Roots of Polynomials

Quadratics

$ax^2 + bx + c = a(x-\alpha)(x-\beta)$ where $a, b, c \in \mathbb{R}$ & constant

$x \in \mathbb{C}$ (x can be imaginary)

$$\downarrow$$

$$a(x^2 - (\alpha+\beta)x + \alpha\beta)$$

$$= ax^2 - a(\alpha+\beta)x + a\alpha\beta$$

$b = -a(\alpha+\beta)$ $\quad c = a\alpha\beta$ $\quad \rightarrow$

$\alpha + \beta = -\frac{b}{a}$ sum of roots

$\alpha\beta = \frac{c}{a}$ product of roots

Cubics

$ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$ where $a, b, c, d \in \mathbb{R}$ & constant

$x \in \mathbb{C}$

$$\downarrow$$

$$a(x^2 - (\alpha+\beta)x + \alpha\beta)(x-\gamma)$$

$$= a(x^3 - (\alpha+\beta)x^2 + \alpha\beta x - \gamma x^2 + (\alpha+\beta)\gamma x - \alpha\beta\gamma)$$

$$= ax^3 - a(\alpha+\beta+\gamma)x^2 + a(\alpha\beta + \alpha\gamma + \beta\gamma)x - a\alpha\beta\gamma$$

$b = -a(\alpha+\beta+\gamma)$

$c = a(\alpha\beta + \alpha\gamma + \beta\gamma)$ $\quad \rightarrow$

$d = -a\alpha\beta\gamma$

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$\alpha\beta\gamma = -\frac{d}{a}$

Quartics

$ax^4 + bx^3 + cx^2 + dx + e = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ where $a, b, c, d, e \in \mathbb{R}$ & constant

$x \in \mathbb{C}$

$$\downarrow$$

$$a(x^3 - (\alpha+\beta)x^2 + \alpha\beta x - \gamma x^2 + (\alpha+\beta)\gamma x - \alpha\beta\gamma)(x-\delta)$$

$$= a(x^4 - (\alpha+\beta)x^3 + \alpha\beta x^2 - \gamma x^3 + (\alpha\gamma+\beta\gamma)x^2 - \alpha\beta\gamma x - \delta x^3 + (\alpha\delta+\beta\delta)x^2 - \alpha\beta\delta x + \delta\gamma x^2 - (\alpha\gamma\delta + \beta\gamma\delta)x + \alpha\beta\gamma\delta)$$

$$= ax^4 - a(\alpha+\beta+\gamma+\delta)x^3 + a(\alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \gamma\delta)x^2 - a(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)x + a\alpha\beta\gamma\delta$$

$b = -a(\alpha+\beta+\gamma+\delta)$

$c = a(\alpha\beta+\alpha\gamma+\beta\gamma+\alpha\delta+\beta\delta+\gamma\delta)$

$d = -a(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)$

$e = a\alpha\beta\gamma\delta$

$\rightarrow$

$\alpha+\beta+\gamma+\delta = -\frac{b}{a}$

$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a}$

$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -\frac{d}{a}$

$\alpha\beta\gamma\delta = \frac{e}{a}$

Exercise 4C

1. a) $\alpha+\beta+\gamma+\delta = -\frac{b}{a} = -\frac{3}{4}$ b) $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a} = \frac{1}{2}$

c) $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -\frac{d}{a} = -\frac{(-5)}{4} = \frac{5}{4}$ e) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta} = \frac{\alpha+\beta+\gamma+\delta}{\alpha\beta\gamma\delta} = -\frac{b}{a} \div \frac{e}{a} = \left(-\frac{3}{4}\right)\left(-\frac{4}{4}\right)$

$= \frac{3}{4}$

3. a) $\alpha+\beta+\gamma+\delta = -\frac{b}{a} = -3$ b) $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a} = 2$

c) $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -\frac{d}{a} = 1$ d) $\alpha\beta\gamma\delta = \frac{e}{a} = 4$ e) $\alpha^2\beta^2\gamma^2\delta^2 = (\alpha\beta\gamma\delta)^2 = 4^2 = 16$

5. $ax^4+bx^3+cx^2+dx+e$ $\alpha = -\frac{3}{2}$ $\beta = -\frac{1}{2}$ $\gamma = -2$ $\delta = \frac{2}{3}$

$\frac{b}{a} = -\left(-\frac{3}{2}-\frac{1}{2}-2+\frac{2}{3}\right) = \frac{10}{3}$

$\frac{c}{a} = \frac{3}{4}+3-1+1-\frac{1}{3}-\frac{4}{3} = \frac{25}{12}$

$\frac{d}{a} = -\left(-\frac{3}{2}+\frac{1}{2}+2+\frac{2}{3}\right) = -\frac{5}{3}$

