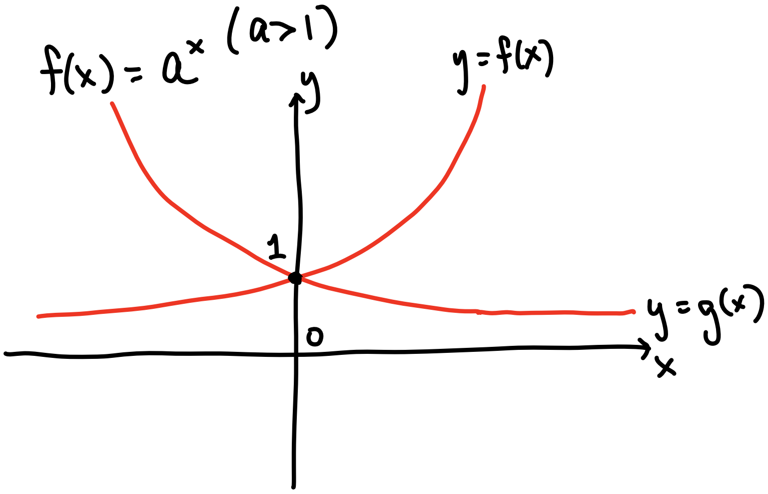
Exponentials

$f(x) = a^x \; (a>1)$

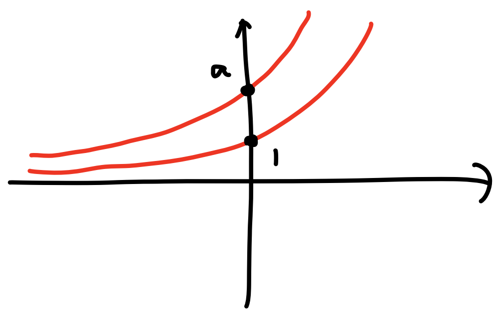


$g(x) = \left(\frac{1}{a}\right)^x = a^{-x}$

reflect along y-axis

translation 1 to the left.

$f(x) = a^x \longmapsto f(x) = a \cdot a^x = a^{x+1}$



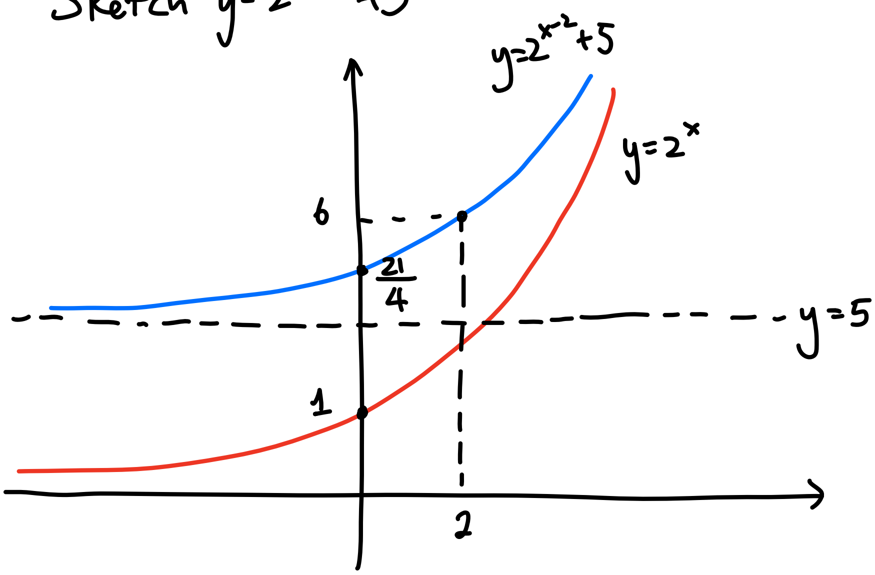
$f(x) = a^x$ for $a>0$,

$\lim_{x \to -\infty} f(x) = 0$

$\lim_{x \to \infty} f(x) = \infty$

Challenge (p.314)

