Book 1 Chapter 1 :

COMPLEX NUMBERS

$\mathbb{N}$ natural numbers

$\mathbb{Z}$ integers

$\mathbb{Q}$ rational numbers

$\mathbb{R}$ real numbers

$\mathbb{C}$ complex numbers

$i = \sqrt{-1}$ ($j = \sqrt{-1}$ in electrical engineering etc)

The Quadratic Equation

$ax^2 + bx + c = 0$ $\quad \Delta > 0$ 2 real roots

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\quad \Delta = 0$ 2 equal real roots

$\Delta < 0$ No real roots (2 complex roots)

The a+bi form (Example 1a)

$\sqrt{-36} = \sqrt{36}\sqrt{-1} = 6i$

$5 + \sqrt{-36} = 5 + 6i$

$$\left.\begin{array}{l} \text{If } b = 0, \text{ real} \\ \text{If } b \neq 0, \text{ imaginary} \\ \text{If } a = 0 \;\&\; b \neq 0, \text{ pure imaginary} \end{array}\right\} \text{Complex}$$

$z = a + bi \in \mathbb{C}$ (The complex unknown) (sometimes w is used)

$Re(z) = a$ $\leftarrow$ Real part of z

$Im(z) = b$ $\leftarrow$ Imaginary part of z

$\Downarrow$

Foreshadowing...

Exercise 1A

1. a) $\sqrt{-9} = 3i$ c) $\sqrt{-121} = 11i$ e) $\sqrt{-225} = 25i$

g) $\sqrt{-12} = 2\sqrt{3}i$ i) $\sqrt{-200} = 10\sqrt{2}i$

2. a) $(5+2i) + (8+9i) = 13 + 11i$

e) $(-4-6i) - (-8-8i) = 4+2i$

i) $(-2-7i) + (1+3i) - (-12+i) = -1-4i + 12 - i = 11-5i$

3. a) $2(7+2i) = 14+4i$ h) $\frac{-8+3i}{4} - \frac{7-2i}{2} = -2 + \frac{3}{4}i - \frac{7}{2} + i$

$$= -\frac{11}{2} + \frac{7}{4}i$$

4. a) $\frac{4-2i}{\sqrt{2}} = \frac{4\sqrt{2} - 2\sqrt{2}i}{2} = 2\sqrt{2} - \sqrt{2}i$

b) $\frac{2-6i}{1+\sqrt{3}} = \frac{(2-6i)(1-\sqrt{3})}{1-3} = \frac{2-2\sqrt{3} - 6i + 6\sqrt{3}i}{-2}$

$= (-1+\sqrt{3}) + (3-3\sqrt{3})i$

6. $z_1 = a+9i$ $z_2 = -3+bi$ $z_2 - z_1 = 7+2i$

$= (-3-a) + (b-9)i$

$\therefore -3-a = 7$ $b-9=2$

$a = -10$ $b = 11$

8. $z = a+bi$ $w = a-bi$

$z+w = 2a = \mathrm{Re}(z+w)$

$z-w = 2bi = \mathrm{Im}(z-w)$

Example 3

$z^2 = -9 \quad z = \pm\sqrt{-9} = \pm 3i$

Example 4

$z^2 + 6z + 25 = 0$

$(z+3)^2 + 25 - 9 = (z+3)^2 + 16 = 0$ Complete square OR Quadratic eqn

$(z+3)^2 = -16$

$z = -3 \pm \sqrt{-16} = -3 \pm 4i$

Conjugate Pairs

If $z = a+bi$,

$z^* = a - bi$

$a \pm bi \quad (a+bi, \ a-bi)$

Always the solutions to quadratic eqn with no real roots

Exercise 1B

1. a) $z^2 + 121 = 0$

$z = \pm 11i$

d) $3z^2 + 150 = 38 - z^2$

$4z^2 = -112$

$z^2 = -28$

$z = \pm 2\sqrt{7}i$

2. c) $2(z-7)^2 + 30 = 6$

$(z-7)^2 = -12$

$z = 7 \pm 2\sqrt{3}i$

5. $z^2 - 8z + 21 = 0$

$z = \frac{8 \pm \sqrt{64-84}}{2} = \frac{8 \pm 2\sqrt{5}i}{2}$

$= 4 \pm \sqrt{5}i$

6. $z^2 + bz + 11 = 0$ has no real roots

$\Delta = b^2 - 44 < 0$

$b^2 < 44$

let $b^2 = 44$

$b = \pm 2\sqrt{11}$

$\therefore -2\sqrt{11} < b < 2\sqrt{11}$

Exercise 1E

5. $z_1 = -5+4i \quad z_2 = -5-4i$

$z^2 + bz + c = 0$

$$z = \frac{-b \pm \sqrt{b^2-4c}}{2} = -\frac{b}{2} \pm \sqrt{\frac{b^2-4c}{4}}$$

$-\frac{b}{2} = -5$

$b = 10$

$\frac{b^2-4c}{4} = -16$

$100 - 4c = -64$

$c = \frac{164}{4} = 41$

$\therefore z^2 + 10z + 41 = 0$

Polynomials

$f(z)$ is a polynomial with real coefficients.

If z_1 is a root, z_1^* is also a root.

