(Not Maclaurin's) Series

sums of natural numbers :)

Sigma Notation

$$\sum_{r=1}^{3}(10r-1) \Rightarrow$$

```
int Sigma()
{
    int sum;
    for(int r=0 ; r≤3 ; r++)
    {
        sum += 10*r -1 ;
    }
    return sum ;
}
```

$$\sum_{r=1}^{n} r = 1+2+3+\ldots+n = \frac{1}{2}n(n+1)$$

FORMULA FOR SUM OF COUNTING NUMBERS

$$\sum_{r=k}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k-1} f(r)$$

$$1\to n \left\{ \begin{array}{l} \left.\begin{array}{l} f(n)\\ \vdots \\ f(k)\end{array}\right\} k\to n \\ \left.\begin{array}{l} f(k-1)\\ \vdots\\ f(2)\\ f(1)\end{array}\right\} 1\to k-1 \end{array}\right.$$

Manipulating the sigma notation:

$$\sum_{r=1}^{n} kf(r) = k\sum_{r=1}^{n} f(r)$$ FACTORISATION

$$\sum_{r=1}^{n}(f(r)+g(r)) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$$ ADDITION / SUBTRACTION

↓ EXAMPLE

$$\sum_{r=1}^{25}(3r+1) = 3\sum_{r=1}^{25} r + \sum_{r=1}^{25} 1 = 3\left(\frac{1}{2}(25)(26)\right) + 25 = 1000$$

SUM OF CONSTANTS $\sum_{r=1}^{n} k = \overbrace{k+k+k+\ldots+k}^{n \text{ times}} = nk$

Sums of Squares and Cubes

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1) \qquad \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Ex3A

1. a) $\sum_{r=0}^{3}(2r+1) = 0+2+4+6+4 = 16$ ✓

c) $\sum_{r=1}^{20} r = \frac{1}{2}(20)(21) = 210$ ✓

e) $\sum_{r=10}^{40} r = \sum_{r=1}^{40} r - \sum_{r=1}^{9} r = \frac{1}{2}(40)(41) - \frac{1}{2}(9)(10) = 775$ ✓

g) $\sum_{r=21}^{40} r = \frac{1}{2}(40)(41) - \frac{1}{2}(20)(21) = 610$ ✓

3. $\sum_{r=1}^{k} r = \frac{1}{2}\sum_{r=1}^{20} r \qquad \frac{1}{2}\sum_{r=1}^{20} r = \frac{1}{2}\left(\frac{1}{2}\right)(20)(21) = 105$

$\sum_{r=1}^{k} r = \frac{1}{2}(k)(k+1) = \frac{1}{2}k^2 + \frac{1}{2}k - 105 = 0$

$k = 14, -15 = 14$ ✓

rej.

5. $\sum_{r=n-1}^{2n} r = \sum_{r=1}^{2n} r - \sum_{r=1}^{n-2} r = \frac{1}{2}(2n)(2n+1) - \frac{1}{2}(n-2)(n-1)$

$3n \quad -1$
$n \quad +2$

$= 2n^2 + n - \frac{1}{2}n^2 + \frac{3}{2}n - 1 = \frac{3}{2}n^2 + \frac{5}{2}n - 1 = \frac{1}{2}(3n^2 + 5n - 2) = \frac{1}{2}(n+2)(3n-1)$ ✓

7. a) $\sum_{r=1}^{55}(3r-1) = 3\sum_{r=1}^{55} r - 55 = \frac{3}{2}(55)(56) - 55 = 4565$ ✓

b) $\sum_{r=1}^{90}(2-7r) = 180 - 7\sum_{r=1}^{90} = 180 - \frac{7}{2}(90)(91) = -28485$ ✓

c) $\sum_{r=1}^{46}(9+2r) = 9(46) + 2\sum_{r=1}^{46} r = 414 + (46)(47) = 2576$ ✓

13. $\sum_{r=1}^{n} f(r) = n^2 + 4n = n^2 + n + 3n = 2\left(\frac{1}{2}n(n+1)\right) + 3n = 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 3 = \sum_{r=1}^{n}(2r+3)$

$\therefore f(r) = 2r + 3$ ✓

Mixed Ex 3

1. a) $\sum_{r=1}^{10} r = \frac{1}{2}(10)(11) = \frac{110}{2} = 55$ c) $\sum_{r=1}^{10} r^2 = \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}(10)(11)(21) = 385$

d) $\sum_{r=1}^{10} r^3 = \frac{1}{4}n^2(n+1)^2 = \frac{1}{4}(100)(121) = 3025$ g) $\sum_{r=1}^{60} r + \sum_{r=1}^{60} r = \frac{1}{2}(60)(61) + \frac{1}{6}(60)(61)(121)$

