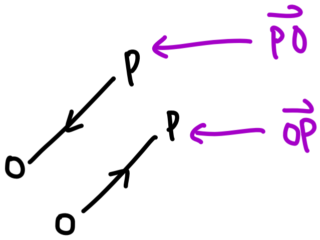
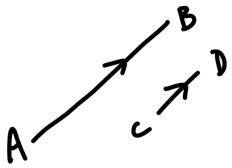
Vector: A directional line segment



$\underbrace{|\overrightarrow{PO}|}_{\text{"magnitude of" ie length}} = |\overrightarrow{OP}|$

$\overrightarrow{PO} = \underbrace{-\overrightarrow{OP}}_{\text{reverse direction}}$

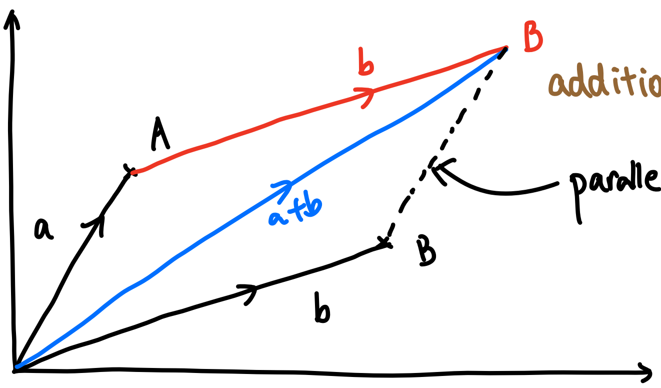


$\overrightarrow{CD} = \lambda \overrightarrow{AB} \quad (0 < \lambda < 1)$

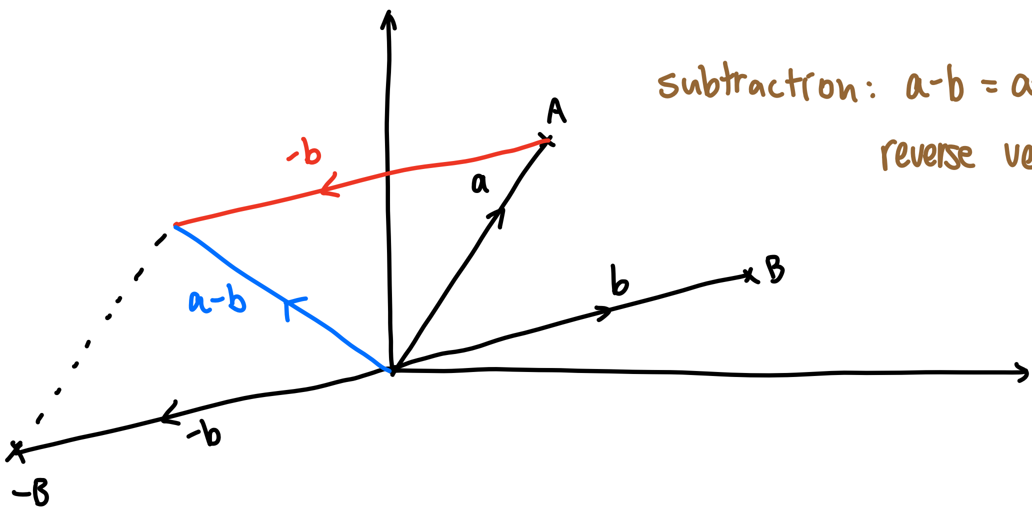
↑ shows that $\overrightarrow{AB} \parallel \overrightarrow{CD}$, $|\overrightarrow{AB}| > |\overrightarrow{CD}|$

Commonly used constants: λ, ϕ, ω, μ

Consider the 2 vectors: $a = \overrightarrow{OA}$
$b = \overrightarrow{OB}$



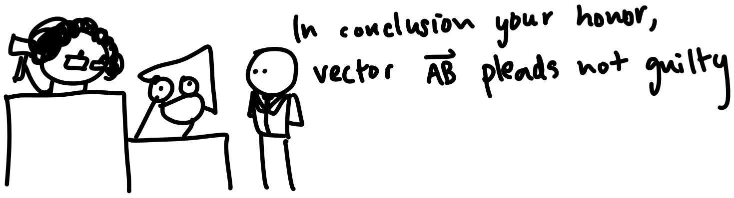
addition: add vectors end-to-end: resultant

