Balanced Trees

Part One

Balanced Trees

- Balanced search trees are among the most useful and versatile data structures.
- Many programming languages ship with a balanced tree library.
  - C++: `std::map` / `std::set`
  - Java: `TreeMap` / `TreeSet`
  - Python: `OrderedDict`
- Many advanced data structures are layered on top of balanced trees.
  - We'll see them used to build *y*-Fast Tries later in the quarter. (They’re really cool, trust me!)

Where We're Going

- ***B-Trees (Today)***
  - A simple type of balanced tree developed for block storage.
- ***Red/Black Trees (Today)***
  - The canonical balanced binary search tree.
- ***Augmented Search Trees (Tuesday)***
  - Adding extra information to balanced trees to supercharge the data structure.
- ***Two Advanced Operations (Tuesday)***
  - Splitting and joining BSTs.

Outline for Today

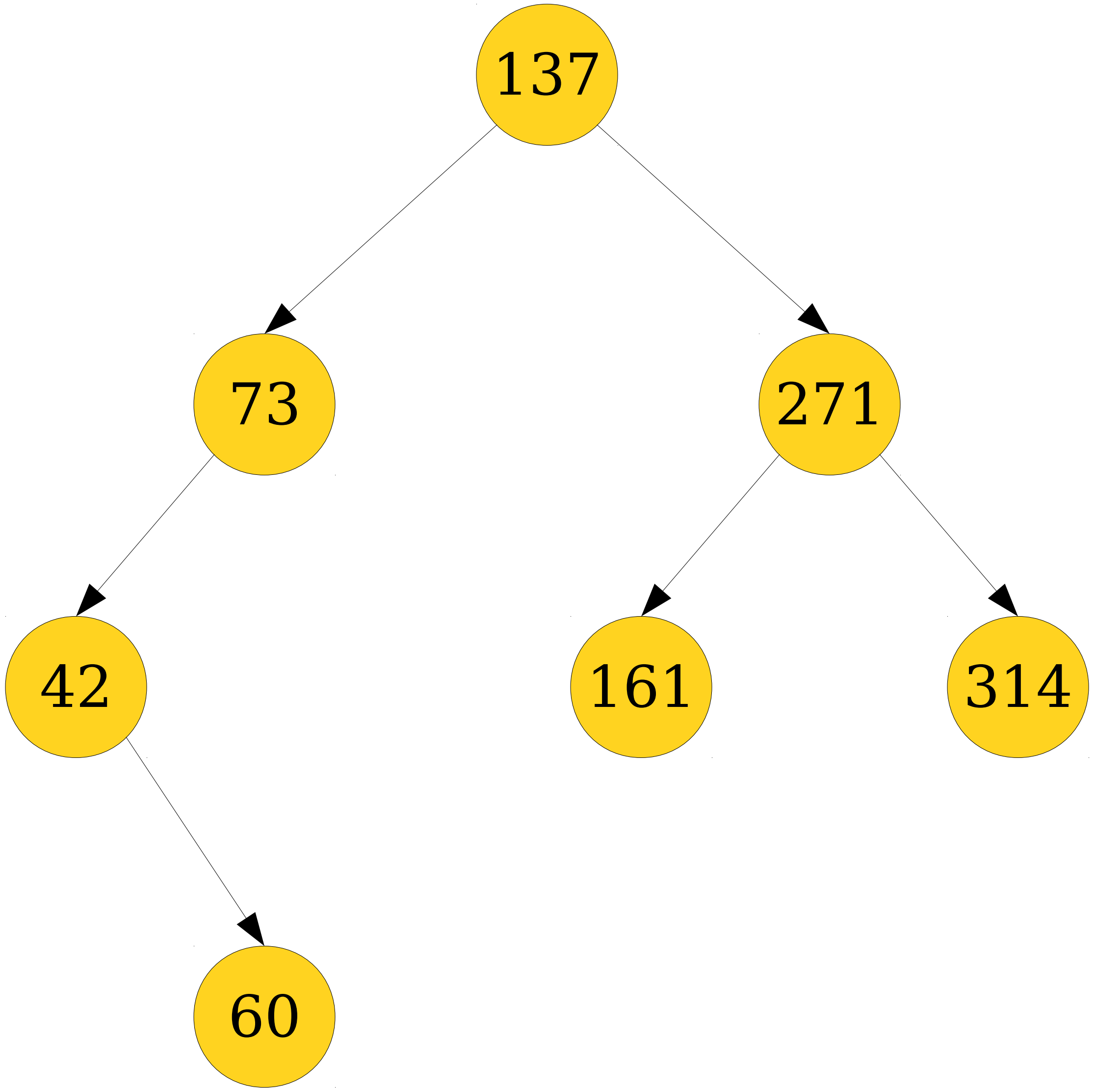
- ***BST Review***
  - Refresher on basic BST concepts and runtimes.
- ***Overview of Red/Black Trees***
  - What we're building toward.
- ***B-Trees and 2-3-4 Trees***
  - A simple balanced tree in depth.
- ***Intuiting Red/Black Trees***
  - A much better feel for red/black trees.

A Quick BST Review

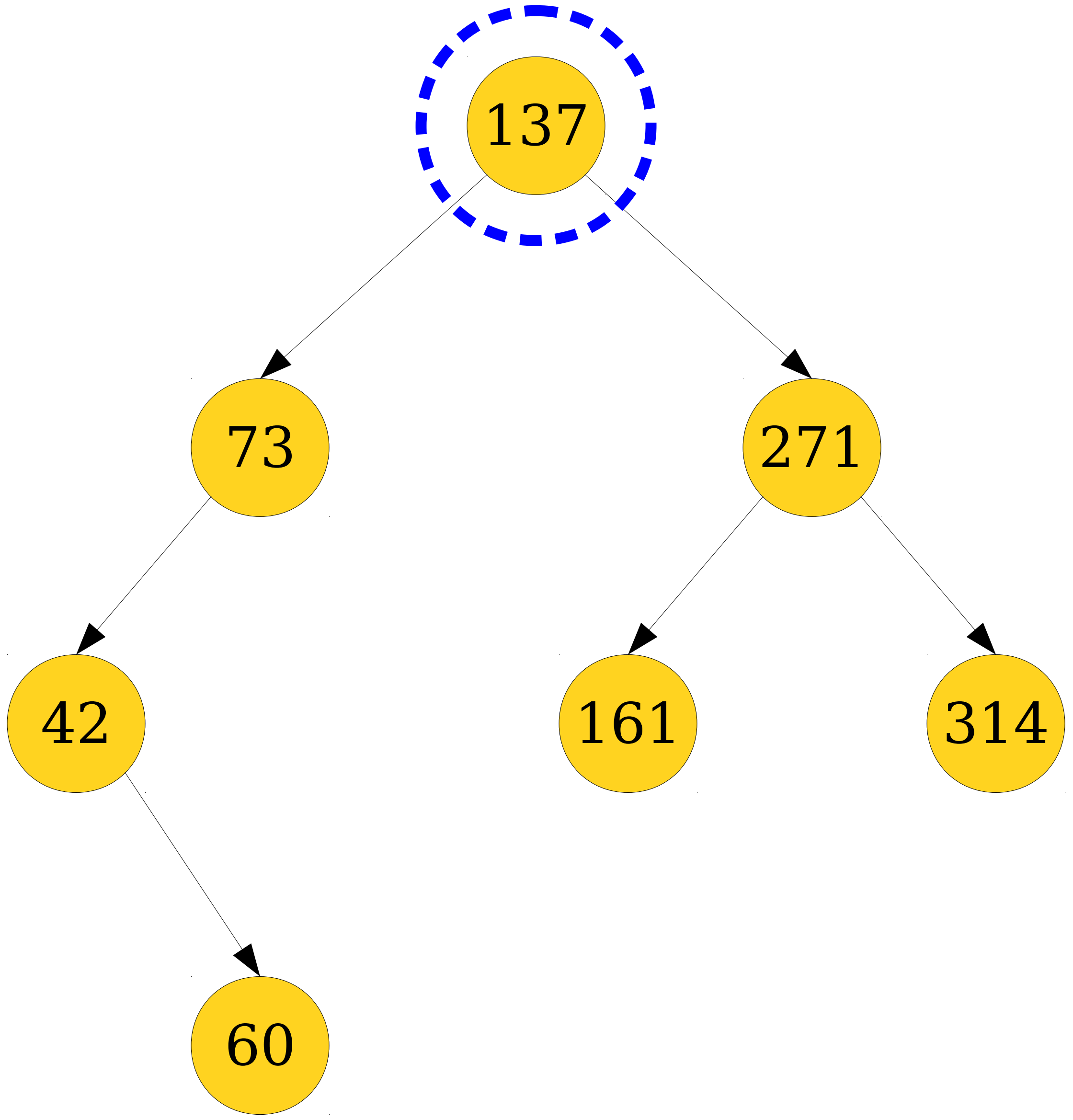
Binary Search Trees

- A ***binary search tree*** is a binary tree with the following properties:
  - Each node in the BST stores a ***key***, and optionally, some auxiliary information.
  - The key of every node in a BST is strictly greater than all keys to its left and strictly smaller than all keys to its right.
- The ***height*** of a binary search tree is the length of the longest path from the root to a leaf, measured in the number of *edges*.
  - A tree with one node has height 0.
  - A tree with no nodes has height -1, by convention.

