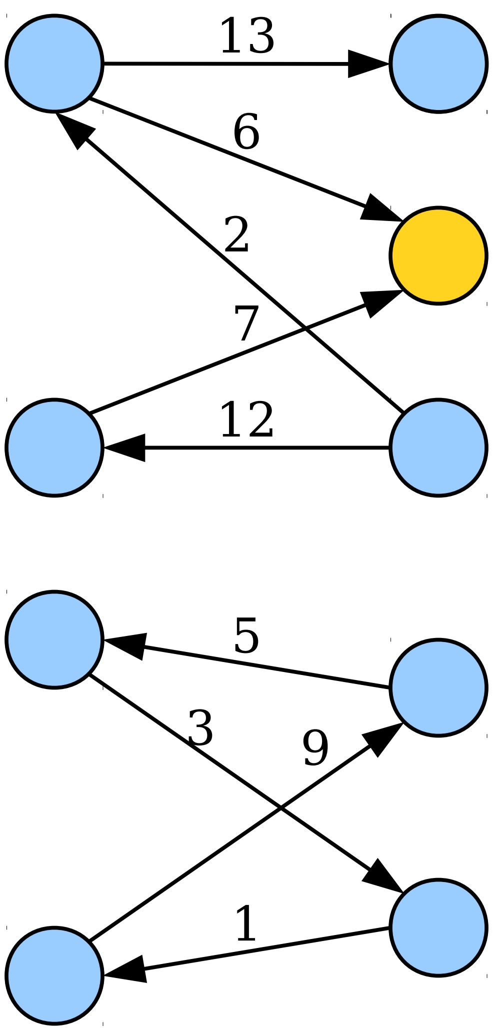

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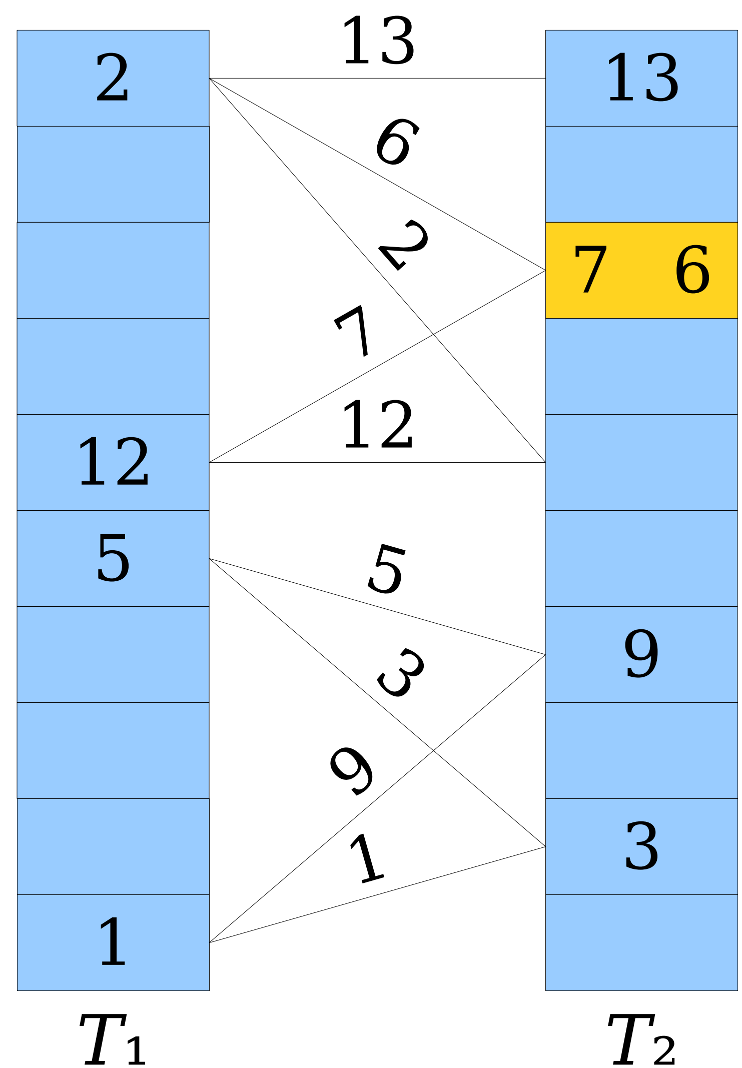

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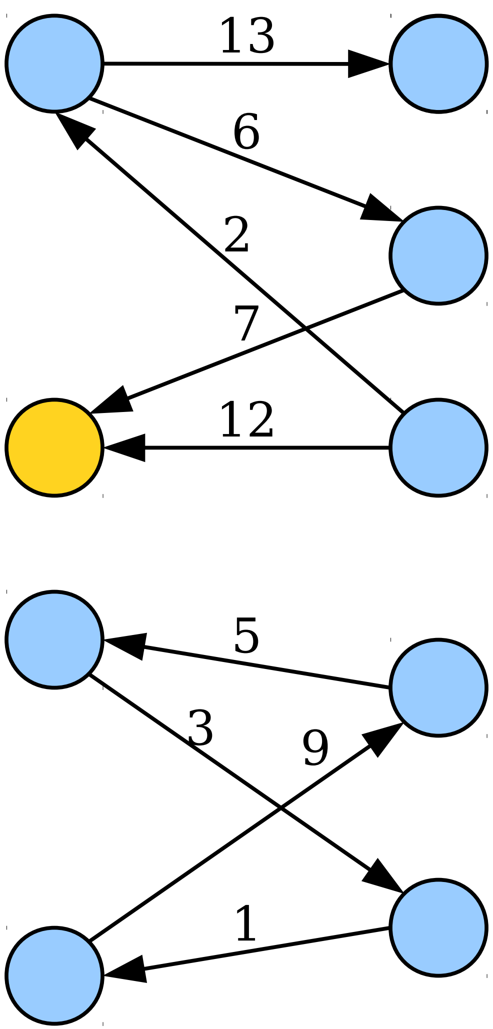

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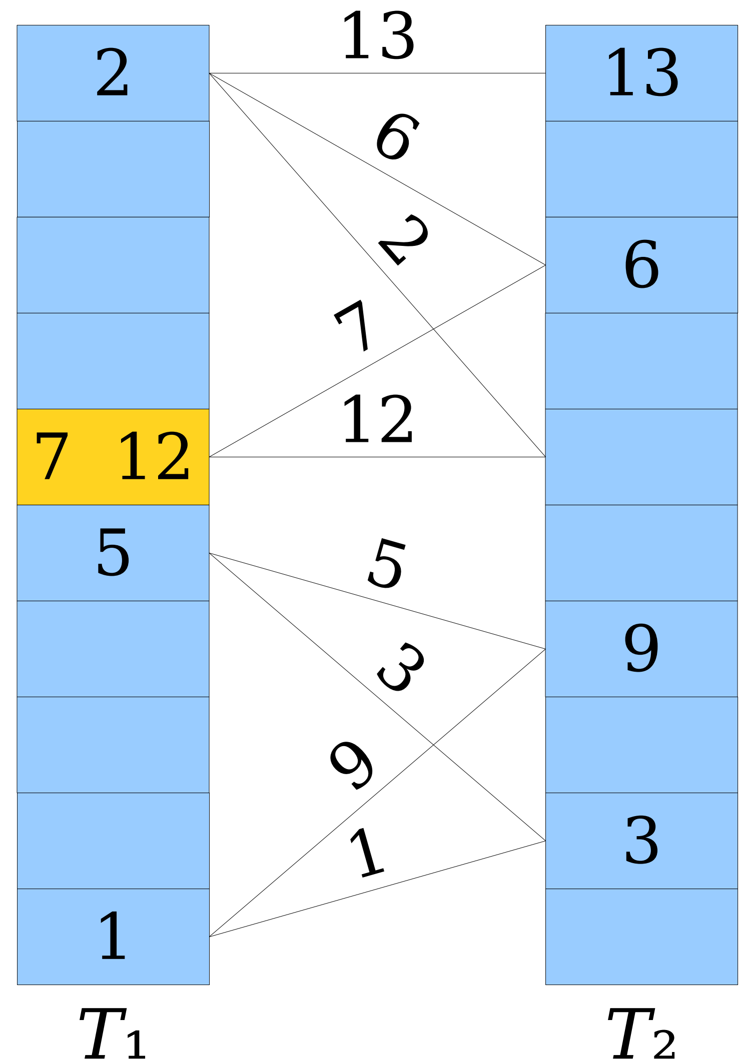

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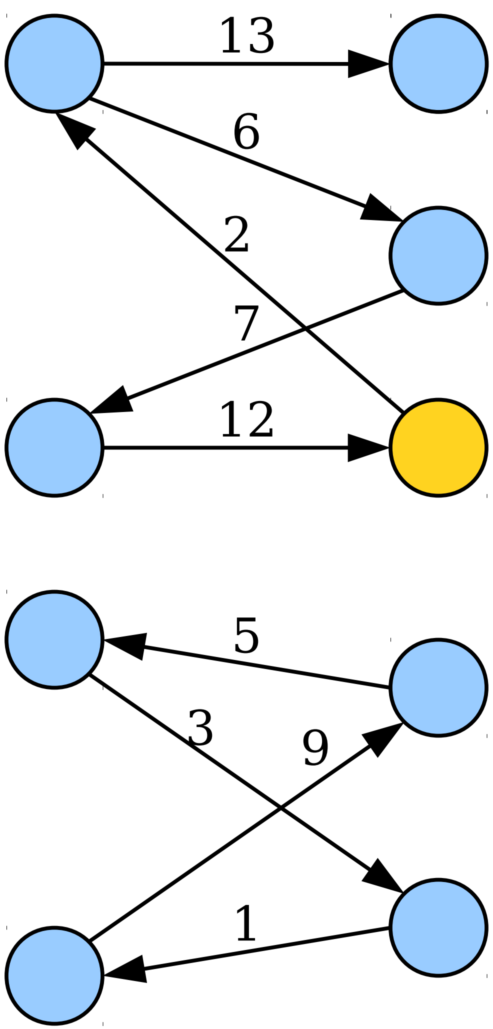

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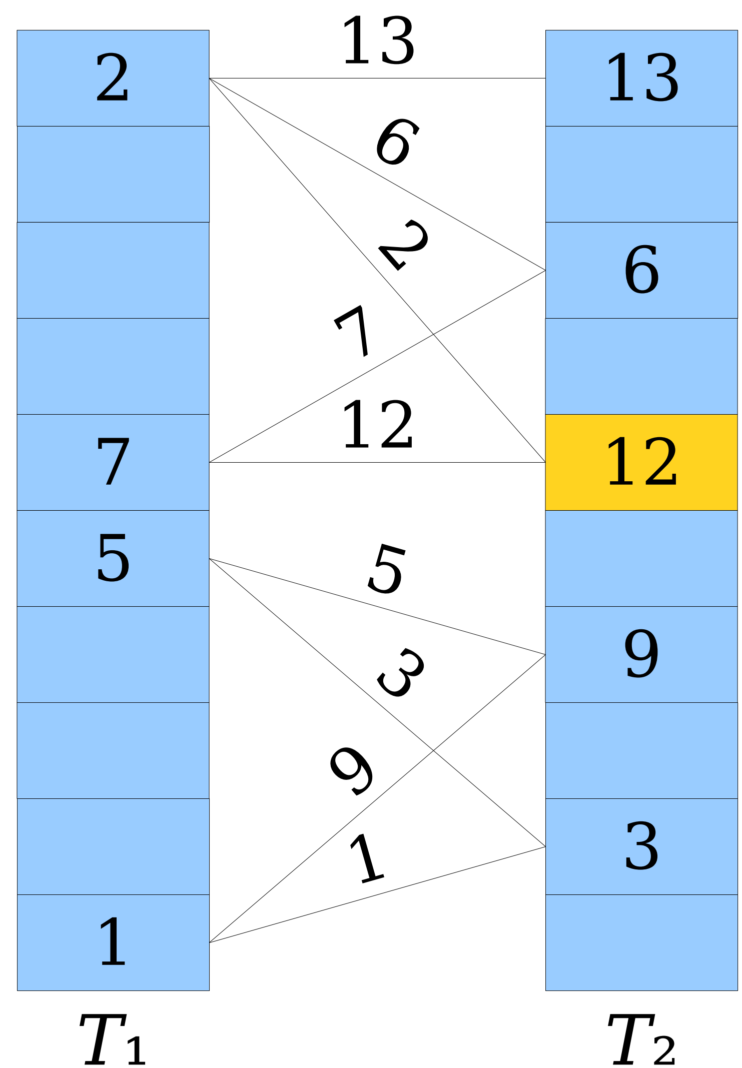

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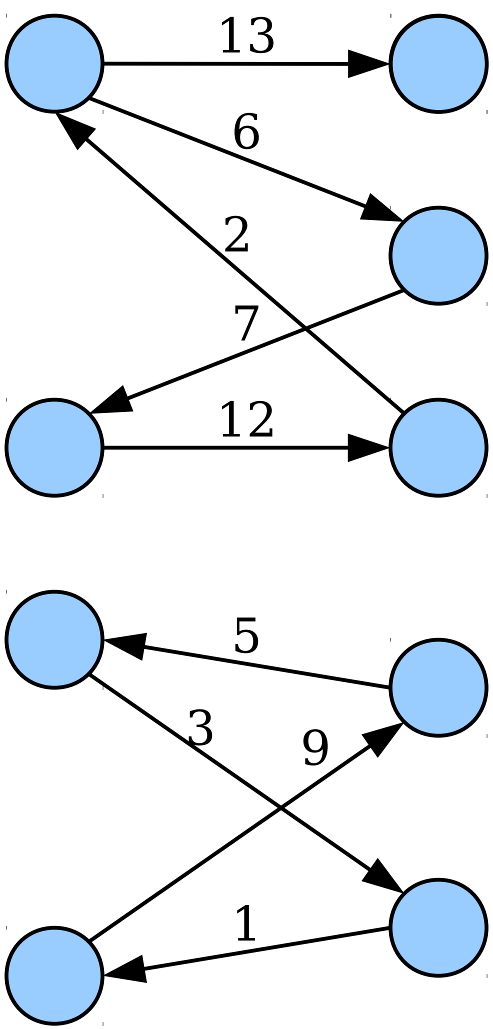

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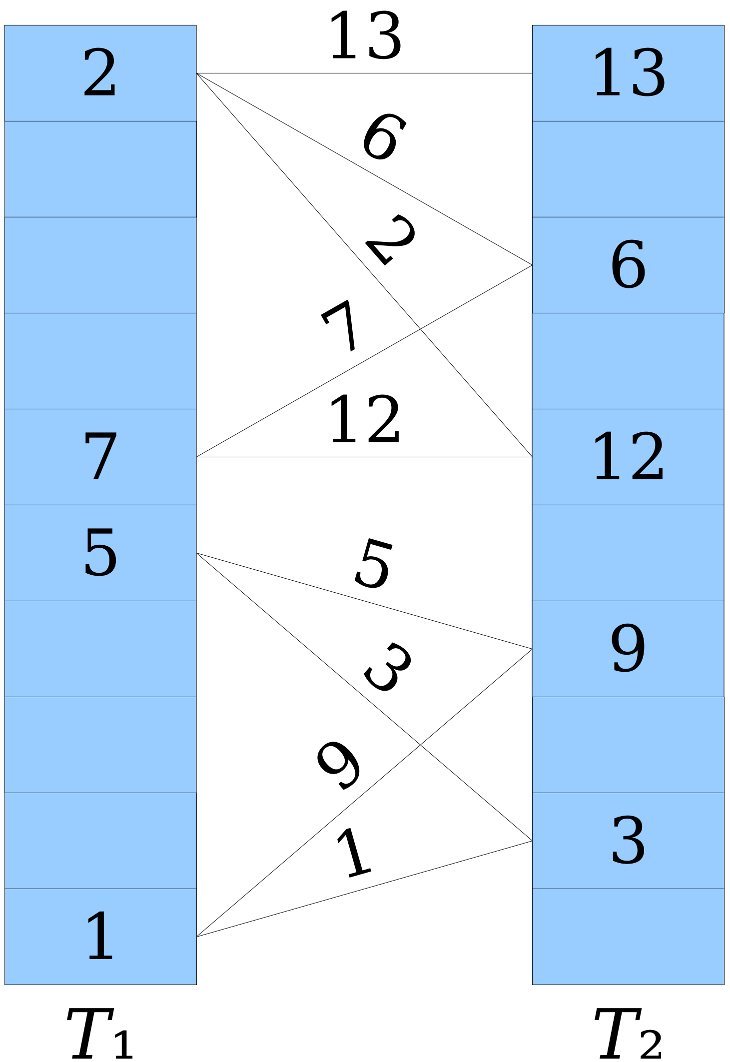