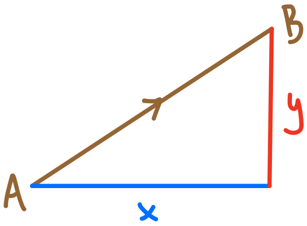


subtraction: $a - b = a + (-b)$

reverse vector, then add them

Representing a vector





$\vec{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$

column vector

adding 2 column vectors:

$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$

multiplying by a scalar: $\lambda \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}$

unit vectors

a vector of magnitude/length 1

special unit vectors:

$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

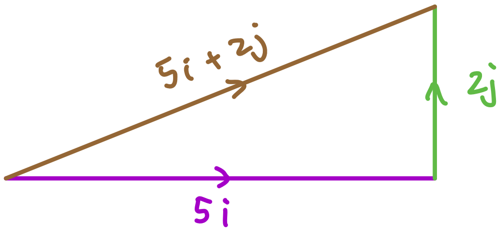
$j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Another form: $pi + qj$

$\begin{pmatrix} p \\ q \end{pmatrix} = pi + qj$

Adding: $ai + bj + ci + dj = (a+c)i + (b+d)j$

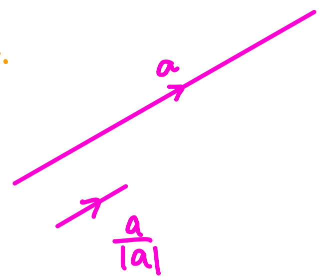
Multiplying: $\lambda(ai + bj) = \lambda ai + \lambda bj$



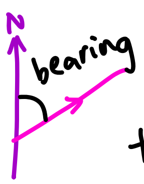
Magnitude & Direction

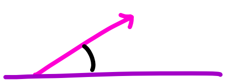
if $a = \begin{pmatrix} x \\ y \end{pmatrix}$, $|a| = \sqrt{x^2 + y^2}$ Pythagoras again?!

a unit vector in the direction of $a = \frac{a}{|a|}$

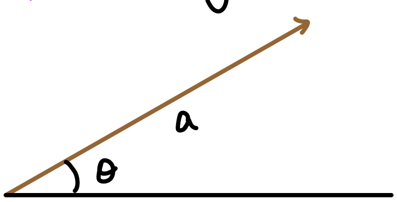


direction: use trigonometry!





they will tell you which angle to find!



if $|a| = r$,

$a = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$

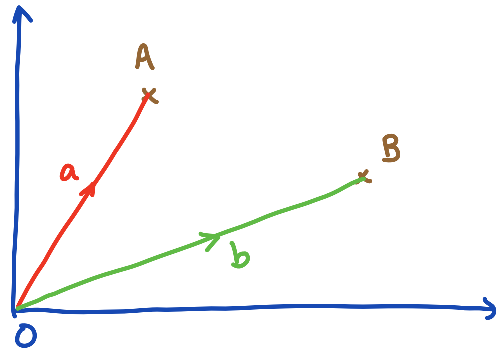
Position vectors

O = origin

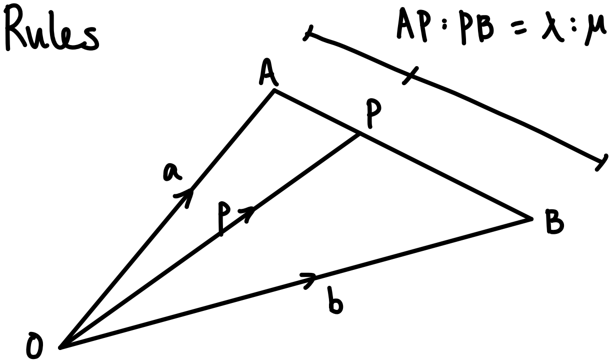
$\overrightarrow{OA} = a$ = position vector of A

$\overrightarrow{OB} = b$ = position vector of B

that's it folks!