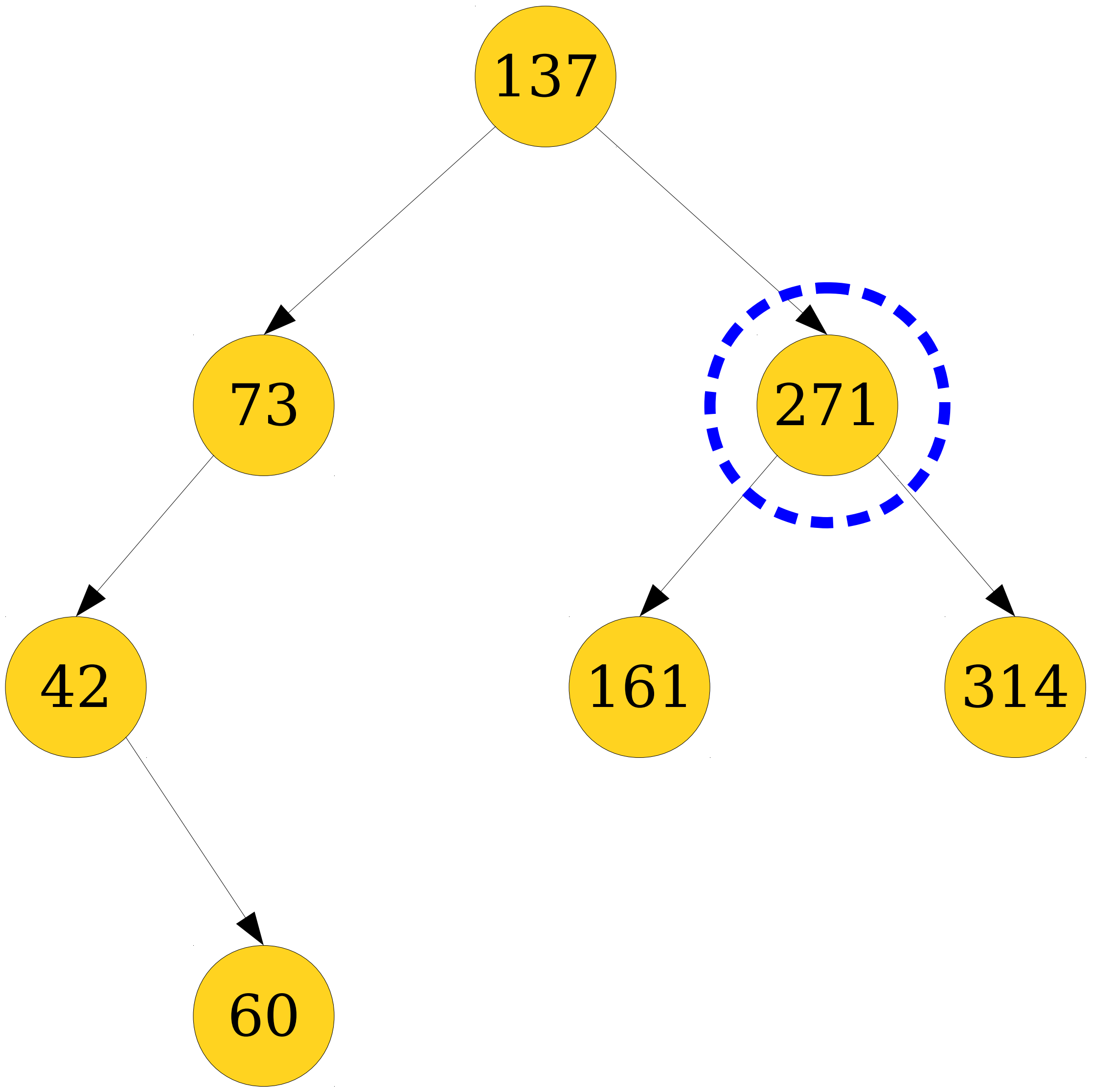
Searching a BST



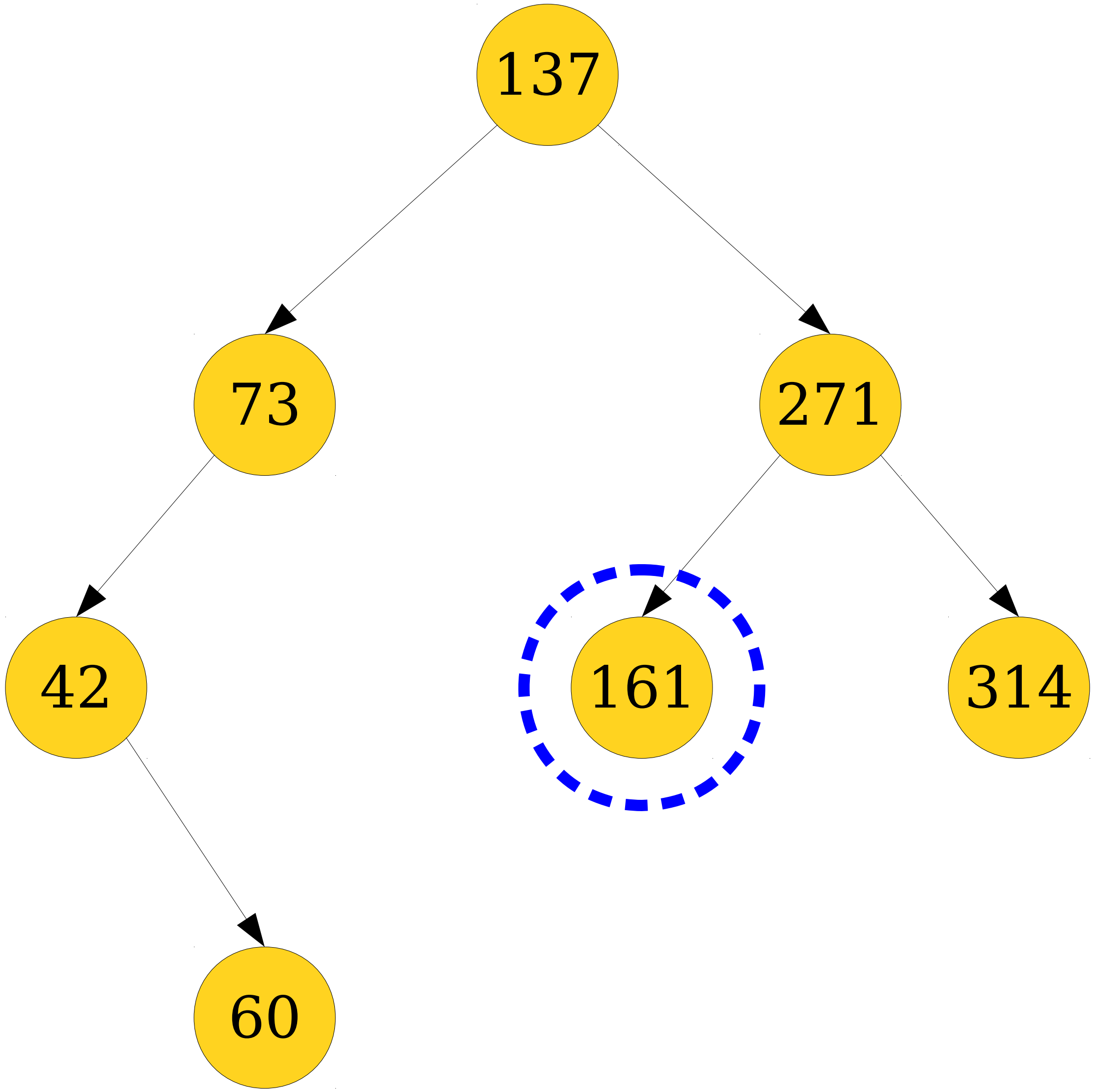
Searching a BST



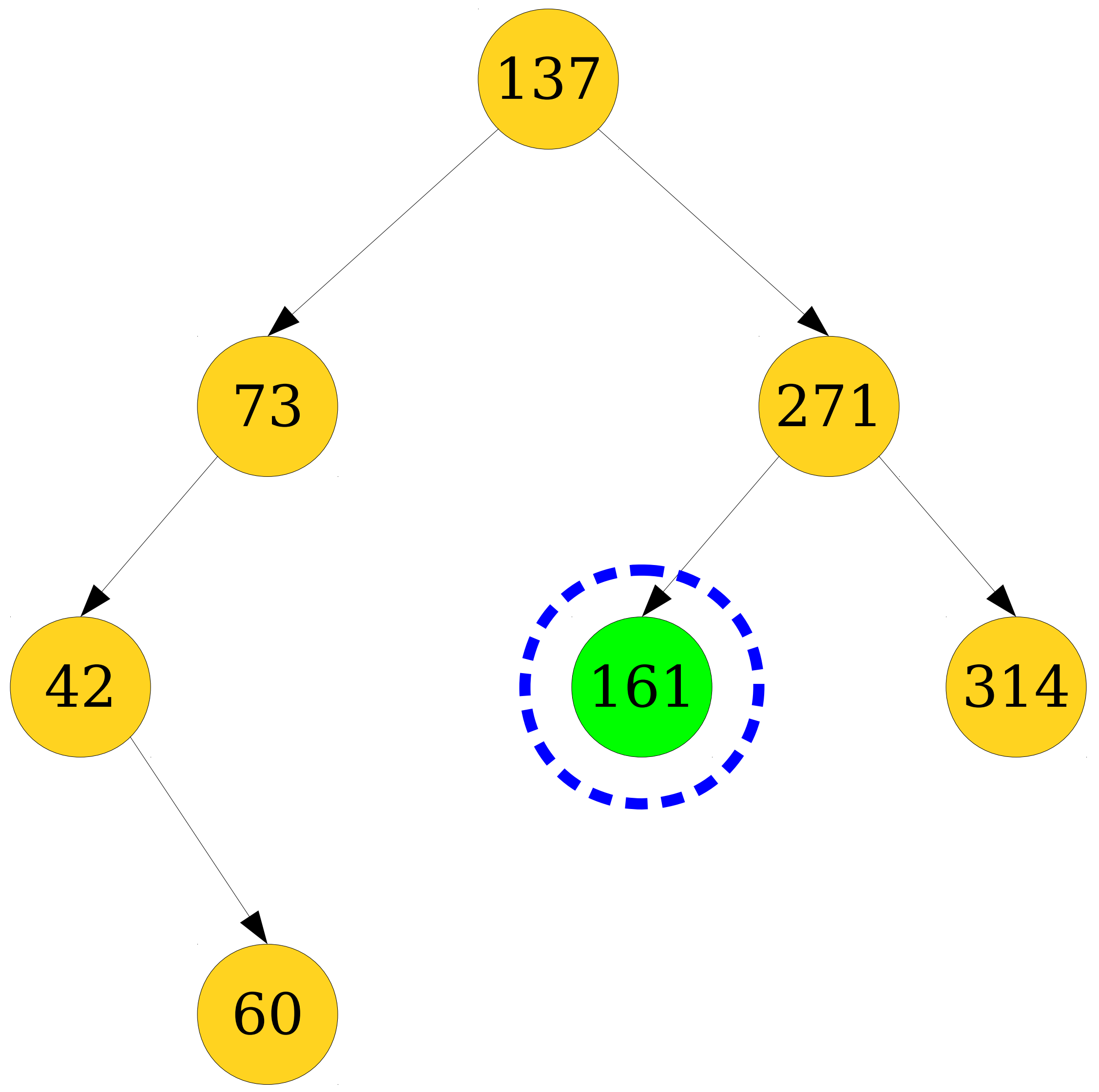
Searching a BST



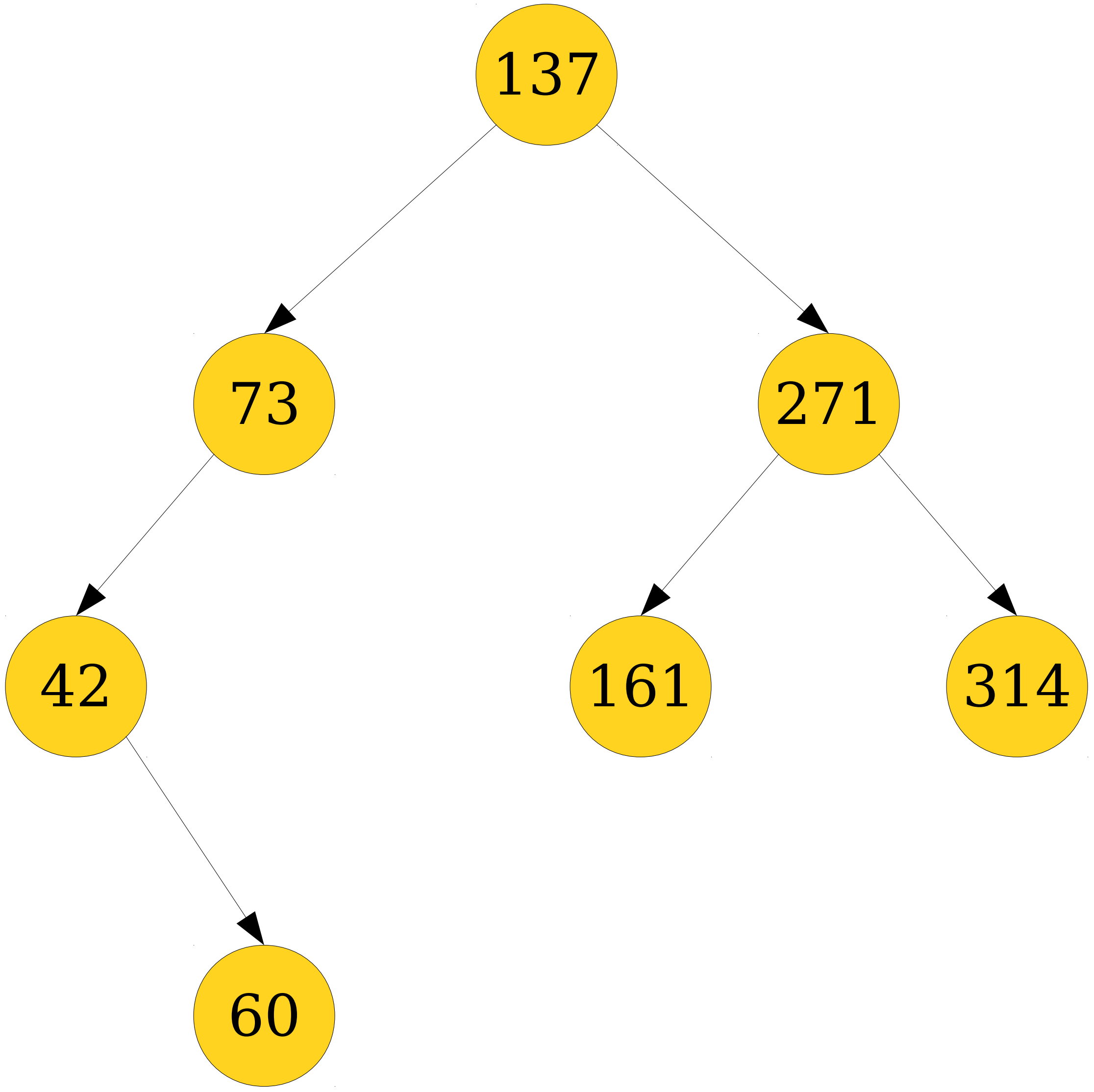
Searching a BST



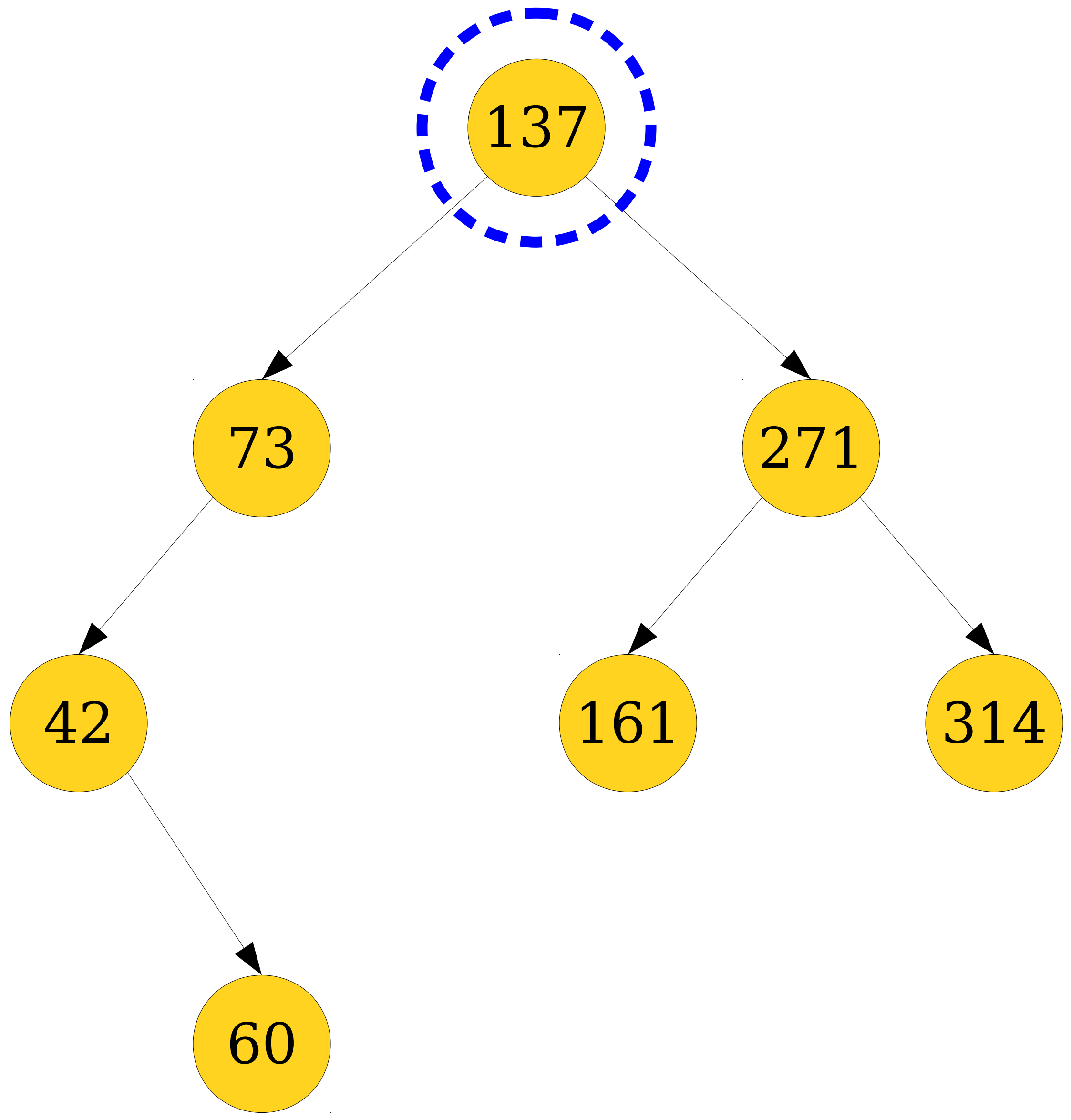
Searching a BST



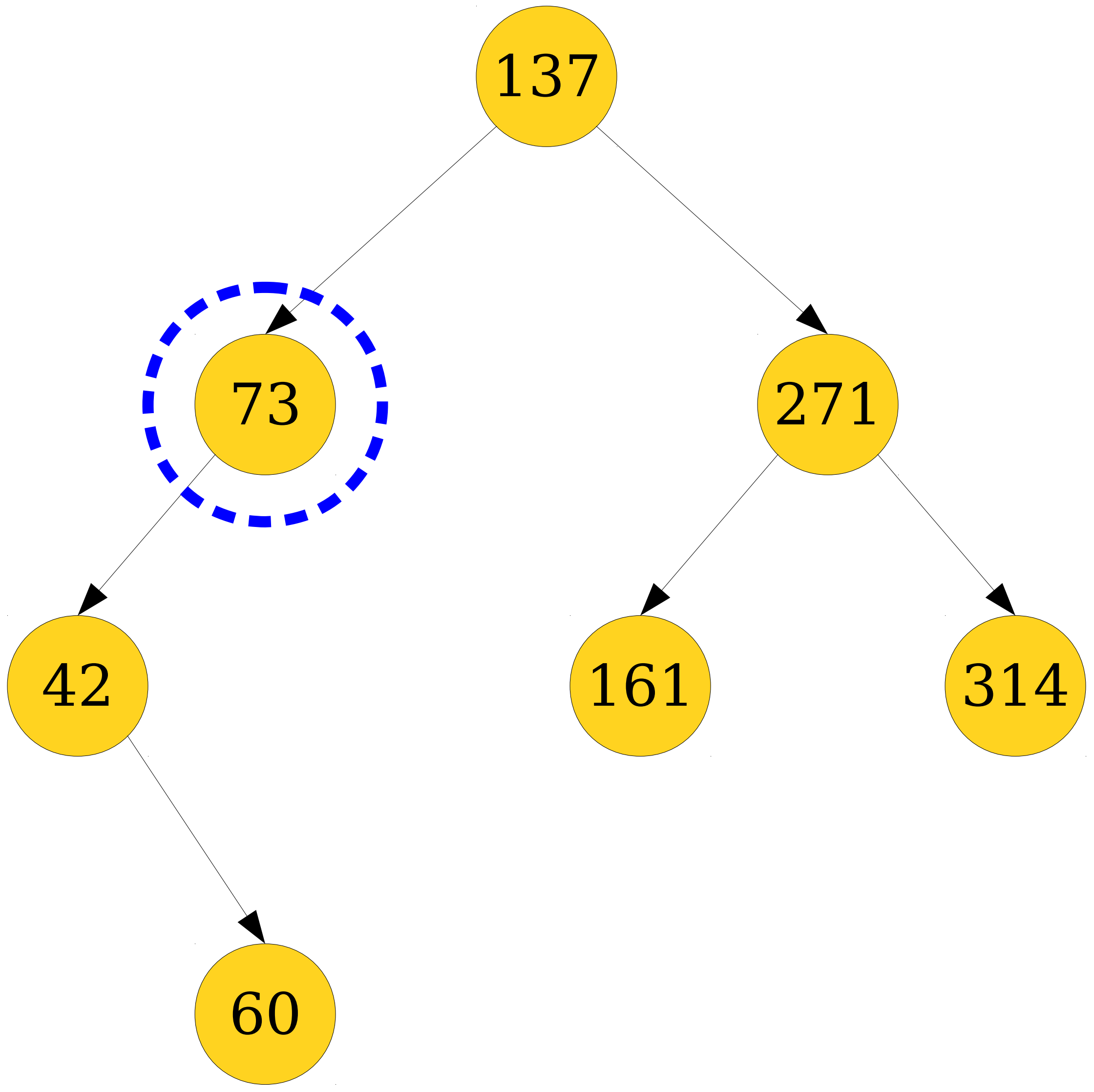
Searching a BST



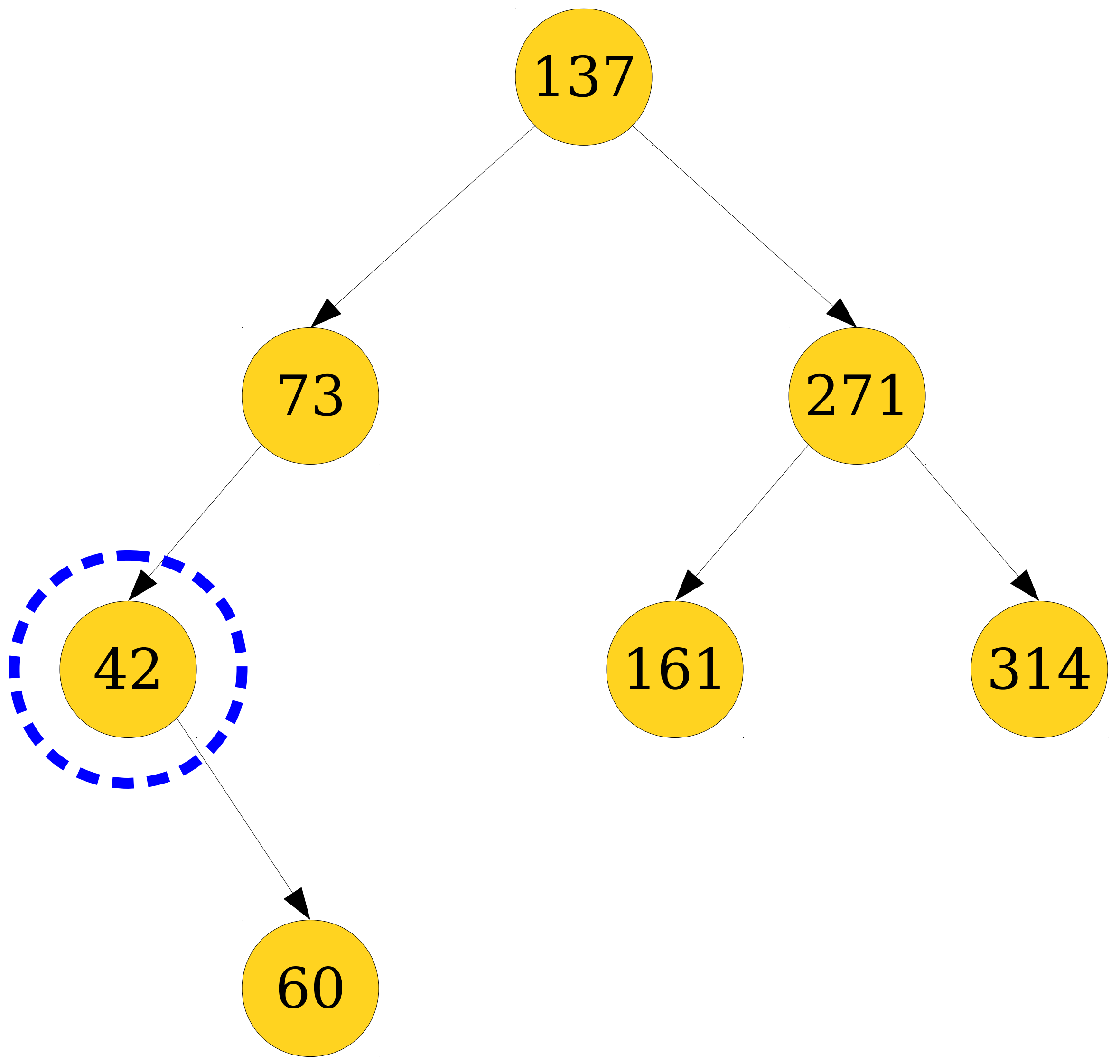
Searching a BST



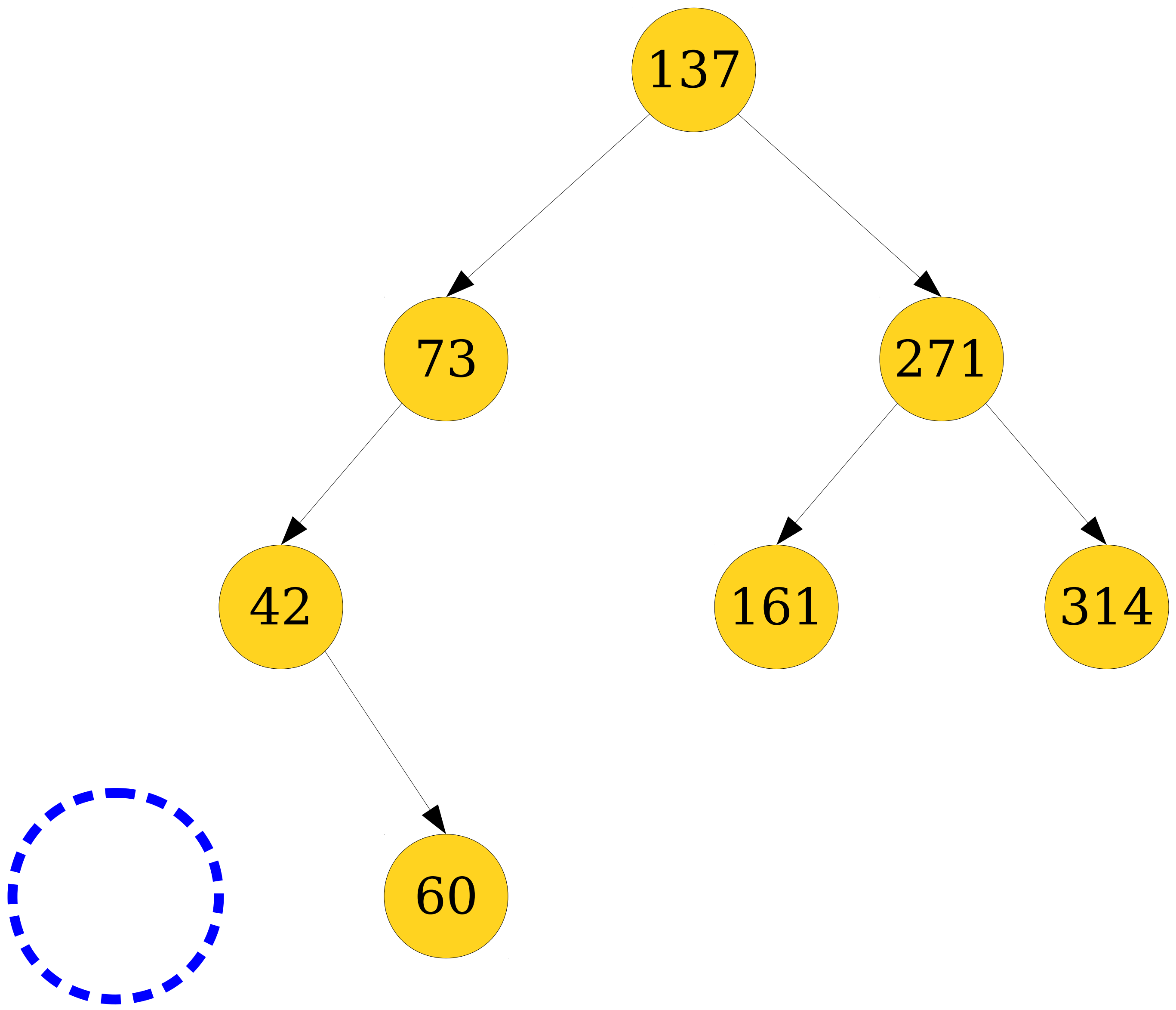
Searching a BST



Searching a BST

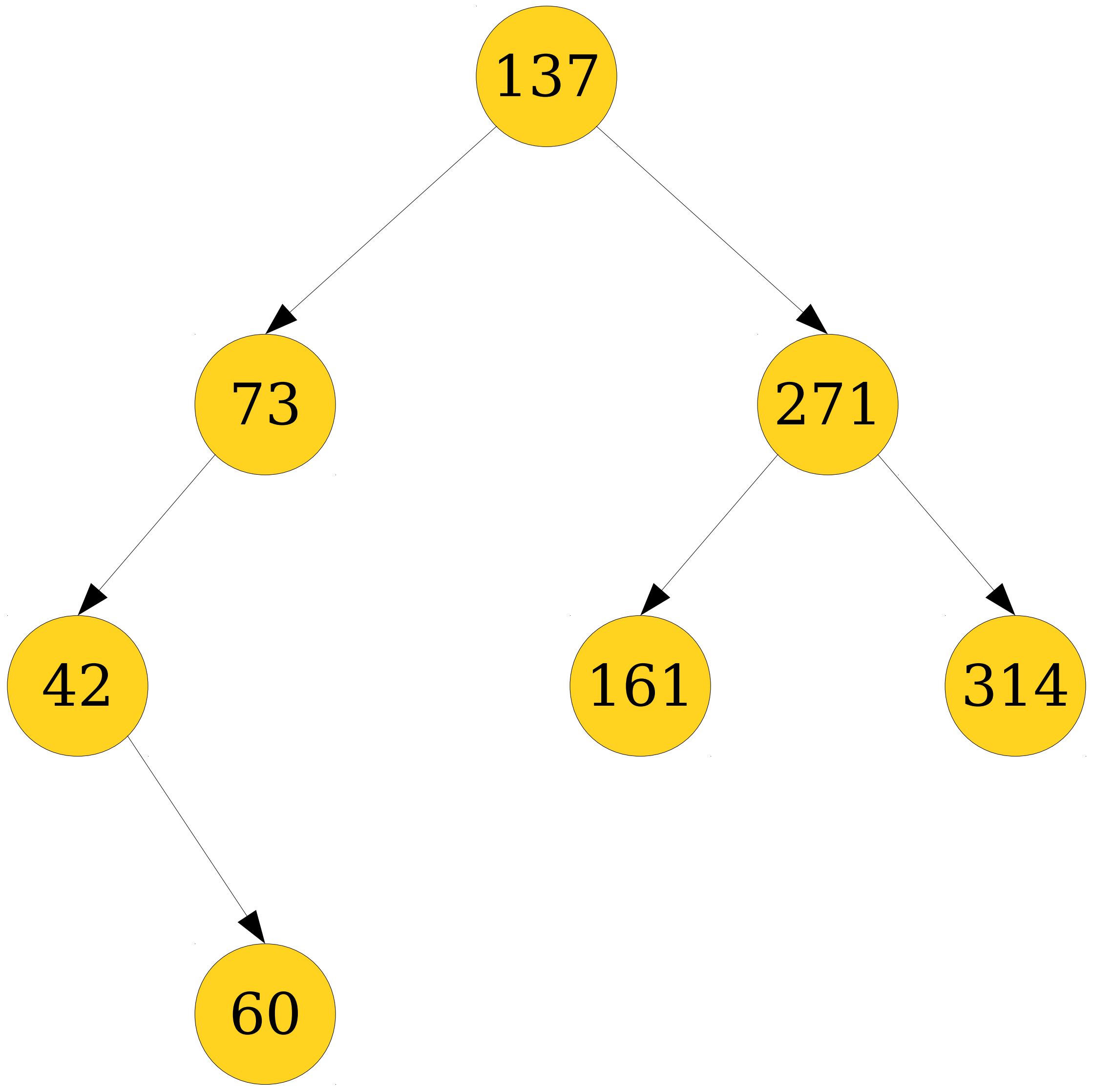


Searching a BST

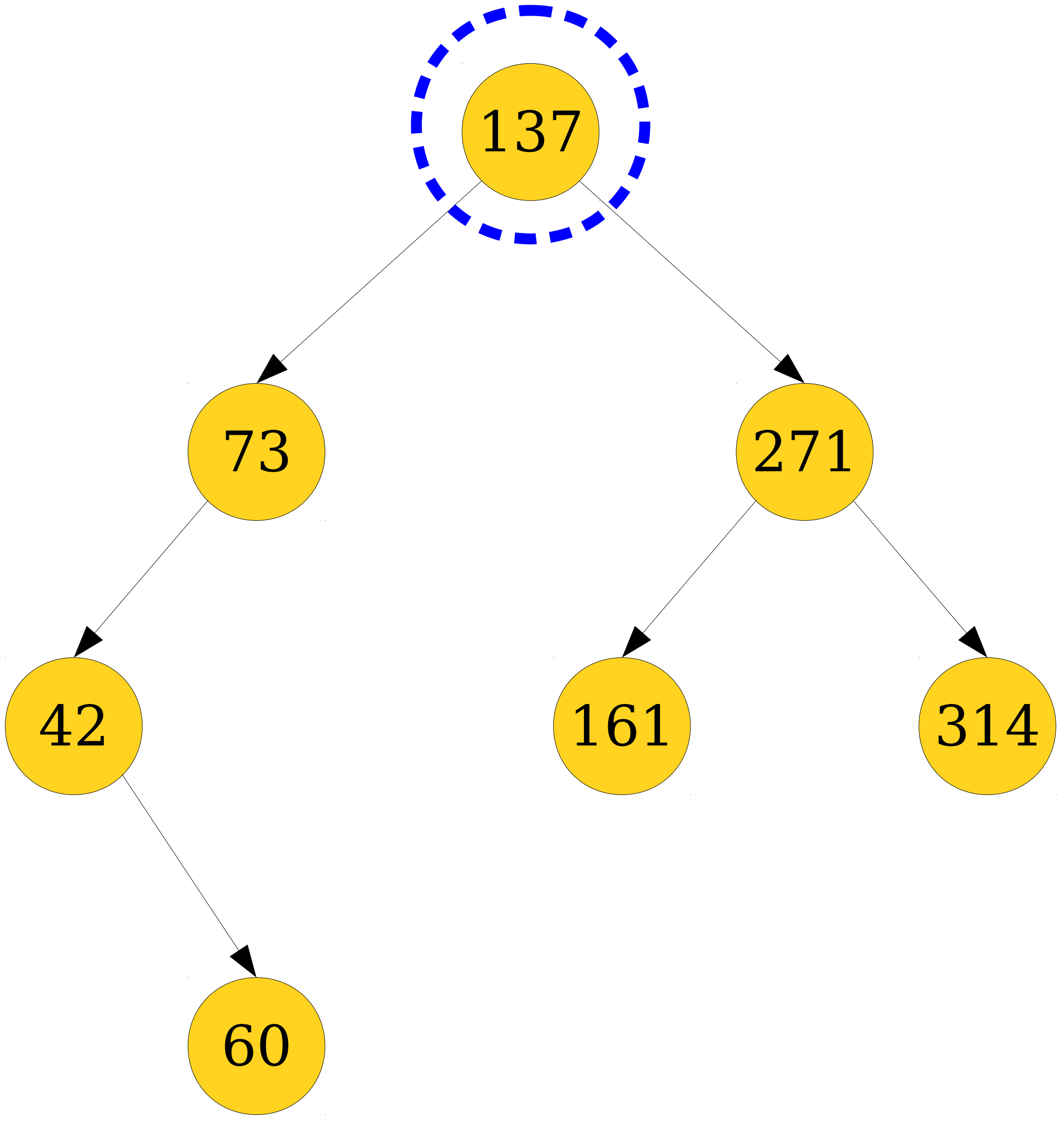


Inserting into a BST

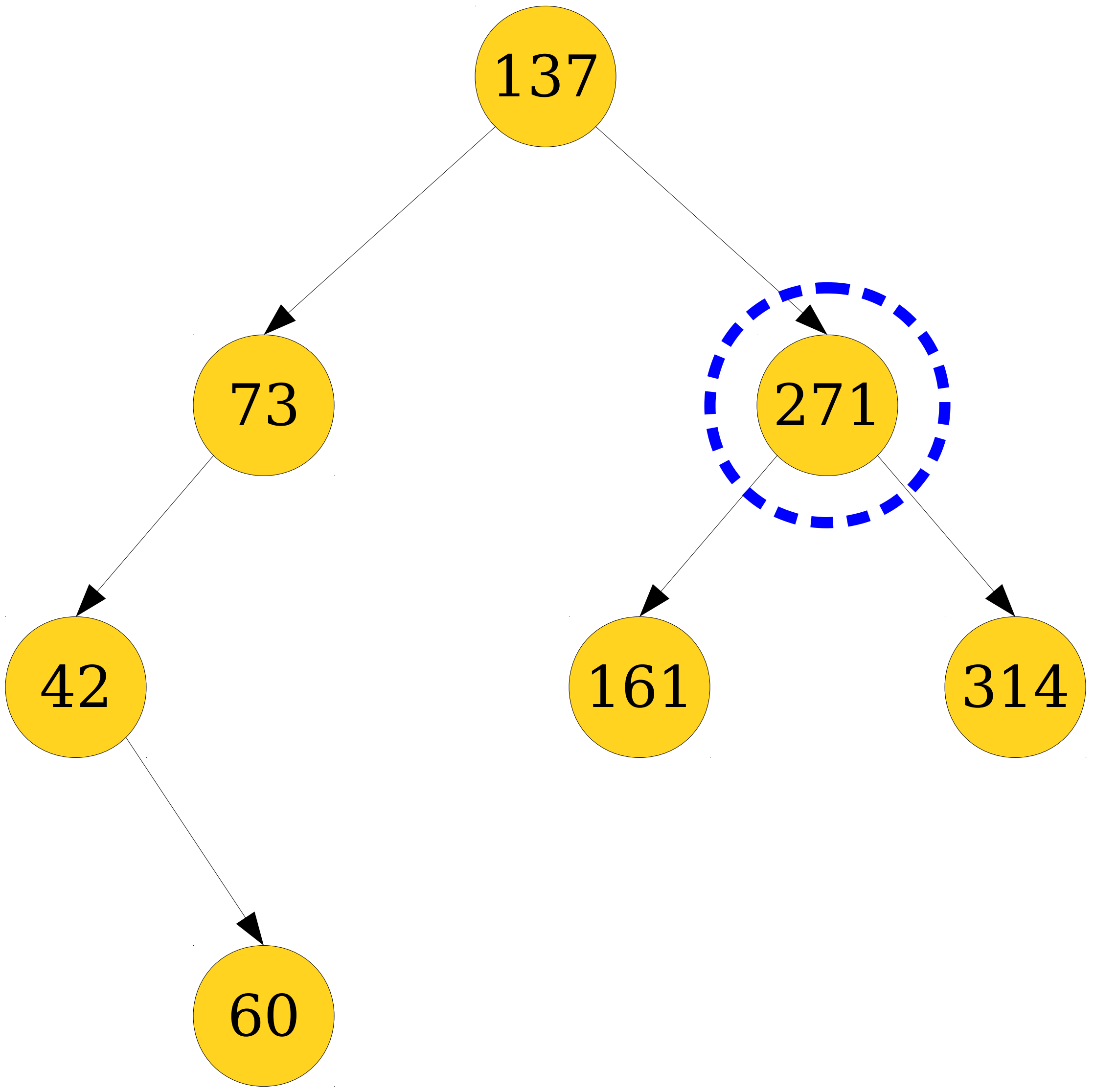
Inserting into a BST



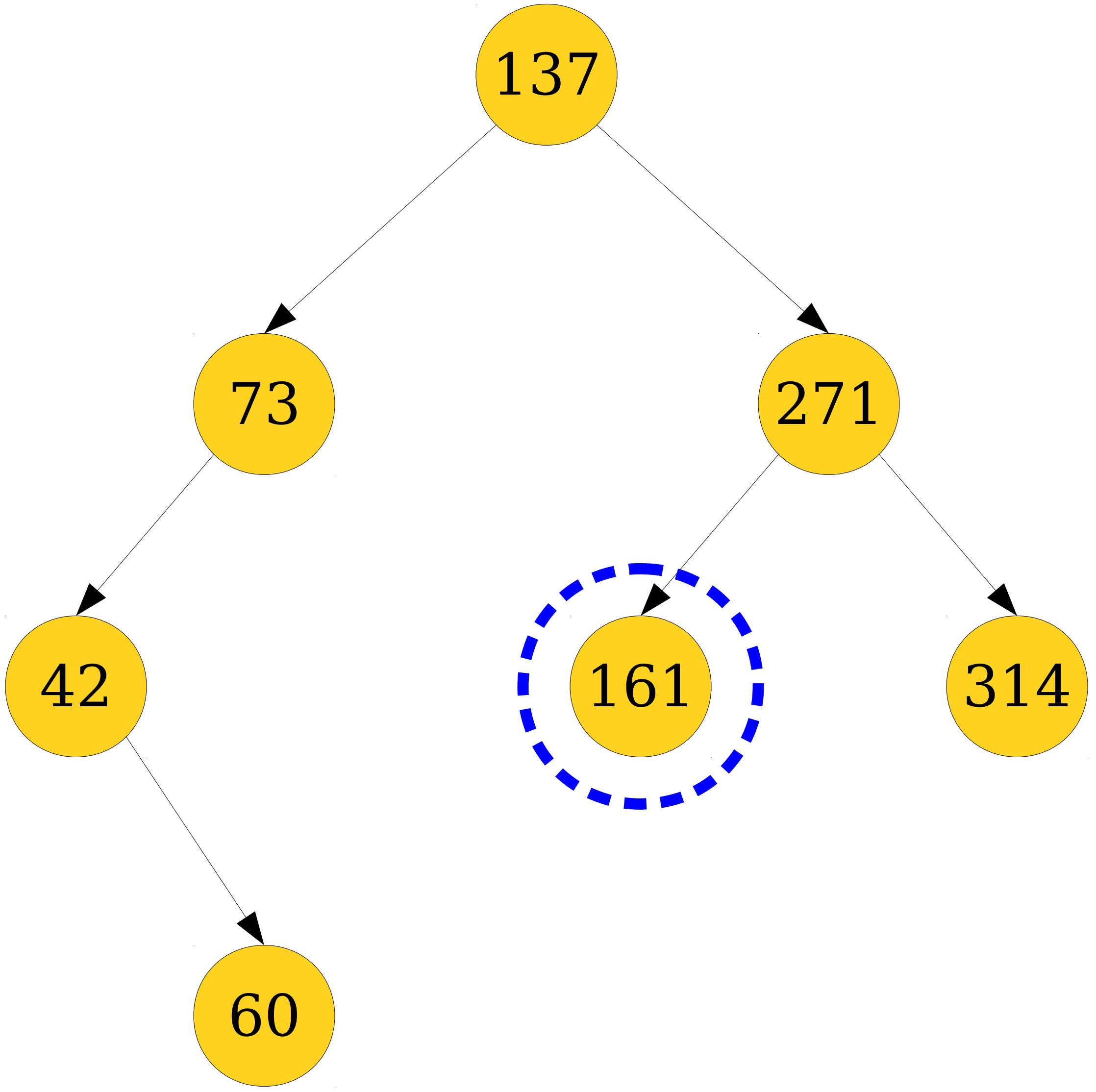
Inserting into a BST



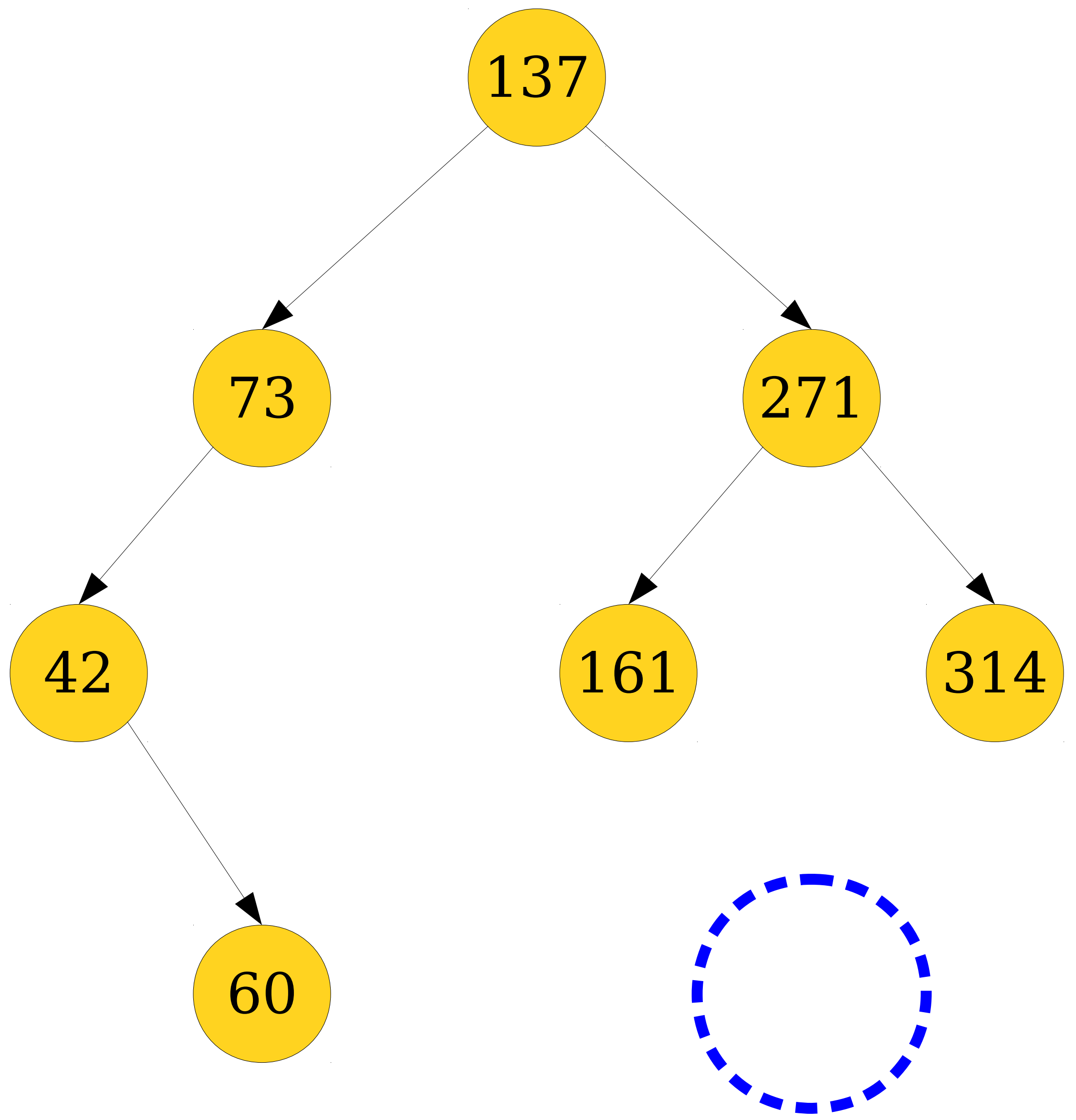
Inserting into a BST



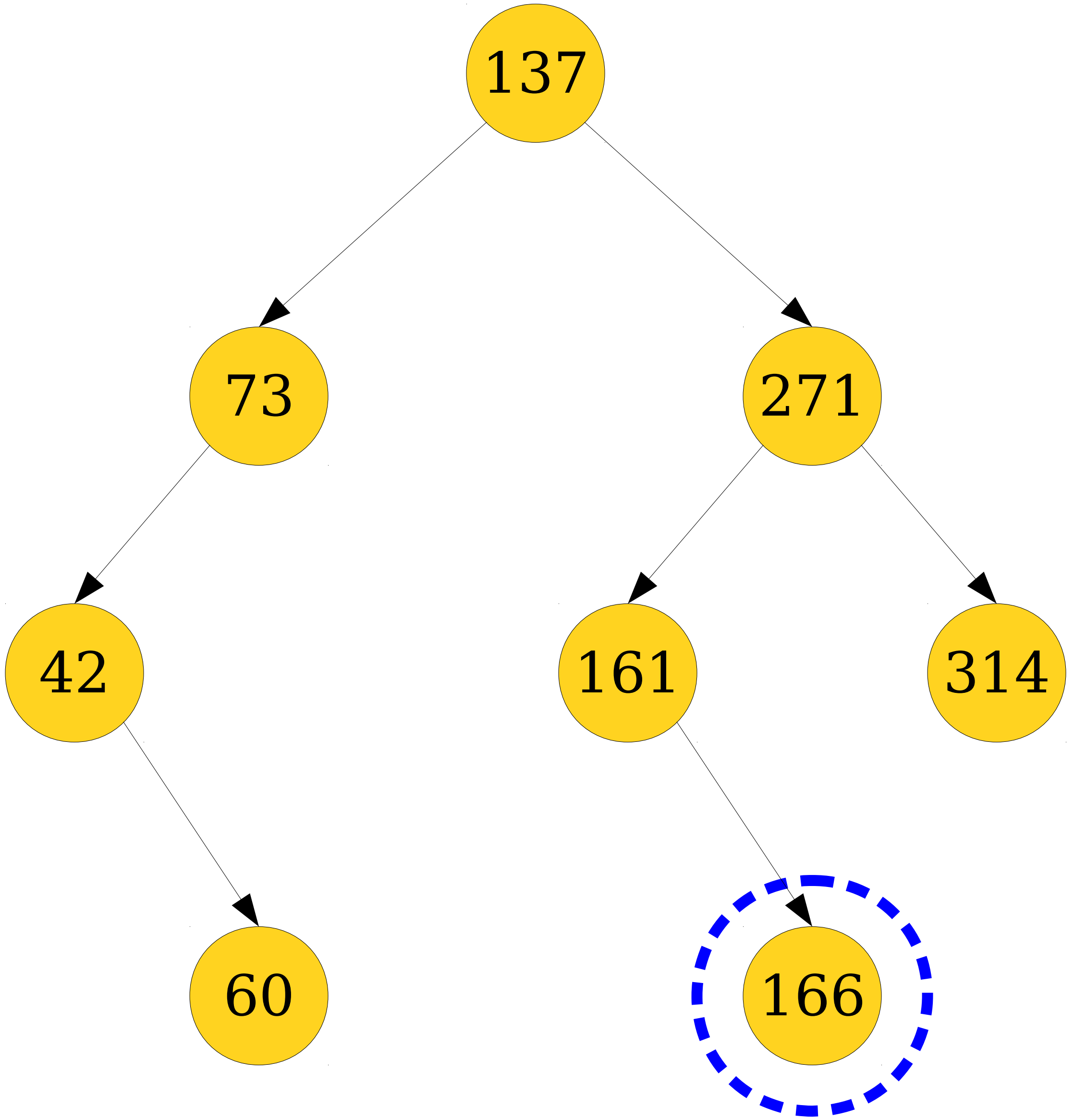
Inserting into a BST



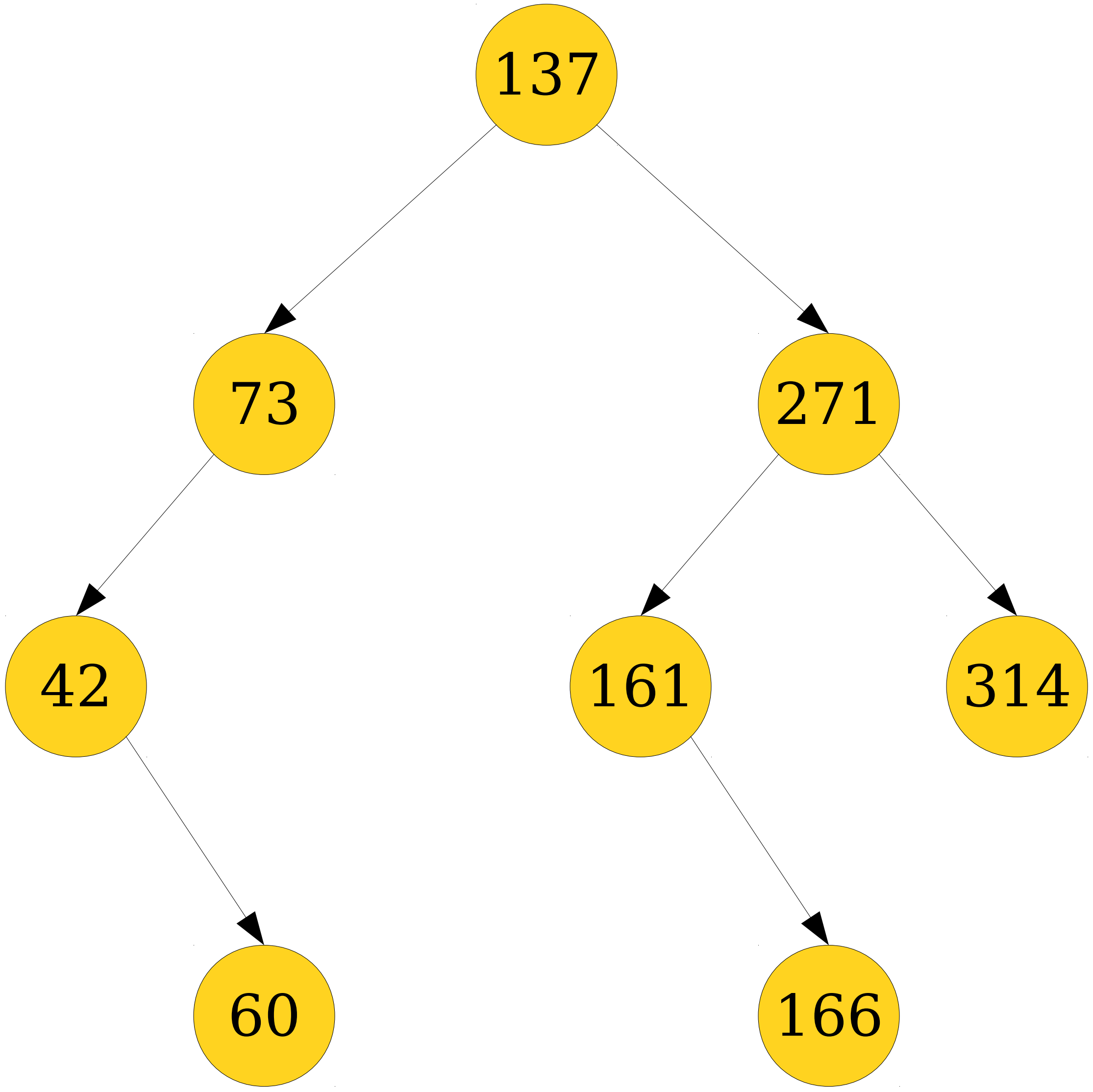
Inserting into a BST



Inserting into a BST

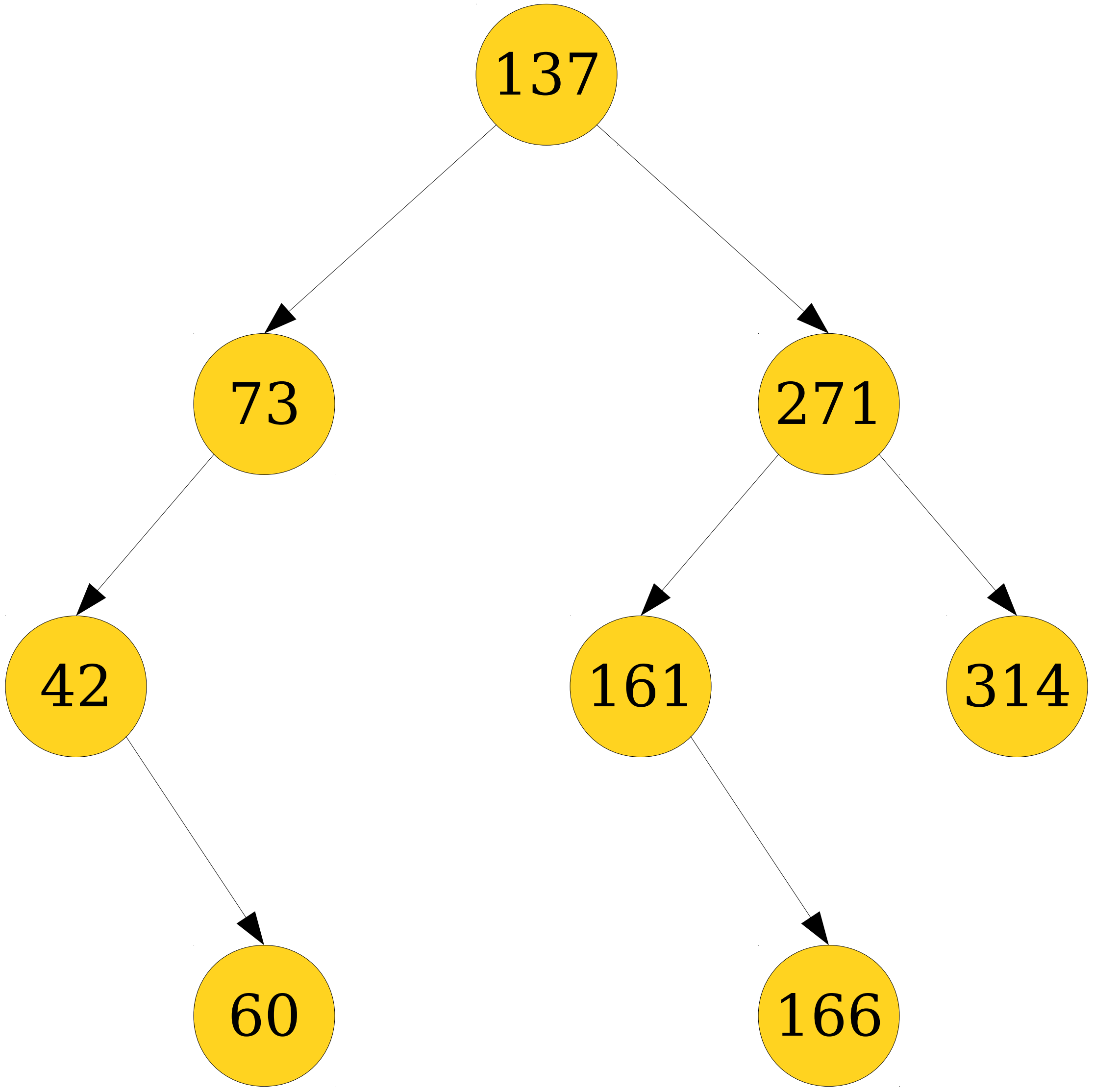


Inserting into a BST

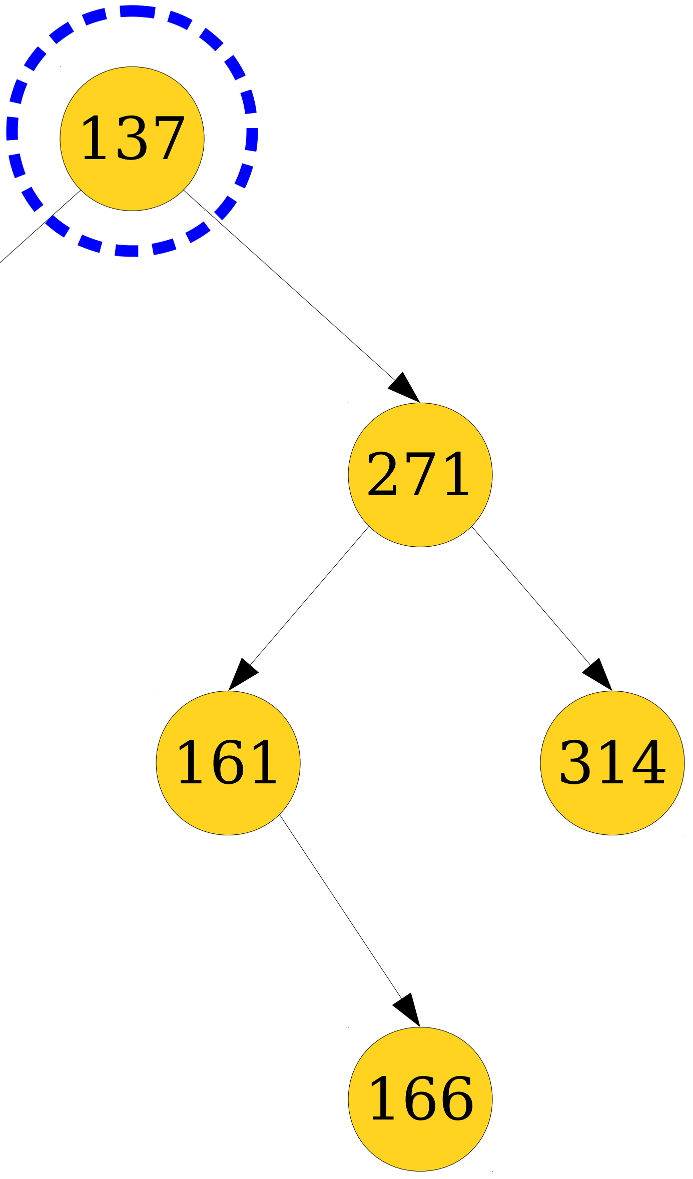


Deleting from a BST

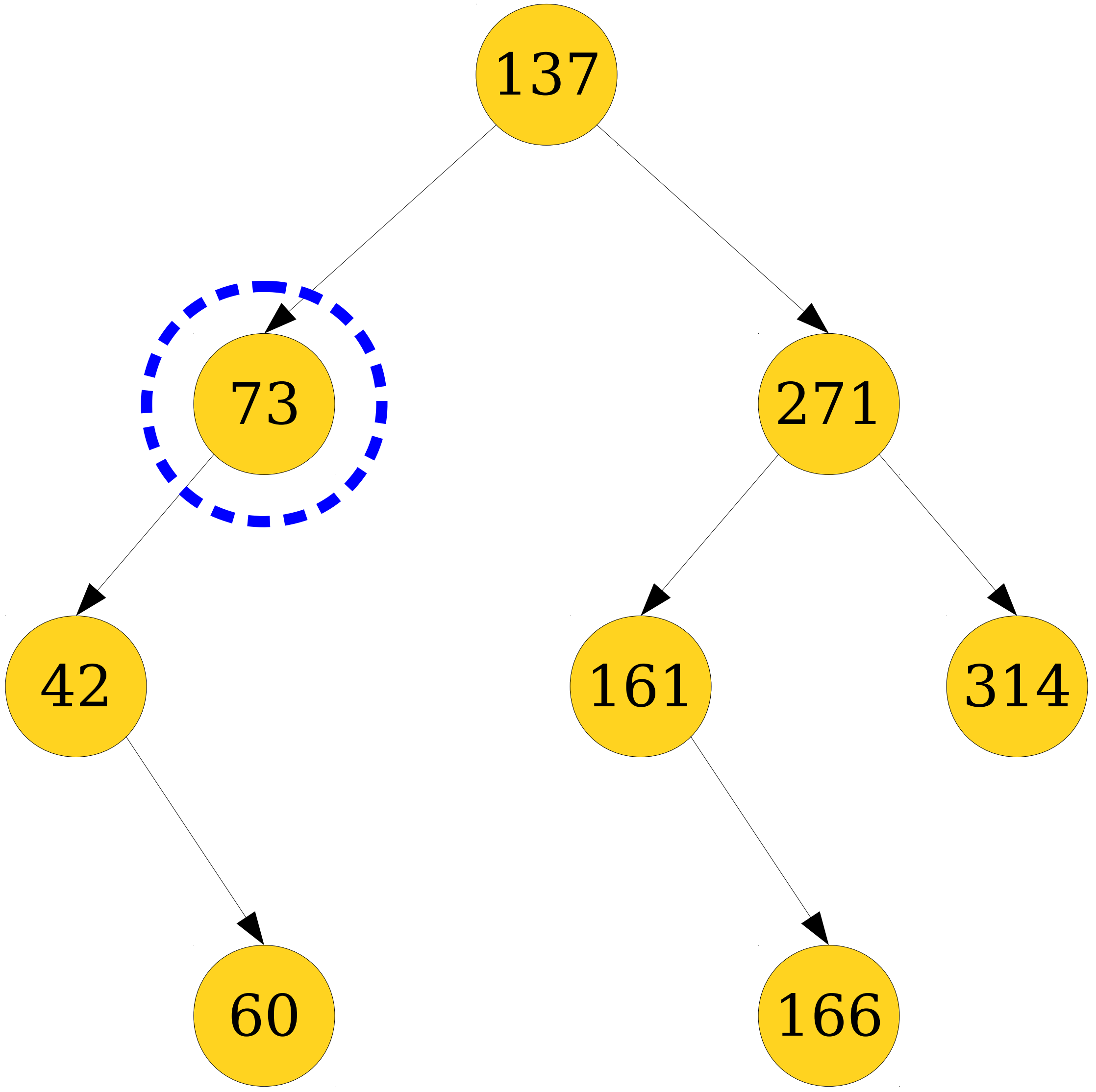
Deleting from a BST



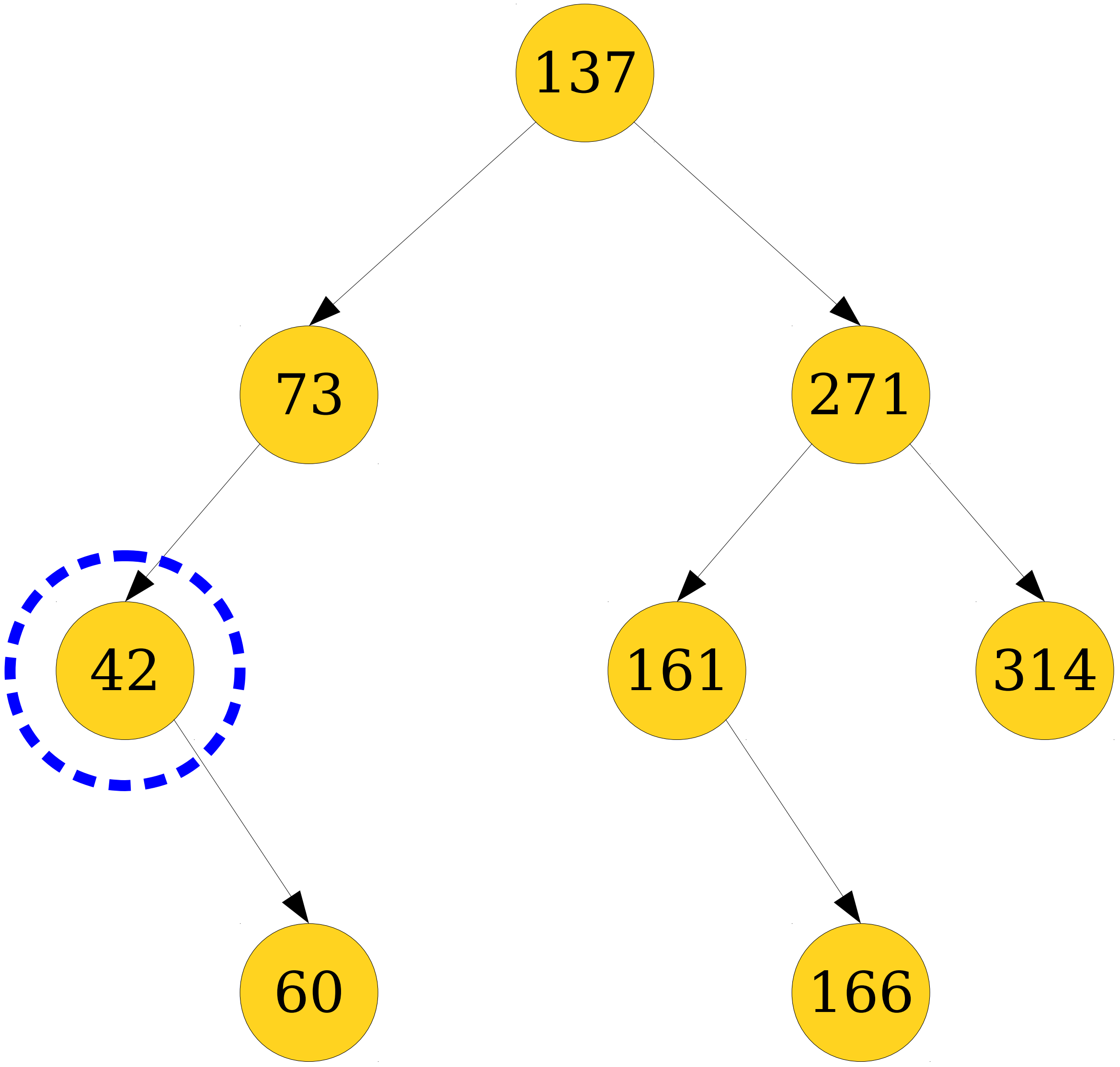
Deleting from a BST



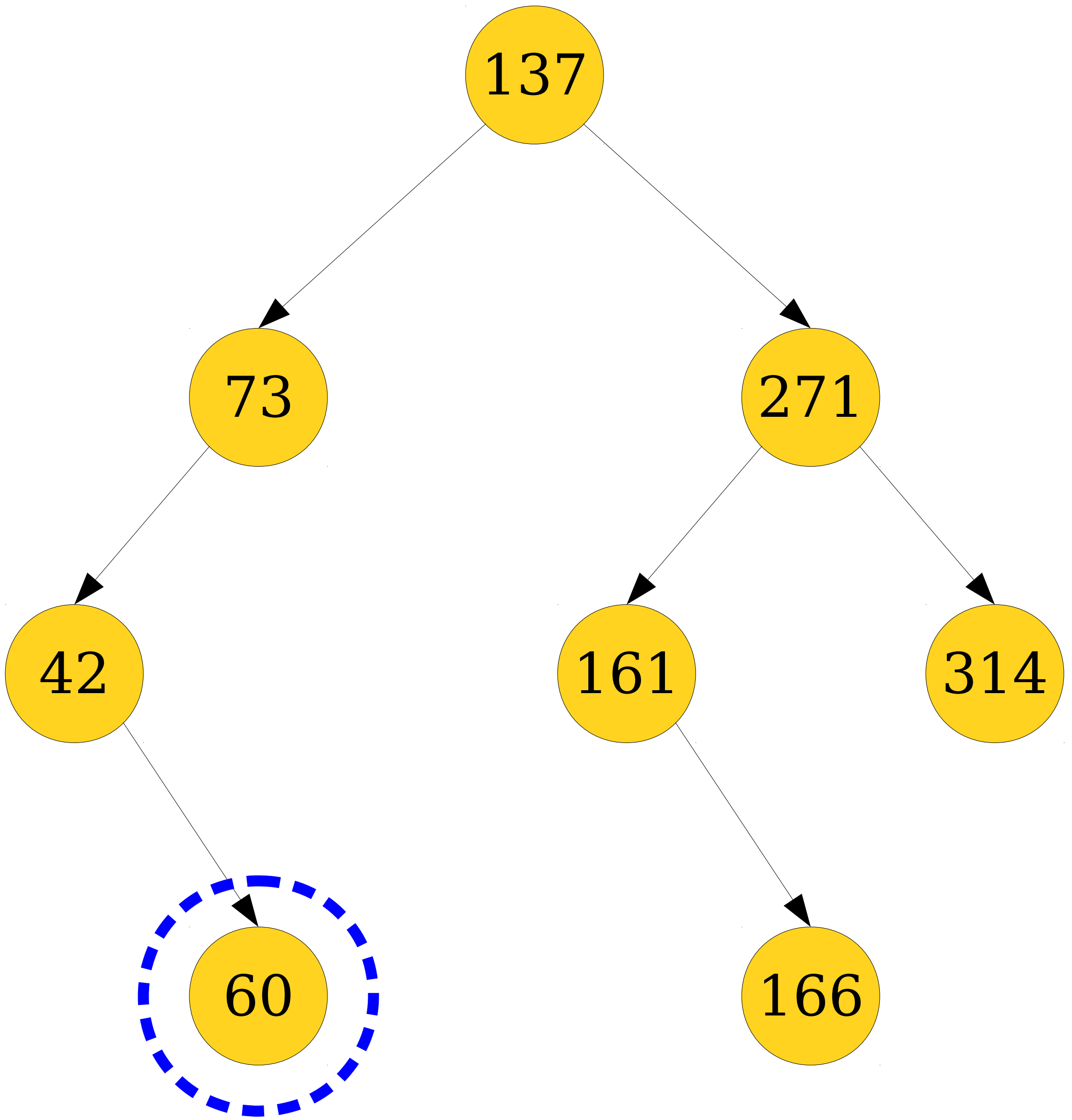
Deleting from a BST



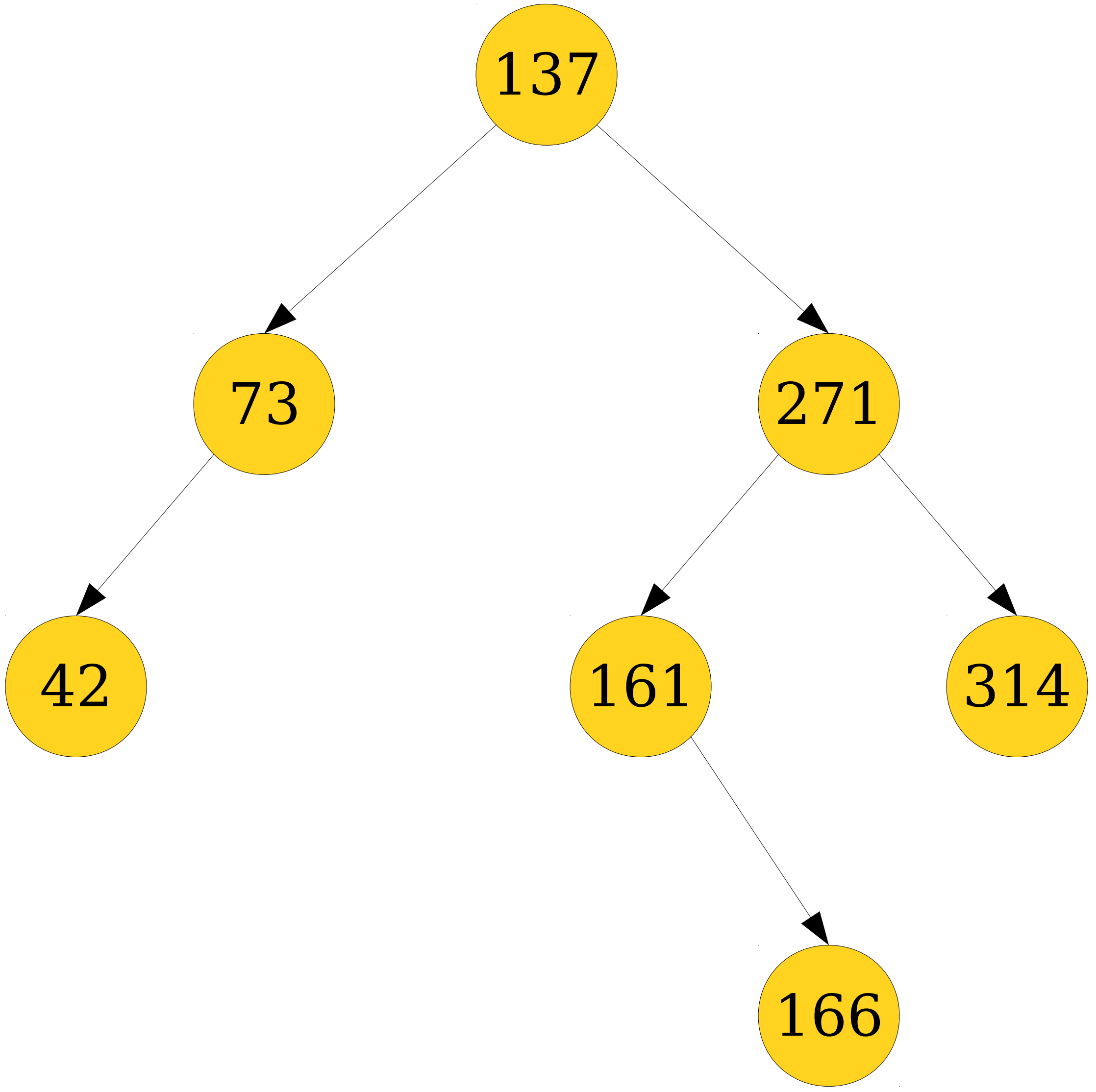
Deleting from a BST



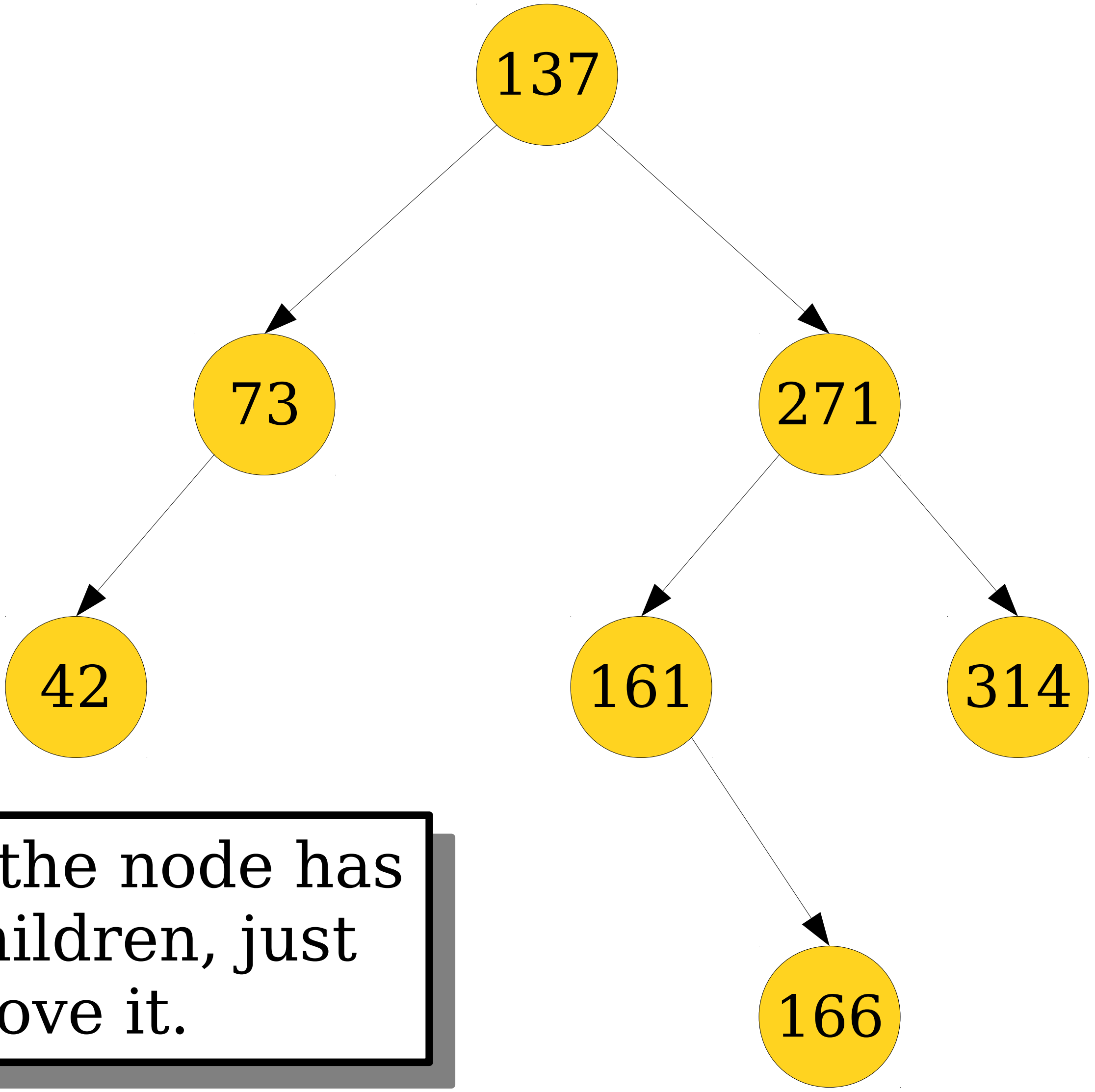
Deleting from a BST



Deleting from a BST

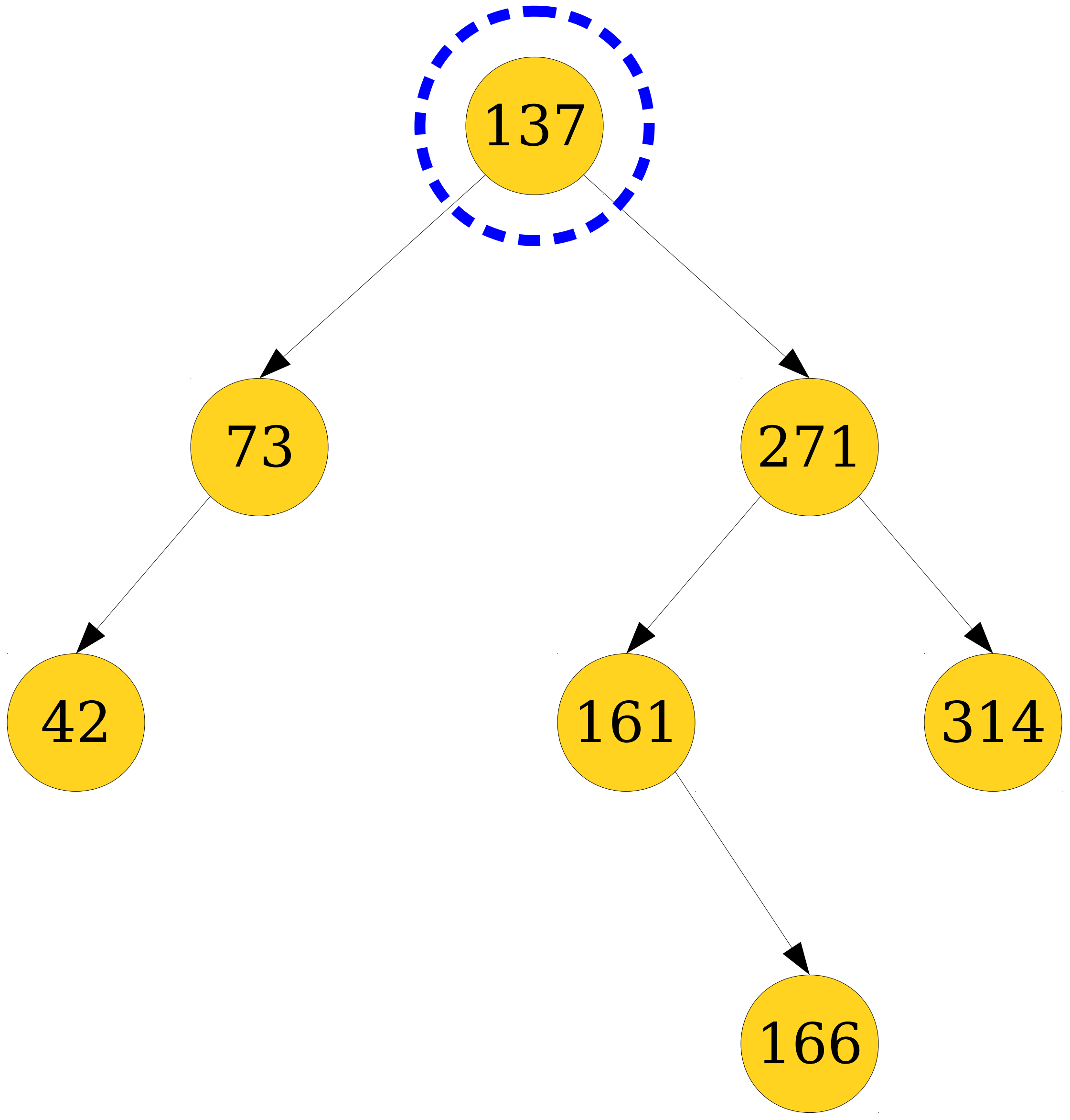


Deleting from a BST

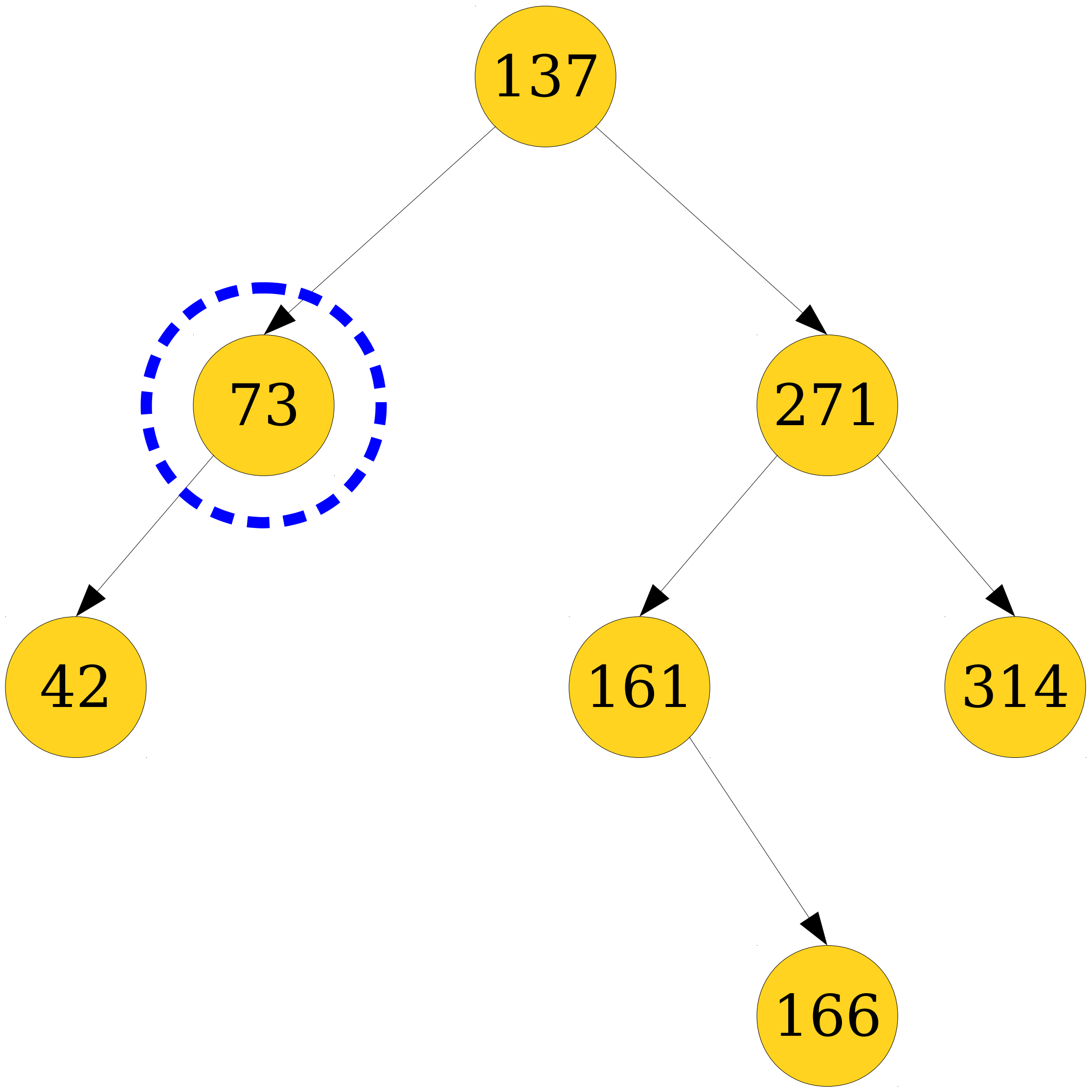


Case 1: If the node has just no children, just remove it.

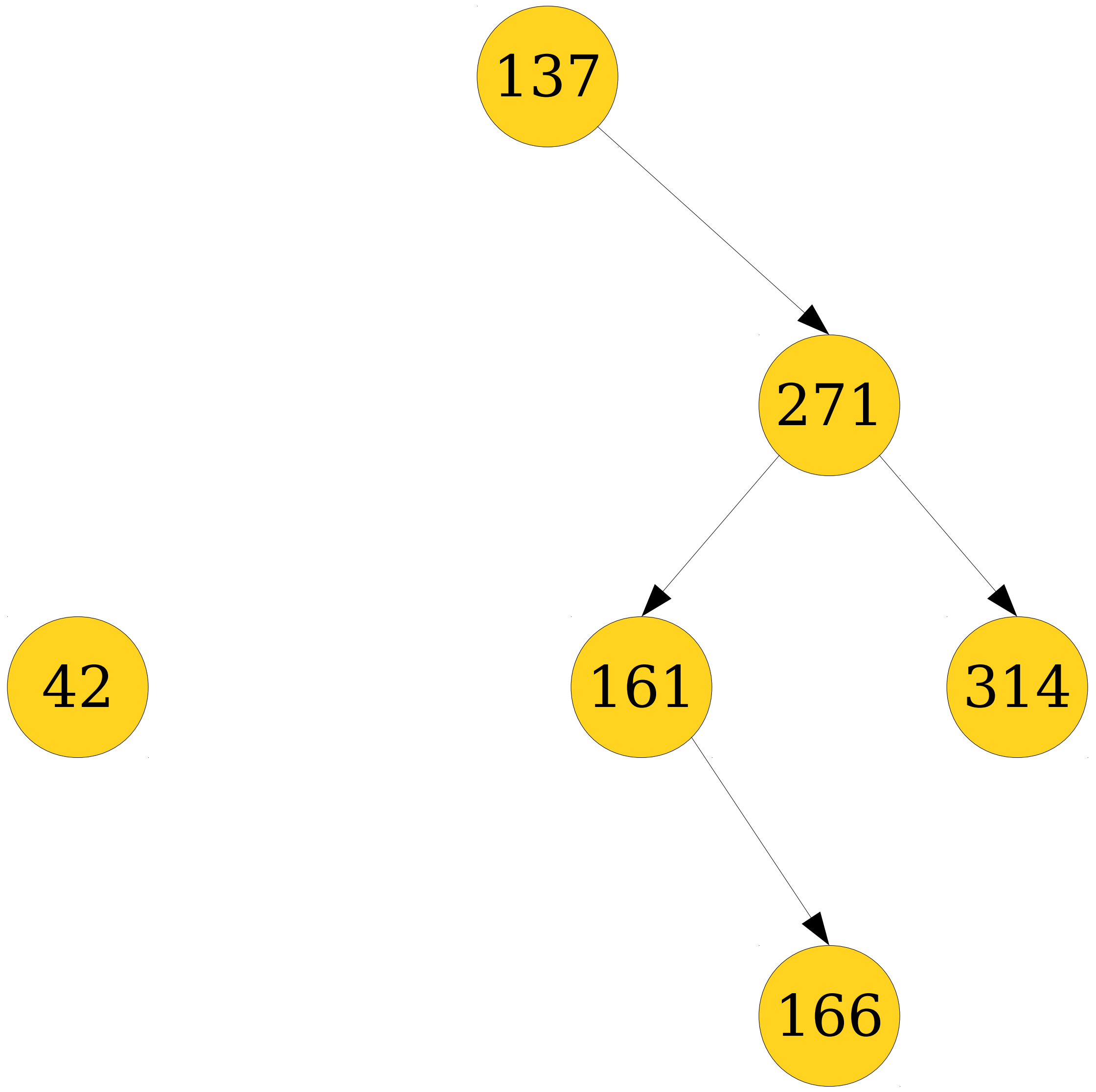
Deleting from a BST



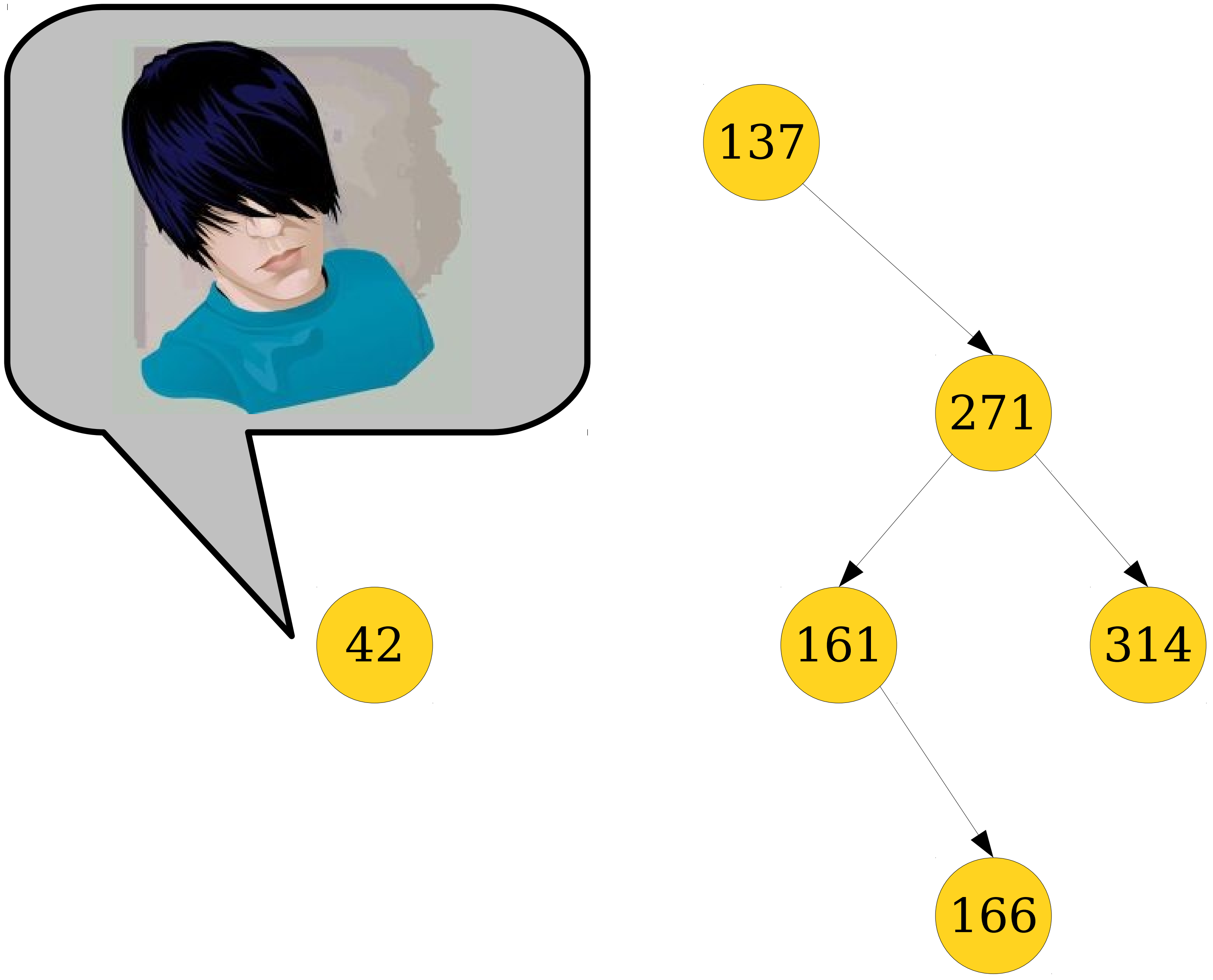
Deleting from a BST



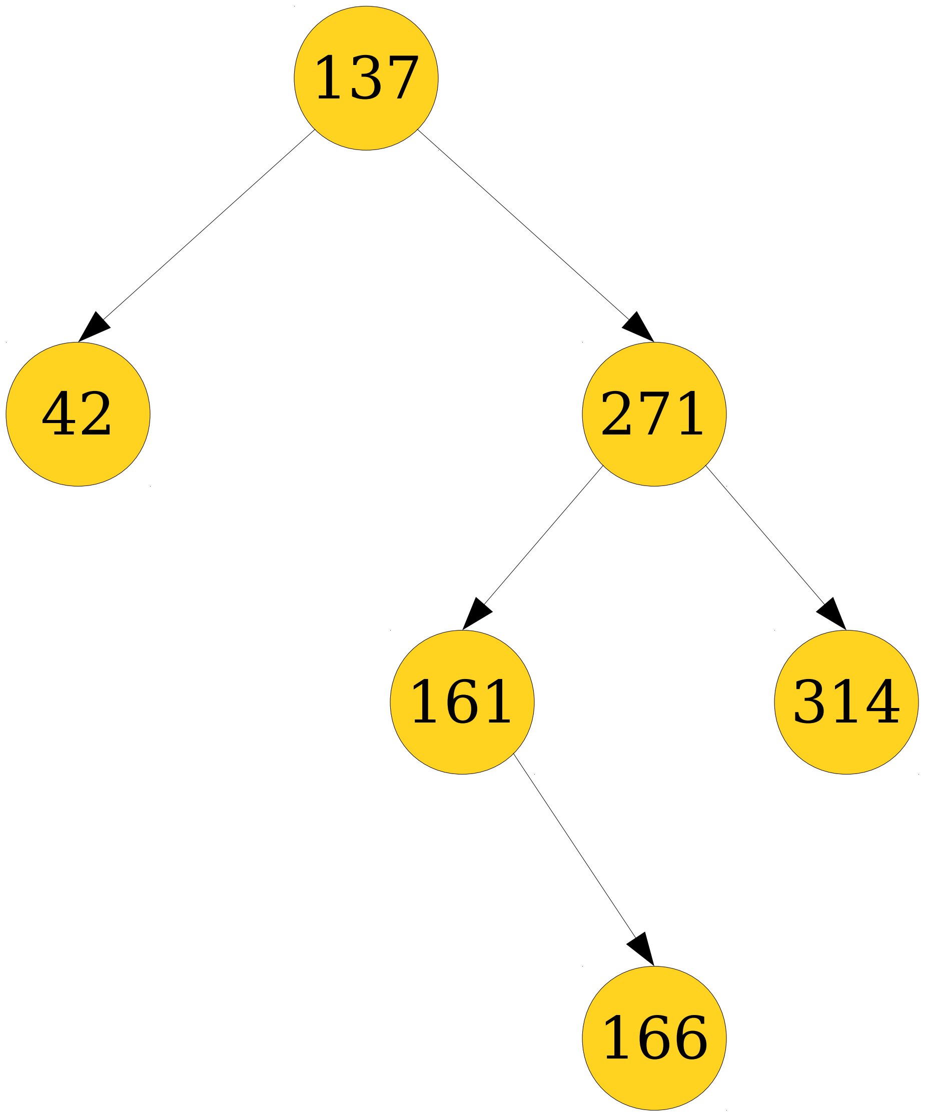
Deleting from a BST



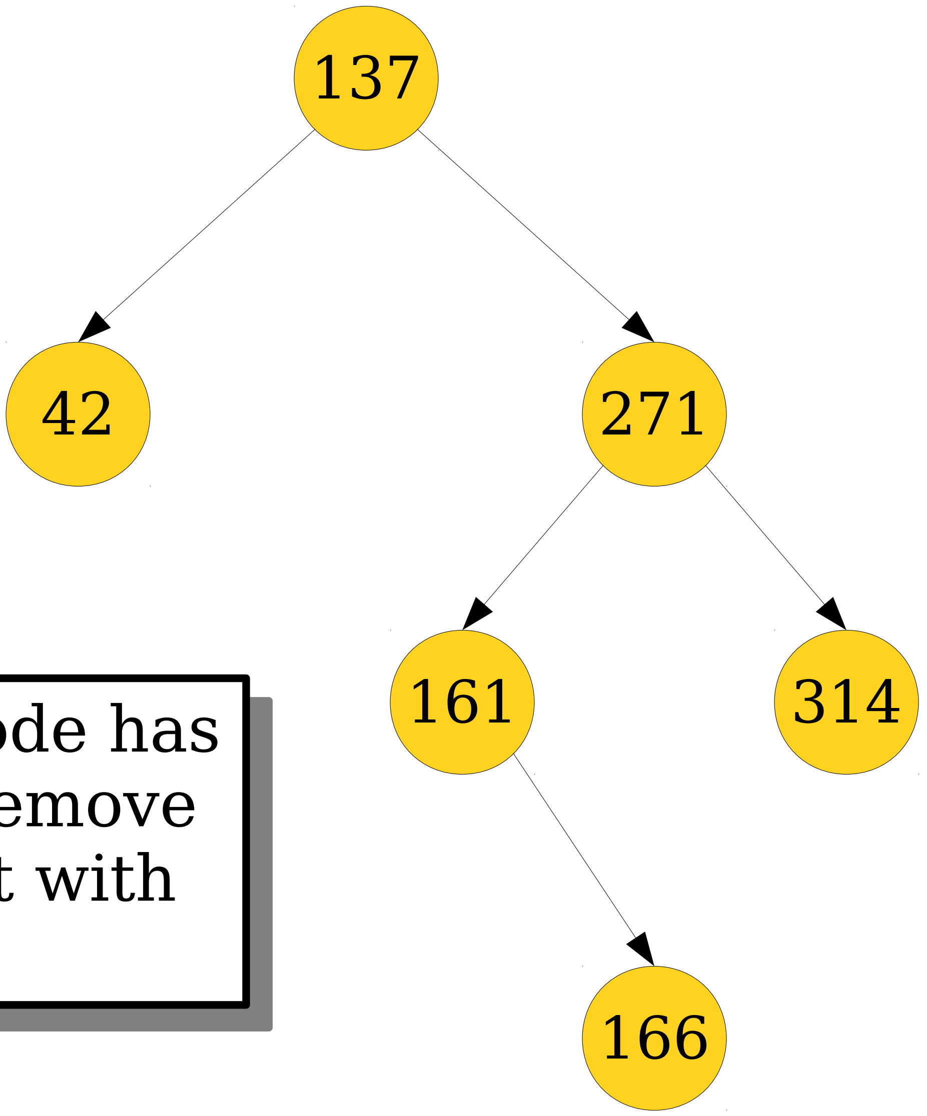
Deleting from a BST



Deleting from a BST

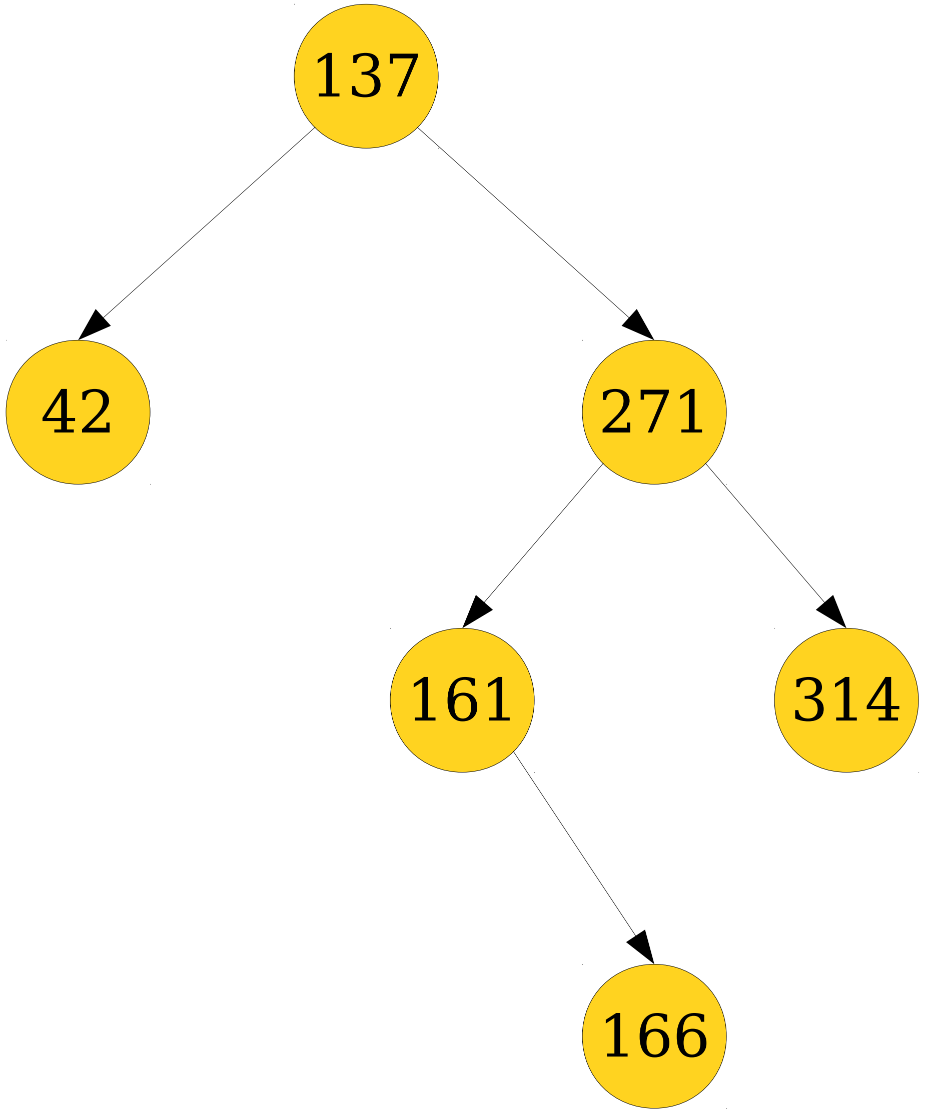


Deleting from a BST

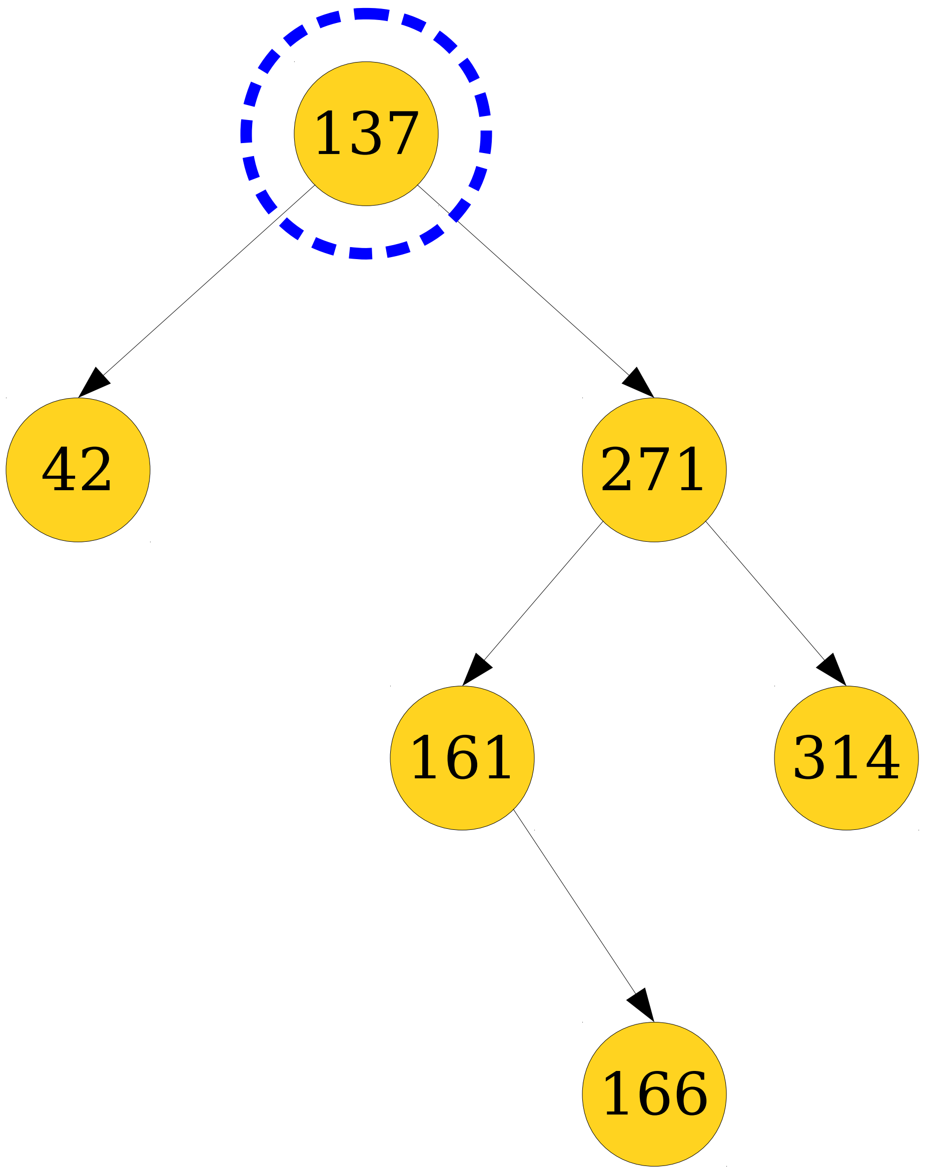


Case 2: If the node has just one child, remove it and replace it with its child.

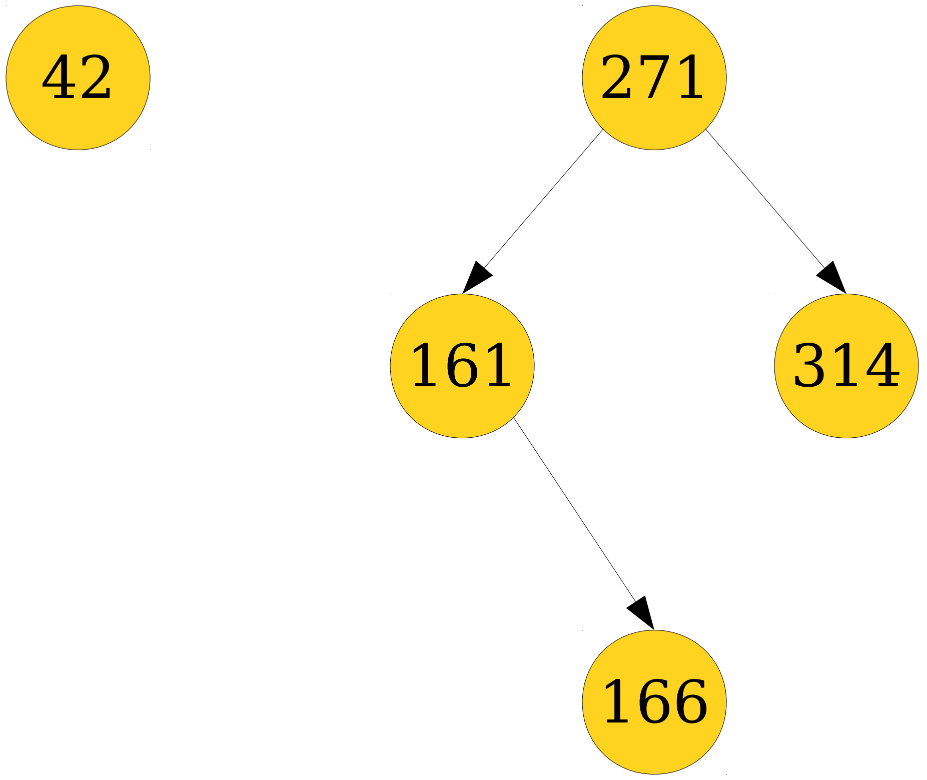
Deleting from a BST



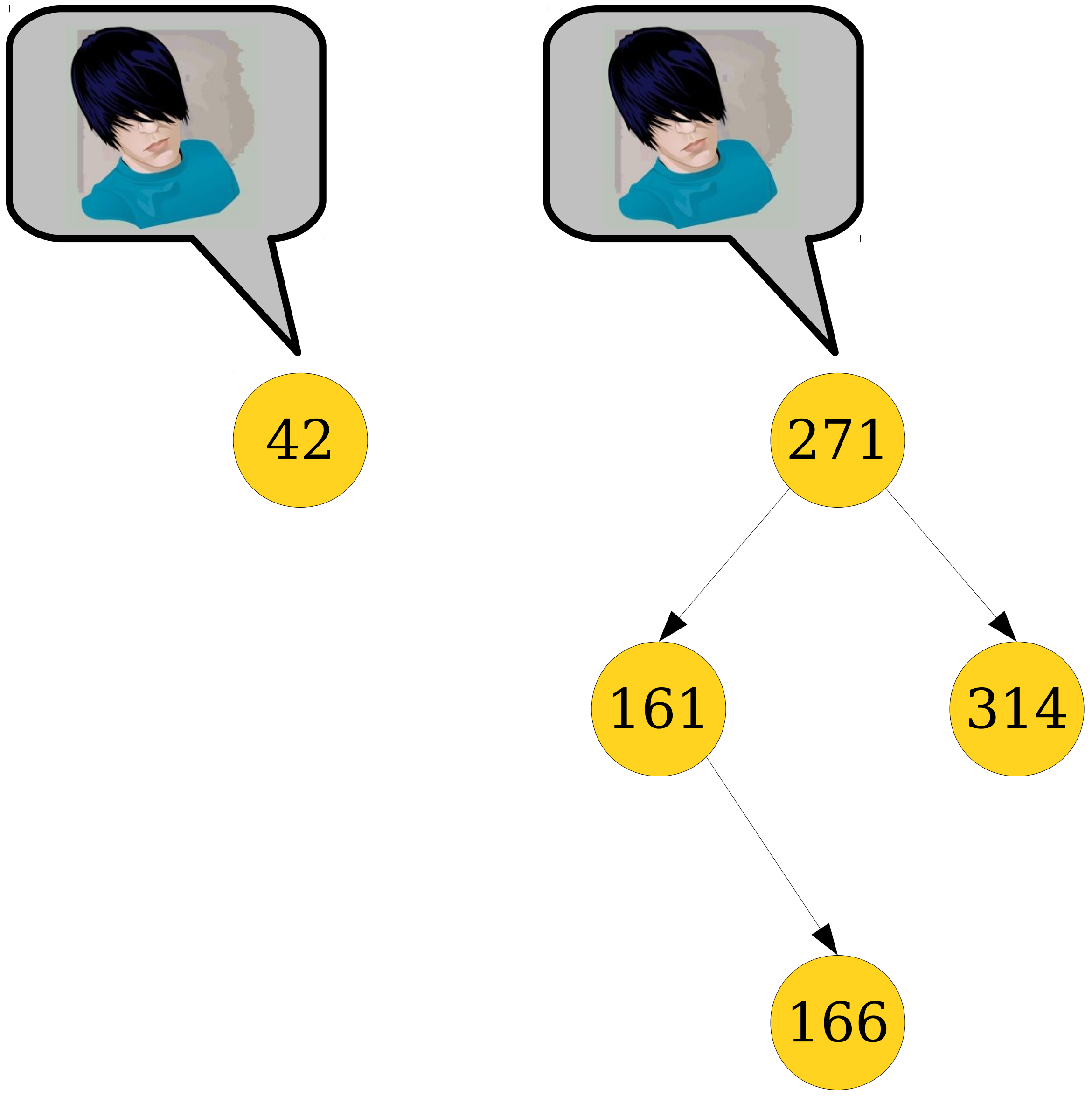
Deleting from a BST



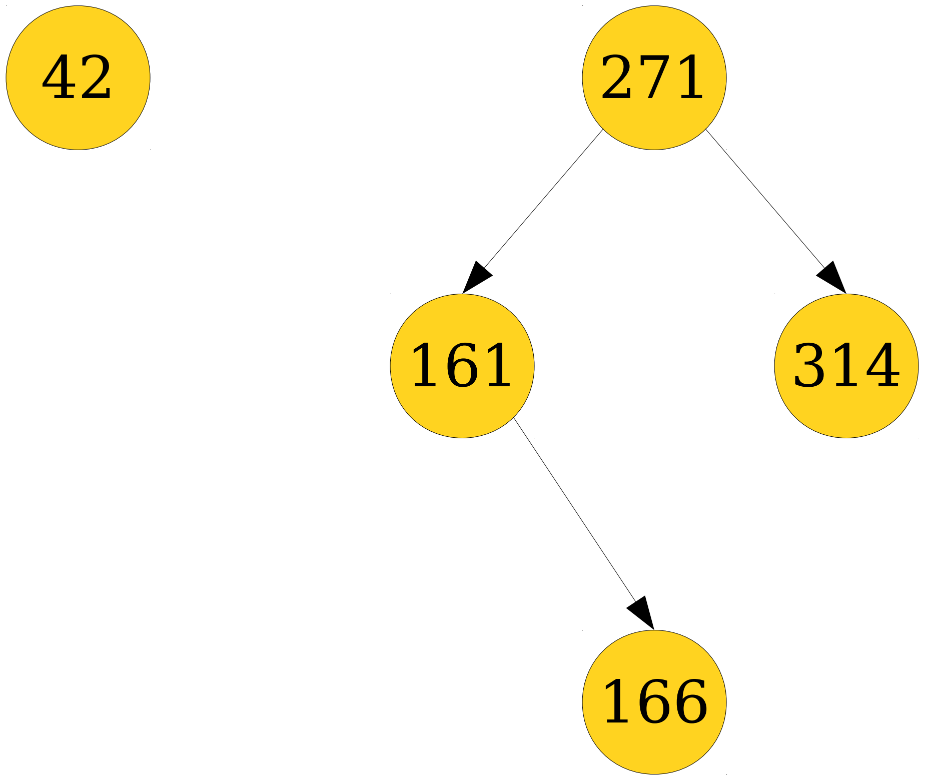
Deleting from a BST



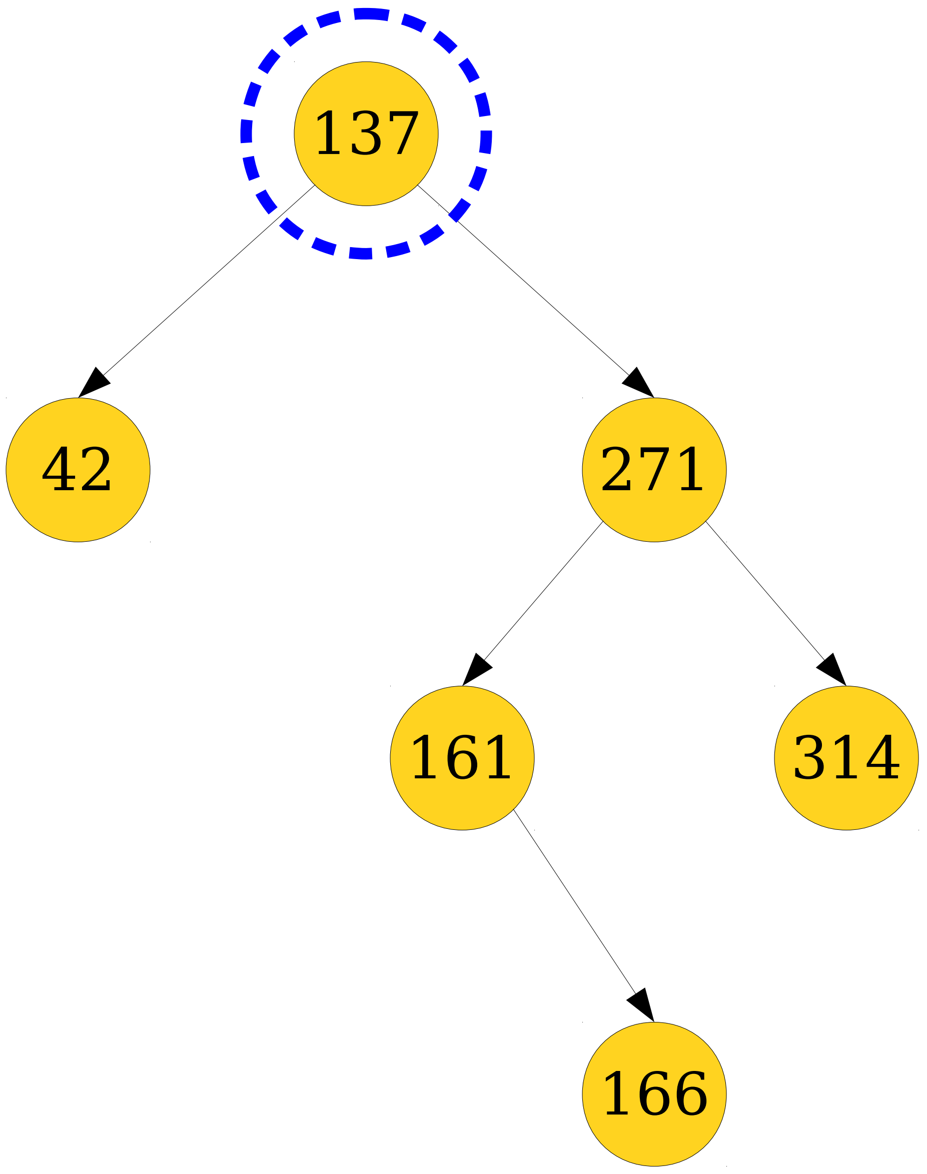
Deleting from a BST



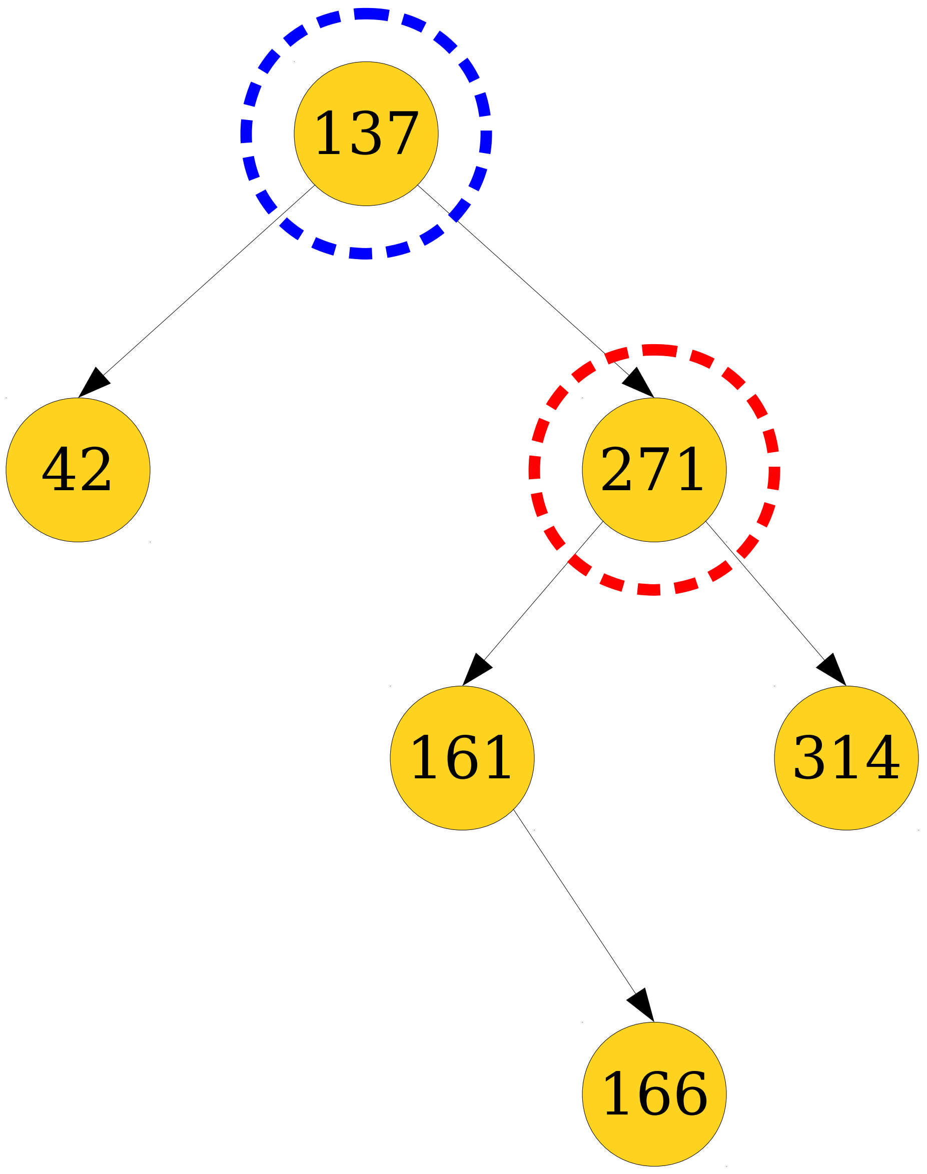
Deleting from a BST



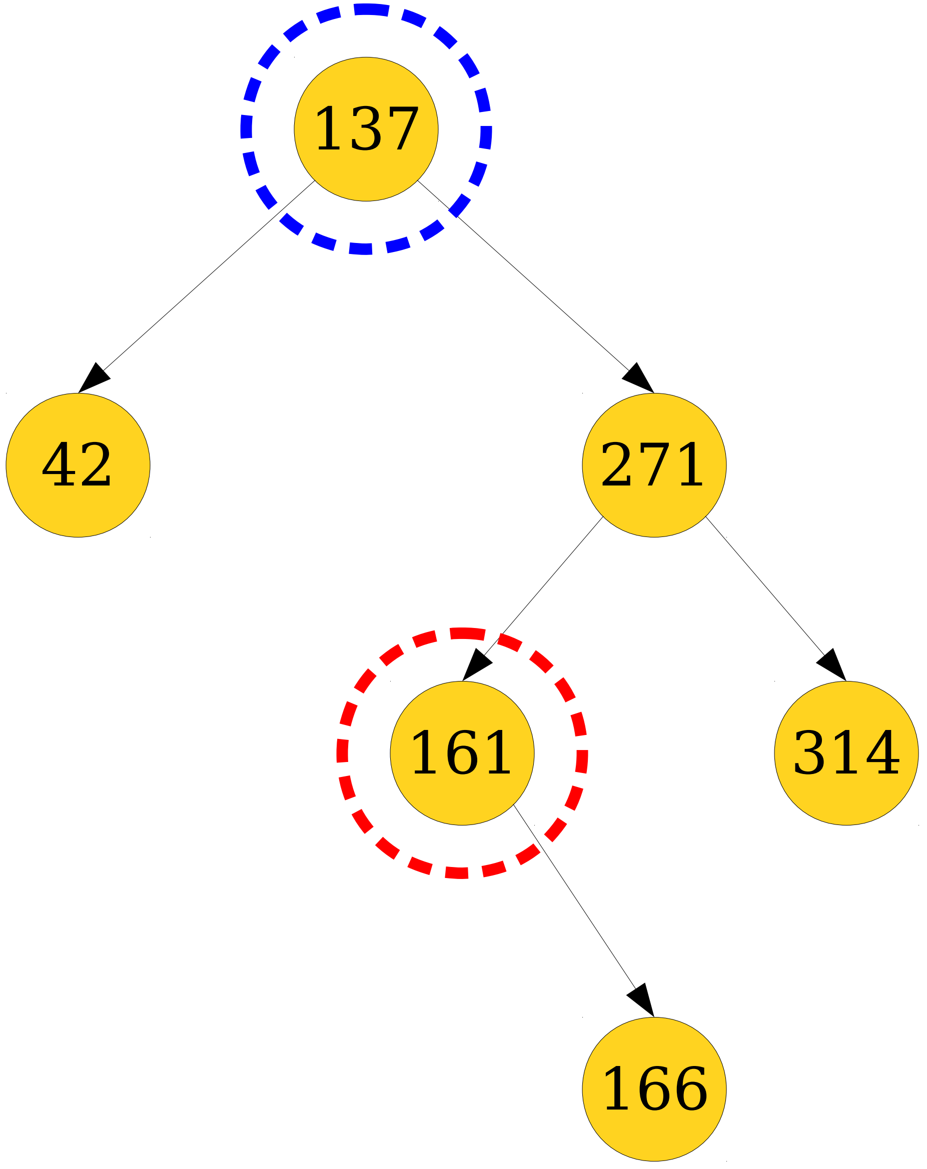
Deleting from a BST



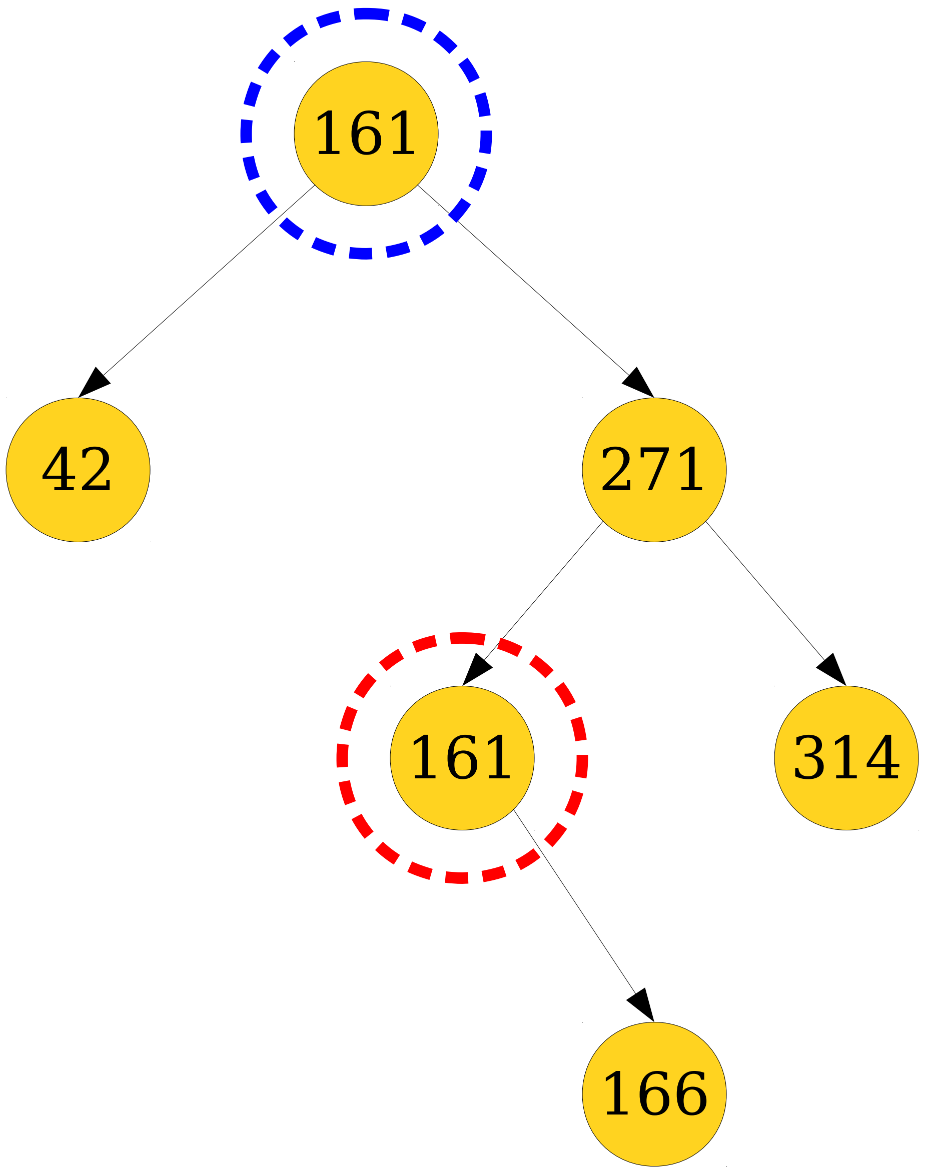
Deleting from a BST



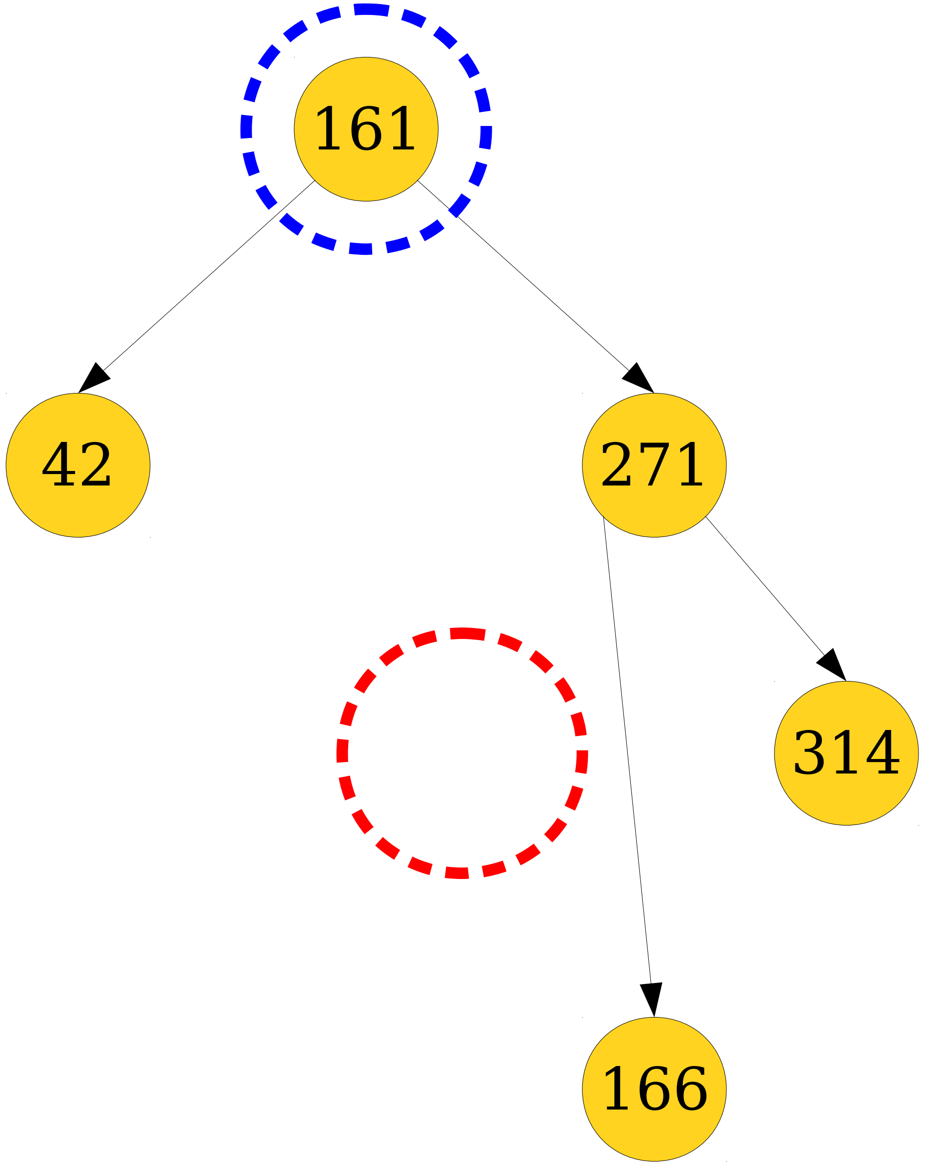
Deleting from a BST



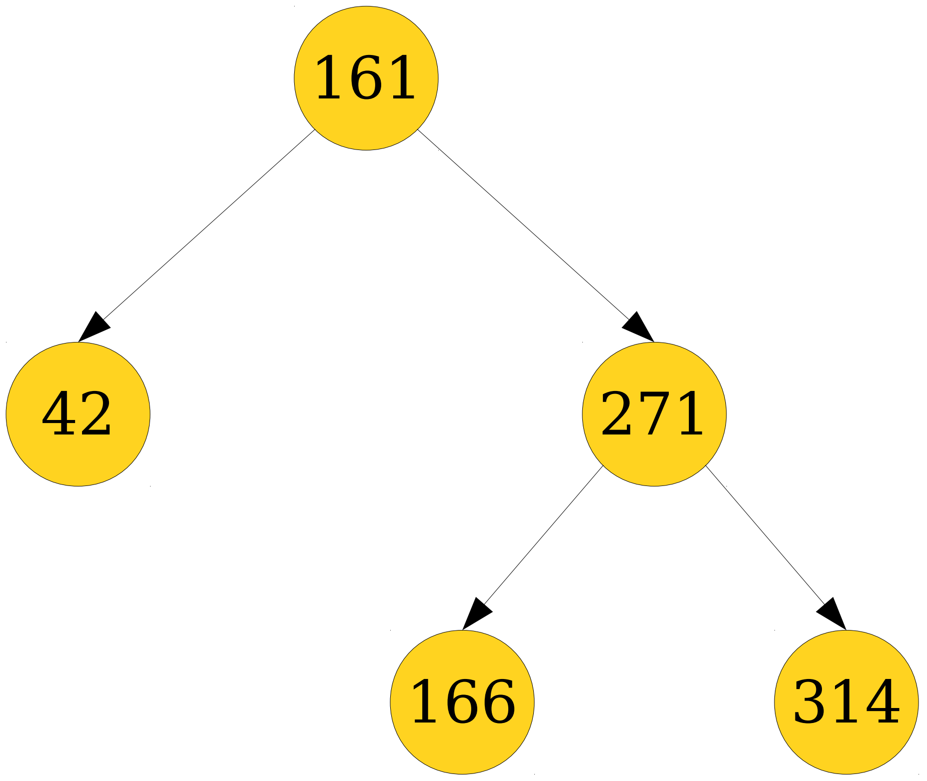
Deleting from a BST



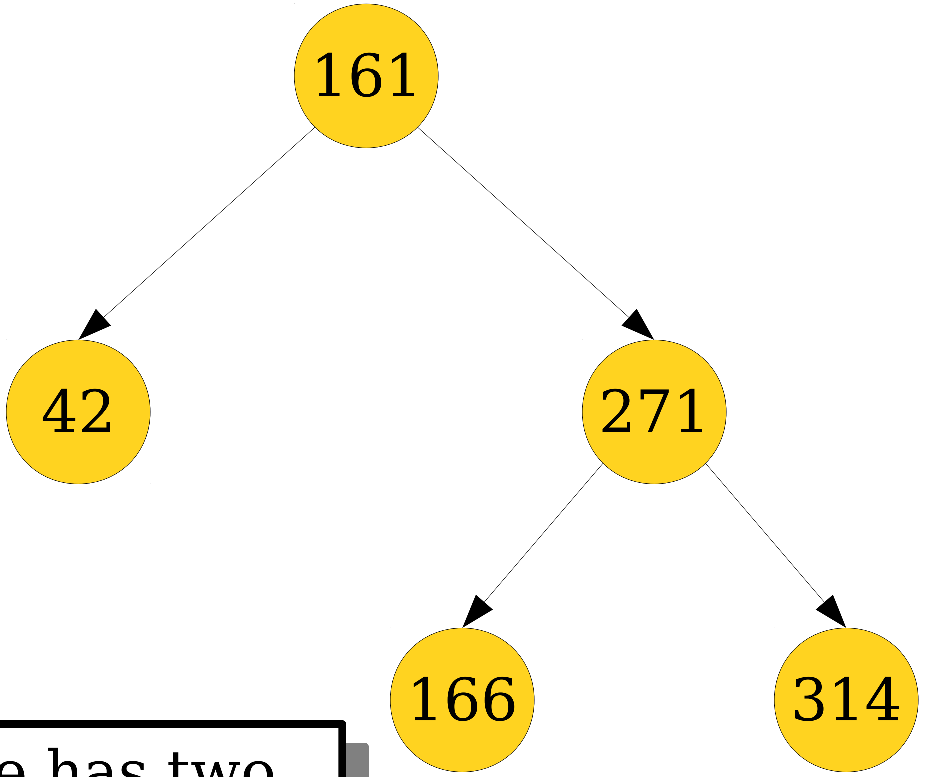
Deleting from a BST



Deleting from a BST



Deleting from a BST



Case 3: If the node has two children, find its inorder successor (which has zero or one child), replace the node's key with its successor's key, then delete its successor.

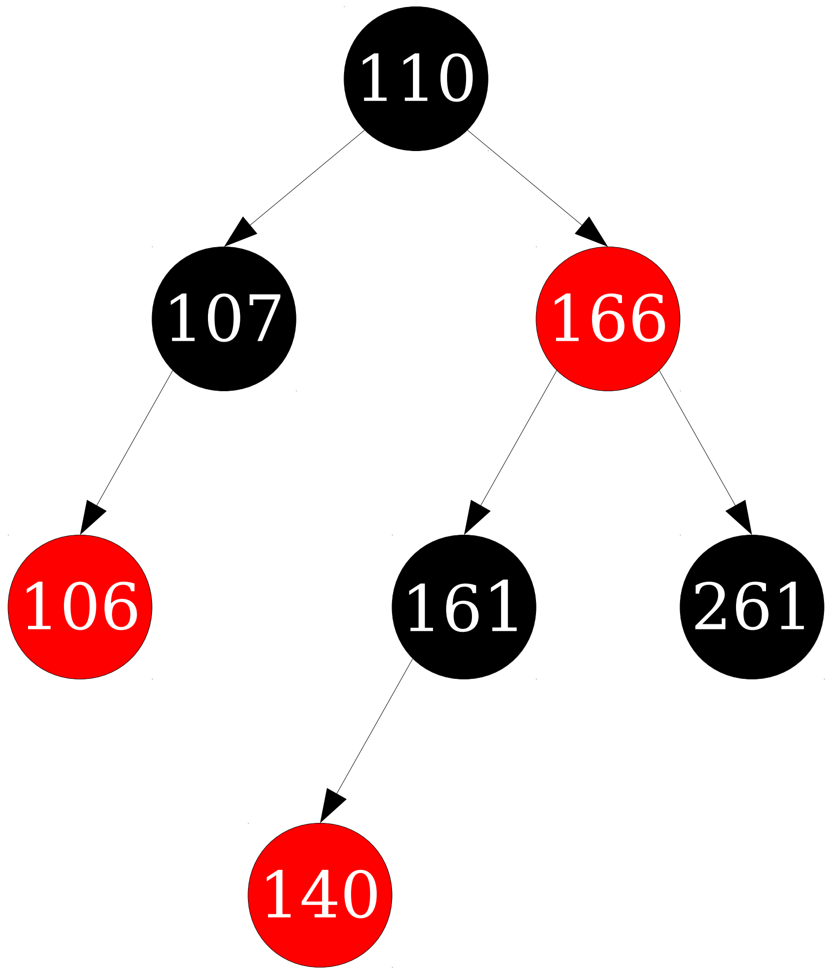
Runtime Analysis

- The time complexity of all these operations is $O(h)$, where h is the height of the tree.
  - That's the longest path we can take.
- In the best case, $h = O(\log n)$ and all operations take time $O(\log n)$.
- In the worst case, $h = \Theta(n)$ and some operations will take time $\Theta(n)$.
- ***Challenge:*** How do you efficiently keep the height of a tree low?

A Glimpse of Red/Black Trees

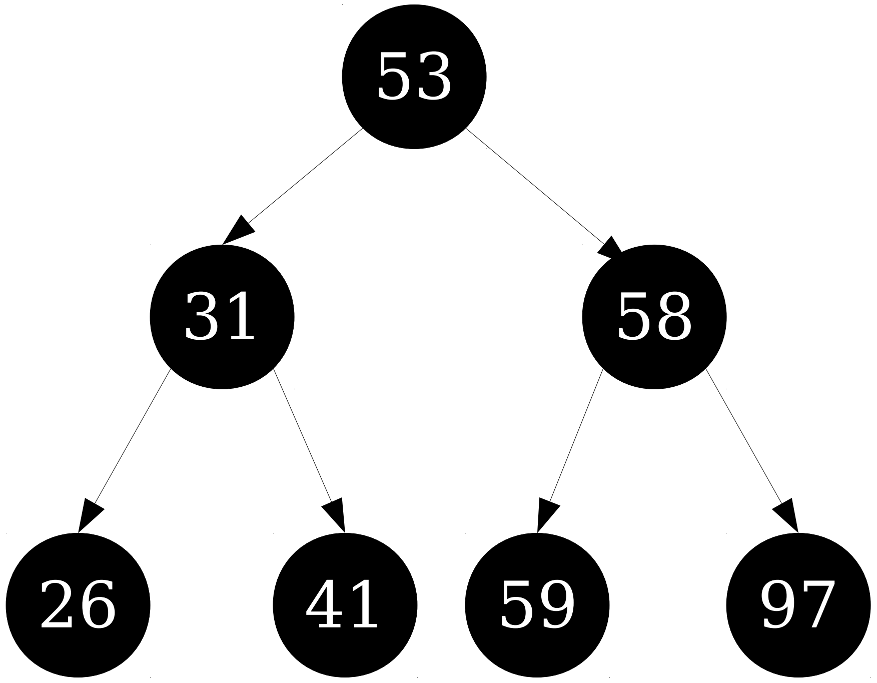
Red/Black Trees

- A ***red/black tree*** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.



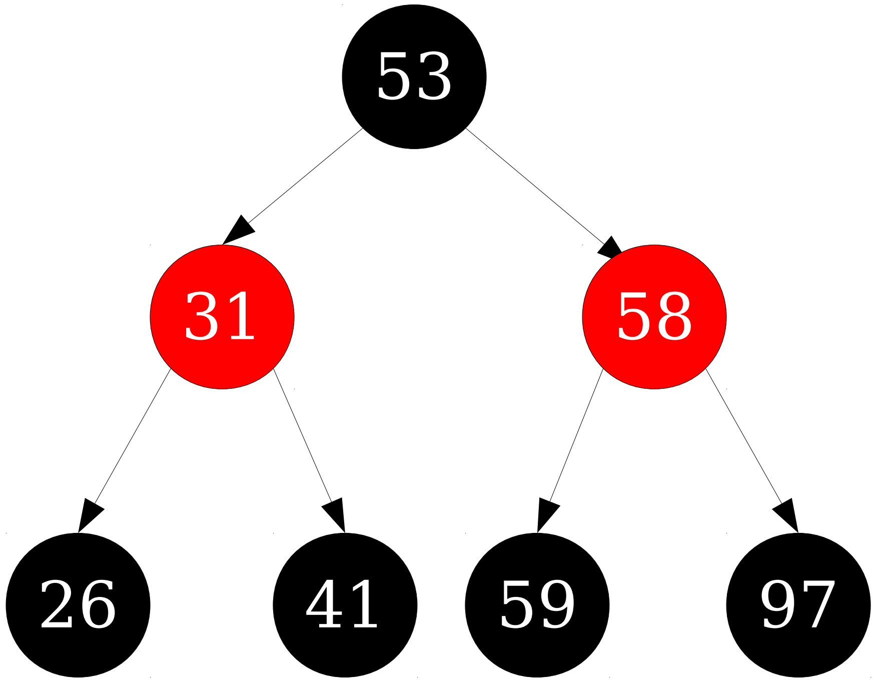
Red/Black Trees

- A ***red/black tree*** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.



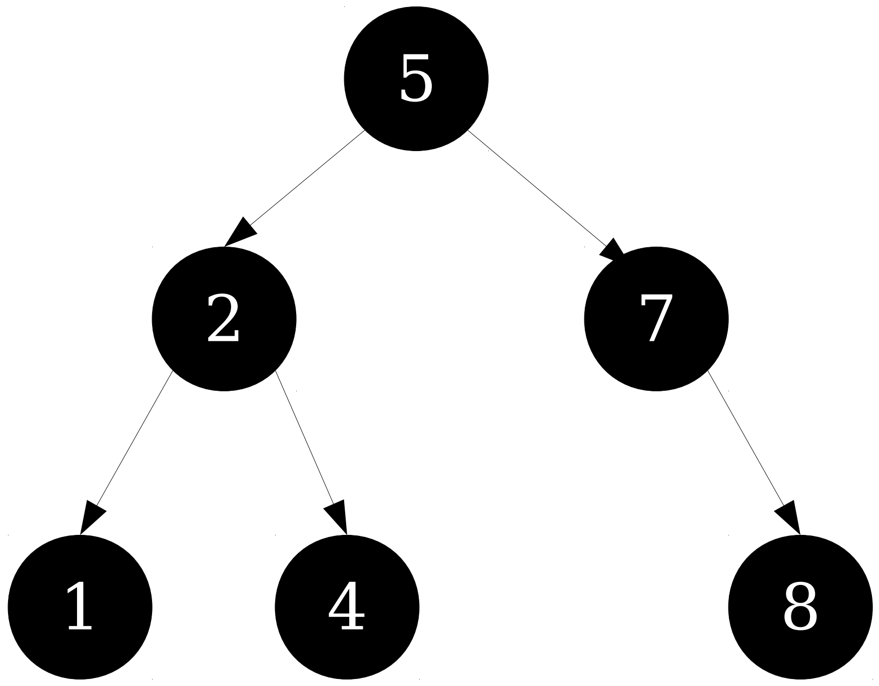
Red/Black Trees

- A ***red/black tree*** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.



