Volumes of Revolution (AGAIN?!)

Recap: $V = \pi \int_a^b y^2\,dx$ or $\pi \int_a^b x^2\,dy$ ← Just more of this sh*t but with more complex functions ☹ (first 2 subchapters anyways)

Parametric eqns:

what's new, right?

$$V = \pi \int_{x=a}^{x=b} y^2\,dx = \pi \int_{t=p}^{t=q} y^2 \frac{dx}{dt}\,dt$$

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \qquad \frac{dx}{dt} = f'(t) \qquad \frac{dy}{dt} = g'(t)$$

$$V = \pi \int_{y=a}^{y=b} x^2\,dy = \pi \int_{t=p}^{t=q} x^2 \frac{dy}{dt}\,dt$$

Sooooooooooooooo
Stuuuuuppppiiiiiddddd

Ex 4A

1\. a) $y = \frac{2}{x+1}$ x-axis from 0 to 2

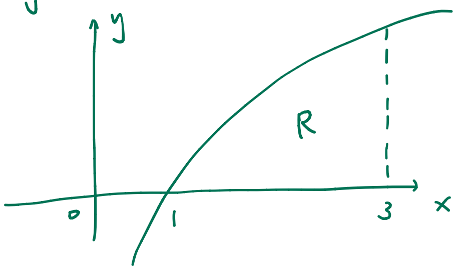
$$y^2 = \frac{4}{(x+1)^2} \qquad V = \pi \int_0^2 \frac{4}{(x+1)^2}\,dx = \pi \left[-\frac{4}{x+1} \right]_0^2 = \pi \left(-\frac{4}{3} + \frac{4}{1} \right) = \frac{8}{3}\pi$$ ✓

e) $y = \frac{\sqrt{\ln x}}{x}$ x-axis from 1 to 2

$$y^2 = \frac{\ln x}{x^2} = \ln x \cdot \frac{1}{x^2} \qquad \int \ln x \cdot \frac{1}{x^2}\,dx = -\frac{1}{x}\ln x - \int -\frac{1}{x^2}\,dx = -\frac{1}{x}\ln x - \frac{1}{x} + c$$

$$V = \pi \int_1^2 \ln x \cdot \frac{1}{x^2}\,dx = \pi \left[-\frac{1}{x}\ln x - \frac{1}{x} \right]_1^2 = \pi \left(-\frac{1}{2}\ln 2 - \frac{1}{2} + \ln 1 + 1 \right) = \left(\frac{1}{2} - \ln\sqrt{2} \right)\pi = \frac{\pi}{2}(1 - \ln 2)$$ ✓

③ $y=\ln x$



$y^2=(\ln x)^2$

$\int (\ln x)^2\,dx$ let $u=\ln x$

$du=\frac{1}{x}dx$

$dx=x\,du=e^u du$

$=\int u^2\cdot e^u\,du$

$=u^2\cdot e^u-\int 2ue^u\,du$

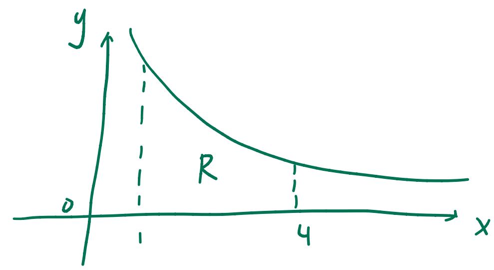
$=u^2\cdot e^u-\left(2ue^u-\int 2e^u du\right)=u^2e^u-2ue^u+2e^u+c$

$=x(\ln x)^2-2x\ln x+2x+C$

$\therefore V=\pi\int_1^3 (\ln x)^2dx=\pi\left[x(\ln x)^2-2x\ln x+2x\right]_1^3=\pi\left(\left(3(\ln 3)^2-6\ln 3+6\right)-\left((\ln 1)^2-2\ln 1+2\right)\right)$

$=\pi\left(3(\ln 3)^2-6\ln 3+4\right)$ ✓

⑤ $y^2=\frac{4x+3}{(x+2)(2x-1)}=\frac{A}{x+2}+\frac{B}{2x-1}=\frac{(2A+B)x+(-A+2B)}{(x+2)(2x-1)}$



$4A+2B=8$ — ①

$-A+2B=3$ — ②

①-②: $5A=5$

$A=1$

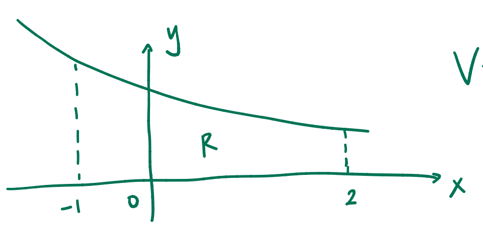
$B=\frac{3+A}{2}=2$

$\therefore y^2=\frac{1}{x+2}+\frac{2}{2x-1}$

$\int y^2dx=\int\left(\frac{1}{x+2}+\frac{2}{2x-1}\right)dx=\ln(x+2)+\ln(2x-1)+C$