$= 75640$

2. a) $\sum_{r=1}^{n}(3r-5) = 3\sum_{r=1}^{n} r - \sum_{r=1}^{n} 5 = 3\left(\frac{1}{2}n(n+1)\right) - 5n = \frac{3}{2}n^2 + \frac{3}{2}n - 5n = \frac{3}{2}n^2 - \frac{7}{2}n = \frac{1}{2}n(3n-7)$

c) $\sum_{r=1}^{n}(3r^2+7r) = 3\sum_{r=1}^{n} r^2 + 7\sum_{r=1}^{n} r = 3\left(\frac{1}{6}n(n+1)(n^2+1)\right) + 7\left(\frac{1}{2}n(n+1)\right)$

$= \frac{1}{2}n(n+1)(2n+1) + \frac{7}{2}n(n+1) = \frac{1}{2}n(n+1)((2n+1)+7) = \frac{1}{2}n(n+1)(2n+8)$

$= n(n+1)(n+4)$

e) $\sum_{r=1}^{n}(r^2-2r) = \sum_{r=1}^{n} r^2 - 2\sum_{r=1}^{n} r = \frac{1}{6}n(n+1)(2n+1) - 2\left(\frac{1}{2}n(n+1)\right)$

$= \frac{1}{6}n(n+1)(2n+1) - n(n+1) = \frac{1}{6}n(n+1)(2n+1-6) = \frac{1}{6}n(n+1)(2n-5)$

g) $\sum_{r=1}^{n}(r^2-5) = \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} 5 = \frac{1}{6}n(n+1)(2n+1) - 5n = \frac{1}{6}n\left[(n+1)(2n+1) - 30\right]$

$= \frac{1}{6}n(2n^2+3n+1-30) = \frac{1}{6}n(2n^2+3n-29)$

3. $\sum_{r=1}^{30} r(3r-1) = \sum_{r=1}^{30}(3r^2-r) = 3\sum_{r=1}^{30} r^2 - \sum_{r=1}^{30} r = 3\left(\frac{1}{6}(30)(31)(61)\right) - \frac{1}{2}(30)(31)$

$= 28365 - 465 = 27900$

4. a) $\sum_{r=1}^{n} r^2(r-3) = \sum_{r=1}^{n}(r^3-3r^2) = \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r^2 = \frac{1}{4}n^2(n+1)^2 - \frac{1}{2}n(n+1)(2n+1)$

$= \frac{1}{4}n(n+1)(n^2+n-4n-2) = \frac{1}{4}n(n+1)(n^2-3n-2)$

$a = -3, \quad b = -2$

b) $\sum_{r=1}^{20} r^2(r-3) = \frac{1}{4}(20)(21)(400-60-2) = \frac{1}{4}(420)(338) = 35490$ ✓

5. a) $\sum_{r=1}^{n}(2r-1)^2 = \sum_{r=1}^{n}(4r^2-4r+1) = 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$= \frac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n = \frac{1}{3}n\left(2(2n^2+3n+1) - 6n - 6 + 3\right)$

$= \frac{1}{3}n(4n^2+6n+2-6n-3) = \frac{1}{3}n(4n^2-1) = \frac{1}{3}n(2n+1)(2n-1)$

b) $\sum_{r=1}^{2n}(2r-1)^2 = \frac{1}{3}(2n)(4n+1)(4n-1) = \frac{2}{3}n(4n+1)(4n-1)$

6. a) $\sum_{r=1}^{n} r(r+2) = \sum_{r=1}^{n}(r^2+2r) = \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r = \frac{1}{6}n(n+1)(2n+1) + n(n+1)$

$= \frac{1}{6}n(n+1)(2n+1+6) = \frac{1}{6}n(n+1)(2n+7)$ $a=2,\ b=7$ ✓

b) $\sum_{r=15}^{30} r(r+2) = \sum_{r=1}^{30} r(r+2) - \sum_{r=1}^{14} r(r+2) = \frac{1}{6}(30)(31)(67) - \frac{1}{6}(14)(15)(35)$

$= 9160$ ✓

7. a) $\sum_{r=n+1}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n} r^2 = \frac{1}{6}(2n)(2n+1)(4n+1) - \frac{1}{6}n(n+1)(2n+1)$