Sketch $y = 2^{x-2} + 5$



Differentiating a^x from 1^{st} principles

$f(x) = a^x$

$$f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} = \lim_{\delta \to 0} \frac{a^{x+\delta} - a^x}{\delta} = a^x \lim_{\delta \to 0} \frac{a^\delta - 1}{\delta} = a^x f'(0)$$

$$f'(0) = \lim_{\delta \to 0} \frac{f(\delta) - f(0)}{\delta} = \lim_{\delta \to 0} \frac{a^\delta - 1}{\delta}$$

For $f(x) = a^x$, $f'(x) = a^x f'(0)$, $\therefore f'(x) \propto a^x$

So when is $f'(0) = 1$ (ie $f'(x) = a^x$)

$\underline{a=2}$ $f'(0) = \dfrac{2^{0.0000001} - 1}{0.0000001} = 0.6931472\ldots$

$\underline{a=3}$ $f'(0) = \dfrac{3^{0.0000001} - 1}{0.0000001} = 1.0986123\ldots$

$\underline{a=2.5}$ $f'(0) = 0.9162907\ldots\ldots$

$a = 2.75$ $f'(0) = 1.0116009\ldots\ldots$

There exists a value of a where $f'(0) = 1$
$(a = e = 2.71828\ldots)$

$$\frac{d}{dx}(e^{kx}) = e^{kx}\frac{d}{dx}(kx) = ke^{kx}$$

this means that if $f(x) = e^{kx}$, $f'(x) \propto f(x)$ where k is the proportionality constant $(f'(x) = kf(x))$

we can use e to model growth of different things (bacteria, rabbits etc.)

Ex14C (p.318) GUIDE!

(4) # of rabbits R after m months: $R = 12e^{0.2m}$ ← equation: usually against time.

a) i) $R|_{m=1} = 12e^{0.2}$ ii) $R|_{m=12} = 12e^{0.2\times12} = 12e^{2.4}$

b) when $m=0$, $R=12$ ∴ 12 = initial population of rabbits

interpret in context

c) $\frac{dR}{dm} = 0.2\times12e^{0.2m} = 2.4e^{0.2m}$

$\frac{dR}{dm}\Big|_{m=6} = 2.4e^{1.2} = 7.968\ldots \approx 8$

rate of change of something = derivative!
(ie rabbits/month)

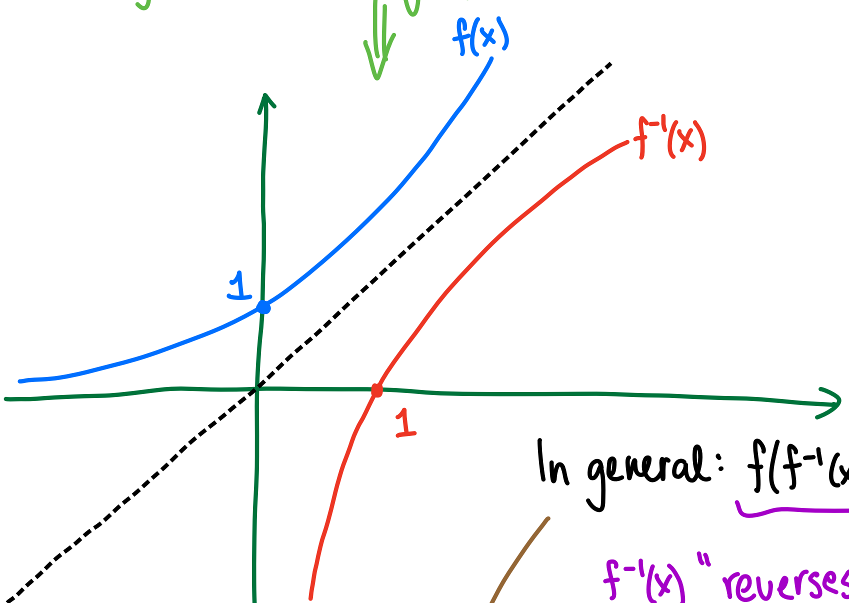
d) Under the model, as $m\to\infty$, $R\to\infty$.