(Complex roots ALWAYS come in pairs)

Exercise 1F

6. a) $3+i$ is a root $\therefore$ Other complex root $= 3-i$

b) $g(z) = z^3 - 12z^2 + cz + d$

$(z-3-i)(z-3+i) = z^2 - 3z + zi - 3z + 9 - 3i - iz + 3i + 1$

$= z^2 - 6z + 10$

$(z-6)(z^2-6z+10)$

$= z^3 - 6z^2 + 10z - 6z^2 + 36z - 60$

$= z^3 - 12z^2 + 46z - 60$

$c = 46 \quad d = -60$

12. $f(z) = z^4 - 10z^3 + 72z^2 + Qz + 442 \qquad z = 2-3i$ is a root to $f(z) = 0$

a) other root $= 2+3i \quad (z-2-3i)(z-2+3i) = z^2 - 2z + 3iz - 2z + 4 - 6i - 3iz - 6i + 9$

$= z^2 - 4z + 13$

$f(z) = (z^2-4z+13)(z^2+az+b)$

$= z^4 + az^3 + bz^2 - 4z^3 - 4az^2 - 4bz + 13z^2 + 13az + 13b$

$= z^4 + (a-4)z^3 + (b-4a+13)z^2 + (-4b+13a)z + 13b$

$a-4 = -10$ $\quad\quad$ $13b = 442$

$a = -6$ $\quad\quad$ $b = 34$ $\quad\quad$ $\therefore (z^2-6z+34)$ is a root

b) $Q = -4b + 13a = -4(34) + 13(-6) = -136 - 78 = -214$

c) $z^2 - 6z + 34 = (z-3)^2 + 25 = 0$

$z - 3 = \pm 5i$

$z = 3 \pm 5i$

$\therefore z = 2 \pm 3i,\ 3 \pm 5i$

Mixed Ex 1 $\quad\quad$ 2, 8, 14, 22, ch

2. $\Delta = b^2 - 56 < 0$

$b^2 < 56$

$-2\sqrt{14} < b < 2\sqrt{14}$ ✓

8. $\frac{4-7i}{z} = 3+i$

$z = \frac{4-7i}{3+i} = \frac{(4-7i)(3-i)}{3+i} = \frac{12-4i-21i-7}{9+1} = \frac{5-25i}{10} = \frac{1}{2} - \frac{5}{2}i$ ✓

14. $z = \frac{q+3i}{4+qi} = \frac{(q+3i)(4-qi)}{16+q^2} = \frac{4q - q^2 i + 12i + 3q}{16+q^2} = \frac{7q + (12-q^2)i}{16+q^2}$

$= \frac{7}{16+q^2} + \frac{12-q^2}{16+q^2} i$ ✓

22. a) $z-4$ is a factor $\quad f(z) = (z-4)(z^3+bz^2+cz+d)$

$= z^4 + bz^3 + cz^2 + dz - 4z^3 - 4bz^2 - 4cz - 4d$

$= z^4 + (b-4)z^3 + (c-4b)z^2 + (d-4c)z - 4d$

Compare coefficients: $b-4=-2$ $\quad$ $c-4b=c-8=-5$ $\quad$ $-4d=24$

$b=2$ $\quad$ $c=3$ $\quad$ $d=-6$

$p=d-4c=-6-12=-18$ ✓

b) $f(z)=(z-4)(z^3+2z^2+3z-6)$ $\quad$ $f(1)=0$ $\quad$ $z-1$ is a factor

$=(z-4)(z-1)(z^2+3z+6)$

$z^2+3z+6=0$

$z=\frac{-3\pm\sqrt{9-24}}{2}=-\frac{3}{2}\pm\frac{\sqrt{15}}{2}i$, 1, 4 ✓

Challenge

a) At least 1 real root ✓

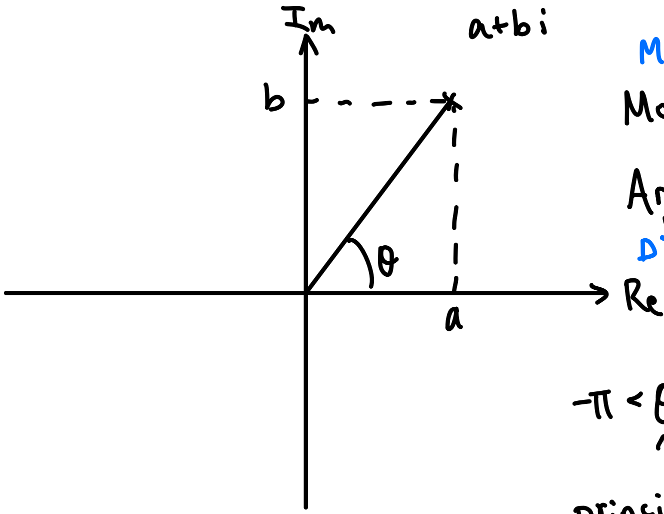
2 complex roots $\longrightarrow$ $a+bi$ $\quad b\neq 0$

$\therefore$ Complex conjugate pair $a+bi$ and $a-bi$ (from $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$)

b) $x^4+2x^2+1=(x^2+1)(x^2+1)=0$ ✓

$x=\pm i$ (no real roots)