$\frac{e}{a} = -1$

since $a, b, c, d, e \in \mathbb{Z}$,

$a = 12$, $b = 40$, $c = 25$, $d = -20$, $e = -12$

$12x^4+40x^3+25x^2-20x-12$

Mixed Ex 4

1. $-\frac{b}{a} = \alpha+\beta+\gamma+\delta = \frac{1}{5}-\frac{2}{5}-\frac{3}{5}-\frac{1}{2} = -\frac{13}{10}$ $\frac{b}{a} = \frac{13}{10}$

$\frac{c}{a} = \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \left(\frac{1}{5}\right)\left(-\frac{2}{5}\right)+\left(\frac{1}{5}\right)\left(-\frac{3}{5}\right)+\left(\frac{1}{5}\right)\left(-\frac{1}{2}\right)+\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)+\left(\frac{2}{5}\right)\left(\frac{1}{2}\right)+\left(\frac{3}{5}\right)\left(\frac{1}{2}\right)$

$= -\frac{2}{25}-\frac{3}{25}-\frac{1}{10}+\frac{6}{25}+\frac{1}{5}+\frac{3}{10} = \frac{11}{25}$

$-\frac{d}{a} = \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = \left(\frac{1}{5}\right)\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)+\left(\frac{1}{5}\right)\left(\frac{2}{5}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{5}\right)\left(\frac{3}{5}\right)\left(\frac{1}{2}\right)-\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)\left(\frac{1}{2}\right)$

$= \frac{6}{125}+\frac{1}{25}+\frac{3}{50}-\frac{3}{25} = \frac{7}{250}$ $\frac{d}{a} = -\frac{7}{250}$

$\frac{e}{a} = \alpha\beta\gamma\delta = \left(\frac{1}{5}\right)\left(-\frac{2}{5}\right)\left(-\frac{3}{5}\right)\left(-\frac{1}{2}\right) = -\frac{3}{125}$

Multiply through 250

$\therefore 250x^4+325x^3+110x^2-7x-6 = 0$

2. a) $\alpha\beta+\beta\gamma+\alpha\gamma=\frac{c}{a}=37$ $\quad \alpha\beta\gamma=-\frac{d}{a}=52$ $\quad p=-\alpha-\beta-\gamma$

b) $\alpha=3-2i$ $\quad \beta=3+2i$ $\quad \gamma(9+4)=13\gamma=52$

$\gamma=4$

$-p=\alpha+\beta+\gamma=4+3+3=10$

$p=-10$

3. a) $\alpha=-2+i$ $\quad \beta=-2-i$ $\quad \alpha+\beta+\gamma=\gamma-4=-\frac{5}{2}$

$\gamma=\frac{3}{2}$

b) $-\frac{q}{2}=\alpha\beta\gamma$

$q=-2\alpha\beta\gamma=-2(4+1)\left(\frac{3}{2}\right)=-15$

4. $\alpha,\beta,\gamma,\delta$ $\quad \beta=\alpha+2k$ $\quad \gamma=\alpha+4k$ $\quad \delta=\alpha+6k$

$\alpha+\beta+\gamma+\delta=4\alpha+12k=40$

$\alpha+3k=10$ — ①

$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\alpha(\beta+\gamma+\delta)+\alpha^2+6\alpha k+8k^2+\alpha^2+8\alpha k+12k^2+\alpha^2+10\alpha k+24k^2$

$=3\alpha^2+12\alpha k+3\alpha^2+24\alpha k+44k^2=6\alpha^2+36\alpha k+44k^2=510$

$3\alpha^2+18\alpha k+22k^2=255$ — ②

①^2: $\alpha^2+6\alpha k+9k^2=100$

↳ $\times 3$: $3\alpha^2+18\alpha k+27k^2=300$ — ①′

①′ − ②: $5k^2=45$

$k^2=9$

$k=\pm 3=3,-3$

$\alpha=10\mp 9=1,19$

$x=1,7,13,19$

5. a) $\alpha=\frac{1}{2}$ $\quad \beta=-\frac{1}{3}$ $\quad \gamma=2$ $\quad \alpha+\beta+\gamma+\delta=\delta+\frac{13}{6}=\frac{58}{24}=\frac{29}{12}$

$\delta=\frac{29}{12}-\frac{26}{12}=\frac{1}{4}$

b) $-\frac{d}{a}=\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta$

$=\left(\frac{1}{2}\right)\left(-\frac{1}{3}\right)(2)+\left(\frac{1}{2}\right)\left(-\frac{1}{3}\right)\left(\frac{1}{4}\right)+\left(\frac{1}{2}\right)(2)\left(\frac{1}{4}\right)+\left(-\frac{1}{3}\right)\left(\frac{1}{4}\right)(2)=-\frac{1}{3}-\frac{1}{24}+\frac{1}{4}-\frac{1}{6}=-\frac{7}{24}$