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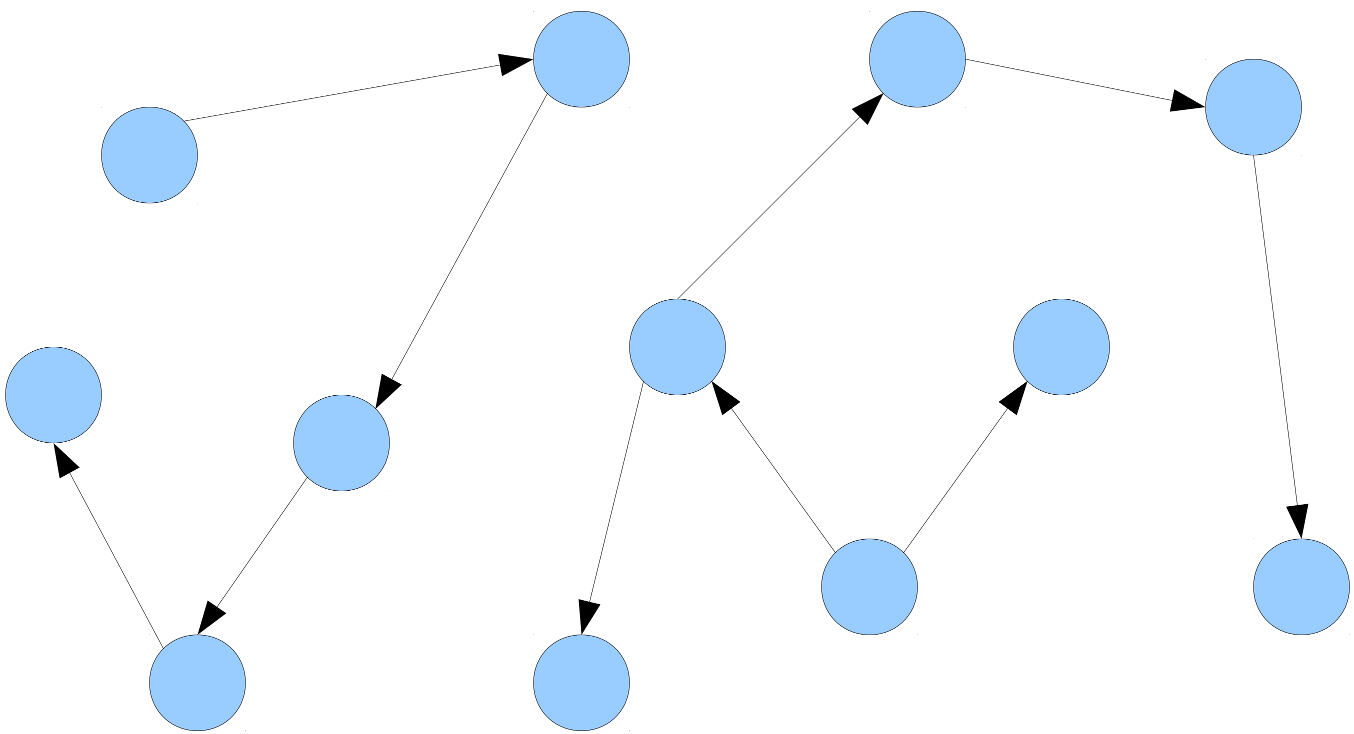

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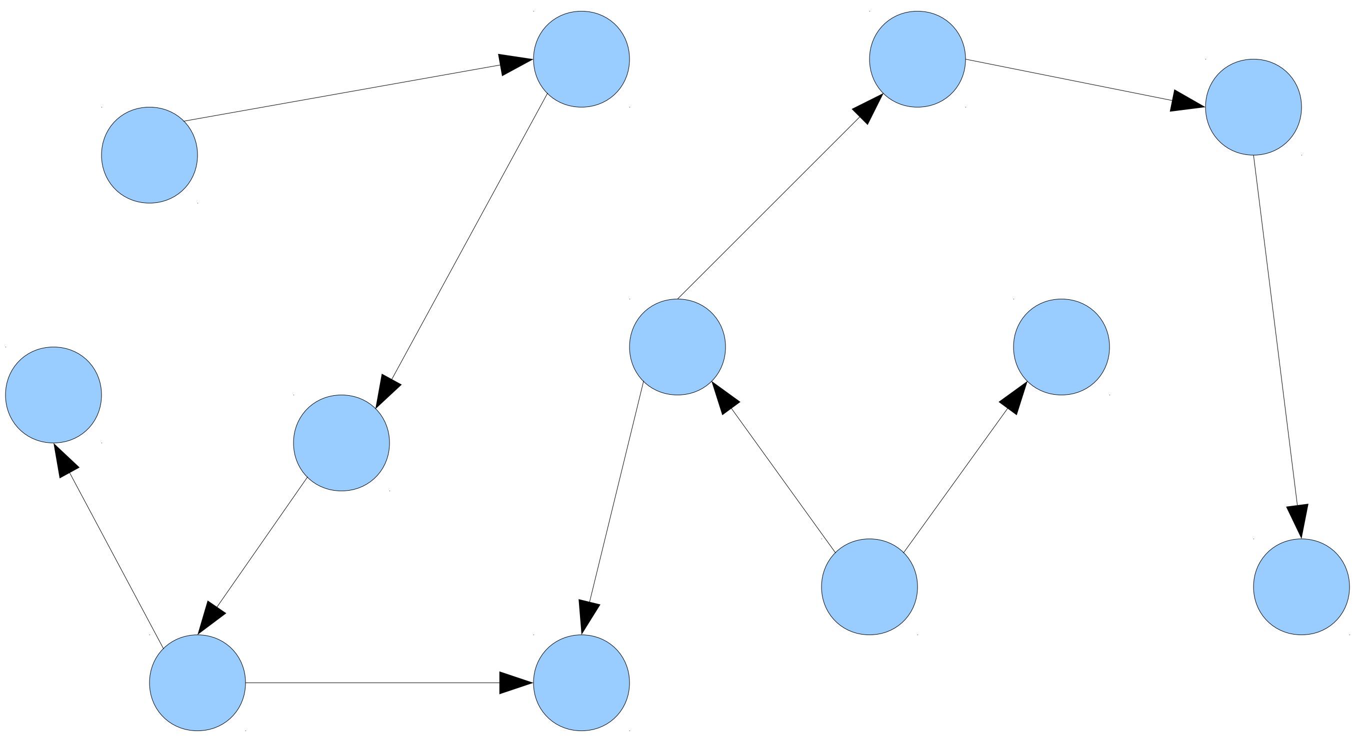

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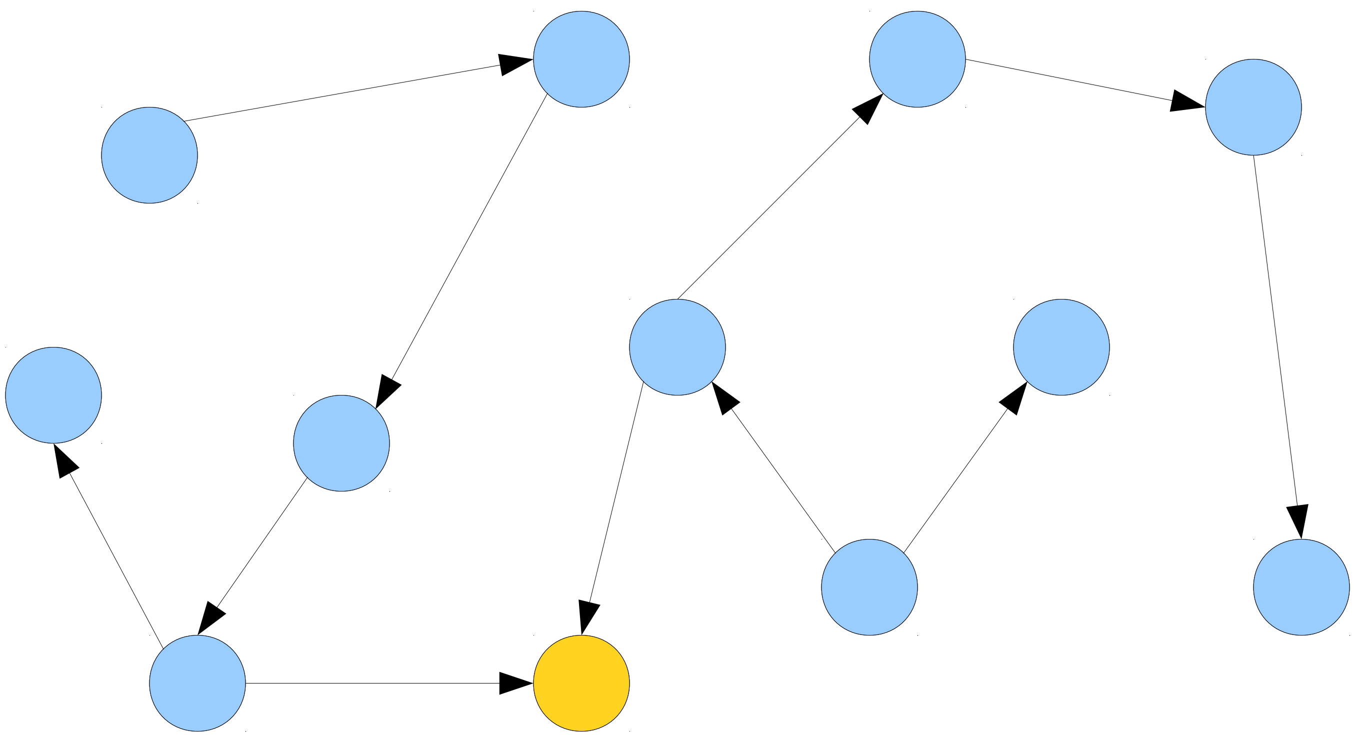

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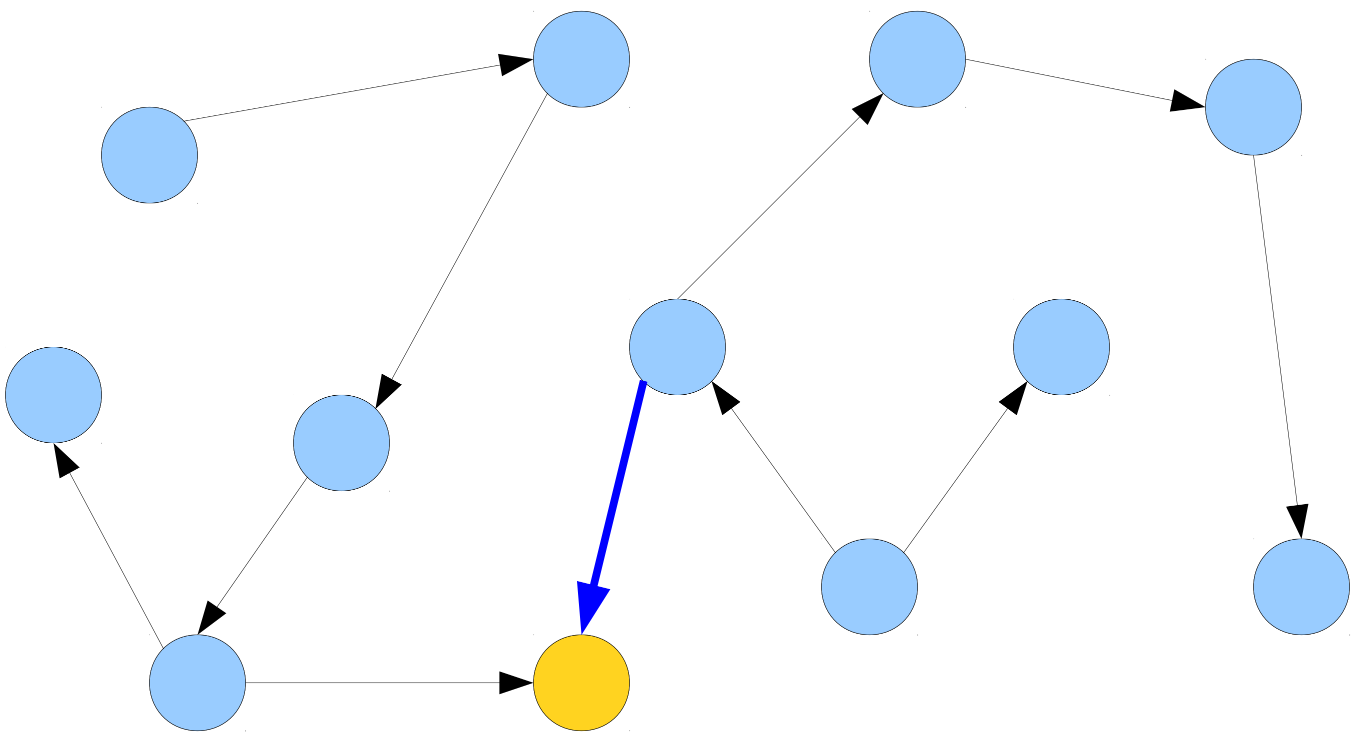

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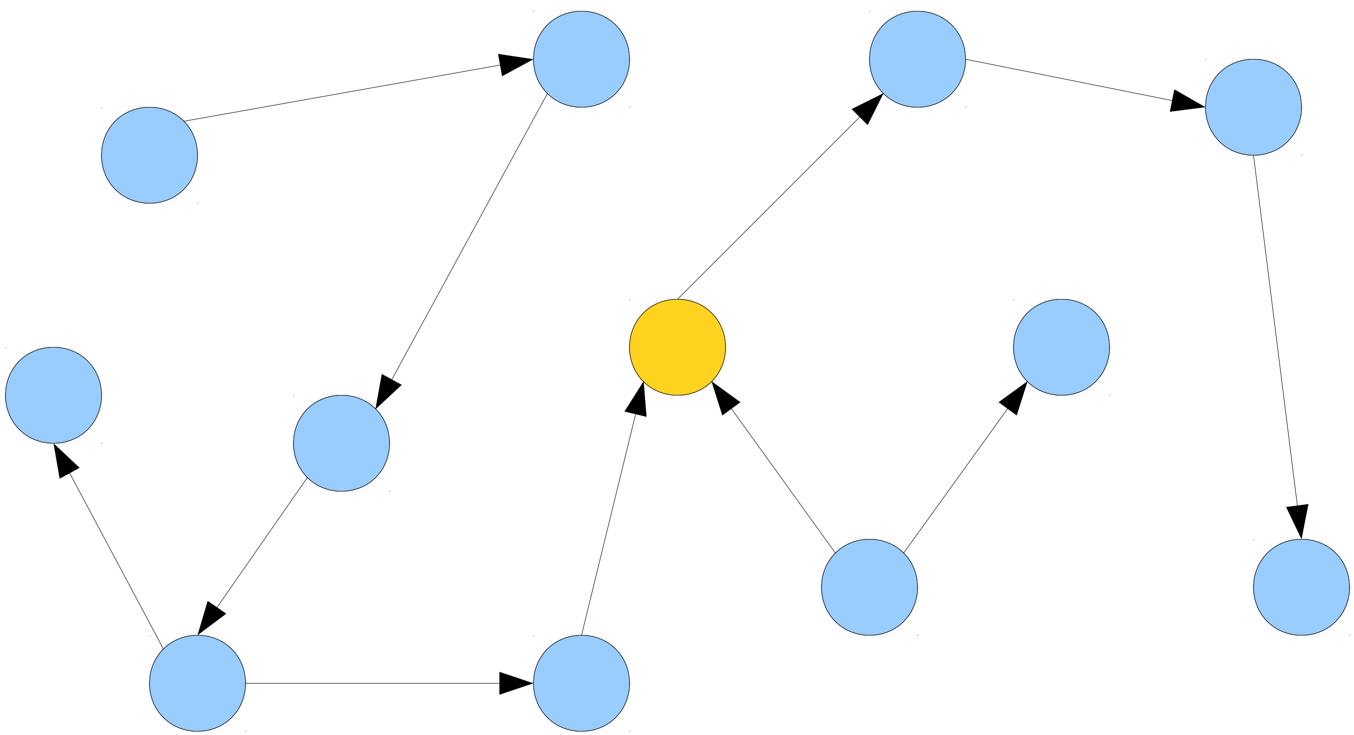