Argand Diagram



Modulus

$\text{Mod}(a+bi) = \sqrt{a^2+b^2}$ = magnitude

$\text{Arg}(a+bi) = \tan^{-1}\left(\frac{a}{b}\right)$ = direction

Direction

$-\pi < \theta \leq \pi$ (in radians obviously)

$\uparrow$

principal argument

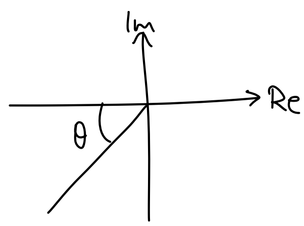
z can be written as $r(\cos\theta + i\sin\theta)$ ⇒ Modulus-Argument form

Example 7

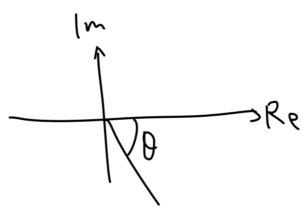
$z = -1 - i = -(1+i)$ $\qquad \cos\theta = \sin\theta$

$= -\frac{2}{\sqrt{2}}\left(\cos\left(-\frac{3}{4}\pi\right) + i\sin\left(-\frac{3}{4}\pi\right)\right)$ $\qquad \theta = \frac{\pi}{2}$ $\qquad \cos\theta = \frac{\sqrt{2}}{2} = \sin\theta$

$= -\sqrt{2}\left(\cos\left(-\frac{3}{4}\pi\right) + i\sin\left(-\frac{3}{4}\pi\right)\right)$



Ex 2C



2 a) $\frac{3}{1+i\sqrt{3}} = \frac{3(1-i\sqrt{3})}{1+3} = \frac{3-i3\sqrt{3}}{4} = \frac{3}{4} - i\frac{3}{4}\sqrt{3}$

$\tan^{-1}(\sqrt{3}) = \frac{1}{3}\pi$ $\qquad = \frac{3}{4}(1-i\sqrt{3}) = \frac{3}{2}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= \frac{3}{2}\left(\cos\frac{1}{3}\pi - i\sin\frac{1}{3}\pi\right)$

$\cos\frac{1}{3}\pi = \frac{1}{2}$

$\sin\frac{1}{3}\pi = \frac{\sqrt{3}}{2}$ $\qquad = \frac{3}{2}\left(\cos\left(-\frac{1}{3}\pi\right) + i\sin\left(-\frac{1}{3}\pi\right)\right)$

Show That:

① $|z|^2 = zz^*$

$$|z|^2 = \left(\sqrt{a^2+b^2}\right)^2 = a^2+b^2$$

$$zz^* = (a+bi)(a-bi) = a^2+b^2 = |z|^2$$

② $\frac{1}{z} = \frac{z^*}{|z|^2}$

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{z^*}{|z|^2}$$

③ $\operatorname{Re} z = \frac{1}{2}(z+z^*)$

$$\frac{1}{2}(z+z^*) = \frac{1}{2}(a+a) = a = \operatorname{Re} z$$

④ $\operatorname{Im} z = \frac{1}{2i}(z-z^*)$

$$\frac{1}{2i}(z-z^*) = \frac{1}{2i}(2bi) = b = \operatorname{Im} z$$

⑤ $(z+\omega)^* = z^* + \omega^*$

$$(a+bi+c+di)^* = (a+c)-(b+d)i$$

$$z^* + \omega^* = a-bi+c-di = (a+c)-(b+d)i = (z+\omega)^*$$

⑥ $(z\omega)^* = z^*\omega^*$

$$(z\omega)^* = (ac+adi+cbi-bd)^* = (ac-bd)-(ad+bc)i$$

$$z^*\omega^* = (a-bi)(c-di) = ac-adi-bci-bd = (ac-bd)-(ad+bc)i = (z\omega)^*$$

⑦ $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

$$\left|\frac{z}{w}\right| = \left|\frac{(a+bi)(c-di)}{c^2+d^2}\right| = \left|\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\right| = \sqrt{\left(\frac{ac+bd}{c^2+d^2}\right)^2+\left(\frac{bc-ad}{c^2+d^2}\right)^2}$$

$$= \sqrt{\frac{a^2c^2+2abcd+b^2d^2+b^2c^2-2abcd+a^2d^2}{(c^2+d^2)^2}}$$

$$= \sqrt{\frac{c^2(a^2+b^2)+d^2(a^2+b^2)}{(c^2+d^2)^2}}$$

$$= \sqrt{\frac{a^2+b^2}{c^2+d^2}}$$

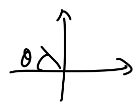
$$\frac{|z|}{|w|} = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} = \sqrt{\frac{a^2+b^2}{c^2+d^2}} = \left|\frac{z}{w}\right|$$