Rules



$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{\lambda}{\lambda+\mu}\overrightarrow{AB}$$

$$p = a + \frac{\lambda}{\lambda+\mu}(b-a)$$

If a & b are not parallel, $pa + qb = ra + sb$

$$\Downarrow$$

$p = r$, $q = s$ (compare coefficients)

If $\overrightarrow{OA} = a$, $\overrightarrow{OB} = b$, $\overrightarrow{AB} = b - a$ (destination − original position)

When trying to represent a vector with other vectors, find 2 different paths, equate & compare coefficients

(example on next page, Q4a)

Parallel vectors: e.g. $\begin{pmatrix} p \\ q \end{pmatrix} \parallel \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\Downarrow$$

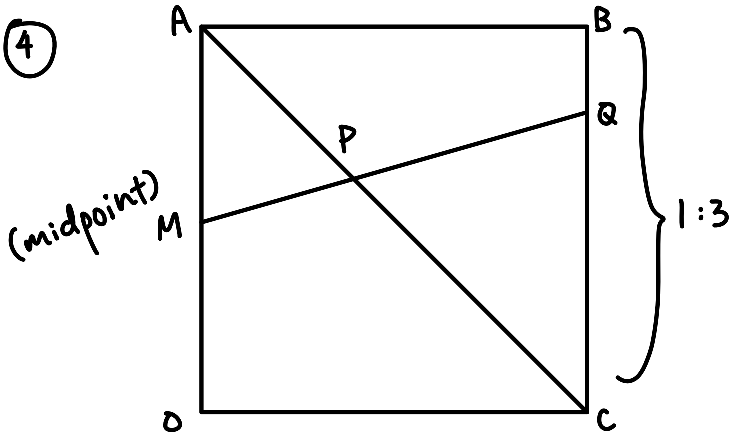
$$\begin{pmatrix} p \\ q \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ -\lambda \end{pmatrix} \Rightarrow \overbrace{2\lambda}^{x} = -2(\overbrace{-\lambda}^{y})$$

$$\Downarrow$$

$$p = -2q$$

Ex 11E (p.246)

4



a) $\overrightarrow{OA} = a \quad \overrightarrow{OC} = c \quad \overrightarrow{OP} = ?$

$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} \quad (1)$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} \quad (2)$

$\overrightarrow{AB} = c \quad \overrightarrow{BQ} = -\frac{1}{4}a$

(1): $\overrightarrow{OP} = \frac{1}{2}a + \lambda\overrightarrow{MQ}$

$= \frac{1}{2}a + \lambda\left(\frac{1}{2}a + c - \frac{1}{4}a\right)$

$= \left(\frac{1}{2} + \frac{\lambda}{4}\right)a + \lambda c$

(2): $\overrightarrow{OP} = a + \phi\overrightarrow{AC} = a + \phi(c-a)$

$= (1-\phi)a + \phi c$

equating (1) & (2),

$\left(\frac{1}{2} + \frac{\lambda}{4}\right)a + \lambda c = (1-\phi)a + \phi c$

comparing coefficients, $\lambda = \phi$

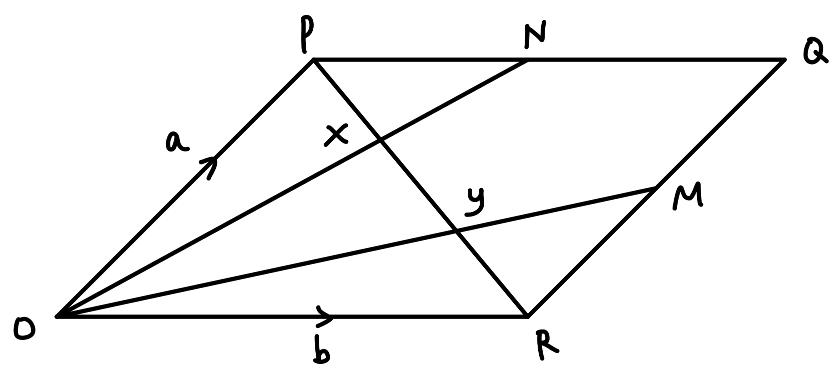
$\therefore \frac{1}{2} + \frac{\lambda}{4} = 1 - \lambda$