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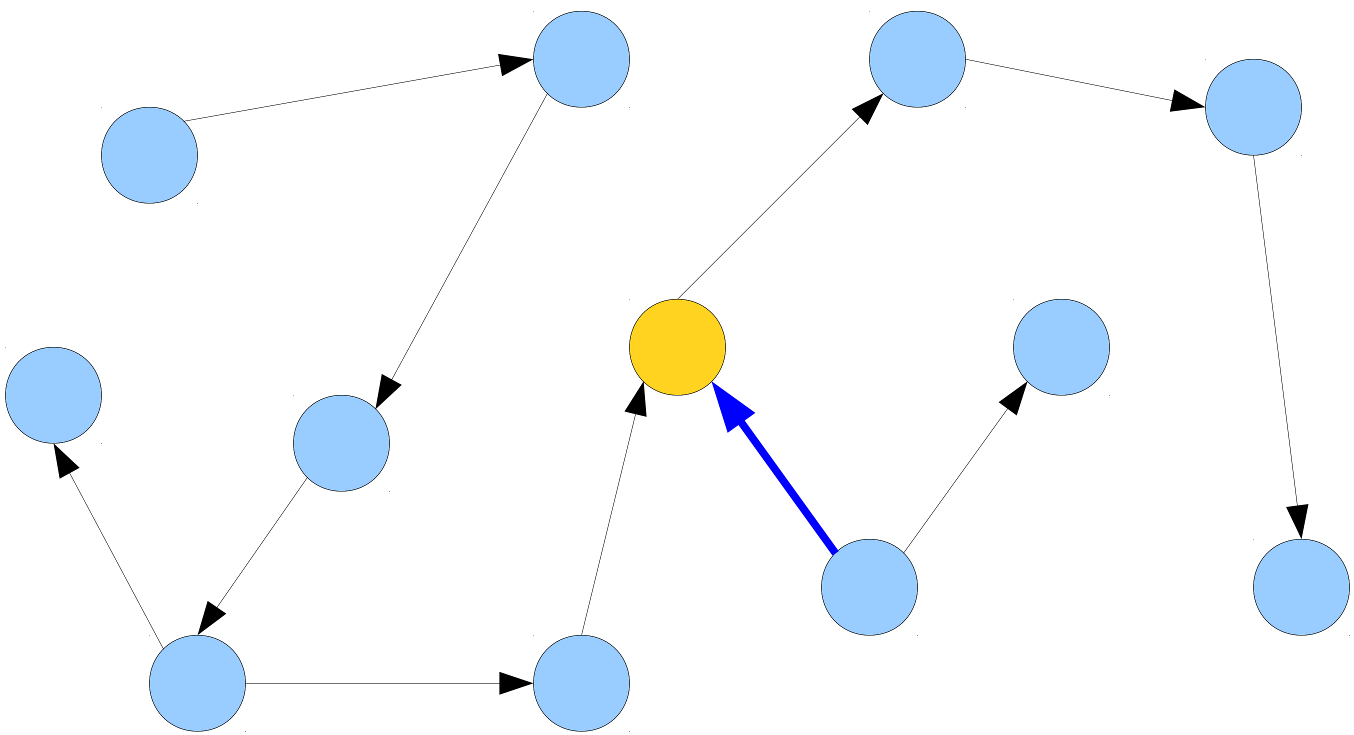

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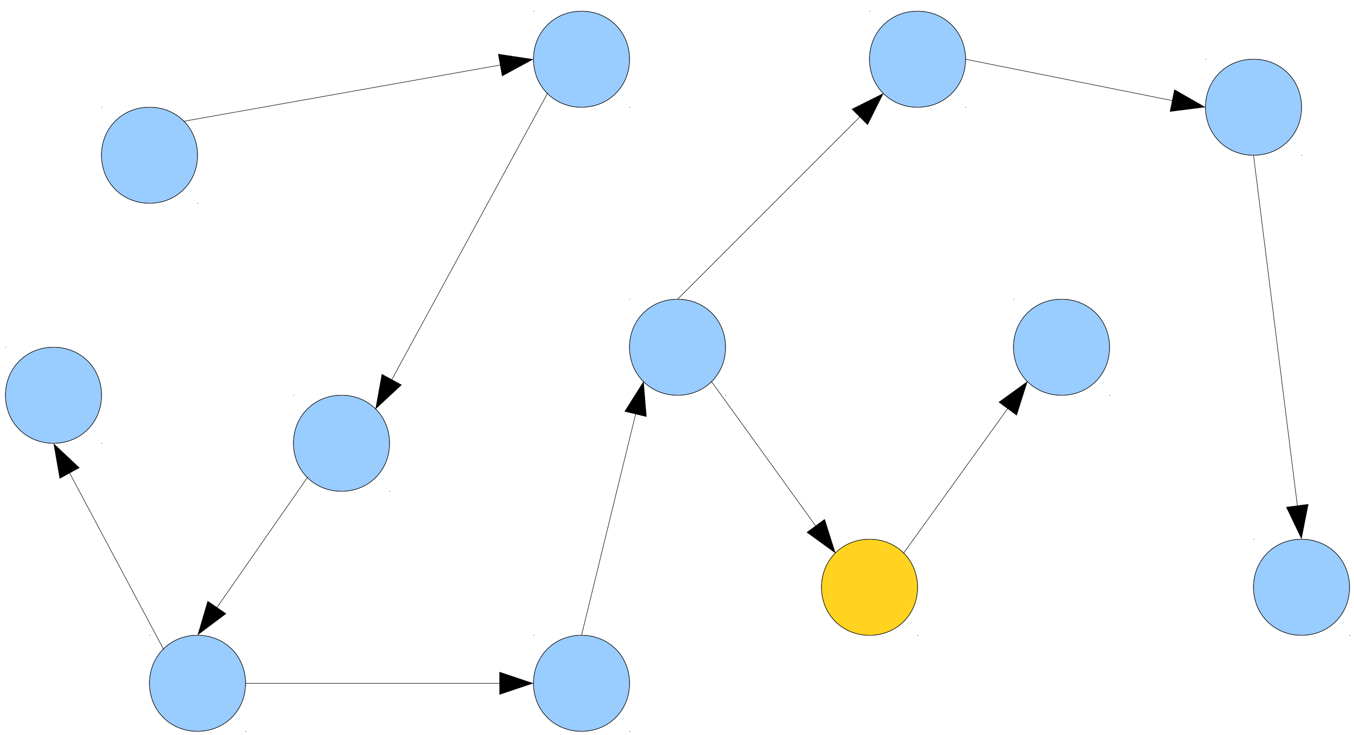

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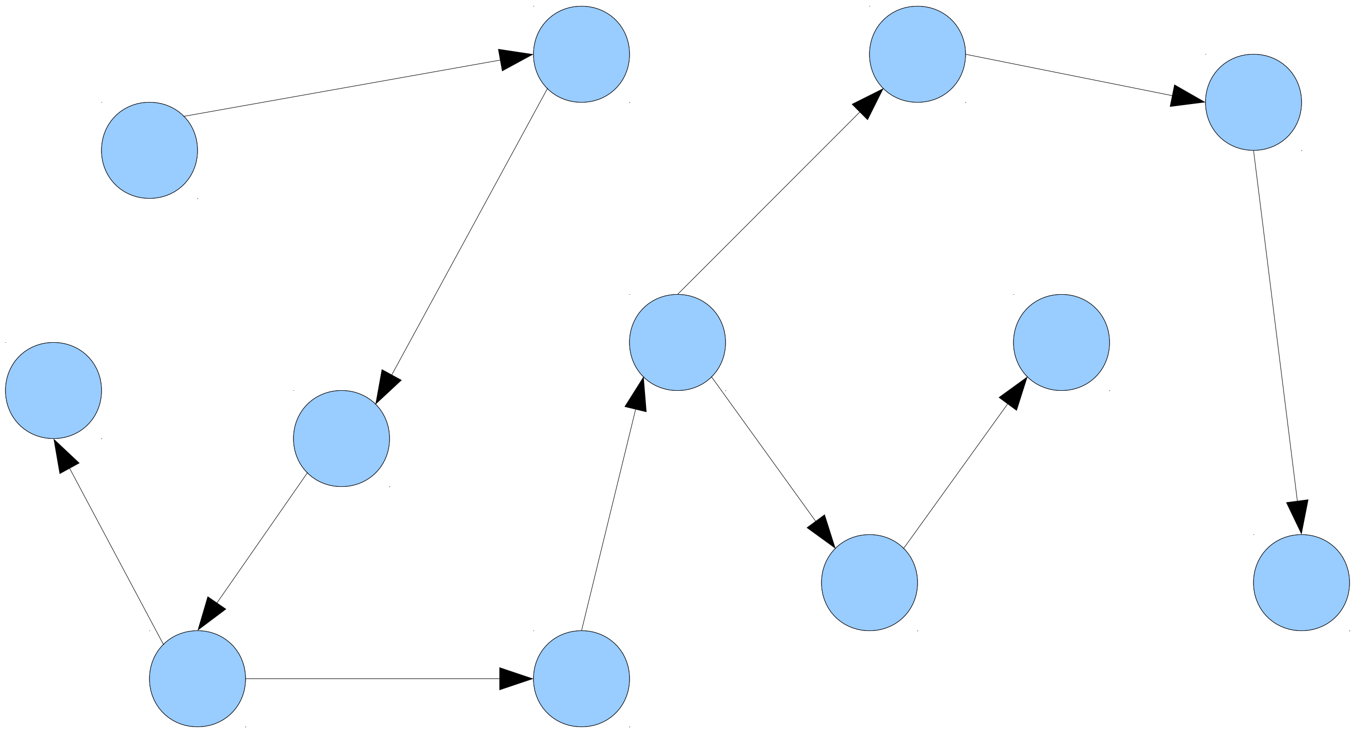

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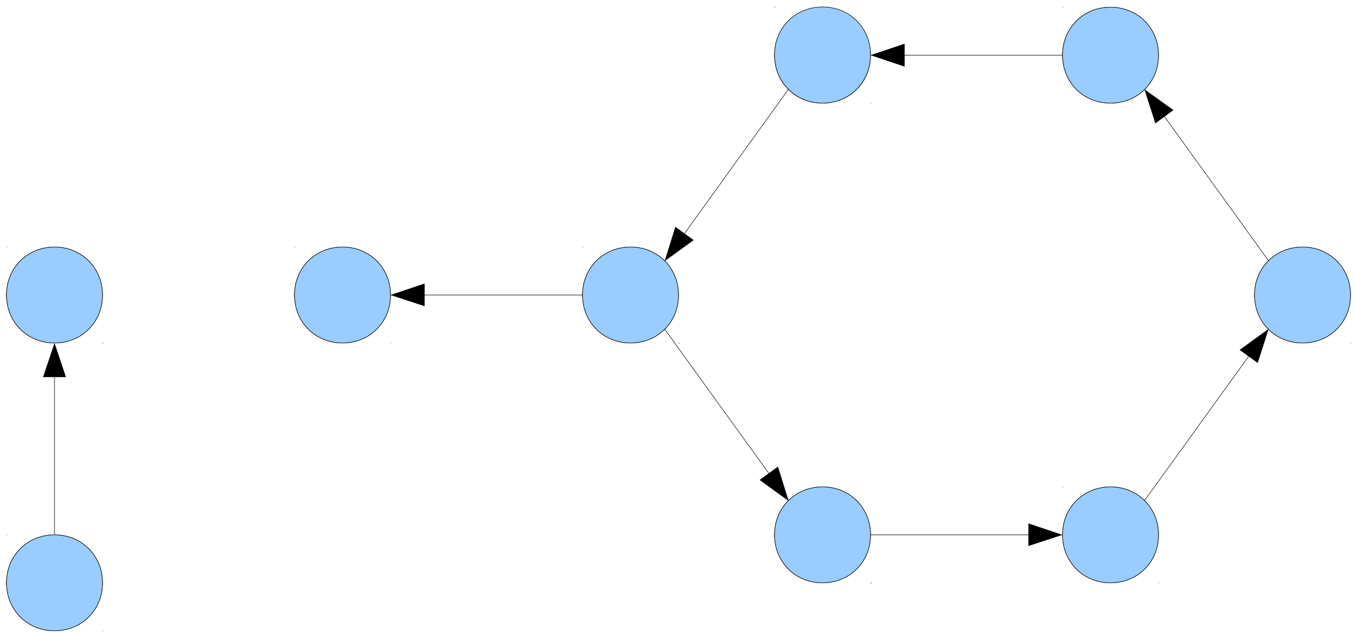

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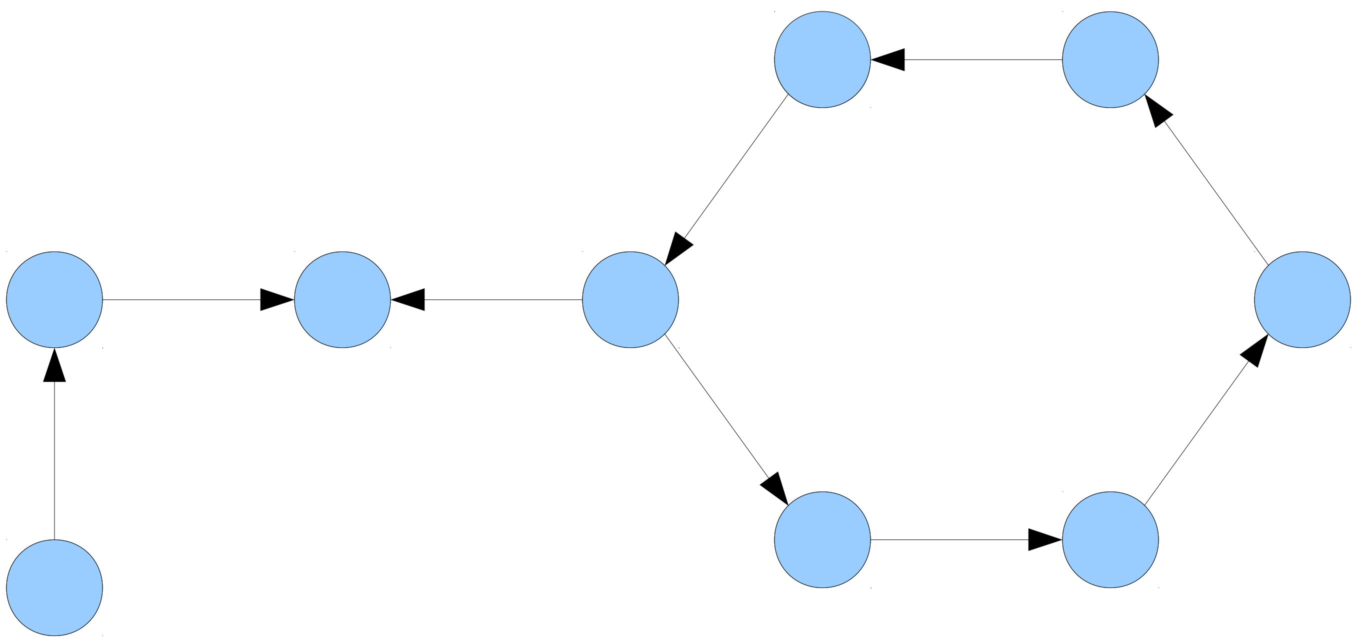

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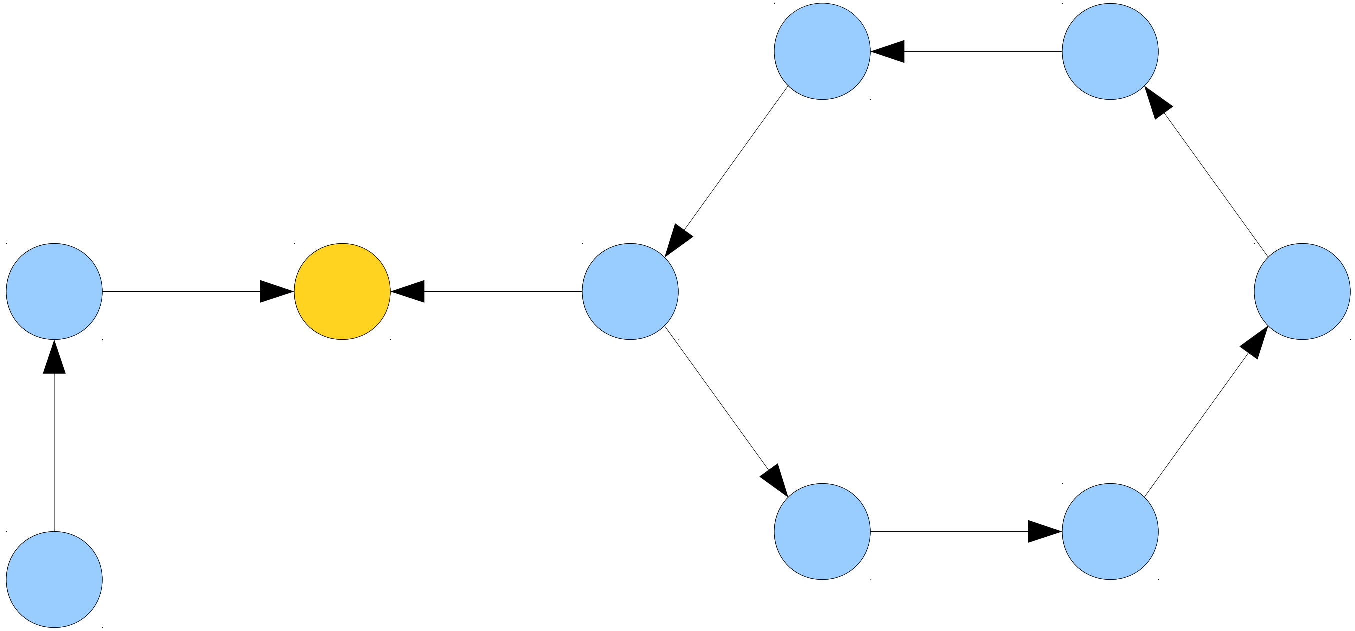