⑧ $|z^*| = |z|$

$$|z^*| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2} = |z|$$

Ex 2D

5. $z = -9+3i\sqrt{3}$ $\tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = \frac{1}{6}\pi$



a) $|z| = \sqrt{81+27} = 6\sqrt{3}$

$$z = 6\sqrt{3}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}\right) = 6\sqrt{3}\left(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi\right)$$

b) $|w| = \sqrt{3}$ $\arg w = \frac{7\pi}{12}$

i) $w = \sqrt{3}\left(\cos\left(\frac{7}{12}\pi\right)+i\sin\left(\frac{7}{12}\pi\right)\right)$

ii) $zw = 18\left(\cos\left(\frac{7}{12}\pi+\frac{5}{6}\pi\right)+i\sin\left(\frac{7}{12}\pi+\frac{5}{6}\pi\right)\right)$

$$= 18\left(\cos\left(-\frac{7}{12}\pi\right)+i\sin\left(-\frac{7}{12}\pi\right)\right)$$

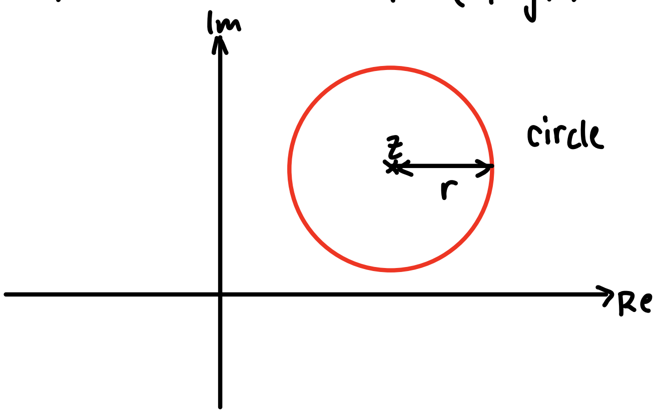
iii) $\frac{z}{w} = 6\left(\cos\left(\frac{5}{6}\pi-\frac{7}{12}\pi\right)+i\sin\left(\frac{5}{6}\pi-\frac{7}{12}\pi\right)\right)$

$$= 6\left(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi\right)$$

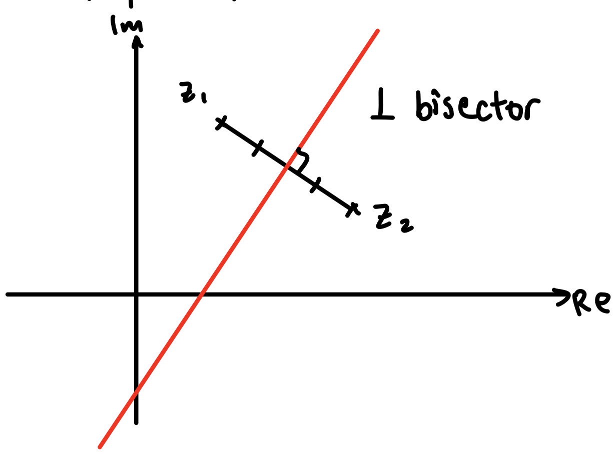
Loci on the Argand Diagram

$z_1 = x_1 + y_1 i$ $\quad$ z is a varying point $\quad$ $z_2 = x_2 + y_2 i$

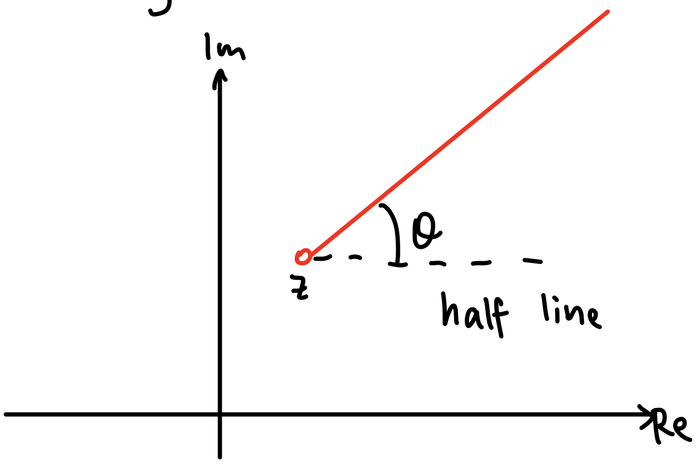
if $|z - z_1| = r$ or $|z - (x_1 + y_1 i)| = r$



if $|z - z_1| = |z - z_2|$



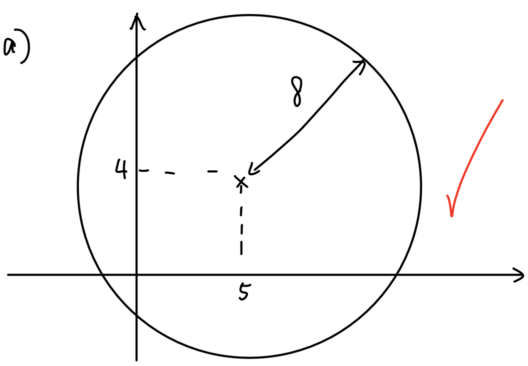
if $\arg(z - z_1) = \theta$



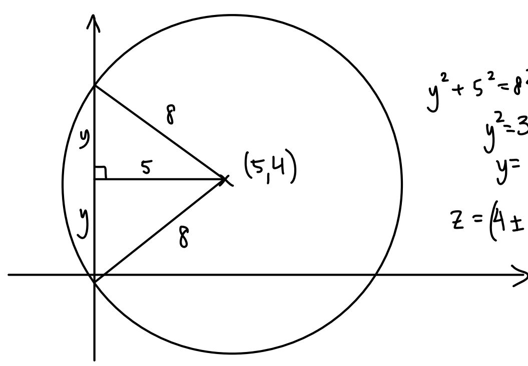
Ex 2E

2. $|z - 5 - 4i| = 8$

a)



b) i)