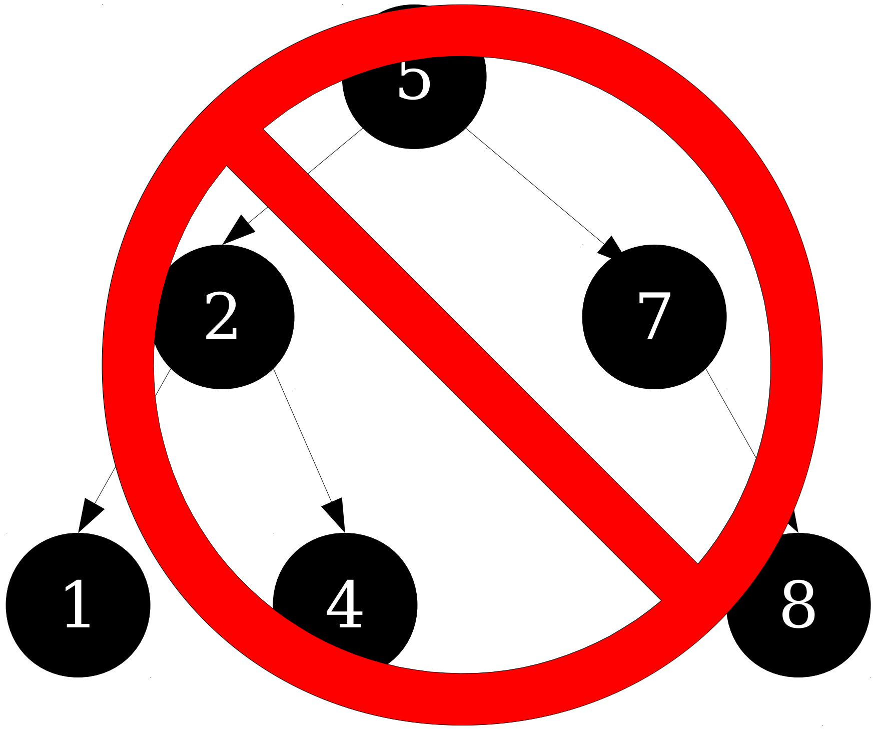
Red/Black Trees

- A ***red/black tree*** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.



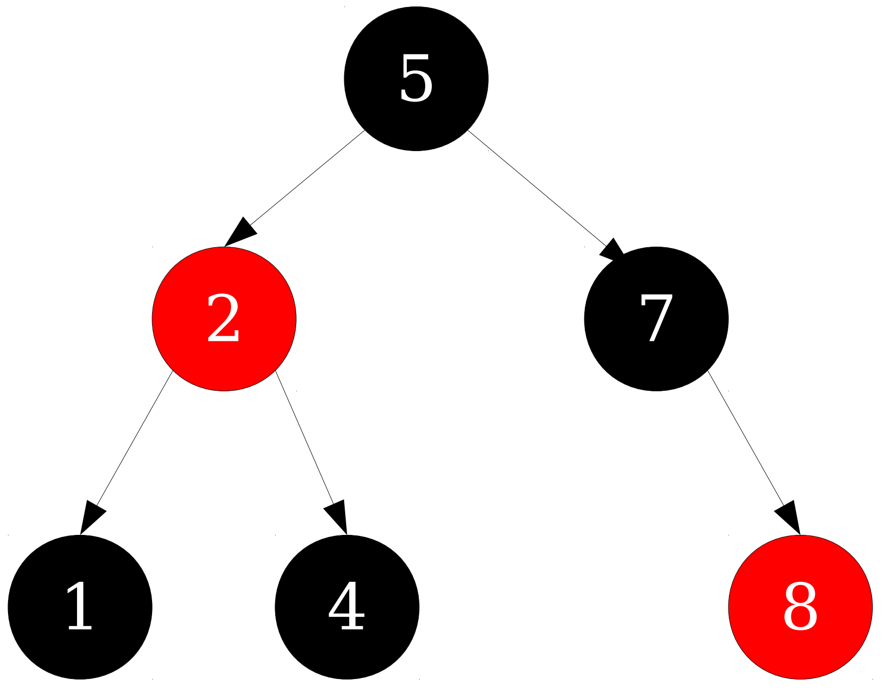
Red/Black Trees

- A ***red/black tree*** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.



Red/Black Trees

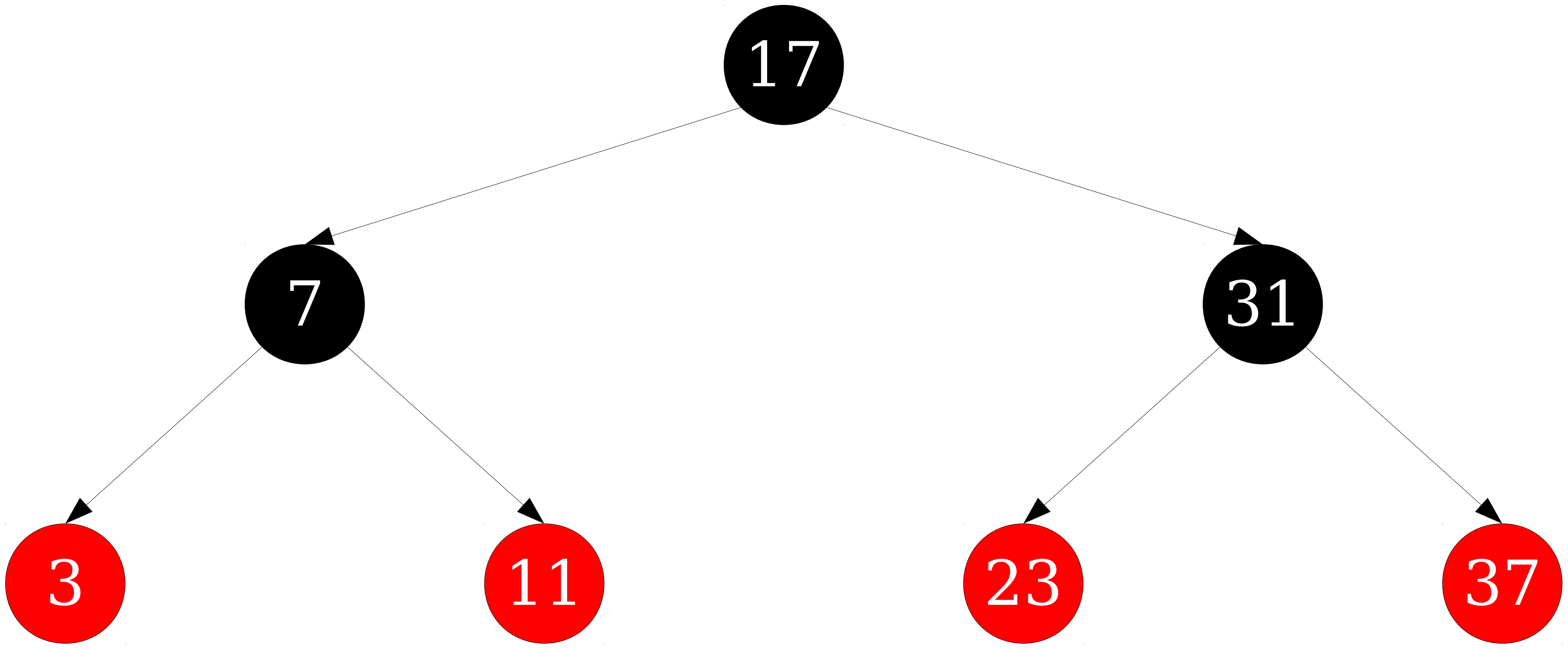
- A ***red/black tree*** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.



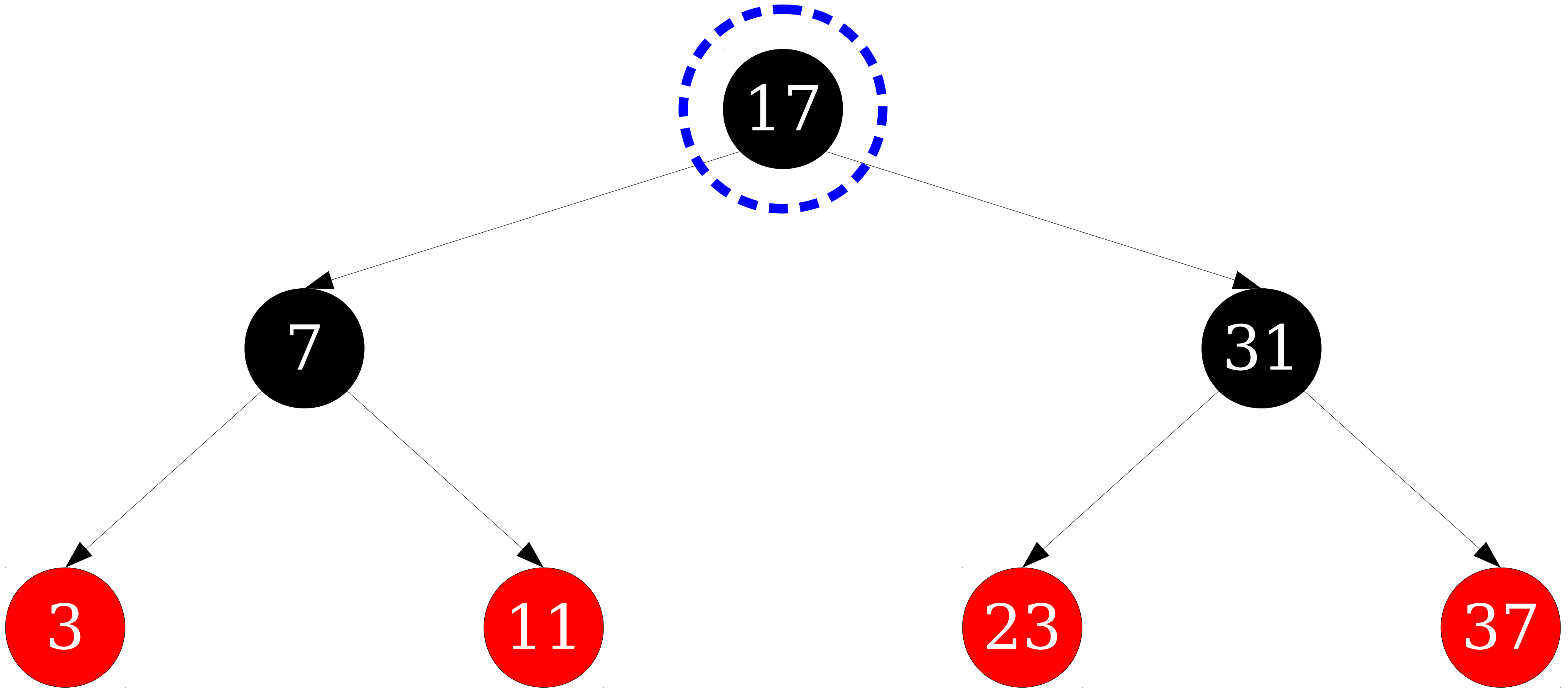
Red/Black Trees

- ***Theorem:*** Any red/black tree with n nodes has height $O(\log n)$.
  - We could prove this now, but there's a *much* simpler proof of this we'll see later on.
- Given a fixed red/black tree, lookups can be done in time $O(\log n)$.

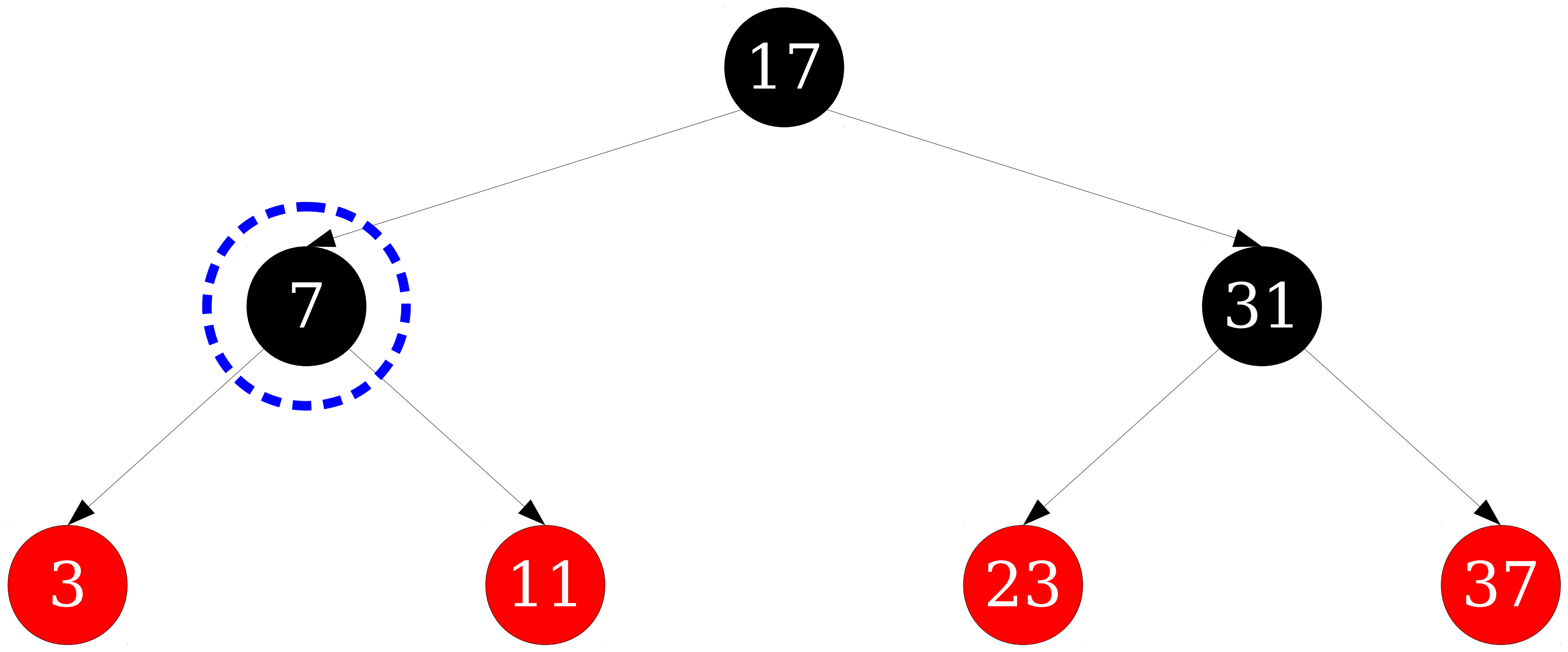
Mutating Red/Black Trees



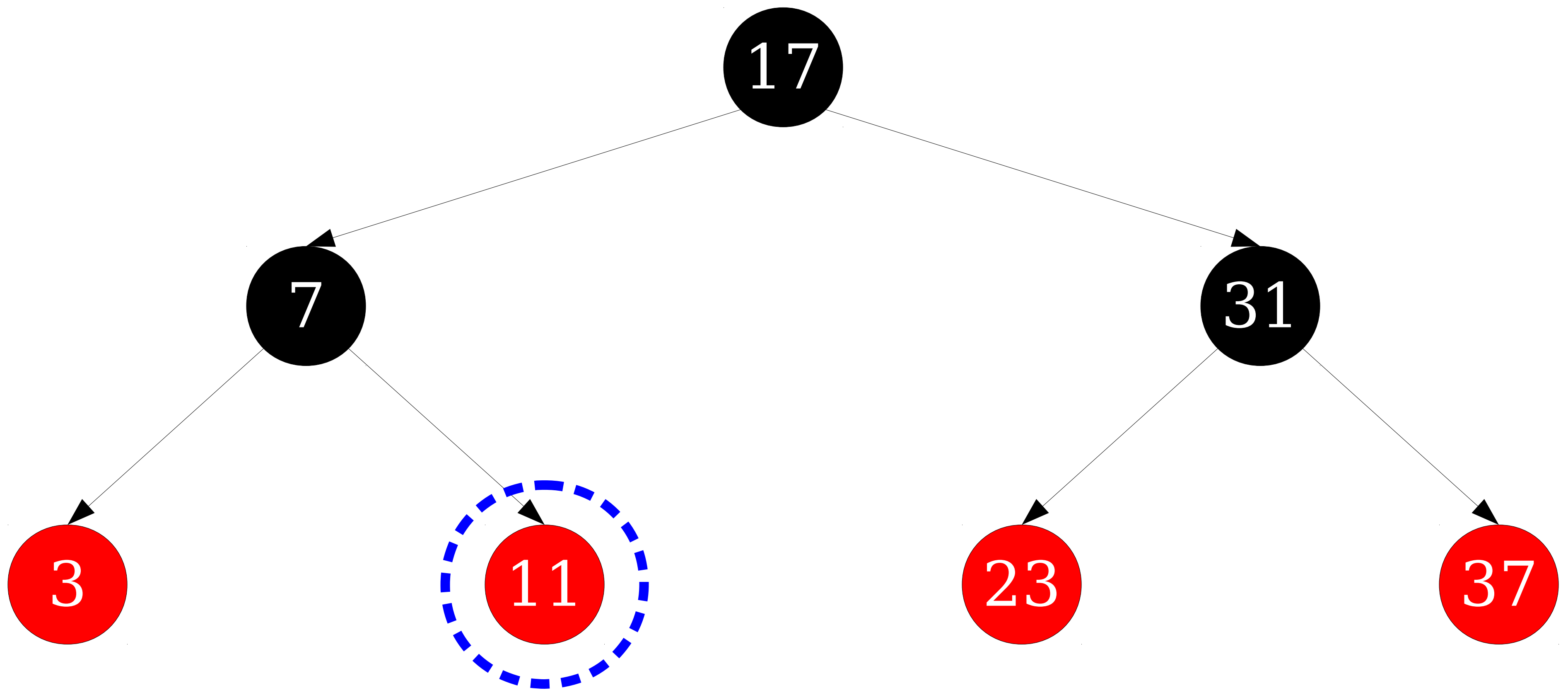
Mutating Red/Black Trees



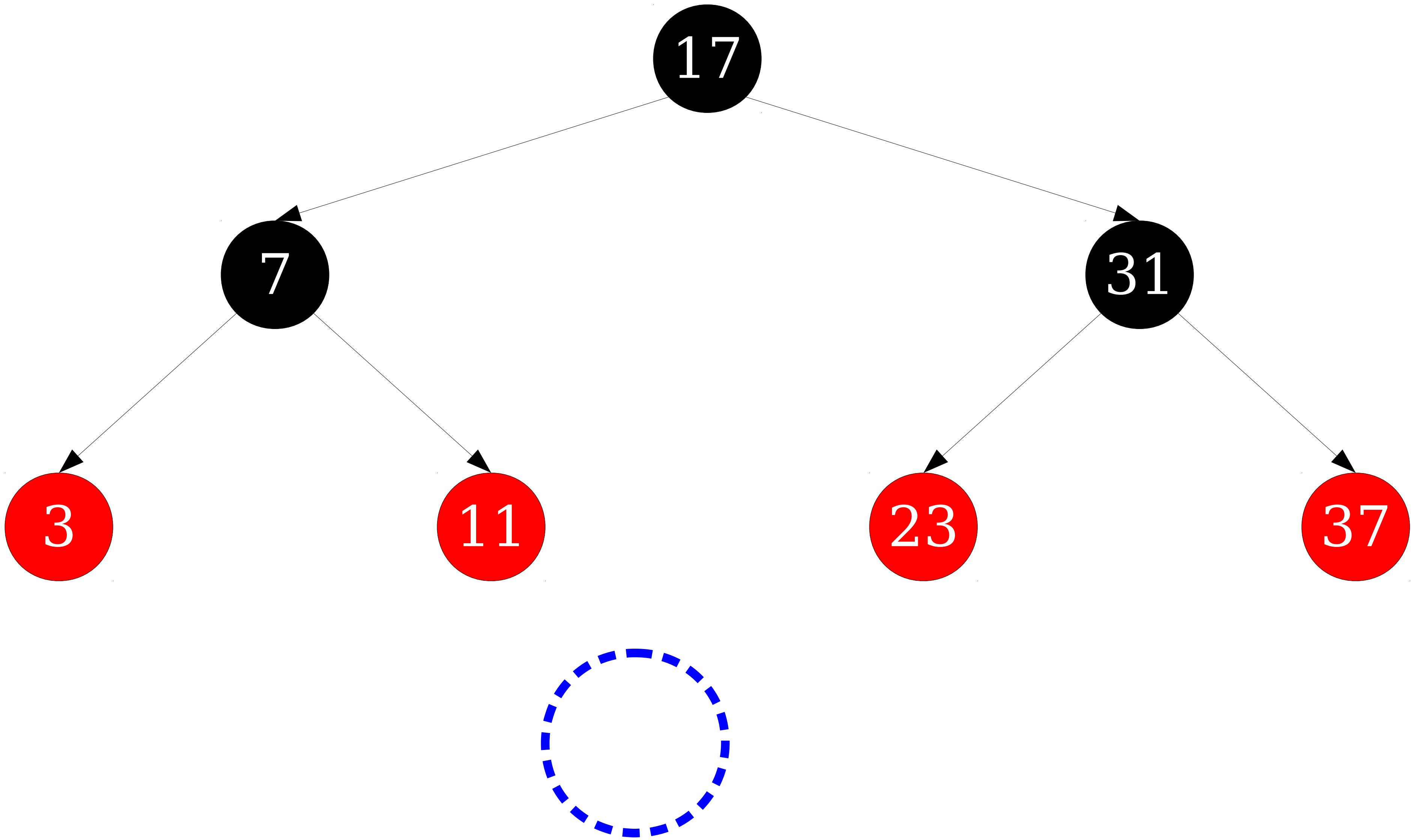
Mutating Red/Black Trees



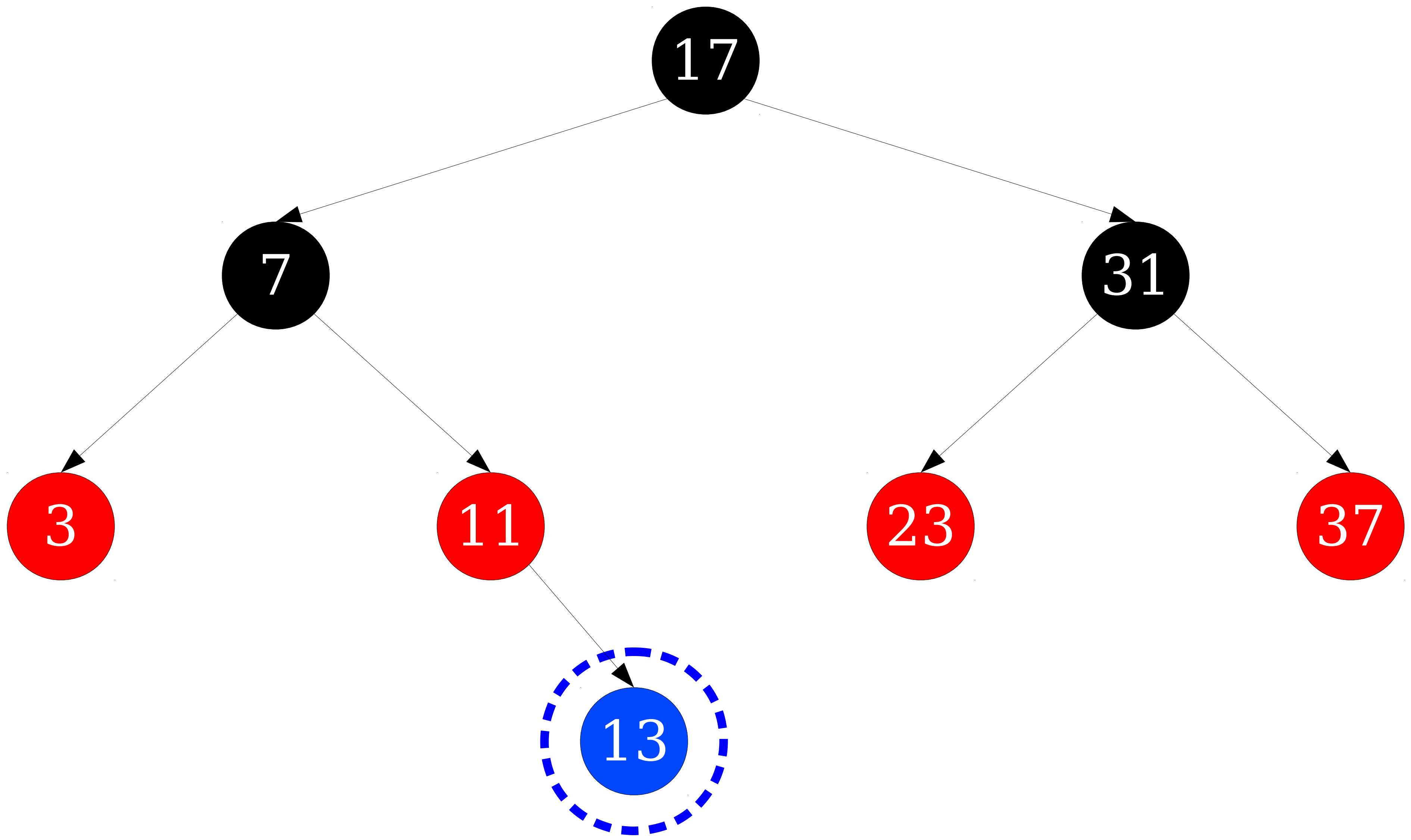
Mutating Red/Black Trees



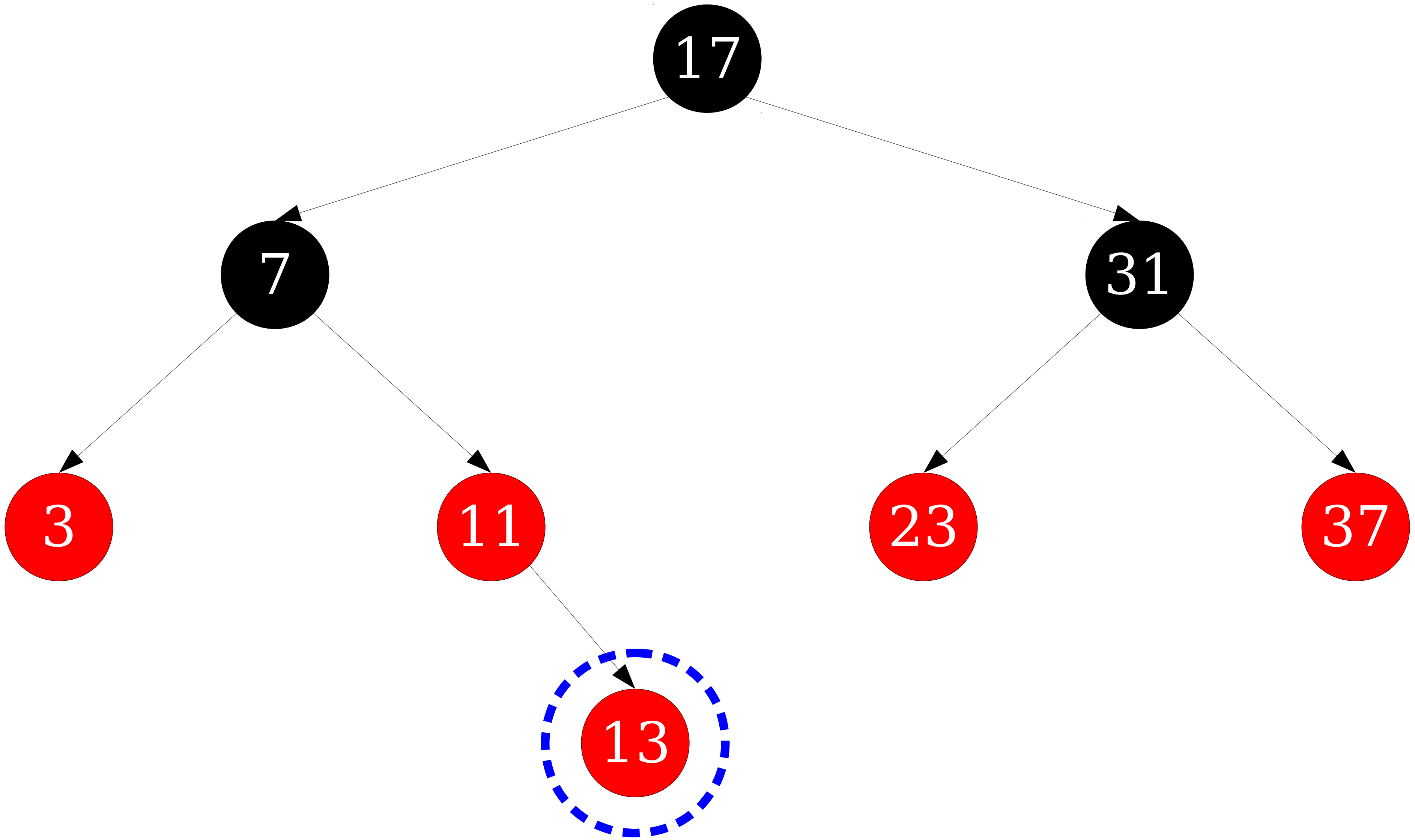
Mutating Red/Black Trees



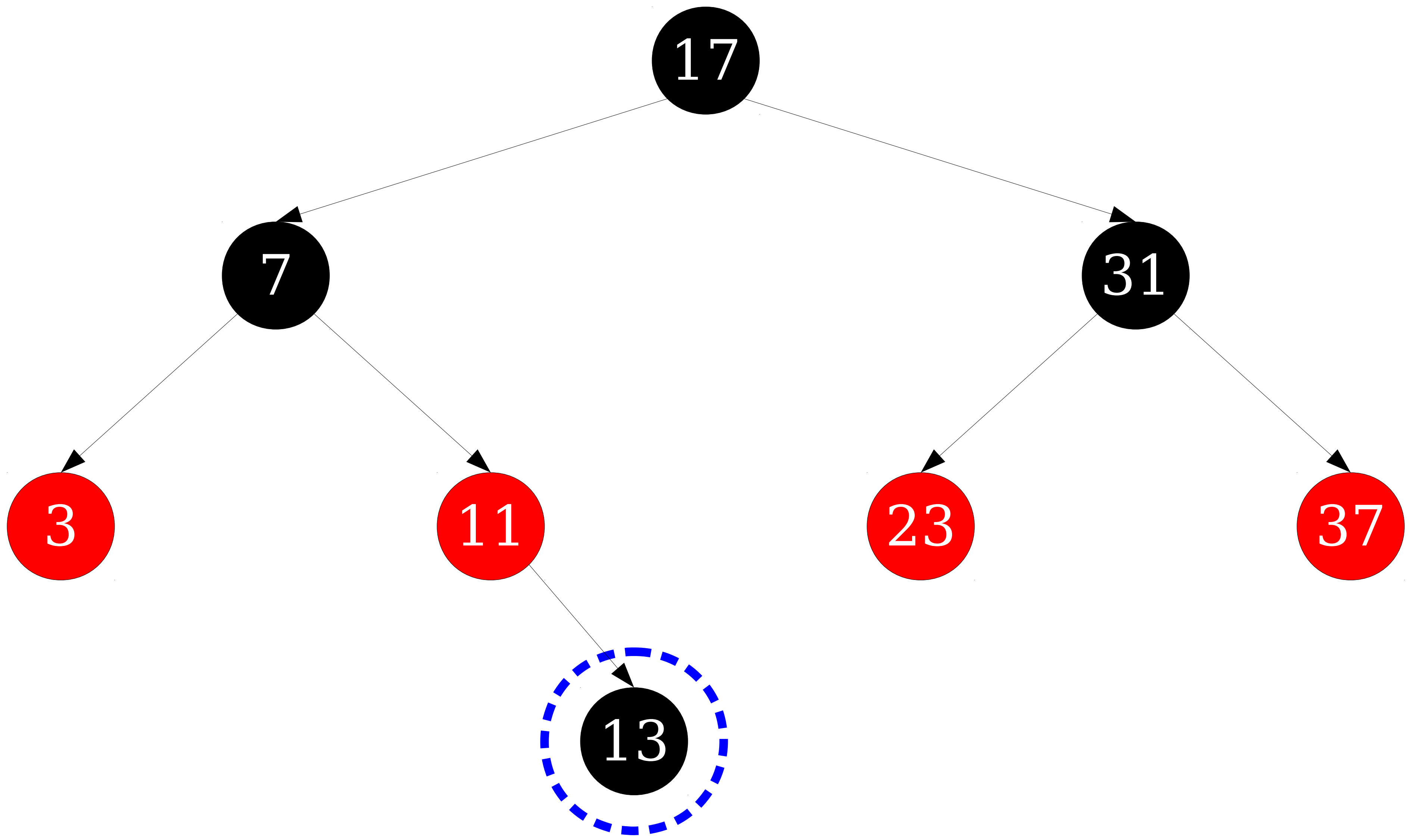
Mutating Red/Black Trees



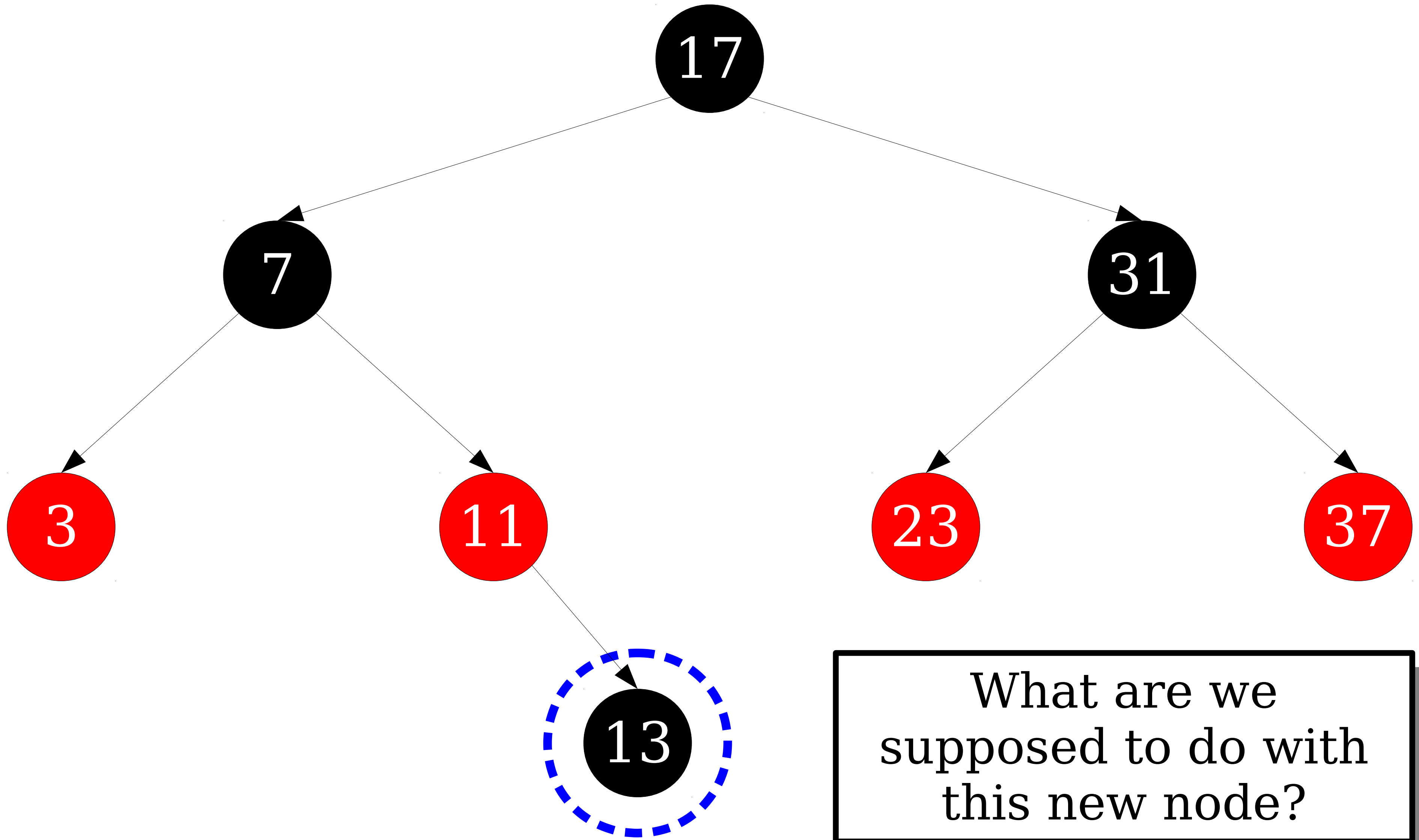
Mutating Red/Black Trees



Mutating Red/Black Trees



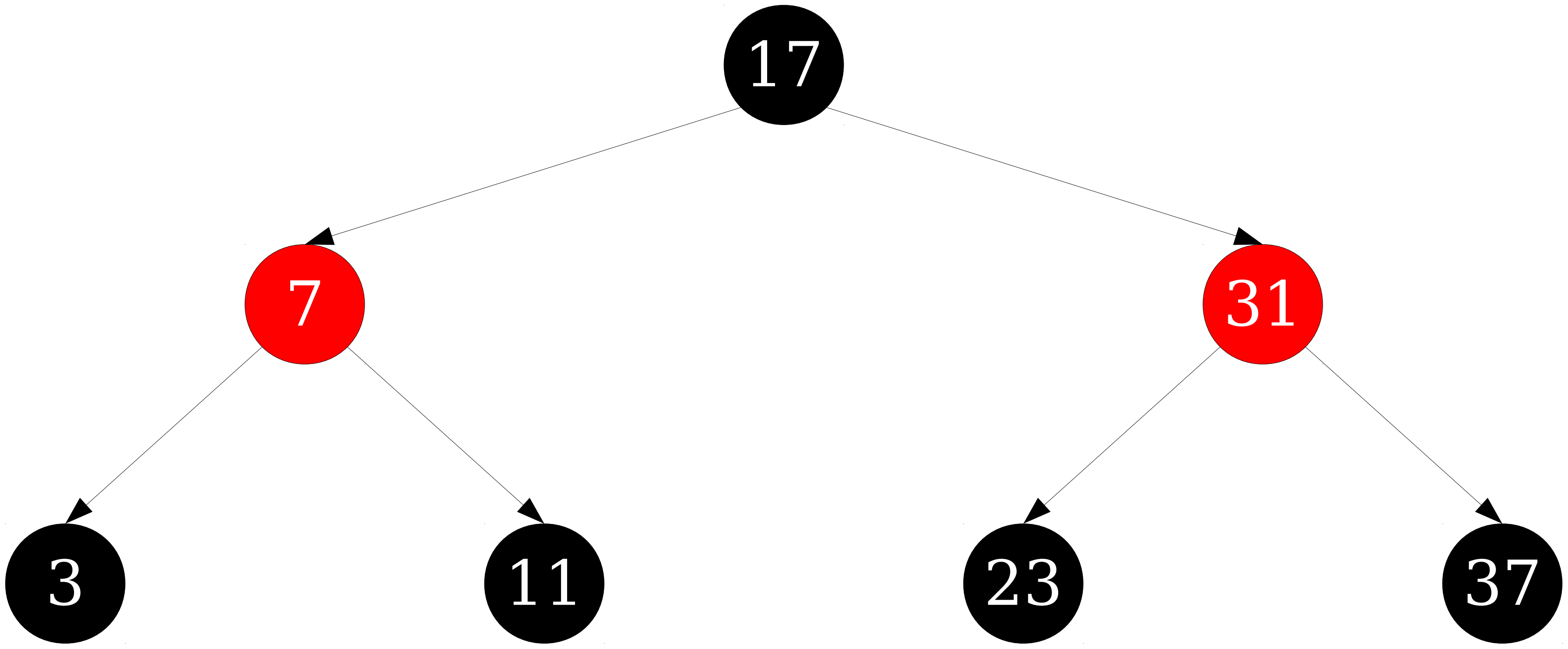
Mutating Red/Black Trees



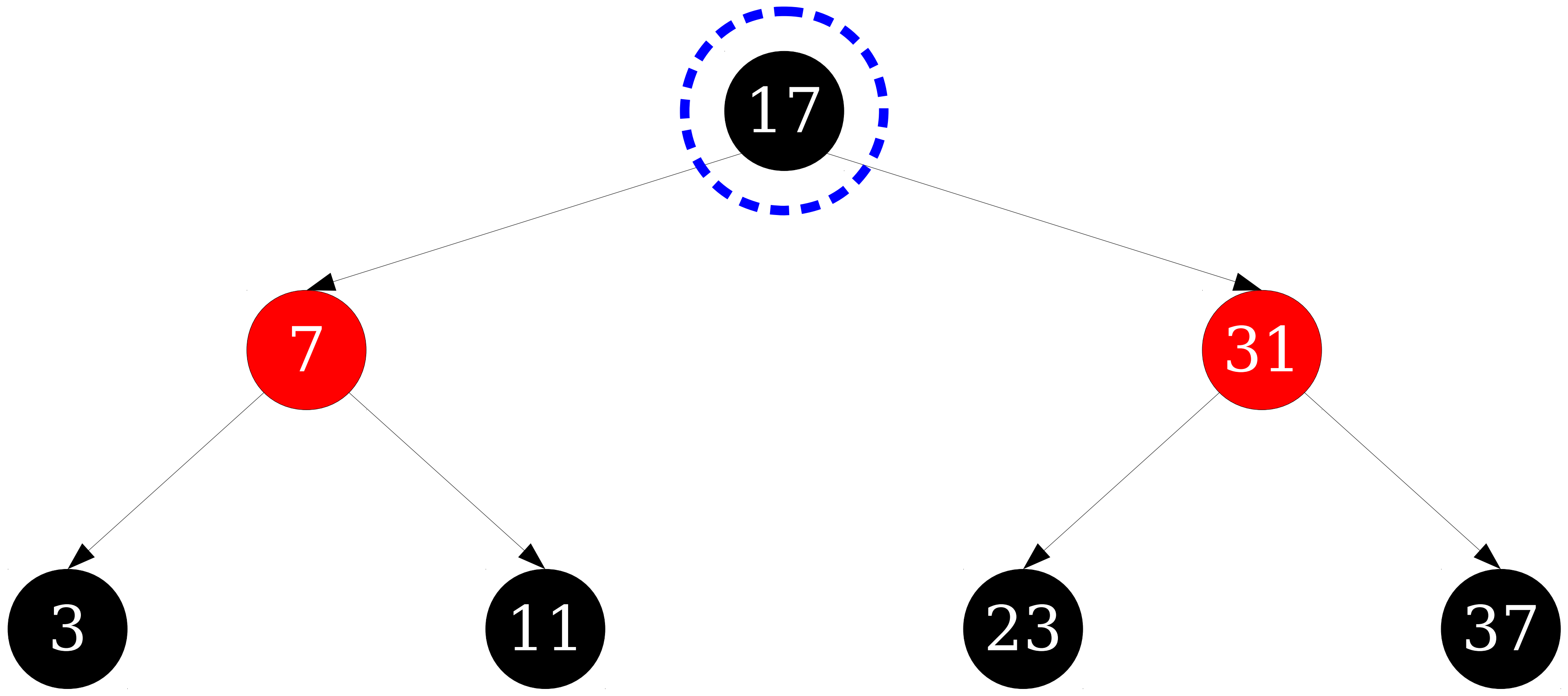
What are we supposed to do with this new node?

Mutating Red/Black Trees

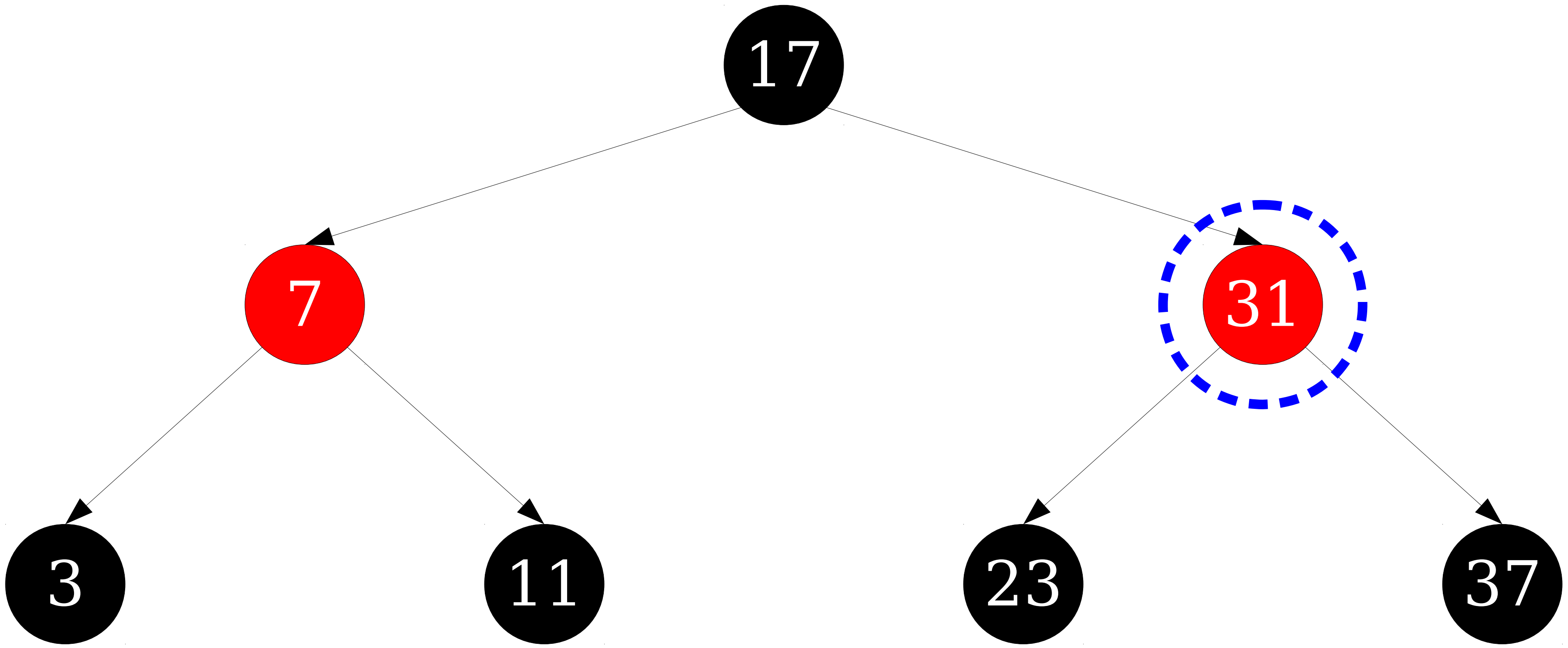
Mutating Red/Black Trees



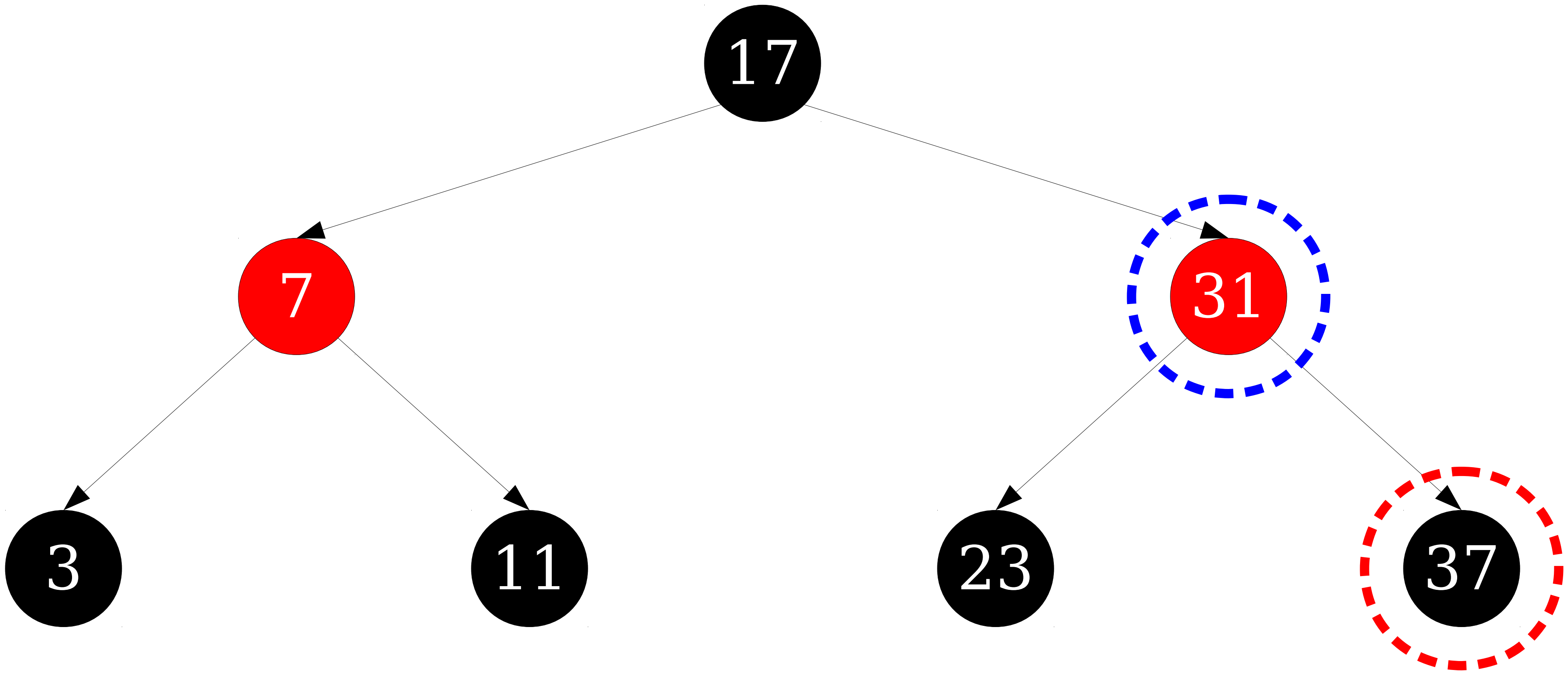
Mutating Red/Black Trees



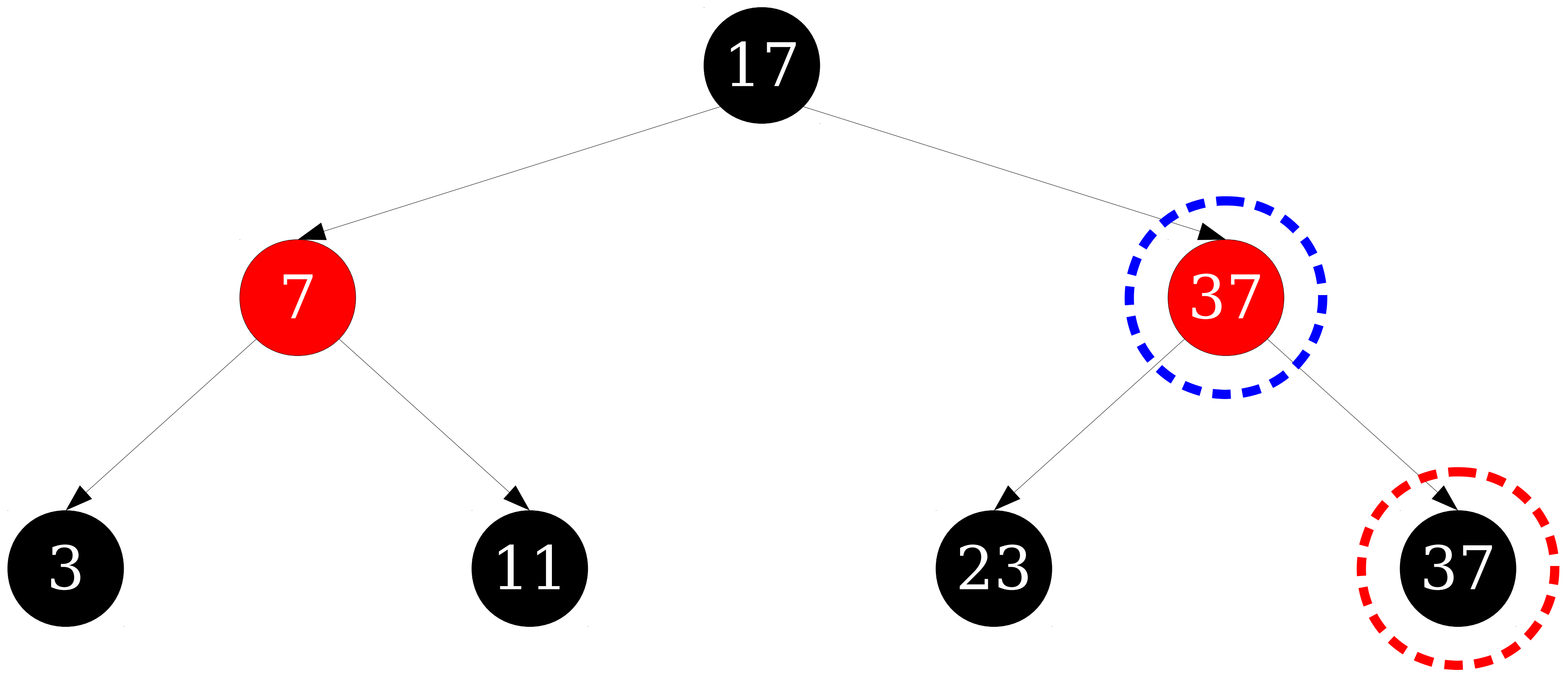
Mutating Red/Black Trees



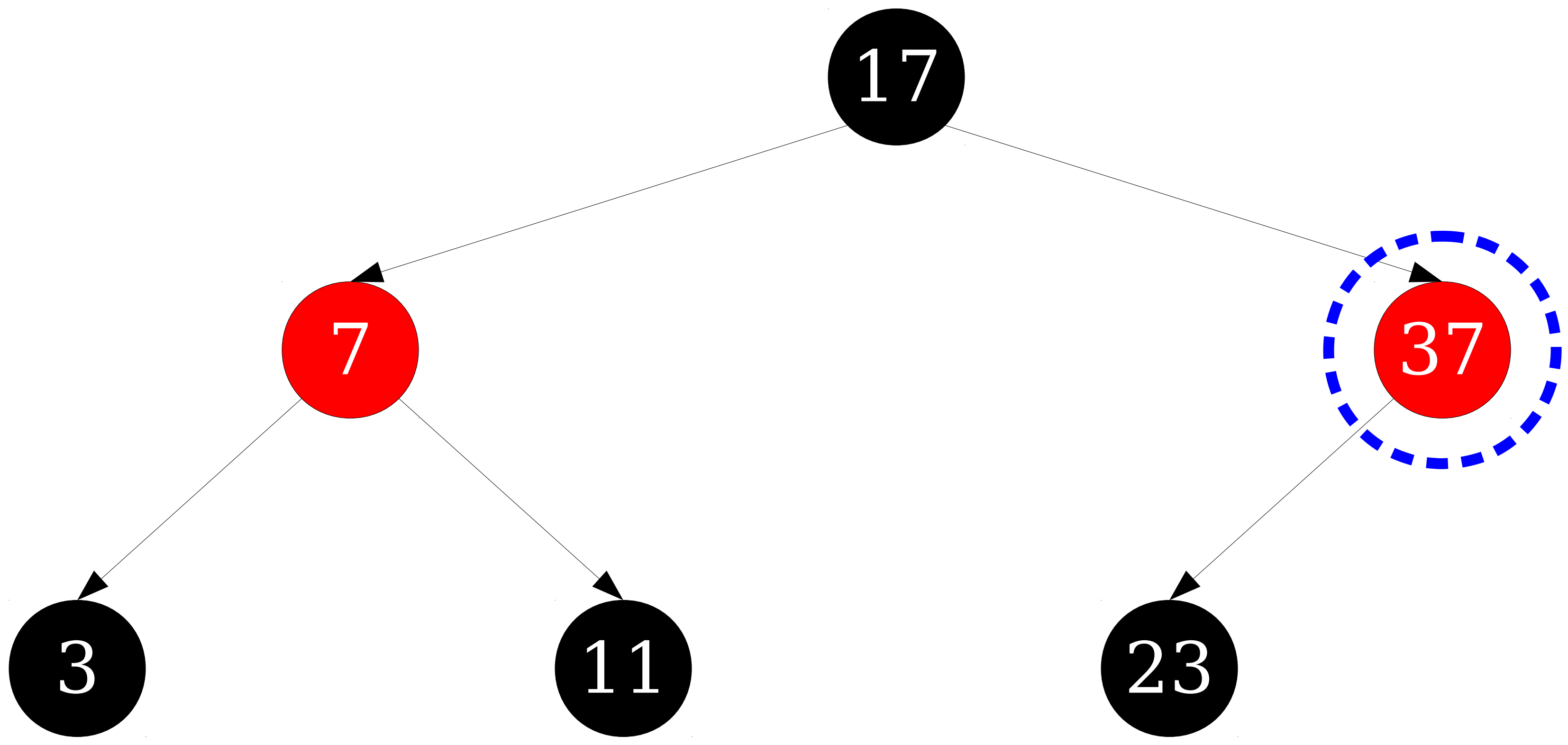
Mutating Red/Black Trees



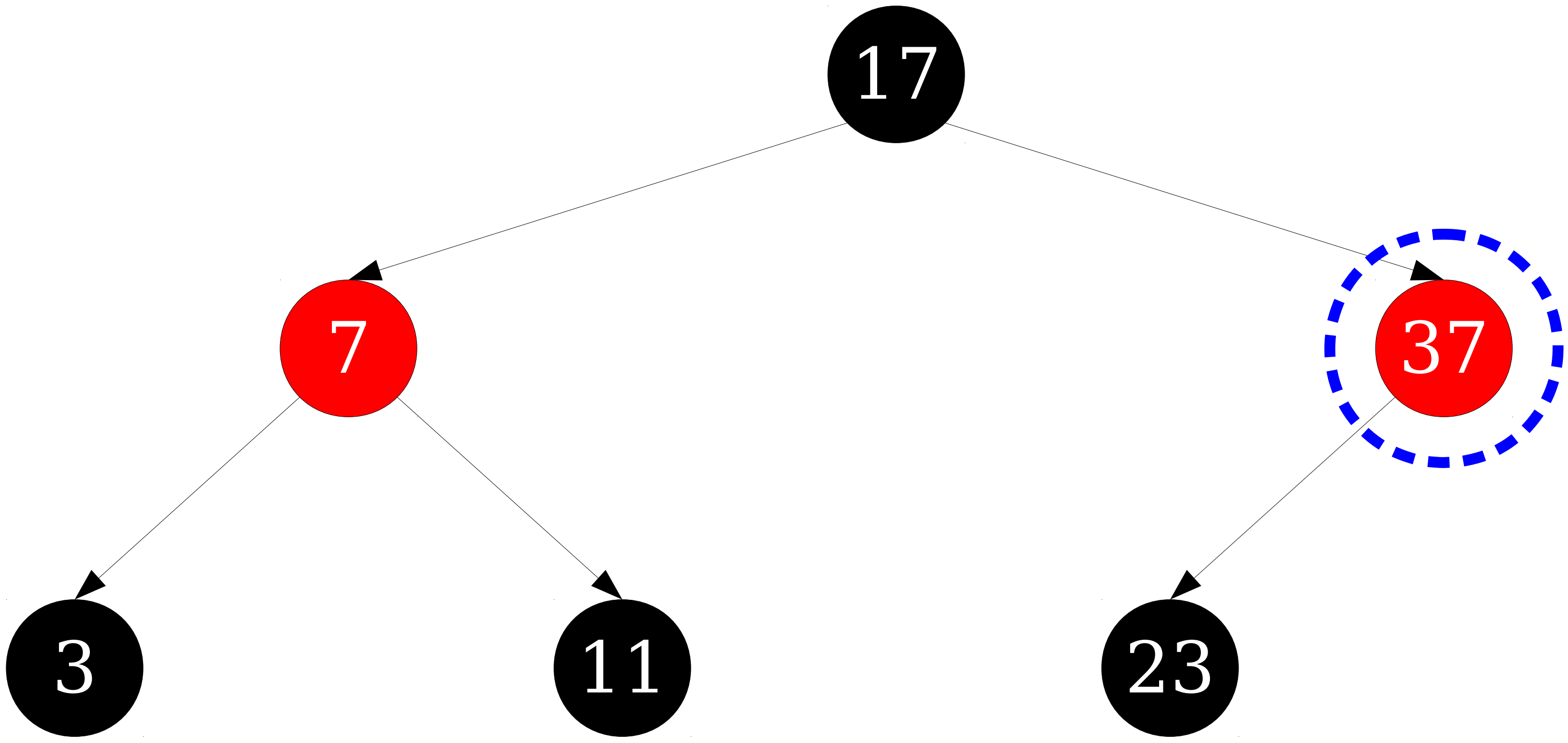
Mutating Red/Black Trees



Mutating Red/Black Trees



Mutating Red/Black Trees



How do we fix up the black-height property?

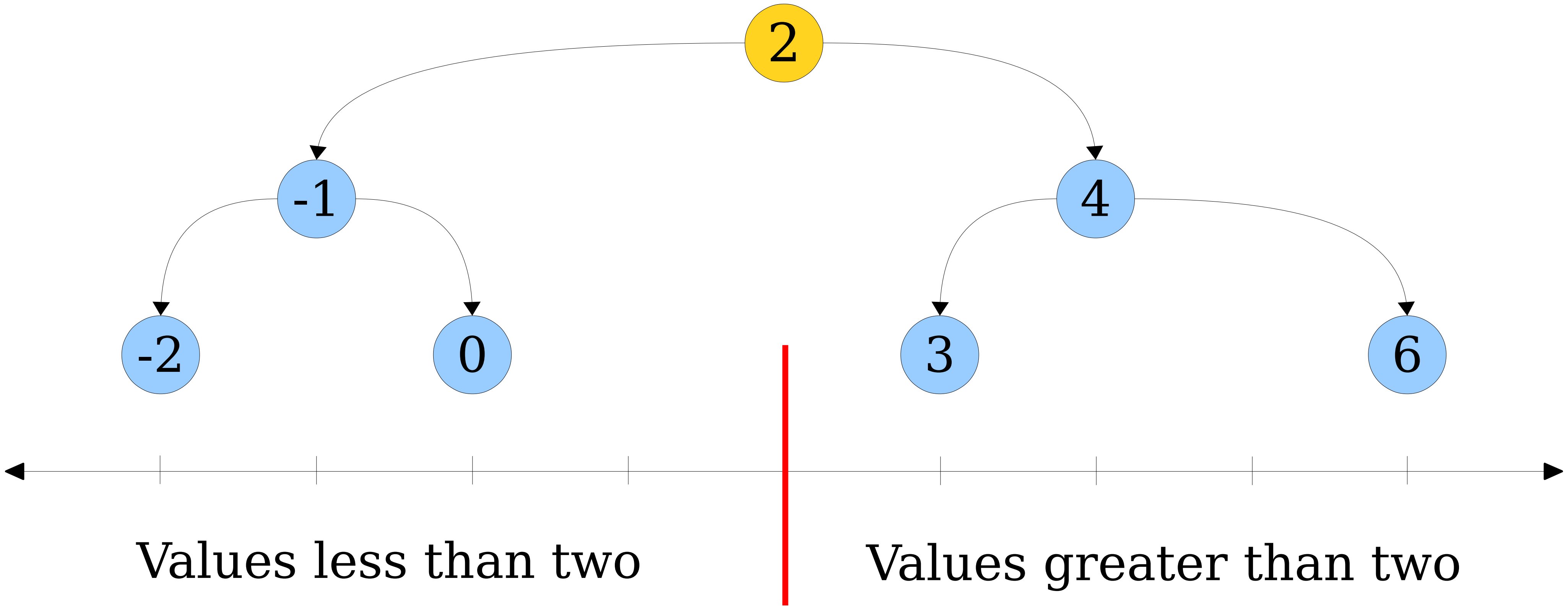
Fixing Up Red/Black Trees

- ***The Good News:*** After doing an insertion or deletion, can locally modify a red/black tree in time O(log *n*) to fix up the red/black properties.
- ***The Bad News:*** There are a *lot* of cases to consider and they're not trivial.
- Some questions:
    - How do you memorize / remember all the different types of rotations?
    - How on earth did anyone come up with red/black trees in the first place?

B-Trees

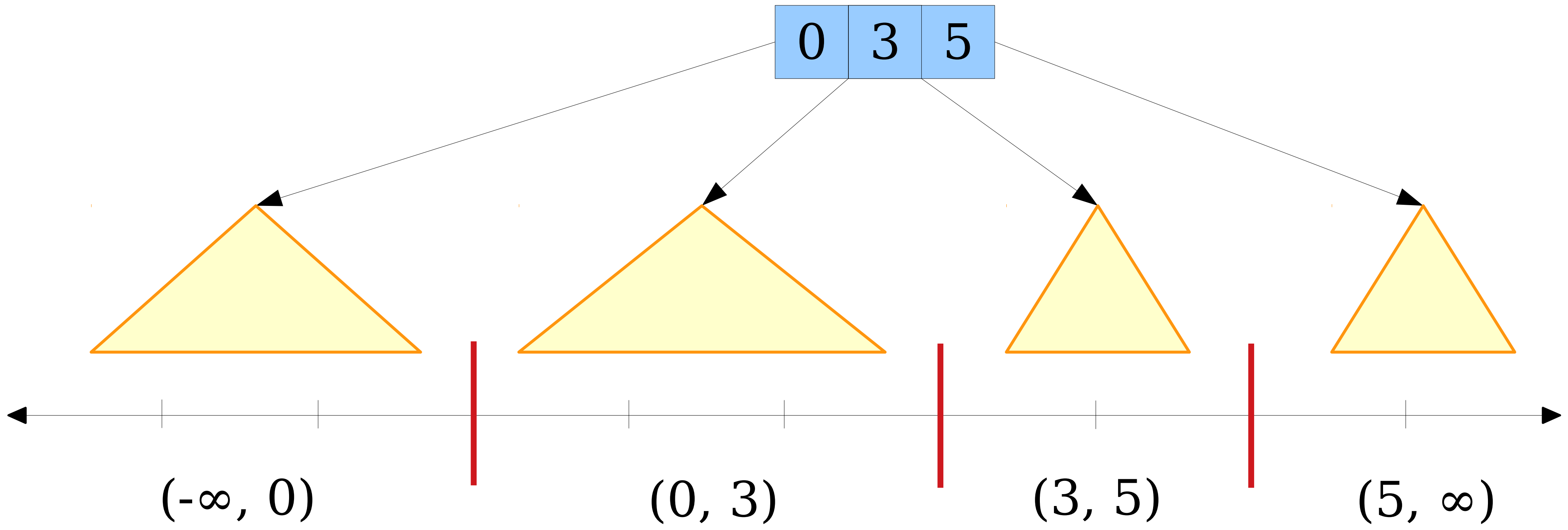
Generalizing BSTs

- In a binary search tree, each node stores a single key.
- That key splits the "key space" into two pieces, and each subtree stores the keys in those halves.



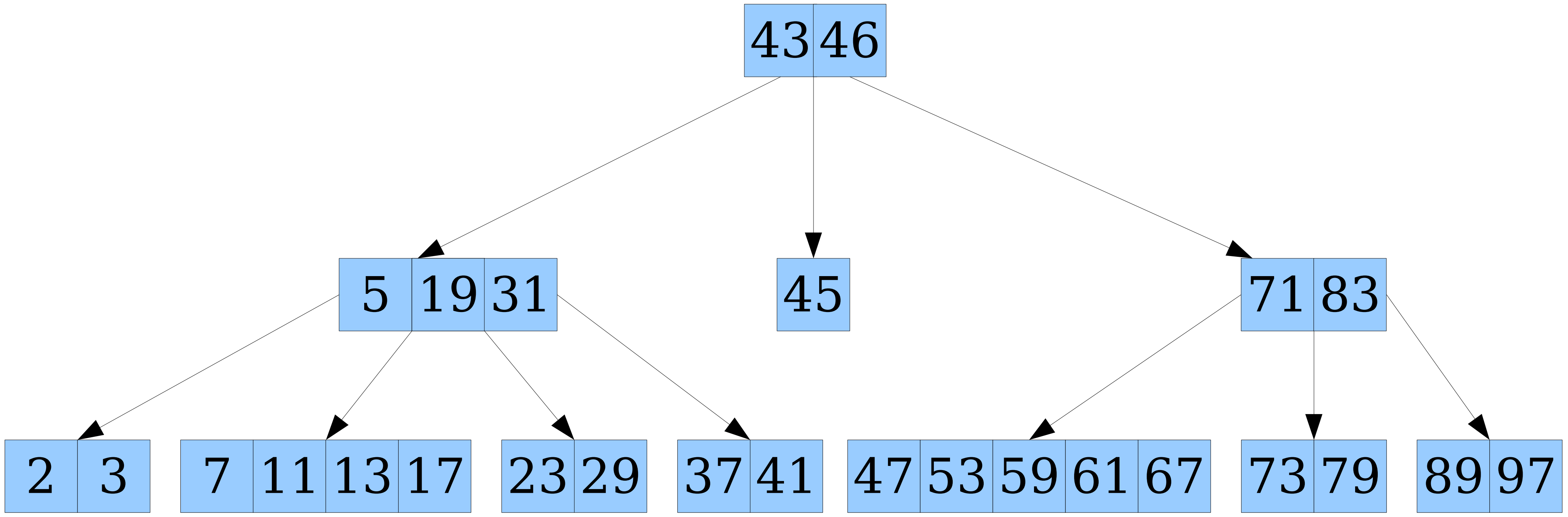
Generalizing BSTs

- In a ***multiway search tree***, each node stores an arbitrary number of keys in sorted order.
- A node with k keys splits the key space into $k+1$ regions, with subtrees for keys in each region.



Generalizing BSTs

- In a ***multiway search tree***, each node stores an arbitrary number of keys in sorted order.



- Surprisingly, it's a bit easier to build a balanced multiway tree than it is to build a balanced BST. Let's see how.

Balanced Multiway Trees

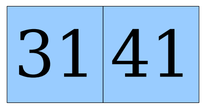
- In some sense, building a balanced multiway tree isn’t all that hard.
- We can always just cram more keys into a single node!

Balanced Multiway Trees

- In some sense, building a balanced multiway tree isn’t all that hard.
- We can always just cram more keys into a single node!

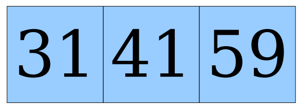
Balanced Multiway Trees

- In some sense, building a balanced multiway tree isn't all that hard.
- We can always just cram more keys into a single node!



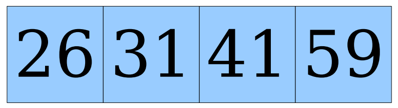
Balanced Multiway Trees

- In some sense, building a balanced multiway tree isn’t all that hard.
- We can always just cram more keys into a single node!



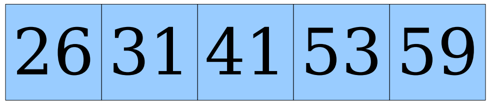
Balanced Multiway Trees

- In some sense, building a balanced multiway tree isn’t all that hard.
- We can always just cram more keys into a single node!



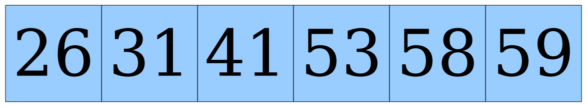
Balanced Multiway Trees

- In some sense, building a balanced multiway tree isn’t all that hard.
- We can always just cram more keys into a single node!



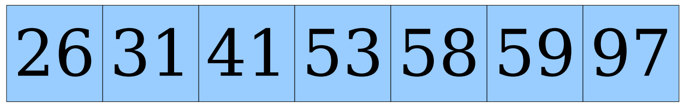
Balanced Multiway Trees

- In some sense, building a balanced multiway tree isn't all that hard.
- We can always just cram more keys into a single node!



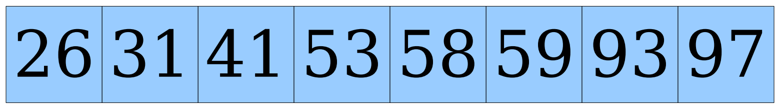
Balanced Multiway Trees

- In some sense, building a balanced multiway tree isn't all that hard.
- We can always just cram more keys into a single node!



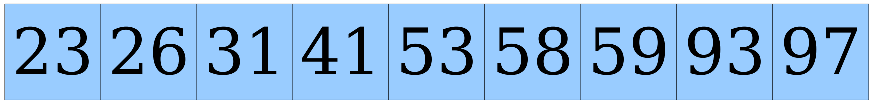
Balanced Multiway Trees

- In some sense, building a balanced multiway tree isn’t all that hard.
- We can always just cram more keys into a single node!



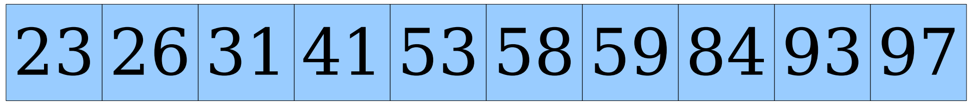
Balanced Multiway Trees

- In some sense, building a balanced multiway tree isn't all that hard.
- We can always just cram more keys into a single node!



Balanced Multiway Trees

- In some sense, building a balanced multiway tree isn’t all that hard.
- We can always just cram more keys into a single node!



Balanced Multiway Trees

- In some sense, building a balanced multiway tree isn’t all that hard.
- We can always just cram more keys into a single node!

23	26	31	41	53	58	59	62	84	93	97

Balanced Multiway Trees

- In some sense, building a balanced multiway tree isn’t all that hard.
- We can always just cram more keys into a single node!

23	26	31	41	53	58	59	62	84	93	97

- At a certain point, this stops being a good idea – it’s basically just a sorted array.

Balanced Multiway Trees

- What could we do if our nodes get too big?

Balanced Multiway Trees

- What could we do if our nodes get too big?
- ***Option 1:*** Push keys down into new nodes.

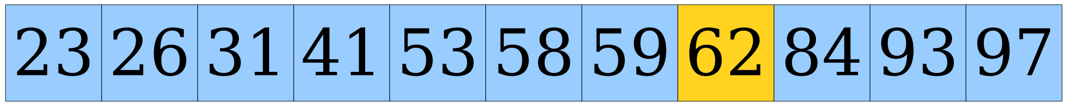
Balanced Multiway Trees

- What could we do if our nodes get too big?
- ***Option 1:*** Push keys down into new nodes.

23	26	31	41	53	58	59	84	93	97

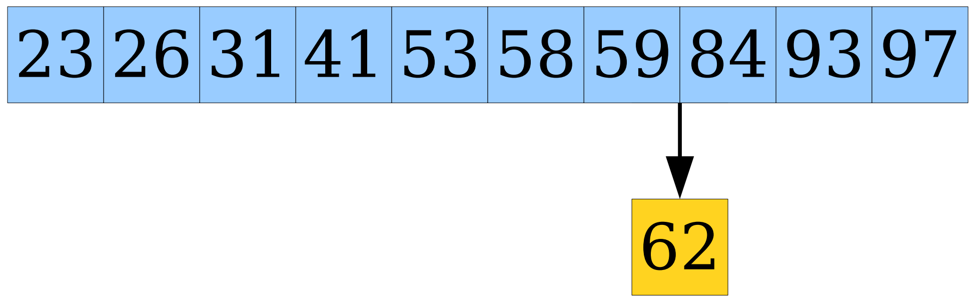
Balanced Multiway Trees

- What could we do if our nodes get too big?
- ***Option 1:*** Push keys down into new nodes.



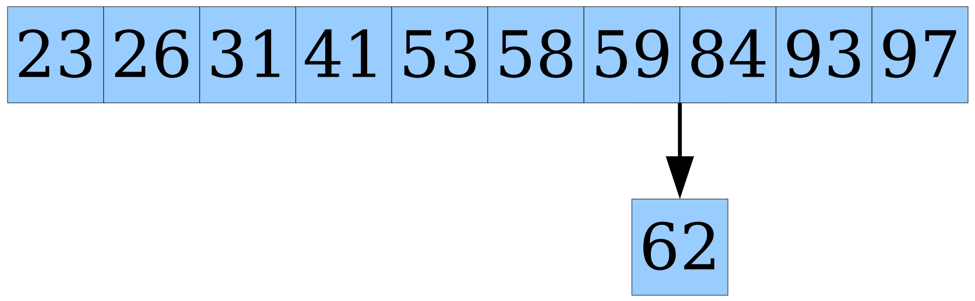
Balanced Multiway Trees

- What could we do if our nodes get too big?
- ***Option 1:*** Push keys down into new nodes.



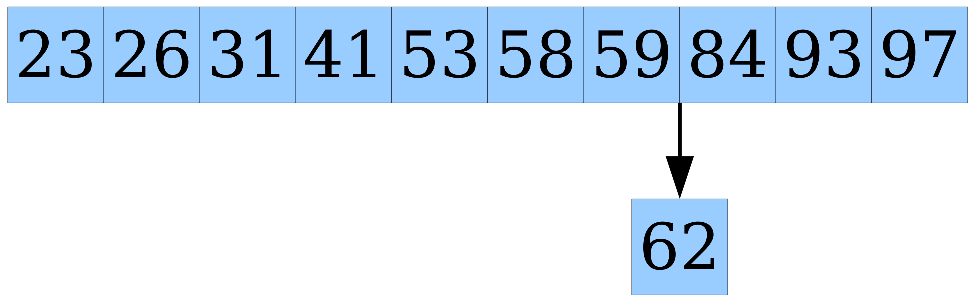
Balanced Multiway Trees

- What could we do if our nodes get too big?
- ***Option 1:*** Push keys down into new nodes.



