LINEAR TRANSFORMATIONS

$\begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{pmatrix}x+2\\y-3\end{pmatrix}$ ← This is not linear

original　　image

only a linear transformation if its linear

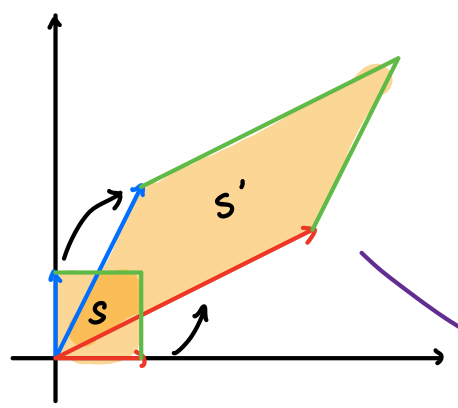
$\begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{pmatrix}2x-y\\x+y\end{pmatrix}$ ← this is linear (in the form $ax+by$)

Linear transformations map the ORIGIN into ITSELF　$\begin{pmatrix}0\\0\end{pmatrix} \mapsto \begin{pmatrix}0\\0\end{pmatrix}$

AS A MATRIX: $\begin{pmatrix}x\\y\end{pmatrix} \mapsto \underbrace{\begin{pmatrix}ax+by\\cx+dy\end{pmatrix}}_{\begin{pmatrix}a & b\\c & d\end{pmatrix}}$

the $\hat{i}$ and $\hat{j}$ vectors go from:

$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}a\\c\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}b\\d\end{pmatrix}$ respectively (watch 3Blue1Brown's videos)



$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}3\\1.5\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\2\end{pmatrix}$　$\Rightarrow$　This linear transformation is $\begin{pmatrix}3 & 1\\1.5 & 2\end{pmatrix}$

The unit square S becomes the image S'

Determinants: area scale factor
(-ve determinant means the x and y axis is flipped)

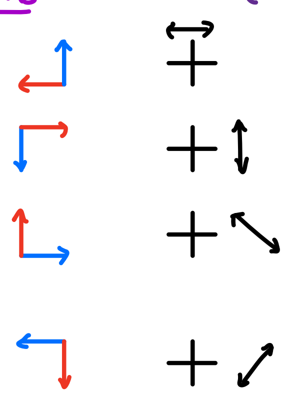
Some Common Linear Transformations:

$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$ reflect in y-axis

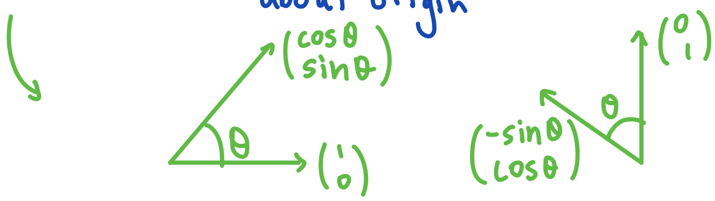
$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ reflect in x-axis

$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ reflect in $y=x$

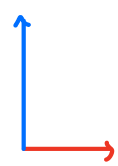
$\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$ reflect in $y=-x$



$\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}$ rotate θ anticlockwise about origin



$\begin{pmatrix}n & 0\\0 & m\end{pmatrix}$ stretch by n along x, stretch by m along y



Applying Linear Transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{original}} = \underbrace{\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}}_{\text{image}}$$

Successive transformations (transforming after transforming):

Matrix PQ means transforming with Q, then transforming with P

$$PQ\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{P \cdot \Big(\underbrace{Q\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{transform with } Q}\Big)}_{\text{then transform with } P} \longrightarrow \text{resulting transformation is } (PQ) \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Invariant Things

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is always invariant (maps onto itself) ie. an invariant point

$y=x$ is an invariant line in the transformation $\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{reflect in } y=x}$

↓ the line maps onto itself!