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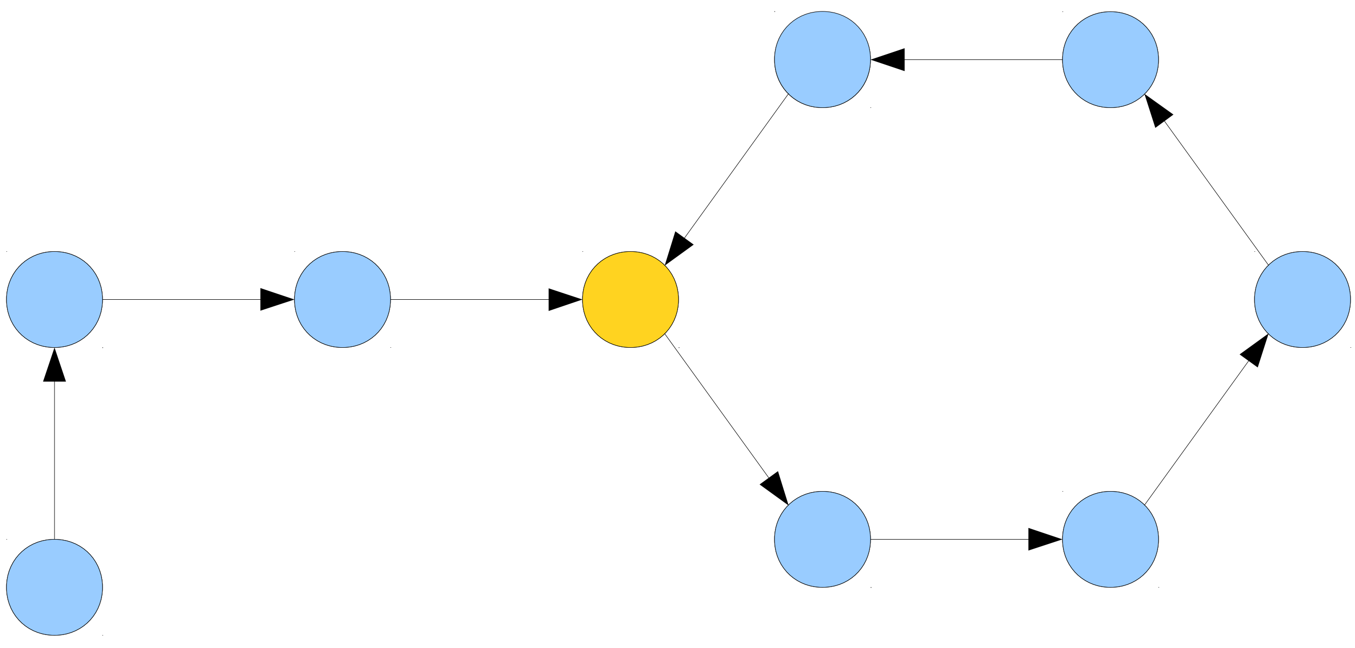

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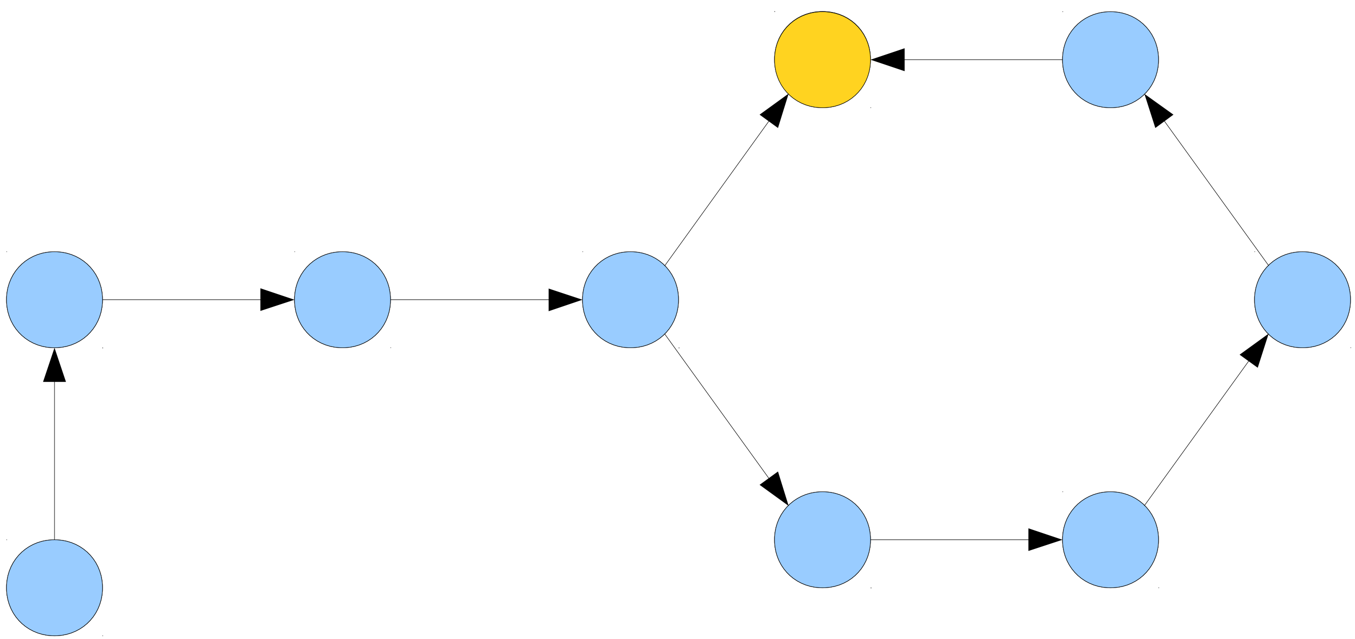

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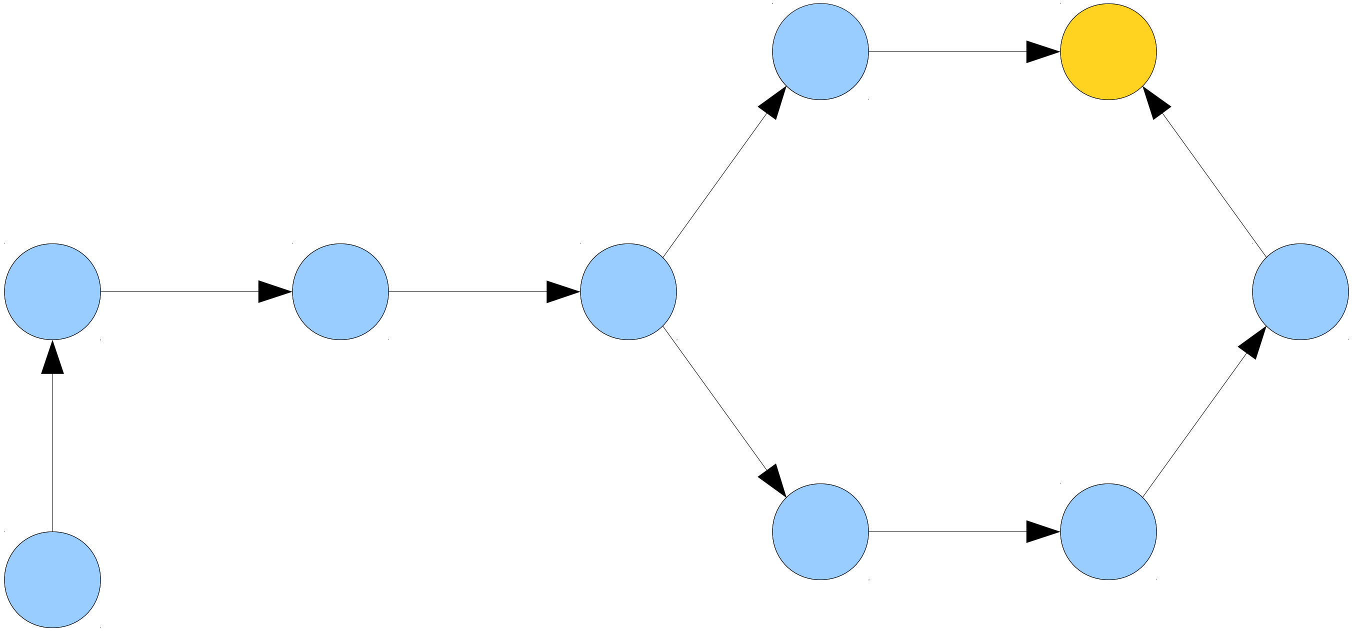

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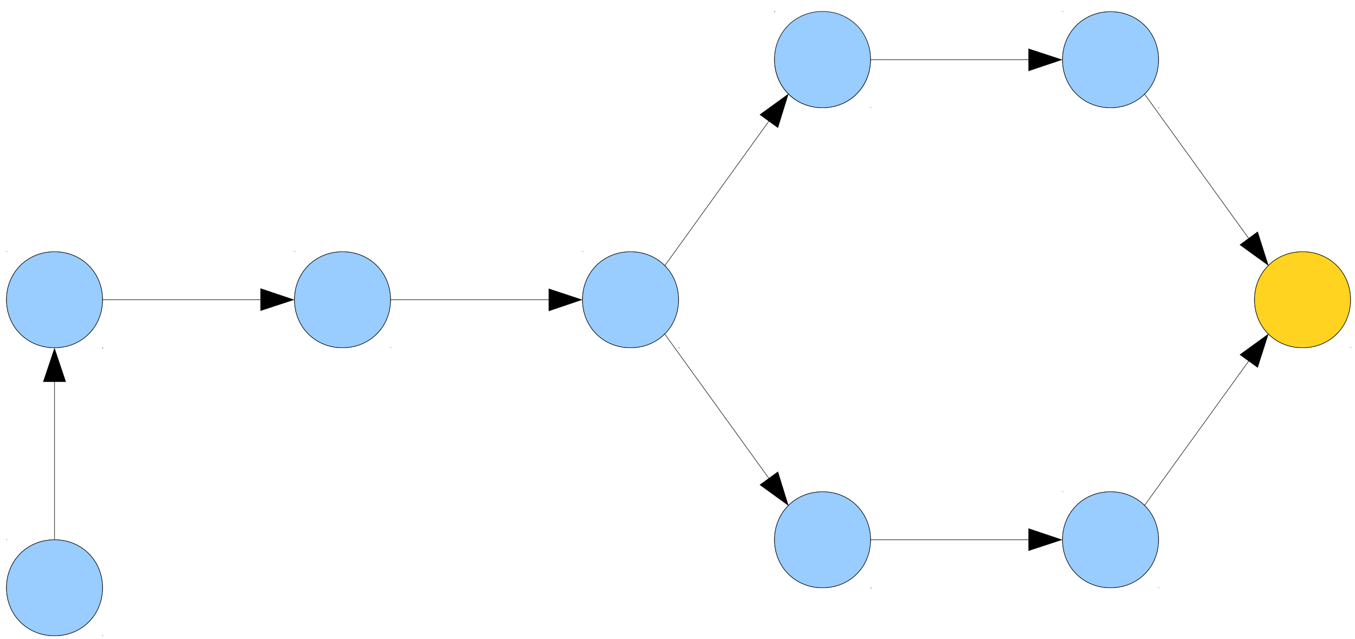

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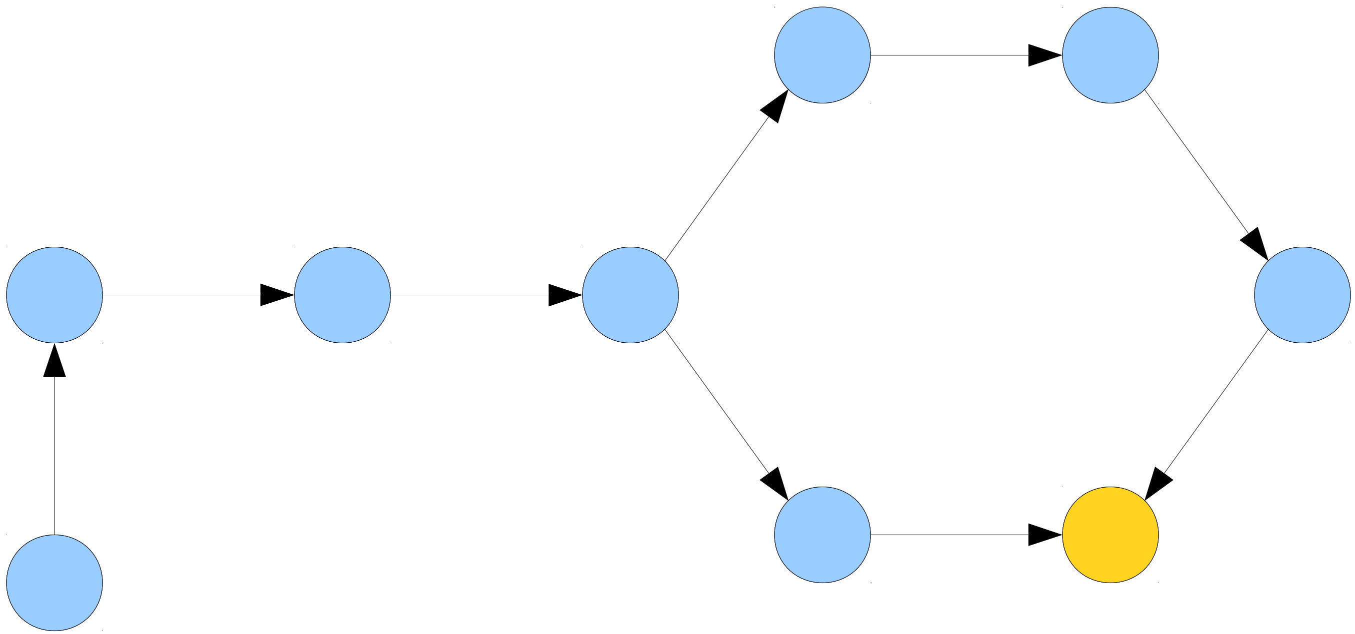

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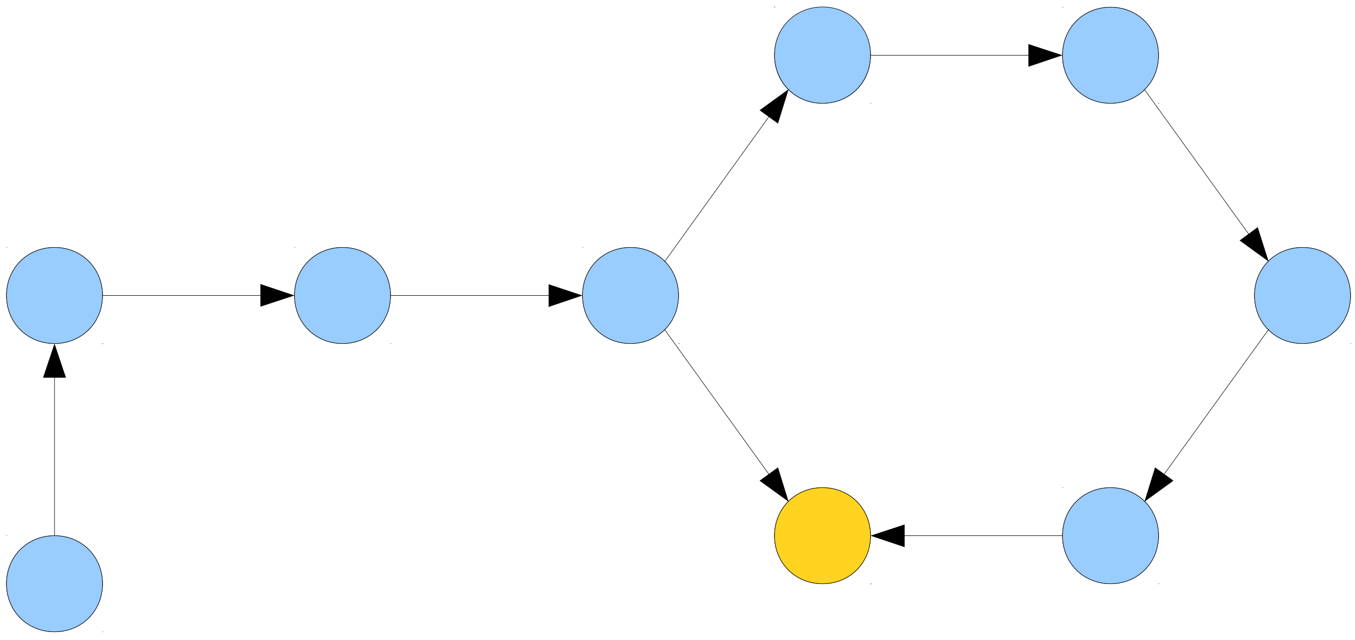