$2 + \lambda = 4 - 4\lambda$

$\lambda = \frac{2}{5}$

$\therefore \overrightarrow{OP} = \frac{3}{5}a + \frac{2}{5}c$

Challenge (p. 247)



$\vec{PX} = \phi\vec{PR} = \phi(\vec{PO} + \vec{OR}) = \phi(-a+b) = -\phi a + \phi b$ (a)

$\vec{PX} = \vec{PO} + \vec{OX} = -a + \lambda\vec{ON} = -a + \lambda(\vec{OP} + \vec{PN}) = -a + \lambda(a + \frac{1}{2}b) = (\lambda - 1)a + \frac{\lambda}{2}b$ (b)

comparing coefficients: $\begin{cases} -\phi = \lambda - 1 & — ① \\ \phi = \frac{\lambda}{2} & — ② \end{cases}$ (c)

①+②: $0 = \frac{3}{2}\lambda - 1$

$\lambda = \frac{2}{3} \quad \phi = \frac{1}{3}$ (d)

$\vec{PX} = \phi\vec{PR} = \frac{1}{3}\vec{PR}$ (we know that) (e)

What about $\vec{Py}$?

$\vec{Py} = \mu\vec{PR} = \mu(\vec{PO} + \vec{OR}) = \mu(-a+b) = -\mu a + \mu b$

$\vec{Py} = \vec{PO} + \vec{Oy} = \vec{PO} + \omega\vec{OM} = -a + \omega(\vec{OR} + \vec{RM}) = -a + \omega(b + \frac{1}{2}a) = (\frac{\omega}{2} - 1)a + \omega b$

$\begin{cases} -\mu = \frac{\omega}{2} - 1 & — ③ \\ \mu = \omega & — ④ \end{cases}$

④+③: $0 = \frac{3}{2}\omega - 1$

$\omega = \frac{2}{3} \quad \mu = \frac{2}{3} \Rightarrow \vec{Py} = \frac{2}{3}\vec{PR}$

$\therefore$ $\vec{ON}$ & $\vec{OM}$ divide $\vec{PR}$ into 3 equal parts

(trisect)

Ex 11F (p. 250)

① b) $\left|\begin{pmatrix}24\\-7\end{pmatrix}\right| = \sqrt{24^2+7^2} = 25$ kmh^{-1}

② a) $s = v \times t$

$= \begin{pmatrix}8\\6\end{pmatrix} \times 5 = \begin{pmatrix}40\\30\end{pmatrix}$

$|s|$ = distance $= \sqrt{40^2+30^2} = 50$ km

c) $s = v \times t$

$= \begin{pmatrix}6\\2\end{pmatrix} \times \frac{45}{60} = \begin{pmatrix}4.5\\1.5\end{pmatrix}$

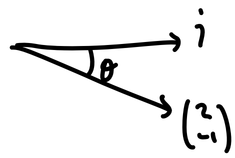
distance $= \sqrt{4.5^2+1.5^2} = 4.74$ km

④ $a_p = \frac{v-u}{t} = \frac{16i-5j-2i-3j}{2} = \frac{14i-8j}{2} = 7i-4j$ ms^{-2}

⑥ $F_1 = \begin{pmatrix}3\\-4\end{pmatrix}$N $F_2 = \begin{pmatrix}p\\q\end{pmatrix}$ N $R = F_1+F_2 = \begin{pmatrix}3+p\\q-4\end{pmatrix} /\!/ \begin{pmatrix}2\\-1\end{pmatrix}$

a) θ between R & vector i

$\because R /\!/ \begin{pmatrix}2\\-1\end{pmatrix} \therefore$ Direction $= \begin{pmatrix}2\\-1\end{pmatrix}$