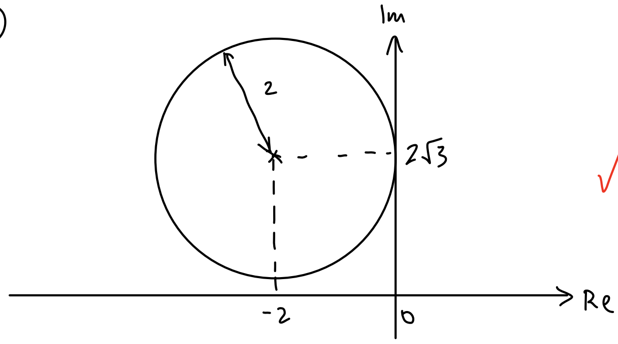
$y^2 + 5^2 = 8^2$

$y^2 = 39$

$y = \sqrt{39}$

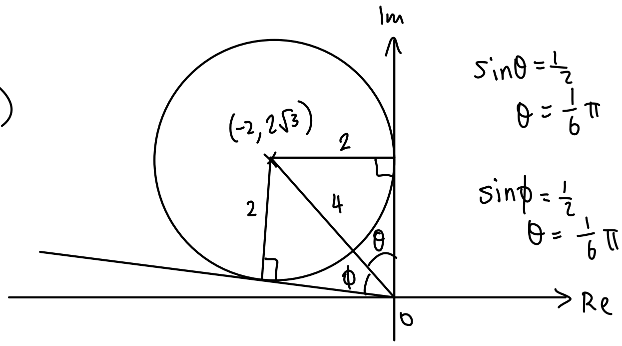
$z = (4 \pm \sqrt{39})i$

5. a)



b) $\frac{\pi}{2}$

c)



$\sin\theta = \frac{1}{2}$

$\theta = \frac{1}{6}\pi$

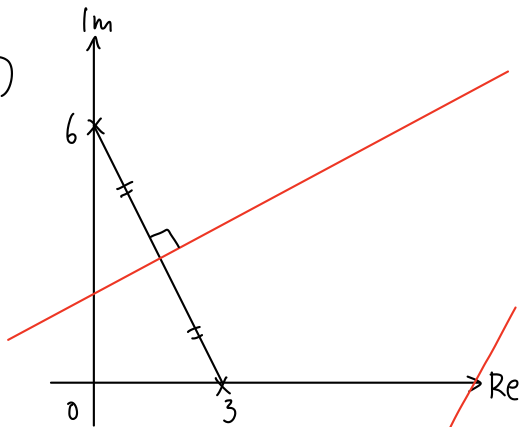
$\sin\phi = \frac{1}{2}$

$\theta = \frac{1}{6}\pi$

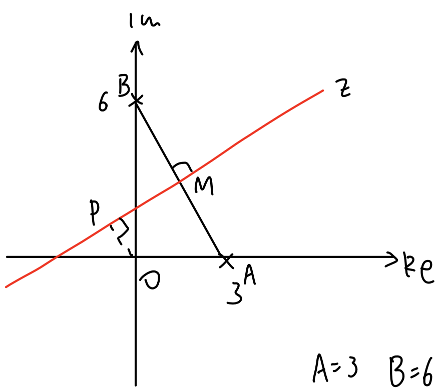
Max $\arg z = \frac{5}{6}\pi$

7. $|z-3| = |z-6i|$

a)



b)



$A = 3 \quad B = 6i \quad |z-A| = |z-B|$

$M = \left(\frac{3}{2}, 3\right)$ (midpoint of A & B)

$m_{AB} = -\frac{6}{3} = -2 \qquad m_z = \frac{1}{2}$

z in cartesian form: $y - 3 = \frac{1}{2}\left(x - \frac{3}{2}\right)$

$y = \frac{1}{2}x + \frac{9}{4}$

OP ⊥ z, passes through origin

$\therefore m_{OP} = -2 \qquad$ OP in cartesian form: $y = -2x$

$$\begin{cases} y = \frac{1}{2}x + \frac{9}{4} & (1) \\ y = 2x & (2) \end{cases}$$

$(1) - (2): \frac{5}{2}x + \frac{9}{4} = 0$

$x = \left(-\frac{9}{4}\right)\left(\frac{2}{5}\right) = -\frac{9}{10}$

$y = -\frac{9}{5}$

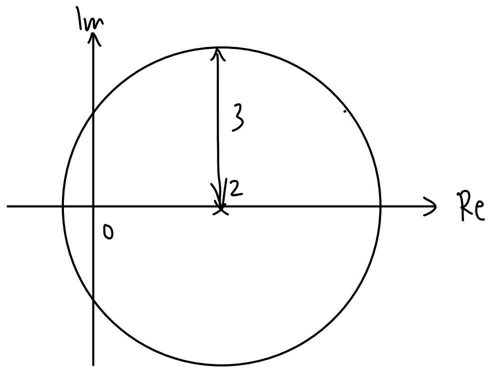
$P = -\frac{9}{10} - \frac{9}{5}i$

$|P| = \min|z| = \sqrt{\frac{81}{100} + \frac{81}{25}} = \sqrt{\frac{81}{20}} = \frac{9}{10}\sqrt{5}$