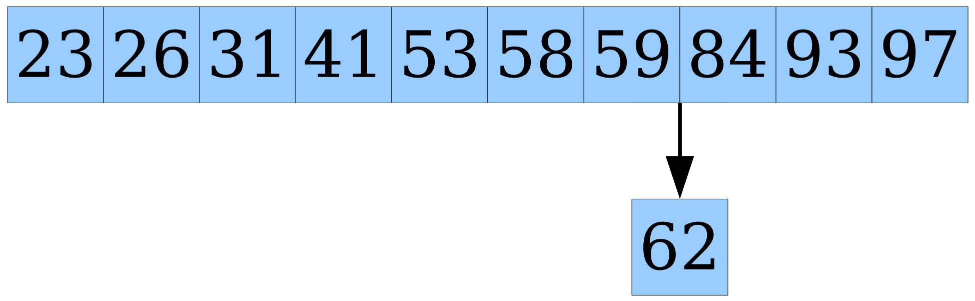
Balanced Multiway Trees

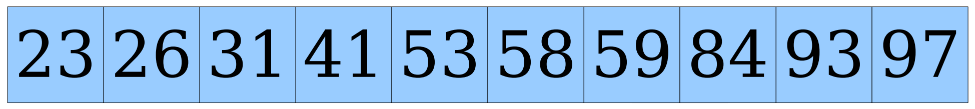
- What could we do if our nodes get too big?
- ***Option 1:*** Push keys down into new nodes.
- ***Option 2:*** Split big nodes, kicking keys higher up.



Balanced Multiway Trees

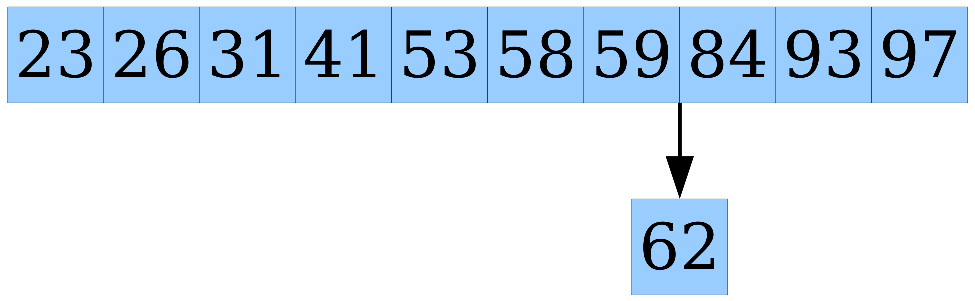
- What could we do if our nodes get too big?
- ***Option 1:*** Push keys down into new nodes.
- ***Option 2:*** Split big nodes, kicking keys higher up.

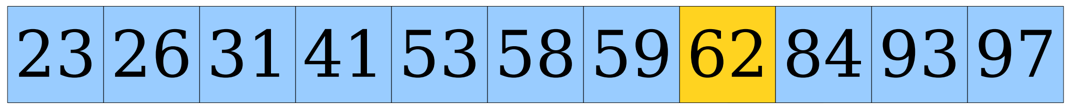




Balanced Multiway Trees

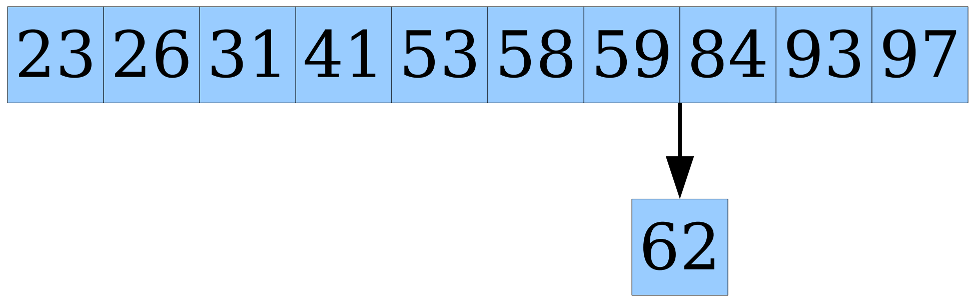
- What could we do if our nodes get too big?
- ***Option 1:*** Push keys down into new nodes.
- ***Option 2:*** Split big nodes, kicking keys higher up.

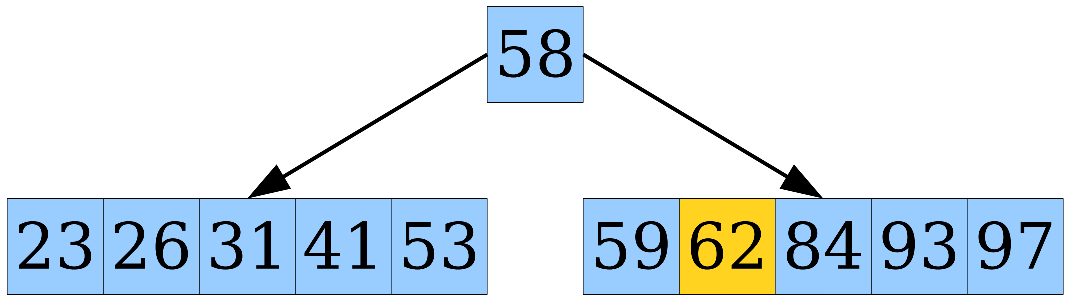




Balanced Multiway Trees

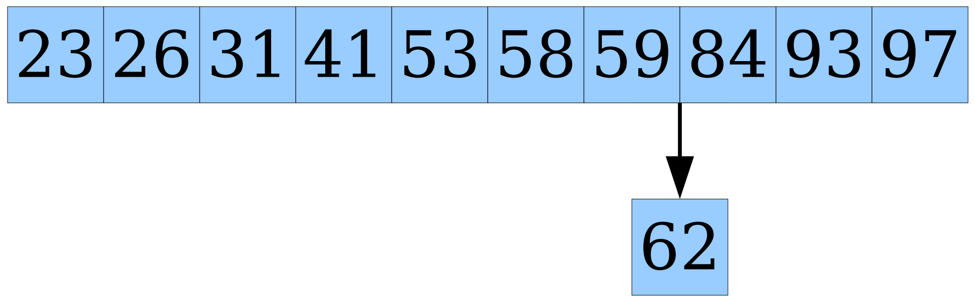
- What could we do if our nodes get too big?
- ***Option 1:*** Push keys down into new nodes.
- ***Option 2:*** Split big nodes, kicking keys higher up.

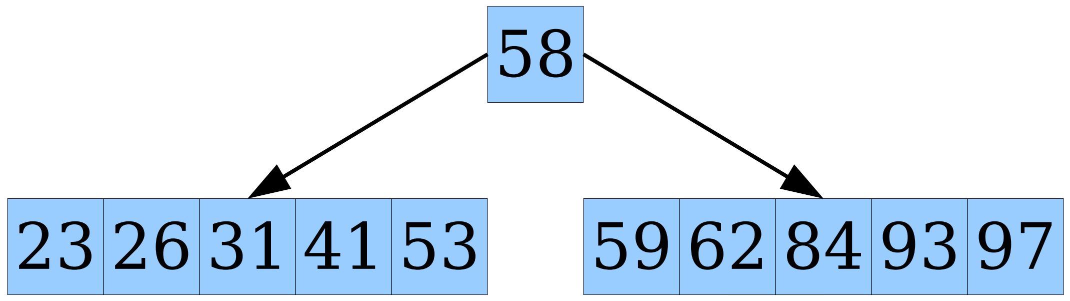




Balanced Multiway Trees

- What could we do if our nodes get too big?
- ***Option 1:*** Push keys down into new nodes.
- ***Option 2:*** Split big nodes, kicking keys higher up.
- What are some advantages of each approach?
- What are some disadvantages?





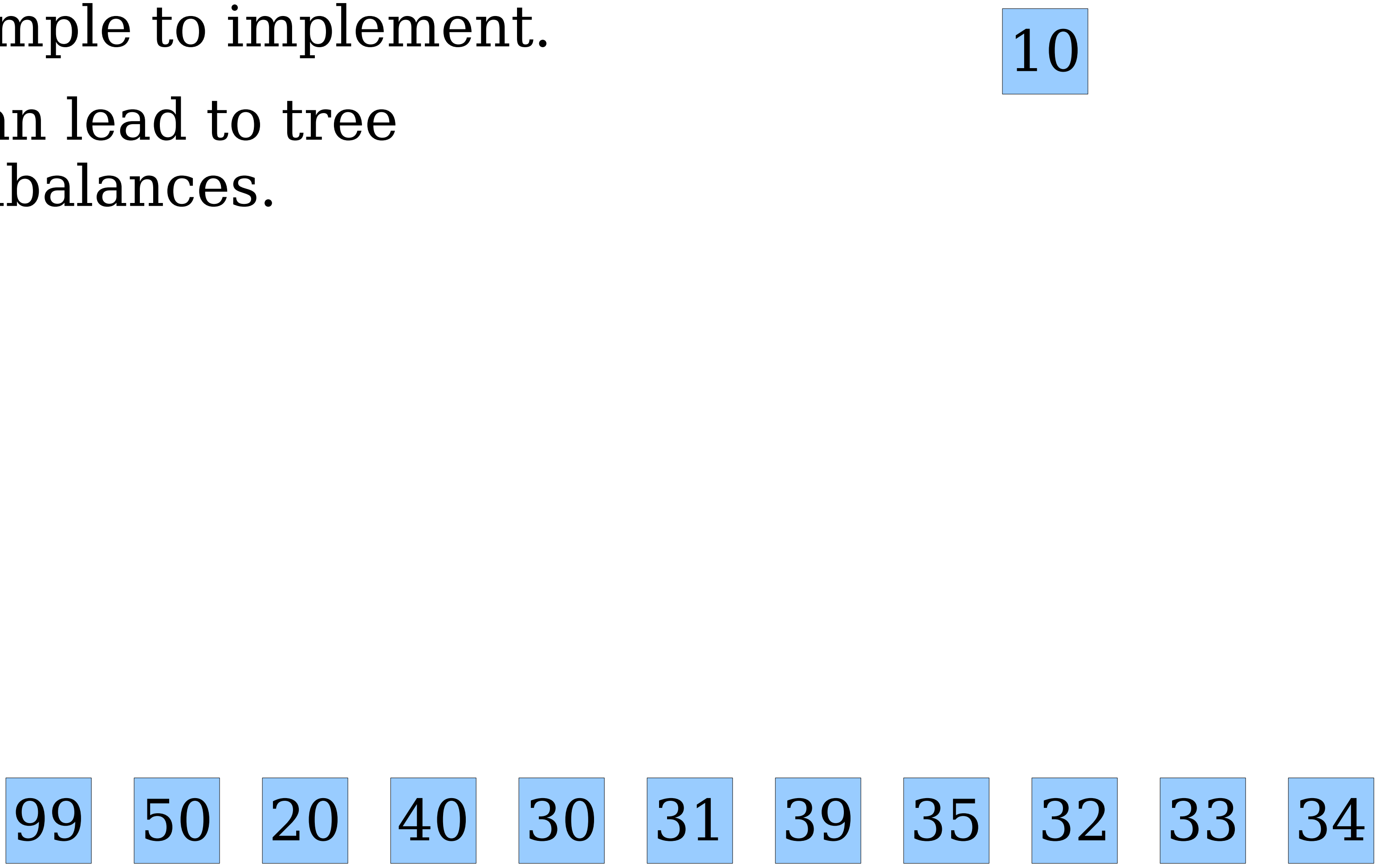
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.

10 99 50 20 40 30 31 39 35 32 33 34

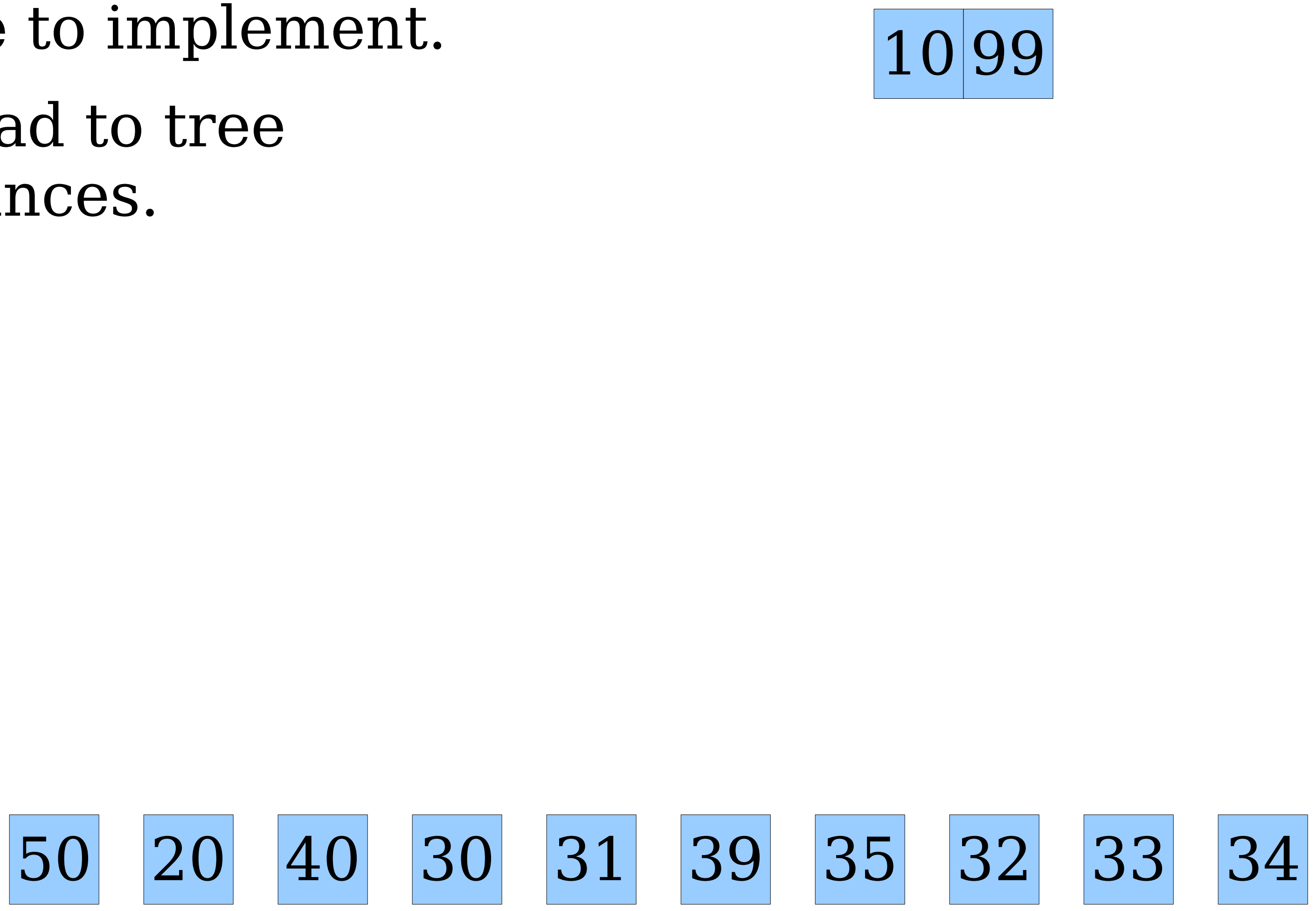
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



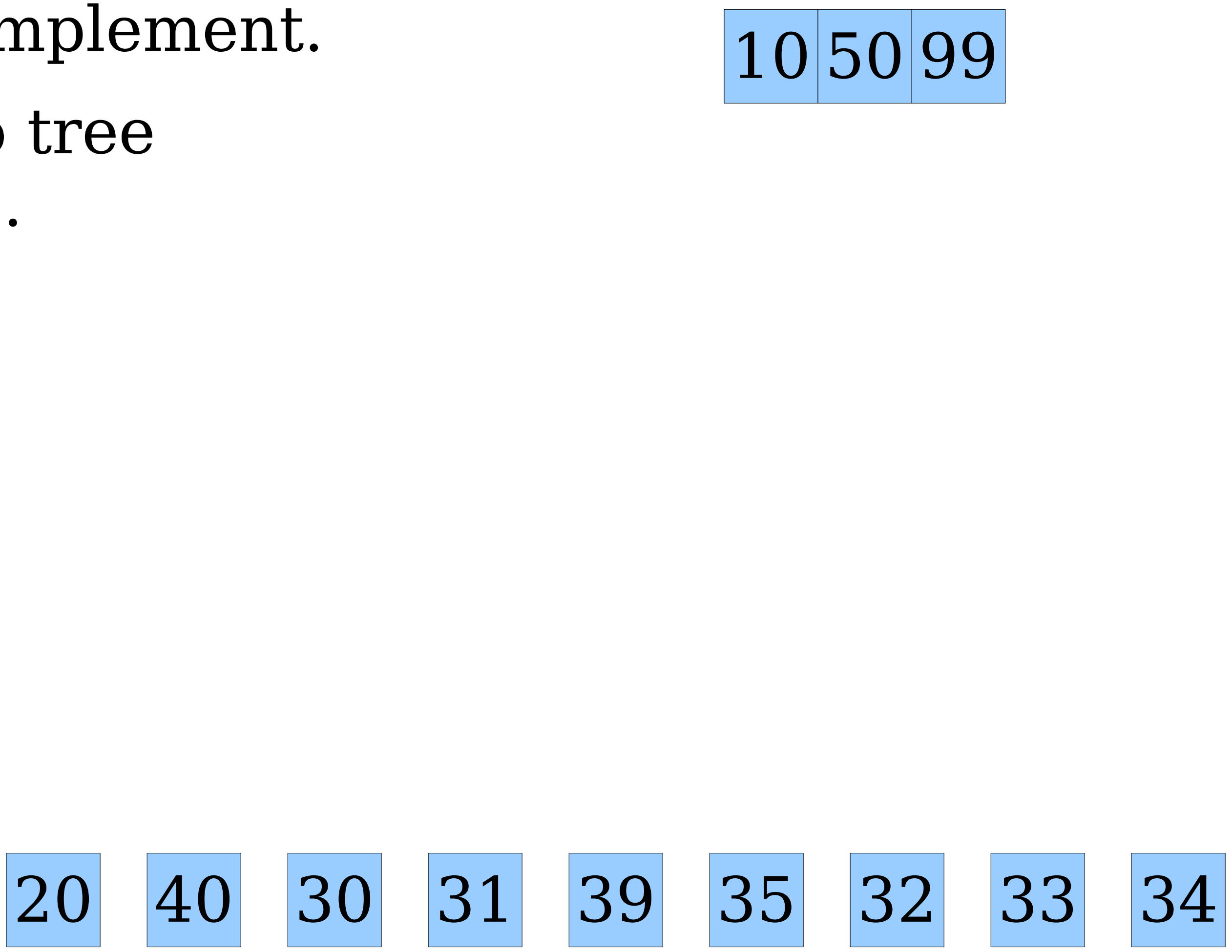
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



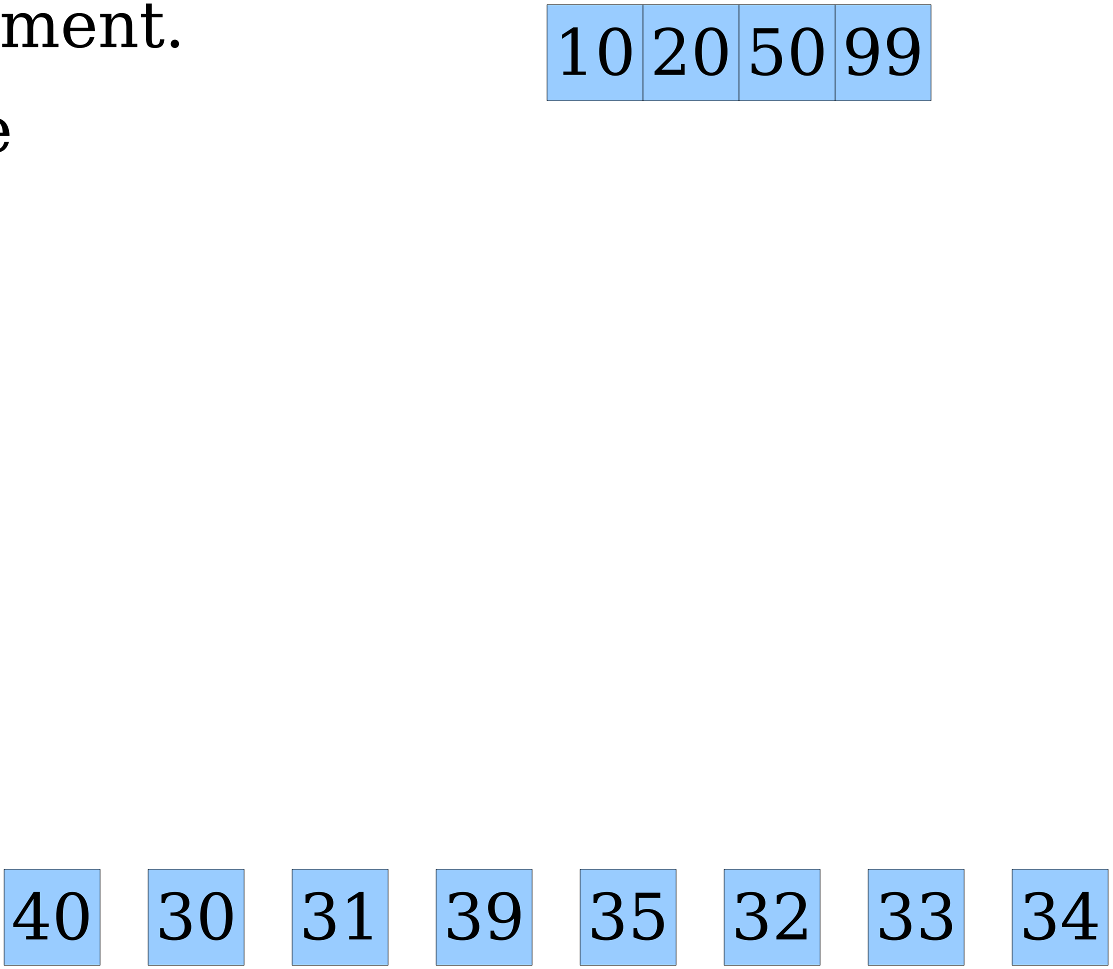
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
    - Simple to implement.
    - Can lead to tree imbalances.



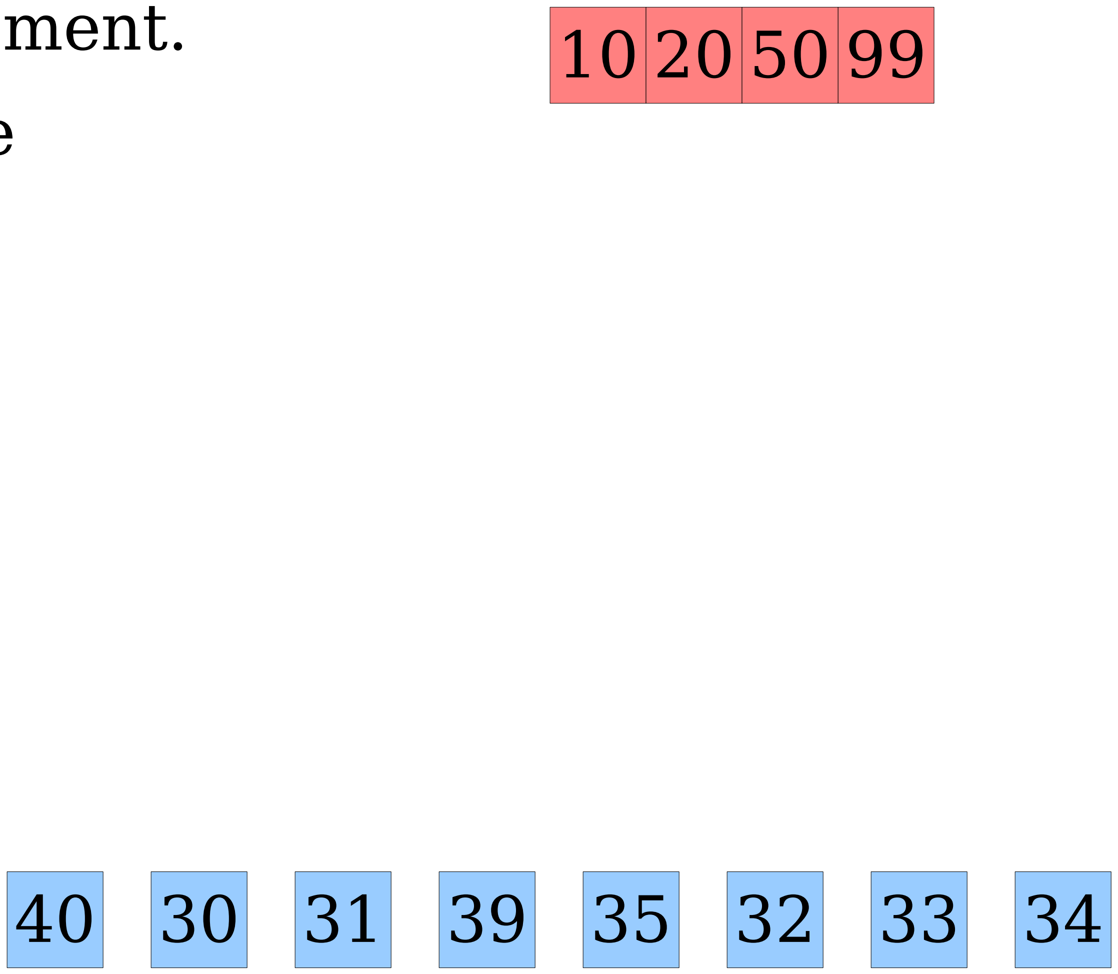
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



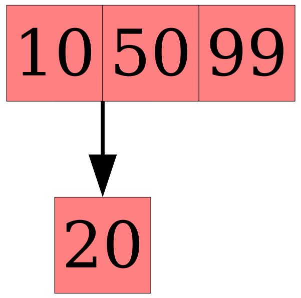
Balanced Multiway Trees

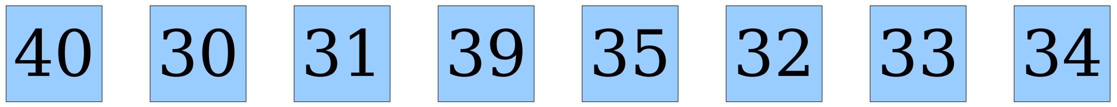
- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



Balanced Multiway Trees

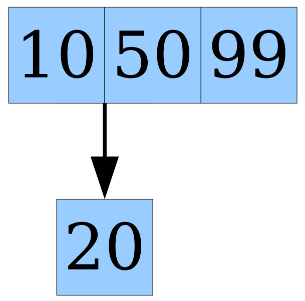
- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.

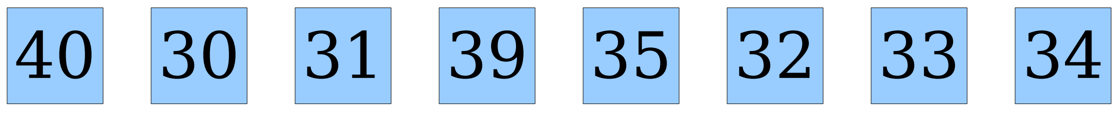




Balanced Multiway Trees

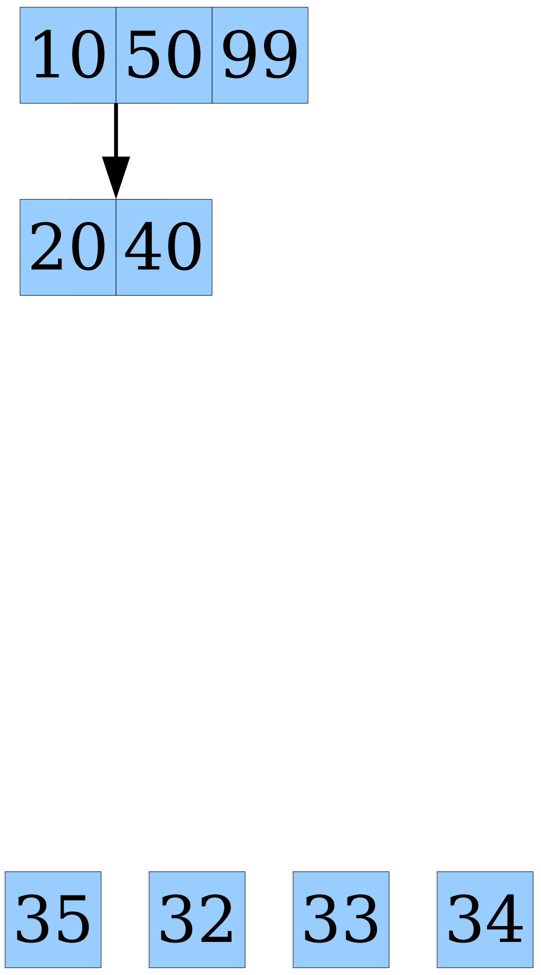
- ***Option 1:*** Push keys down into new nodes.
    - Simple to implement.
    - Can lead to tree imbalances.





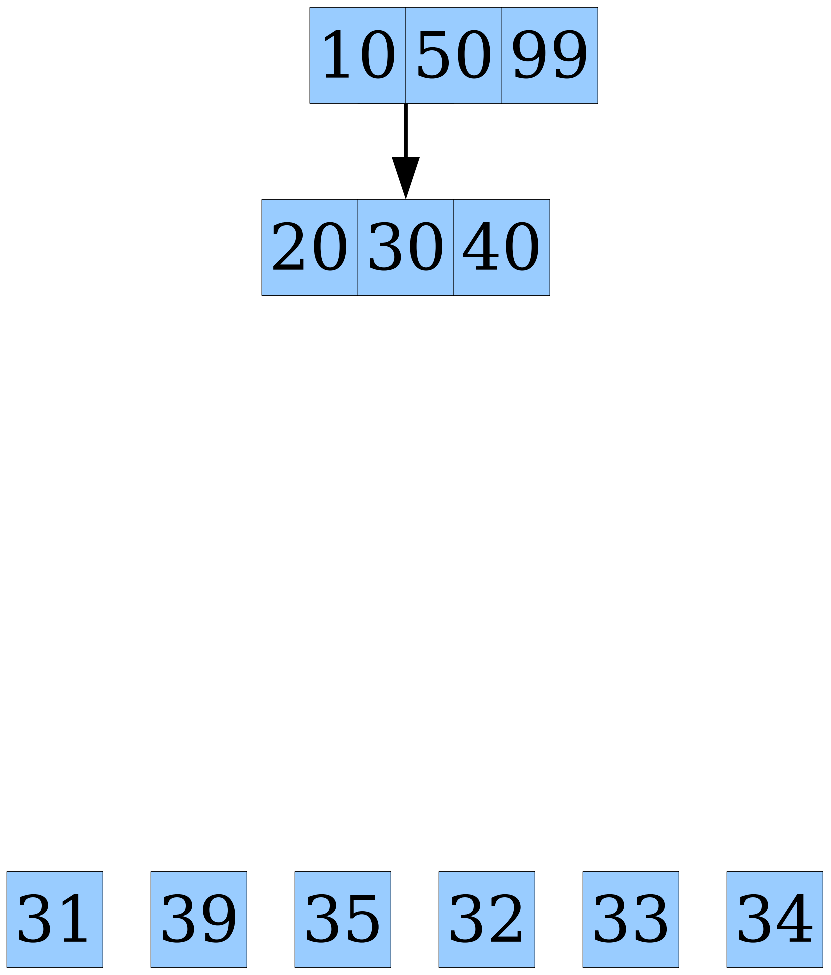
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



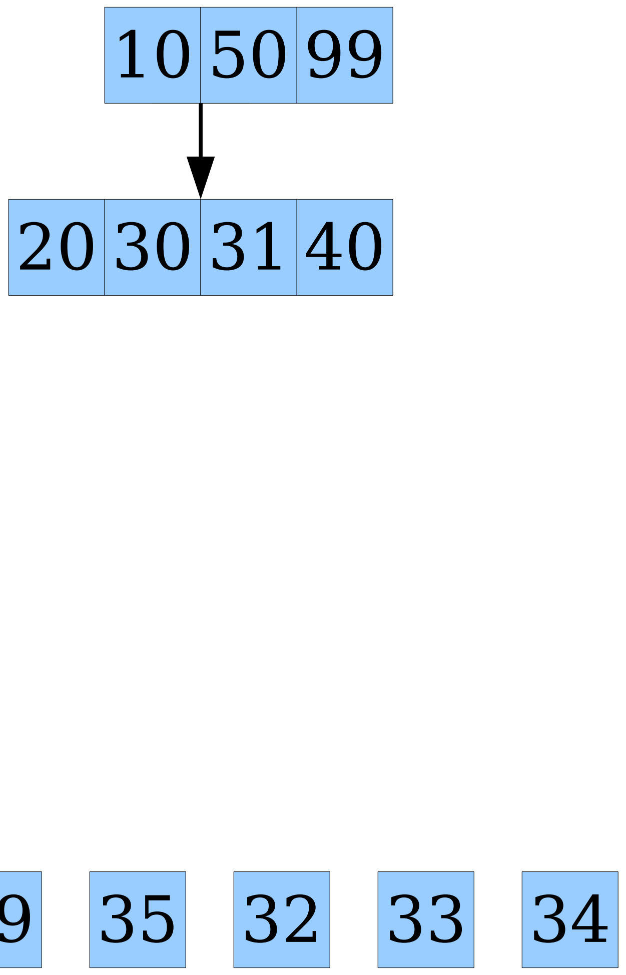
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



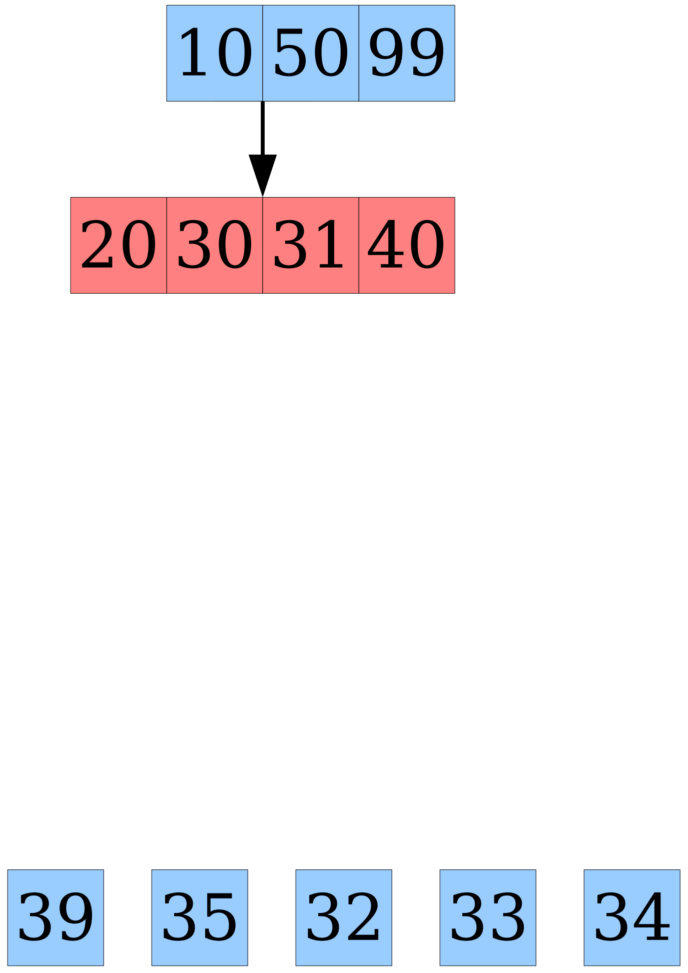
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



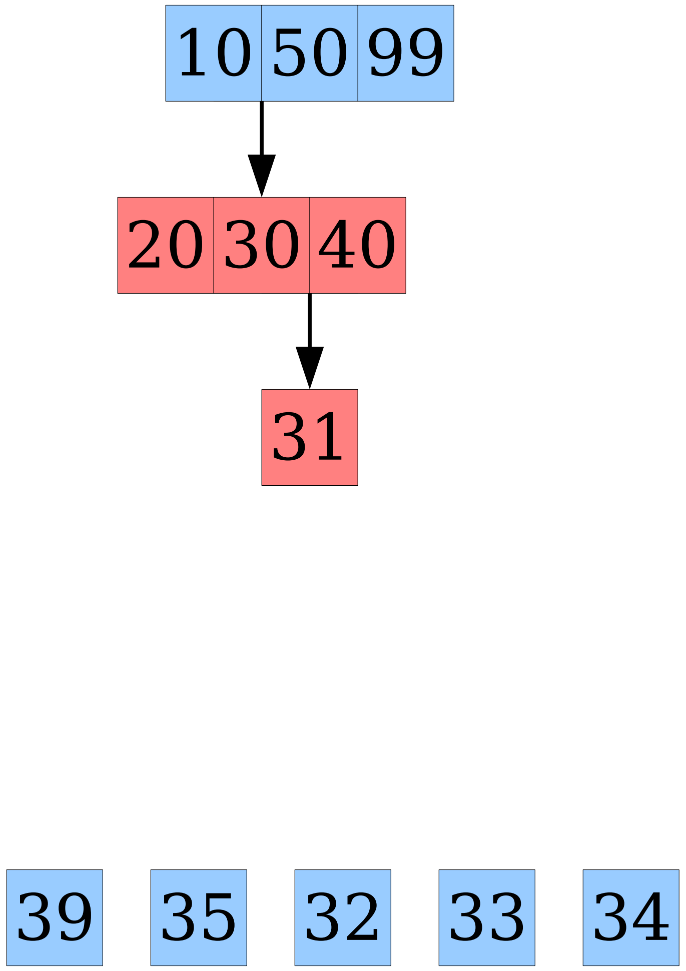
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



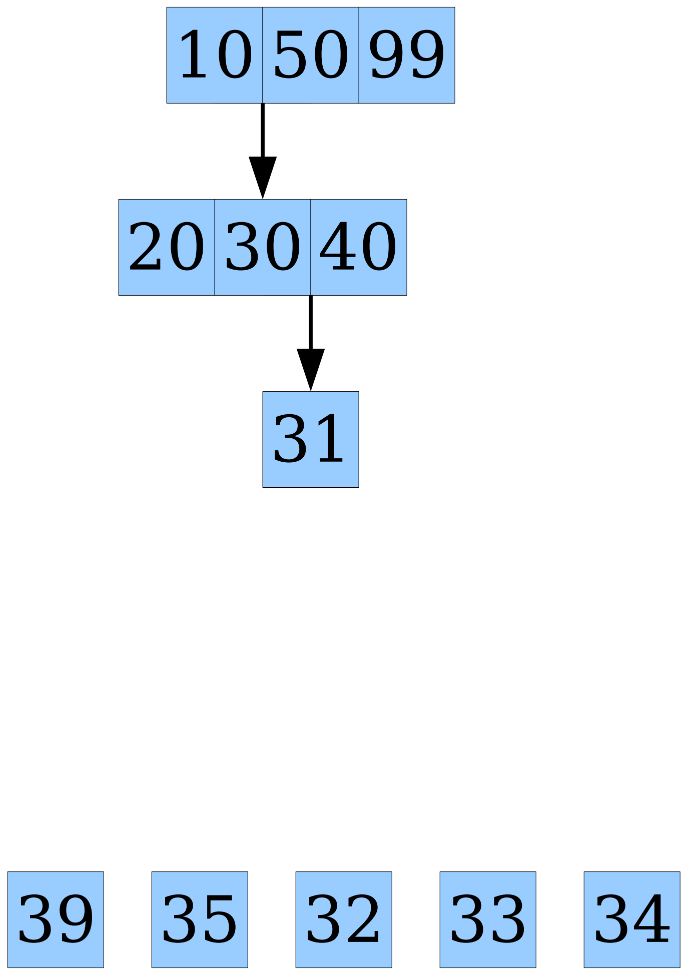
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



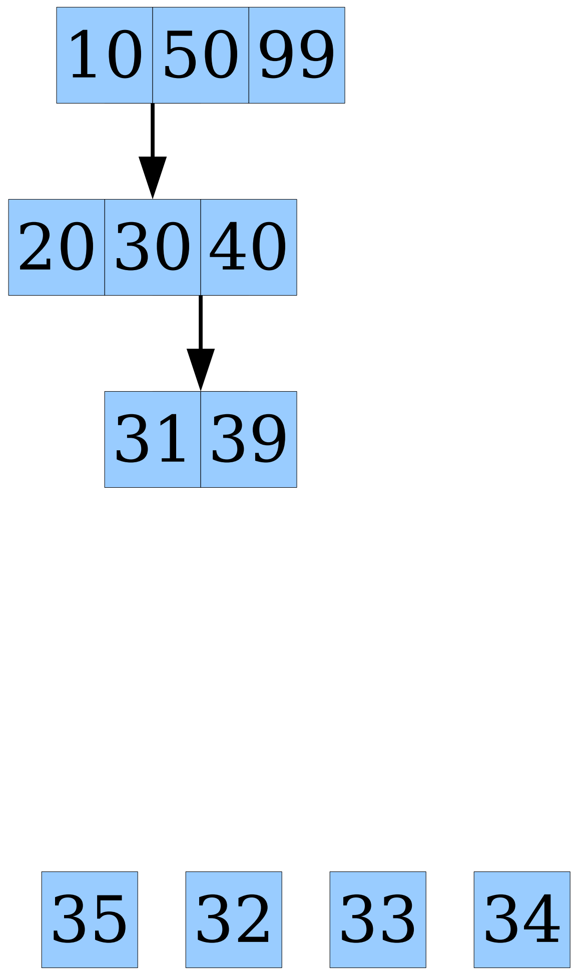
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



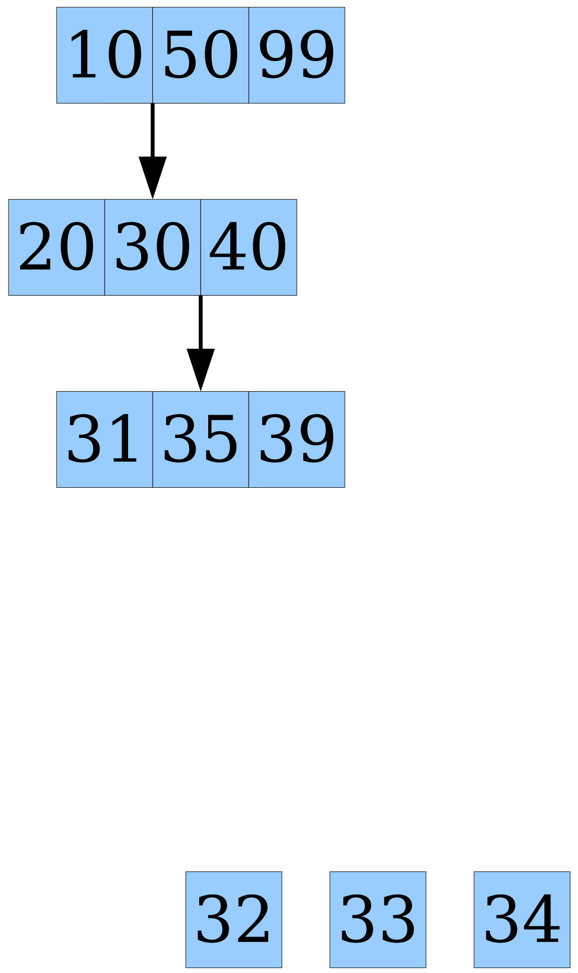
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



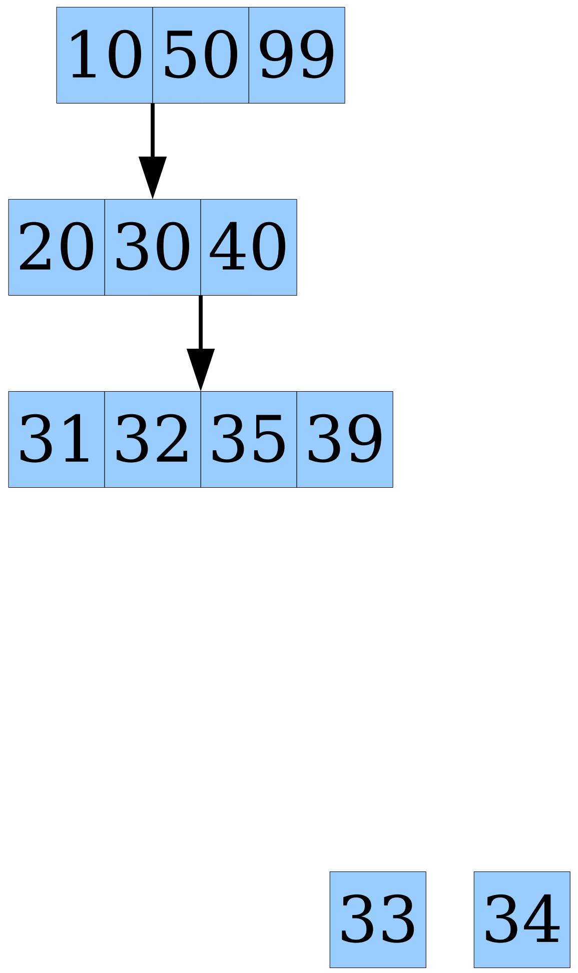
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



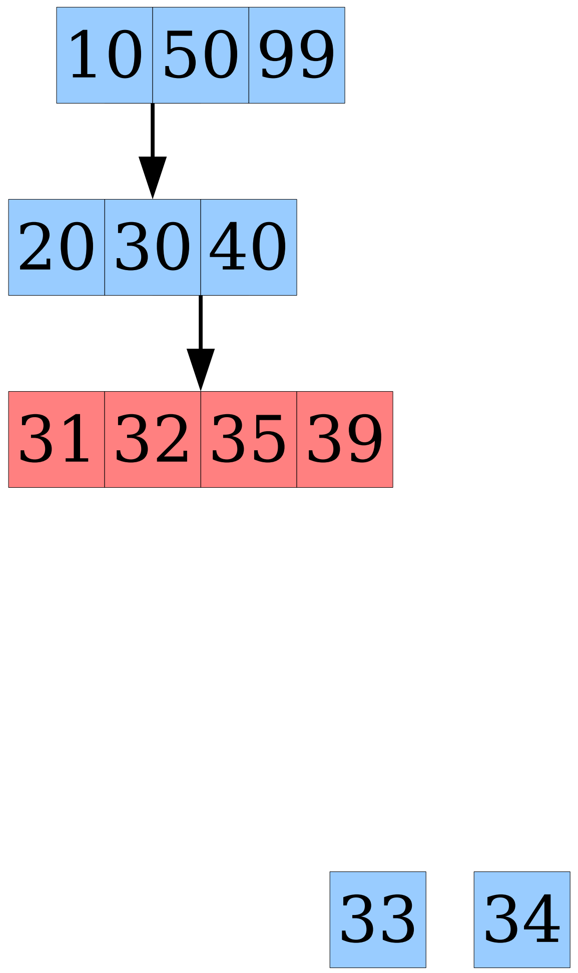
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
    - Simple to implement.
    - Can lead to tree imbalances.



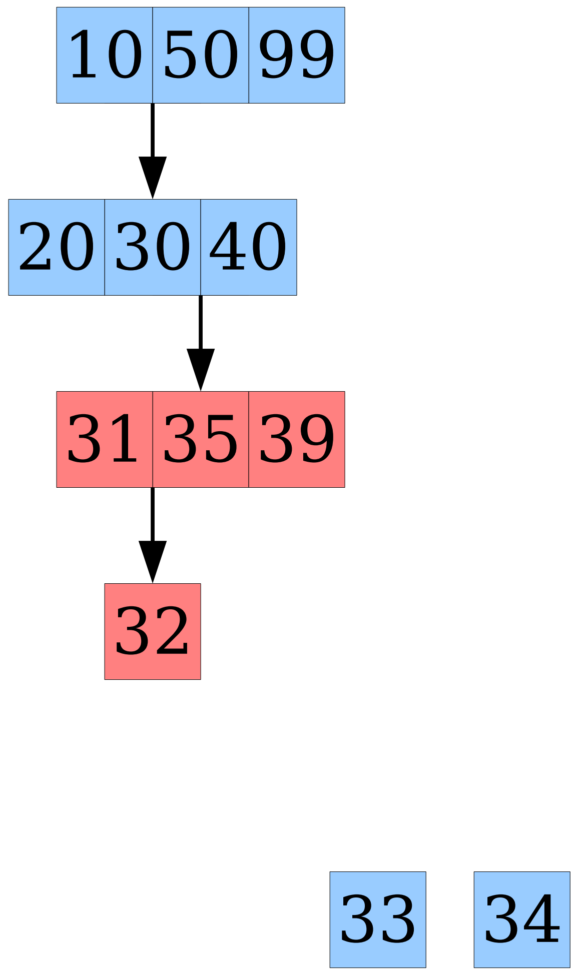
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
    - Simple to implement.
    - Can lead to tree imbalances.



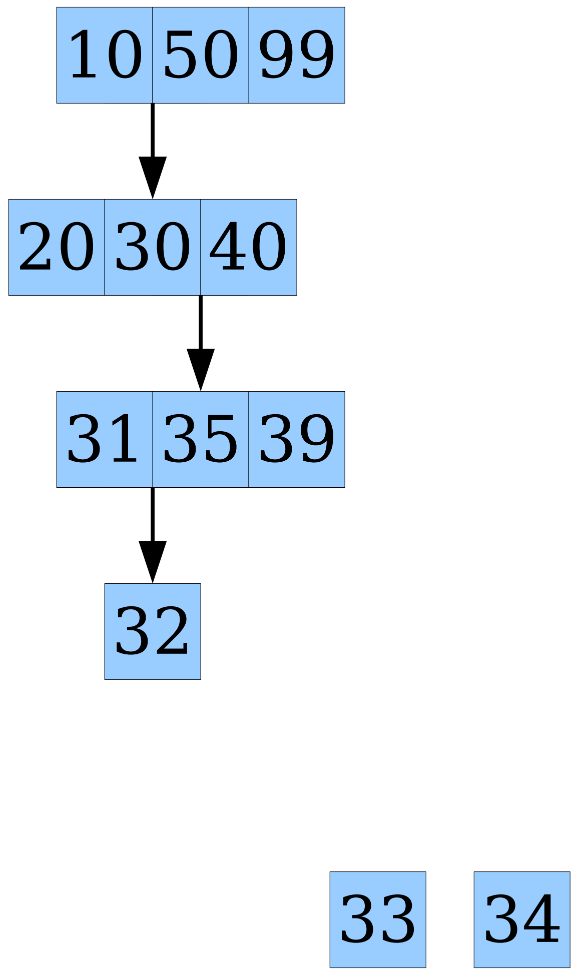
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



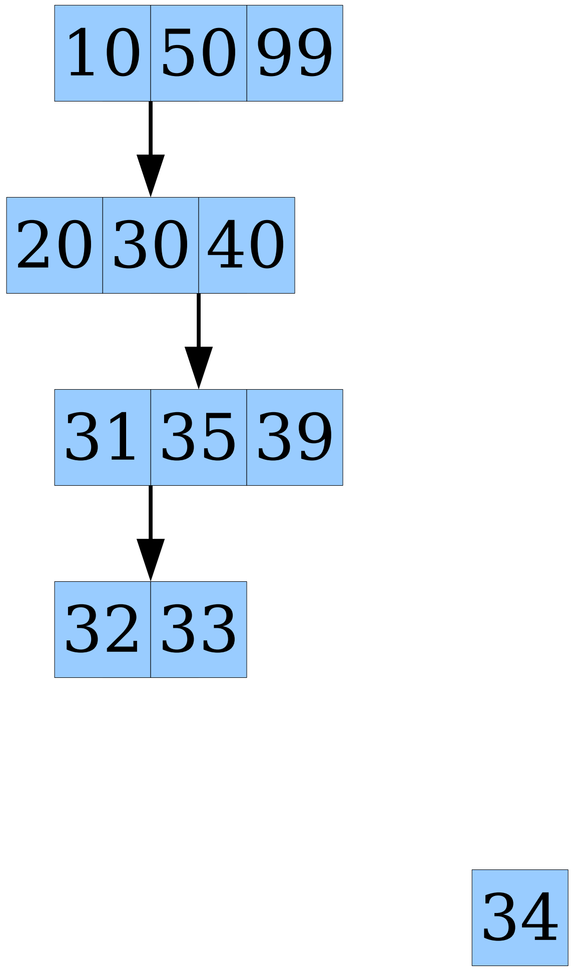
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



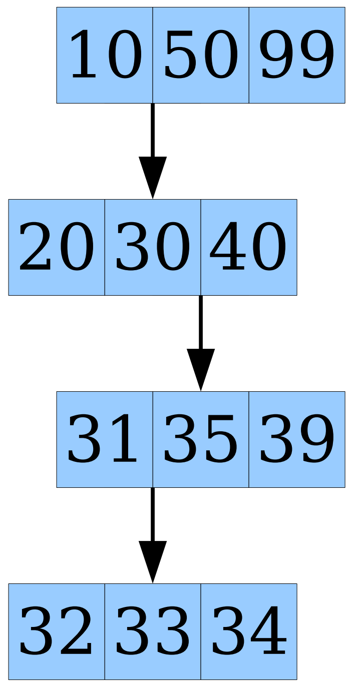
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



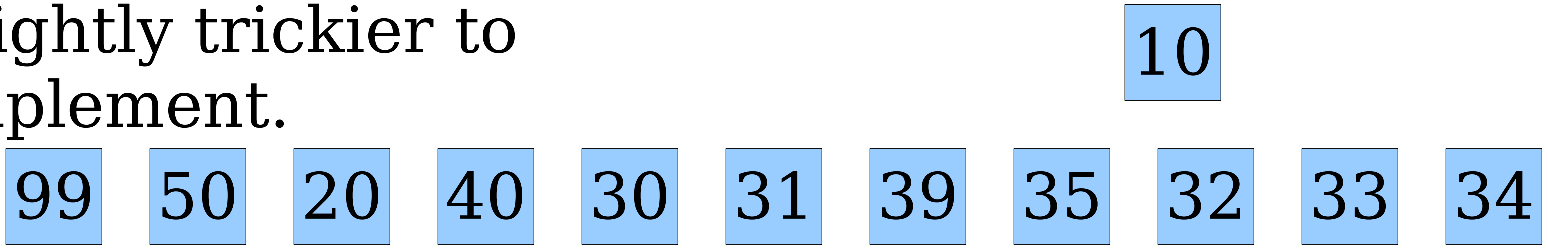
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.

10	99	50	20	40	30	31	39	35	32	33	34

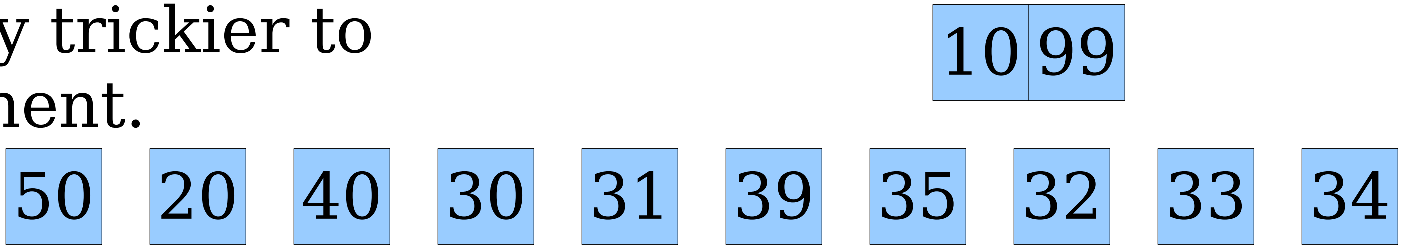
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



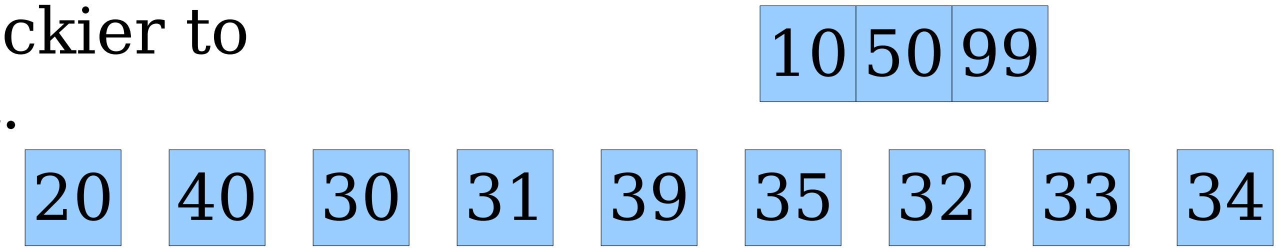
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
    - Simple to implement.
    - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
    - Keeps the tree balanced.
    - Slightly trickier to implement.



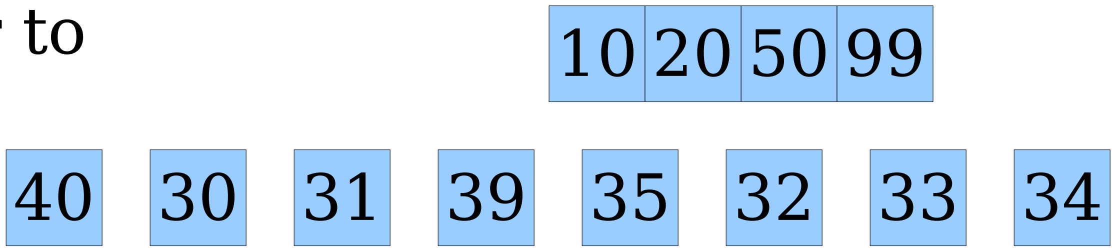
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



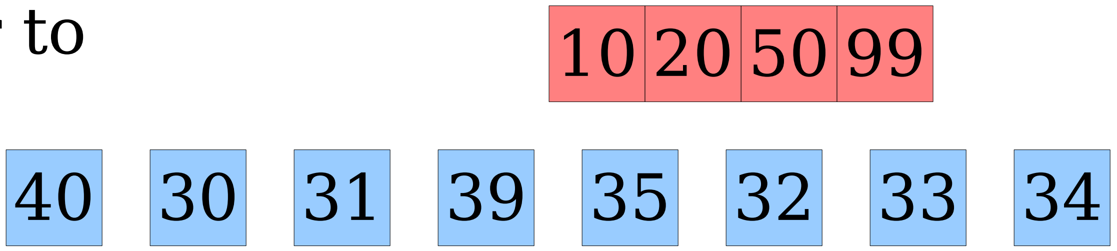
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



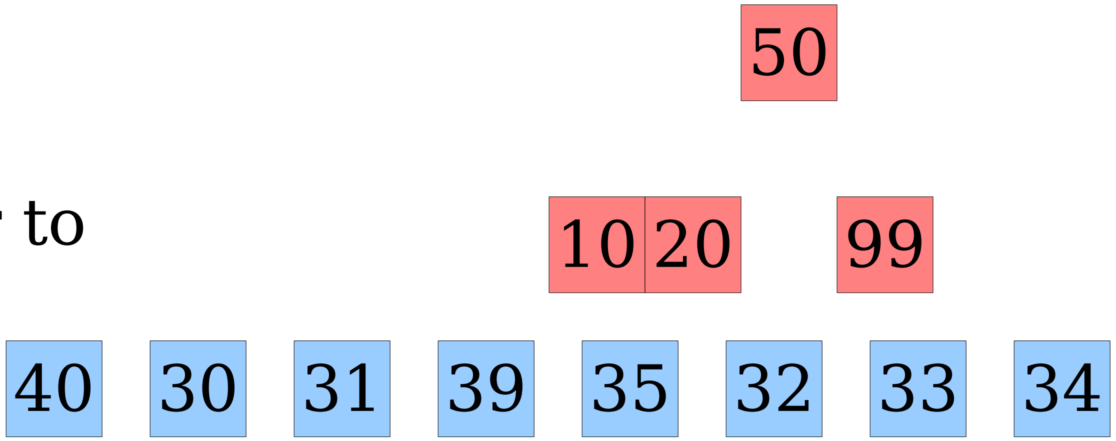
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



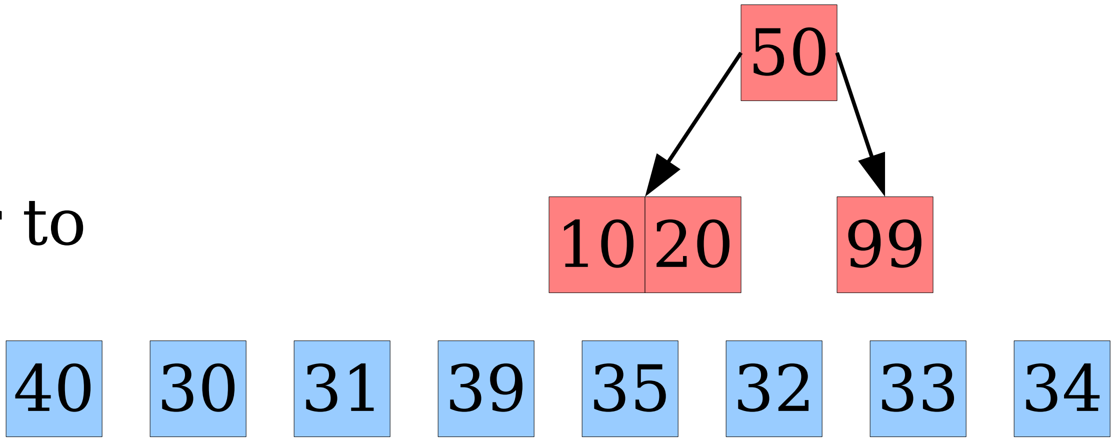
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



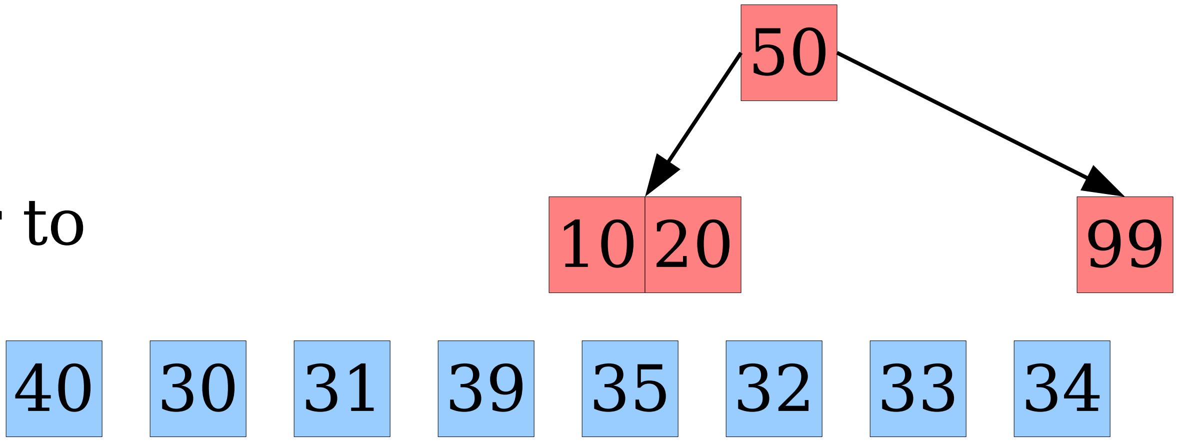
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



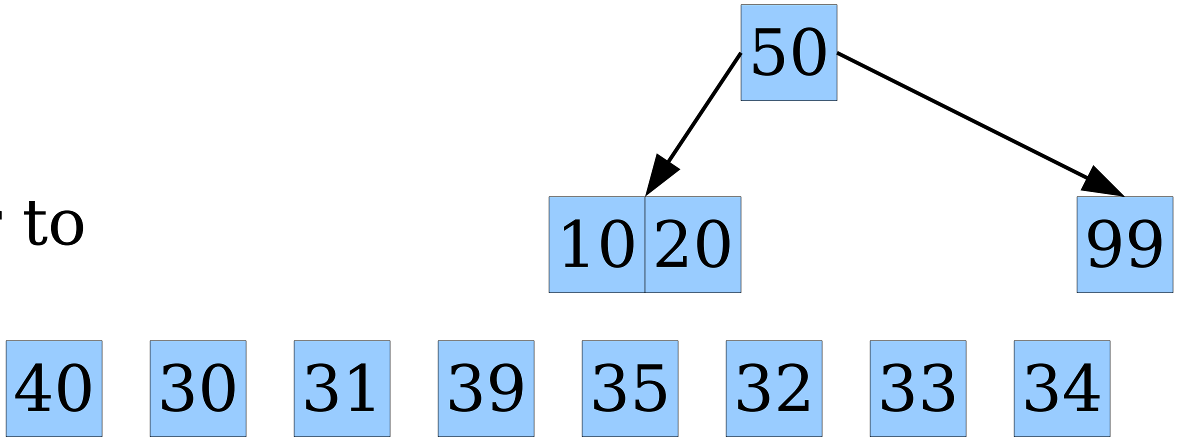
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
    - Simple to implement.
    - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
    - Keeps the tree balanced.
    - Slightly trickier to implement.



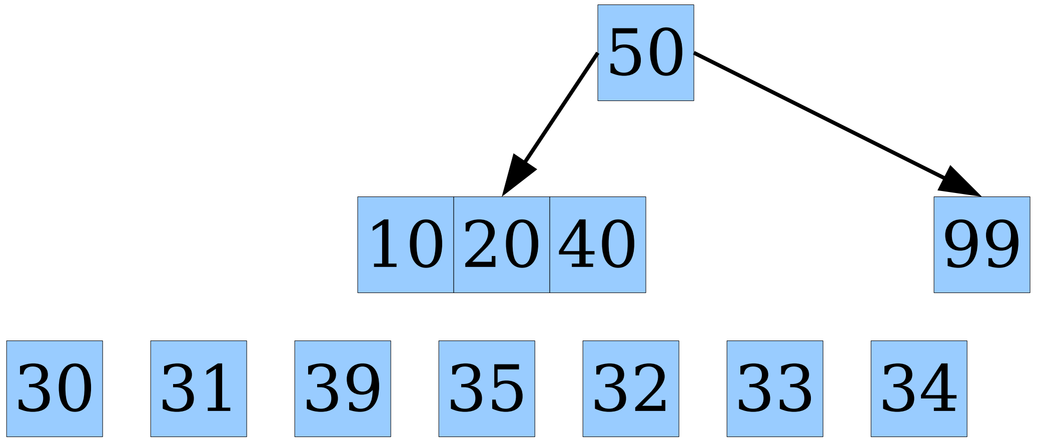
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



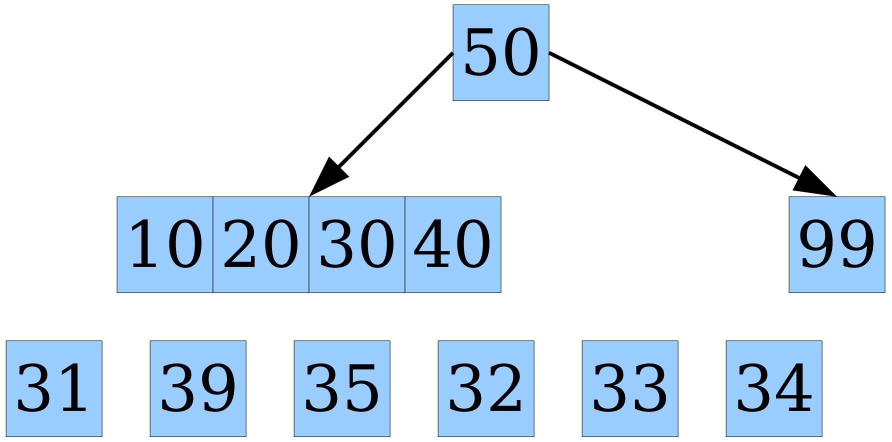
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
    - Simple to implement.
    - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
    - Keeps the tree balanced.
    - Slightly trickier to implement.



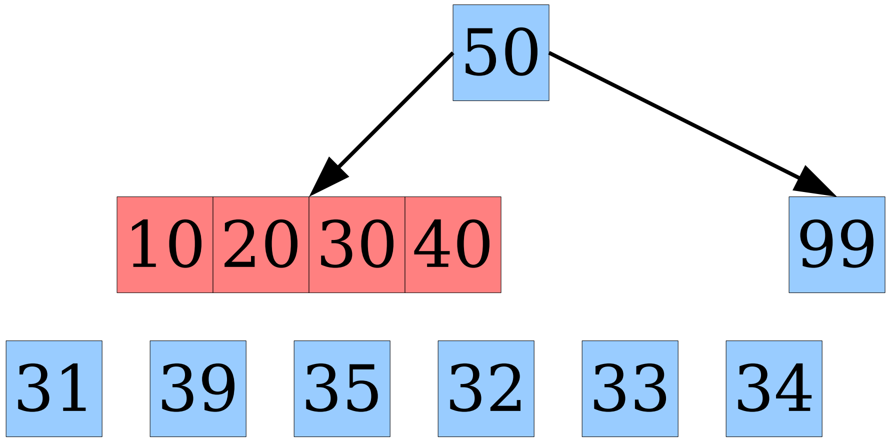
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



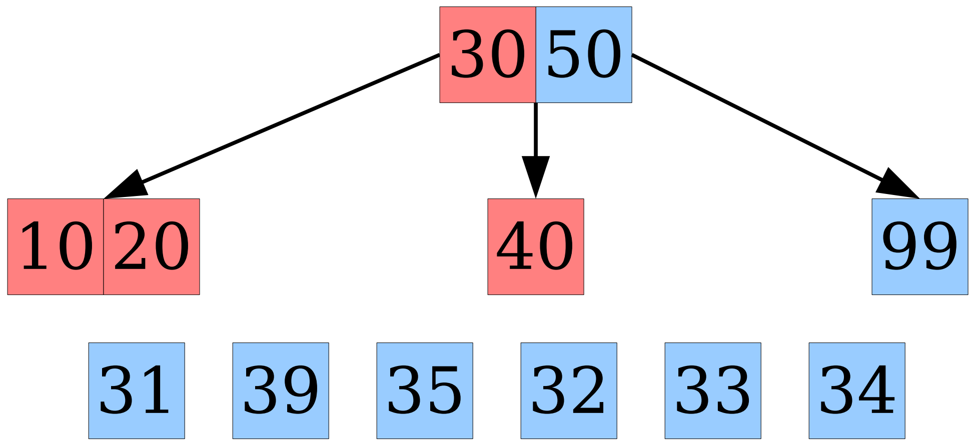
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
    - Simple to implement.
    - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
    - Keeps the tree balanced.
    - Slightly trickier to implement.



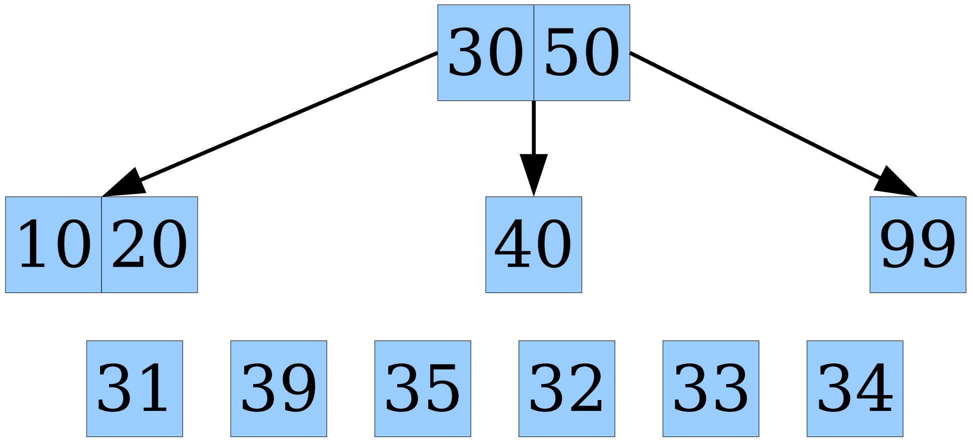
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



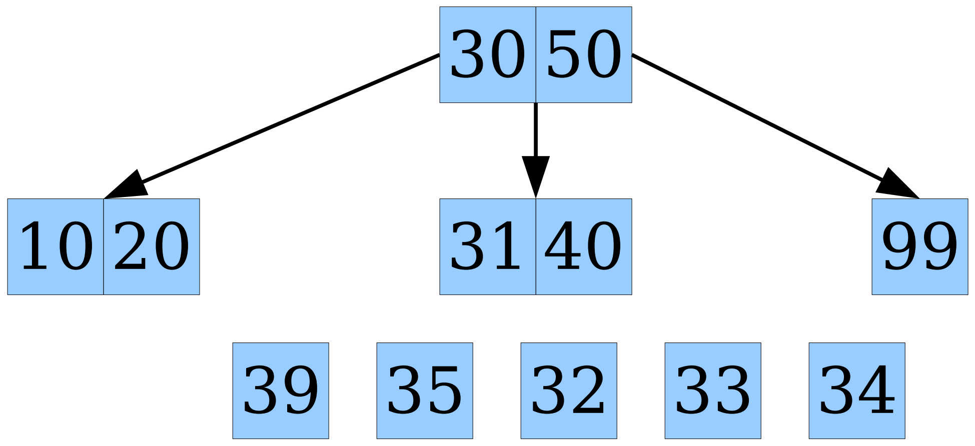
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



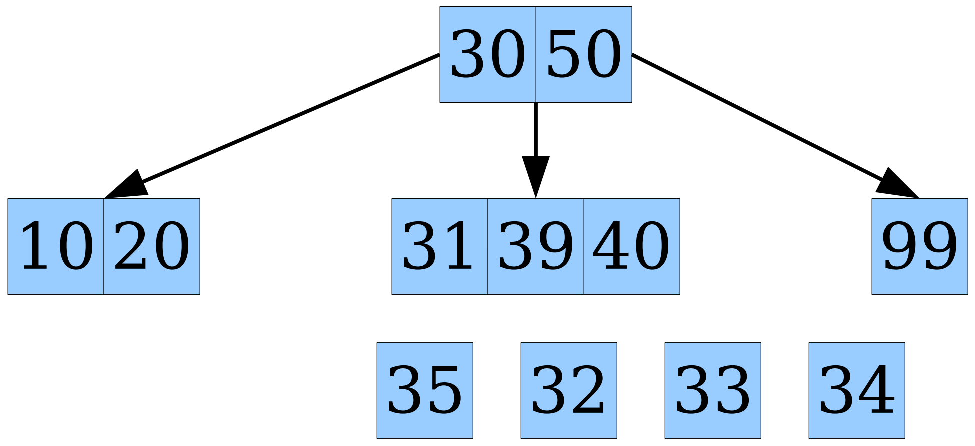
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



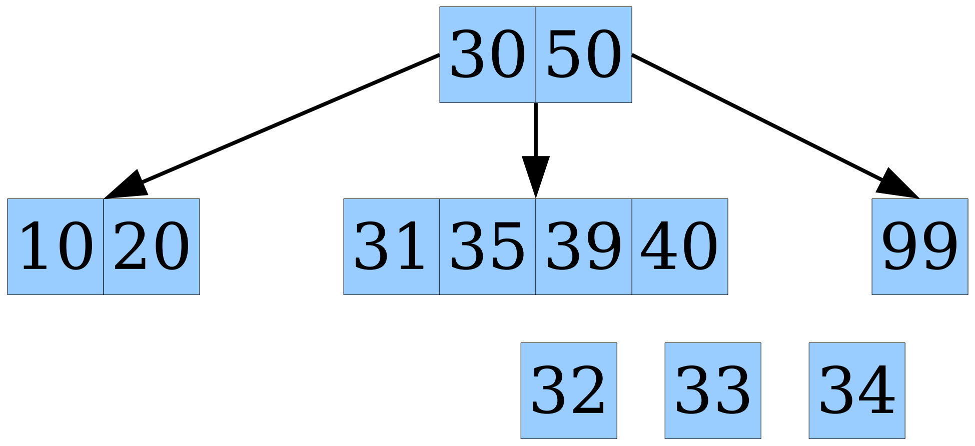
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



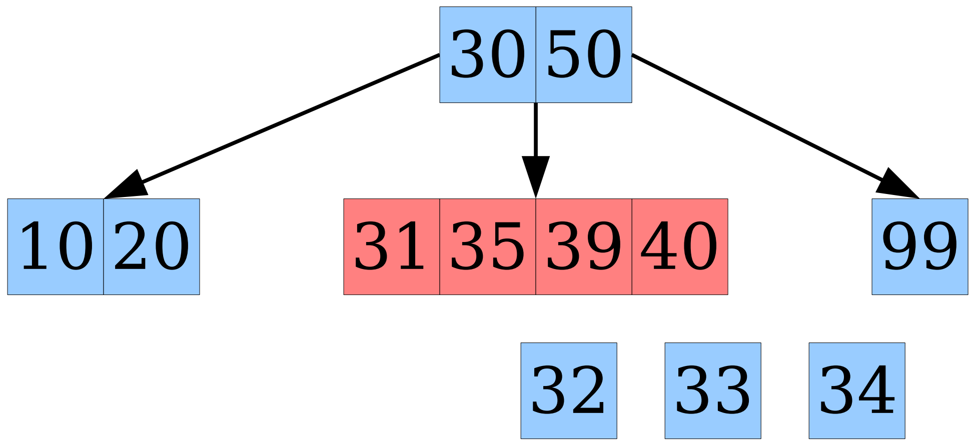
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



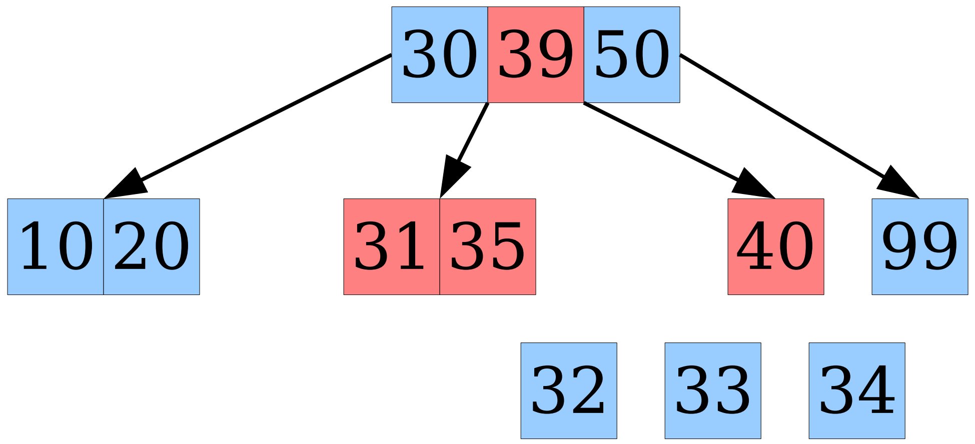
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



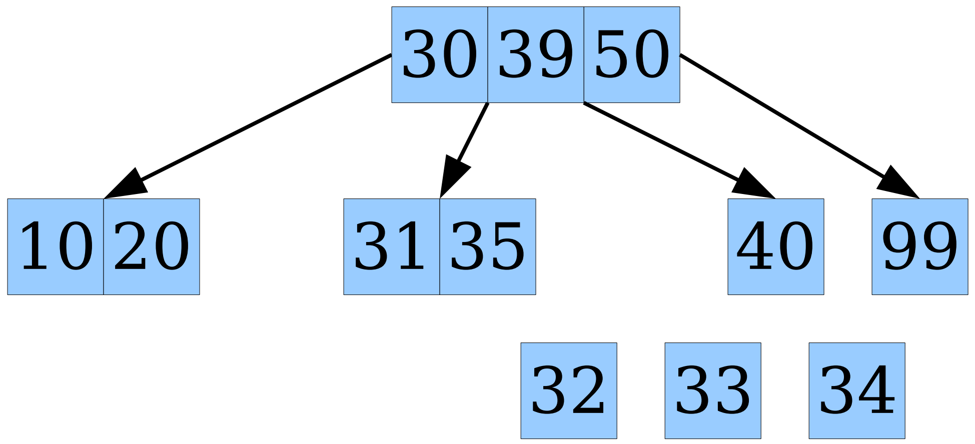
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



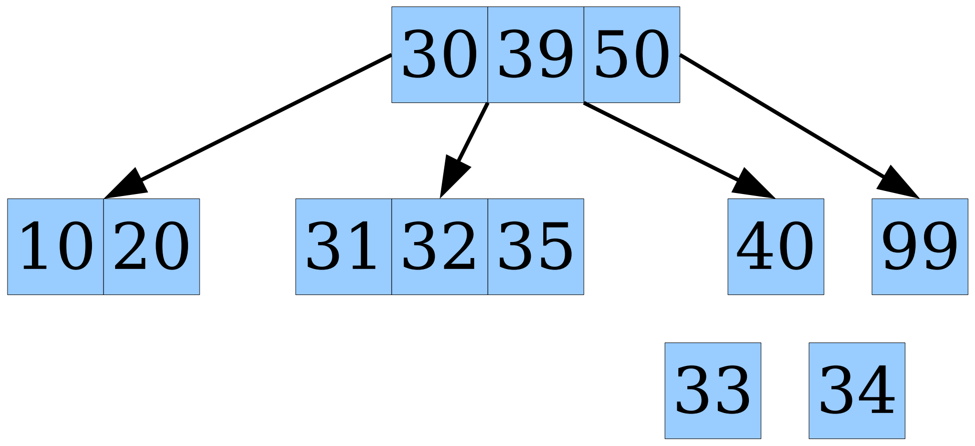
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



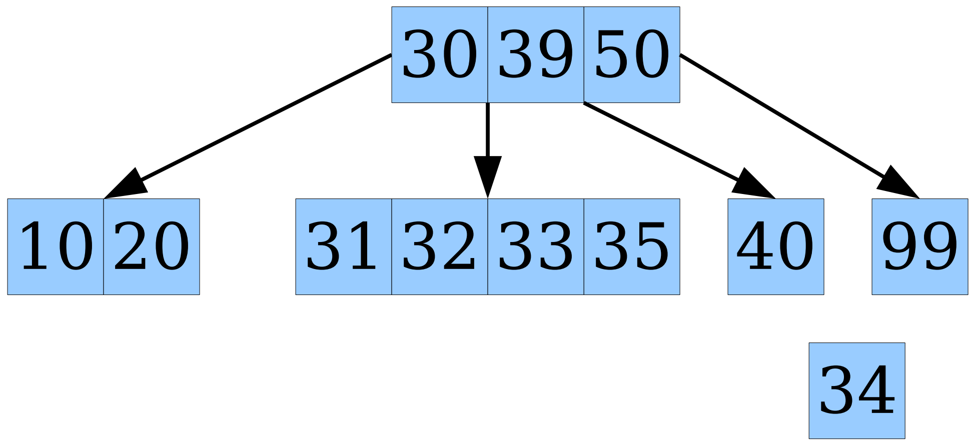
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
    - Simple to implement.
    - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
    - Keeps the tree balanced.
    - Slightly trickier to implement.