$= \frac{1}{6}n\left(2(8n^2+6n+1) - (2n^2+3n+1)\right) = \frac{1}{6}n(16n^2+12n+2-2n^2-3n-1)$

$= \frac{1}{6}n(14n^2+9n+1) = \frac{1}{6}n(2n+1)(7n+1)$ ✓

$\begin{matrix} 7n & & 1 \\ & \times & \\ 2n & & 1 \end{matrix}$

b) $\sum_{r=16}^{30} r^2 = \frac{1}{6}(15)(31)(7\times15+1) = \frac{1}{6}(15)(31)(106) = 8215$ ✓

8. a) $\sum_{r=1}^{n}(r^2-r-1) = \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 = \frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) - n$

$= \frac{1}{6}n(2n^2+3n+1-3n-3-6) = \frac{1}{6}n(2n^2-8) = \frac{1}{3}n(n^2-4)$ ✓

b) $\sum_{r=10}^{40}(r^2-r-1)=\sum_{r=1}^{40}(r^2-r-1)-\sum_{r=1}^{9}(r^2-r-1)=\frac{1}{3}(40)(1600-4)-\frac{1}{3}(9)(81-4)$

$=21049$

c) $\sum_{r=1}^{2n} r=\frac{1}{2}(2n)(2n+1)=n(2n+1)=\frac{1}{3}n(n+1)(n-1)$

$3(2n+1)=(n+1)(n-1)$

$6n+3=n^2-4$

$n^2-6n-7=0$

$(n-7)(n+1)=0$

$n=-1,7$

$n \quad -7$
$n \quad +1$

$\therefore \sum_{r=1}^{n}(r^2-r-1)=\sum_{r=1}^{2n} r$

when $n=7$.

9. a) $\sum_{r=1}^{n} r(2r^2+1)=\sum_{r=1}^{n}(2r^3+r)=2\sum_{r=1}^{n} r^3+\sum_{r=1}^{n} r=\frac{1}{2}n^2(n+1)^2+\frac{1}{2}n(n+1)$

$=\frac{1}{2}n(n+1)(n^2+n+1)$

b) $\sum_{r=1}^{n}(100r^2-r)=100\sum_{r=1}^{n} r^2-\sum_{r=1}^{n} r=\frac{100}{6}n(n+1)(2n+1)-\frac{1}{2}n(n+1)$

$=\frac{1}{6}n(n+1)(200n+100-3)=\frac{1}{6}n(n+1)(200n+97)=\frac{1}{2}n(n+1)(n^2+n+1)$

$200n+97=3n^2+3n+3$

$3n^2-197n-94=0$

$n=66.14, -0.47$ no integer value of n satisfies

$\Delta=39937$, which isn't square

10. a) $\sum_{r=1}^{n} r(r+1)^2=\sum_{r=1}^{n}(r^3+2r^2+r)=\sum_{r=1}^{n} r^3+2\sum_{r=1}^{n} r^2+\sum_{r=1}^{n} r$

$=\frac{1}{4}n^2(n+1)^2+\frac{1}{3}n(n+1)(2n+1)+\frac{1}{2}n(n+1)$

$=\frac{1}{12}n(n+1)(3(n^2+n)+4(2n+1)+6)=\frac{1}{12}n(n+1)(3n^2+11n+10)$

$=\frac{1}{12}n(n+1)(n+2)(3n+5)$

$3n \quad 5$
$n \quad 2$

b) $\sum_{r=1}^{n} 70r=35n(n+1)=\frac{1}{12}n(n+1)(n+2)(3n+5)$

$420=3n^2+11n+10$

$3n^2+11n-410=0 \longrightarrow n=10,-\frac{41}{3} \quad \therefore n=10$

11. $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n+1}(9r+1) = \frac{9}{2}(n+1)(n+2) + n+1 = \frac{9}{2}(n+1)(n+2+\frac{2}{9})$

$= \frac{9}{2}(n+1)(n+\frac{20}{9})$

$\frac{1}{6}n(n+1)(2n+1) = \frac{9}{2}(n+1)(n+\frac{20}{9})$

$\frac{1}{6}(2n^2+n) = \frac{9}{2}n+10$

$2n^2+n = 27n+60$

$2n^2-26n-60=0$

$n^2-13n-30=0$

$n=15,-2$

$\therefore n=15$ ✓