$\theta = \tan^{-1}\left(\frac{1}{2}\right) = 26.6°$

b) $R = \begin{pmatrix}p+3\\q-4\end{pmatrix} = \lambda\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}2\lambda\\-\lambda\end{pmatrix}$

since $2\lambda = -2(-\lambda)$,

$p+3 = -2(q-4)$

$\therefore p+2q = 5$

c) given $p=1$, $p = \frac{5-1}{2} = 2$

$\therefore R = \begin{pmatrix}4\\-2\end{pmatrix}$

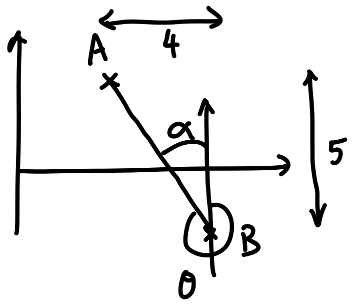
$\therefore |R| = \sqrt{4^2+2^2} = \sqrt{20} = 2\sqrt{5}$

8) A = boat B = buoy

$\vec{OA} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\vec{OB} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$

a) $|\vec{AB}| = |\vec{OB} - \vec{OA}| = \left|\begin{pmatrix} 6-2 \\ -4-1 \end{pmatrix}\right| = \left|\begin{pmatrix} 4 \\ -5 \end{pmatrix}\right| = \sqrt{16 + 25} = \sqrt{41}$

b)



$\alpha = \tan^{-1}\frac{4}{5}$

$\theta = 360 - \alpha = 360 - \tan^{-1}\frac{4}{5} = 321.3^{\circ}$

c) boat travels $\begin{pmatrix} 8 \\ -10 \end{pmatrix}$ kmh^{-1}

$\vec{AB} = \underbrace{\begin{pmatrix} 4 \\ -5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 8 \\ -10 \end{pmatrix}}$

$\therefore$ travelling towards buoy

d) speed $= \left|\begin{pmatrix} 8 \\ -10 \end{pmatrix}\right| = \sqrt{8^2 + 10^2} = \sqrt{164} = 2\sqrt{41}$

e) time taken:

$t = \frac{d}{s} = \frac{\sqrt{41}}{2\sqrt{41}} = \frac{1}{2} = 0.5$ hours or 30 mins

Mixed Ex 11

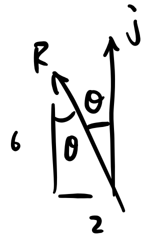
① $F_1 = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$ N $\quad R = F_1 + F_2$

$F_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ N $\quad = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$ N

a) $|R| = \sqrt{2^2 + 6^2} = \sqrt{40}$

$= 2\sqrt{10}$ N

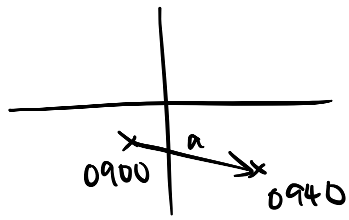
b)



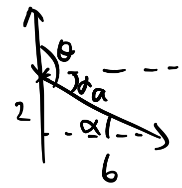
$\theta = \tan^{-1}\frac{2}{6} = 18.4°$

② 0900 : $\overrightarrow{OS} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

0940 : $\overrightarrow{OS} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$



a)



$a = \begin{pmatrix} 4 \\ -6 \end{pmatrix} - \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

$\alpha = \tan^{-1}\frac{2}{6} = 18.4°$

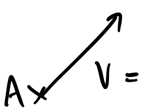
$\theta = 180 + \alpha = 198.4°$

b) $v = \frac{s}{t} = \frac{a}{(40/60)} = \frac{3}{2}\begin{pmatrix} 6 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 9 \\ -3 \end{pmatrix}$ kmh^{-1} $\quad$ speed $= |v| = \sqrt{9^2 + 3^2}$

$= \sqrt{27 + 9} = 6$ kmh^{-1}

③



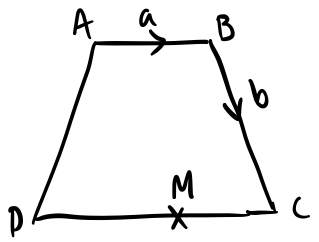
$V = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

a) speed $= |v| = \sqrt{4^2 + 9^2} = \sqrt{97}$

b) $s = vt = \begin{pmatrix} 4 \\ 9 \end{pmatrix} \times 6 = \begin{pmatrix} 24 \\ 54 \end{pmatrix}$ m from A

distance $= |s| = \sqrt{24^2 + 54^2} = 6\sqrt{97}$

c) Not valid ∵ wind / air resistance may alter v

④ 

$AB \parallel DC \quad DC = 4AB$

$DM : MC = 3:2$

a) $\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CM}$

$= a + b + \frac{2}{5}\overrightarrow{CD}$

$= a + b + \frac{2}{5}(-4a)$

$= a + b - \frac{8}{5}a$

$= -\frac{3}{5}a + b$

b) $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = b - \overrightarrow{DC} = b - 4a = -4a + b$

c) $\overrightarrow{MB} = -\overrightarrow{BM} = -(\overrightarrow{BC} + \overrightarrow{CM})$