9. a) $|2-z|=3$

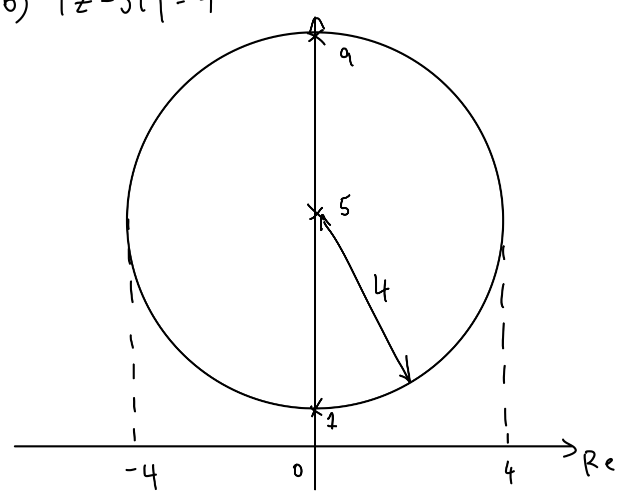
$\downarrow$

$|z-2|=3$



Cartesian: $(x-2)^2+y^2=9$ ✓

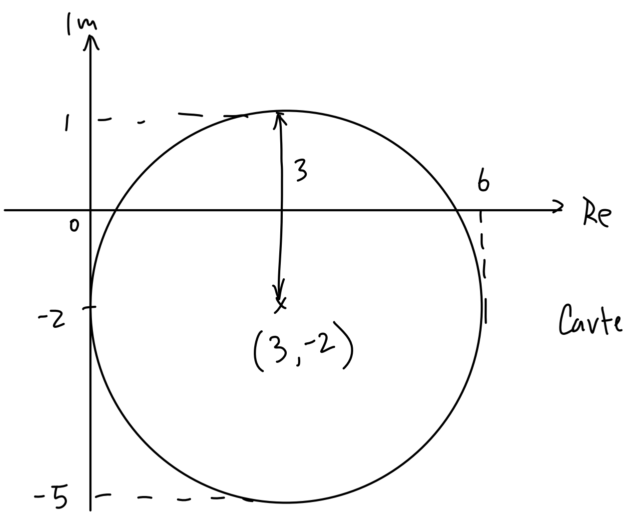
b) $|z-5i|=4$



Cartesian: $x^2+(y-5)^2=16$ ✓

c) $|z-3+2i|=3$

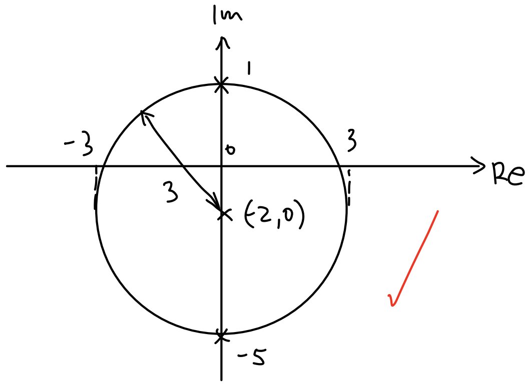
$|z-(3-2i)|=3$



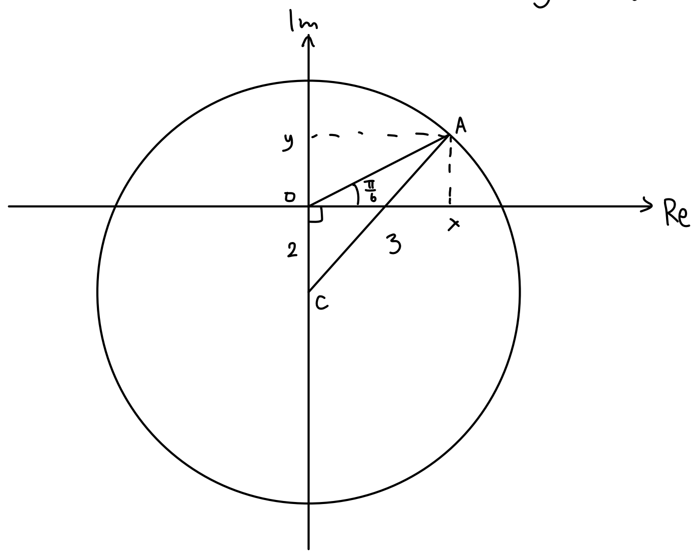
Cartesian: $(x-3)^2+(y+2)^2=9$

11. $|z+2i|=3$

a)



b) satisfies $|z+2i|=3$ and $\arg z=\frac{\pi}{6}$



A satisfies. C = centre = $(0,-2)$

Applying cosine rule: $3^2 = 2^2 + (OA)^2 - 2(2)(OA)\cos\left(\frac{\pi}{6}+\frac{\pi}{2}\right)$