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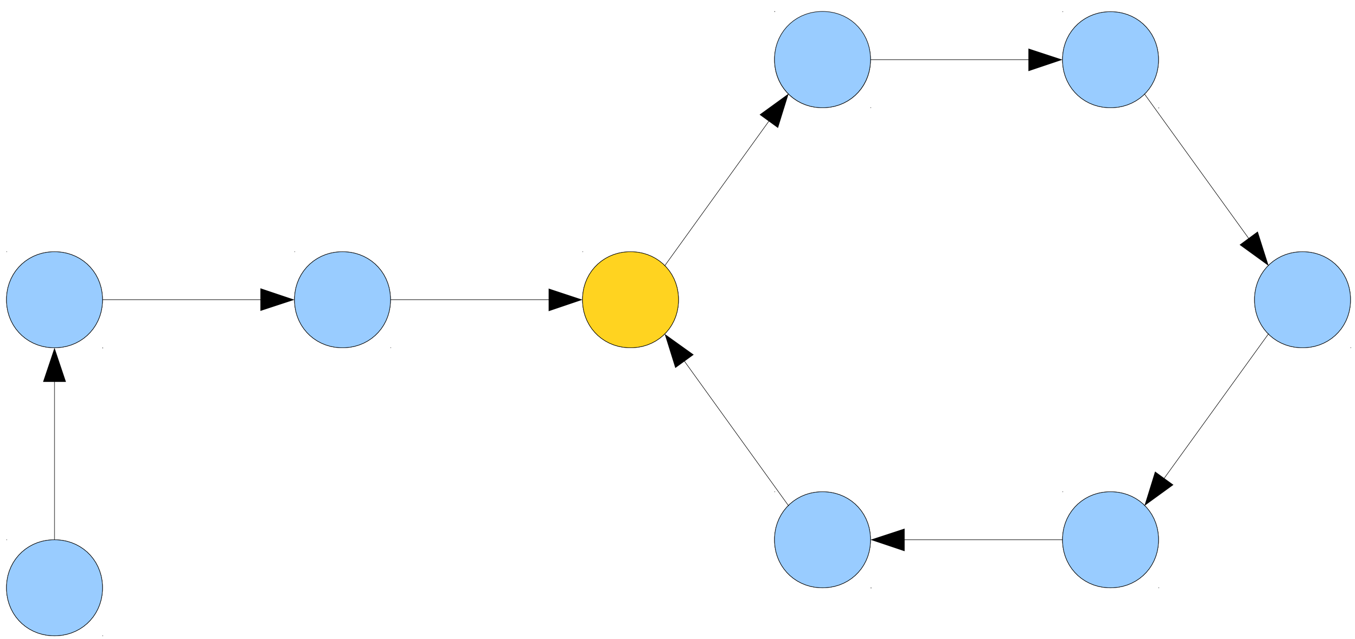

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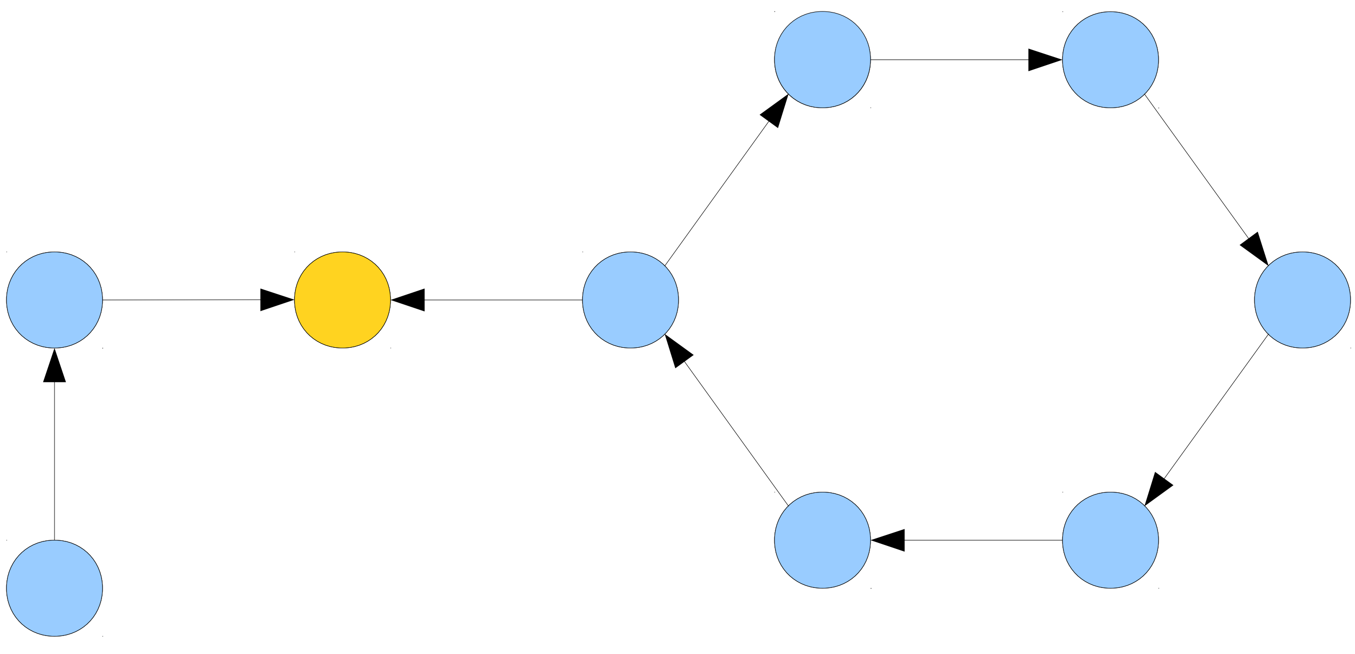

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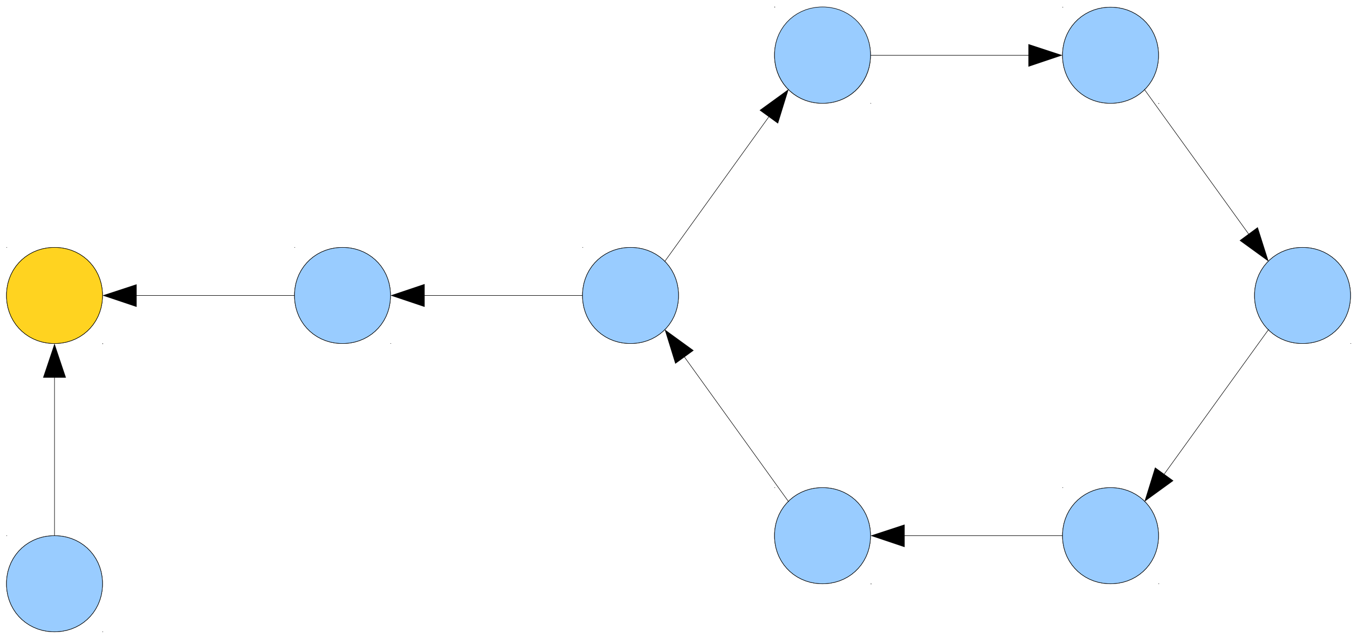

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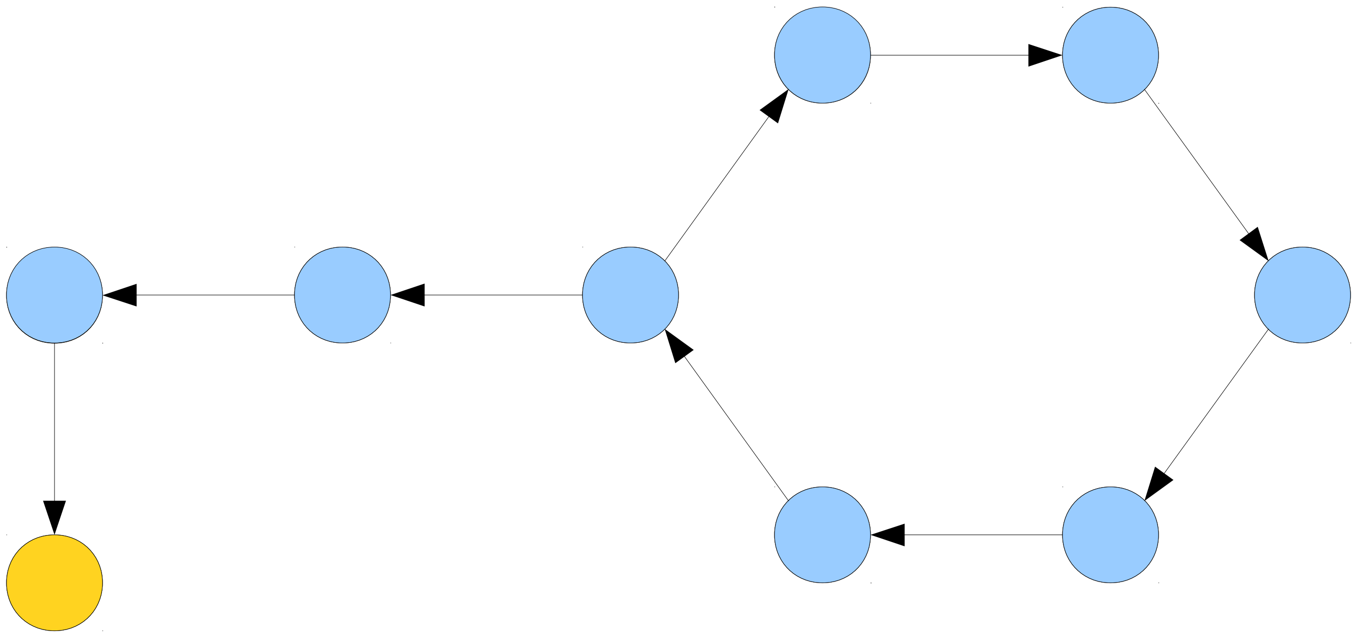

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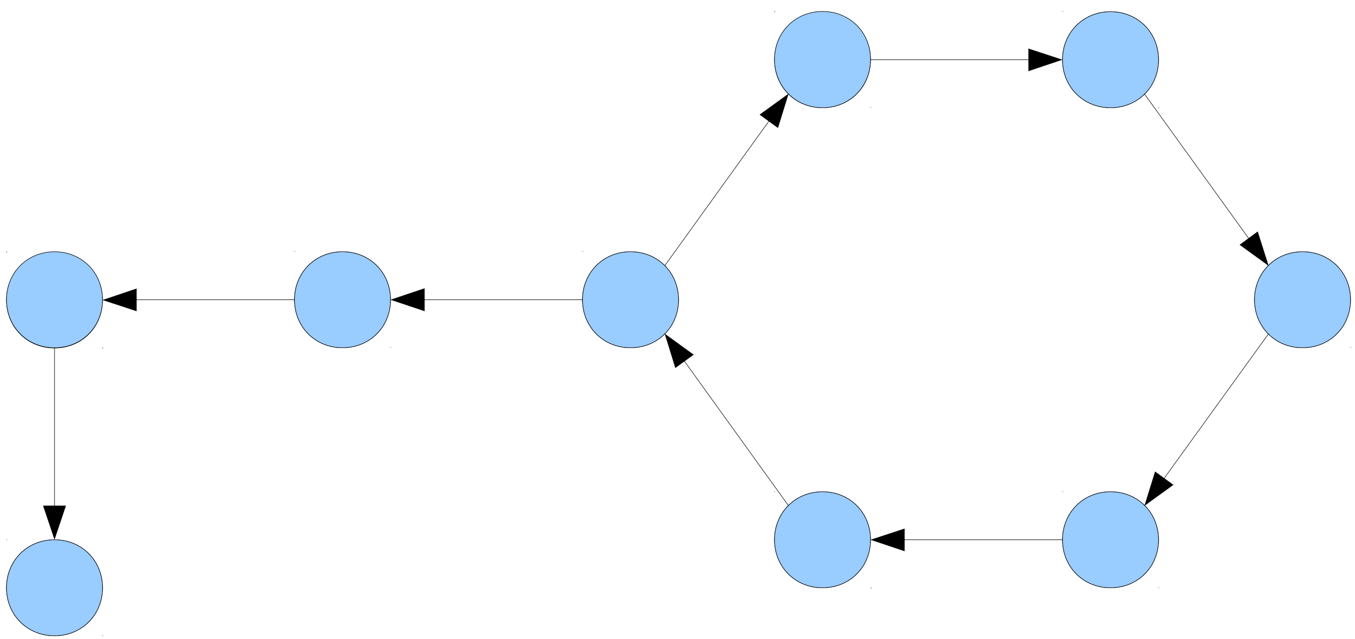

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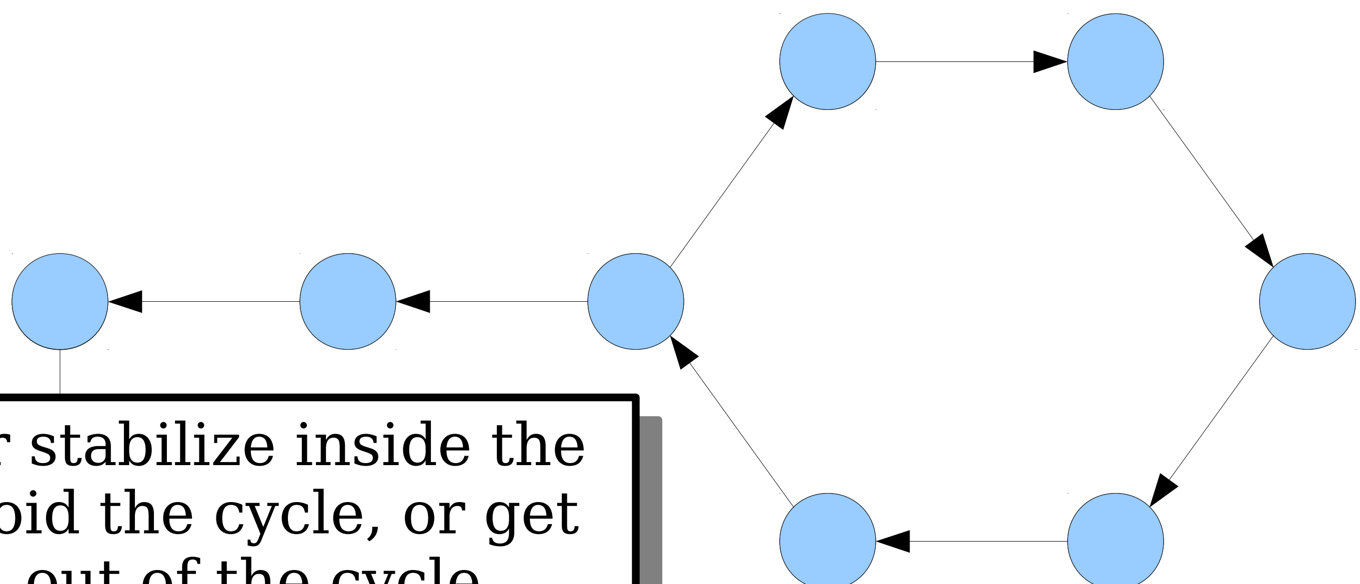
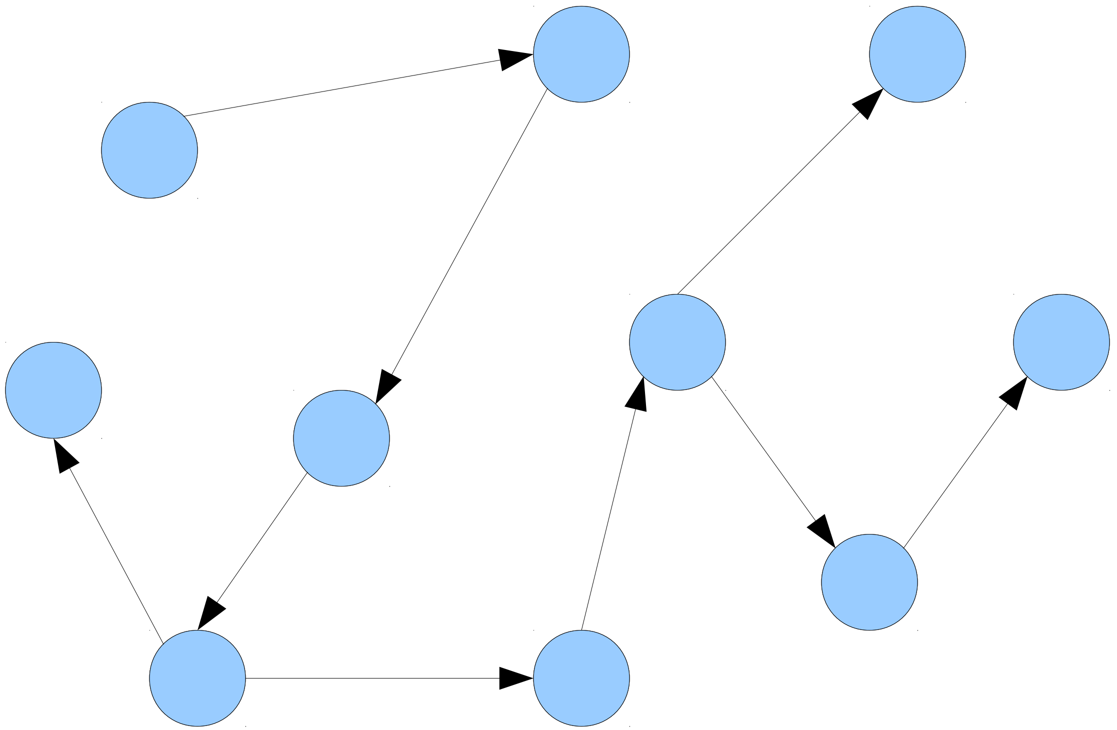


We either stabilize inside the cycle, avoid the cycle, or get kicked out of the cycle.