$= -(b - \frac{8}{5}a) = \frac{8}{5}a - b$

d) $\overrightarrow{DA} = \overrightarrow{DC} + \overrightarrow{CB} + \overrightarrow{BA} = 4a + (-b) + (-a)$

$= 3a - b$

⑤ $\begin{pmatrix} 5 \\ k \end{pmatrix} = \lambda \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 8\lambda \\ 2\lambda \end{pmatrix}$

$8\lambda = 5$

$\lambda = \frac{8}{5} \quad \therefore k = 2\lambda = \frac{16}{5}$

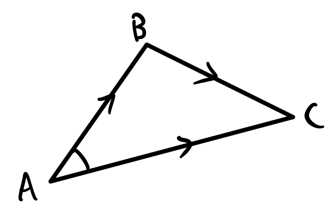
⑥ $a = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \quad b = \begin{pmatrix} 10 \\ -2 \end{pmatrix} \quad c = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$

a) $a + b + c = \begin{pmatrix} 7 + 10 - 5 \\ 4 - 2 - 3 \end{pmatrix} = \begin{pmatrix} 12 \\ -1 \end{pmatrix}$

b) $a - 2b + c = \begin{pmatrix} 7 - 20 - 5 \\ 4 + 4 - 3 \end{pmatrix} = \begin{pmatrix} -18 \\ 5 \end{pmatrix}$

c) $2a + 2b - 3c = \begin{pmatrix} 14 \\ 8 \end{pmatrix} + \begin{pmatrix} 20 \\ -4 \end{pmatrix} - \begin{pmatrix} -15 \\ -9 \end{pmatrix} = \begin{pmatrix} 14 + 20 + 15 \\ 8 - 4 + 9 \end{pmatrix} = \begin{pmatrix} 49 \\ 13 \end{pmatrix}$

⑦ $\triangle ABC$: $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$



a) $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

b) $|AB| = \sqrt{3^2 + 5^2} = \sqrt{34}$

$|AC| = \sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$

$|BC| = \sqrt{3^2 + 2^2} = \sqrt{13}$

$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AC| \cos \angle BAC$

$\angle BAC = \cos^{-1}\left(\frac{|AB|^2 + |AC|^2 - |BC|^2}{2|AB||AC|}\right)$

$= 32.47°$

c) Area $= \frac{1}{2}|AB||AC| \sin \angle BAC$

$= \frac{21}{2} = 10.5$ units squared

⑧ $a=\begin{pmatrix}4\\-3\end{pmatrix}$ $b=\begin{pmatrix}2p\\-p\end{pmatrix}$ $a+b=\begin{pmatrix}2p+4\\-p-3\end{pmatrix} \parallel \begin{pmatrix}2\\-3\end{pmatrix}$

a) $\begin{pmatrix}2p+4\\-p-3\end{pmatrix}=\lambda\begin{pmatrix}2\\-3\end{pmatrix}=\begin{pmatrix}2\lambda\\-3\lambda\end{pmatrix}$ $\begin{cases}2p+4=2\lambda & (1)\\-p-3=-3\lambda & (2)\end{cases}$

$2(2) = -2p-6=-6\lambda \quad (3)$

$(1)+(3) = -2=-4\lambda$

$\lambda=\frac{1}{2} \quad -p-3=-\frac{3}{2}$

$p=-\frac{3}{2}$

b) $\begin{pmatrix}-3+4\\ \frac{3}{2}-3\end{pmatrix}=\begin{pmatrix}1\\-\frac{3}{2}\end{pmatrix}$