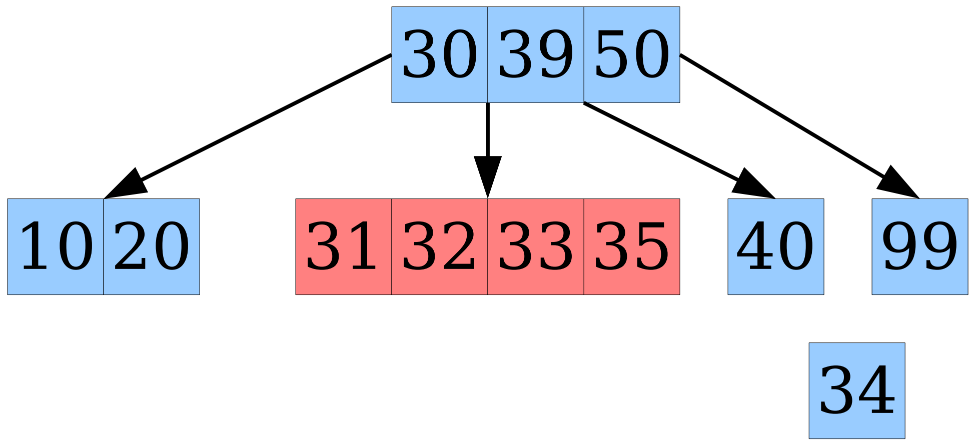
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



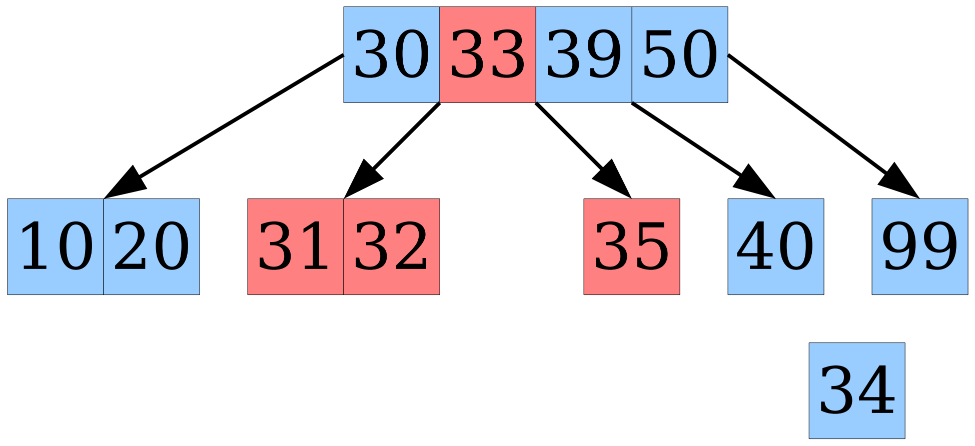
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
    - Simple to implement.
    - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
    - Keeps the tree balanced.
    - Slightly trickier to implement.



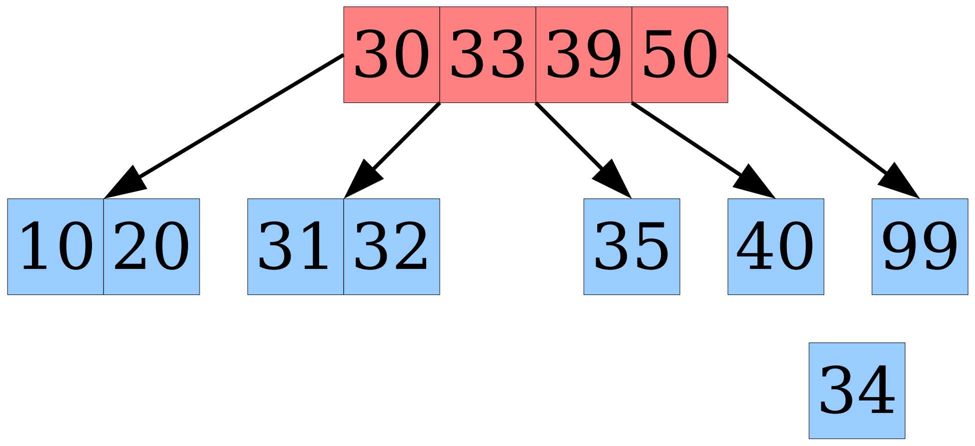
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



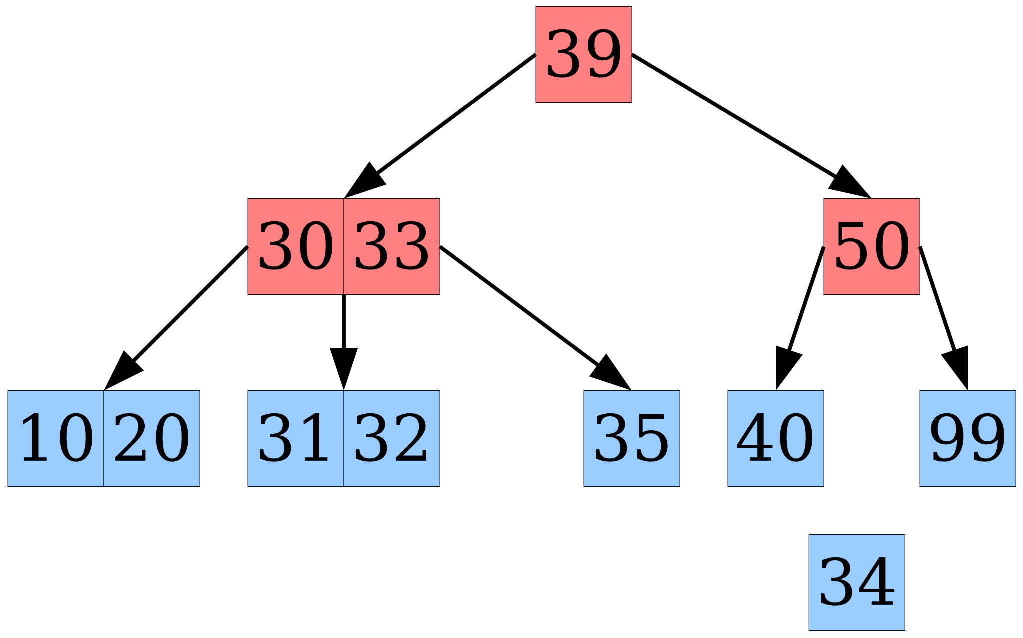
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
    - Simple to implement.
    - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
    - Keeps the tree balanced.
    - Slightly trickier to implement.



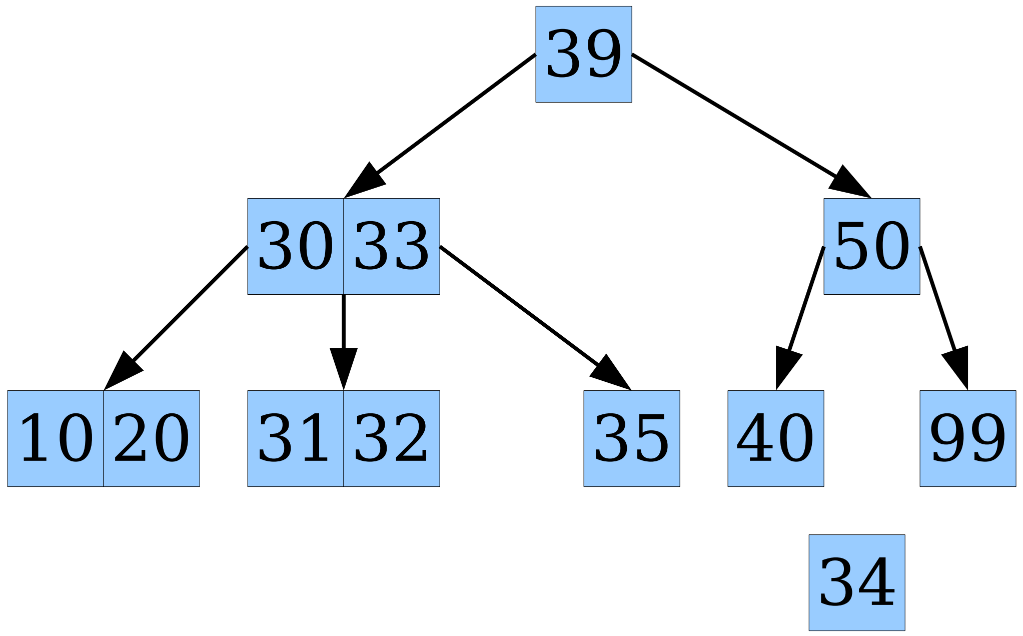
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



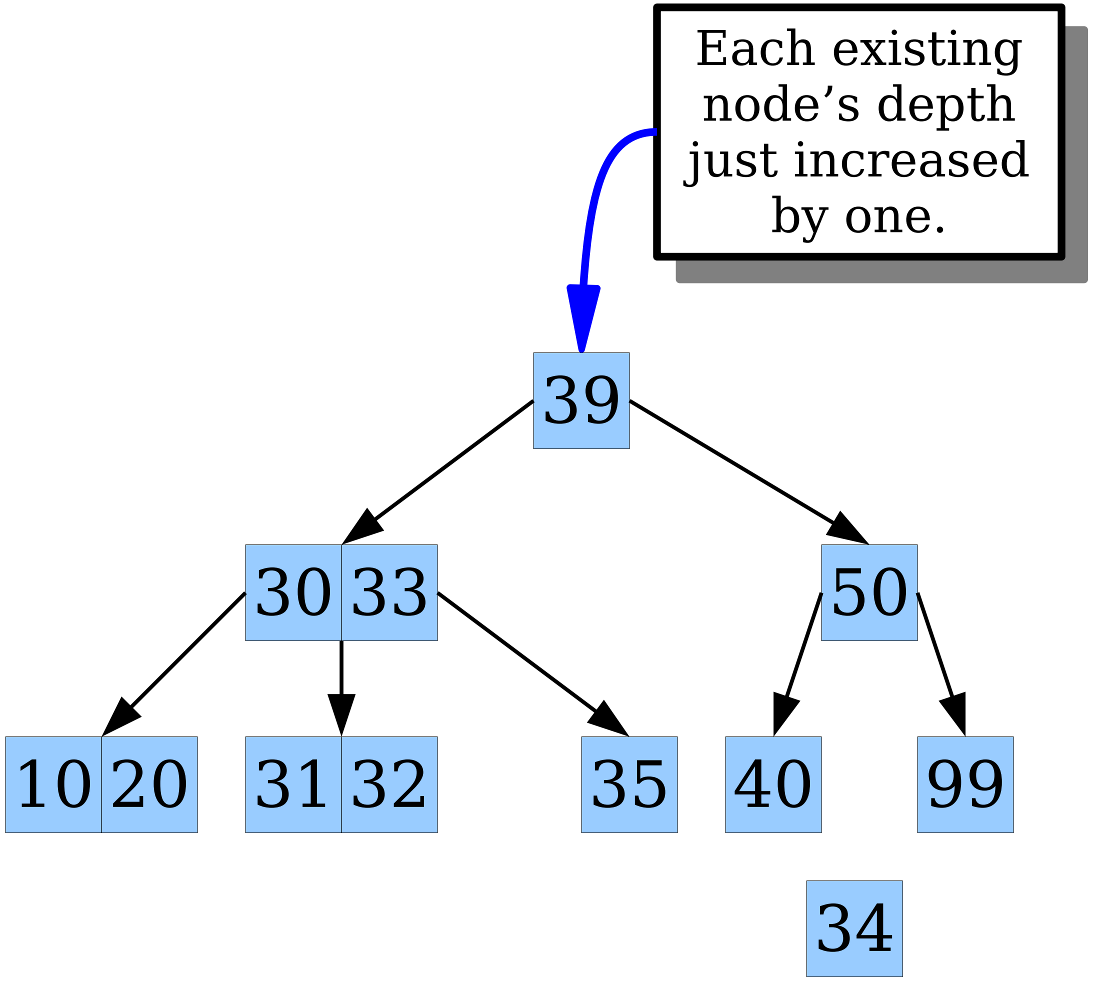
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



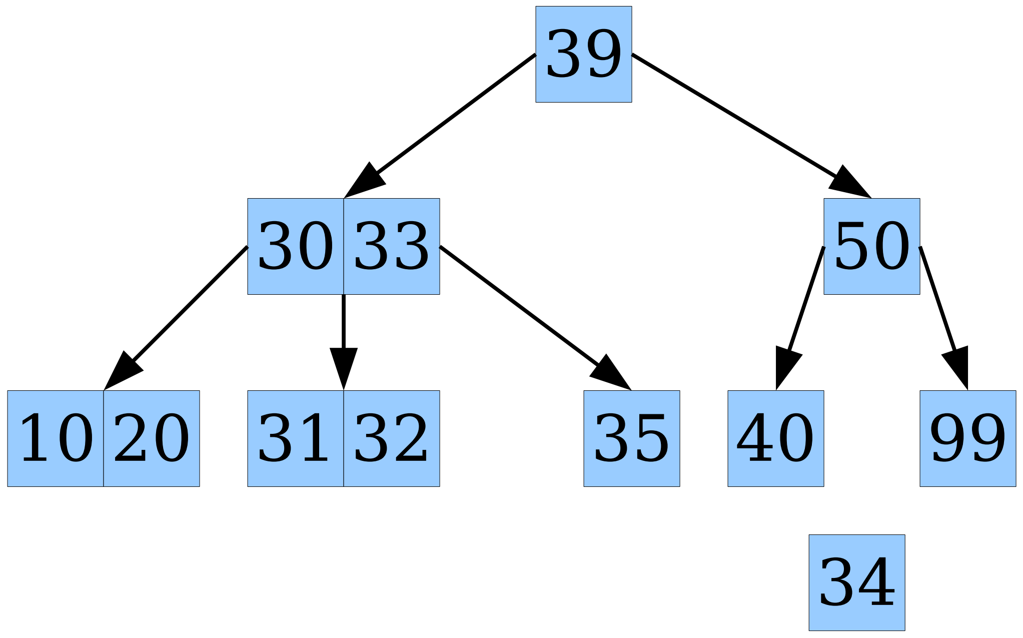
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



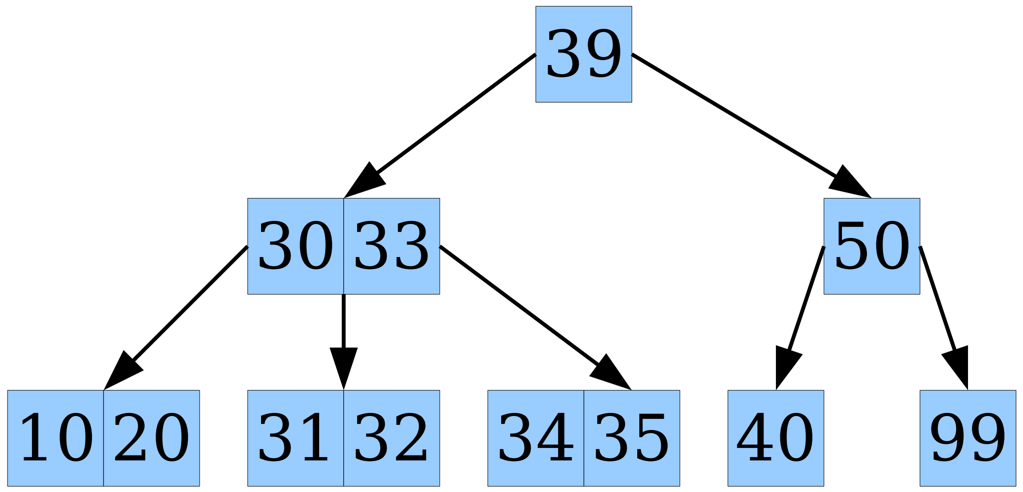
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



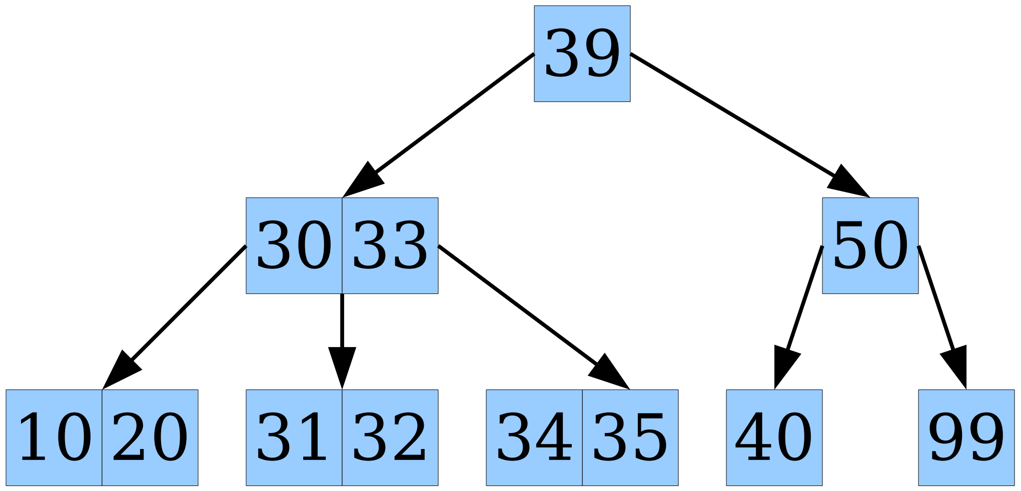
Balanced Multiway Trees

- ***Option 1:*** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- ***Option 2:*** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



Balanced Multiway Trees

- ***General idea:*** Keep nodes holding roughly between b and $2b$ keys, for some parameter b.
  - (Exception: the root node can have fewer keys.)
- If a node gets too big, split it and kick a key higher up.



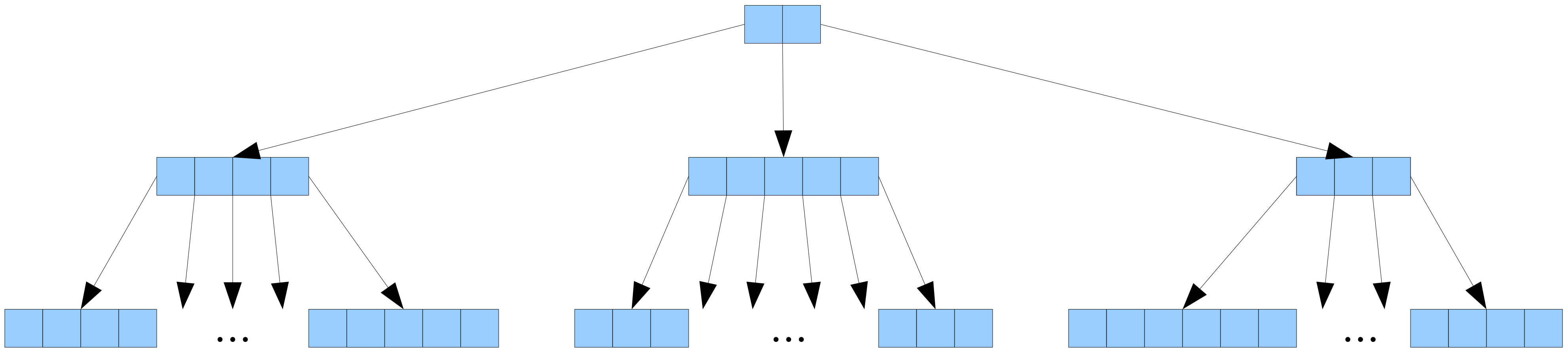
- ***Advantage 1:*** The tree is always balanced.
- ***Advantage 2:*** Insertions and lookups are pretty fast.

Balanced Multiway Trees

- We currently have a ***mechanical definition*** of how these balanced multiway trees work:
  - Nodes should have between roughly b and $2b$ keys in them.
  - Split nodes when they get too big and propagate the splits upward.
- We currently don't have an ***operational definition*** of how these balanced multiway trees work.
  - e.g. "A Cartesian tree for an array is a binary tree that's a min-heap and whose inorder traversal gives back the original array."
  - e.g. "A suffix tree is a Patricia trie with one node for each suffix and branching word of T."

B-Trees

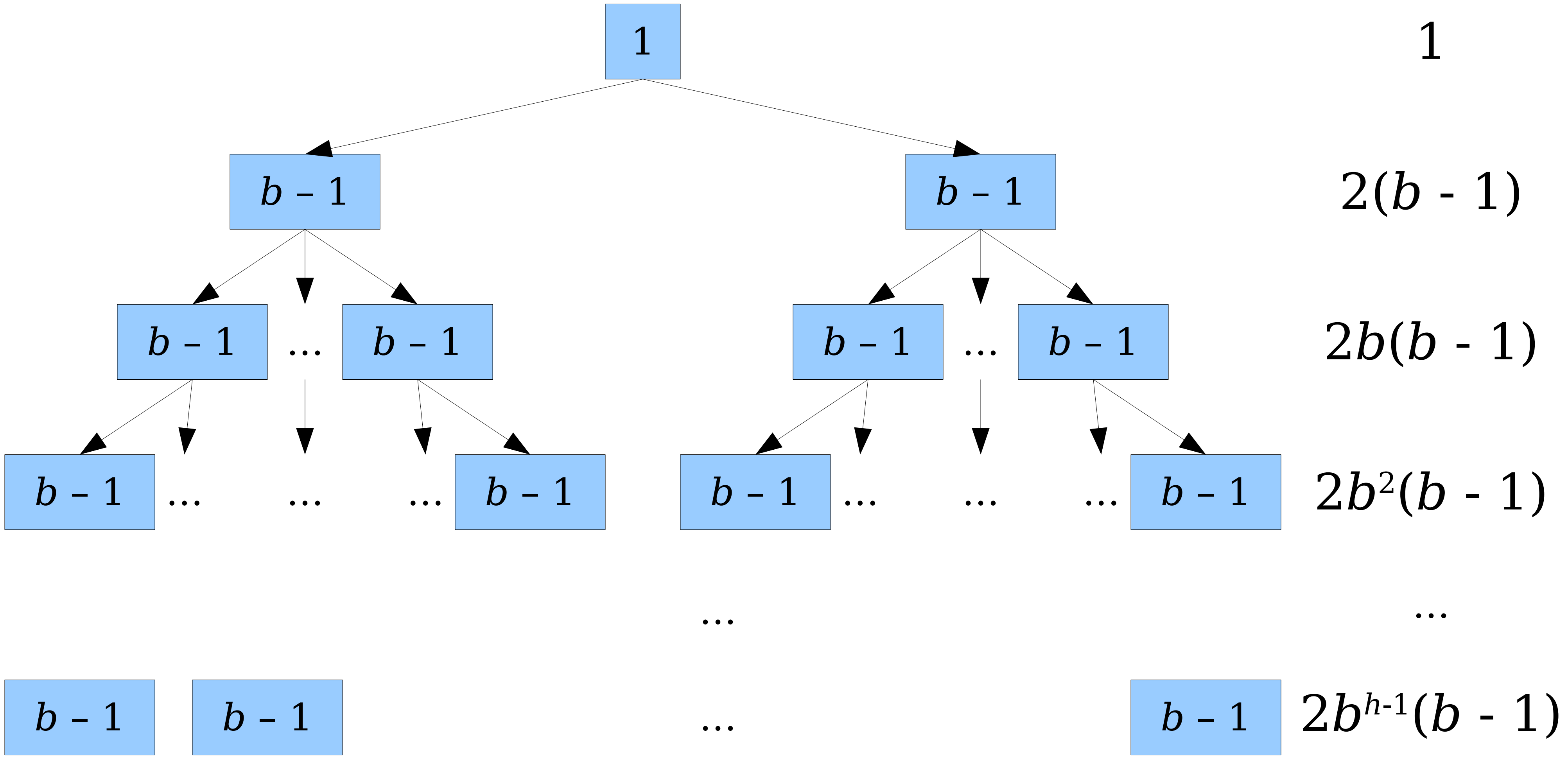
- A ***B-tree of order b*** is a multiway search tree where
  - each node has (roughly) between *b* and 2*b* keys, except the root, which may only have between 1 and *b* keys;
  - each node is either a leaf or has one more child than key; and
  - all leaves are at the same depth.
- Different authors give different bounds on how many keys can be in each node. The ranges are often [*b*–1, 2*b*–1] or [*b*, 2*b*]. For the purposes of today's lecture, we'll use the range [*b*-1, 2*b*-1] for the key limits, just for simplicity.



Analyzing Multiway Trees

The Height of a B-Tree

- What is the maximum possible height of a B-tree of order *b*?



The Height of a B-Tree

- ***Theorem:*** The maximum height of a B-tree of order b containing n keys is $\log_b ((n + 1) / 2)$.

- ***Proof:*** Number of keys n in a B-tree of height h is guaranteed to be at least

$$1 + 2(\boldsymbol{b} - \mathbf{1}) + 2\boldsymbol{b}(\boldsymbol{b} - \mathbf{1}) + 2\boldsymbol{b}^2(\boldsymbol{b} - \mathbf{1}) + \ldots + 2\boldsymbol{b}^{h-1}(\boldsymbol{b} - \mathbf{1})$$

$$= 1 + 2(\boldsymbol{b} - \mathbf{1})(1 + \boldsymbol{b} + \boldsymbol{b}^2 + \ldots + \boldsymbol{b}^{h-1})$$

$$= 1 + 2(\boldsymbol{b} - \mathbf{1})((\boldsymbol{b}^h - 1) / (\boldsymbol{b} - \mathbf{1}))$$

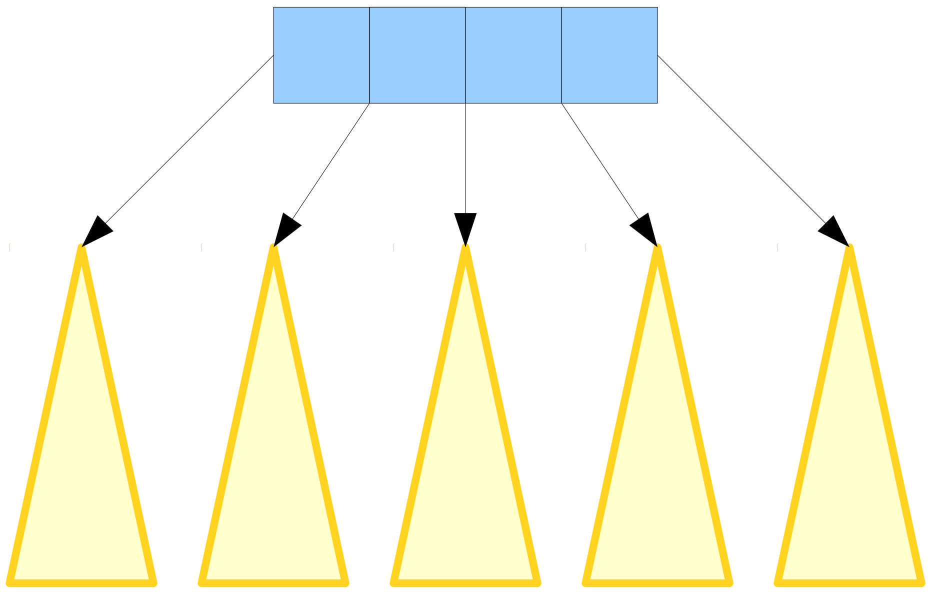
$$= 1 + 2(\boldsymbol{b}^h - 1) = 2\boldsymbol{b}^h - 1.$$

Solving $n = 2b^h - 1$ yields $h = \log_b ((n + 1) / 2)$. ■

- ***Corollary:*** B-trees of order b have height $\Theta(\log_b n)$.

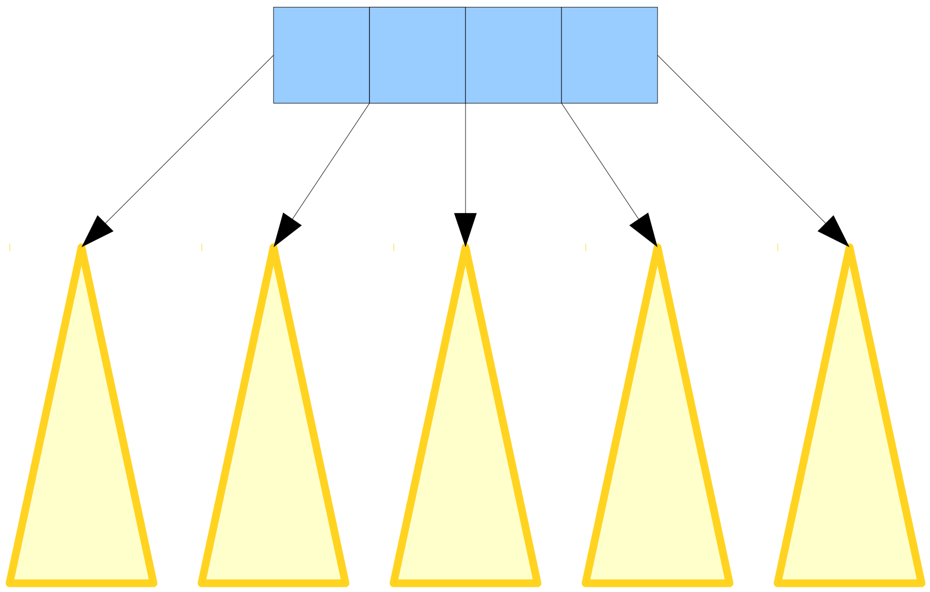
Analyzing Efficiency

- Suppose we have a B-tree of order *b*.
- What is the worst-case runtime of looking up a key in the B-tree?
- ***Answer:*** It depends on how we do the search!



Analyzing Efficiency

- To do a lookup in a B-tree, we need to determine which child tree to descend into.
- This means we need to compare our query key against the keys in the node.
- ***Question:*** How should we do this?

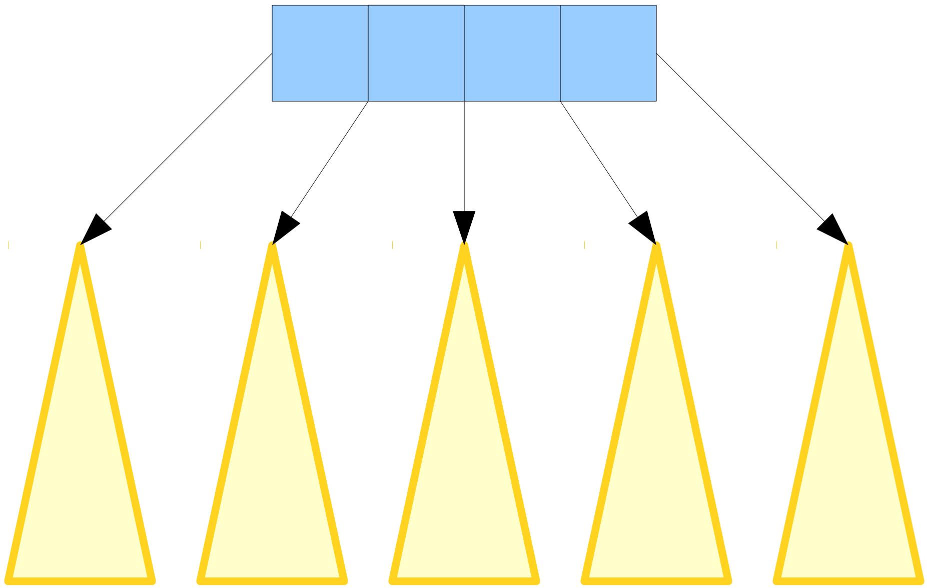


Analyzing Efficiency

- ***Option 1:*** Use a linear search!
- Cost per node: $O(b)$.
- Nodes visited: $O(\log_b n)$.
- Total cost:

$$O(b) \cdot O(\log_b n)$$

$$= \mathbf{O(b \log_b n)}$$



Analyzing Efficiency

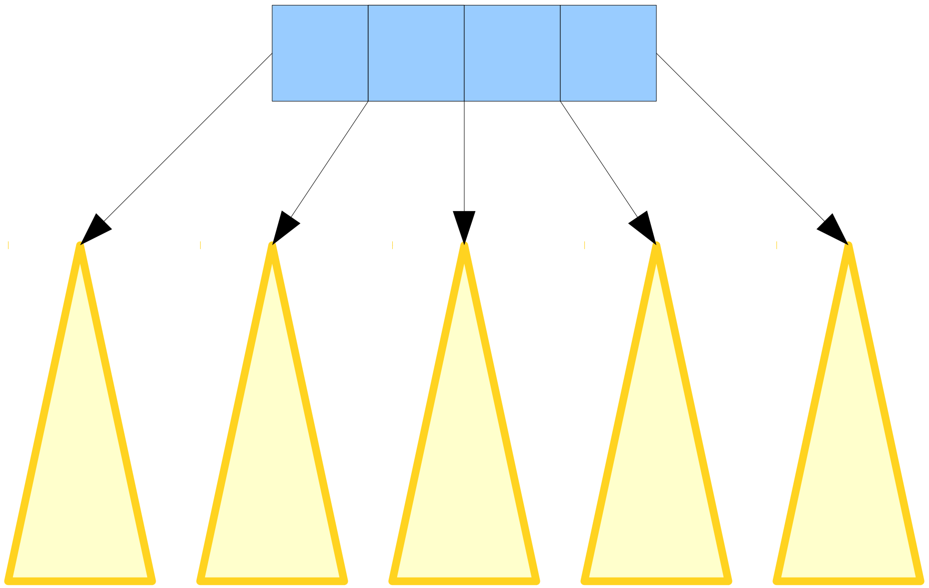
- ***Option 2:*** Use a binary search!
- Cost per node: O(log b).
- Nodes visited: O($\log_b n$).
- Total cost:

$$O(\log b) \cdot O(\log_b n)$$

$$= O(\log b \cdot \log_b n)$$

$$= O(\log b \cdot (\log n) / (\log b))$$

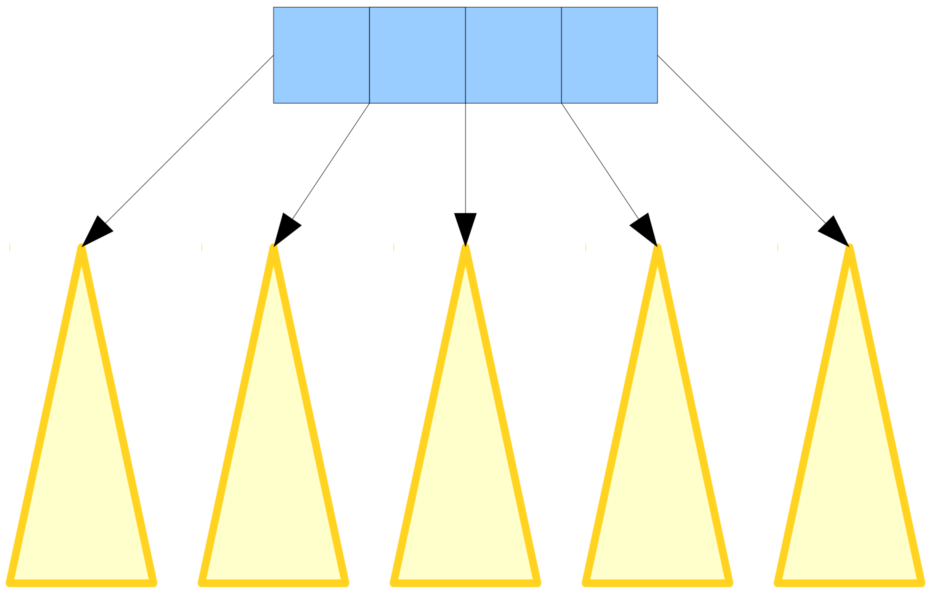
$$= \mathbf{O(\log n)}.$$



Intuition: We can't do better than O(log n) for arbitrary data, because it's the information-theoretic minimum number of comparisons needed to find something in a sorted collection!

Analyzing Efficiency

- Suppose we have a B-tree of order b.
- What is the worst-case runtime of inserting a key into the B-tree?
- Each insertion visits $O(\log_b n)$ nodes, and in the worst case we have to split every node we see.
- ***Answer:*** $O(b \log_b n)$.



Analyzing Efficiency

- The cost of an insertion in a B-tree of order b is $O(b \log_b n)$.
- What's the best choice of b to use here?
- Note that

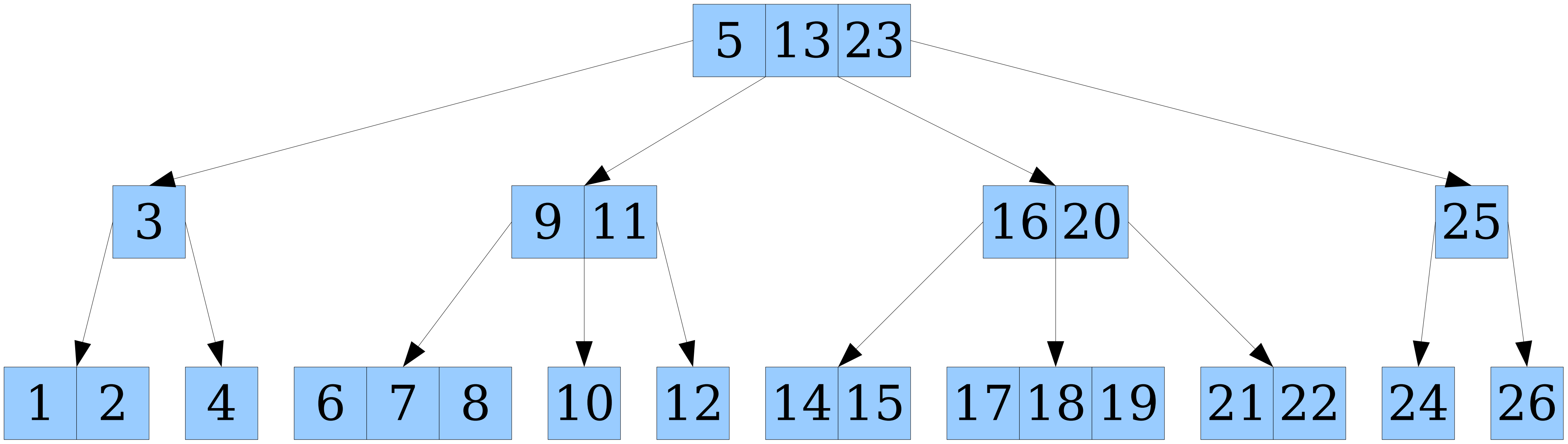
$$\begin{aligned} & b \log_b n \\ = & b (\log n / \log b) \\ = & (b / \log b) \log n. \end{aligned}$$

Fun fact: This is the same time bound you'd get if you used a b-ary heap instead of a binary heap for a priority queue.

- What choice of b minimizes $b / \log b$?
- ***Answer:*** Pick $b = e$.

2-3-4 Trees

- A ***2-3-4 tree*** is a B-tree of order 2. Specifically:
  - each node has between 1 and 3 keys;
  - each node is either a leaf or has one more child than key; and
  - all leaves are at the same depth.
- You actually saw this B-tree earlier! It's the type of tree from our insertion example.



The Story So Far

- A B-tree supports
  - lookups in time O(log n), and
  - insertions in time O($b \log_b n$).
- Picking b to be around 2 or 3 makes this optimal in Theoryland.
  - The 2-3-4 tree is great for that reason.
- ***Plot Twist:*** In practice, you most often see choices of b like 1,024 or 4,096.
- ***Question:*** Why would anyone do that?

Theoryland
IRL

The Memory Hierarchy

Memory Tradeoffs

- There is an enormous tradeoff between *speed* and *size* in memory.
- SRAM (the stuff registers are made of) is fast but very expensive:
  - Can keep up with processor speeds in the GHz.
  - As of 2010, cost is $5/MB. *(Anyone know a good source for a more recent price?)*
  - Good luck buying 1TB of the stuff!
- Hard disks are cheap but very slow:
  - As of 2019, you can buy a 4TB hard drive for about $70.
  - As of 2019, good disk seek times for magnetic drives are measured in ms (about two to four million times slower than a processor cycle!)

The Memory Hierarchy

- ***Idea:*** Try to get the best of all worlds by using multiple types of memory.

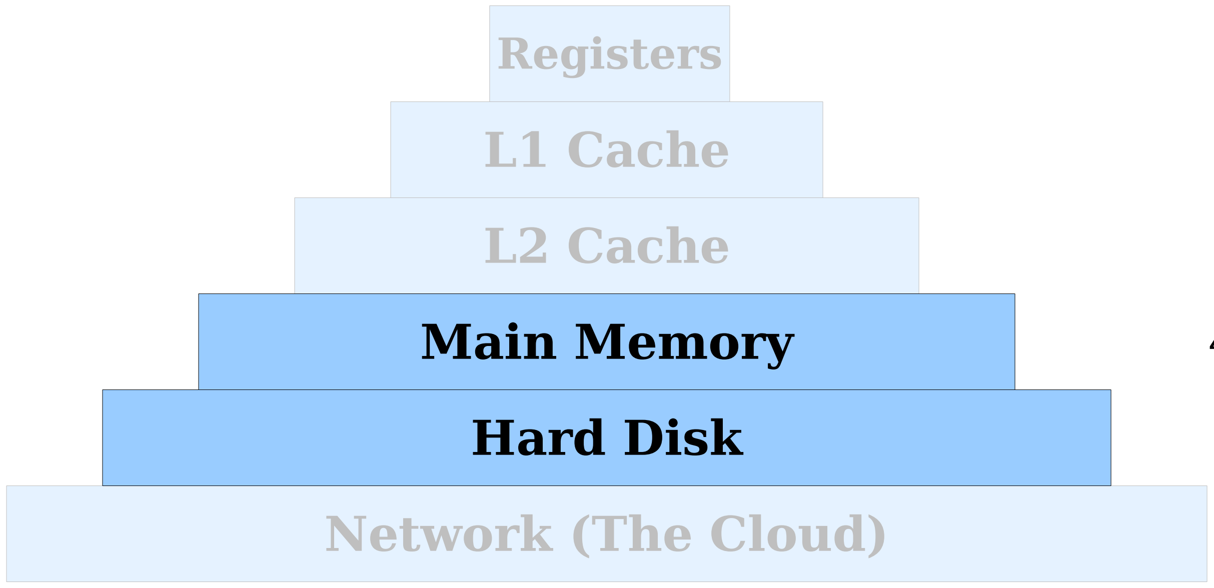
The Memory Hierarchy

- ***Idea:*** Try to get the best of all worlds by using multiple types of memory.

Registers	256B - 8KB	0.25 – 1ns
L1 Cache	16KB – 64KB	1ns – 5ns
L2 Cache	1MB - 4MB	5ns – 25ns
Main Memory	4GB – 256GB	25ns – 100ns
Hard Disk	1TB+	3 – 10ms
Network (The Cloud)	*Lots*	10 – 2000ms

The Memory Hierarchy

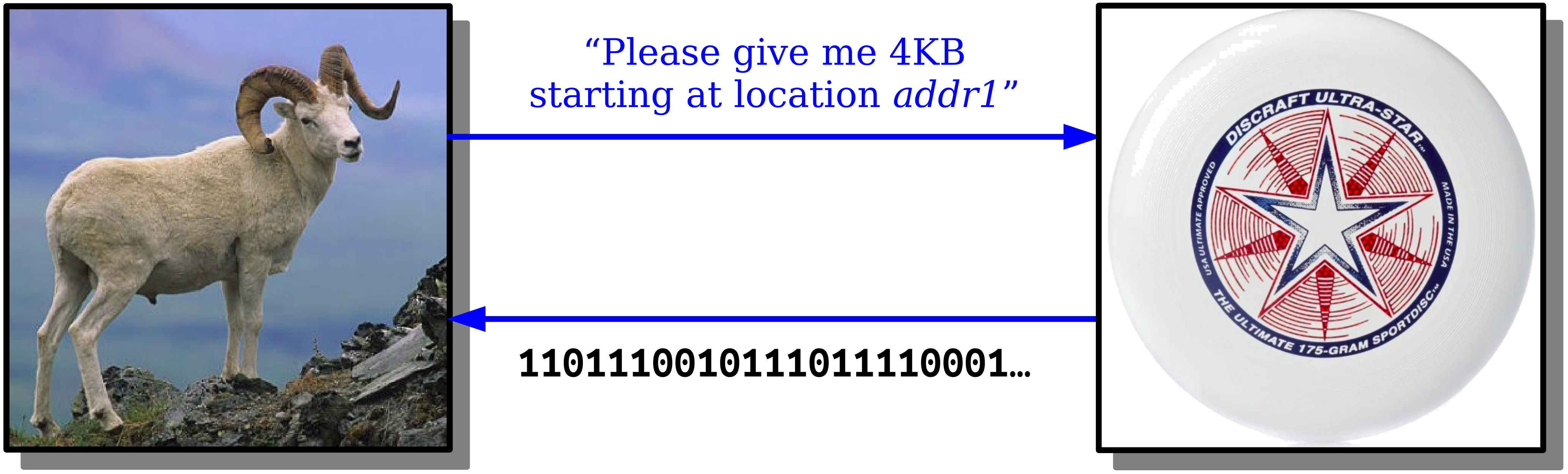
- ***Idea:*** Try to get the best of all worlds by using multiple types of memory.



256B - 8KB	0.25 – 1ns
16KB – 64KB	1ns – 5ns
1MB - 4MB	5ns – 25ns
4GB – 256GB	25ns – 100ns
1TB+	3 – 10ms
Lots	10 – 2000ms

External Data Structures

- Suppose you have a data set that's *way* too big to fit in RAM.
- The data structure is on disk and read into RAM as needed.
- Data from disk doesn't come back one *byte* at a time, but rather one *page* at a time.
- ***Goal:*** Minimize the number of disk reads and writes, not the number of instructions executed.

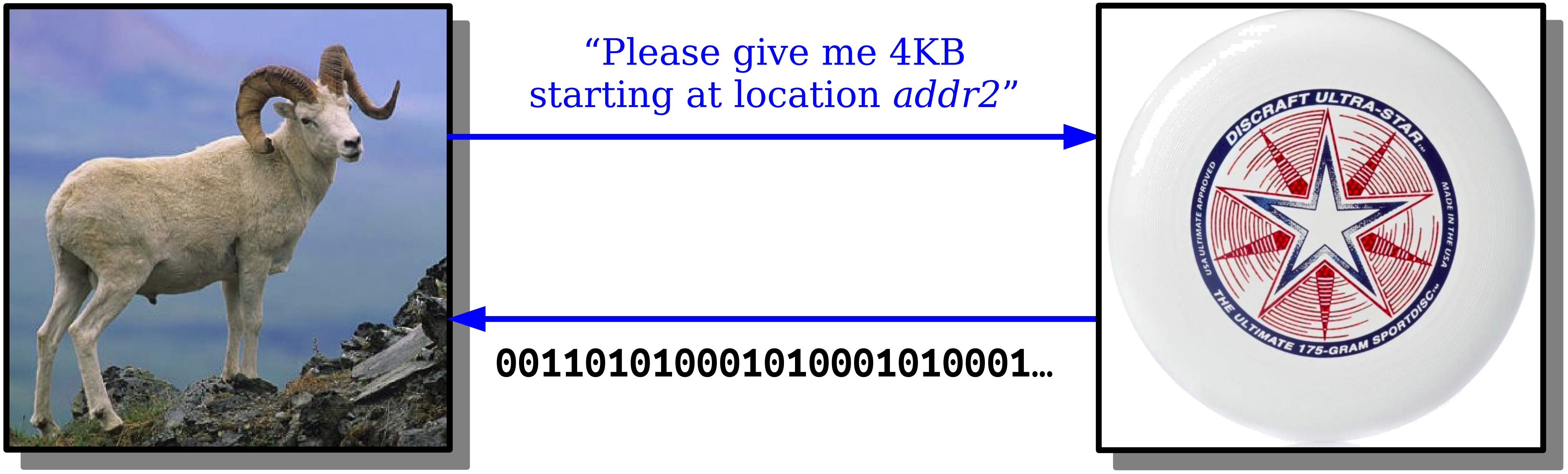

"Please give me 4KB starting at location *addr1*"
1101110010111011110001…
DISCRAFT ULTRA-STAR
USA ULTIMATE APPROVED
MADE IN THE USA
THE ULTIMATE 175-GRAM SPORTDISC

External Data Structures

- Suppose you have a data set that's *way* too big to fit in RAM.
- The data structure is on disk and read into RAM as needed.
- Data from disk doesn't come back one *byte* at a time, but rather one *page* at a time.
- ***Goal:*** Minimize the number of disk reads and writes, not the number of instructions executed.

External Data Structures

- Suppose you have a data set that's *way* too big to fit in RAM.
- The data structure is on disk and read into RAM as needed.
- Data from disk doesn't come back one *byte* at a time, but rather one *page* at a time.
- ***Goal:*** Minimize the number of disk reads and writes, not the number of instructions executed.

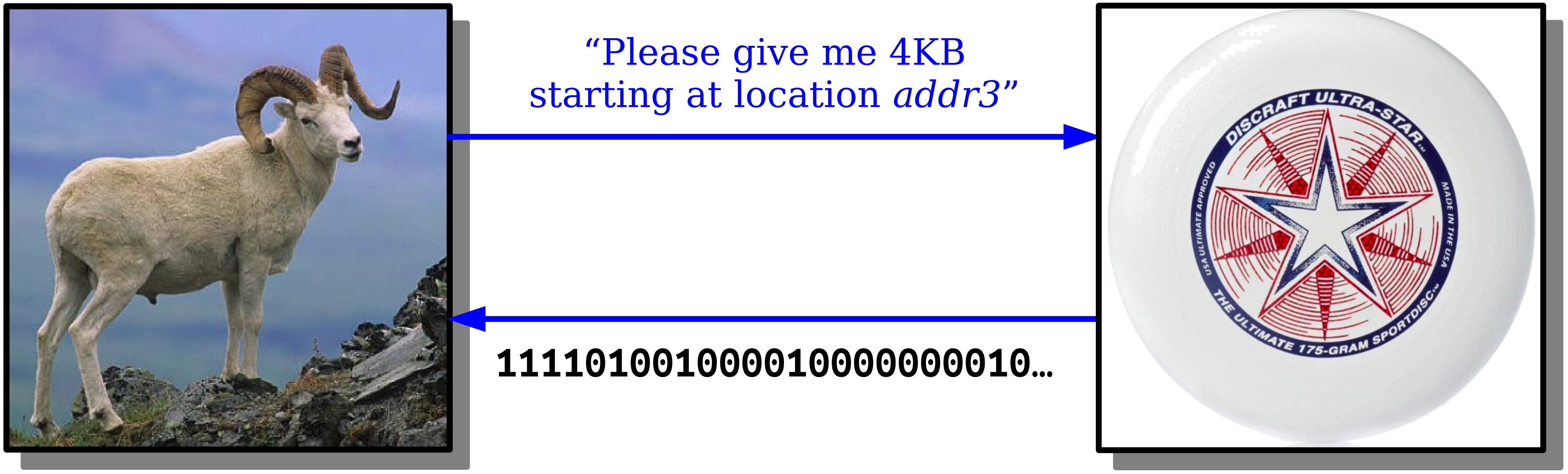


External Data Structures

- Suppose you have a data set that's *way* too big to fit in RAM.
- The data structure is on disk and read into RAM as needed.
- Data from disk doesn't come back one *byte* at a time, but rather one *page* at a time.
- ***Goal:*** Minimize the number of disk reads and writes, not the number of instructions executed.

External Data Structures

- Suppose you have a data set that’s *way* too big to fit in RAM.
- The data structure is on disk and read into RAM as needed.
- Data from disk doesn’t come back one *byte* at a time, but rather one *page* at a time.
- ***Goal:*** Minimize the number of disk reads and writes, not the number of instructions executed.



External Data Structures

- Because B-trees have a huge branching factor, they're great for on-disk storage.
  - Disk block reads/writes are glacially slow.
  - The high branching factor minimizes the number of blocks to read during a lookup.
  - Extra work scanning inside a block offset by these savings.
- Major use cases for B-trees and their variants (B^+-trees, H-trees, etc.) include
  - databases (huge amount of data stored on disk);
  - file systems (ext4, NTFS, ReFS); and, recently,
  - in-memory data structures (due to cache effects).

Analyzing B-Trees

- Suppose we tune b so that each node in the B-tree fits inside a single disk page.
- We *only* care about the number of disk pages read or written.
  - It's so much slower than RAM that it'll dominate the runtime.
- ***Question:*** What is the cost of a lookup in a B-tree in this model?
  - Answer: The height of the tree, $O(\log_b n)$.
- ***Question:*** What is the cost of inserting into a B-tree in this model?
  - Answer: The height of the tree, $O(\log_b n)$.

Analyzing B-Trees

- The cost model we use will change our overall analysis.
- Cost is number of operations:

  O(log n) / lookup, O(b log$_\mathrm{b}$ n) / insertion.

- Cost is number of blocks accessed:

  O(log$_b$ n) / lookup, O(log$_b$ n) / insertion.

- Going forward, we'll use operation counts as our cost model, though looking at caching effects of data structures would make for an awesome final project!

The Story So Far

- We’ve just built a simple, elegant, balanced multiway tree structure.
- We can use them as balanced trees in main memory (2-3-4 trees).
- We can use them to store huge quantities of information on disk (B-trees).
- We’ve seen that different cost models are appropriate in different situations.

Time-Out for Announcements!

CS Townhall

- John Mitchell (CS Department Chair) and Mehran Sahami (CS Associate Chair for Education) are holding a CS townhall event next month.
- What are we doing well? What can we improve on? This is your chance to provide input!
- Held in Gates 104 from 4:30PM – 6:00PM on Tuesday, May 14th.

Problem Sets

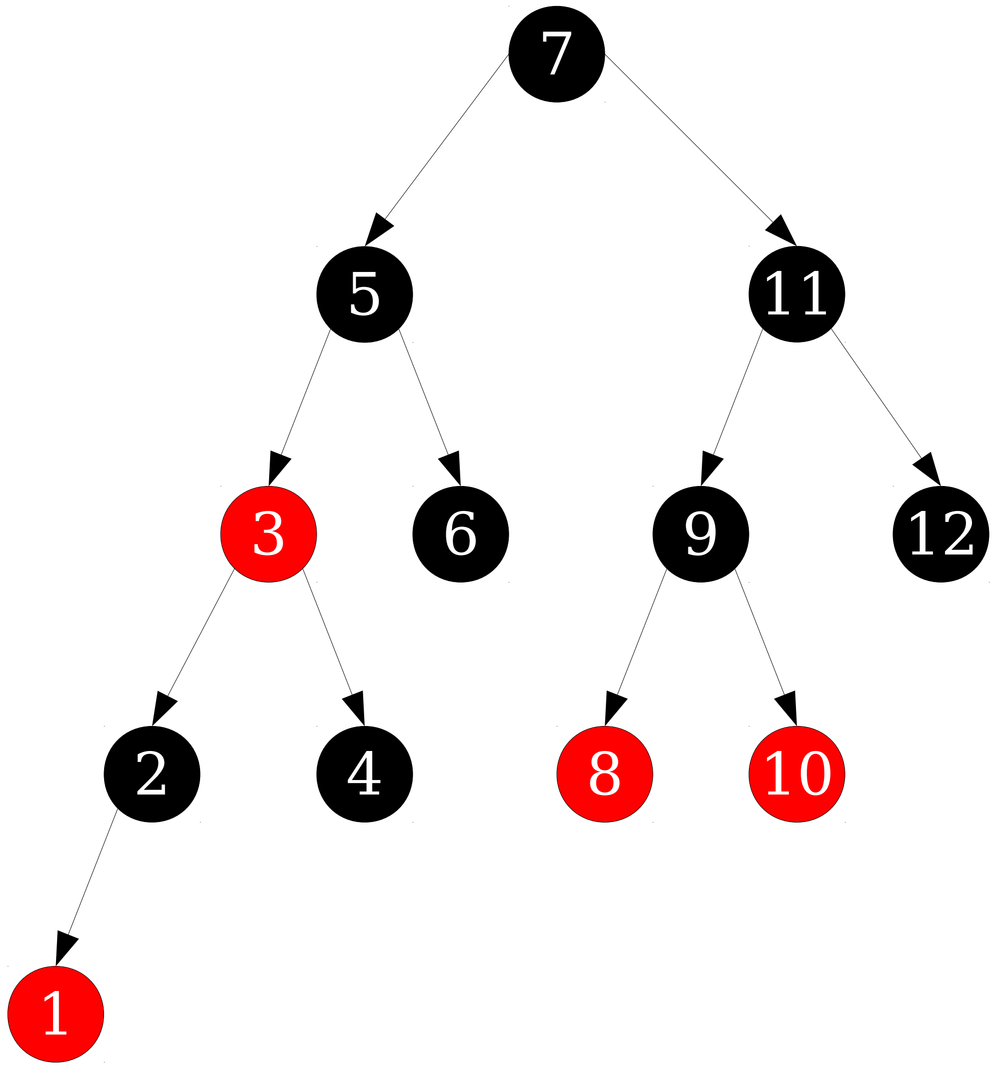
- Problem Set One solutions are now available up on the course website.
  - We’re working on getting them graded – stay tuned!
- Problem Set Two is due next Thursday.
  - Have questions? Ask them on Piazza or stop by our office hours!

Back to CS166!

So… red/black trees?

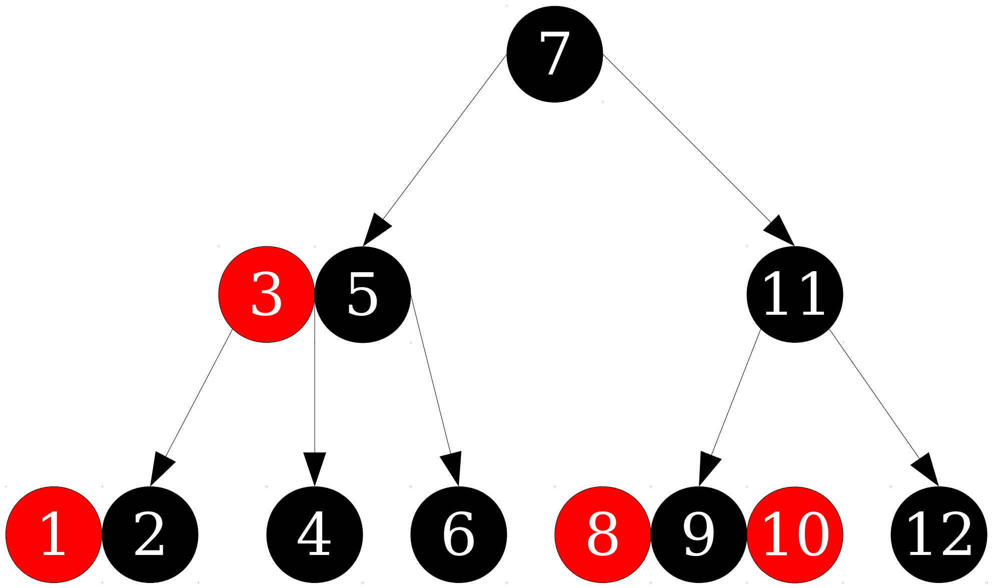
Red/Black Trees

- A ***red/black tree*** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.



Red/Black Trees

- A ***red/black tree*** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.
- After we hoist red nodes into their parents:
  - Each "meta node" has 1, 2, or 3 keys in it. (No red node has a red child.)
  - Each "meta node" is either a leaf or has one more key than node. (Root-null path property.)
  - Each "meta leaf" is at the same depth. (Root-null path property.)



This is a 2-3-4 tree!

Data Structure Isometries

- Red/black trees are an ***isometry*** of 2-3-4 trees; they represent the structure of 2-3-4 trees in a different way.
- Many data structures can be designed and analyzed in the same way.
- ***Huge advantage:*** Rather than memorizing a complex list of red/black tree rules, just think about what the equivalent operation on the corresponding 2-3-4 tree would be and simulate it with BST operations.

The Height of a Red/Black Tree

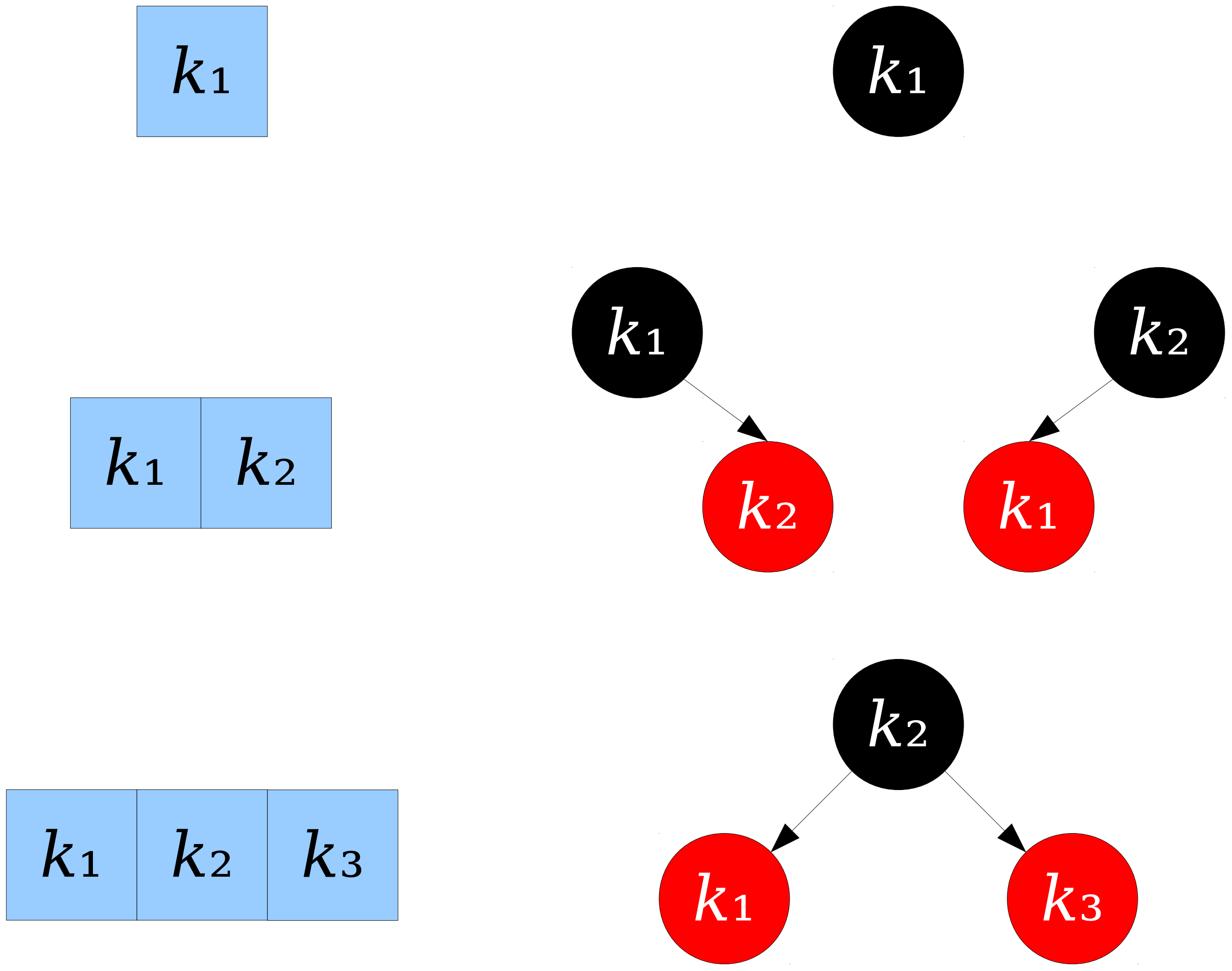
Theorem: Any red/black tree with n nodes has height $O(\log n)$.

Proof: Contract all red nodes into their parent nodes to convert the red/black tree into a 2-3-4 tree. This decreases the height of the tree by at most a factor of two. The resulting 2-3-4 tree has height $O(\log n)$, so the original red/black tree has height $2 \cdot O(\log n) = O(\log n)$. ■

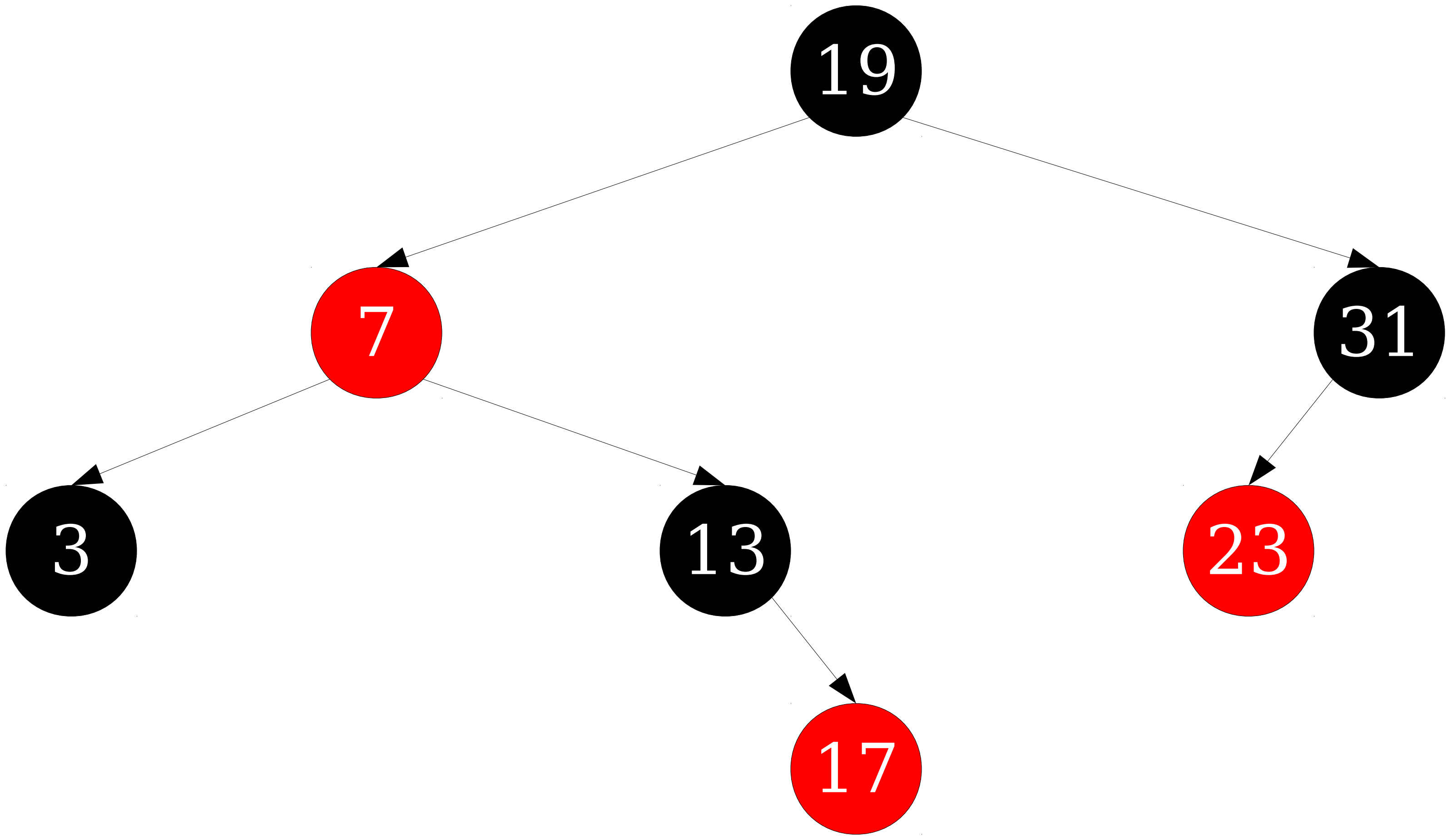
Exploring the Isometry

- Nodes in a 2-3-4 tree are classified into types based on the number of children they can have.
  - ***2-nodes*** have one key (two children).
  - ***3-nodes*** have two keys (three children).
  - ***4-nodes*** have three keys (four children).
- How might these nodes be represented?