$V=\pi\int_1^4 y^2dx=\pi\left[\ln(x+2)+\ln(2x-1)\right]_1^4=\pi\left(\ln 6+\ln 7-\ln 3-\ln 1\right)=\pi\ln\frac{42}{3}=\pi\ln 14$ ✓

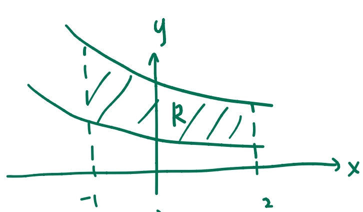
⑦ a) $y=\frac{10}{3(5+2x)}$ $y^2=\frac{100}{9(5+2x)^2}$ $\int y^2dx=\int\frac{100}{9(5+2x)^2}dx=-\frac{50}{9(5+2x)}+c$



$V=\pi\int_{-1}^{2}y^2dx=\pi\left[-\frac{50}{9(5+2x)}\right]_{-1}^{2}=\pi\left(-\frac{50}{81}+\frac{50}{27}\right)$

$=\frac{100}{81}\pi$ ✓

b) $y=\frac{20}{3(5+2x)}$



V= Volume of new curve - volume of old curve

$=\frac{100}{81}\pi\cdot 2^2-\frac{100}{81}\pi=3\cdot\frac{100}{81}\pi=\frac{100}{27}\pi$

Twice the height, same length → double the area
↓
4x the volume. ✓

Ex 4B

① a) $x = e^{2y} - e^{-y}$ y-axis   from 0 to 1

$$x^2 = e^{4y} - 2e^{y} + e^{-2y} \qquad \int x^2\,dy = \int \left(e^{4y} - 2e^{y} + e^{-2y}\right) dy = \frac{1}{4}e^{4y} - 2e^{y} - \frac{1}{2}e^{-2y} + c$$

$$V = \pi \int_0^1 x^2\,dy = \pi\left[\frac{1}{4}e^{4y} - 2e^{y} - \frac{1}{2}e^{-2y}\right]_0^1 = \pi\left(\frac{1}{4}e^{4} - 2e - \frac{1}{2}e^{-2} - \frac{1}{4} + 2 + \frac{1}{2}\right)$$

$$= \pi\left(\frac{1}{4}e^{4} - 2e - \frac{1}{2}e^{-2} + \frac{9}{4}\right)$$

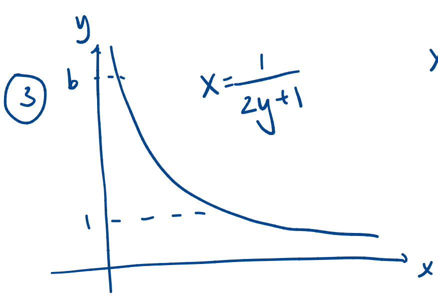
d) $x = \dfrac{1}{\sqrt{y \ln y}}$ y-axis   from e^4 to e^9

$$\int x^2\,dy = \int \frac{1}{y \ln y}\,dy \qquad \text{let } u = \ln y, \quad du = \frac{1}{y}\,dy$$

$$x^2 = \frac{1}{y \ln y}$$

$$= \int \frac{1}{u}\,du = \ln u + c = \ln(\ln y) + c$$

$$V = \pi \int_{e^4}^{e^9} \frac{1}{y \ln y}\,dy = \pi\Big[\ln(\ln y)\Big]_{e^4}^{e^9} = \pi(\ln 9 - \ln 4) = \pi \ln \frac{9}{4}$$

③ 

$$x^2 = \frac{1}{(2y+1)^2} \qquad \int x^2\,dy = \int \frac{1}{(2y+1)^2}\,dy = -\frac{1}{2(2y+1)} + c = -\frac{1}{4y+2} + c$$

$$V = \pi \int_1^b x^2\,dy = \pi\left[-\frac{1}{4y+2}\right]_1^b = \pi\left(-\frac{1}{4b+2} + \frac{1}{6}\right) = \frac{\pi}{10} \quad \text{(given)}$$

$$-\frac{1}{4b+2} + \frac{1}{6} = \frac{1}{10}$$

$$\frac{1}{4b+2} = \frac{1}{15}$$

$$4b + 2 = 15 \qquad \left(b \neq -\frac{1}{2}\right)$$

$$b = \frac{13}{4}$$