The Cuckoo Graph

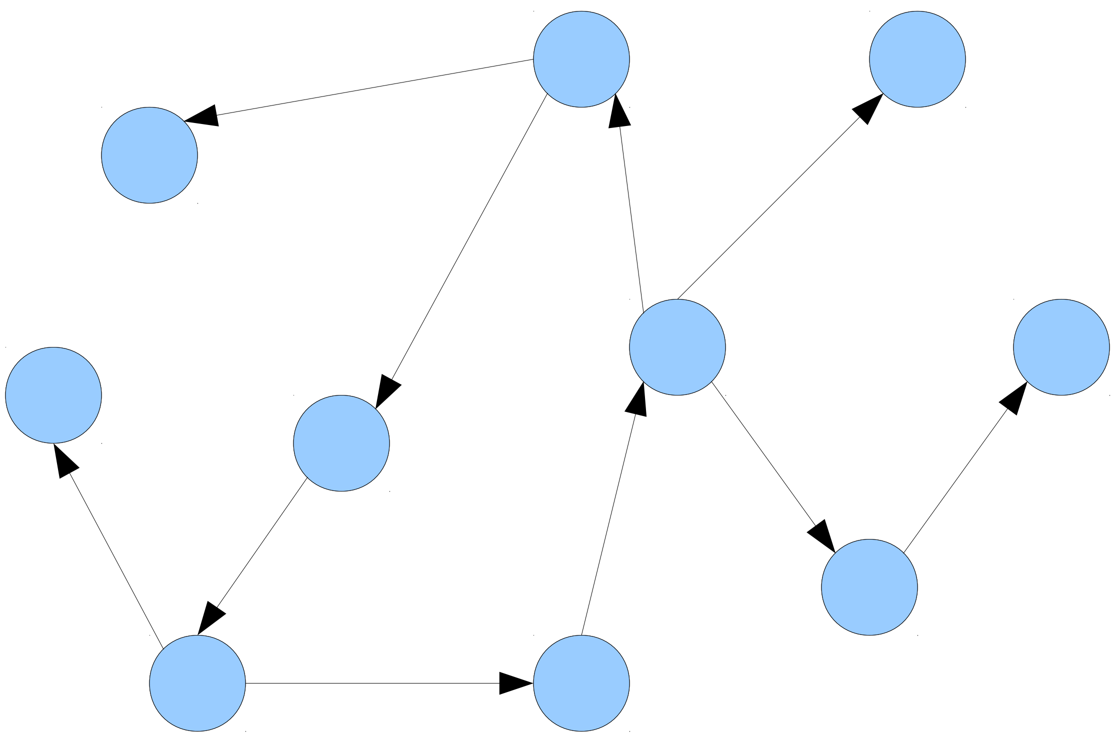
- ***Claim 2:*** If x is inserted into a cuckoo hash table, the insertion fails if the connected component containing x contains more than one cycle.



No cycles: The graph is a directed tree. A tree with k nodes has k – 1 edges.

The Cuckoo Graph

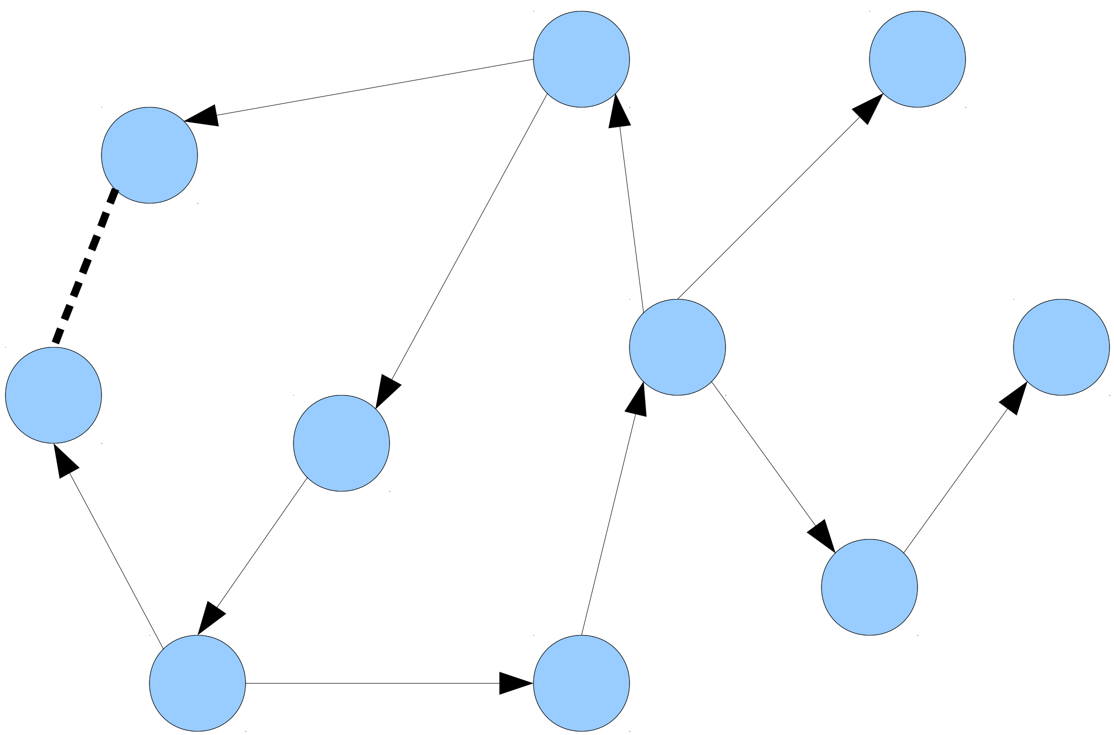
- ***Claim 2:*** If x is inserted into a cuckoo hash table, the insertion fails if the connected component containing x contains more than one cycle.



One cycle: We've added an edge, giving k nodes and k edges.

The Cuckoo Graph

- ***Claim 2:*** If x is inserted into a cuckoo hash table, the insertion fails if the connected component containing x contains more than one cycle.



Two cycles: There are k nodes and $k+1$ edges. There are too many arrowheads to place at most one arrowhead per node.

The Cuckoo Graph

- A connected component of a graph is called ***complex*** if it contains two or more cycles.
- ***Theorem:*** Insertion into a cuckoo hash table succeeds if and only if the resulting cuckoo graph has no complex connected components.
- Questions we still need to answer:
  - The number of nodes in a connected component can be used to bound the cost of an insertion. On average, how big are those connected components?
  - What is the probability that an insertion fails because we create a complex connected component?

Time-Out for Announcements!

Project Checkpoints

- Project checkpoints were due today at 2:30PM.
- We’ll be reviewing them over the weekend with the aim of getting back to you by early next week.
- In general, please feel free to reach out to us if you have any questions about the project or your topic! We’re happy to help out.

Problem Set Four

- Problem Set Four is due next Tuesday at 2:30PM.
- You know the drill! Ask questions if you have them, get in touch with us if there's anything we can assist with, and have a lot of fun working through these exercises!

Back to CS166!

Two Major Questions

How big are the connected components in the cuckoo graph?

How likely is it for an insertion to fail?

Step One: Sizing Connected Components

Analyzing Connected Components

- The cost of inserting x into a cuckoo hash table is proportional to the size of the CC containing x.
- ***Question:*** What is the expected size of a CC in the cuckoo graph?

Idea: Count the number of nodes in a connected component by simulating a BFS.

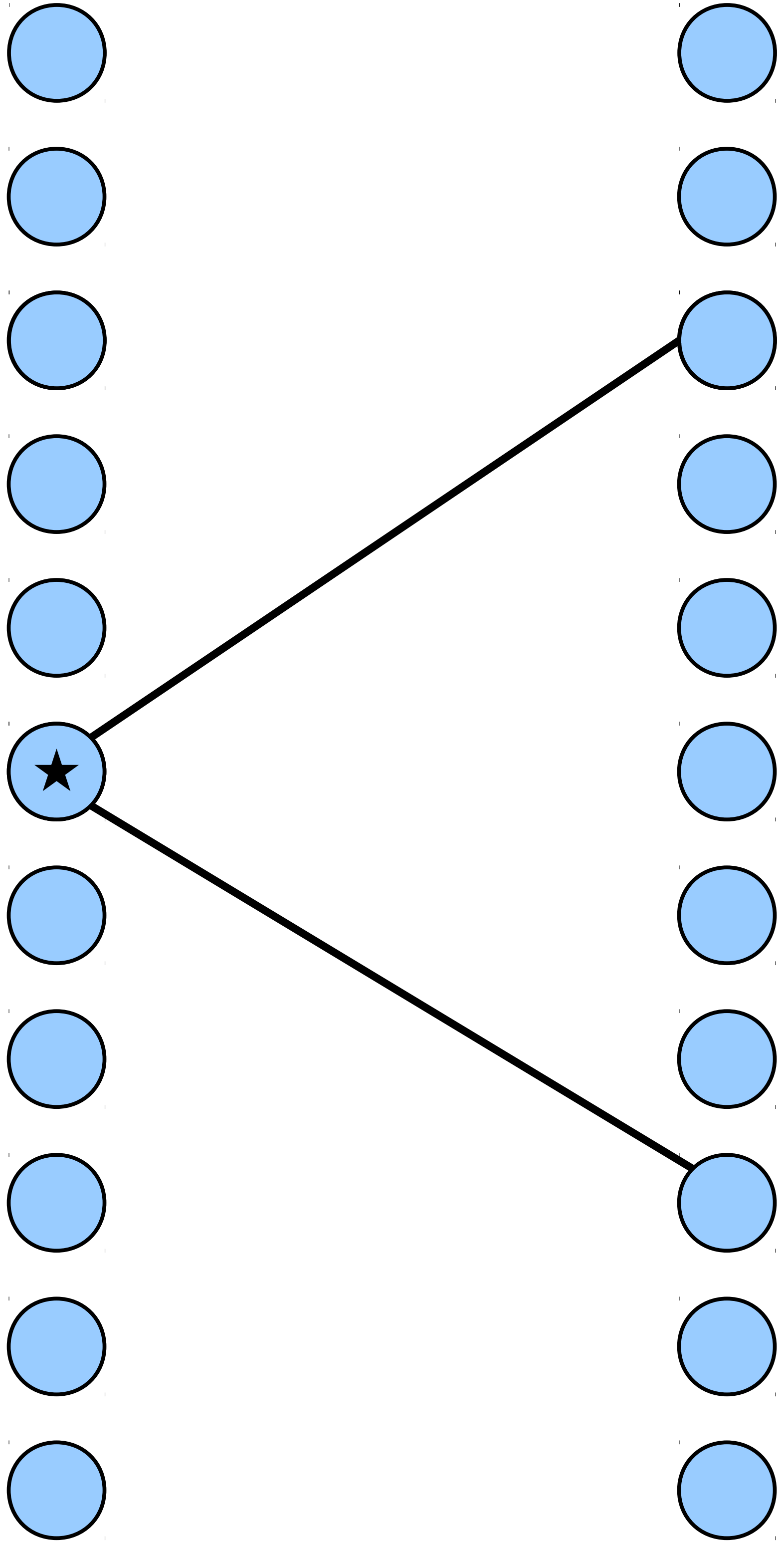
Pick some starting table slot.

There are n elements in the table, so this graph has n edges.

Assume, for now, that our hash functions are truly random.

Each edge has a $^1/_m$ chance of touching this table slot.

The number of adjacent nodes, which will be visited in the next step of BFS, is a Binom(n, $^1/_m$) variable.



Idea: Count the number of nodes in a connected component by simulating a BFS.

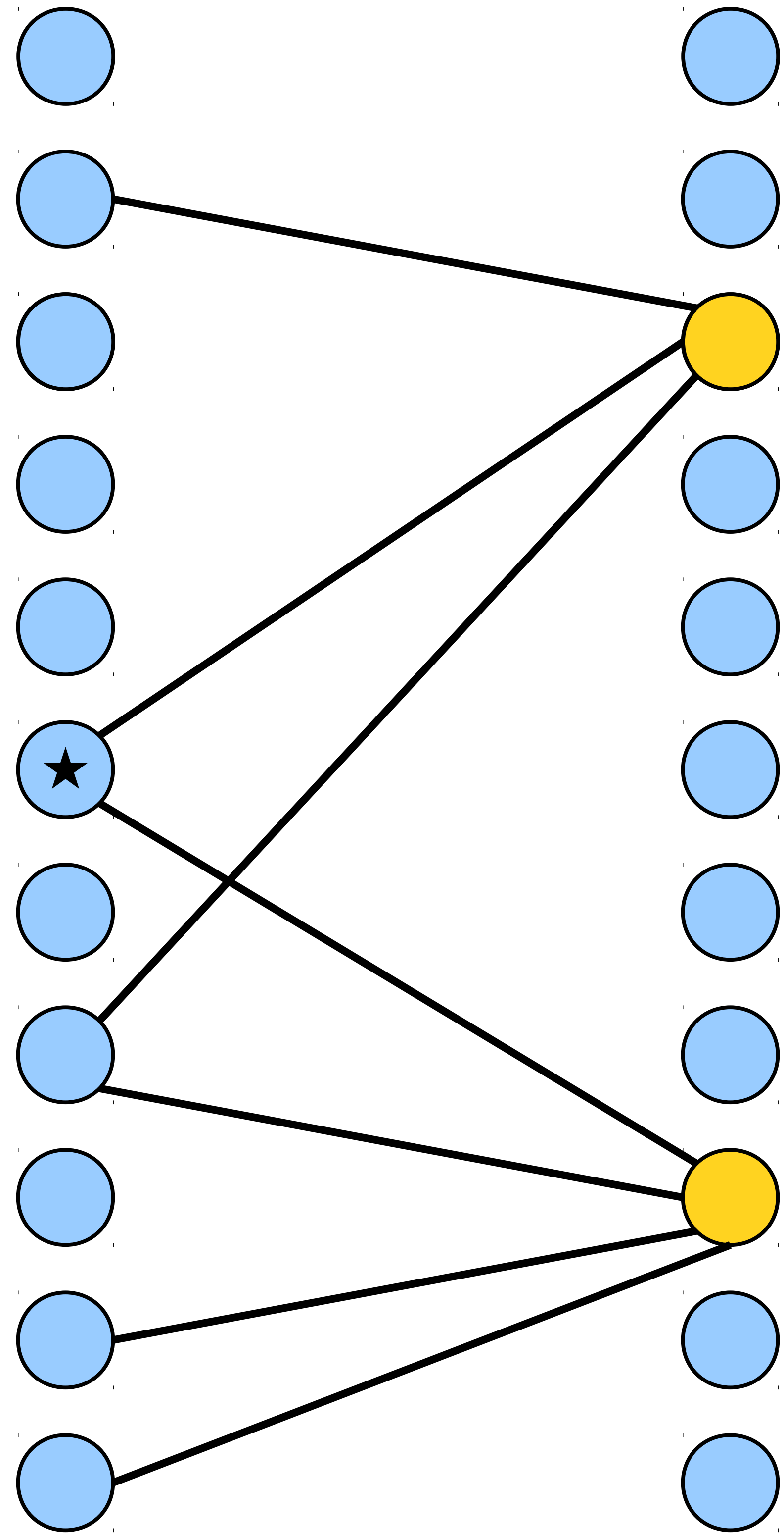
Each new node kinda sorta ish also touches a number of new nodes on the other side that can be modeled as a Binom(n, $^1/_m$) variable.

This ignores double-counting nodes.

This ignores existing edges.