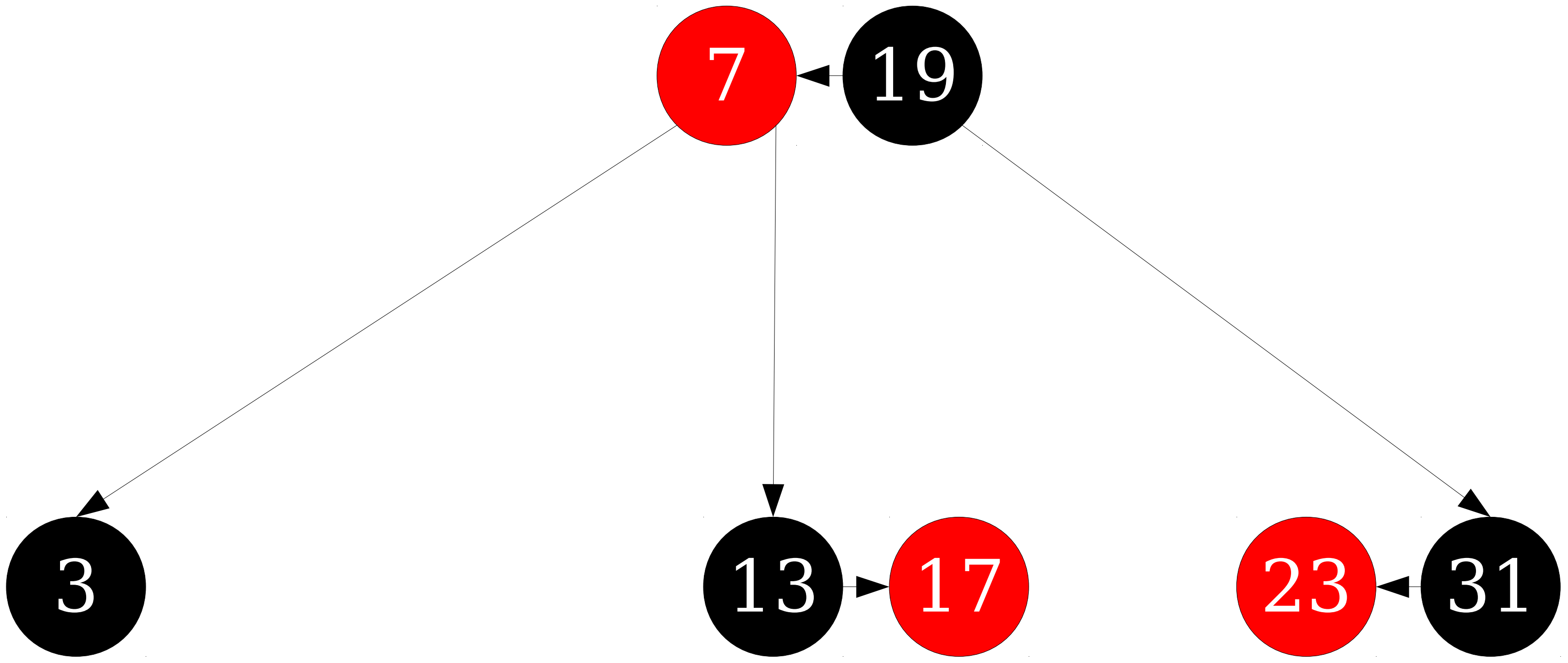
Exploring the Isometry



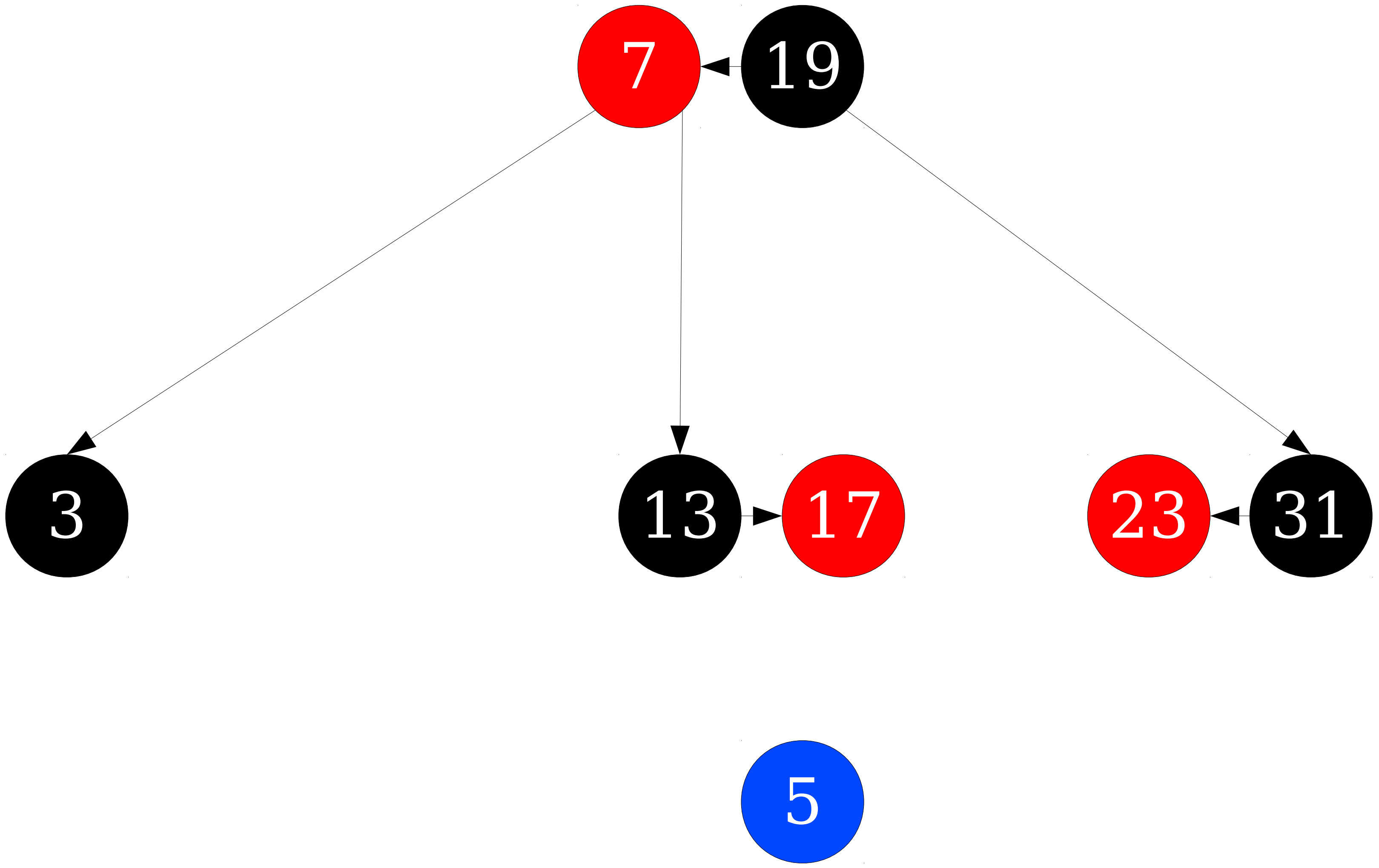
Using the Isometry



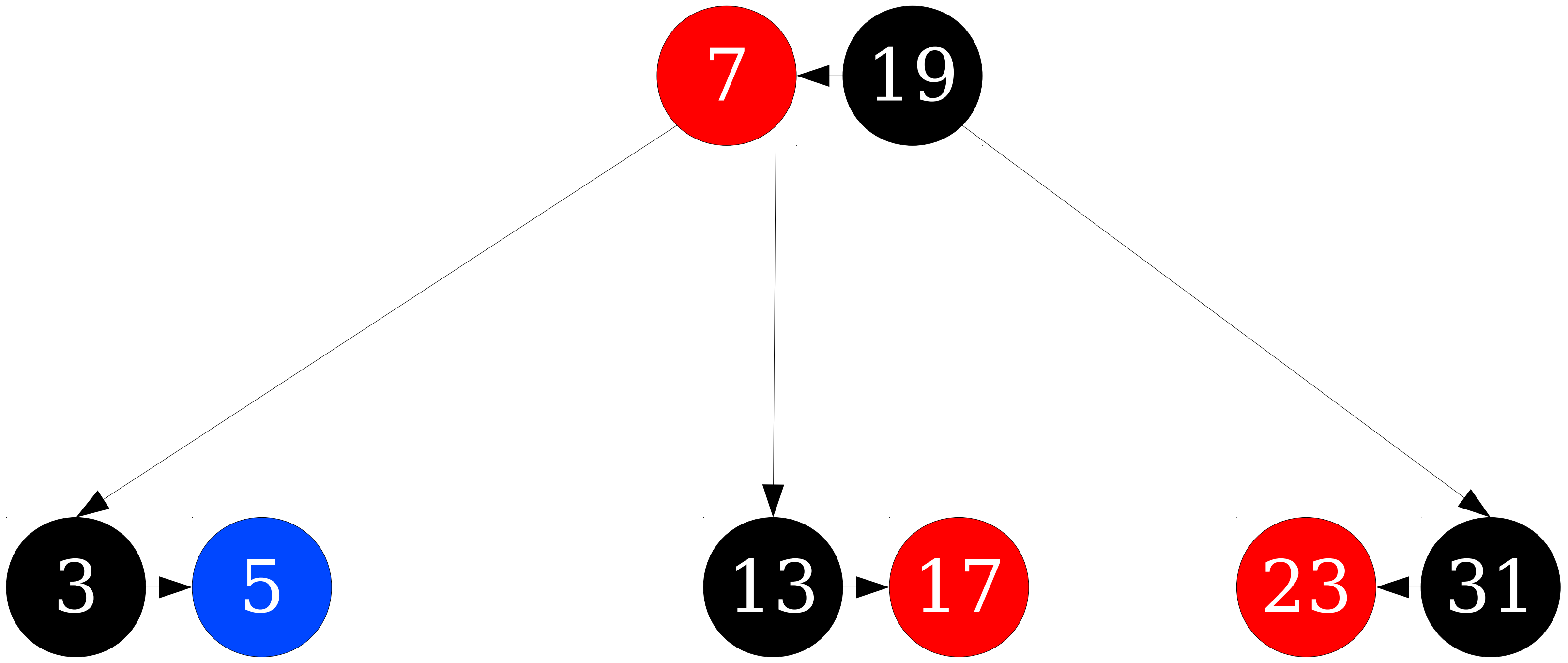
Using the Isometry



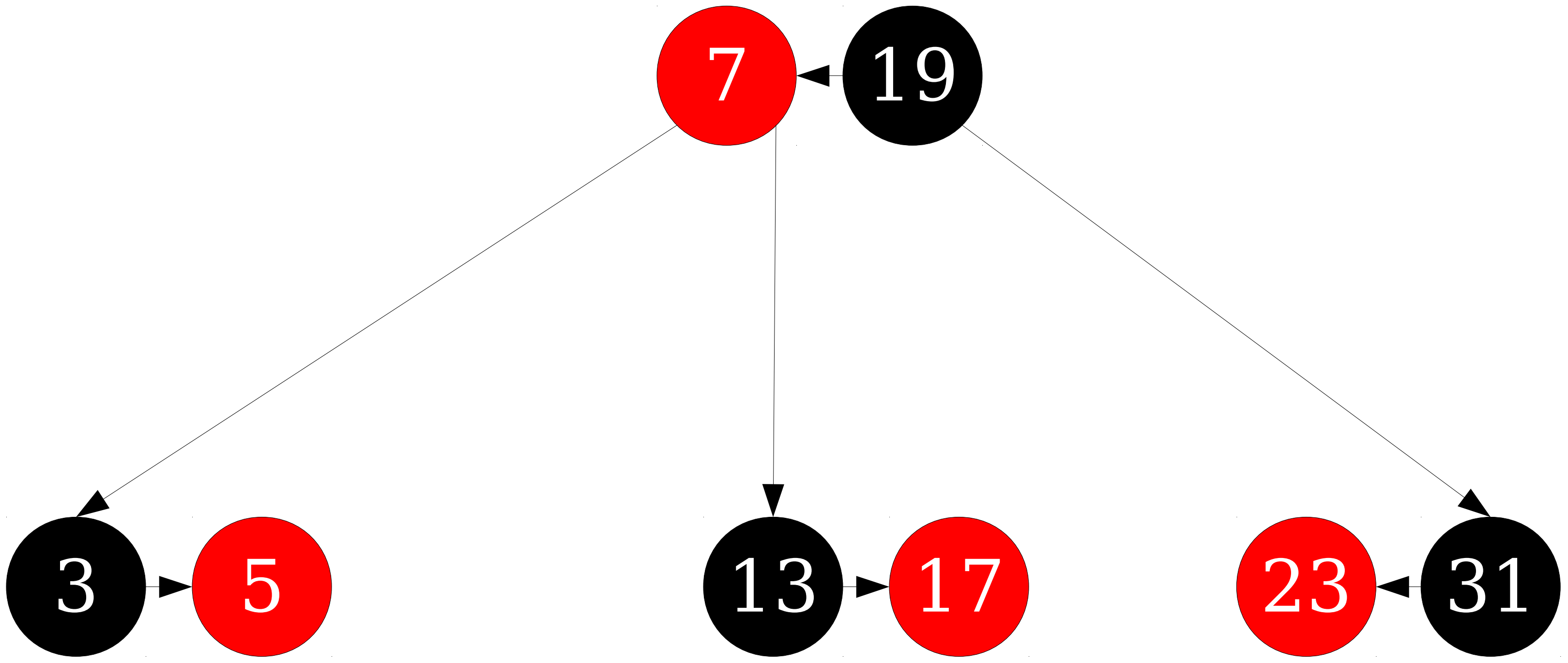
Using the Isometry



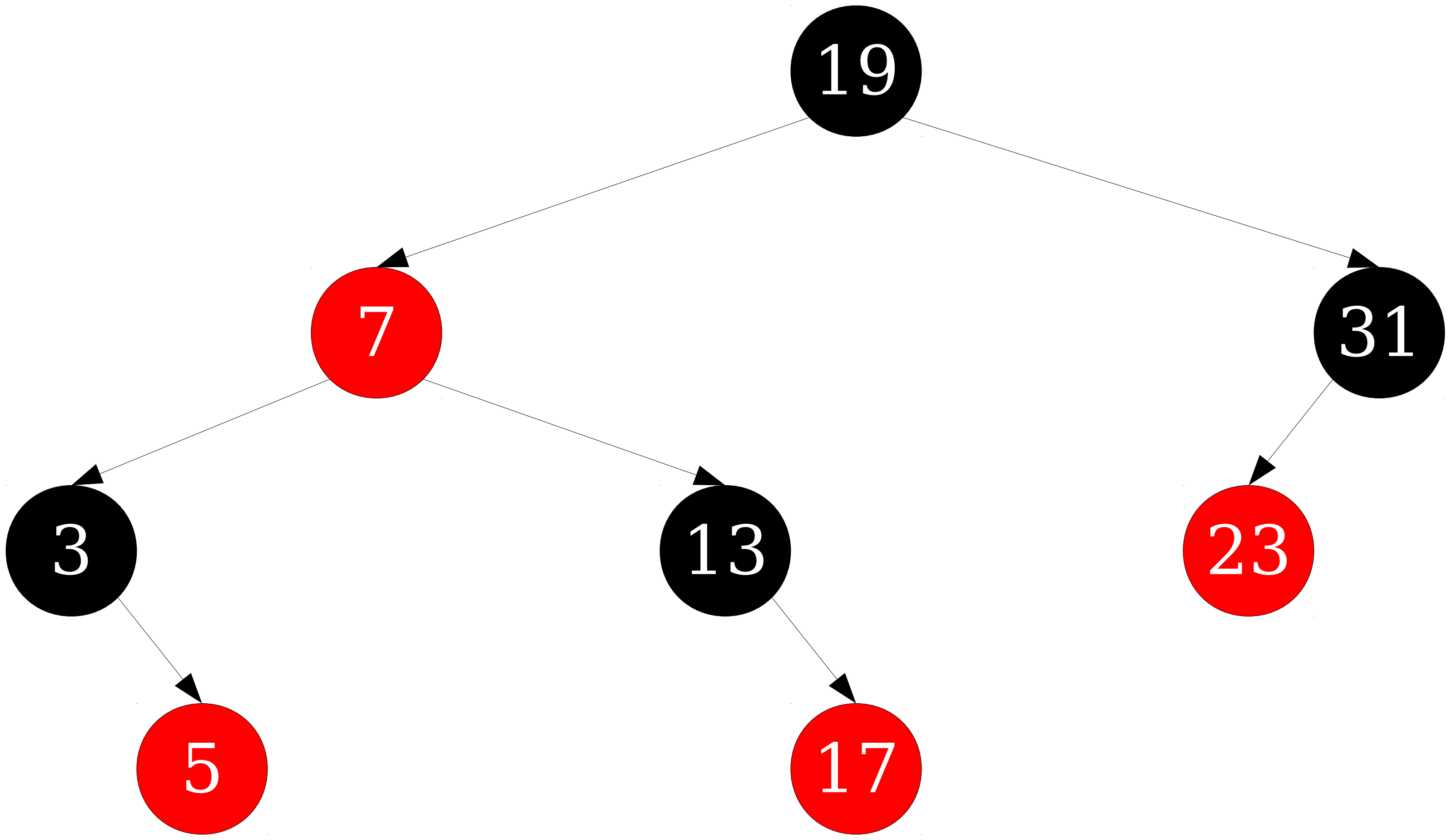
Using the Isometry



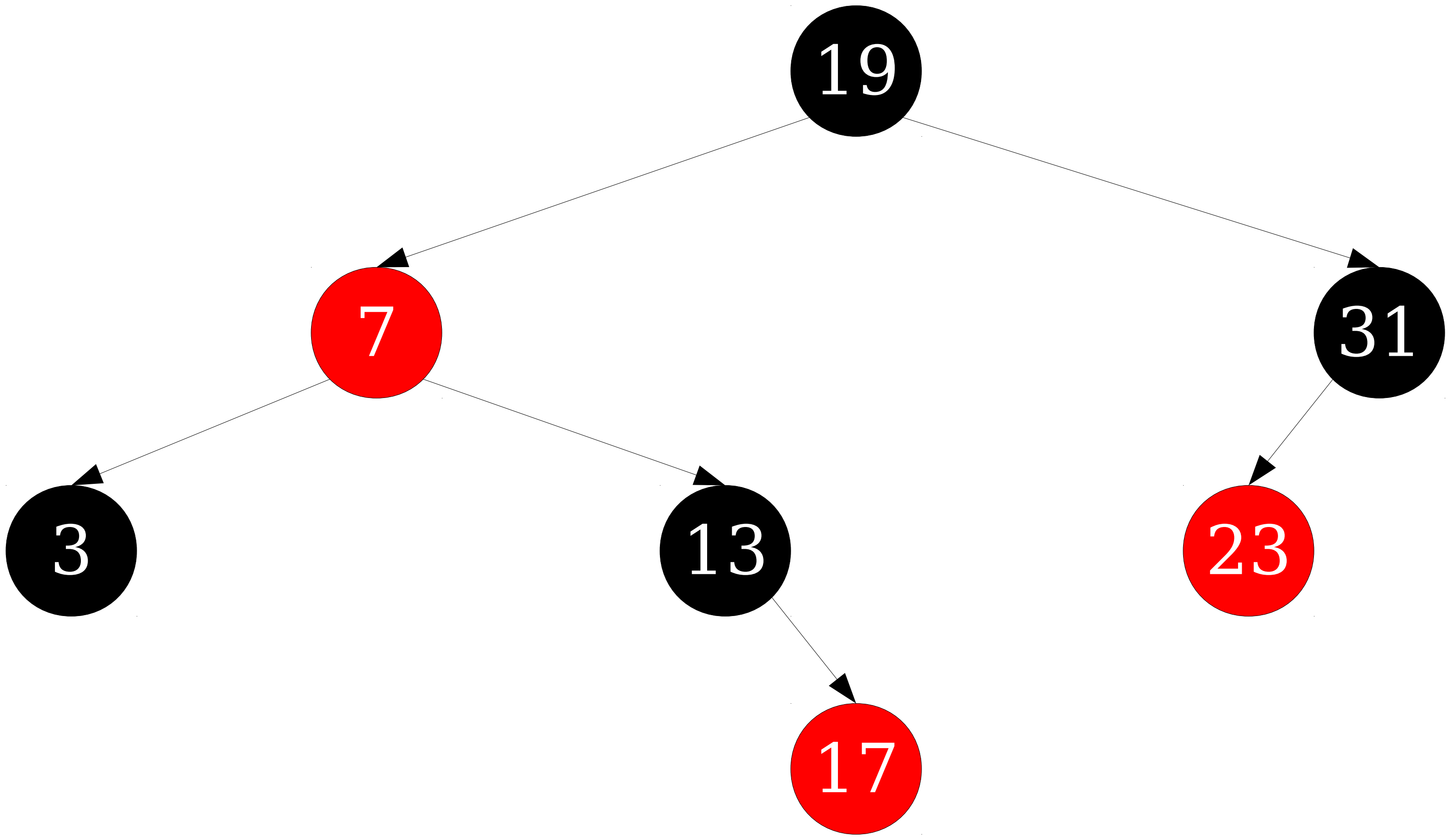
Using the Isometry



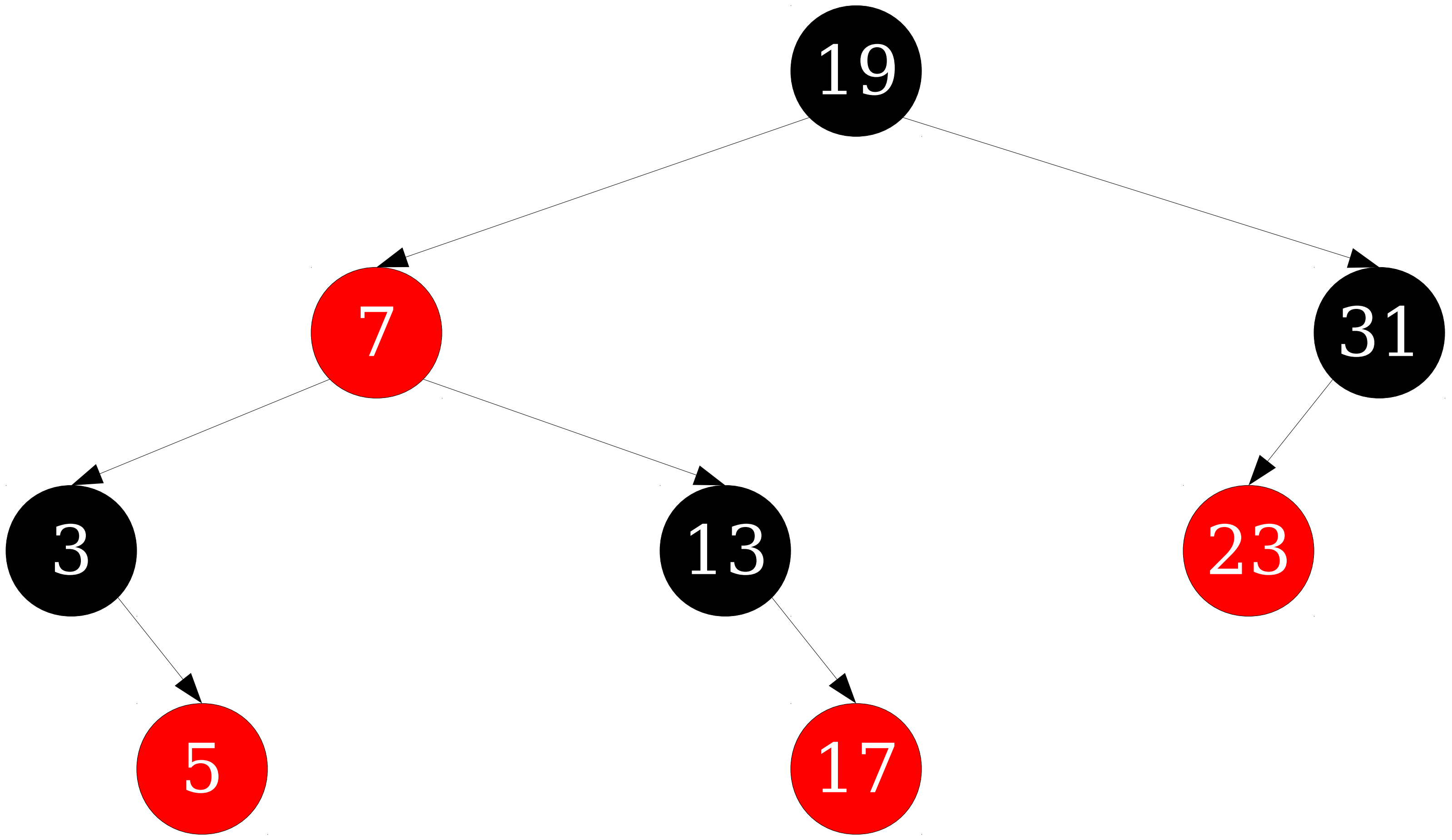
Using the Isometry



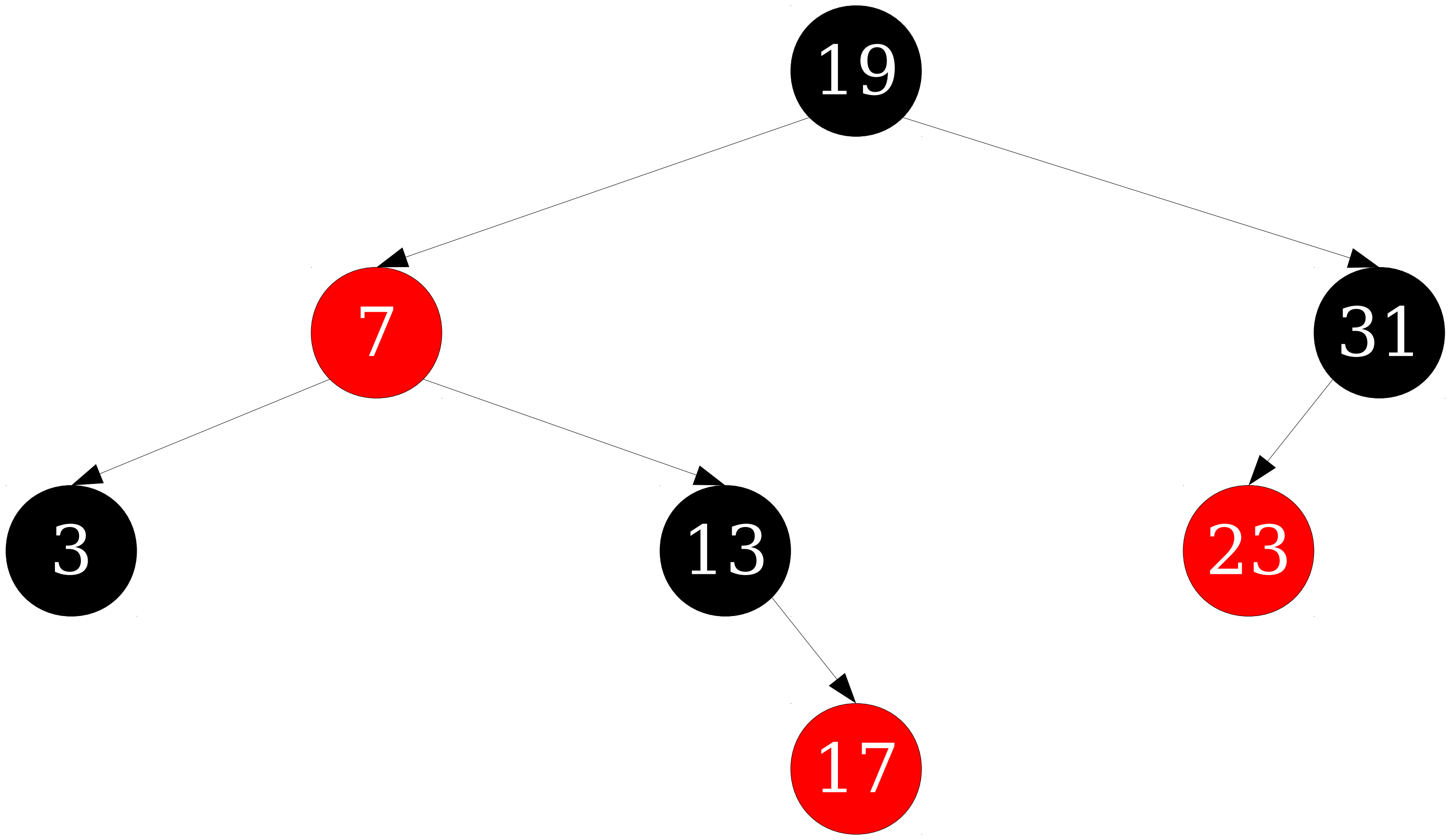
Using the Isometry



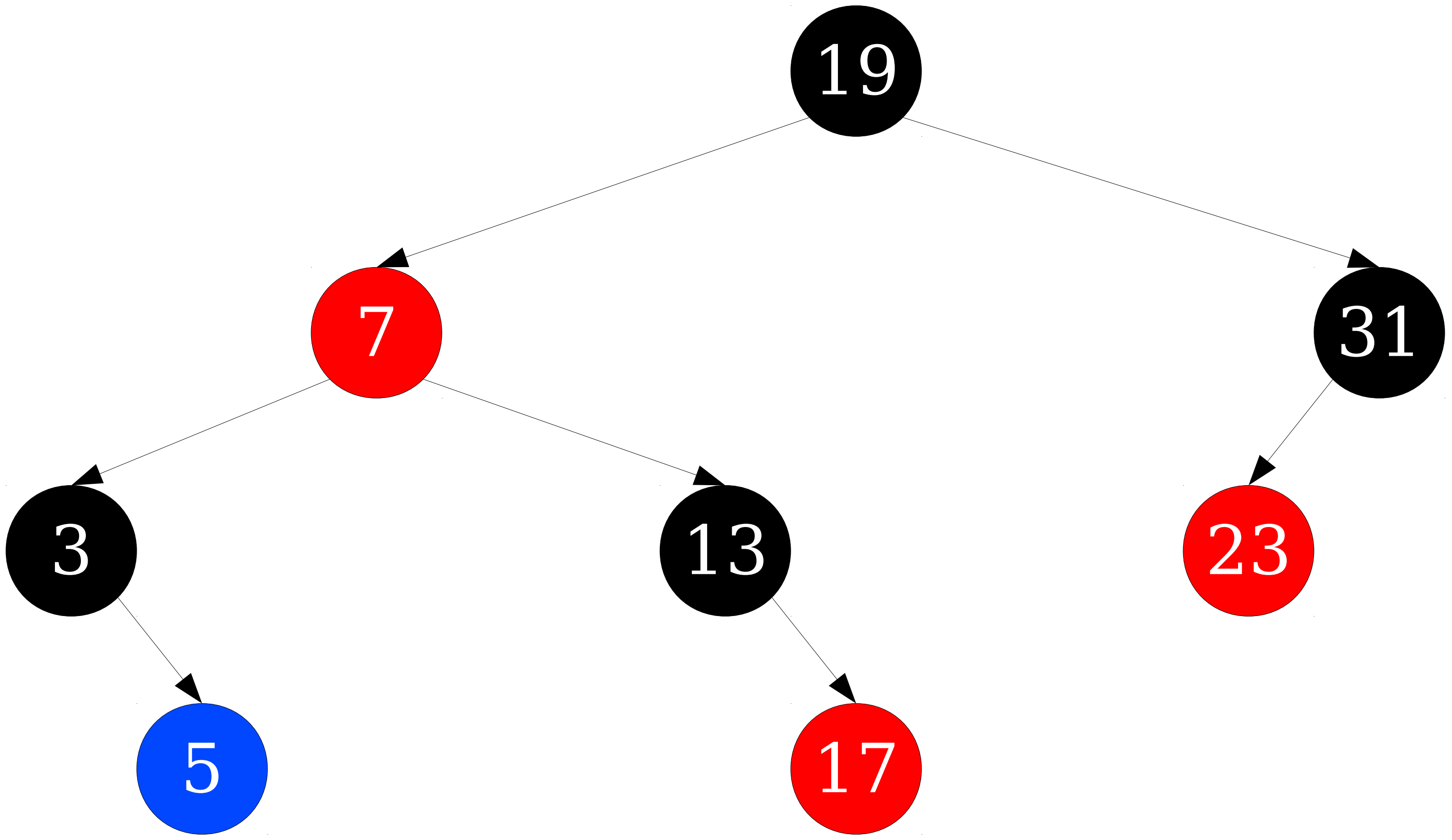
Using the Isometry



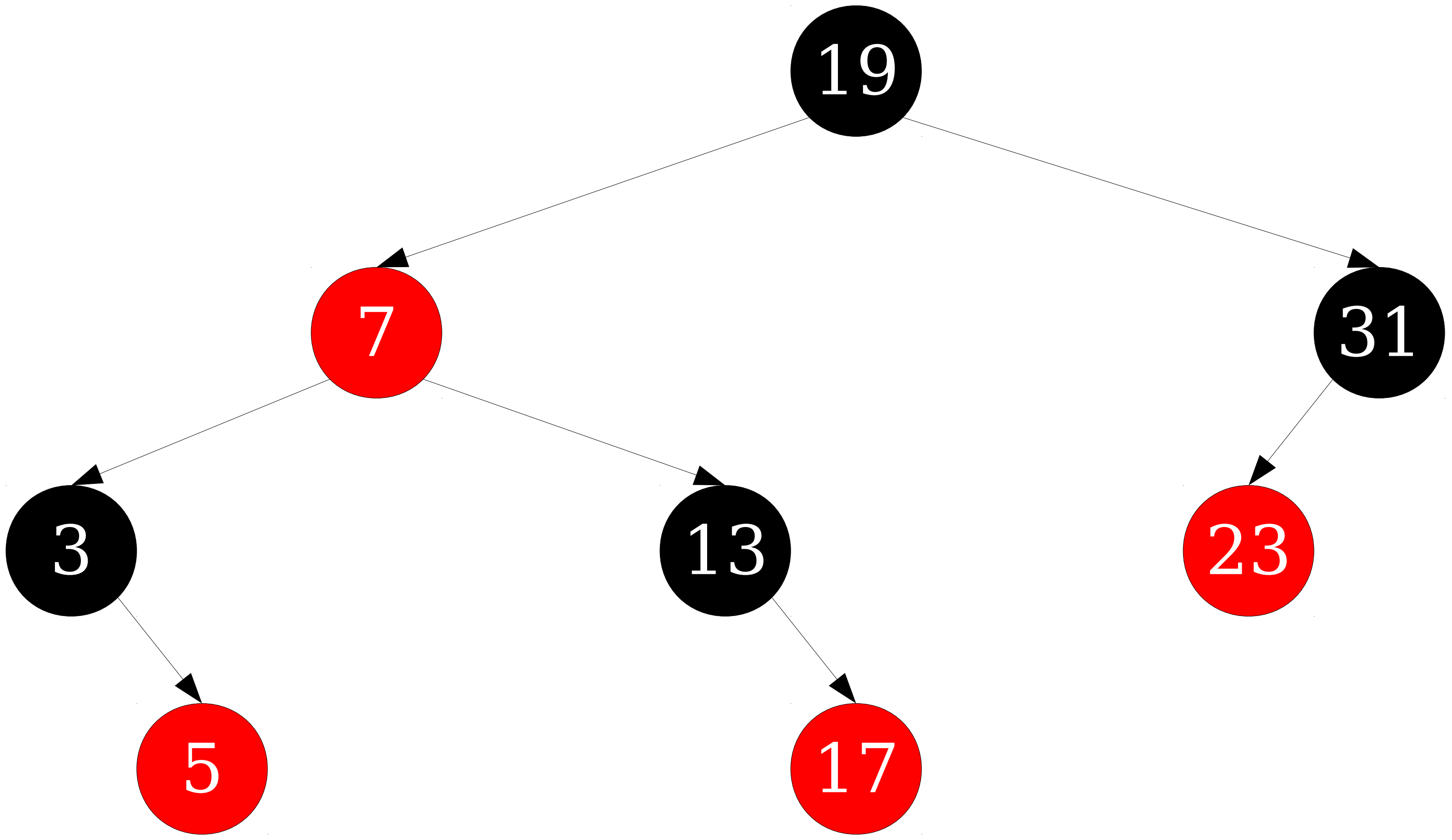
Using the Isometry



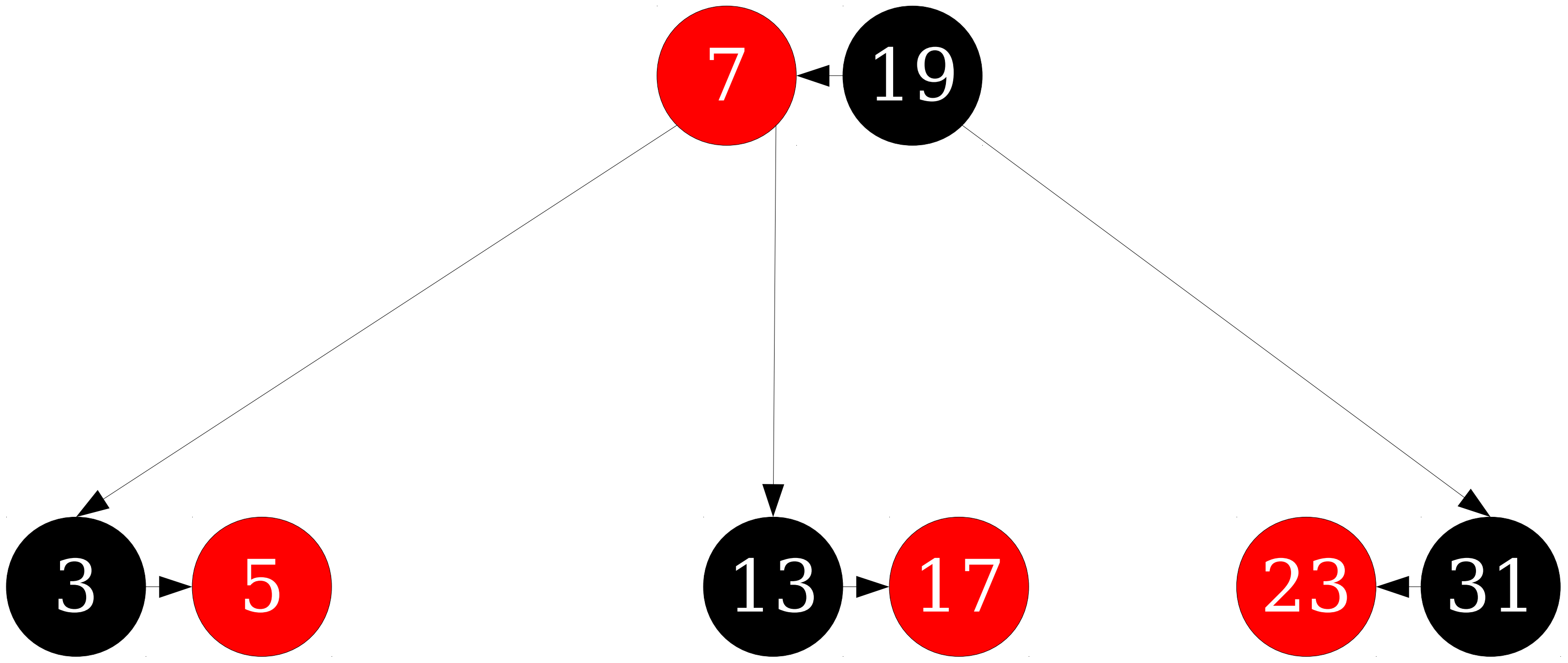
Using the Isometry



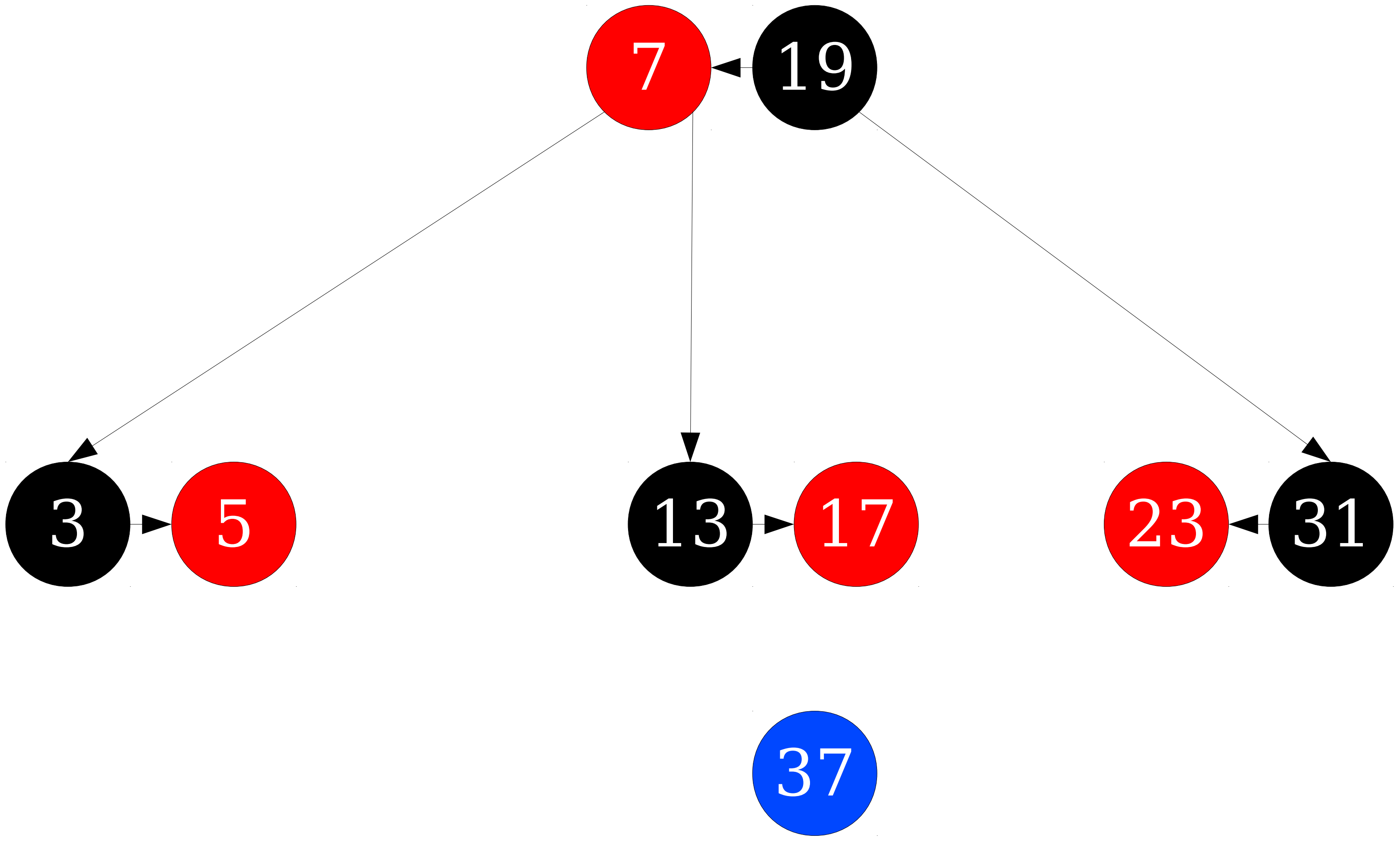
Using the Isometry



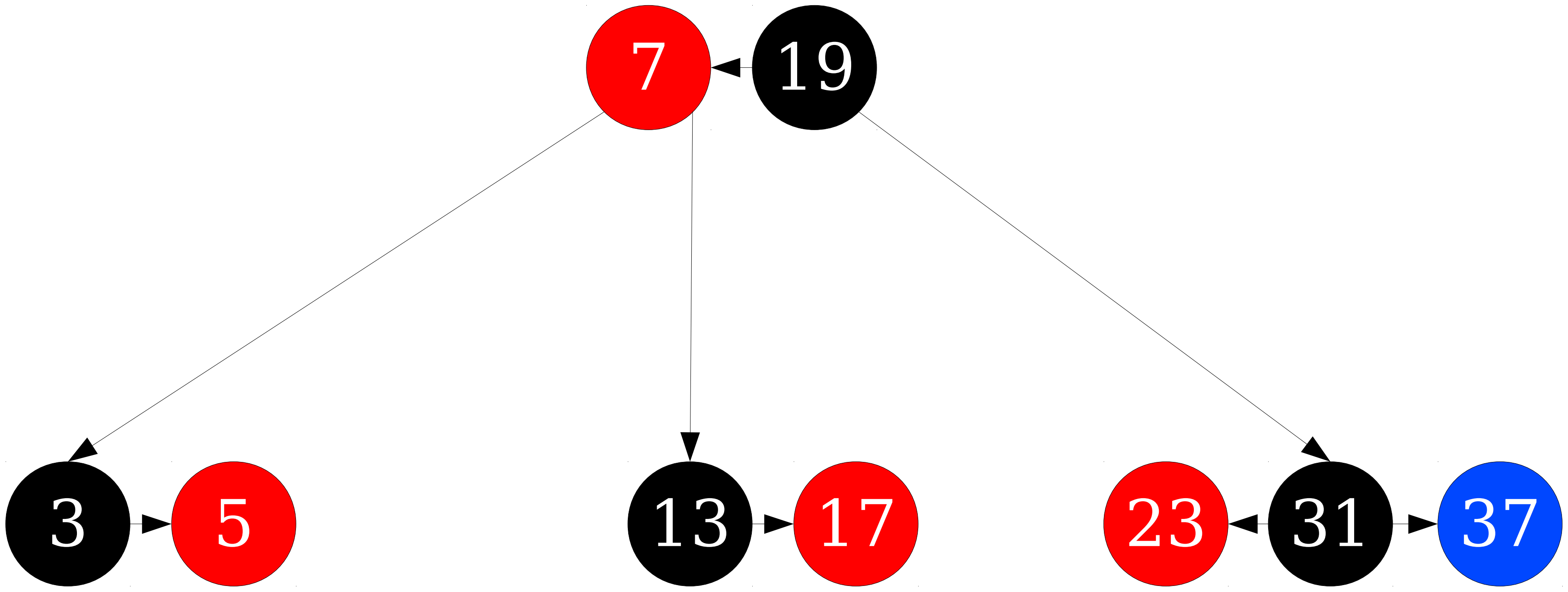
Using the Isometry



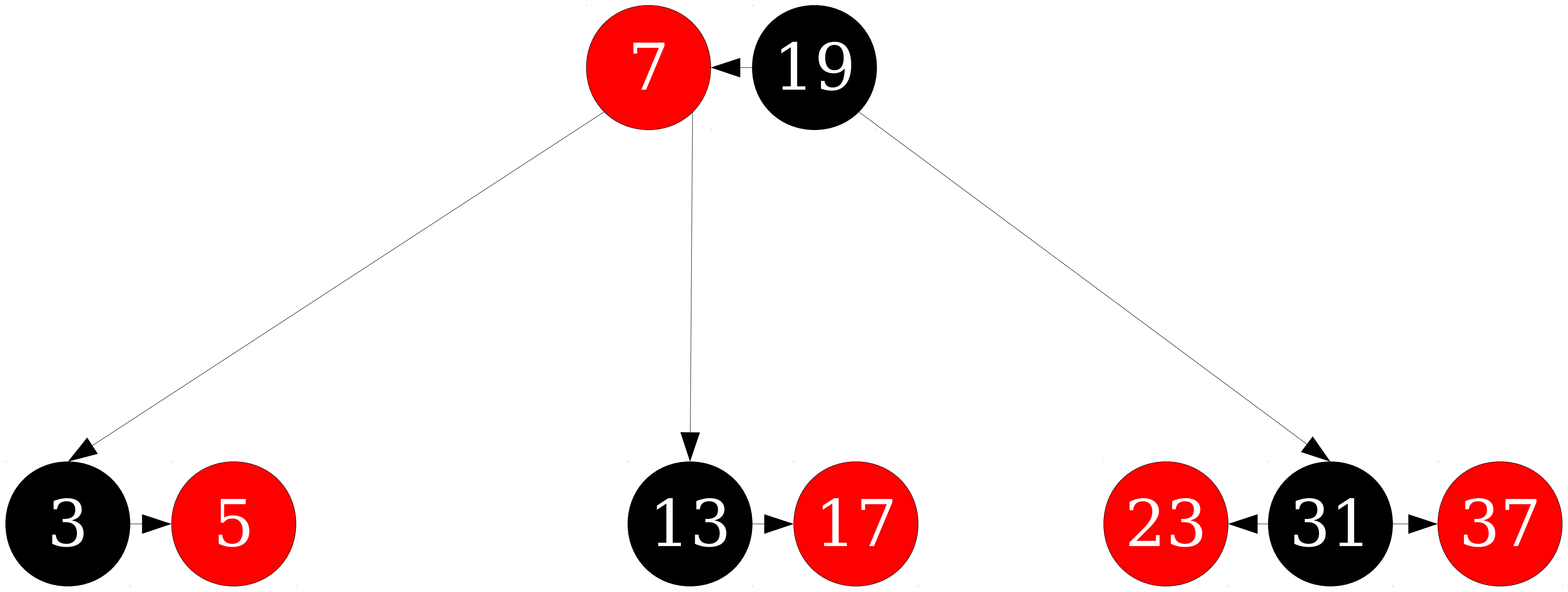
Using the Isometry



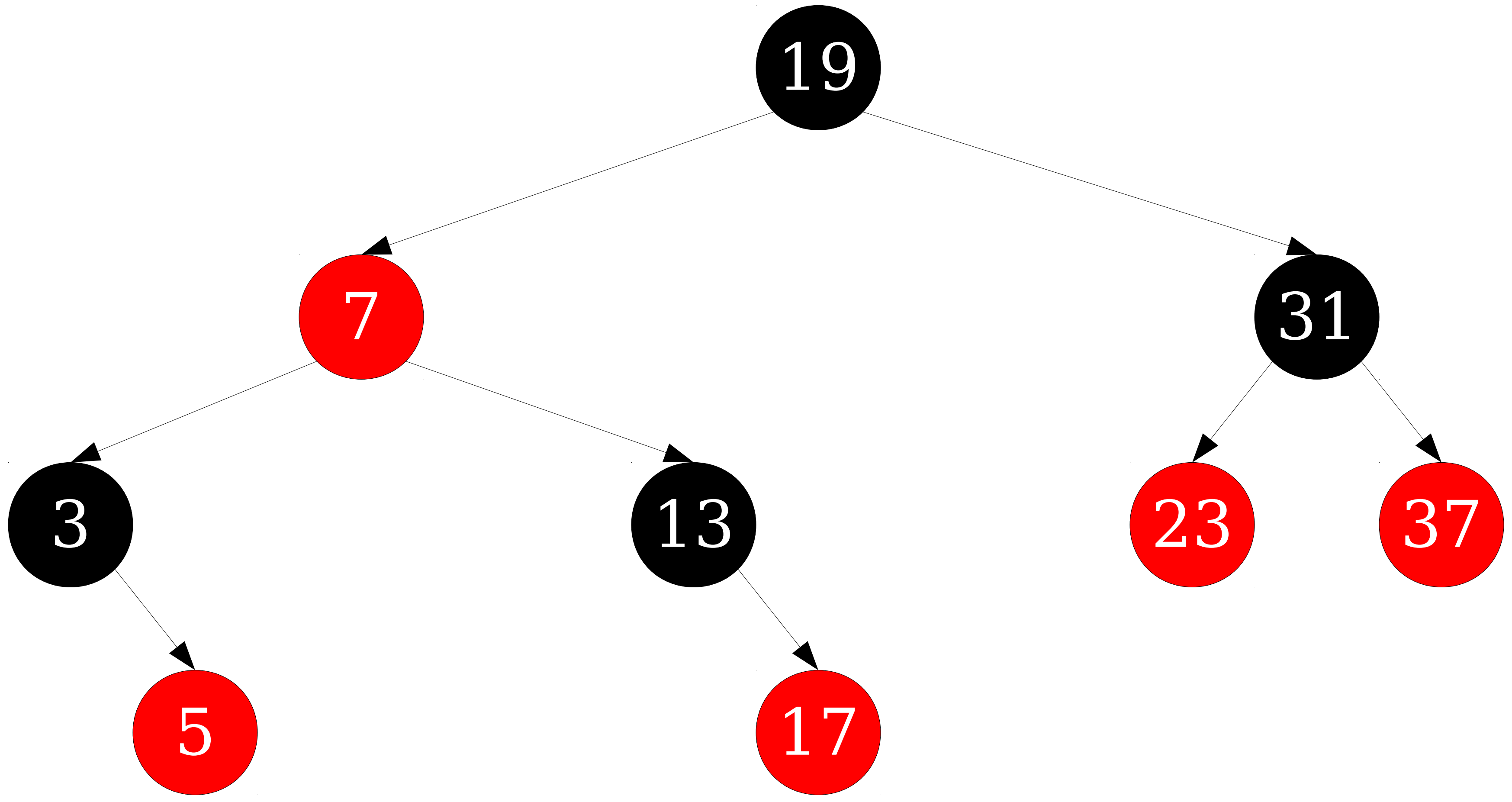
Using the Isometry



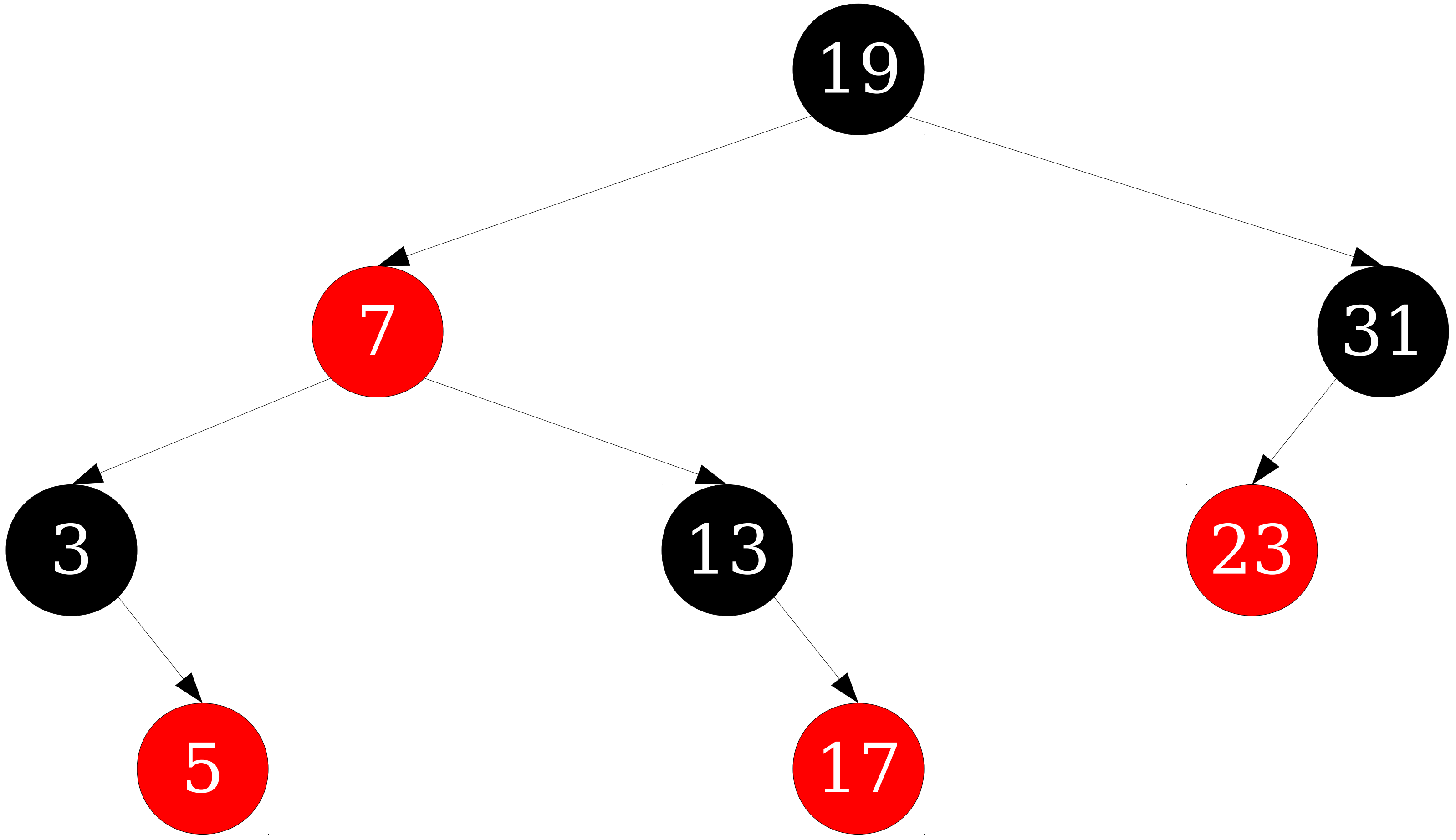
Using the Isometry



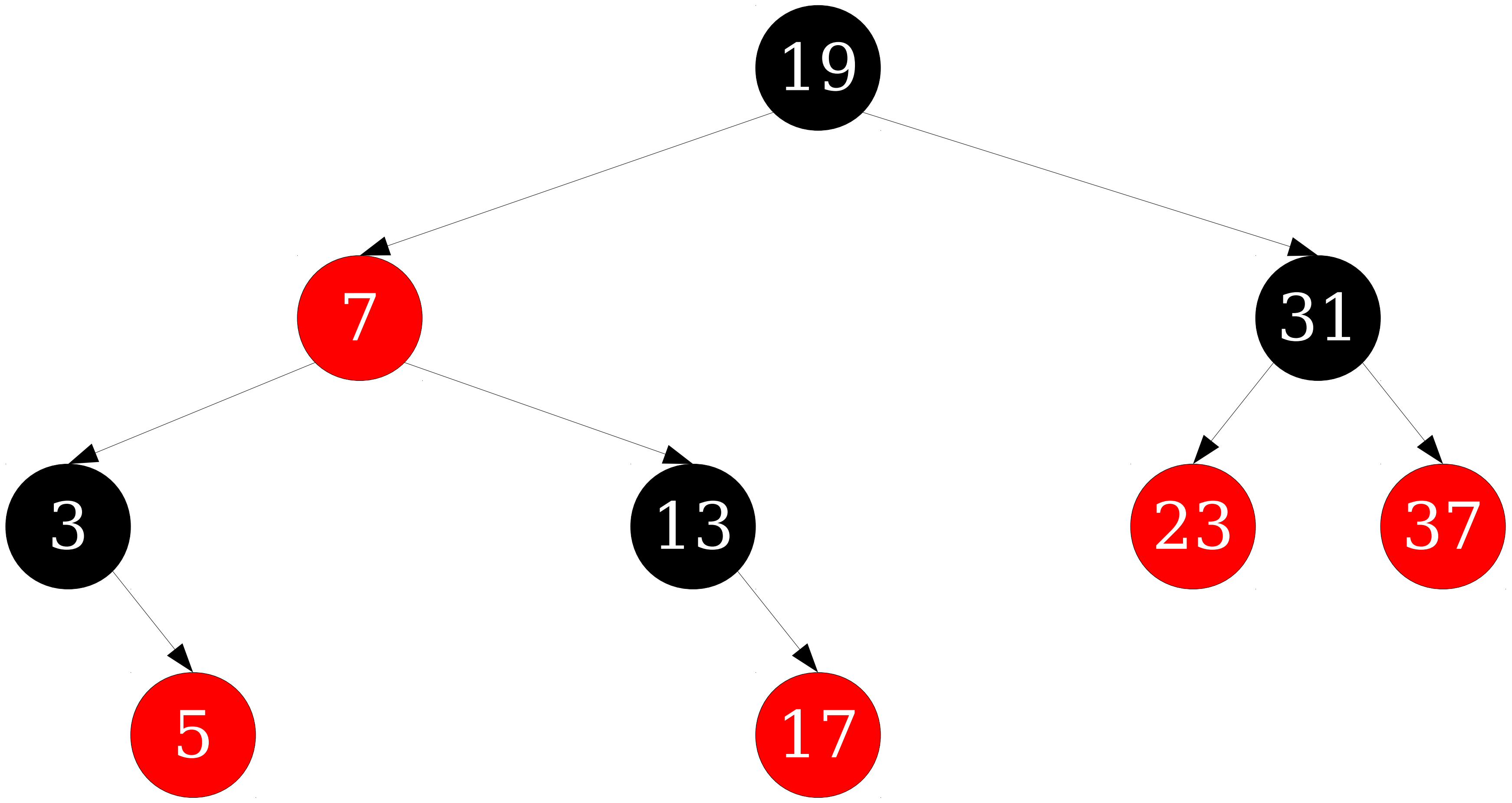
Using the Isometry



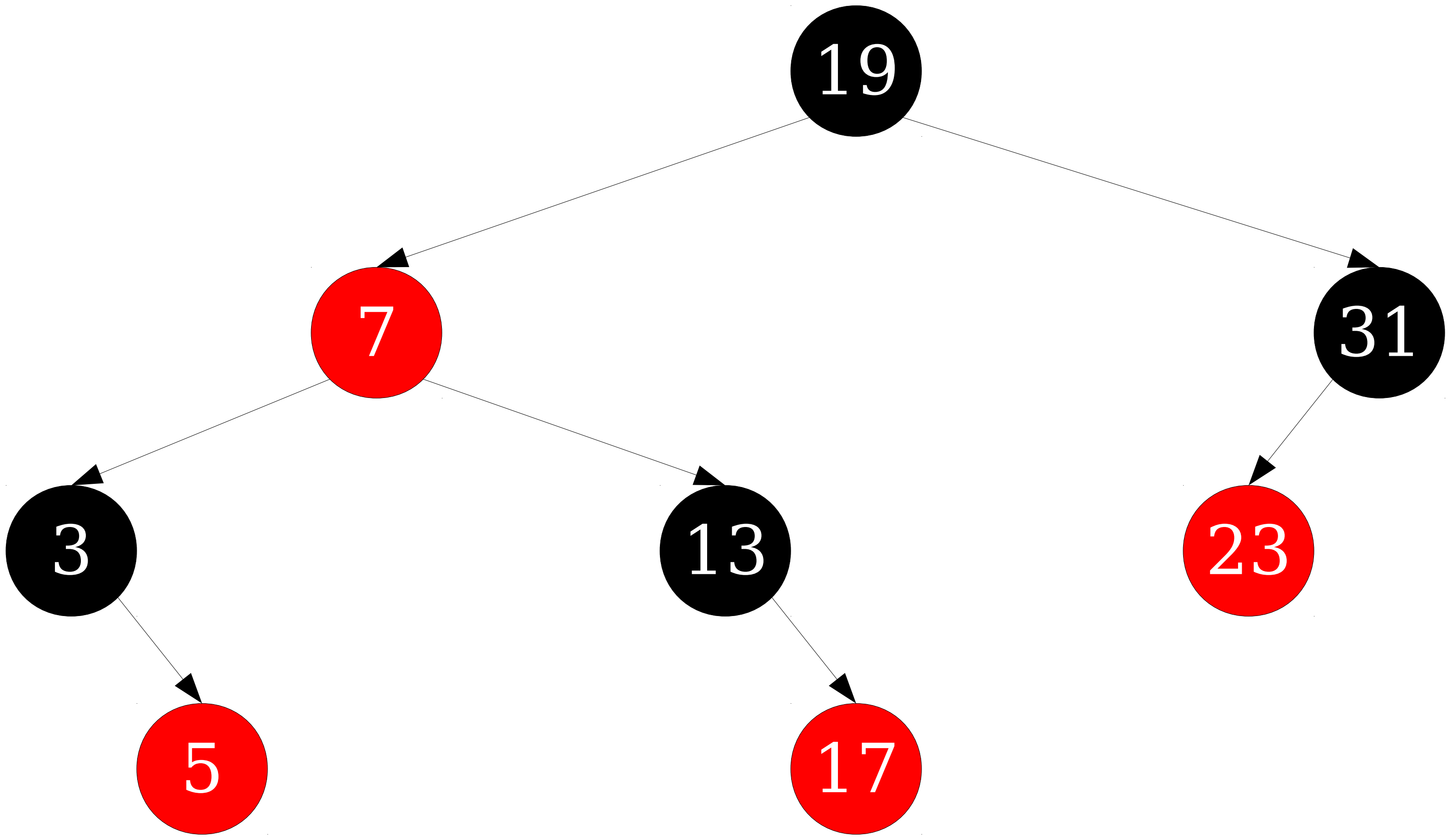
Using the Isometry



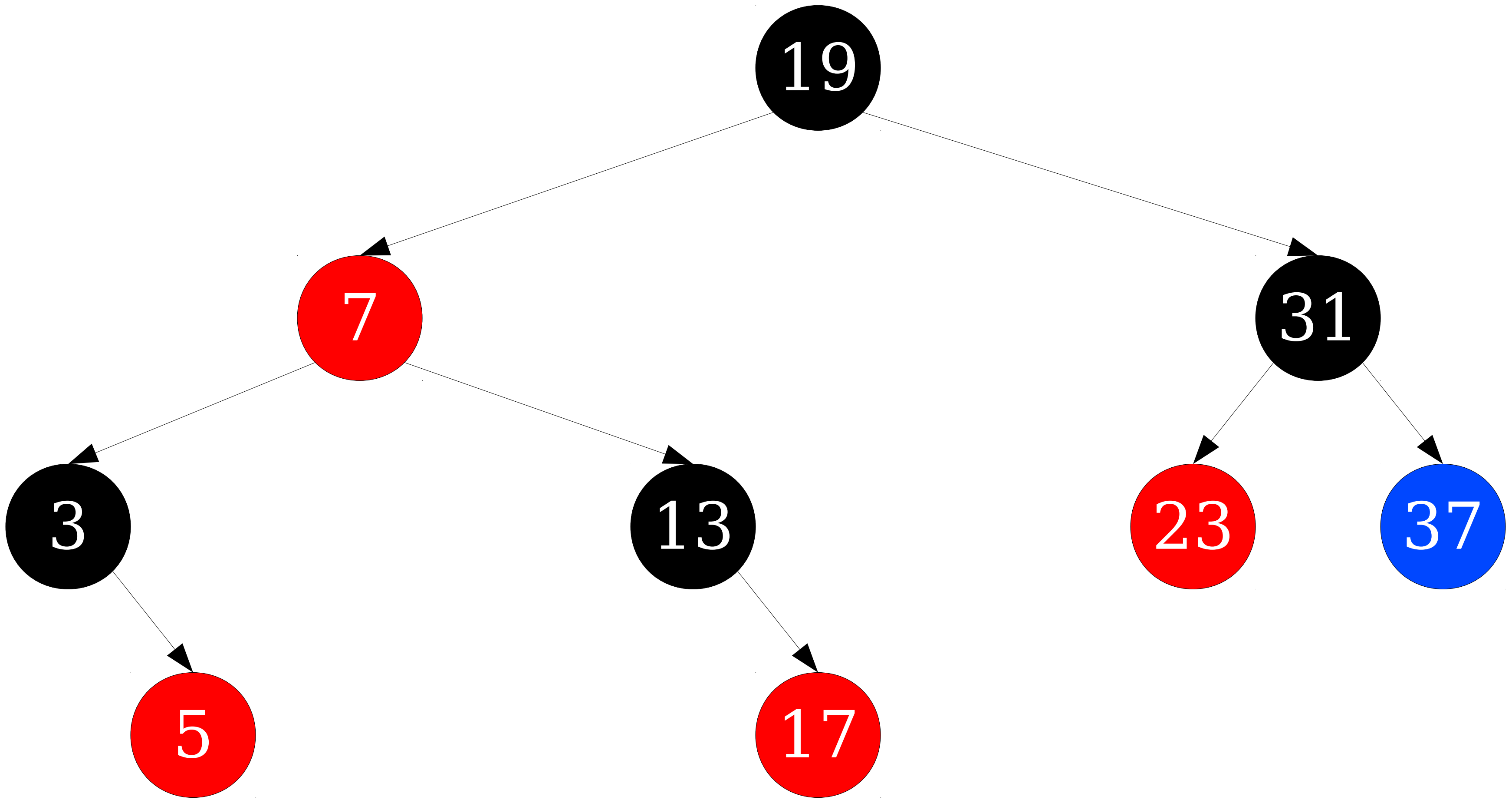
Using the Isometry



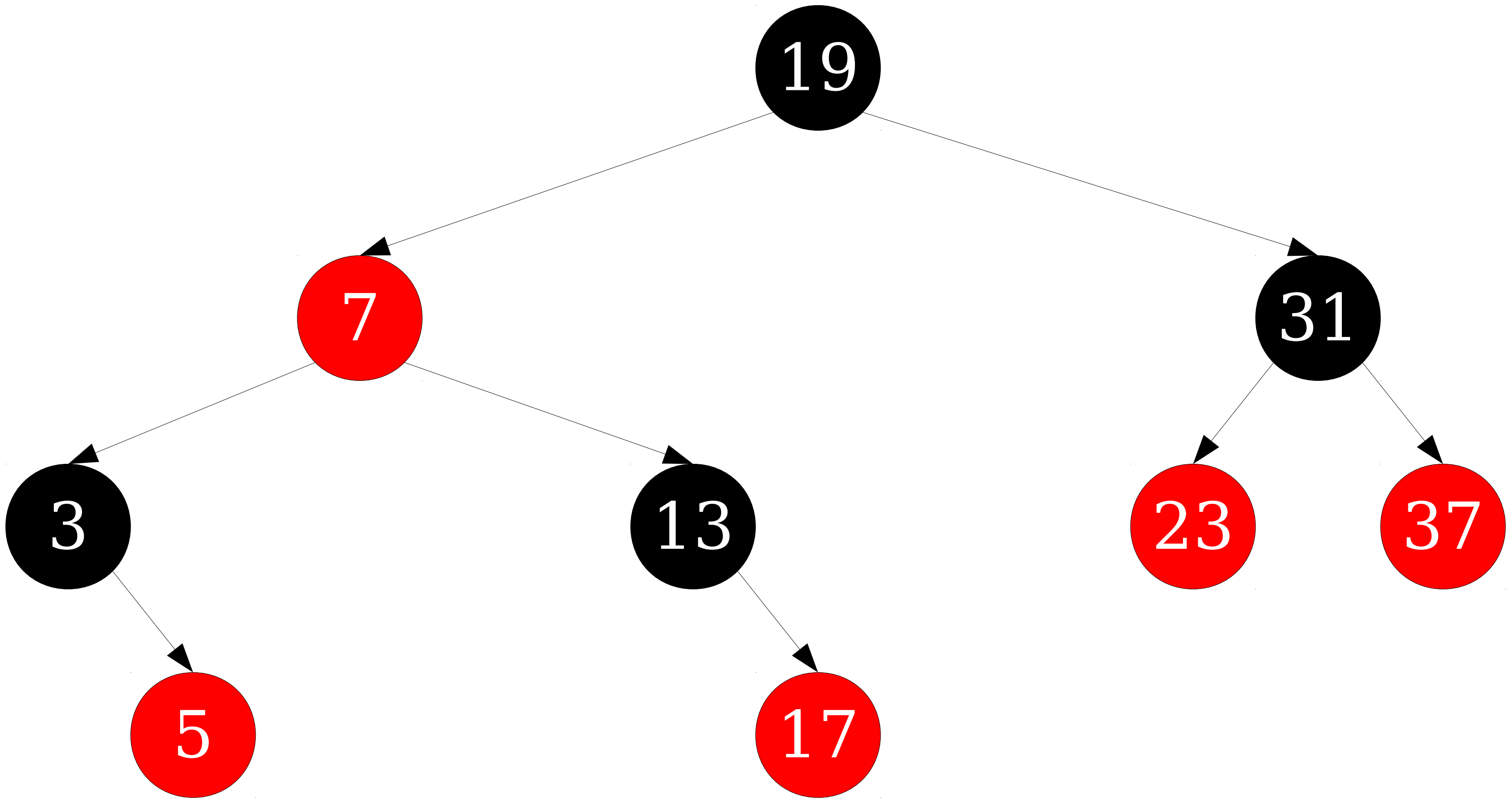
Using the Isometry



Using the Isometry

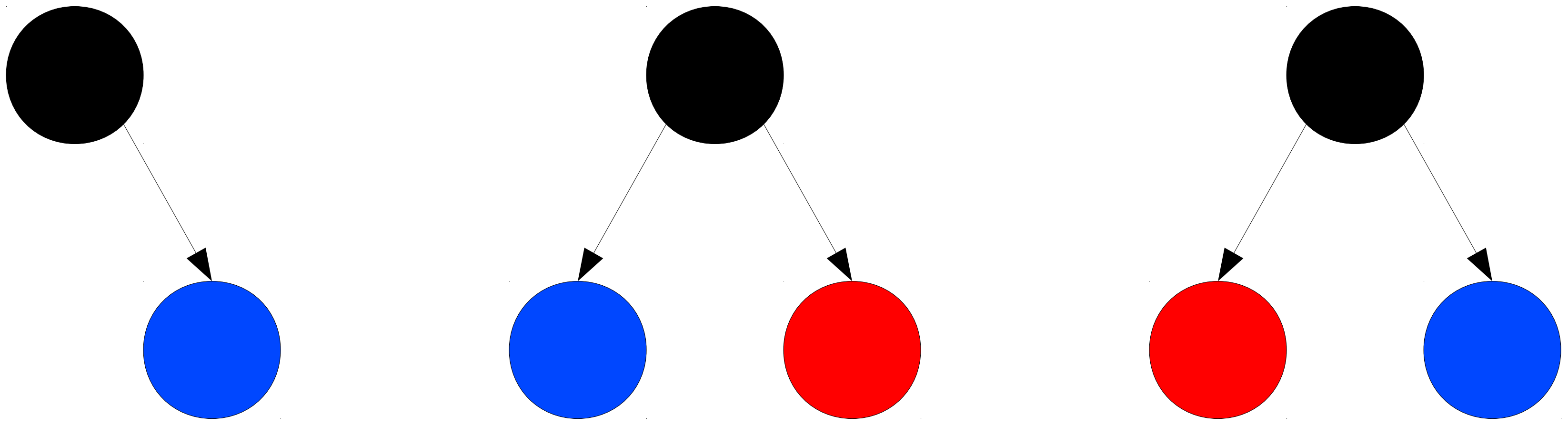


Using the Isometry

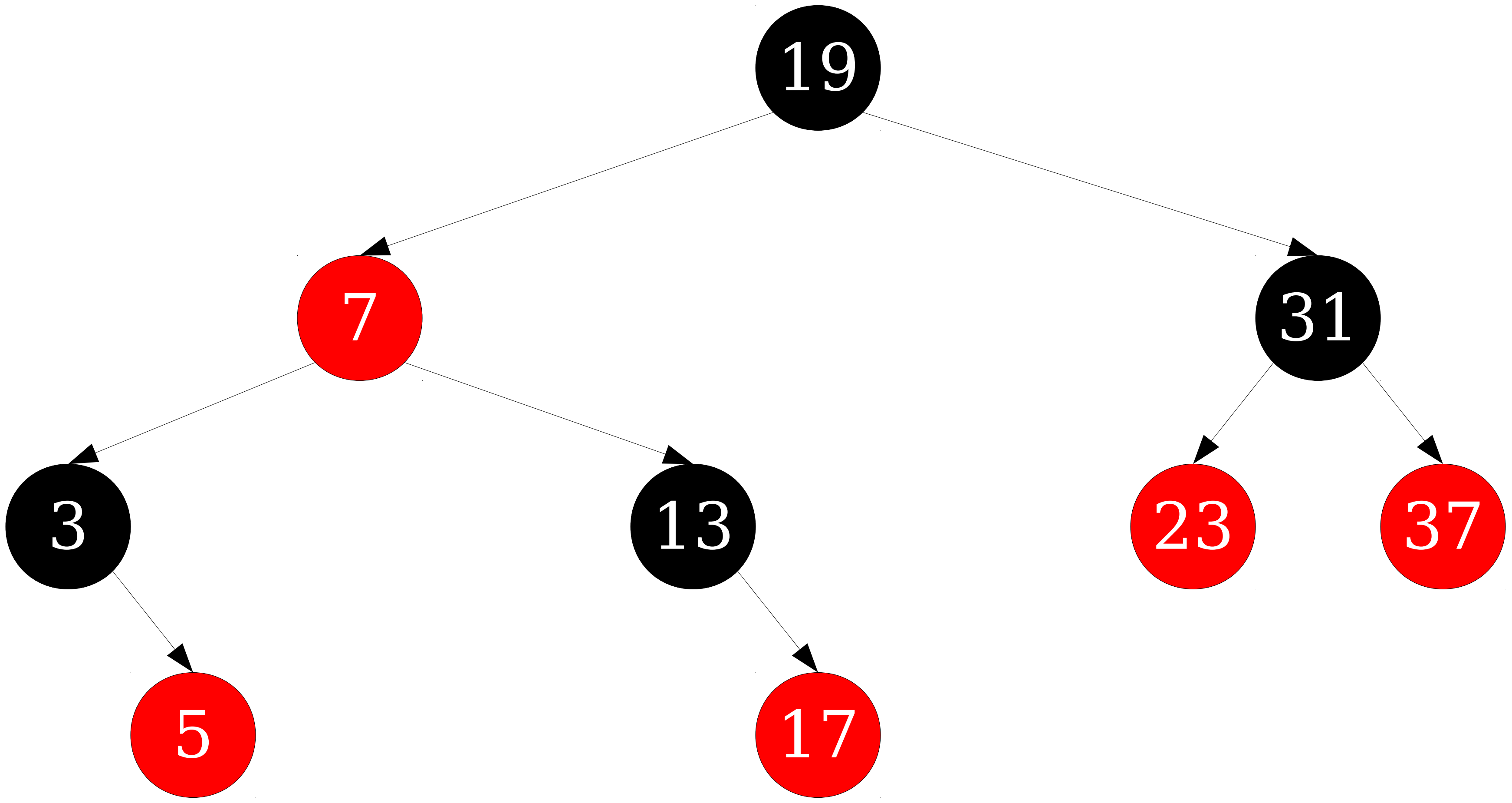


Red/Black Tree Insertion

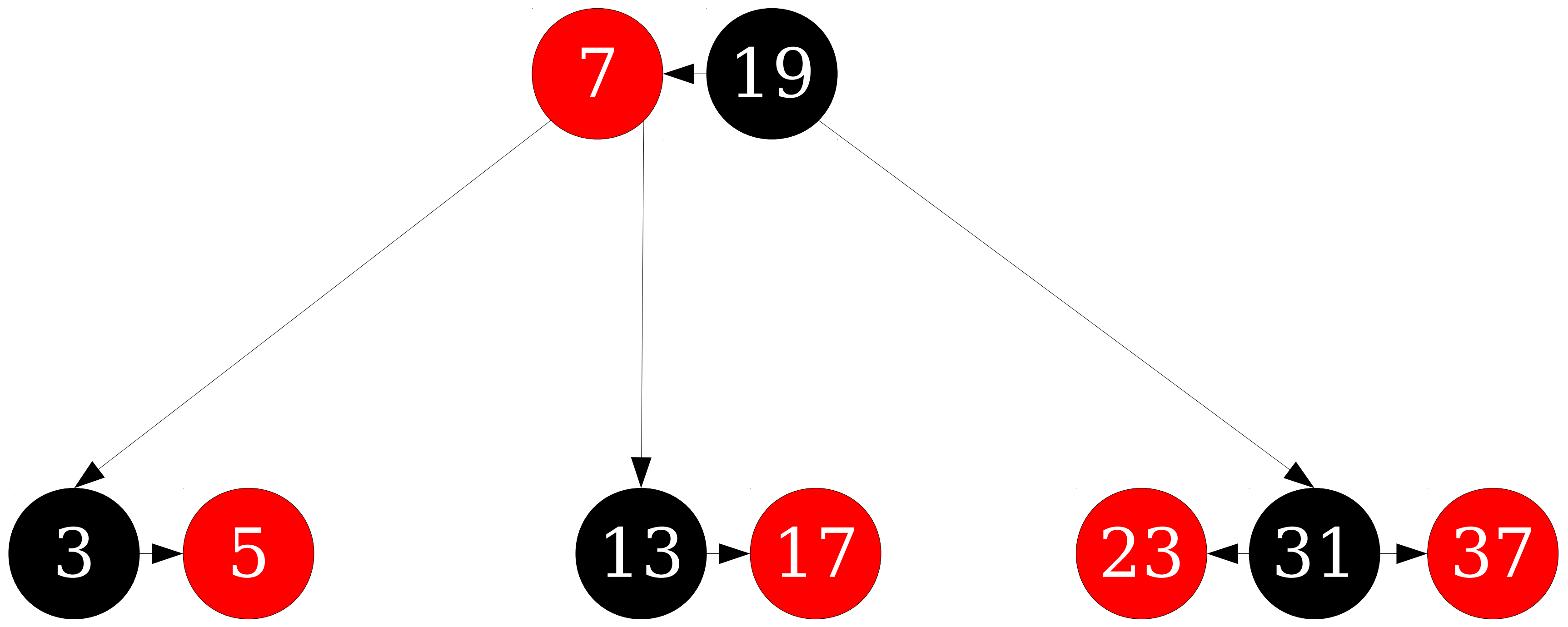
- ***Rule #1:*** When inserting a node, if its parent is black, make the node red and stop.
- ***Justification:*** This simulates inserting a key into an existing 2-node or 3-node.