However, growth ↓ when pop ↑ due to limited resources.

criticize the model
(usually the growth isn't so perfect)

Logarithms

Finding $f^{-1}(x)$ from graph of $f(x)$: reflect along $y=x$



$f(x) = a^x$

$f^{-1}(x) = \log_a x$

In general: $\underbrace{f(f^{-1}(x)) = f^{-1}(f(x)) = x}$

$f^{-1}(x)$ "reverses" the operation of $f(x)$

$\log_a a^x = x$

$a^{\log_a x} = x$

Ex14D (p.320)

① a) $4^4 = 256$
$\log_4 256 = 4$

b) $3^{-2} = \frac{1}{9}$
$\log_3 \frac{1}{9} = -2$

} rewrite using a logarithm

② a) $\log_2 16 = 4$
$2^4 = 16$

b) $\log_5 25 = 2$
$5^2 = 25$

} rewrite using a power

③ a) $2^3 = 8$
$\therefore \log_2 8 = 3$

c) $10^7 = 10{,}000{,}000$
$\therefore \log_{10} 10{,}000{,}000 = 7$

} use power to solve without calculator

⑦ a) i) 1 ii) 1 iii) 1

b) let $x = \log_a a \Rightarrow a^x = a^{\log_a a} = a$
$a^x = a^1$
$\therefore x = 1$

} $\log_a a = 1$ for all +ve values of a $(a \neq 1)$

Challenge (p.324)

$\log_a x = m$ $\log_a y = n$, prove $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$

$x = a^m$ $y = a^n$ $\Rightarrow \frac{x}{y} = \frac{a^m}{a^n} = a^{m-n} \Rightarrow \log_a\left(\frac{x}{y}\right) = \log_a a^{m-n}$
$= m - n = \log_a x - \log_a y$

SOLVING EQNS

EX14F (p.324)

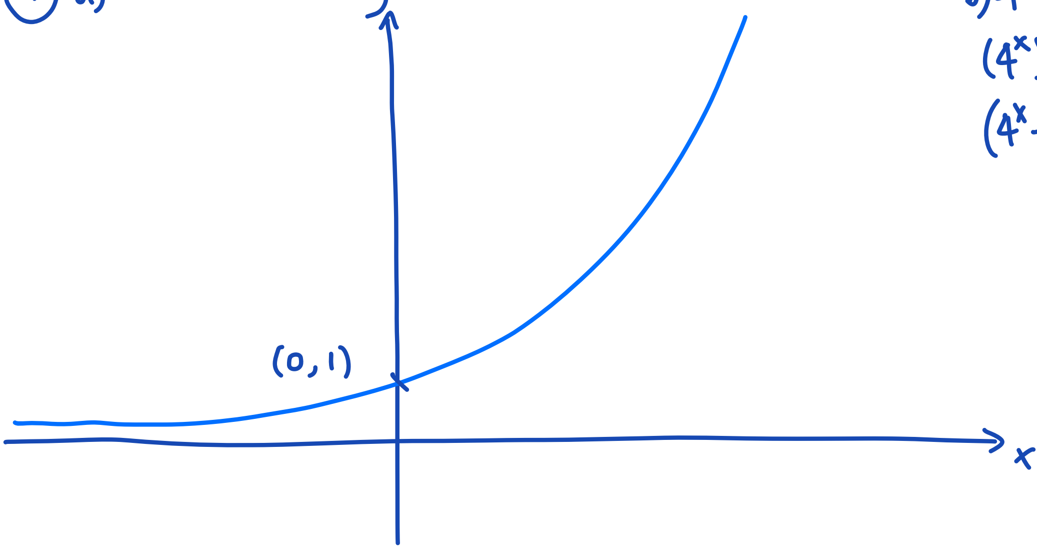
② c) $5^{2x} - 6(5^x) - 7 = 0$
let $y = 5^x$
$y^2 - 6y - 7 = 0$
$(y-7)(y+1) = 0$
$5^x = 7, -1$ ← reject
$x = \log_5 7$

e) $7^{2x} + 12 = 7^{x+1}$
let $y = 7^x$
$y^2 + 12 = 7y$
$y^2 - 7y + 12 = 0$
$(y-3)(y-4) = 0$
$7^x = 3, 4$
$x = \log_7 3, \log_7 4$