$(OA)^2 - 4(OA)\cos\left(\frac{2}{3}\pi\right) - 5 = 0$

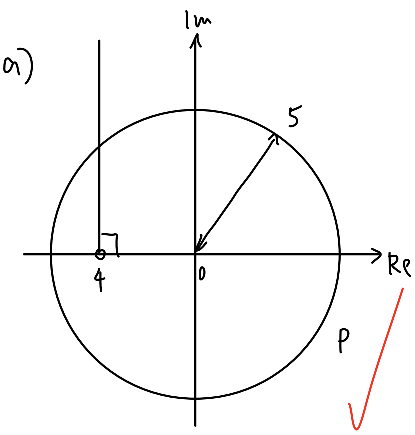
$(OA)^2 + 2(OA) - 5 = 0$

$OA = -1 \pm \sqrt{6} = -1+\sqrt{6}$ (length > 0)

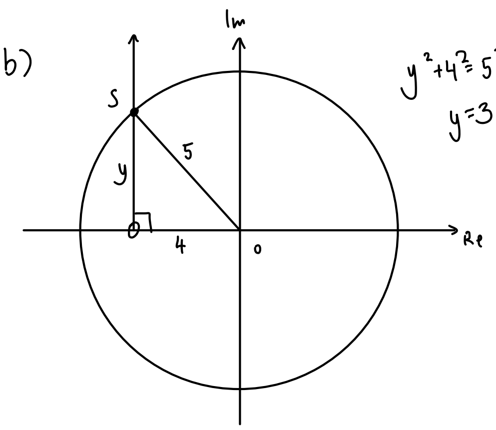
13. $P: |z| = 5$

$Q: \arg(z+4) = \frac{\pi}{2}$

a)



b)



$y^2+4^2=5^2$

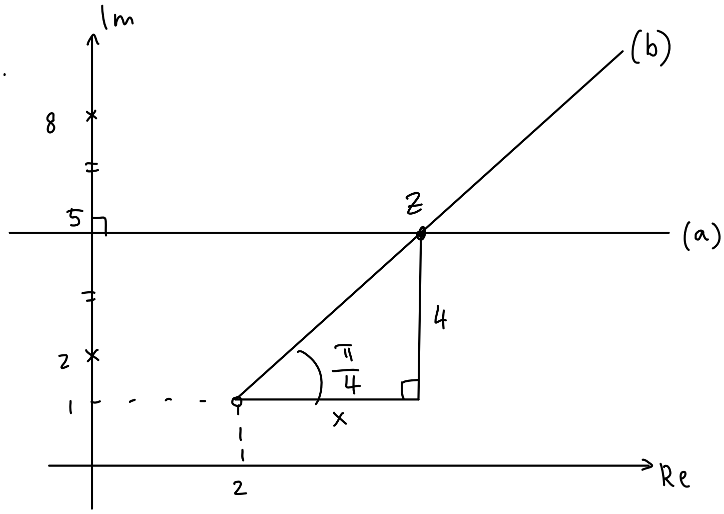
$y=3$

$S=(-4,3)$

$\downarrow$

$z=-4+3i$

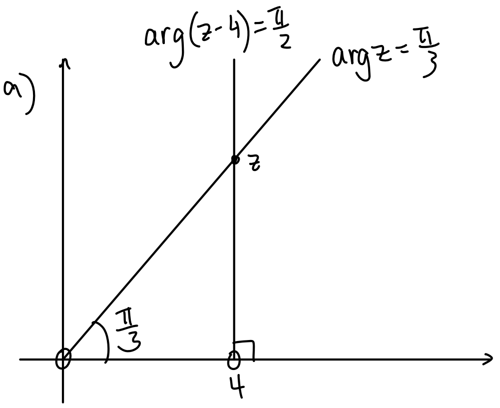
15.



c) $\tan\frac{\pi}{4} = \frac{4}{x}$ $\qquad$ $z = (4+2) + (4+1)i = 6+5i$

$x = \frac{4}{\tan\left(\frac{\pi}{4}\right)} = 4$

17. a)