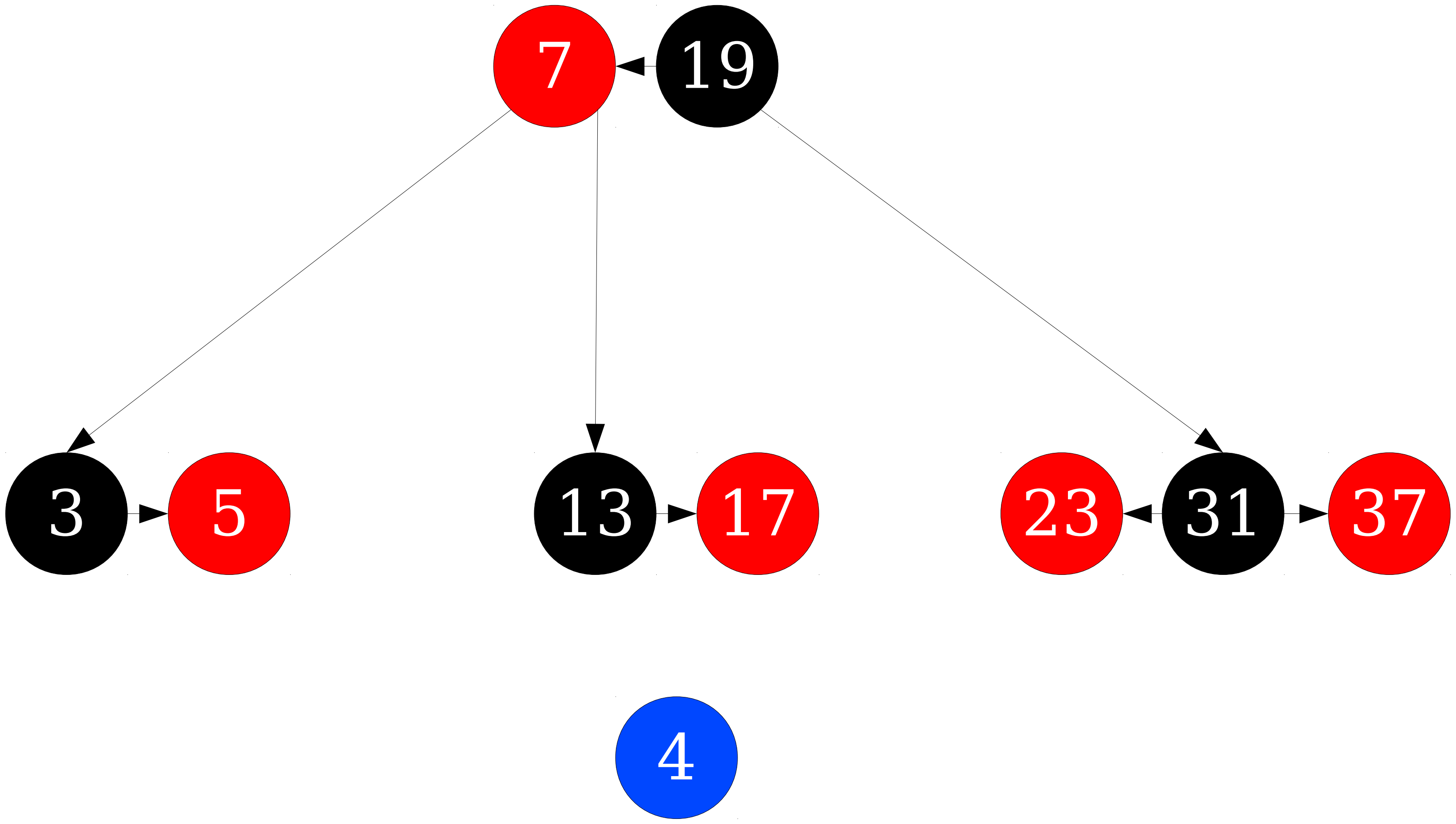
Using the Isometry



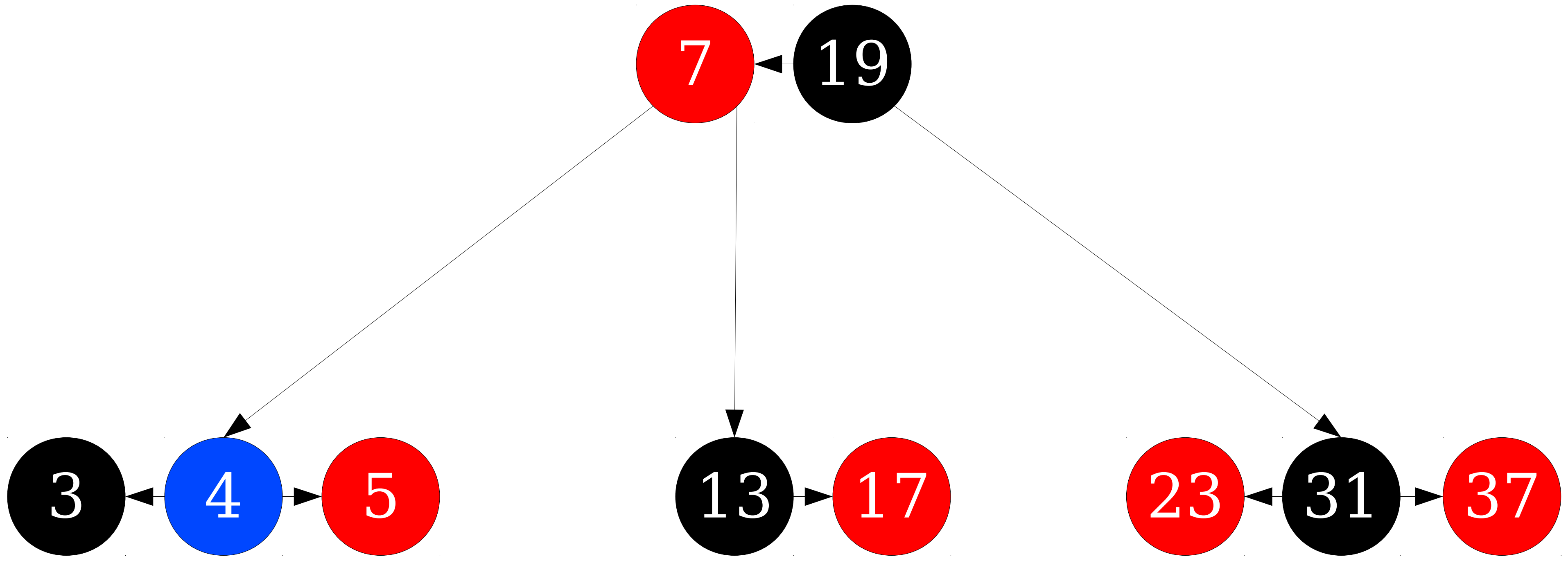
Using the Isometry



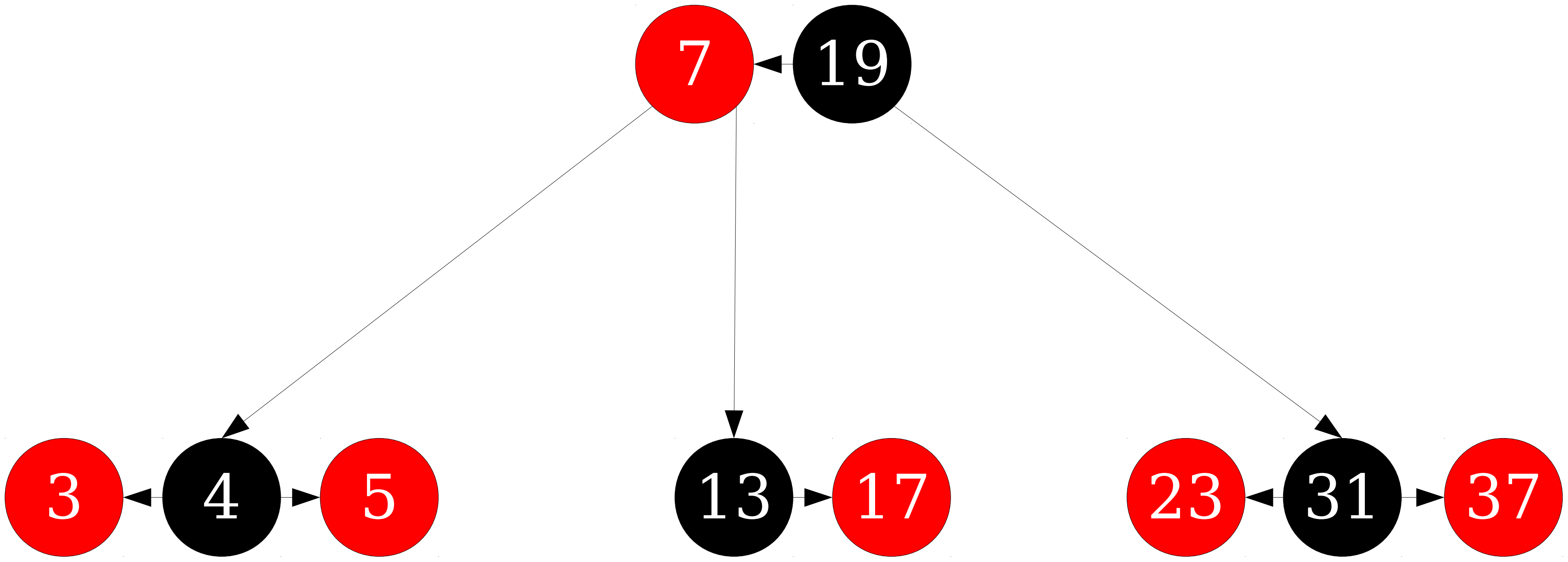
Using the Isometry



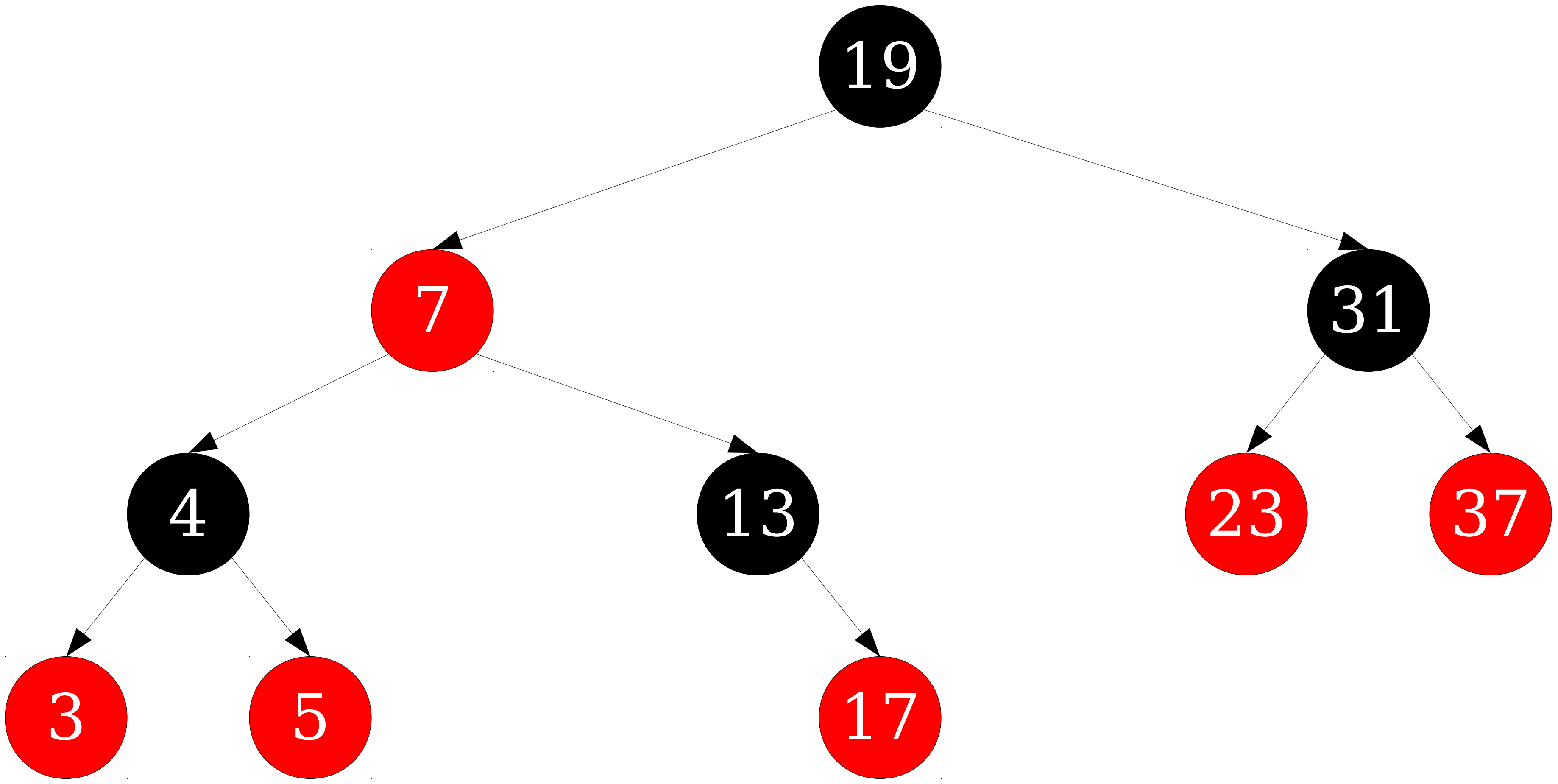
Using the Isometry



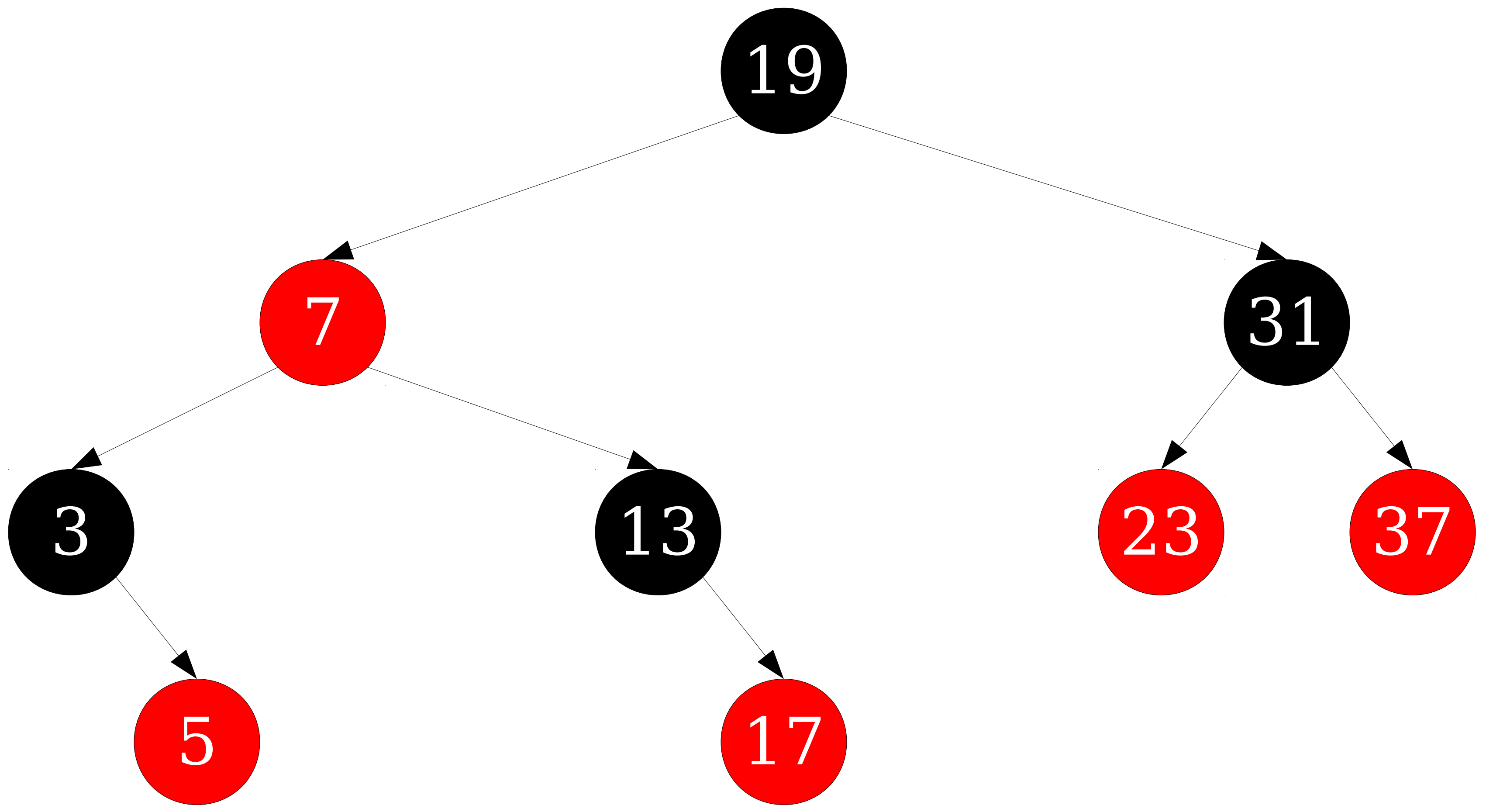
Using the Isometry



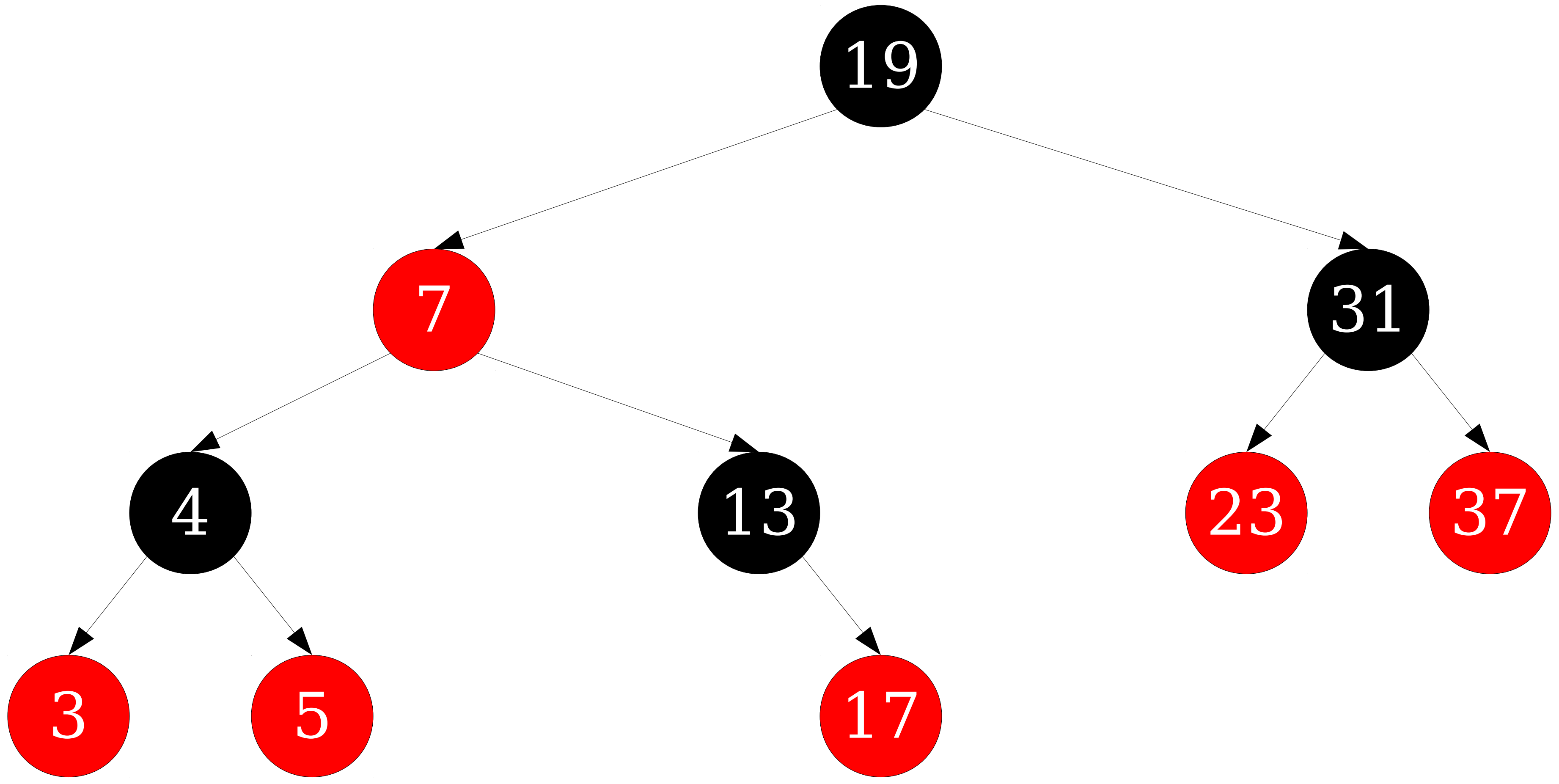
Using the Isometry



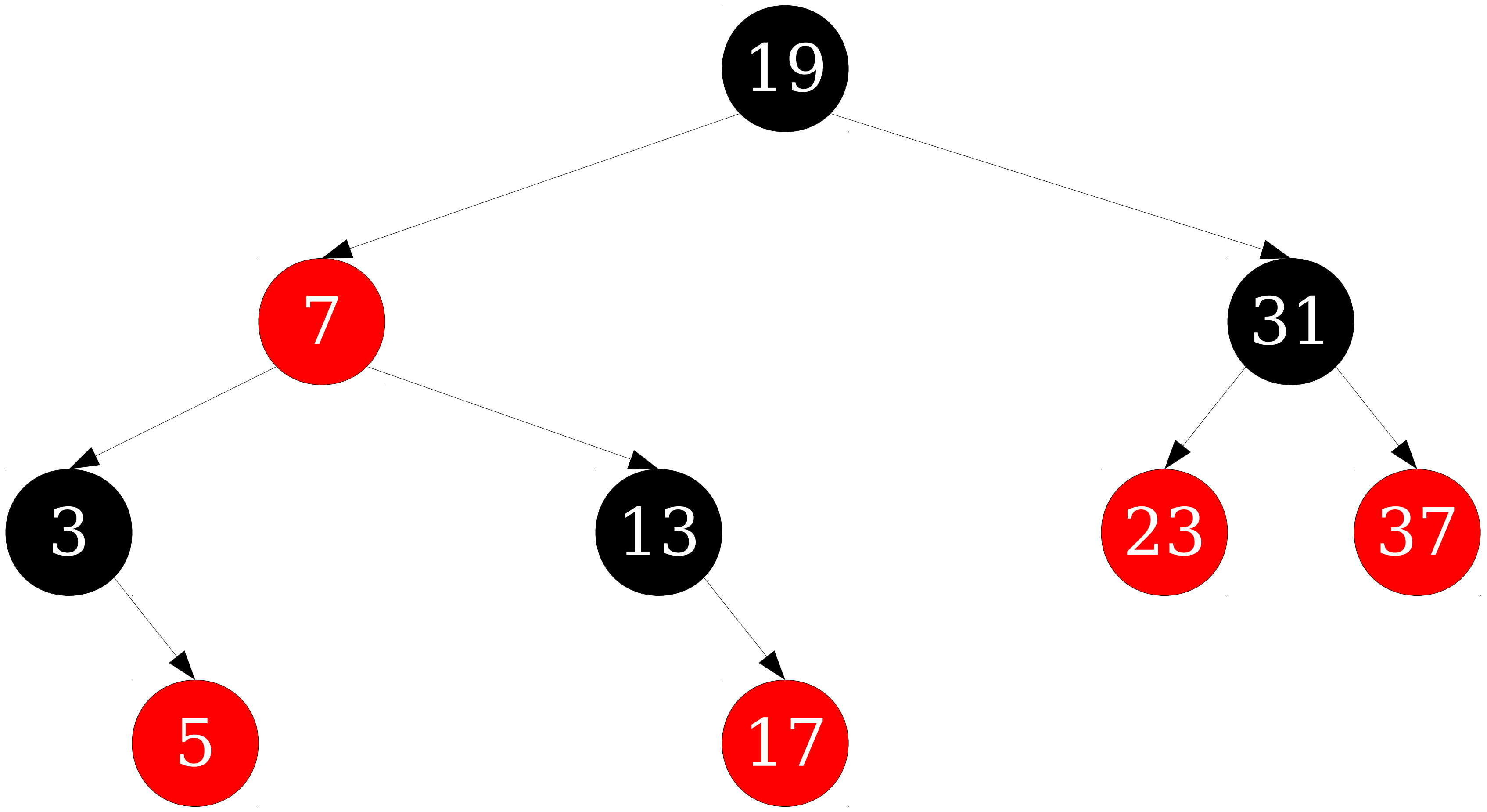
Using the Isometry



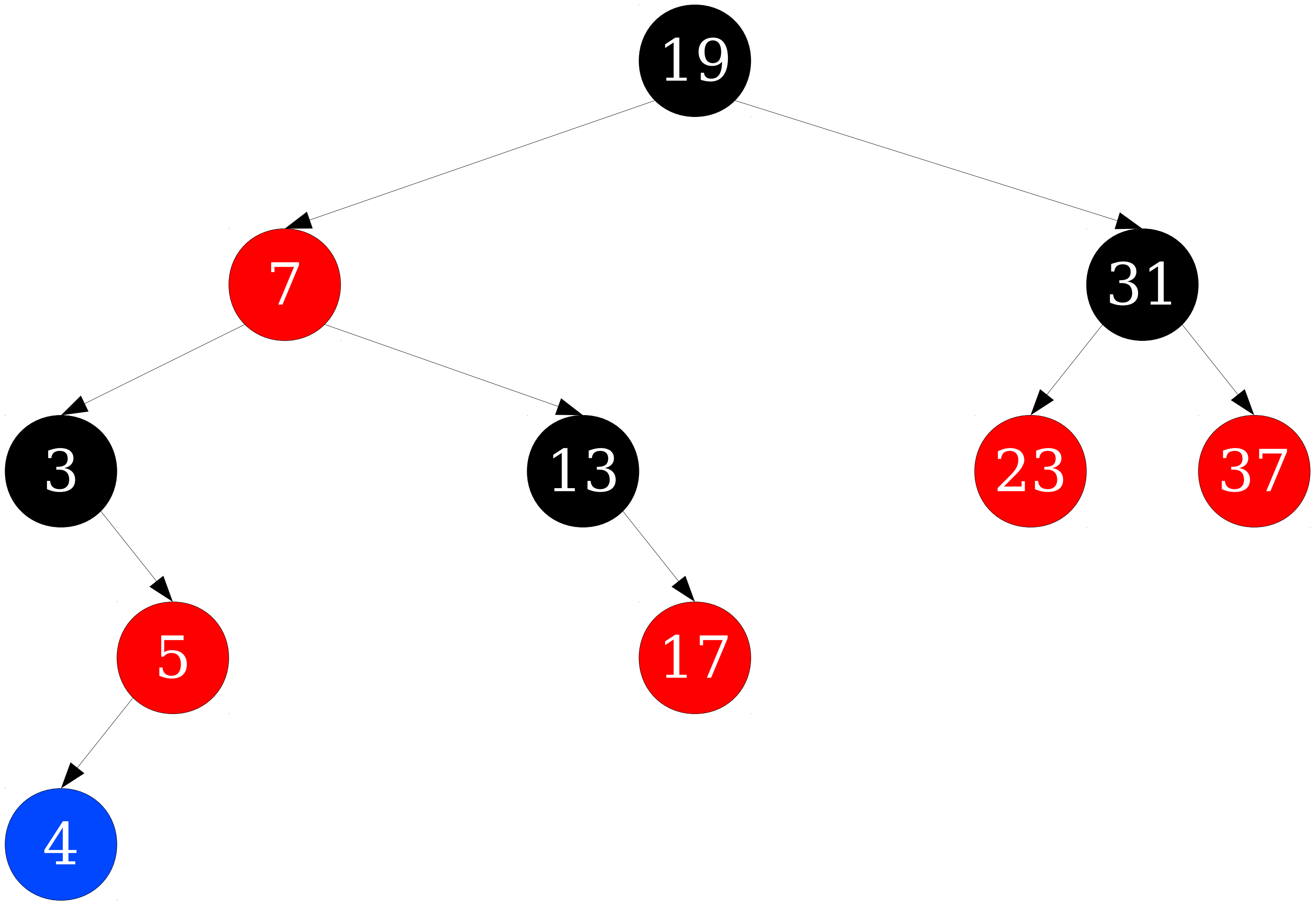
Using the Isometry



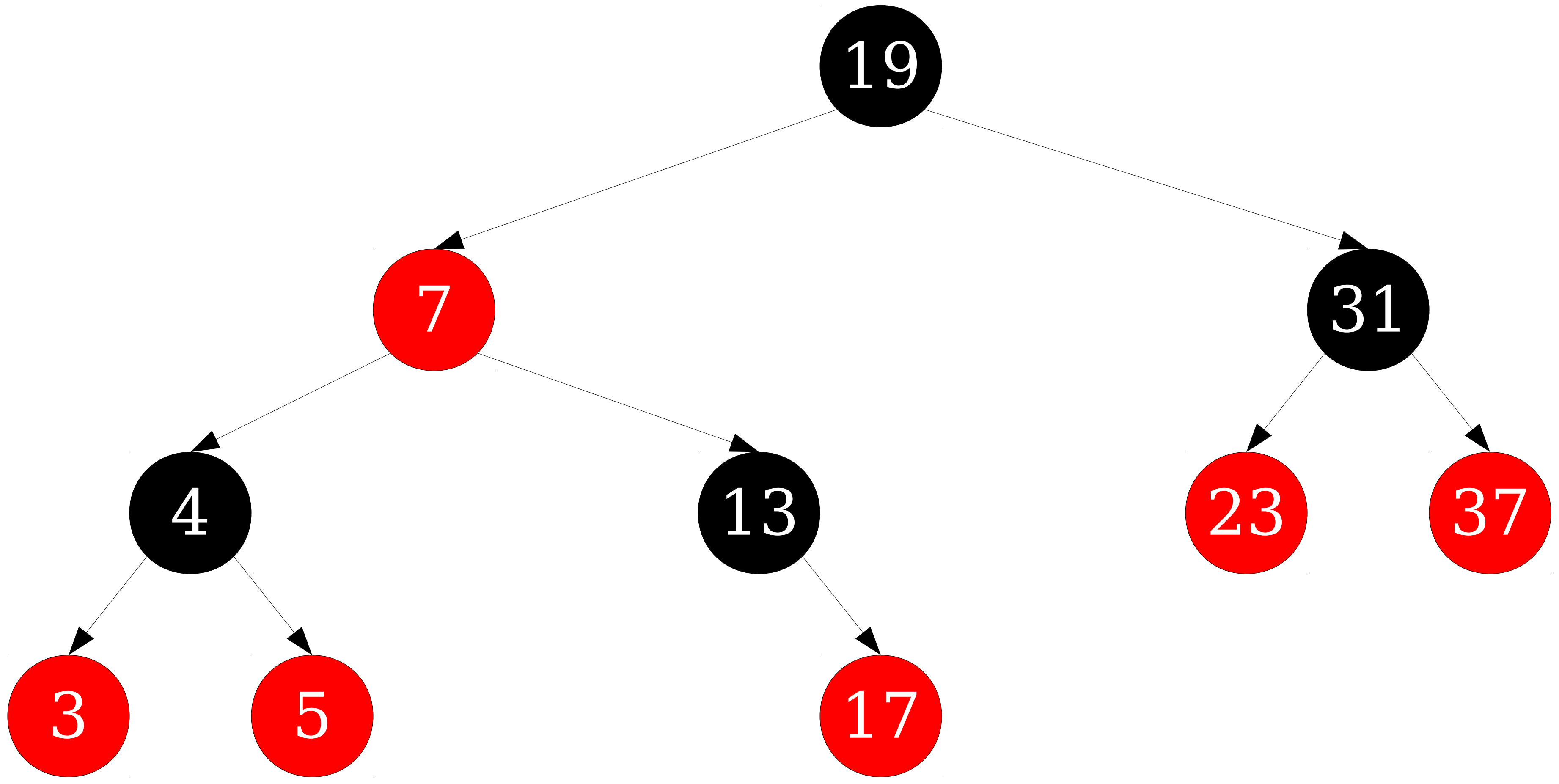
Using the Isometry



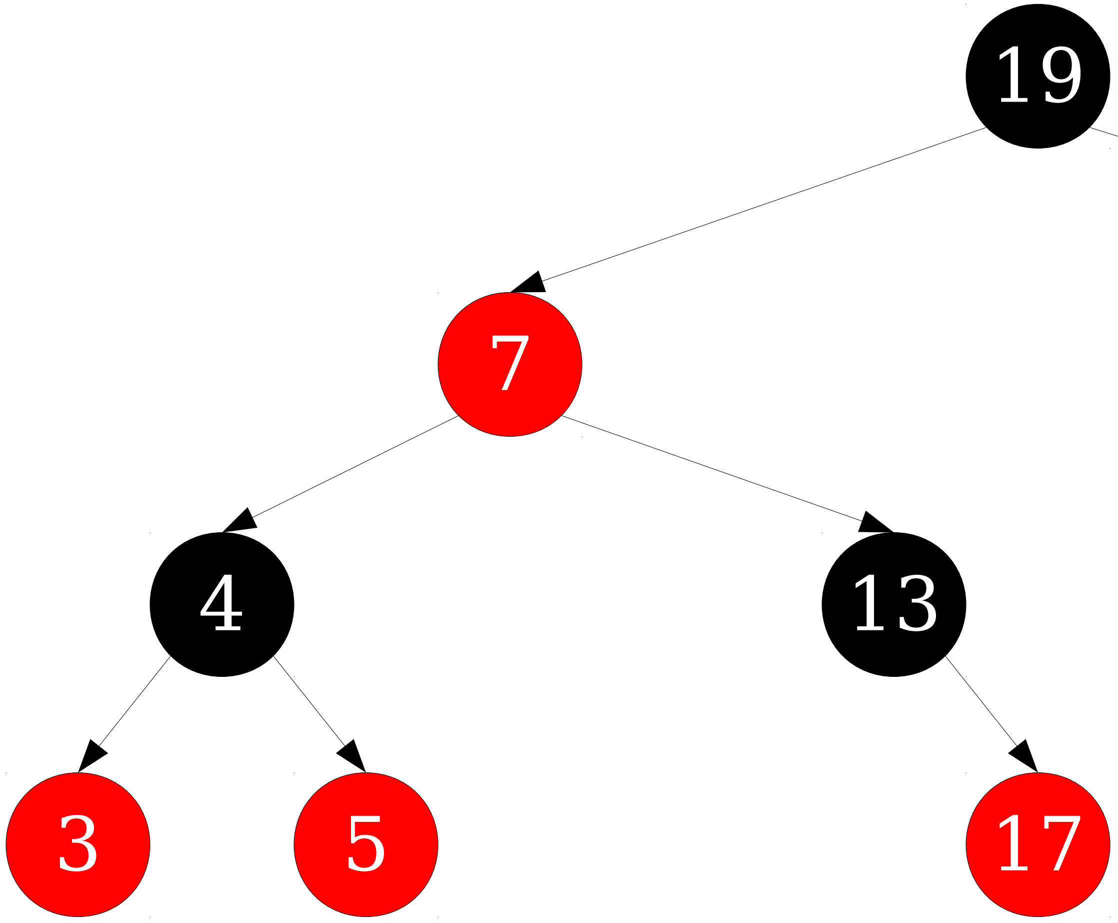
Using the Isometry



Using the Isometry



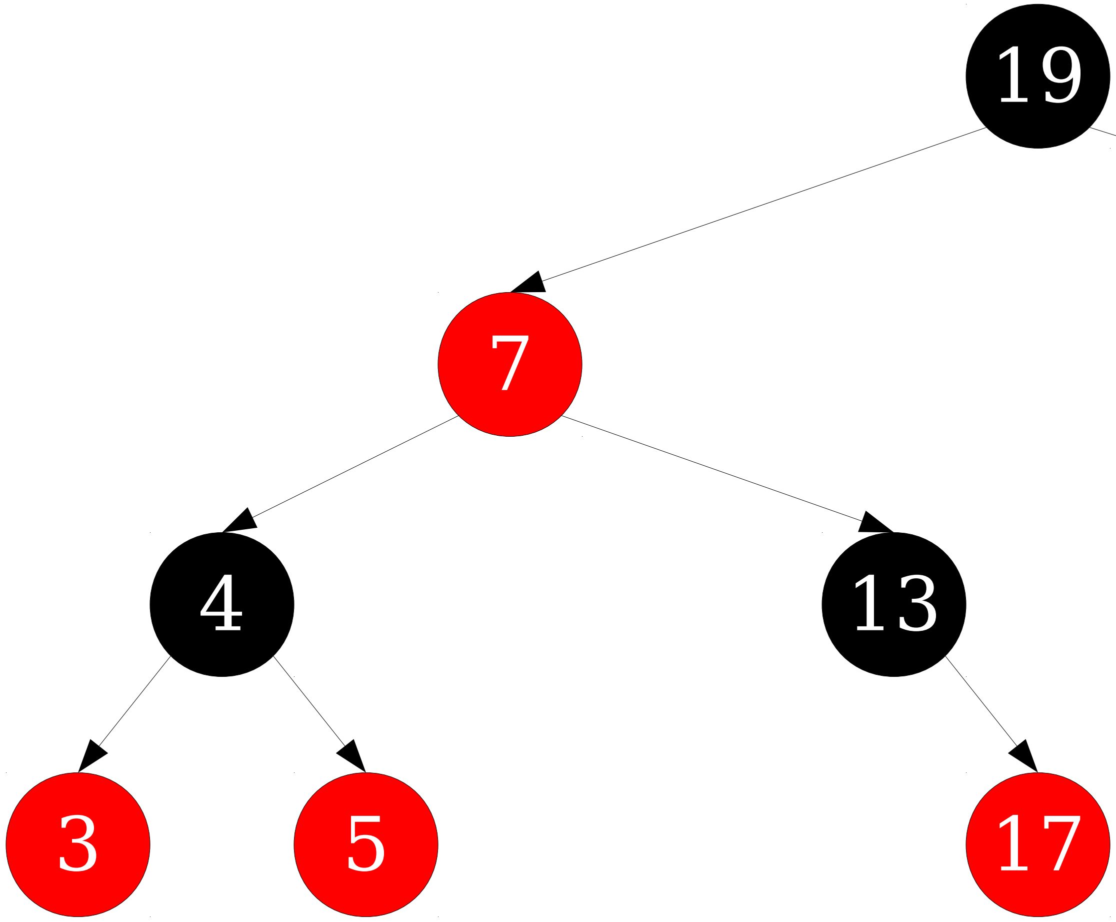
Using the Isometry



We need to

1. Change the colors of the nodes, and
2. Move the nodes around in the tree.

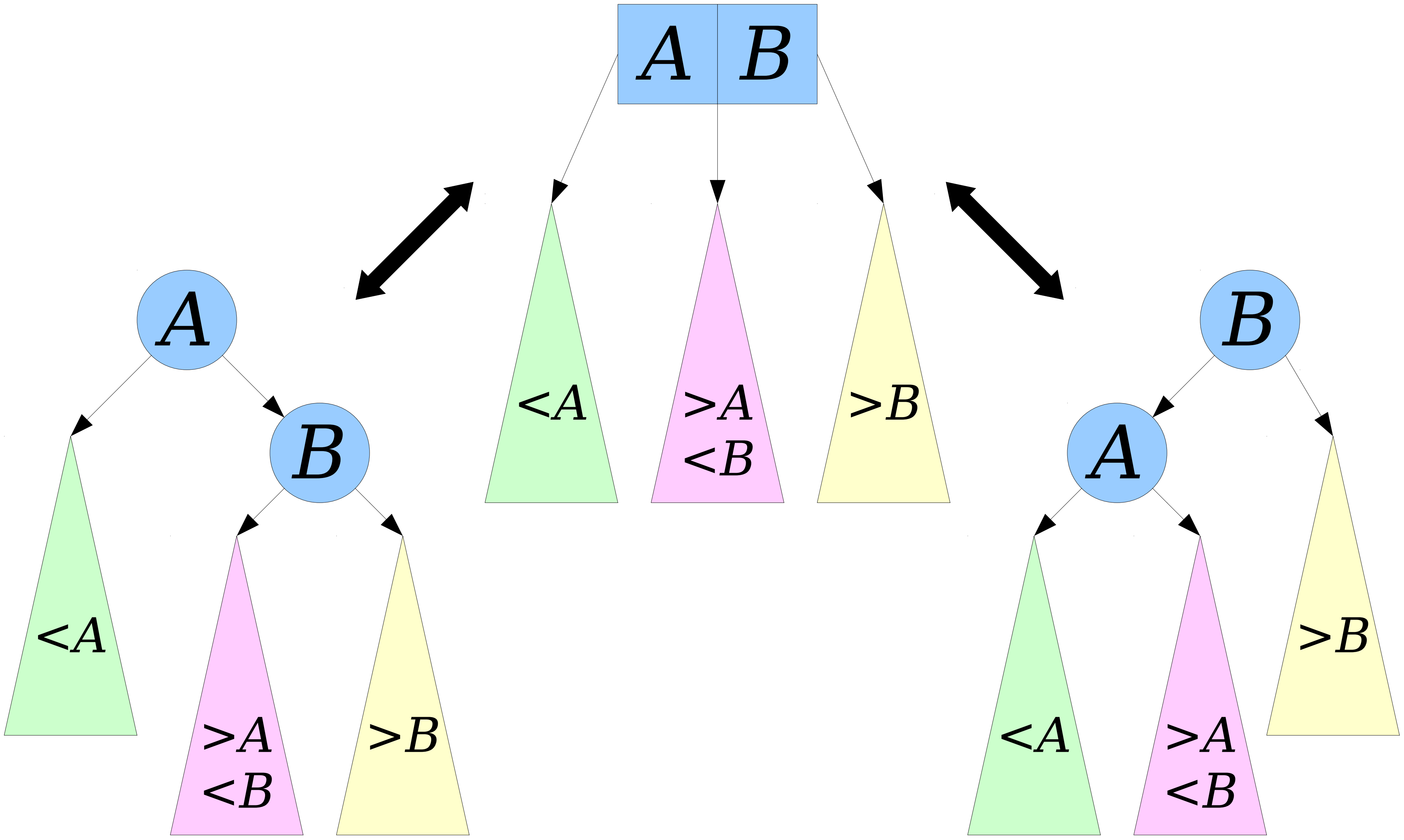
Using the Isometry



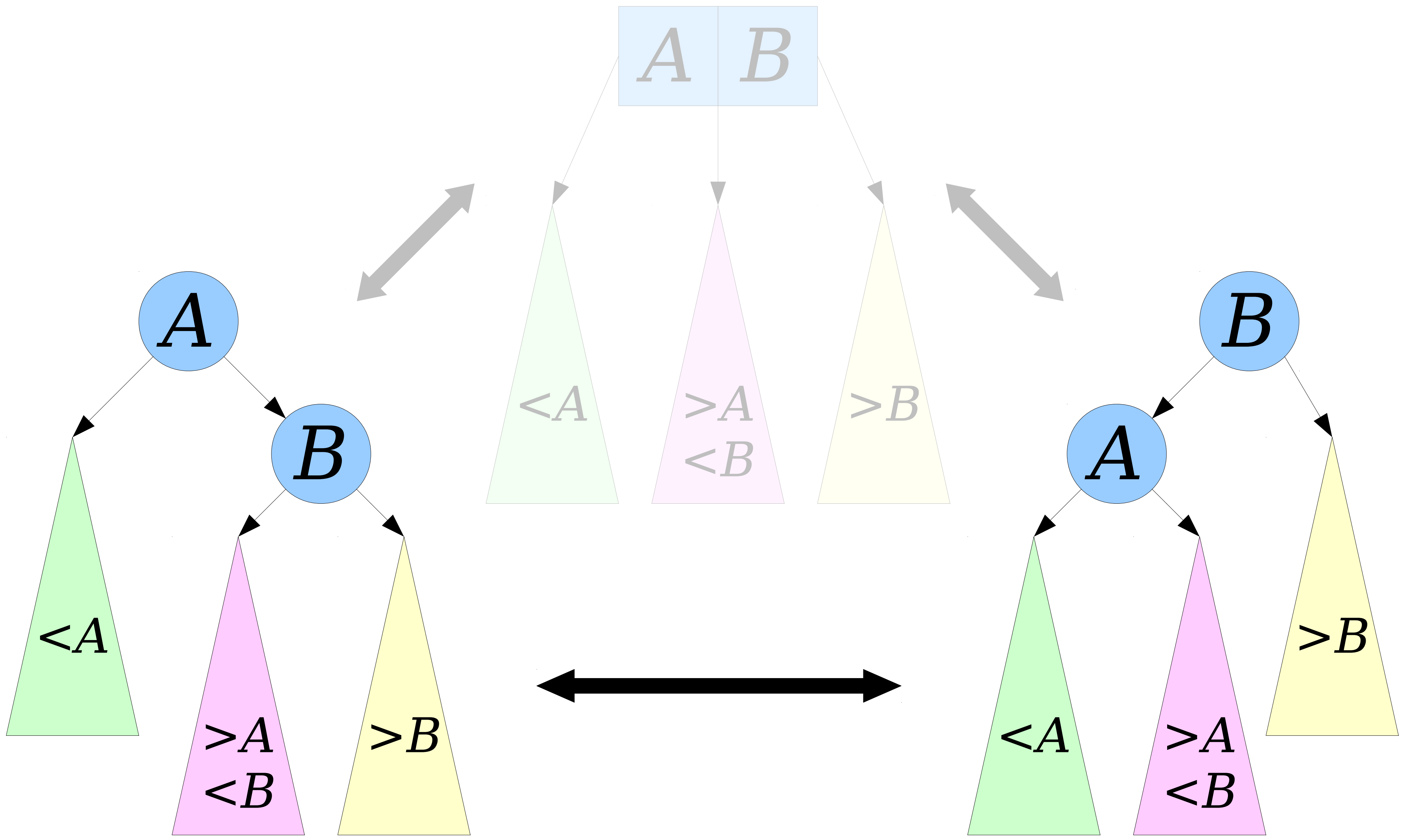
We need to

1. Change the colors of the nodes, and
2. Move the nodes around in the tree.

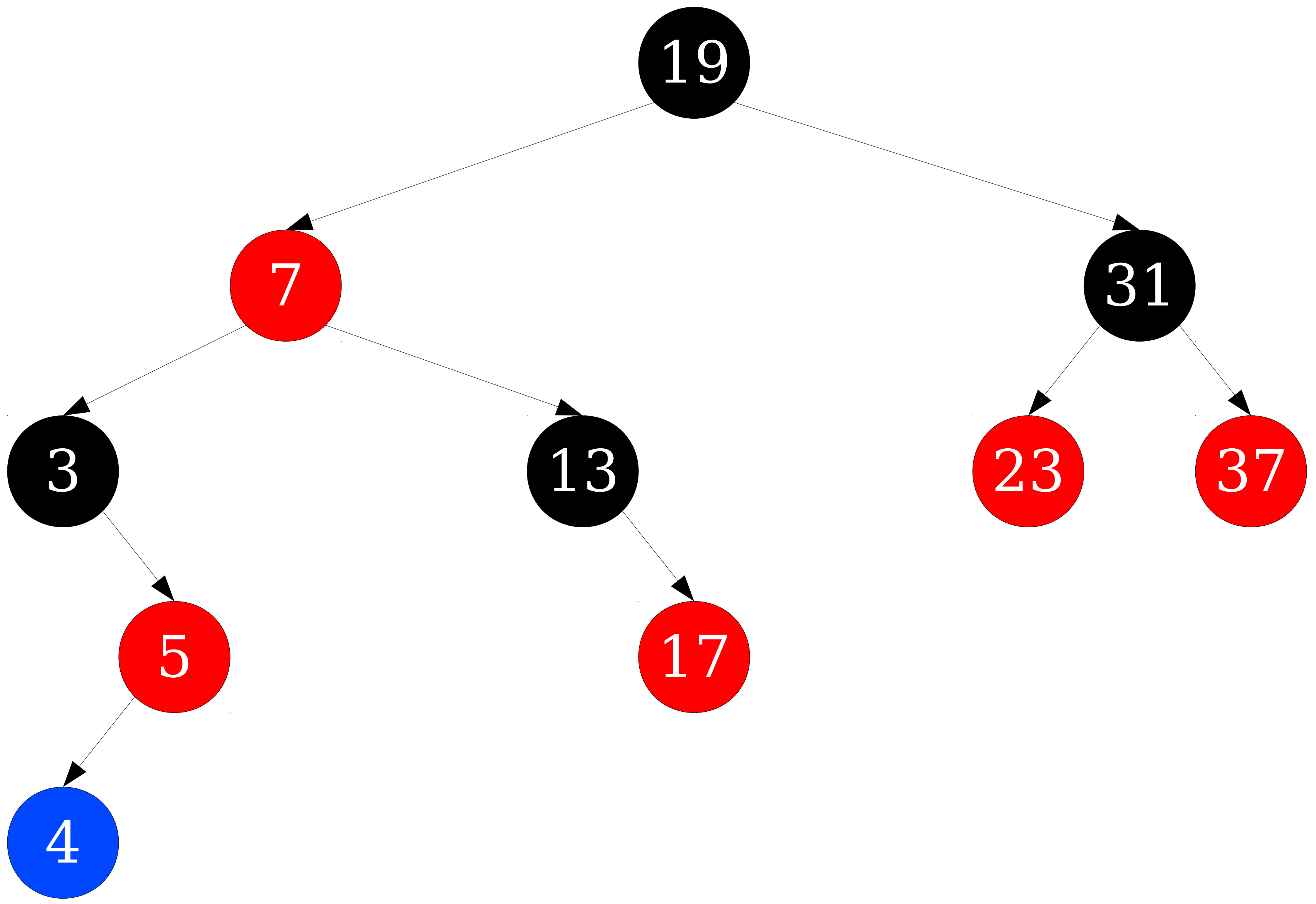
Tree Rotations



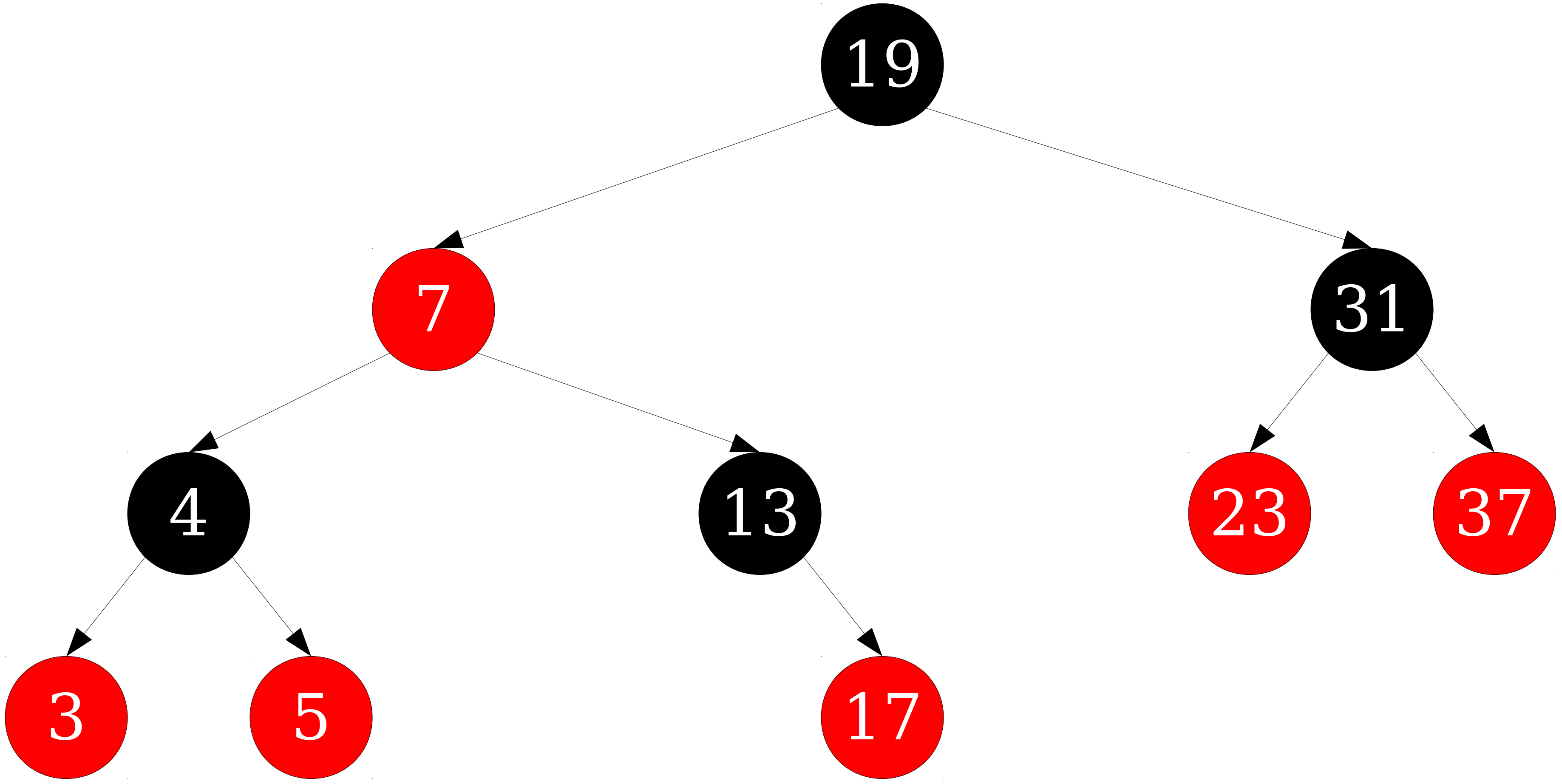
Tree Rotations

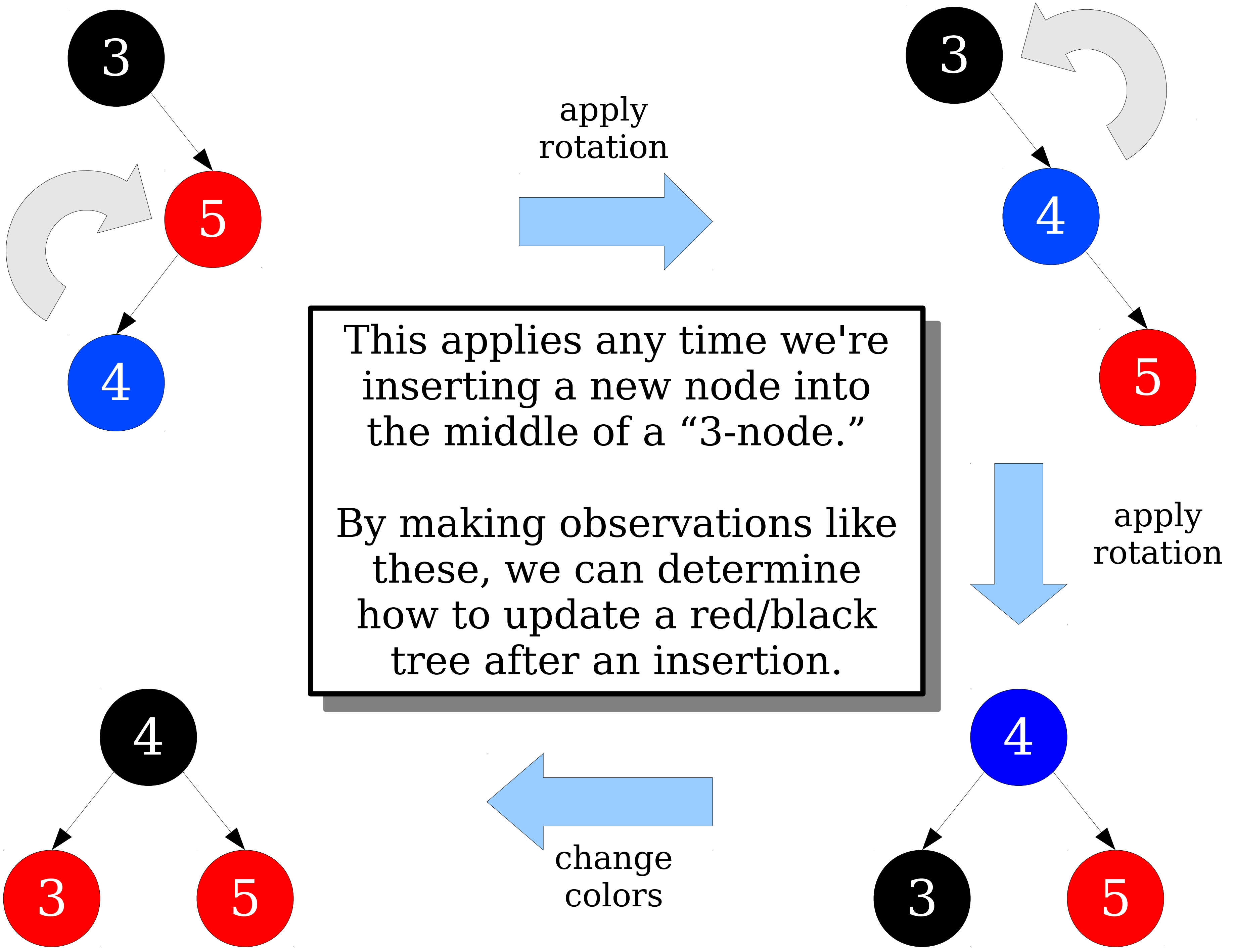


Using the Isometry



Using the Isometry

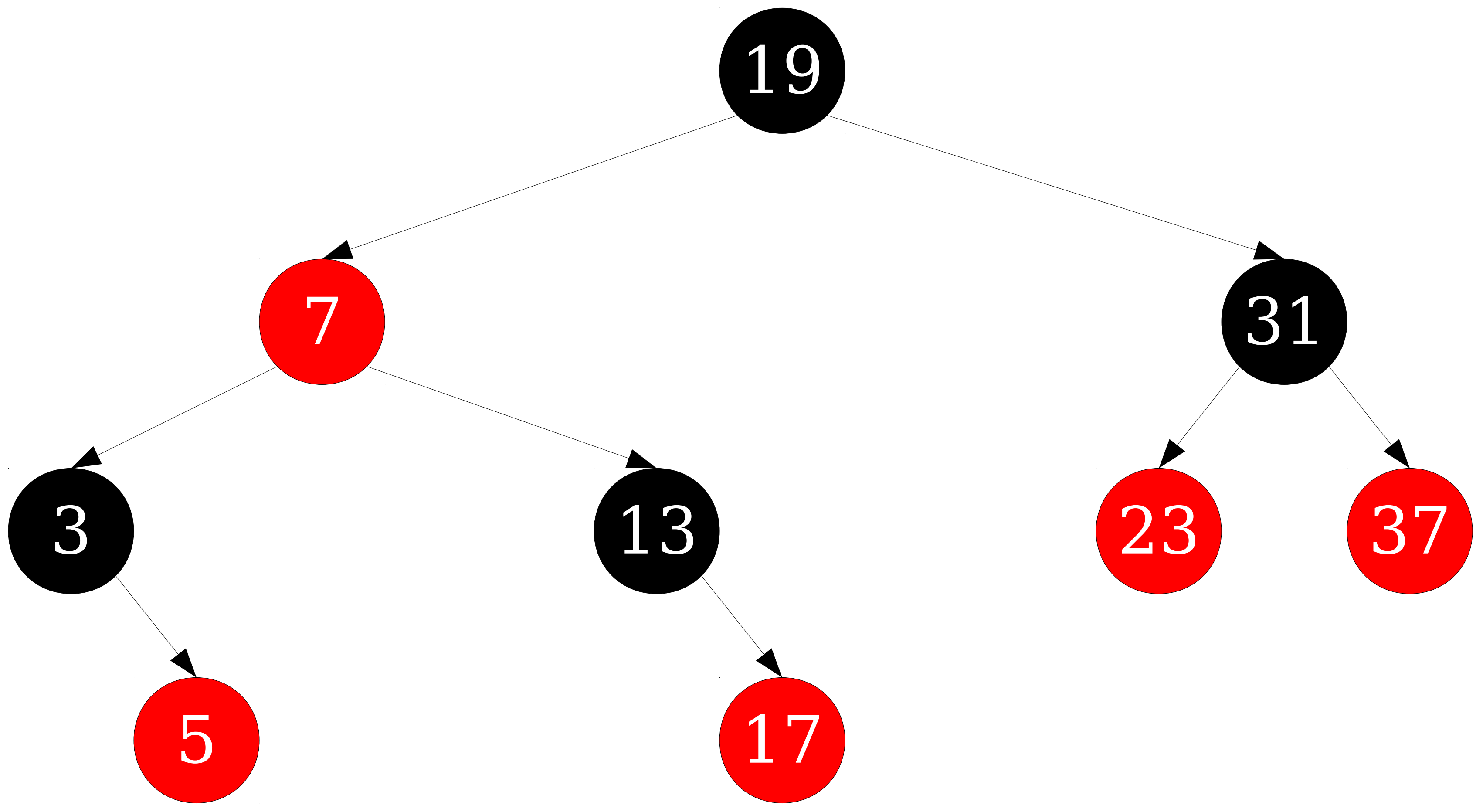




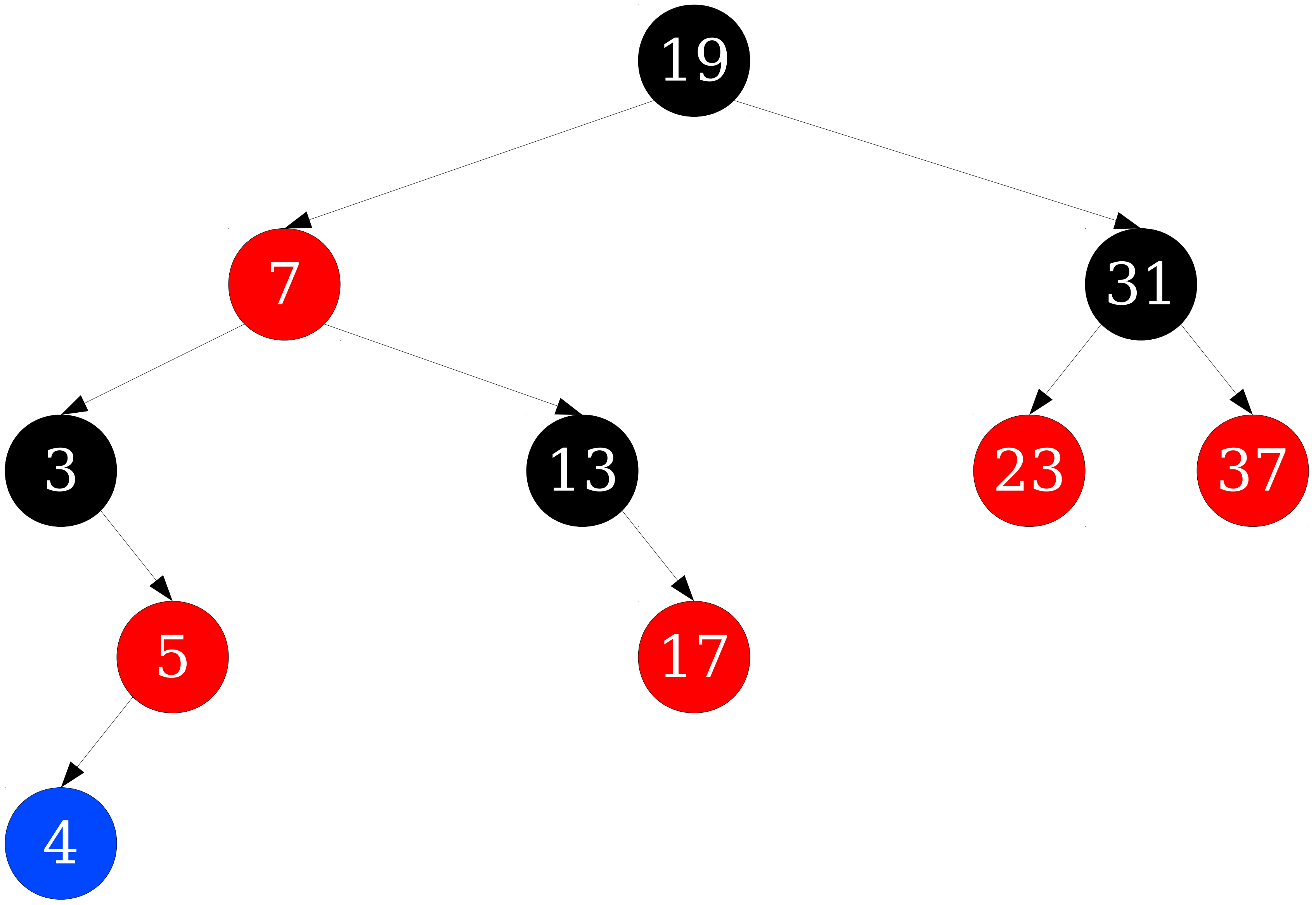
This applies any time we're inserting a new node into the middle of a “3-node.”

By making observations like these, we can determine how to update a red/black tree after an insertion.