} treat power as another variable

continued on next page

4 a)



b) $4^{2x}-10(4^x)+16=0$

$(4^x)^2-10(4^x)+16=0$

$(4^x-8)(4^x-2)=0$

$4^x=2,8$

$x=\log_4 2, \log_4 8$

$=\frac{1}{2}, \frac{3}{2}$

5 a) $5^x=2^{x+1}$

ln both sides

$\ln 5^x=\ln 2^{x+1}$

$x\ln 5=(x+1)\ln 2$

$x(\ln 5-\ln 2)=\ln 2$

$x=\dfrac{\ln 2}{\ln 5-\ln 2}=\dfrac{\ln 2}{\ln(5/2)}$

Reminder: $\ln = \log_e$

$\log$ or $\lg = \log_{10}$

Reminder:

$\ln x+\ln y=\ln(xy)$

$\ln x-\ln y=\ln(x/y)$

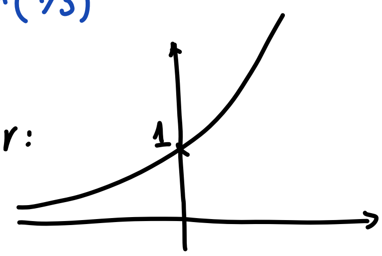
c) $7^{x+1}=3^{x+2}$

$(x+1)\ln 7=(x+2)\ln 3$

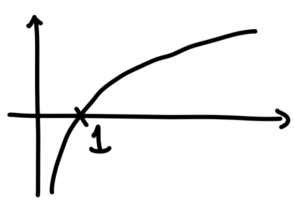
$x(\ln 7-\ln 3)=2\ln 3-\ln 7$

$x=\dfrac{\ln 9-\ln 7}{\ln 7-\ln 3}=\dfrac{\ln(9/7)}{\ln(7/3)}$

All $y=a^x$ graphs look very similar:



All $y=\log_a x$ graphs look very similar:



Ex14G (p.367)

2 a) $\ln x=2$

$x=e^2$

c) $\ln(2x+3)=4$

$2x+3=e^4$

$x=\dfrac{e^4-3}{2}$

e) $\ln(18-x)=\frac{1}{2}$

$18-x=\sqrt{e}$

$x=18-\sqrt{e}$

These are done the same way as "normal" logarithms.

Challenge (p.328)

$g(x) = Ae^{Bx} + C$ passes through $(0,5)$ & $(6,10)$

$y=2$ is an asymptote.

$C=2 \Rightarrow g(x) = Ae^{Bx}+2 \Rightarrow \begin{cases} 5 = A+2 & \text{— ①} \\ 10 = Ae^{6B}+2 & \text{— ②} \end{cases}$

①: $A=3$

②: $10 = 3e^{6B}+2$

$e^{6B} = \frac{8}{3}$

$B = \ln\left(\frac{8}{3}\right)^{\frac{1}{6}}$ or $\frac{1}{6}\ln\left(\frac{8}{3}\right)$

$\therefore g(x) = 3e^{x\ln\left(\frac{8}{3}\right)^{\frac{1}{6}}} + 2$

Non-Linear Data

these logs can be any base — as long as it's consistent!

Case 1: $y = ax^n \xrightarrow[\text{both sides}]{\log} \log y = \log ax^n = \log x^n + \log a$

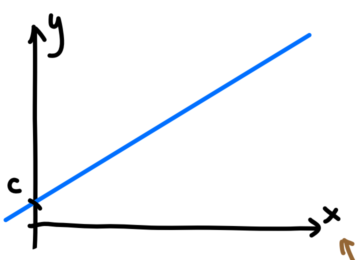
$= n\log x + \log a$

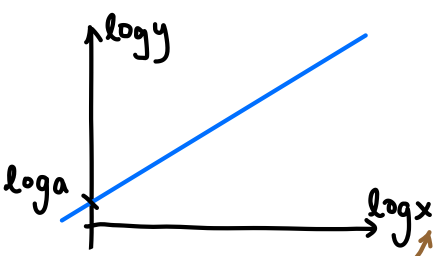
Case 2: $y = ab^x \xrightarrow[\text{both sides}]{\log} \log y = \log ab^x = \log b^x + \log a$

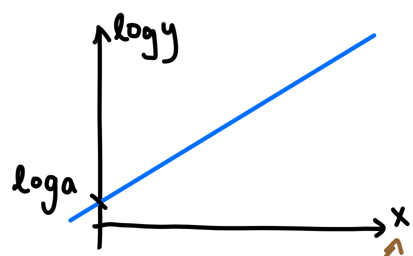
$= x\log b + \log a$

Let's compare!