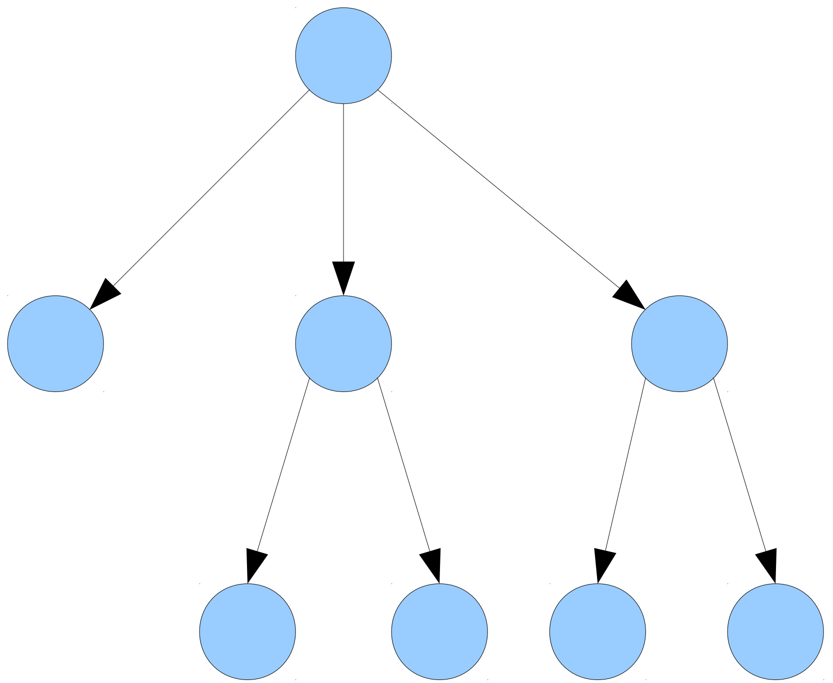
This ignores correlations between edge counts.

However, it conservatively bounds the next BFS step.



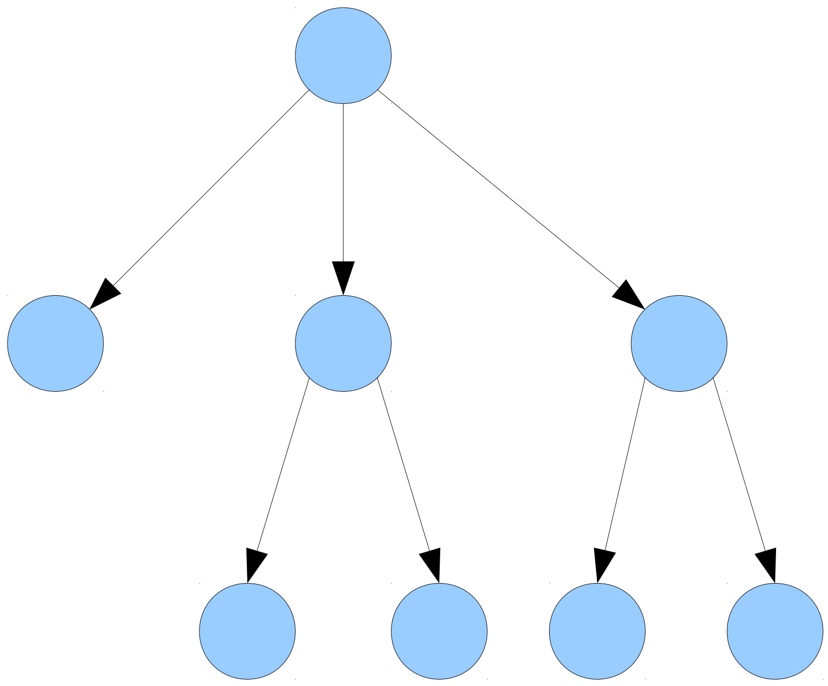
Modeling the BFS

- Simulate a BFS tree using Binom(n, $^1/_m$) variables!
  - Begin with a root node.
  - Each node has children distributed as a Binom(n, $^1/_m$) variable.
- ***Question:*** How many total nodes will this simulated BFS discover before terminating?



Modeling the BFS

- A tree-shaped process where each node has an i.i.d. number of children is called a ***Galton-Watson process***.
- A ***subcritical*** process is one where the expected number of children of each node is less than one.
- ***Question:*** Assuming each node's child count is an i.i.d. copy of some random variable ξ, how many nodes should we expect to see in the tree?



GAZETTE OF FASHION

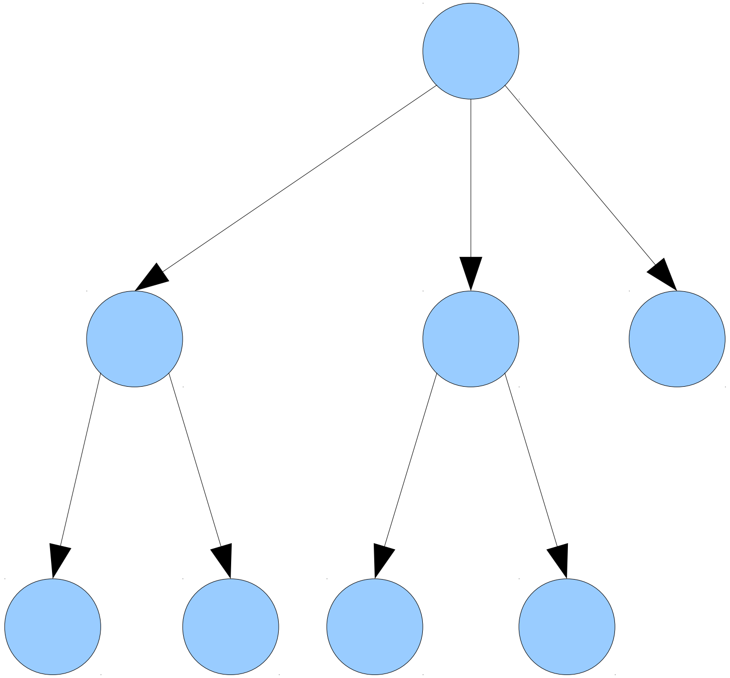
http://1.bp.blogspot.com/-f3gzOHMBM6I/TWcdXn-R1SI/AAAAAAAAAEA/G8uu8scBj1I/s1600/1868-02%2Bgazette%2Bof%2Bfashion.jpg

Subcritical Galton-Watson Processes

- Denote by X_n the number of nodes alive at depth n. This gives a series of random variables X_0, X_1, X_2, … .
- These variables are defined by the following randomized recurrence:

$$X_0 = 1 \qquad X_{n+1} = \sum_{i=1}^{X_n} \xi_{i,n}$$

- Here, each $\xi_{i,n}$ is an i.i.d. copy of ξ.



$X_0 = 1$

$X_1 = 3$

$X_2 = 4$

$X_3 = 0$

Lemma 1: $E[X_i] = E[\xi]^i$.

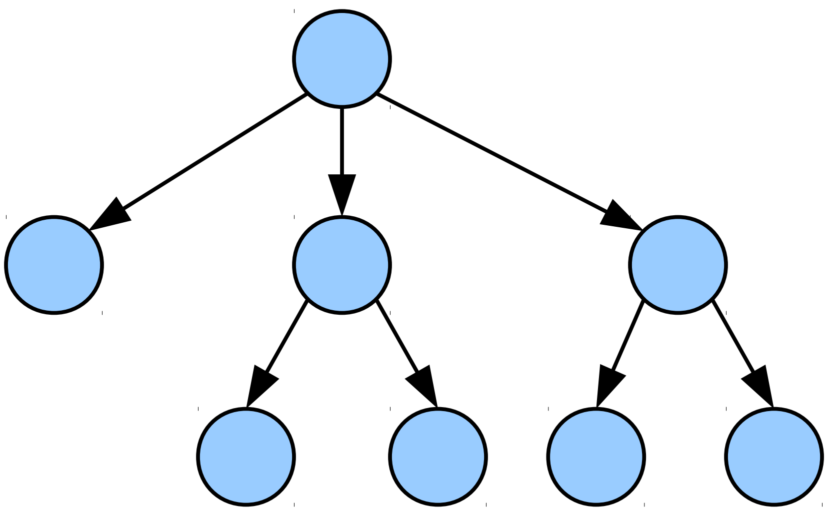
(Induction and conditional expectation.)

Lemma 2: $E[\sum_{i=0}^{\infty} X_i] = \frac{1}{1 - E[\xi]}$

(Linearity of expectation; sum of a geometric series.)

Theorem: The expected number of nodes in a connected component of the cuckoo graph is O(1), assuming that $m = (1+\varepsilon)n$.

Proof: ξ in this case is a Binom(n, $^1/_m$) variable. So $E[\xi] = {^n/_m} = O(1)$.



$$X_0 = 1 \qquad X_{n+1} = \sum_{i=1}^{X_n} \xi_{i,n} \qquad E[\xi] < 1$$

The Story So Far

- The expected size of a connected component in the cuckoo graph is O(1).
- Therefore, each *successful* insertion takes expected time O(1).
- ***Question:*** What happens in an unsuccessful insertion? And what does that do for our expected cost of *any* insertion?

Step Two:

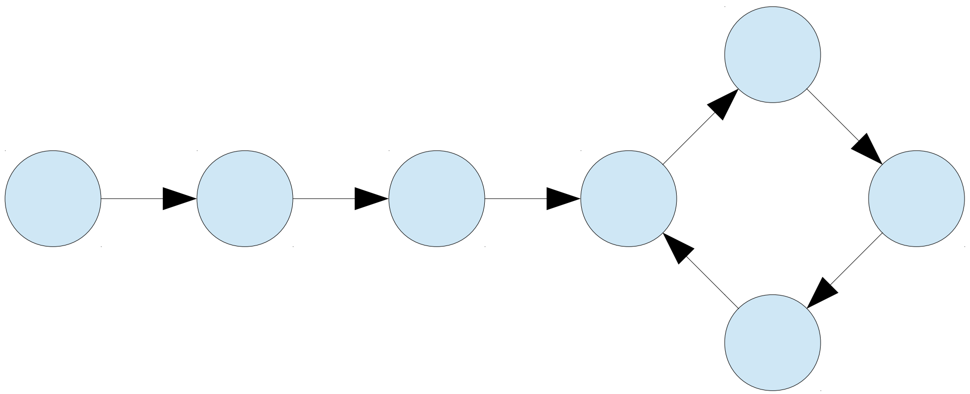
Exploring the Graph Structure

Exploring the Graph Structure

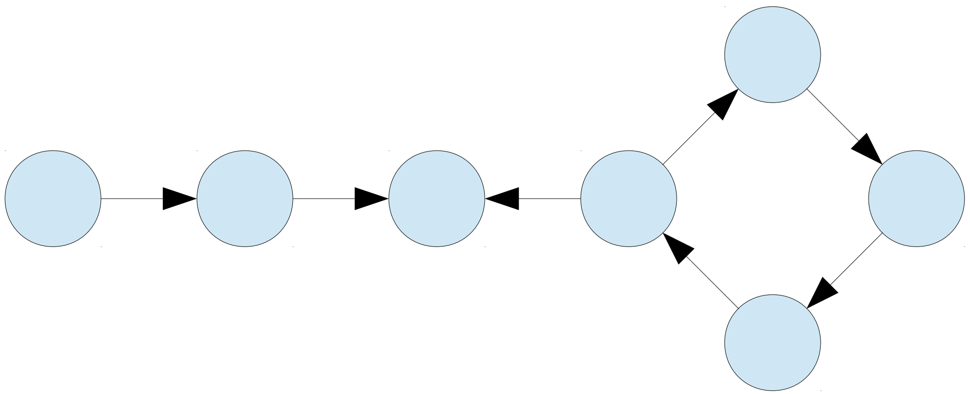
- Cuckoo hashing will always succeed in the case where the cuckoo graph has no complex connected components.
- If there are no complex CC's, then we will not get into a loop and insertion time will depend only on the sizes of the CC's.
- It's reasonable to ask, therefore, how likely we are to not have complex components.

Exploring the Graph Structure

- ***Question:*** What is the probability that a randomly-chosen bipartite multigraph with $2m$ nodes and n edges will contain a complex connected component?
- Directly answering this question is challenging and requires some fairly detailed combinatorics.
- However, there's a very clever technique we can use to bound this probability indirectly.



Insertion fails if we have a complex connected component.
What specifically happens in that case?



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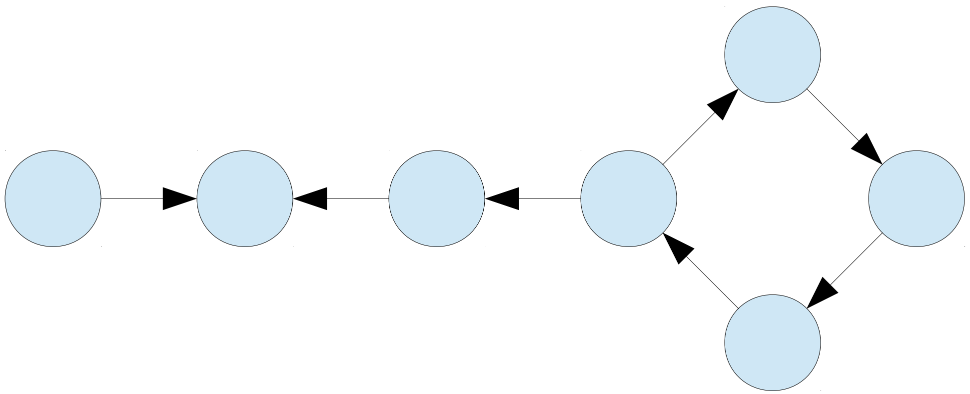

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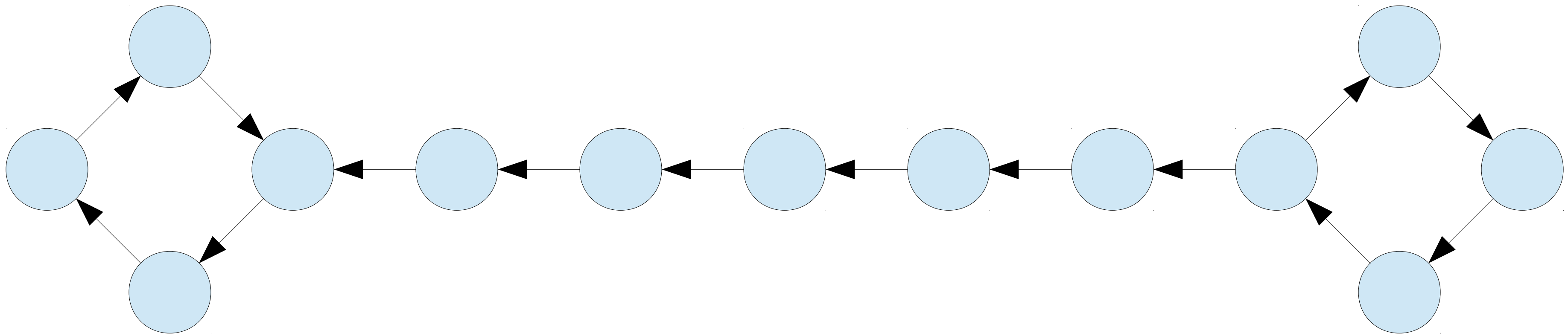

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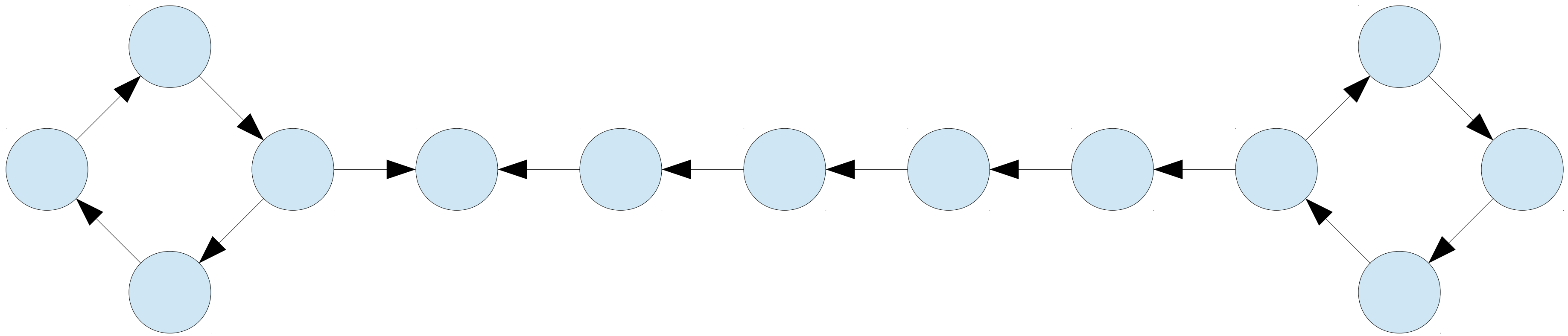