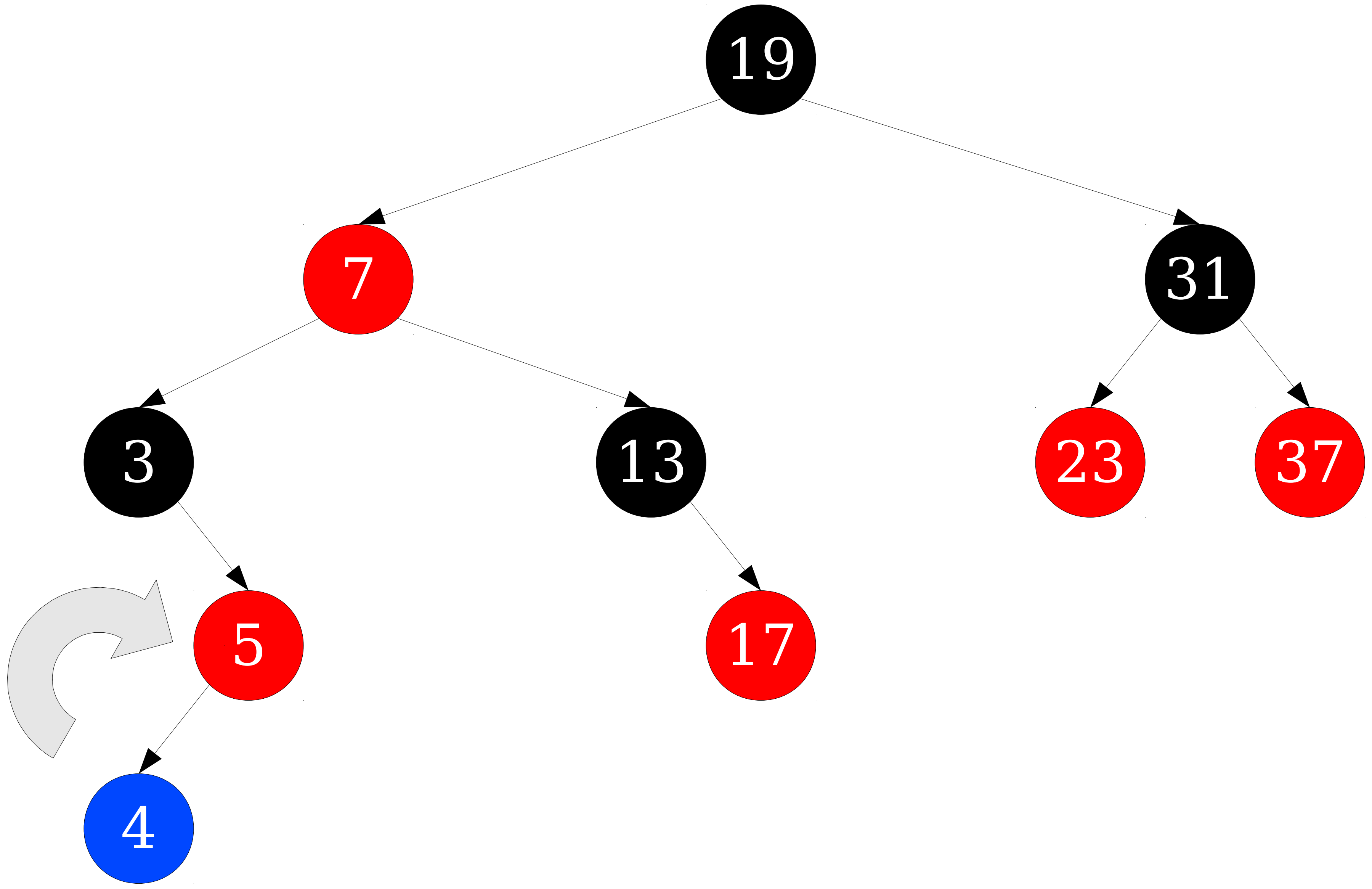
Using the Isometry



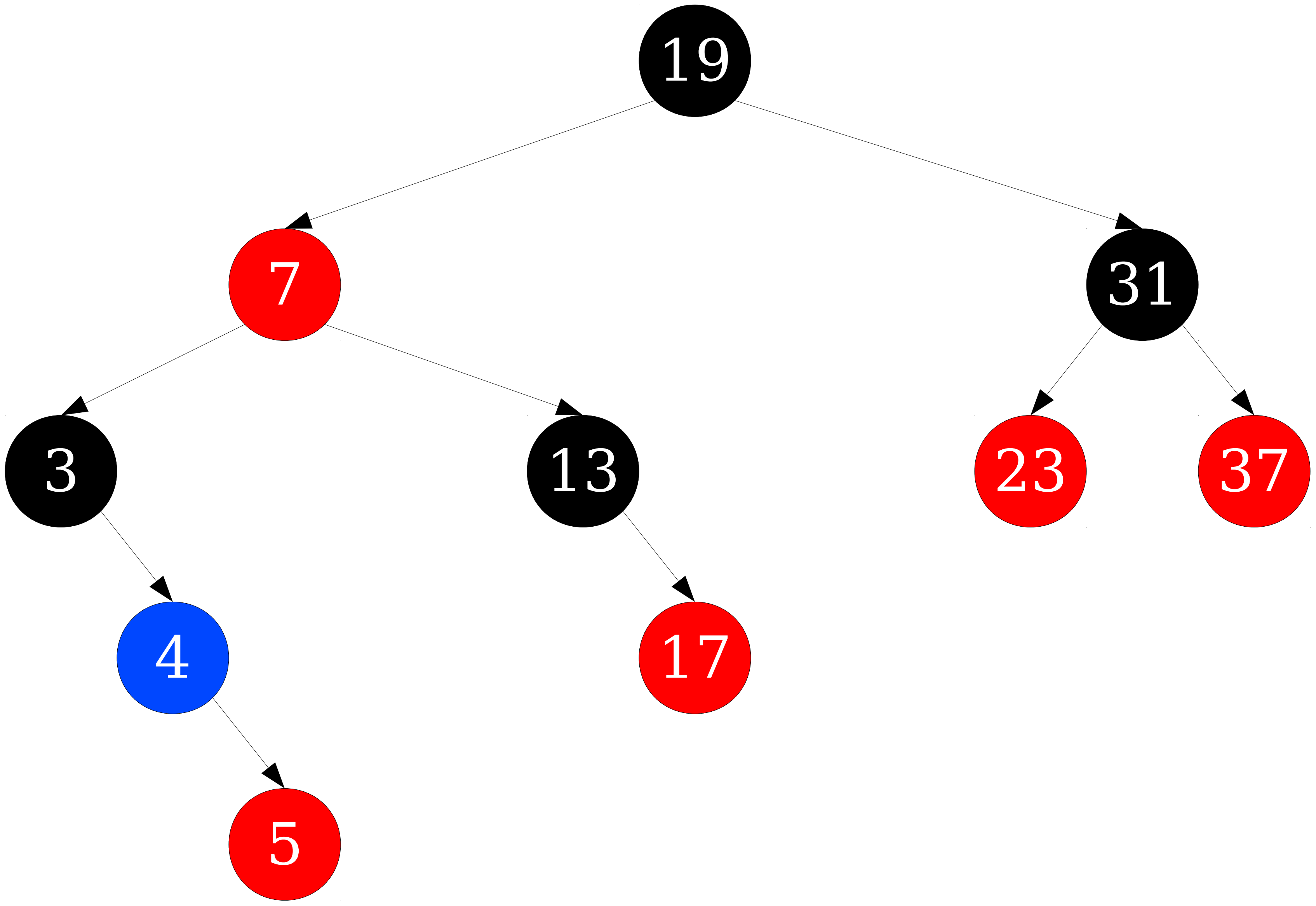
Using the Isometry



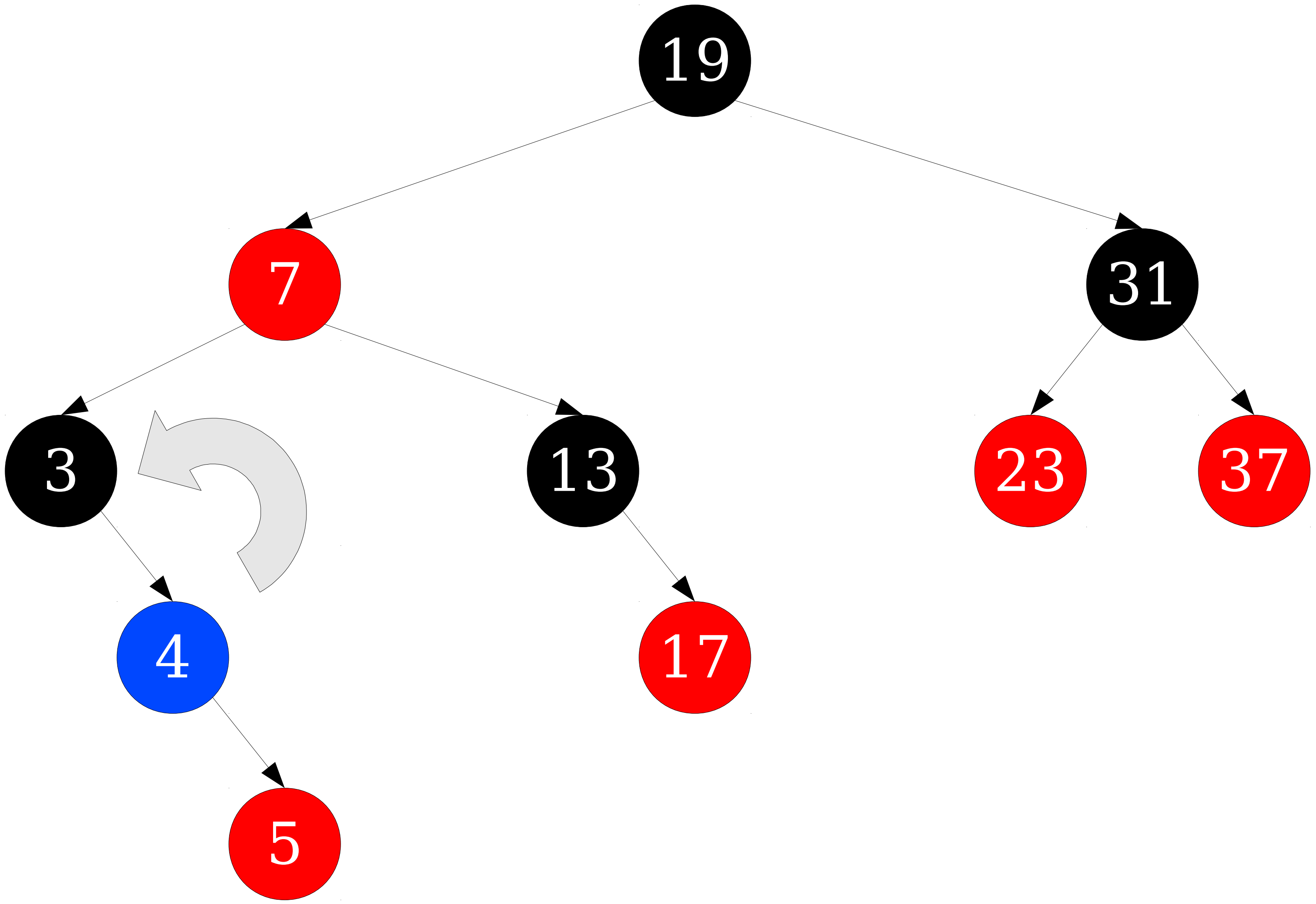
Using the Isometry



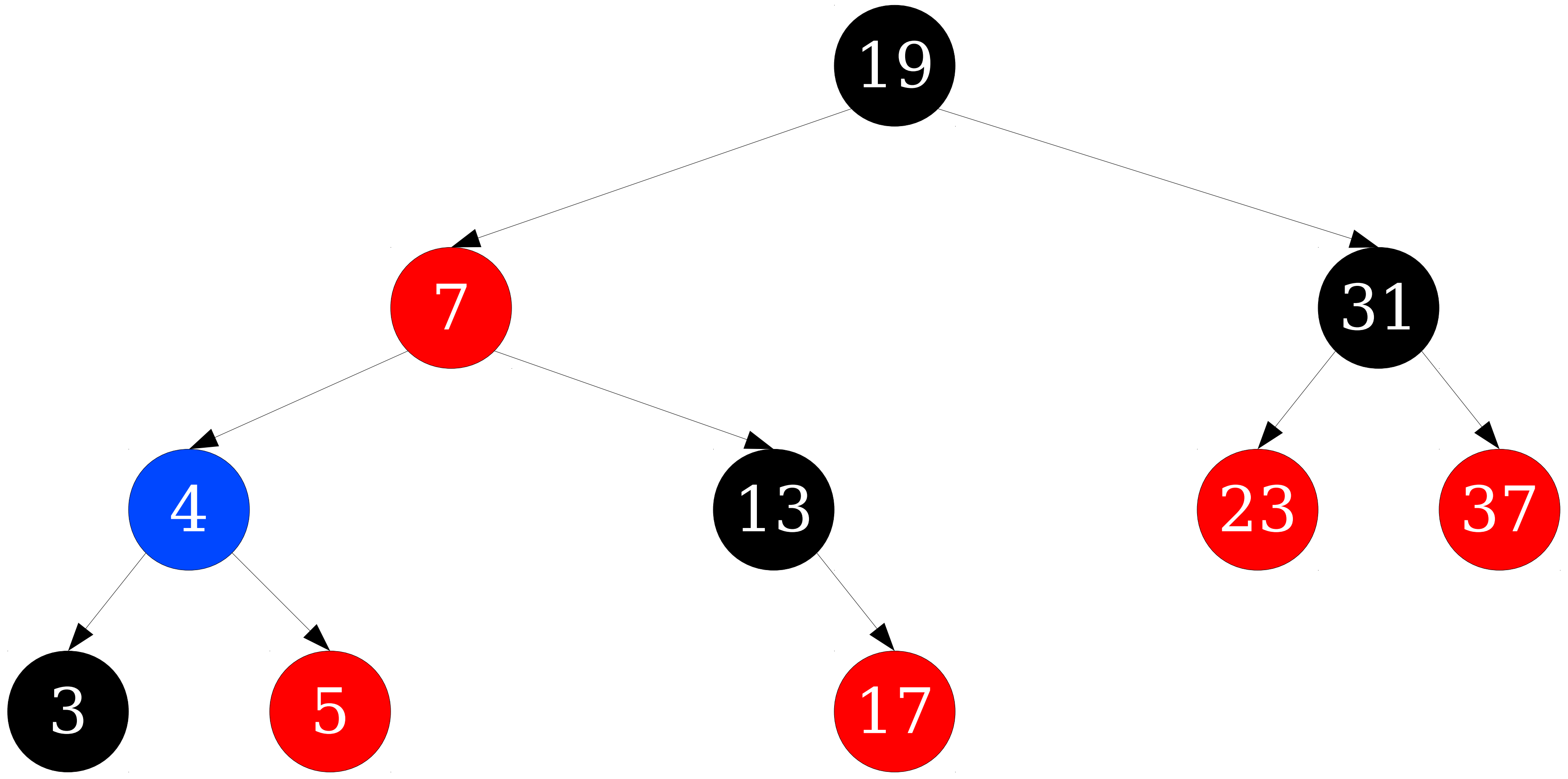
Using the Isometry



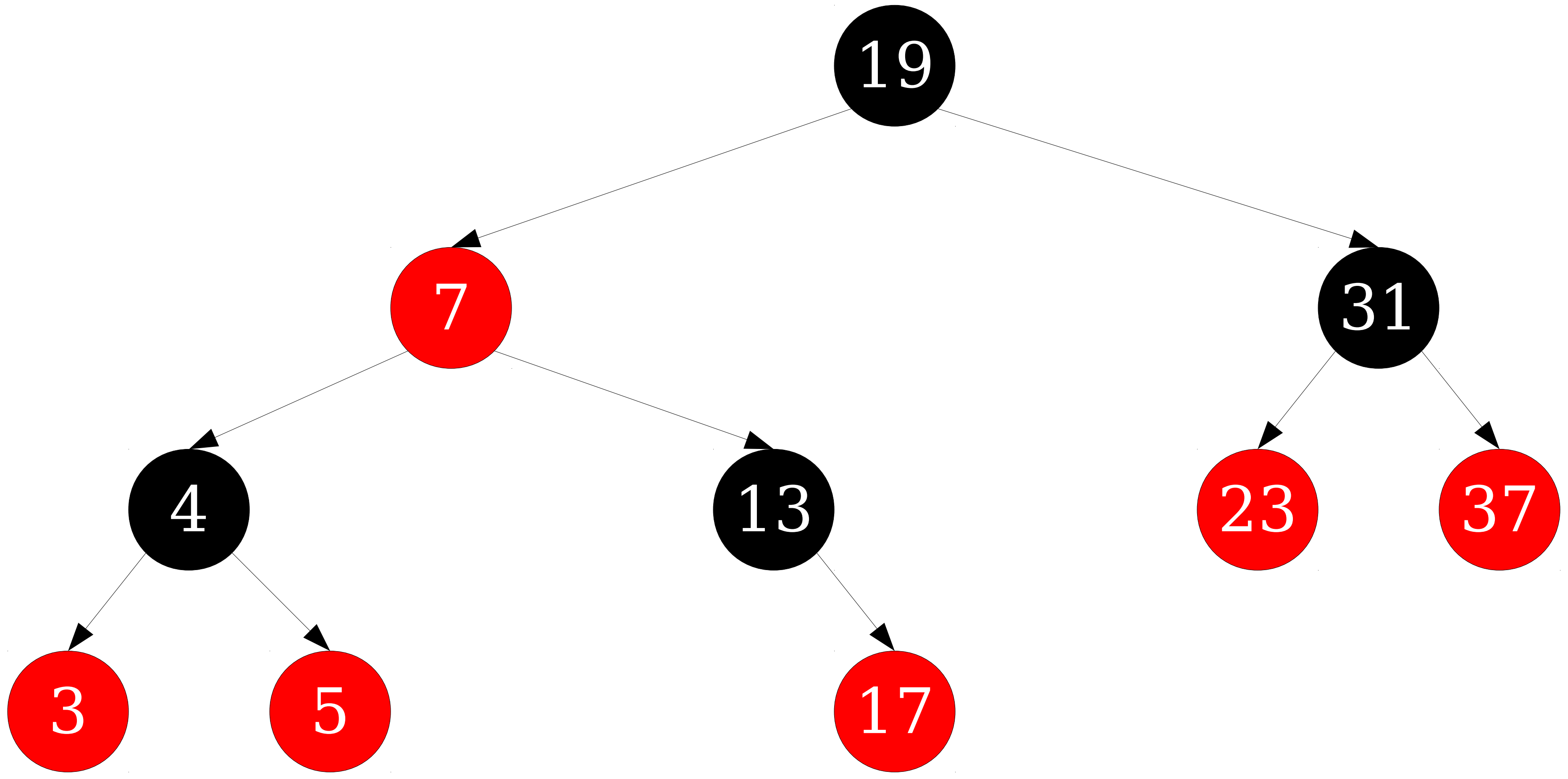
Using the Isometry



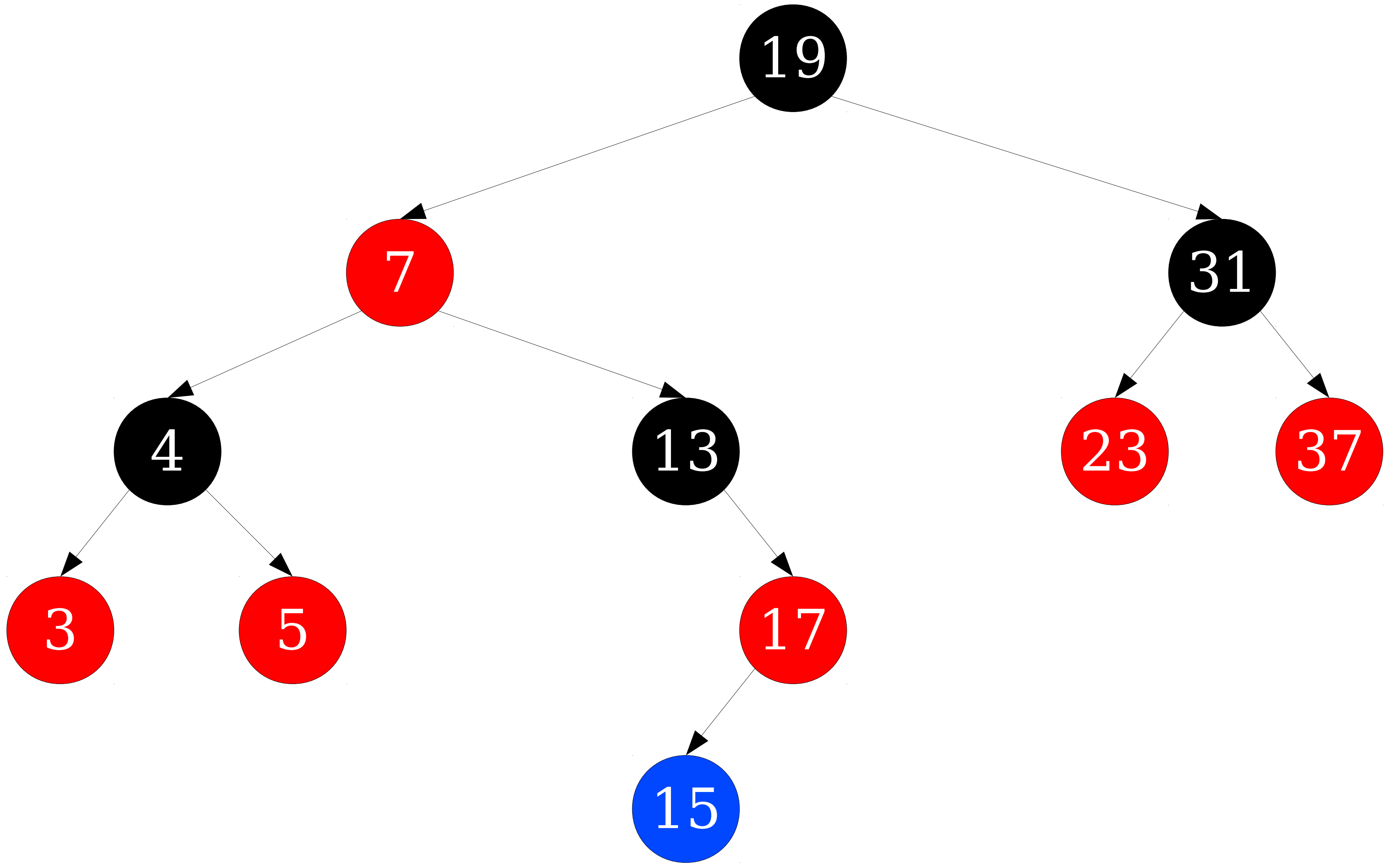
Using the Isometry



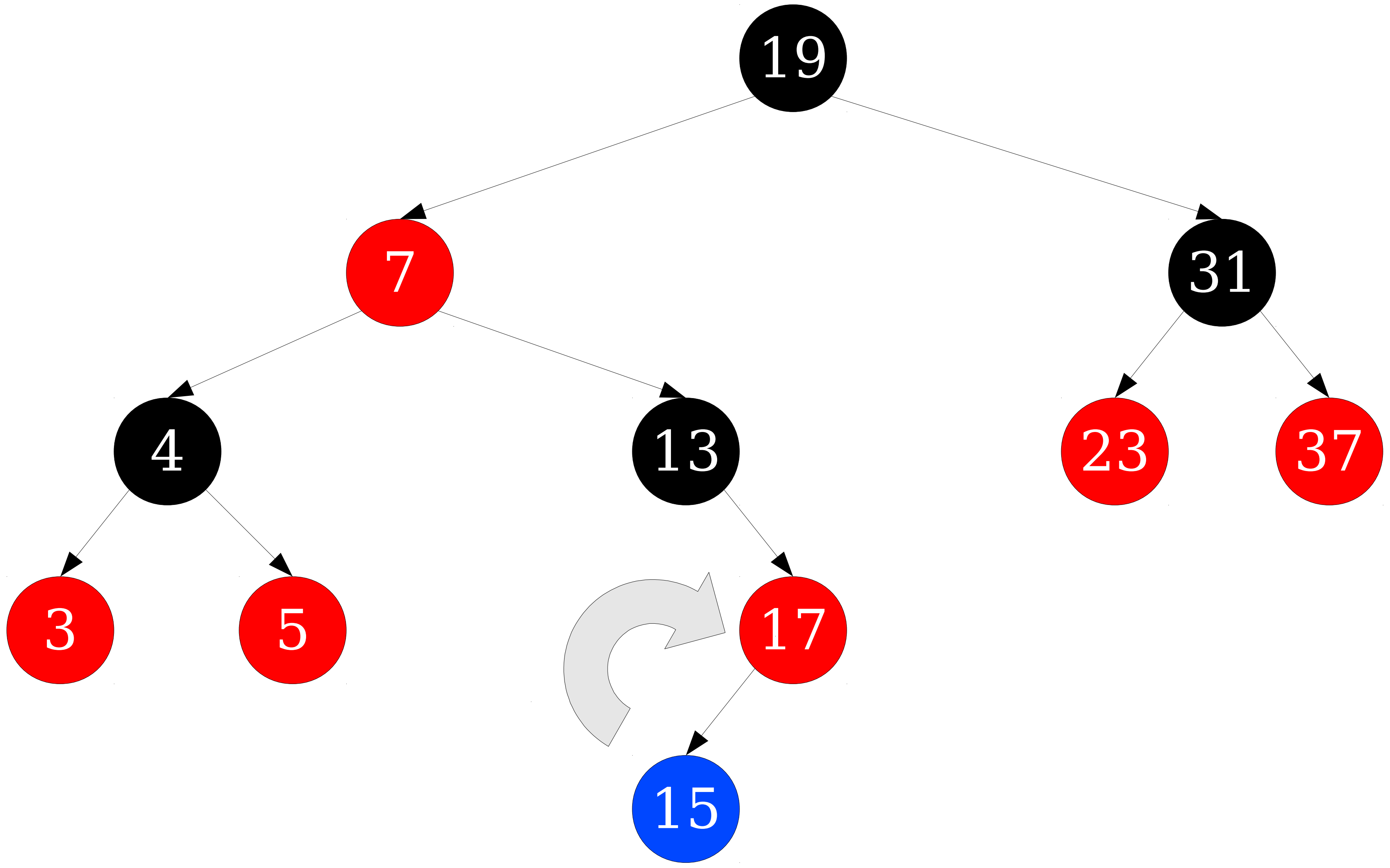
Using the Isometry



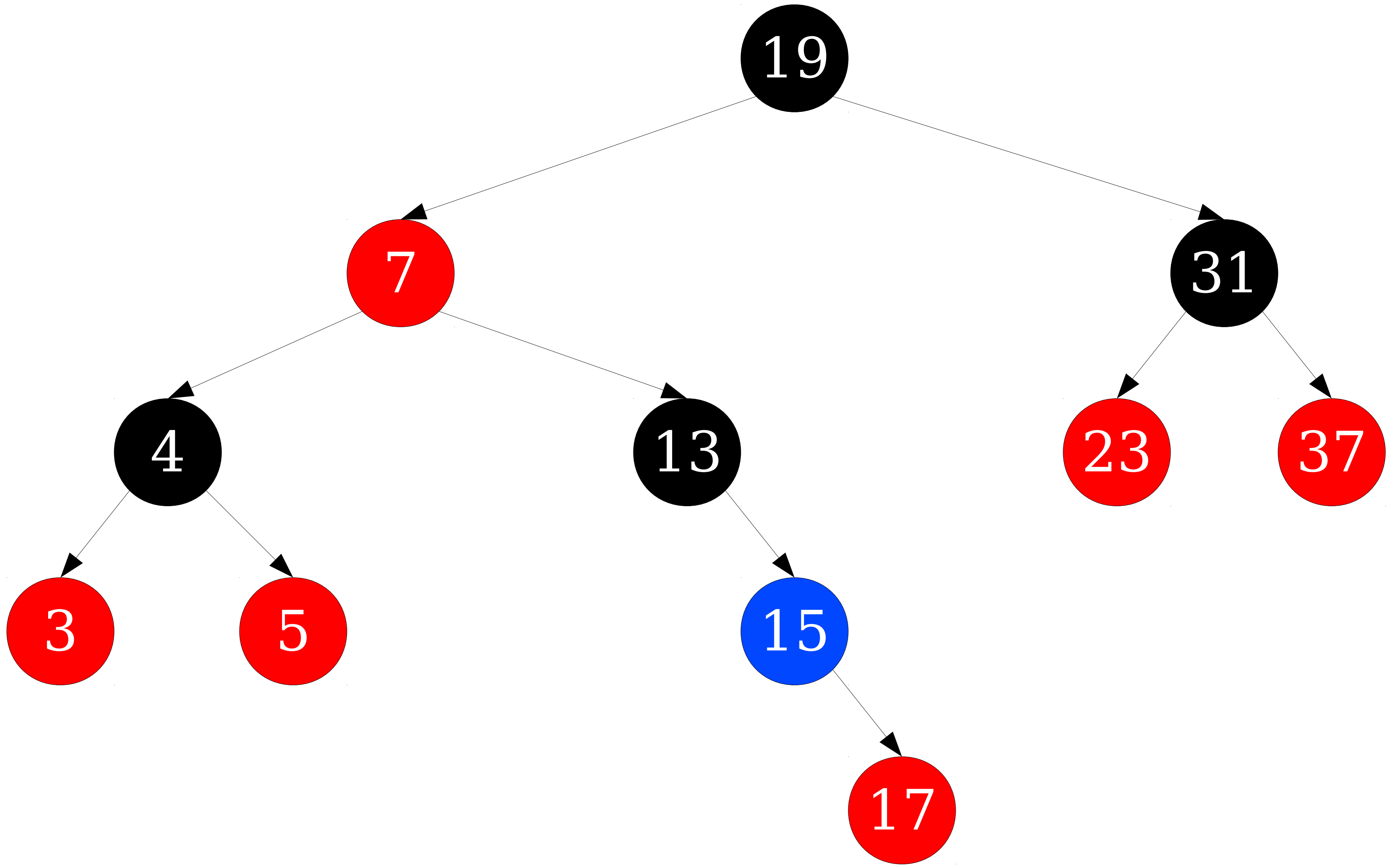
Using the Isometry



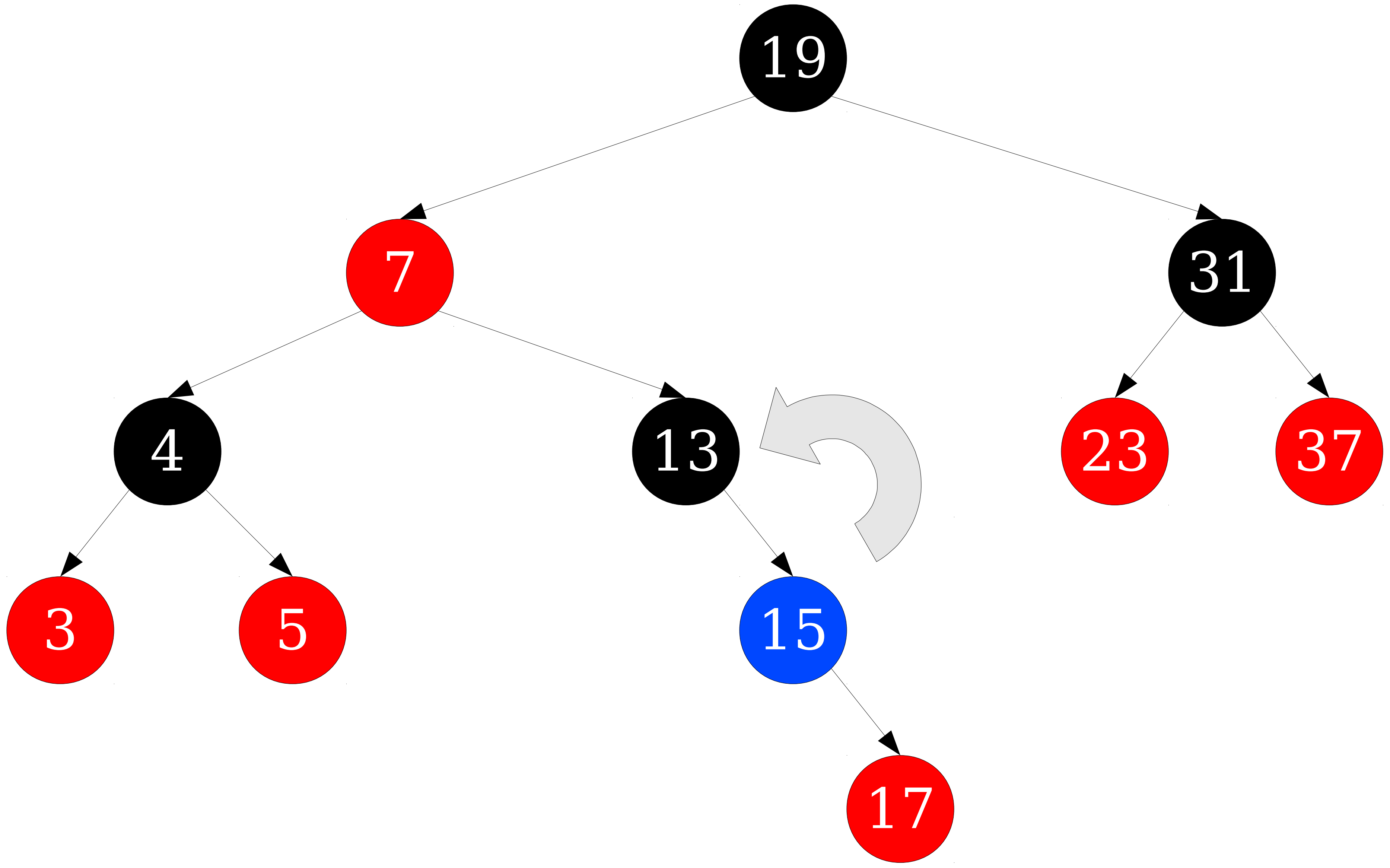
Using the Isometry



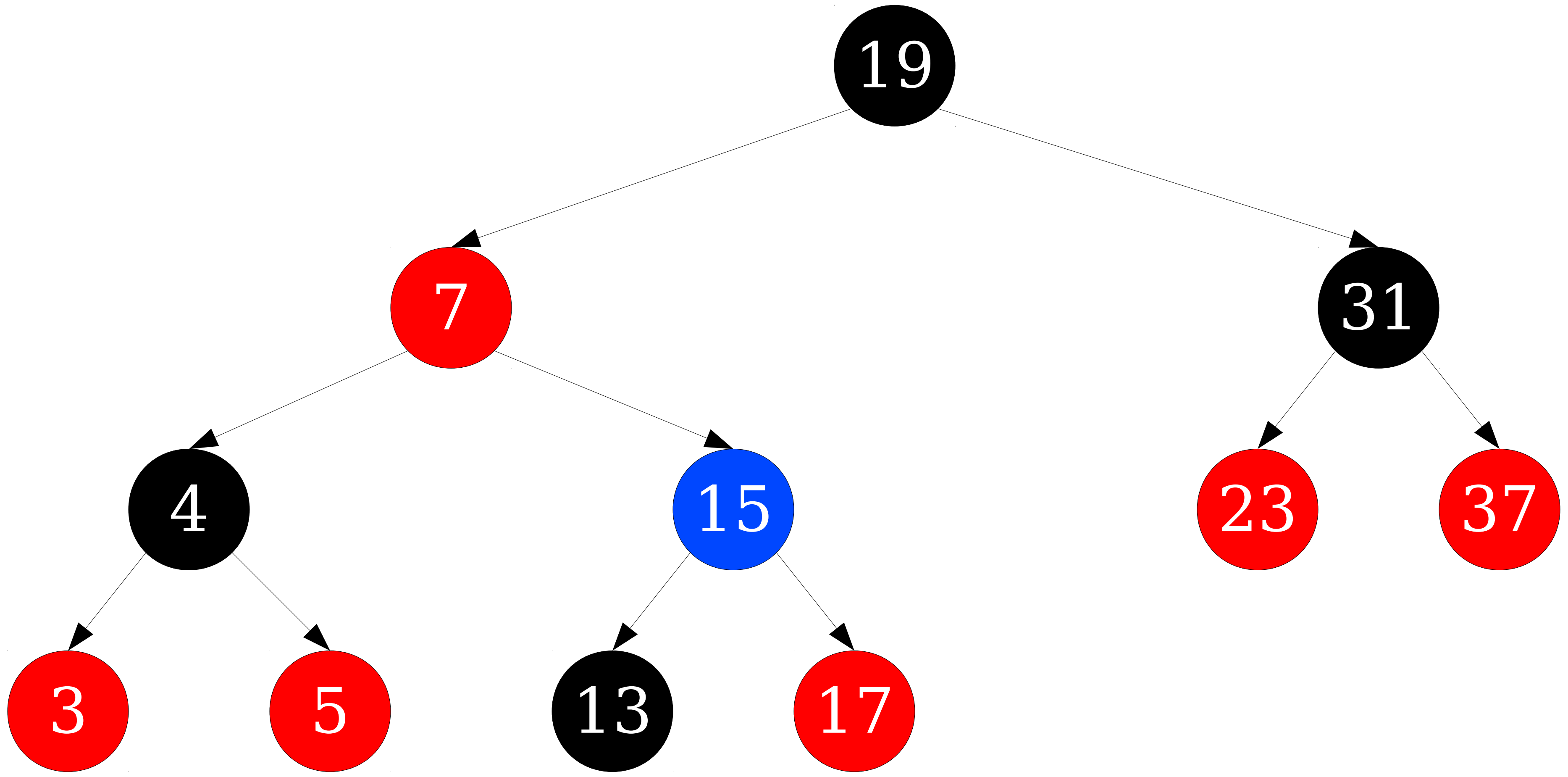
Using the Isometry



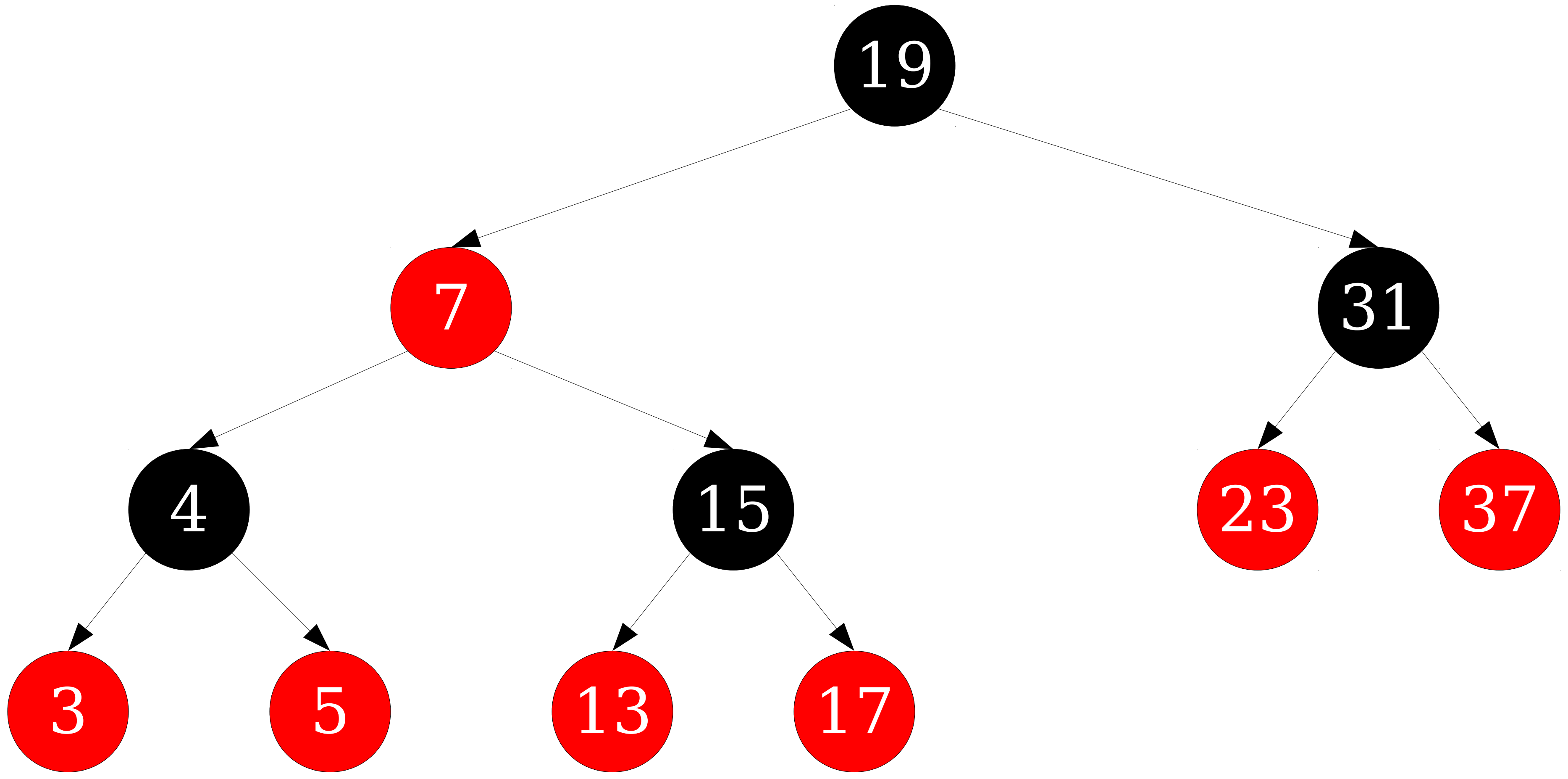
Using the Isometry



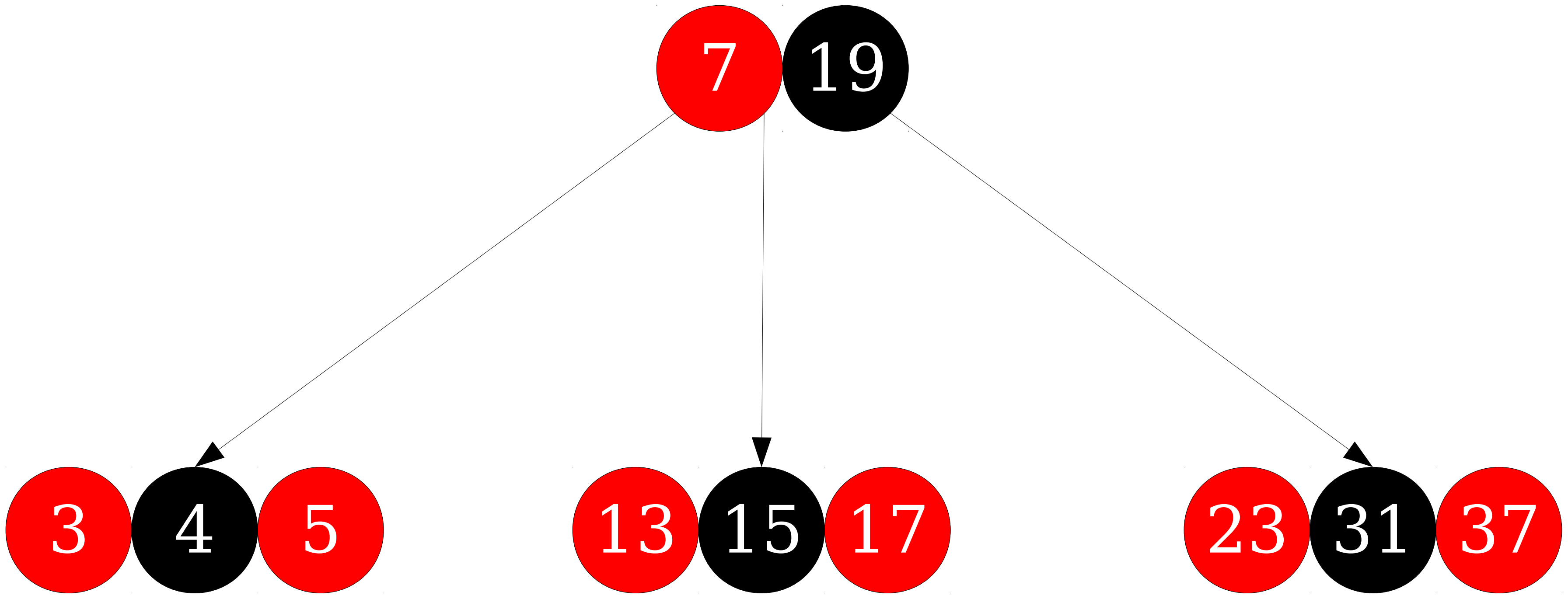
Using the Isometry



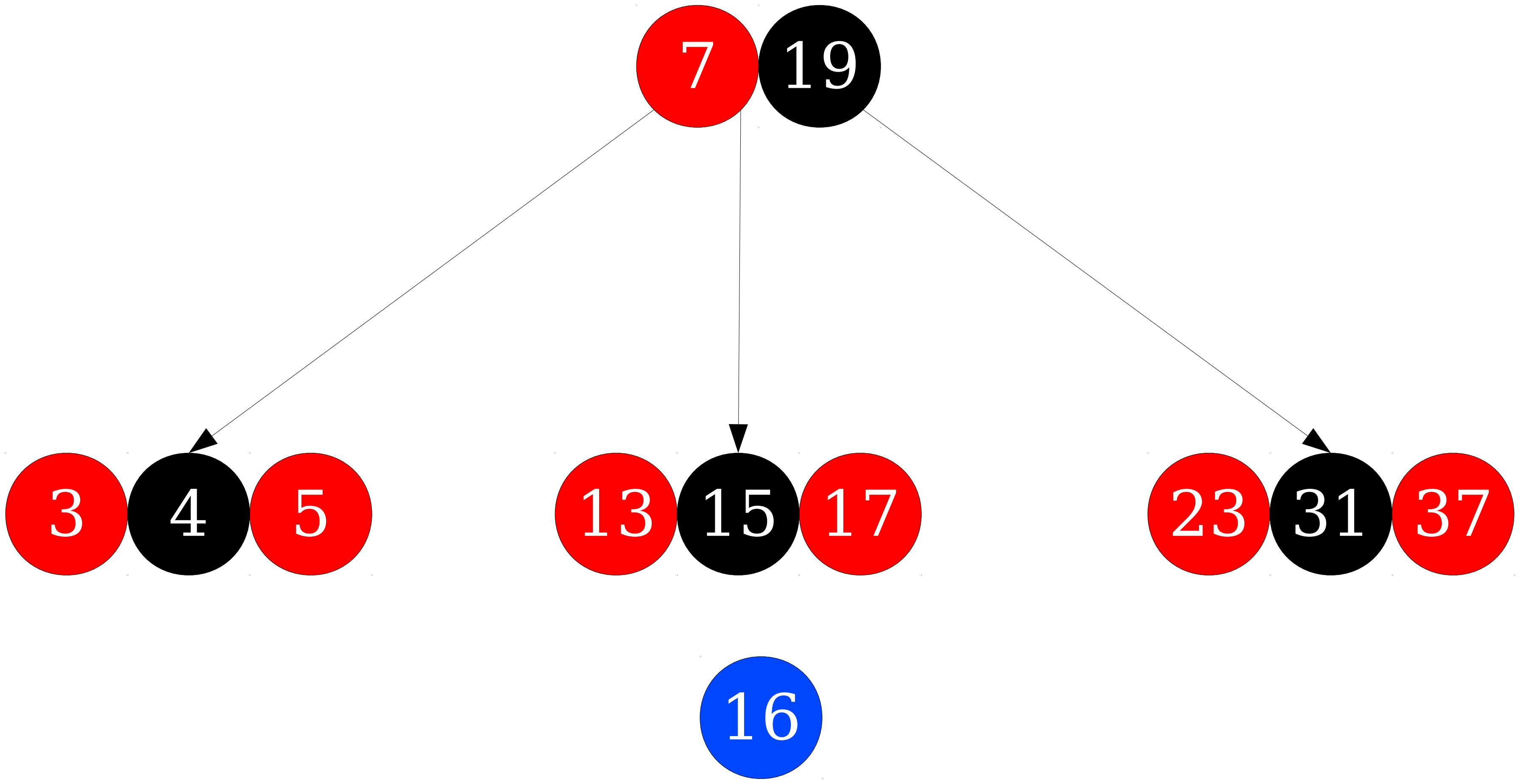
Using the Isometry



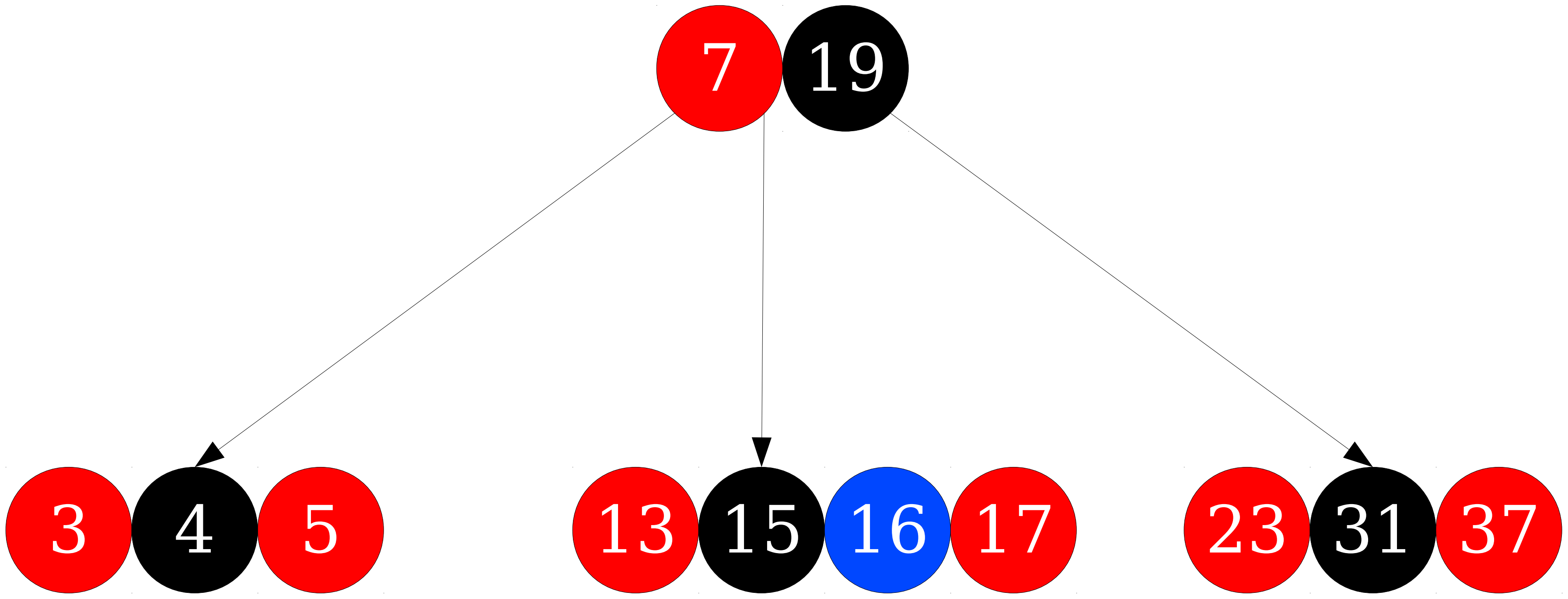
Using the Isometry



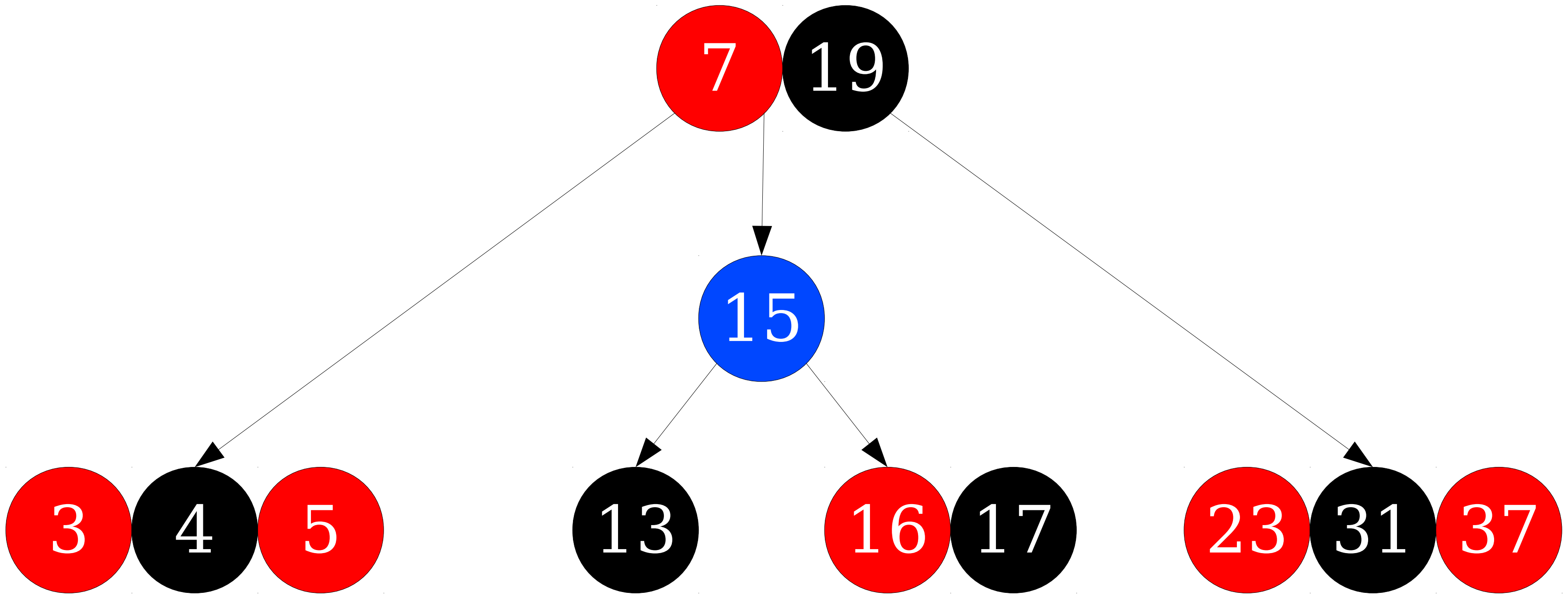
Using the Isometry



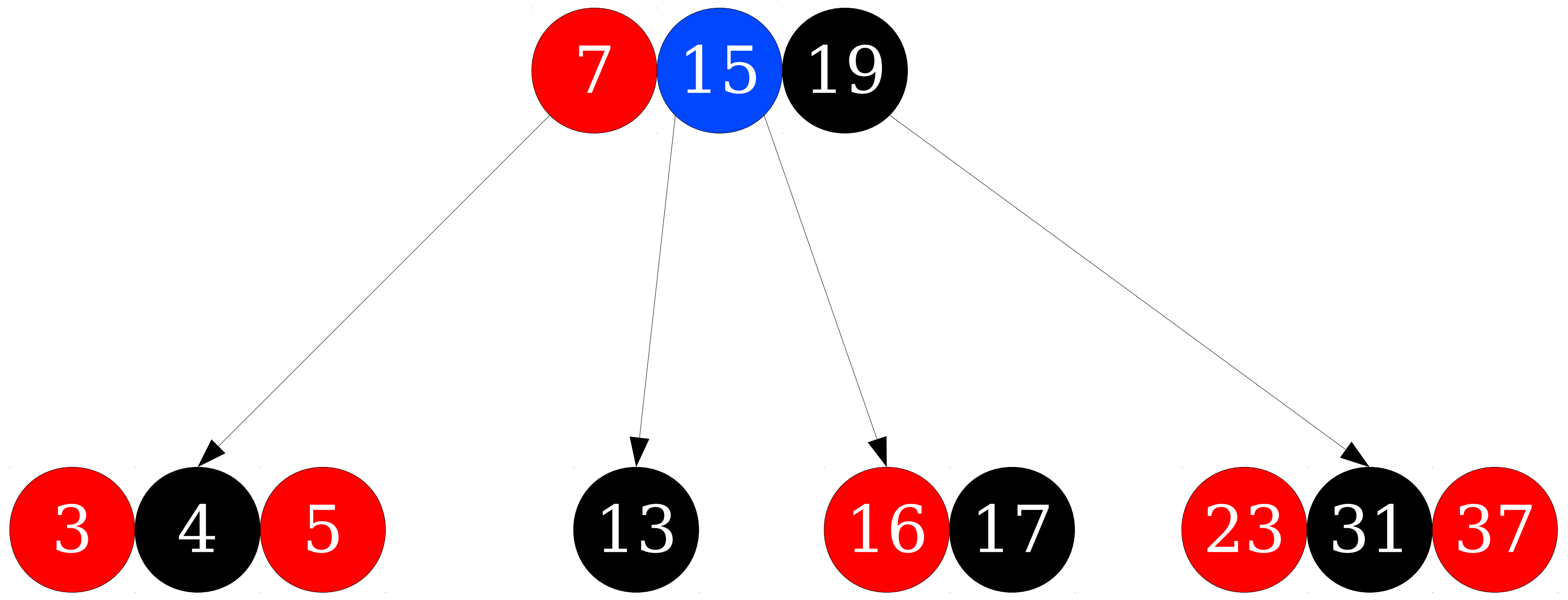
Using the Isometry



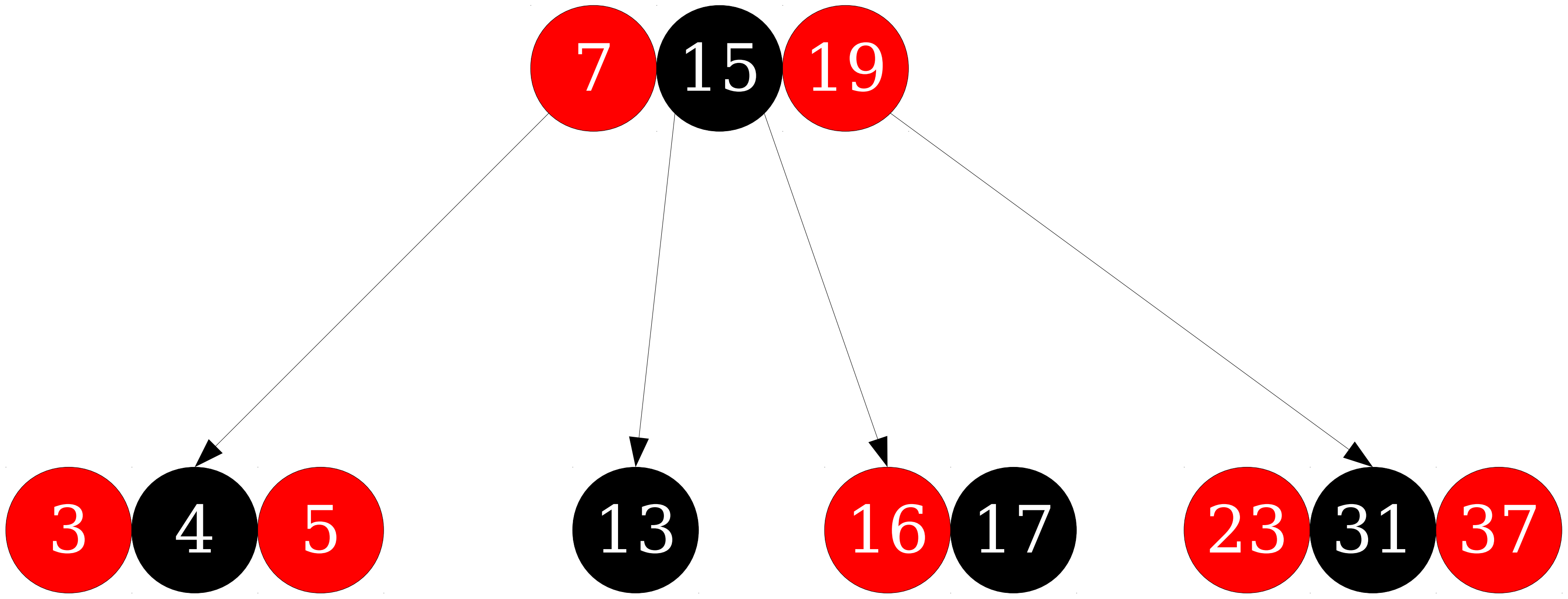
Using the Isometry



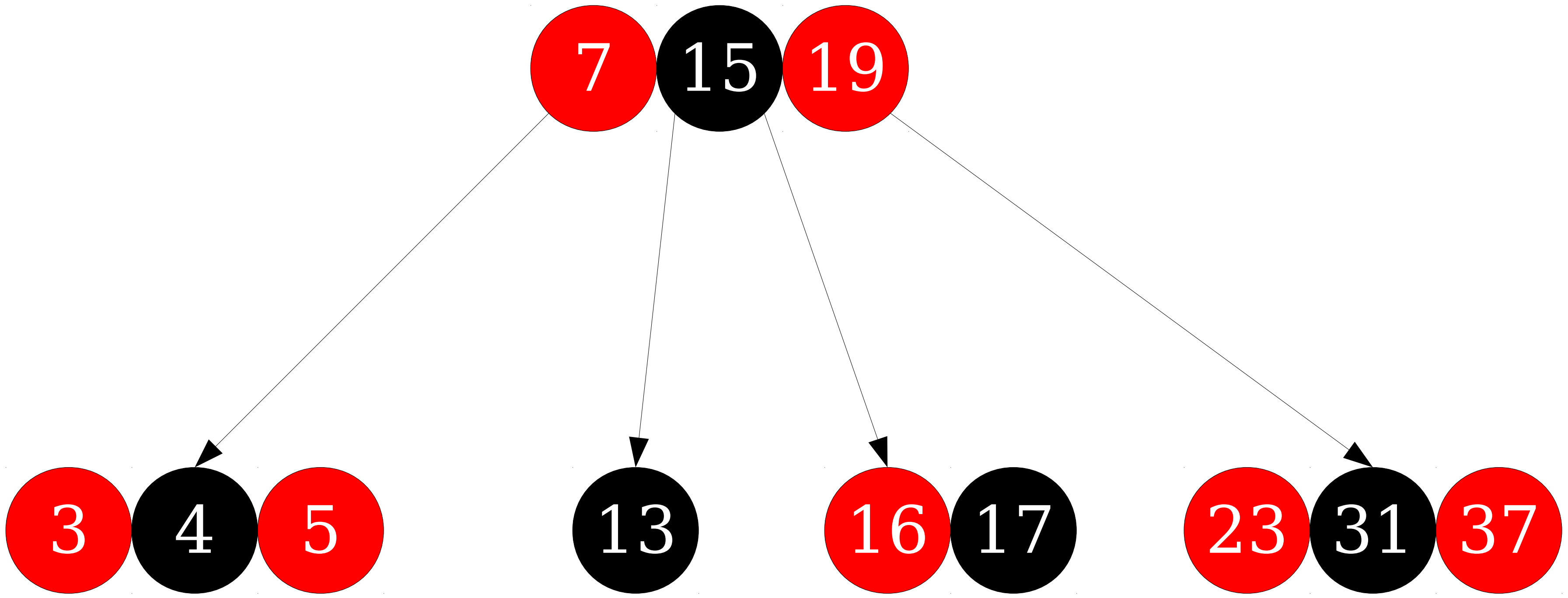
Using the Isometry



Using the Isometry

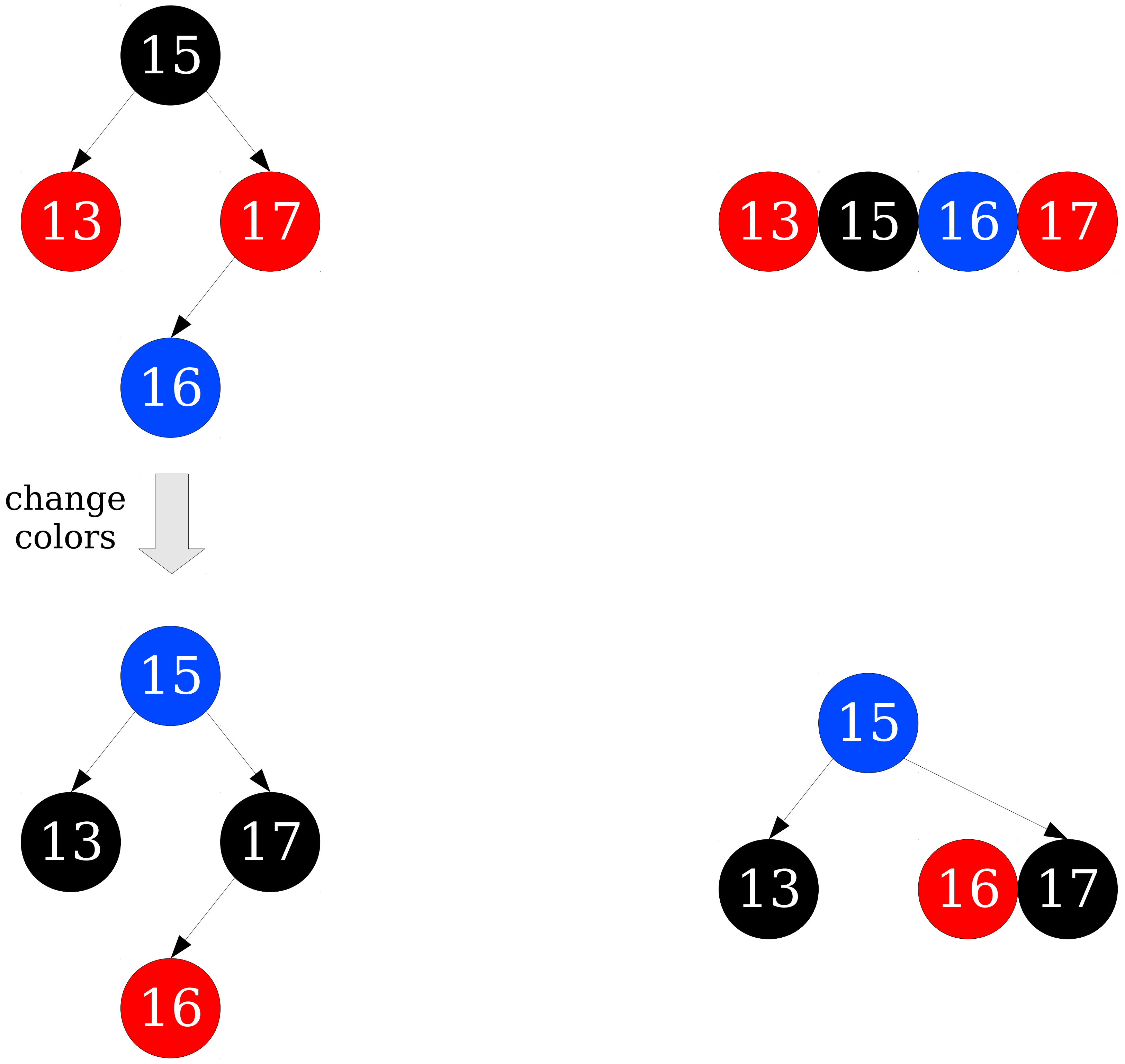


Using the Isometry

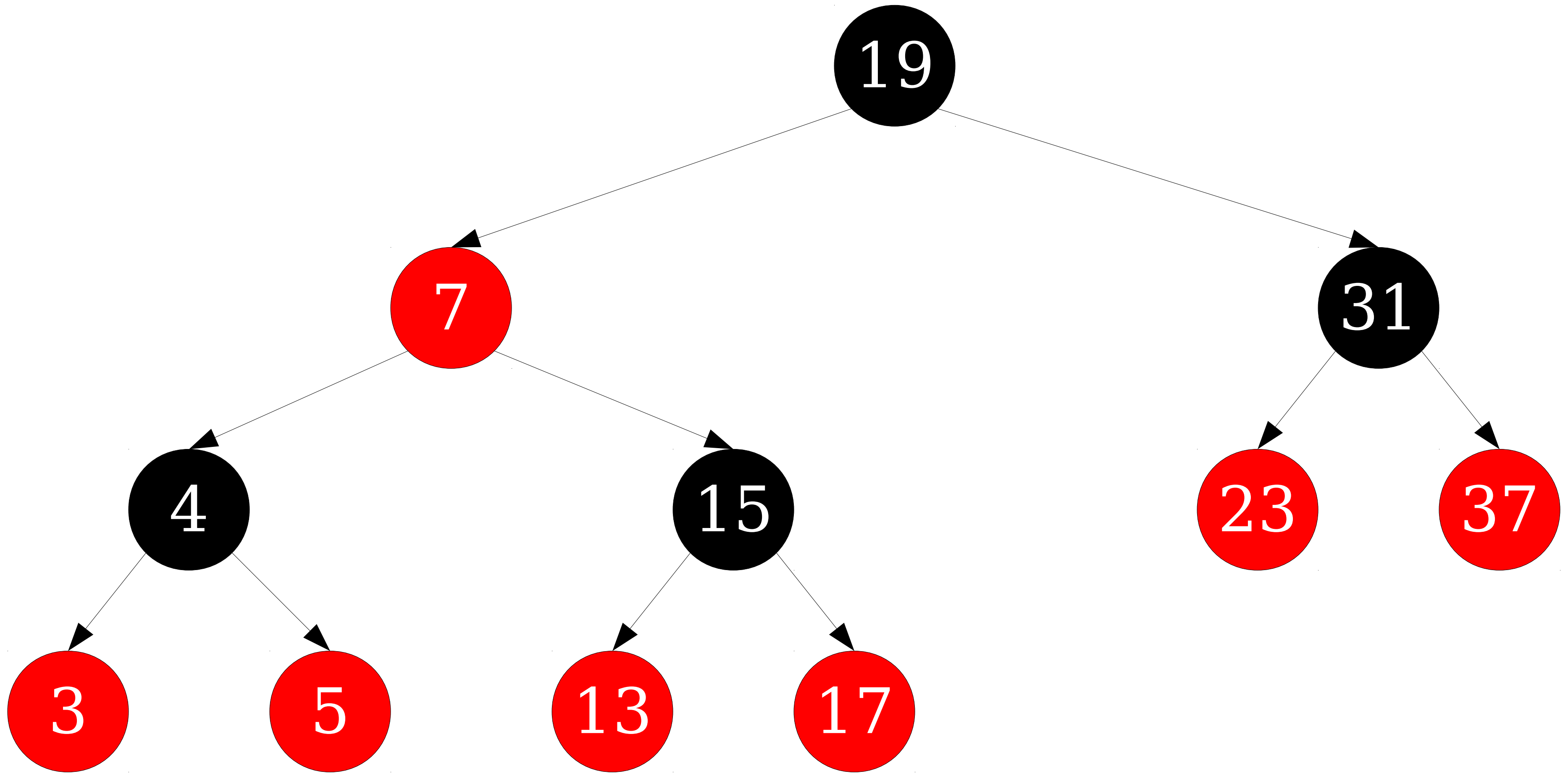


Two steps:

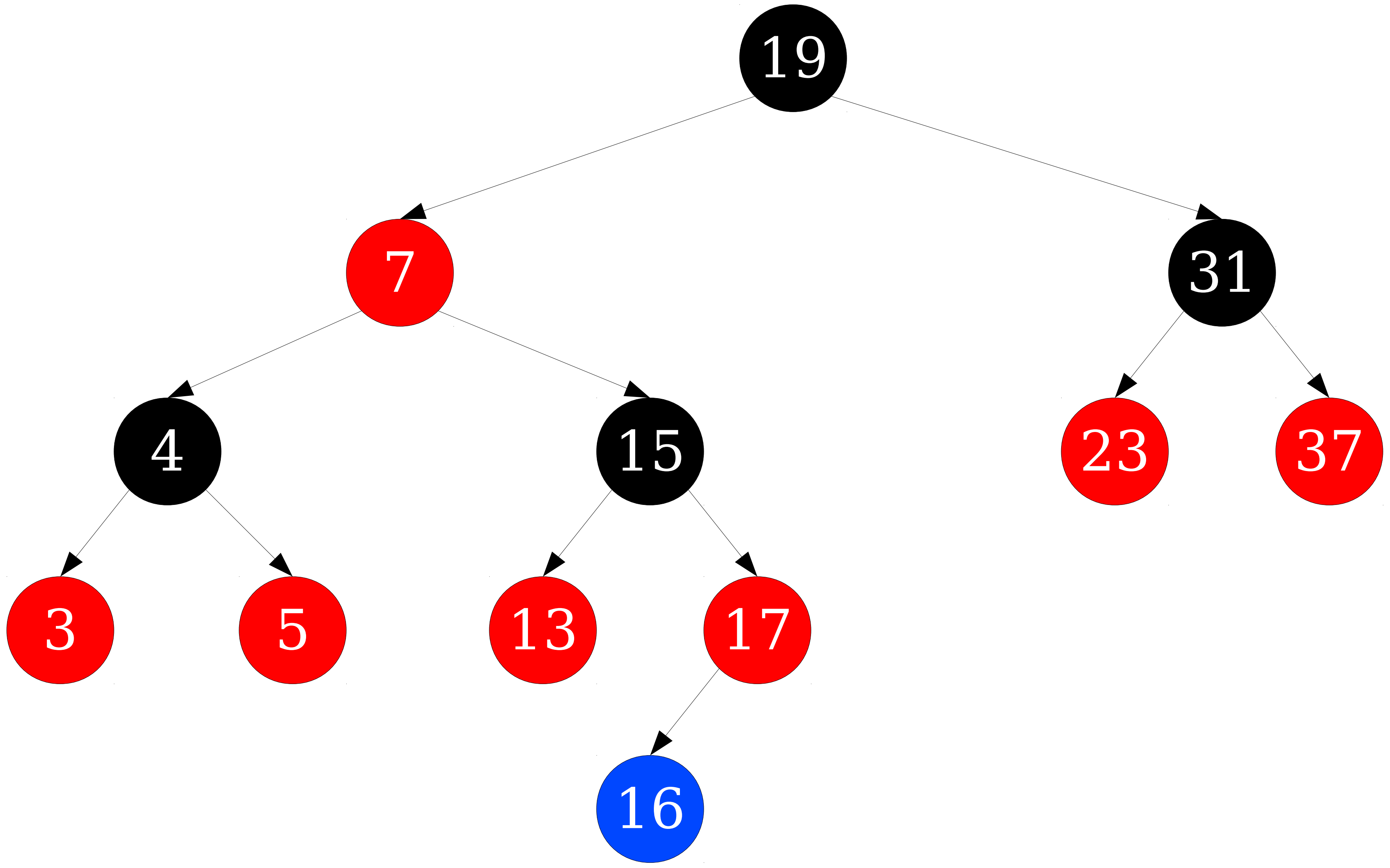
1. Split the “5-node” into a “2-node” and a “3-node.”
2. Insert the new parent of the two nodes into the parent node.


15
13
17
16
change colors
15
13
17
16
13
15
16
17
15
13
16
17

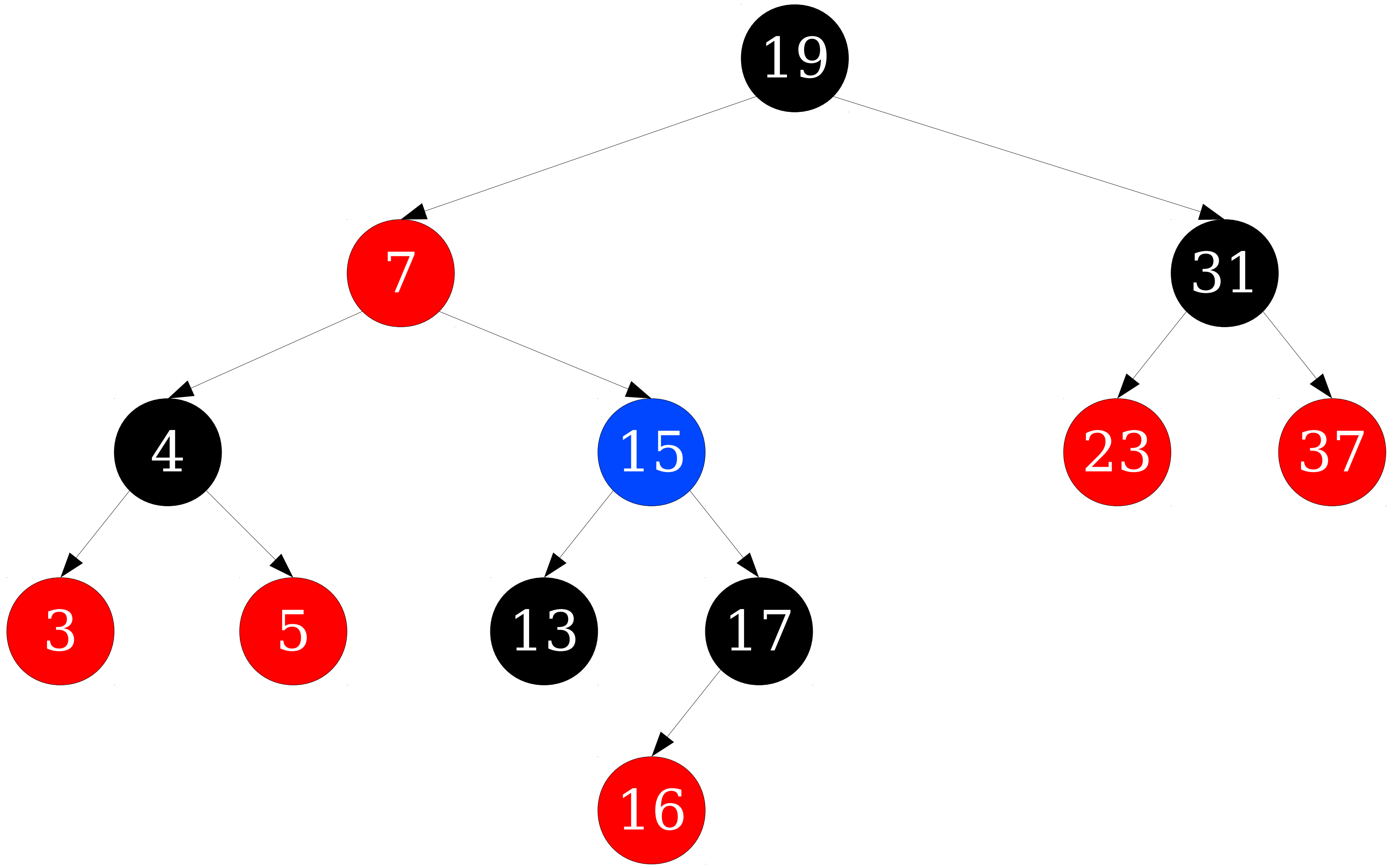
Using the Isometry



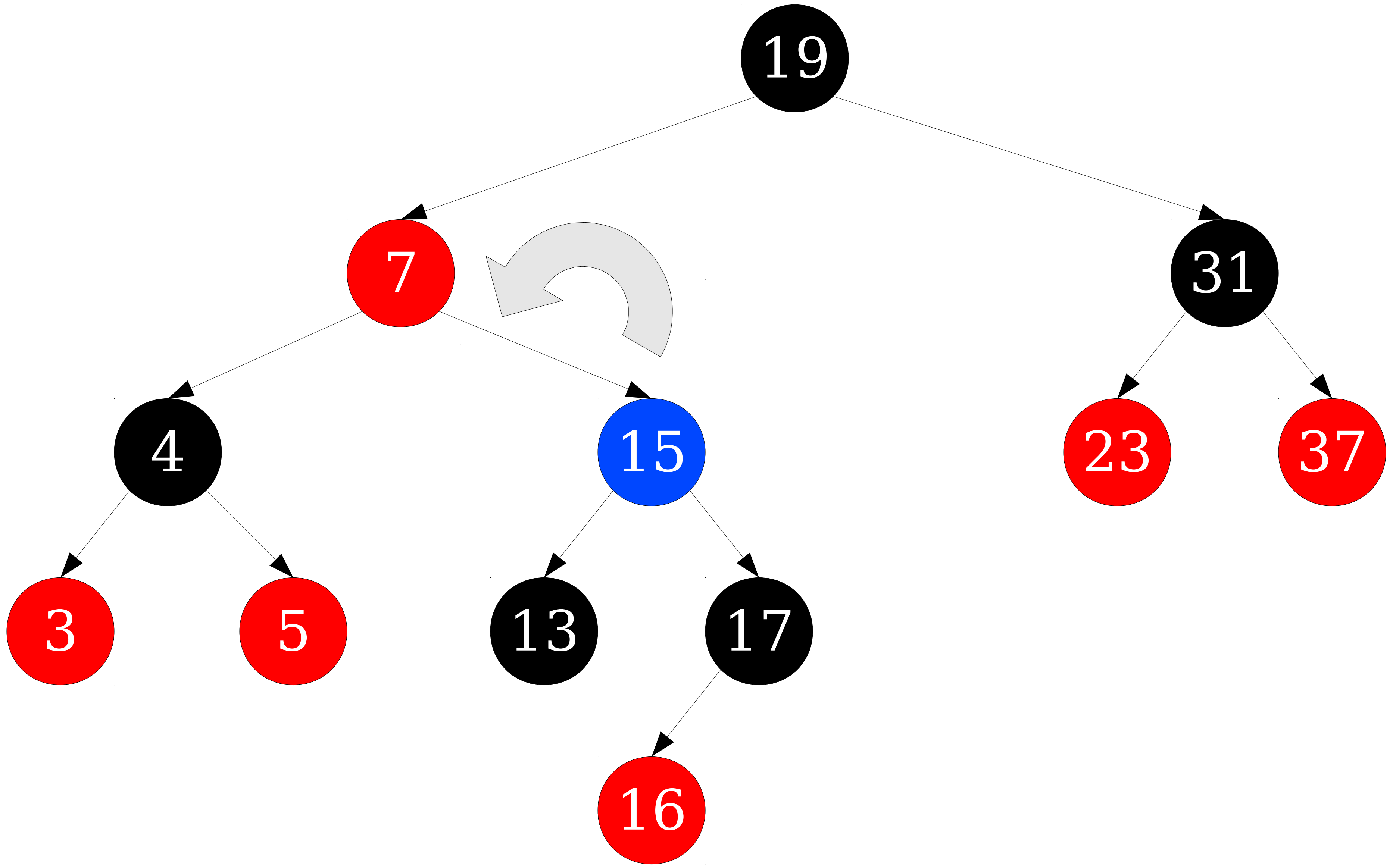
Using the Isometry



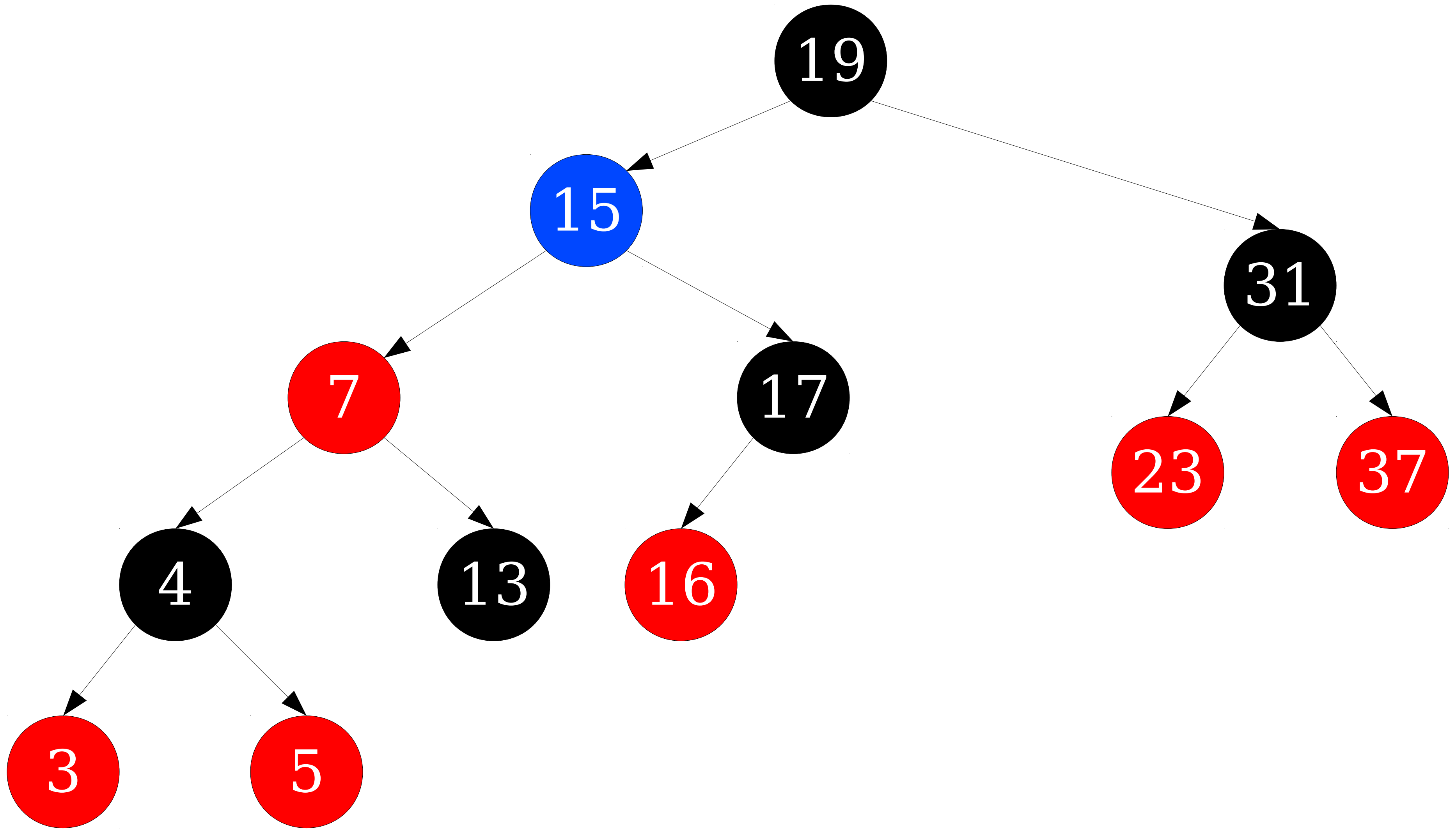
Using the Isometry



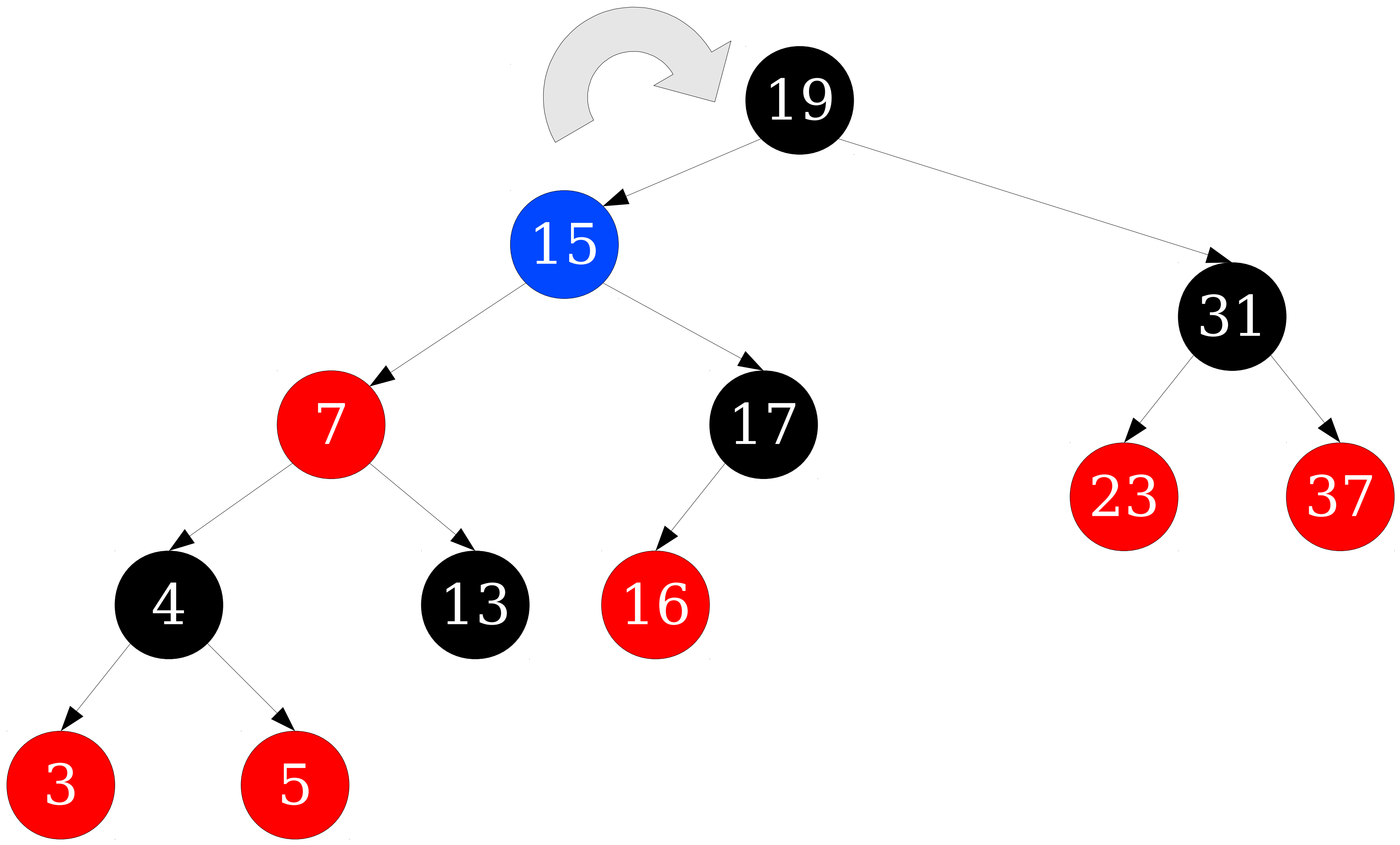
Using the Isometry



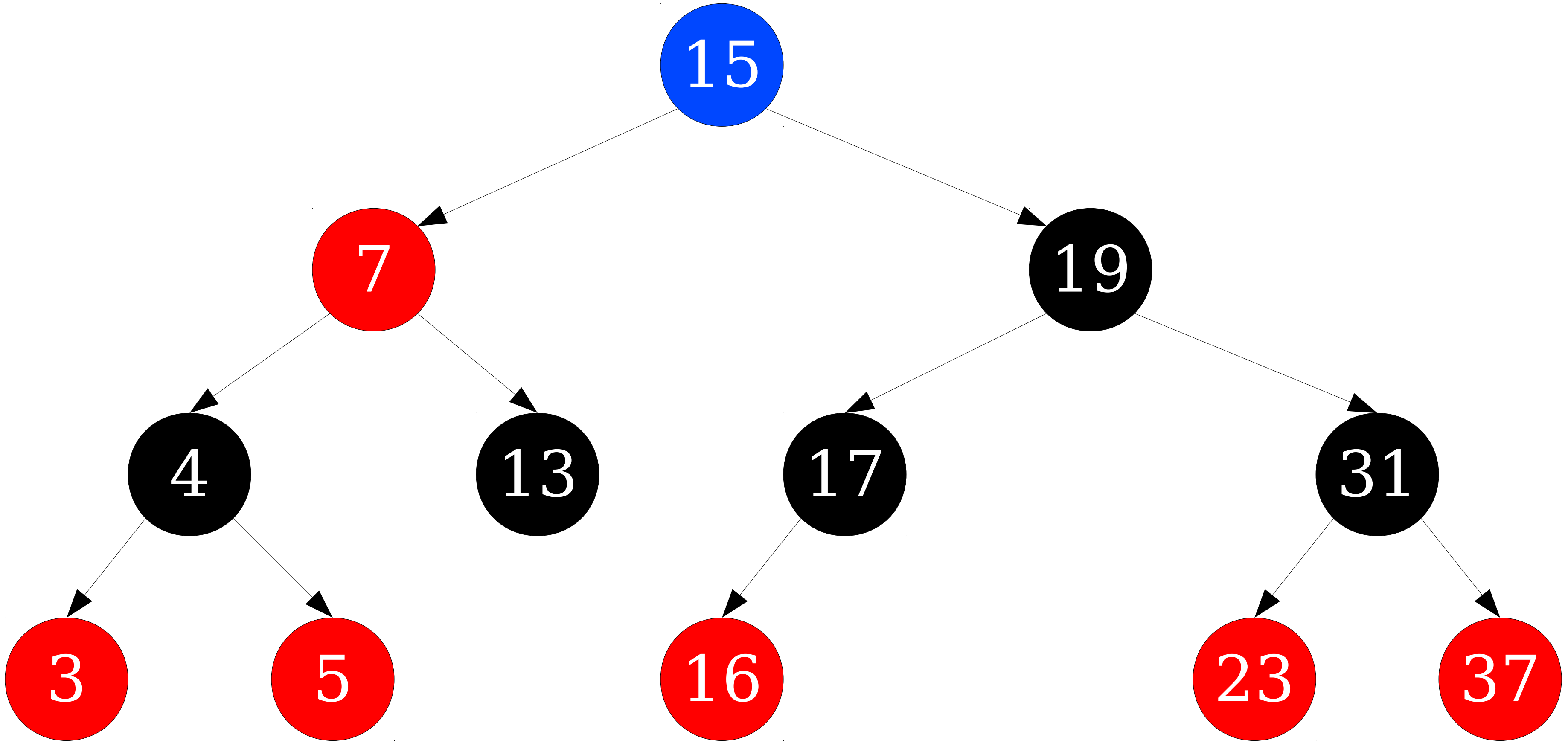
Using the Isometry



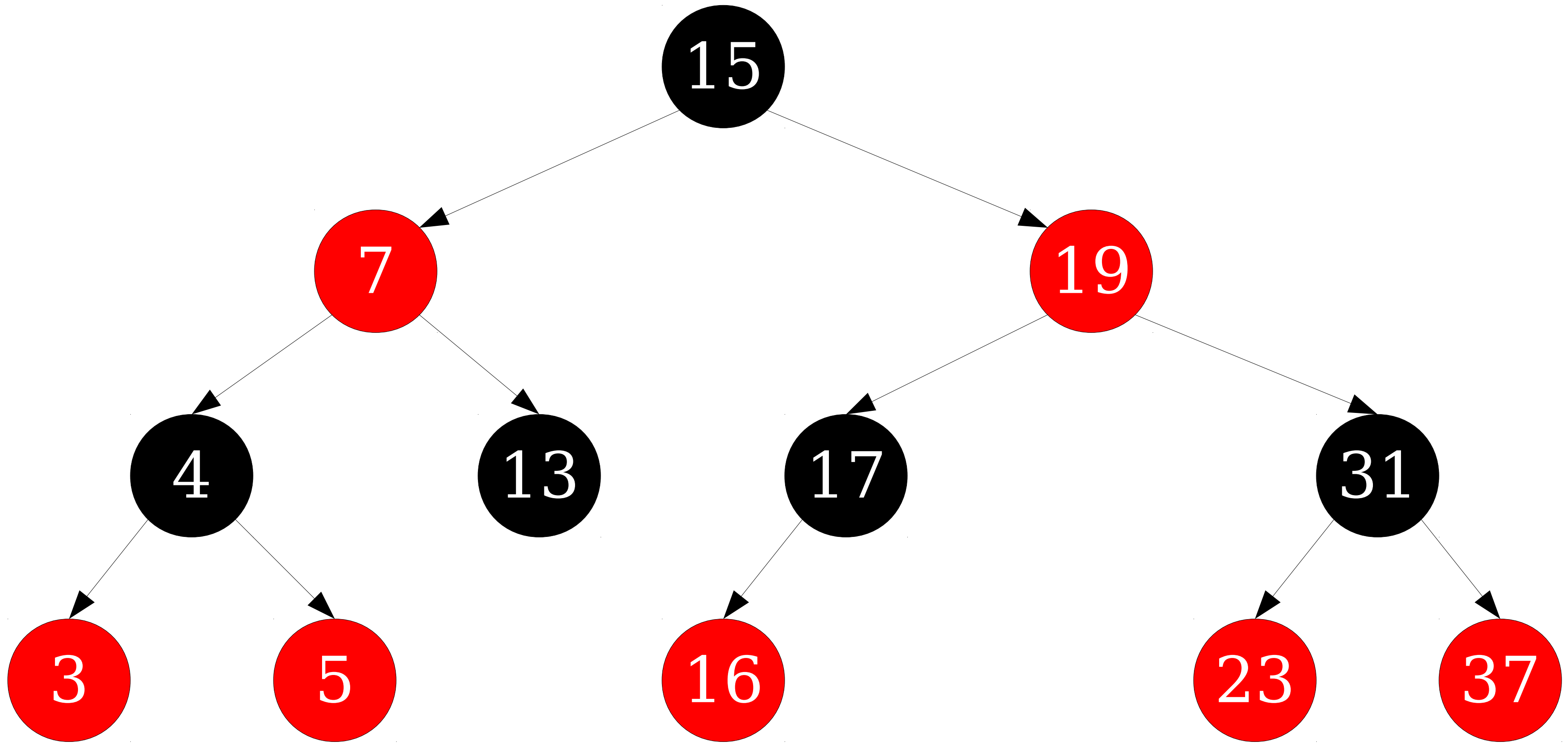
Using the Isometry



Using the Isometry



Using the Isometry



Building Up Rules

- All of the crazy insertion rules on red/black trees make perfect sense if you connect it back to 2-3-4 trees.
- There are lots of cases to consider because there are many different ways you can insert into a red/black tree.
- ***Main point:*** Simulating the insertion of a key into a node takes time O(1) in all cases. Therefore, since 2-3-4 trees support O(log *n*) insertions, red/black trees support O(log *n*) insertions.
- The same is true of deletions.

My Advice

- ***Do*** know how to do B-tree insertions and searches.
  - You can derive these easily if you remember to split nodes.
- ***Do*** remember the rules for red/black trees and B-trees.
  - These are useful for proving bounds and deriving results.
- ***Do*** remember the isometry between red/black trees and 2-3-4 trees.
  - Gives immediate intuition for all the red/black tree operations.
- ***Don't*** memorize the red/black rotations and color flips.
  - This is rarely useful. If you're coding up a red/black tree, just flip open CLRS and translate the pseudocode. ☺

More to Explore

- The ***2-3 tree*** is another simple B-tree that's often used in place of 2-3-4 trees.
  - It gives rise to the ***AA-tree***, another balanced tree data structure.
- The ***left-leaning red/black tree*** is a simplification of red/black trees that forces 3-nodes to be encoded in only one way.
- ***Cache-oblivious data structures*** get the benefit of good cache performance, but without having advance knowledge of the cache size.
- B-tree variants are used in many contexts:
  - The ***B$^+$-Tree*** are used in databases.
  - The ***R-tree*** is used for spatial indexing.
- These might be interesting topics to look into for a final project!

Next Time

- ***Augmented Trees***
  - Building data structures on top of balanced BSTs.
- ***Splitting and Joining Trees***
  - Two powerful operations on balanced trees.