$y = mx + c$ $\qquad$ $\log y = n\log x + \log a$ $\qquad$ $\log y = x\log b + \log a$







pay attention to the axes!

Ex 14H (p. 331)

① a) $S = 4 \cdot 7^x$

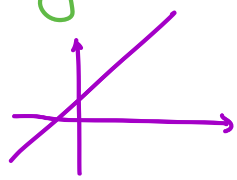
$\log S = \log(4 \cdot 7^x)$

$\log S = \log 4 + \log 7^x$

$\log S = \log 4 + x\log 7 \leftarrow y = c + mx$

b) Gradient: $\log 7$

y Intercept: $\log 4$



⑥ f = how frequently a word is used

R = a word's ranking in a list of the most common words

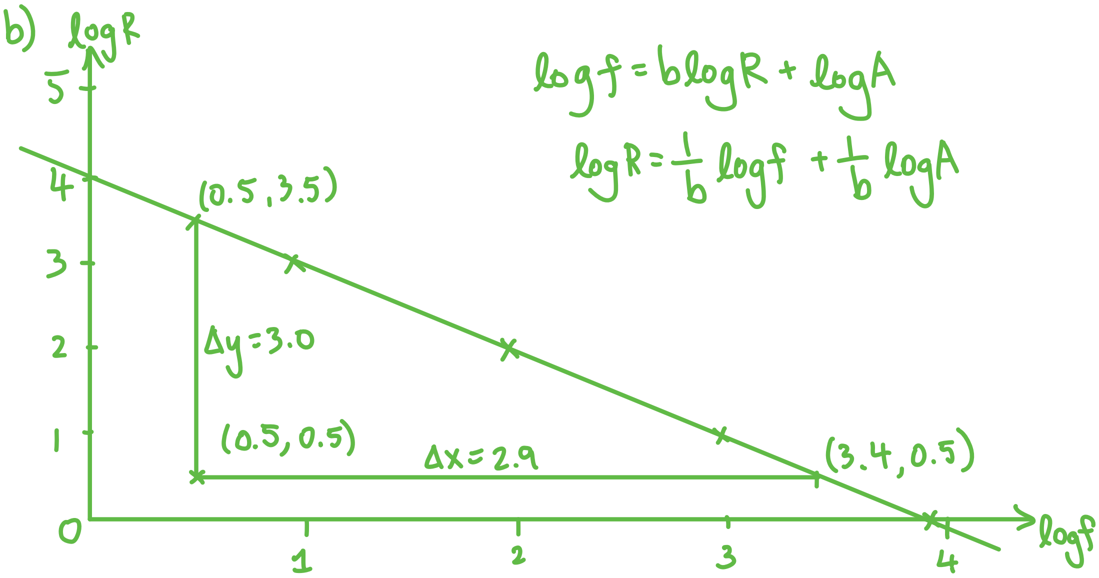
Zipf's law: $f = AR^b$

Word	'the'	'it'	'well'	'detail'
Rank, R	1	10	100	1000
Frequency per 100,000 words, f	4897	861	92	9

a)

$\log R$	0	1	2	3
$\log f$	3.69	2.94	1.96	0.95

b)



$\log f = b\log R + \log A$

$\log R = \frac{1}{b}\log f + \frac{1}{b}\log A$

c) $m = \frac{-3.0}{2.9} = \frac{1}{b}$

$b = \frac{2.9}{-3.0} \approx -1$

$c = 4 = \frac{1}{b}\log A$

$4 = \frac{-3}{2.9}\log A$

$A = 10^{-4\frac{3}{2.9}} \approx 7.3 \times 10^{-5}$

d) $f = 7.3 \times 10^{-5} \times \frac{1}{R}$

'when': R = 57 trilogy has 455125 words

$7.3 \times 10^{-5} \times \frac{1}{57} \times \frac{455125}{100000} \approx 363$