$d=\frac{7}{24}\times 24=7$

$\frac{e}{a} = \alpha\beta\gamma\delta = \left(\frac{1}{2}\right)\left(-\frac{1}{3}\right)\left(2\right)\left(\frac{1}{4}\right) = -\frac{1}{12}$ $\quad e = -\frac{1}{12} \times 24 = -2$ ✓

11. $w = 3x+1$ $\quad 2\left(\frac{(w-1)^3}{27}\right) + 5\left(\frac{(w-1)^2}{9}\right) + 7\left(\frac{w-1}{3}\right) - 2 = 0$

$x = \frac{w-1}{3}$ $\quad \frac{2}{27}(w^3 - 3w^2 + 3w - 1) + \frac{5}{9}(w^2 - 2w + 1) + \frac{7}{3}(w-1) - 2 = 0$

$\frac{2}{27}w^3 + \frac{1}{3}w^2 + \frac{13}{9}w - \frac{104}{27} = 0$

$2w^3 + 9w^2 + 39w - 104 = 0$ ✓

12. a) $w = 2x$ $\quad 6\left(\frac{w^4}{16}\right) - 2\left(\frac{w^3}{8}\right) - 5\left(\frac{w^2}{4}\right) + 7\left(\frac{w}{2}\right) + 8 = 0$

$x = \frac{w}{2}$ $\quad \frac{3}{8}w^4 - \frac{1}{4}w^3 - \frac{5}{4}w^2 + \frac{7}{2}w + 8 = 0$

$3w^4 - 2w^3 - 10w^2 + 28w + 64 = 0$ ✓

b) $\sum\alpha = \frac{1}{3}$ $\quad \sum\alpha\beta = -\frac{5}{6}$ $\quad \sum\alpha\beta\gamma = -\frac{7}{6}$ $\quad \sum\alpha\beta\gamma\delta = \frac{4}{3}$

$(3\alpha-2)+(3\beta-2)+(3\gamma-2)+(3\delta-2) = 3(\alpha+\beta+\gamma+\delta) - 8 = 1 - 8 = -7 = -\frac{b}{a}$

$\frac{b}{a} = 7$

$(3\alpha-2)(3\beta-2) = 9\alpha\beta - 6\alpha - 6\beta + 4$

$\therefore$ Double sum $= 9\sum\alpha\beta - 18\sum\alpha + 24 = -\frac{15}{2} - 6 + 24 = \frac{21}{2} = \frac{c}{a}$

$(3\alpha-2)(3\beta-2)(3\gamma-2) = (9\alpha\beta - 6\alpha - 6\beta + 4)(3\gamma - 2)$

$= 27\alpha\beta\gamma - 18\alpha\gamma - 18\beta\gamma + 12\gamma - 18\alpha\beta + 12\alpha + 12\beta - 8$

$= 27\alpha\beta\gamma - 18\alpha\beta - 18\beta\gamma - 18\alpha\gamma + 12\alpha + 12\beta + 12\gamma - 8$

$\therefore$ Triple sum $= 27\sum\alpha\beta\gamma - 36\sum\alpha\beta + 36\sum\alpha - 32$

$= -\frac{63}{2} + 30 + 12 - 32 = -\frac{43}{2} = -\frac{d}{a}$ $\quad \frac{d}{a} = \frac{43}{2}$

$(3\alpha-2)(3\beta-2)(3\gamma-2)(3\delta-2) = (27\alpha\beta\gamma - 18\alpha\beta - 18\beta\gamma - 18\alpha\gamma + 12\alpha + 12\beta + 12\gamma - 8)(3\delta - 2)$

$= 81\alpha\beta\gamma\delta - 54\alpha\beta\gamma - 54\alpha\beta\delta - 54\alpha\gamma\delta - 54\beta\gamma\delta + 36\alpha\delta + 36\beta\delta + 36\gamma\delta + 36\alpha\beta + 36\beta\gamma + 36\alpha\gamma$
$-24\alpha - 24\beta - 24\gamma - 24\delta + 16$

$= 81\sum\alpha\beta\gamma\delta - 54\sum\alpha\beta\gamma + 36\sum\alpha\beta - 24\sum\alpha + 16 = 108 + 63 - 30 - 8 + 16 = 148 = \frac{e}{a}$

$\rightarrow w^4 + 7w^3 + \frac{21}{2}w^2 + \frac{43}{2}w + 149 = 0$

$\hookrightarrow 2w^4 + 14w^3 + 21w^2 + 43w + 298 = 0$