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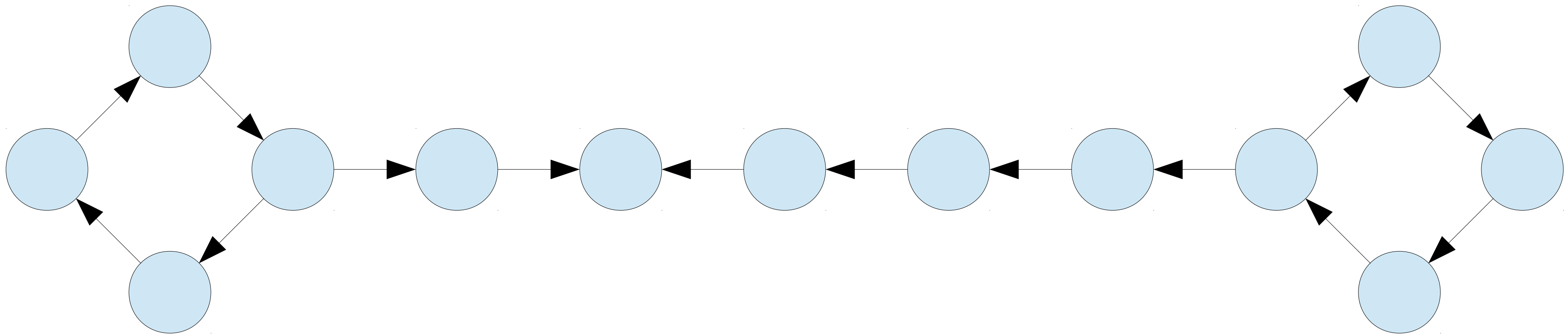

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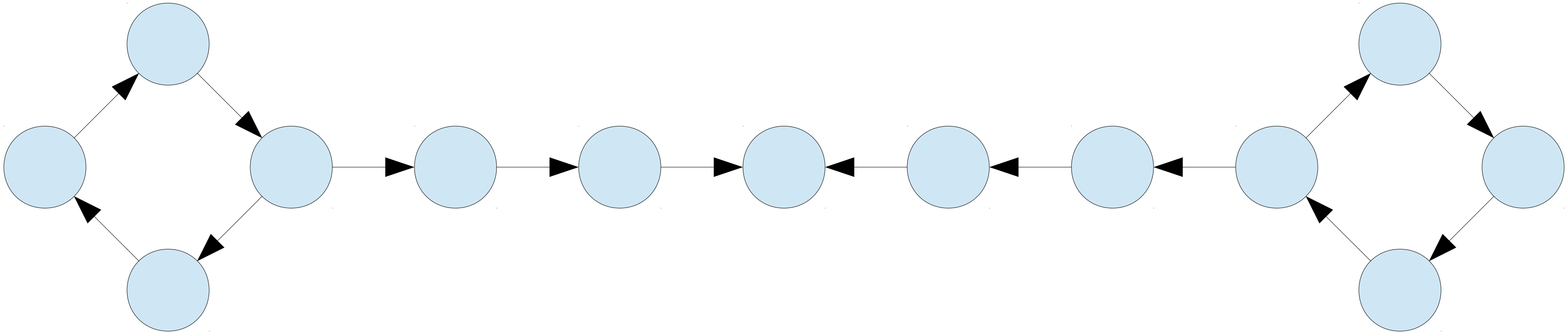
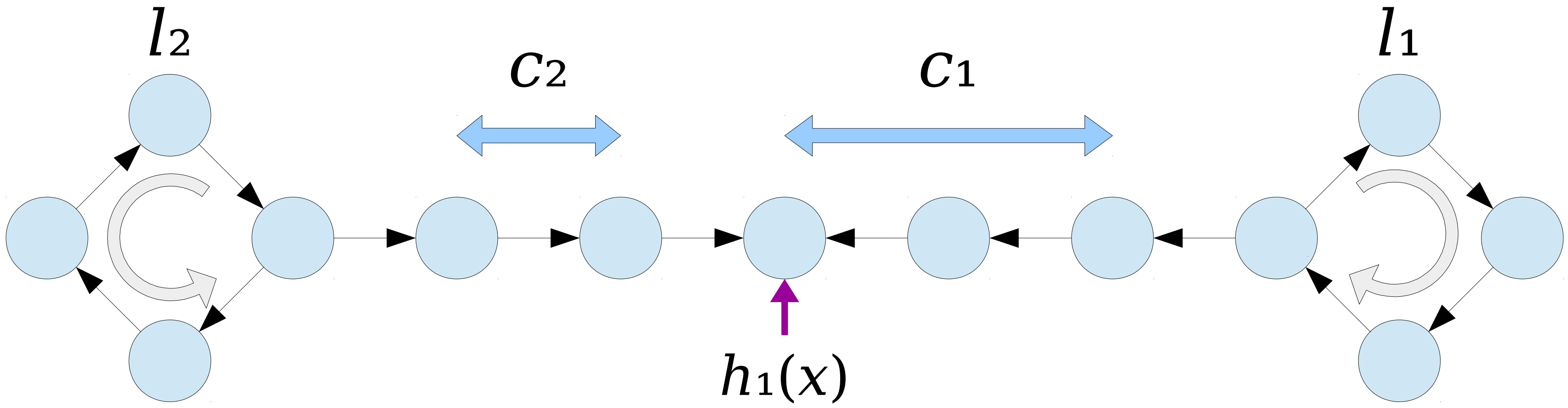


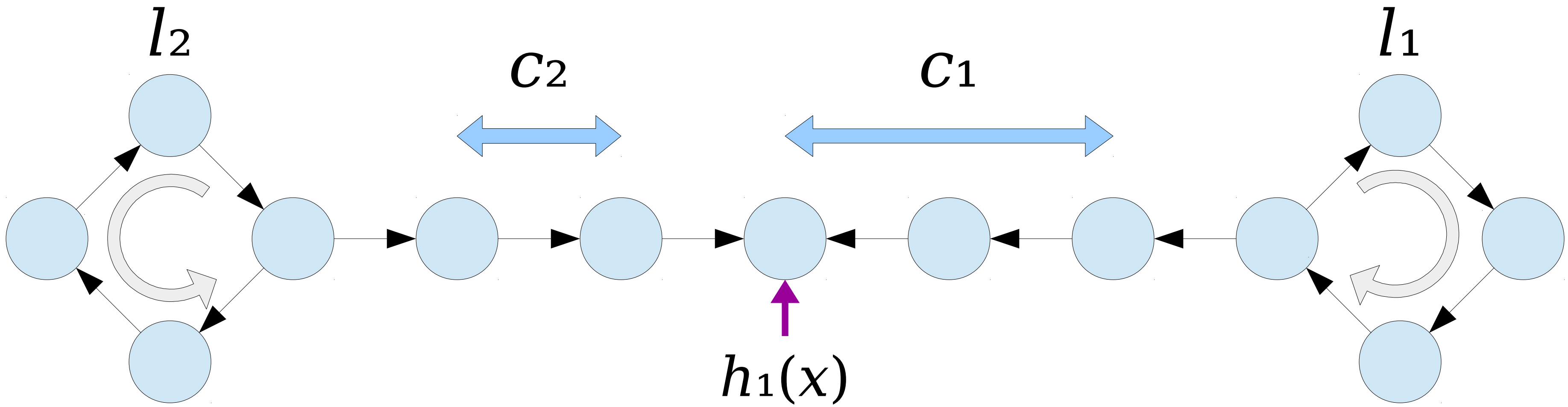
We’re right back where we started. This pattern will continue indefinitely.

Insertion fails if we have a complex connected component.
What specifically happens in that case?



Question: What's the probability that we end up with a configuration like this one?

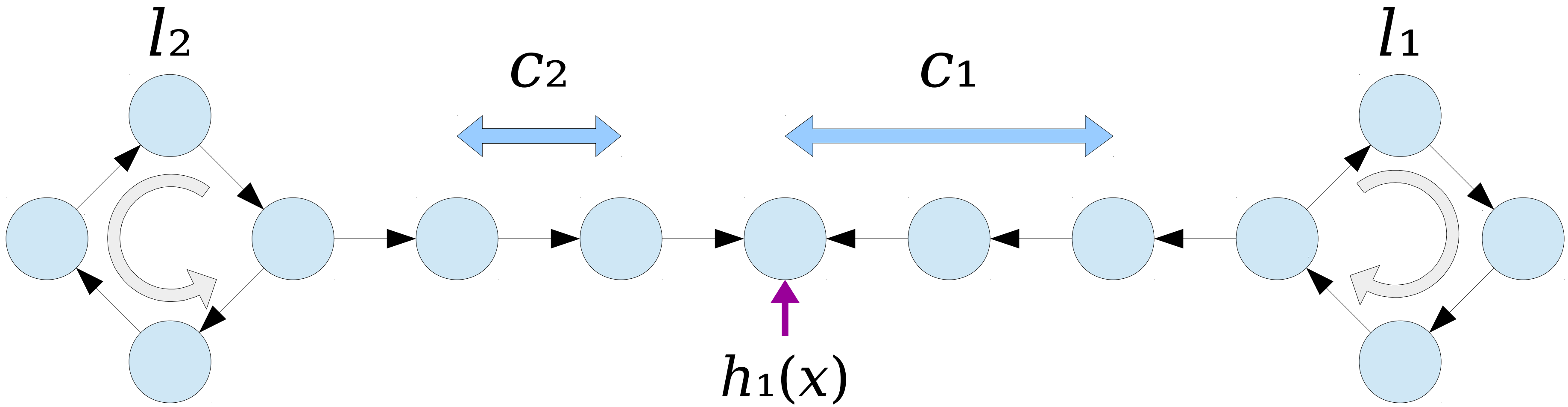
Insertion fails if we have a complex connected component.
What specifically happens in that case?



This next proof comes from a CS166 final project by Noah Arthurs, Joseph Chang, and Nolan Handali. It's inspired by another argument due to Charles Chen (another Stanford student), which is a modification of one by Sanders and Vöcking, which was an improvement of one by Pagh and Rodler.

Key idea: Use a traditional, CS109-style counting argument. Admittedly, it's a *nontrivial* counting argument, but it's a counting argument nonetheless!

Insertion fails if we have a complex connected component. What specifically happens in that case?



Ways to pick k nodes (table slots) given the first is $h_1(x)$. *(upper bound)*

Ways to split k nodes into c_1, l_1, c_2, and l_2. *(upper bound)*

Ways to assign k keys to those slots. *(upper bound)*

Sum over all possible numbers of other keys being displaced.

$$\sum_{k=1}^{n} \left(\frac{(k+1)^4 \, m^{k-1} \, n^k}{m^{2k} \, m} \right)$$

Ways h_1 and h_2 can be chosen for those keys.

Ways $h_2(x)$ can be chosen.

Insertion fails if we have a complex connected component. What specifically happens in that case?

$$\sum_{k=1}^{n} \left(\frac{(k+1)^4 \, m^{k-1} \, n^k}{m^{2k} \, m} \right)$$

$$\sum_{k=1}^{n}\left(\frac{(k+1)^4\, m^{k-1}\, n^k}{m^{2\,k}\, m}\right) \quad = \quad \sum_{k=1}^{n}\left((k+1)^4\, n^k\, m^{k-1-2\,k-1}\right)$$

$$\sum_{k=1}^{n} \left(\frac{(k+1)^4 \, m^{k-1} \, n^k}{m^{2\,k} \, m} \right) \quad = \quad \sum_{k=1}^{n} \left((k+1)^4 \, n^k \, m^{k-1-2\,k-1} \right)$$

$$= \quad \sum_{k=1}^{n} \left((k+1)^4 \, n^k \, m^{k-2} \right)$$

$$\sum_{k=1}^{n}\left(\frac{(k+1)^4\, m^{k-1}\, n^k}{m^{2k}\, m}\right) = \sum_{k=1}^{n}\left((k+1)^4\, n^k\, m^{k-1-2k-1}\right)$$

$$= \sum_{k=1}^{n}\left((k+1)^4\, n^k\, m^{k-2}\right)$$

$$= \frac{1}{m^2}\sum_{k=1}^{n}\left((k+1)^4\, n^k\, m^{-k}\right)$$

$$\sum_{k=1}^{n}\left(\frac{(k+1)^4\, m^{k-1}\, n^k}{m^{2k}\, m}\right) = \sum_{k=1}^{n}\left((k+1)^4\, n^k\, m^{k-1-2k-1}\right)$$

$$= \sum_{k=1}^{n}\left((k+1)^4\, n^k\, m^{k-2}\right)$$

$$= \frac{1}{m^2}\sum_{k=1}^{n}\left((k+1)^4\, n^k\, m^{-k}\right)$$

$$= \frac{1}{m^2}\sum_{k=1}^{n}(k+1)^4\left(\frac{n}{m}\right)^k$$

$$\sum_{k=1}^{n}\left(\frac{(k+1)^4\, m^{k-1}\, n^k}{m^{2k}\, m}\right) = \sum_{k=1}^{n}\left((k+1)^4\, n^k\, m^{k-1-2k-1}\right)$$

$$= \sum_{k=1}^{n}\left((k+1)^4\, n^k\, m^{k-2}\right)$$

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Denominator grows *exponentially* as a function of *k*.

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The probability that a single insertion fails is $O(1 / m^2)$ if $m = (1+\varepsilon)n$.

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If an insertion fails, we ***rehash*** by building a brand-new table, with new hash functions, and inserting all old elements.

It's possible that, when we do a rehash, one of the insertions fails. Therefore, we keep rehashing until we find a working table.

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Let X be a random variable counting the number of rehashes assuming at least one rehash occurs.

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The probability that a series of n insertions fail is $O(1 / m)$.

Question 3: What is the expected cost of an insertion into a cuckoo hash table?

$$\underbrace{\mathbf{O(1)}}_{\text{Expected cost of successful insertion.}} + \underbrace{\mathbf{O(1\ /\ \boldsymbol{m}^2)}}_{\text{Expected number of rehashes.}} \cdot \underbrace{\mathbf{O(\boldsymbol{m})}}_{\text{Cost of doing one rehash.}}$$

The expected number of rehashes on any insertion is $O(1\ /\ m^2)$.

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$$\mathbf{O(1)} + \mathbf{O(1\ /\ \boldsymbol{m}^2)} \cdot \mathbf{O(\boldsymbol{m})}$$

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O(1)

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The Overall Analysis

- Cuckoo hashing gives worst-case lookups and deletions.
- Insertions are expected, amortized O(1).
- The hidden constants are small, and this is a practical technique for building hash tables.

Cuckoo Hashing:

- ***lookup***: O(1)
- ***insert***: O(1)*
- ***delete***: O(1)

* *expected, amortized*

More to Explore

Hash Function Strength

- We analyzed cuckoo hashing assuming our hash functions were truly random. That’s often too strong of an assumption.
- What we know:
  - 6-independent hashing isn't sufficient for expected O(1) insertion time, but that O(log n)-independence is.
  - Some simple classes of hash functions (e.g. 2-independent polynomial) perform poorly for cuckoo hashing.
  - Some simple classes of hash functions (e.g. 3-independent simple tabulation) perform very well.
- ***Open problem:*** Determine the strength of hash function needed for cuckoo hashing to work efficiently.

Multiple Tables

- Cuckoo hashing works well with two tables. So why not 3, 4, 5, …, or k tables?
- In practice, cuckoo hashing with $k \geq 3$ tables tends to perform much better than cuckoo hashing with $k = 2$ tables:
  - The load factor can increase substantially; with $k=3$, it's only around $\alpha = 0.91$ that you run into trouble with the cuckoo graph.
  - Displacements are less likely to chain together; they only occur when all hash locations are filled in.
- ***Open problem:*** Determine where these phase transition thresholds are for arbitrary k.

Increasing Bucket Sizes

- What if each slot in a cuckoo hash table can store multiple elements?
- When displacing an element, choose a random one to move and move it.
- This turns out to work remarkably well in practice, since it makes it really unlikely that you'll have long chains of displacements.
- ***Open problem:*** Quantify the effect of larger bucket sizes on the overall runtime of cuckoo hashing.

Restricting Moves

- Insertions in cuckoo hashing only run into trouble when you encounter long chains of displacements during insertions.
- ***Idea:*** Cap the number of displacements at some fixed factor, then store overflowing elements in a secondary hash table.
- In practice, this works remarkably well, since the auxiliary table doesn't tend to get very large.
- ***Open problem:*** Quantify the effects of “hashing with a stash” for arbitrary stash sizes and displacement limits.

Other Dynamic Schemes

- There is another famous dynamic perfect hashing scheme called ***dynamic FKS hashing***.
- It works by using closed addressing and resolving collisions at the top level with a secondary (static) perfect hash table.
- In practice, it's not as fast as these other approaches. However, it only requires 2-independent hash functions.
- Check CLRS for details!

Lower Bounds?

- ***Open Problem:*** Is there a hash table that supports amortized O(1) insertions, deletions, and lookups?
- You'd think that we'd know the answer to this question, but, sadly, we don't.

Next Time

- ***Approximate Membership Queries***
  - Educated guesses about whether things have been seen before.
- ***Bloom Filters***
  - The original – and one of the most popular – solutions to this problem.
- ***Quotient Filters***
  - Adapting linear probing for AMQ.
- ***Cuckoo Filters***
  - Adapting cuckoo hashing for AMQ.