$y = 4\tan\frac{\pi}{3} = 4\sqrt{3}$

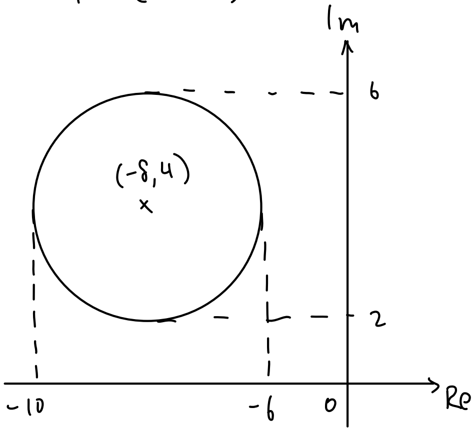
$\therefore z = 4 + 4\sqrt{3}\,i$

b) $z - 8 = -4 + 4\sqrt{3}\,i$

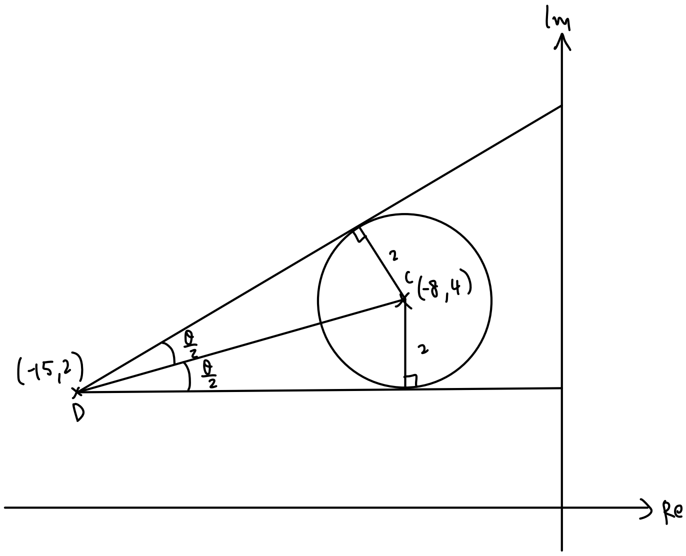
$\operatorname{Arg} z = \frac{\pi}{3}$, $\operatorname{Arg}(z-8) = \pi - \frac{\pi}{3} = \frac{2}{3}\pi$

19. a) $P: |z+8-4i| = 2$

$|z-(-8+4i)| = 2$



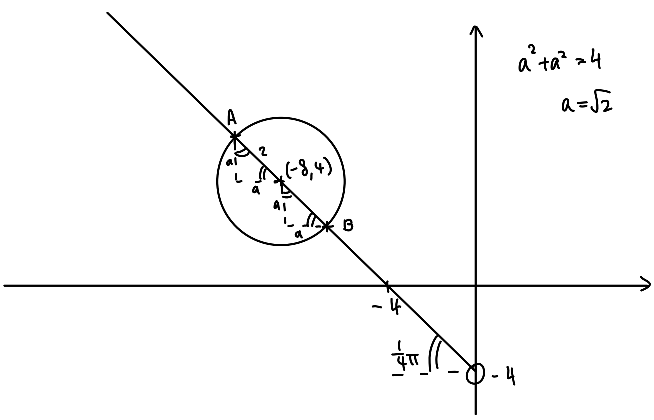
b)



$|CD| = \sqrt{2^2+7^2} = \sqrt{53}$ $\quad \sin\frac{\theta}{2} = \frac{2}{\sqrt{53}}$

$\theta = 2\arcsin\left(\frac{2}{\sqrt{53}}\right)$

c)



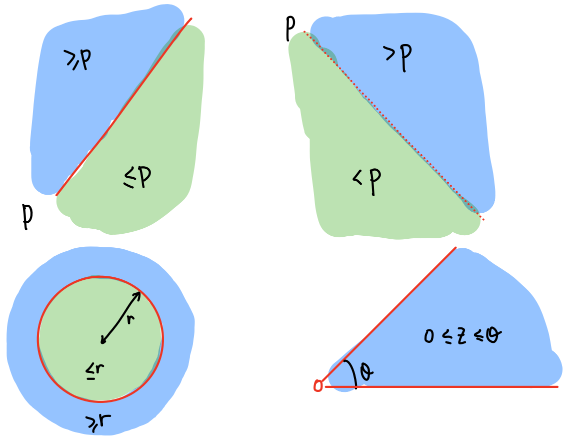
$a^2+a^2 = 4$

$a = \sqrt{2}$

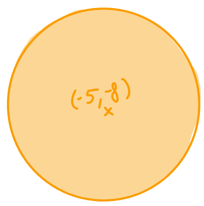
$A = (-8-\sqrt{2}, 4+\sqrt{2}) \quad B = (-8+\sqrt{2}, 4-\sqrt{2})$

$\therefore (-8-\sqrt{2}) + (4+\sqrt{2})i, \ (-8+\sqrt{2}) + (4-\sqrt{2})i$

Regions:



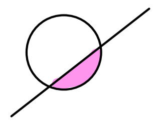
$\{\underbrace{z \in \mathbb{C}}_{\text{z is a complex number}} : \underbrace{|z+5+8i| \le 5}_{\text{Satisfies this inequality}}\}$



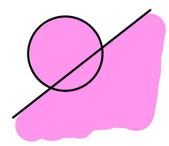
← Set notation

↓

A∩B

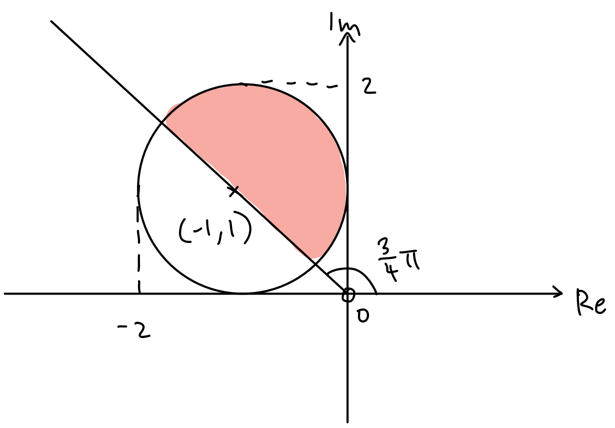


A∪B

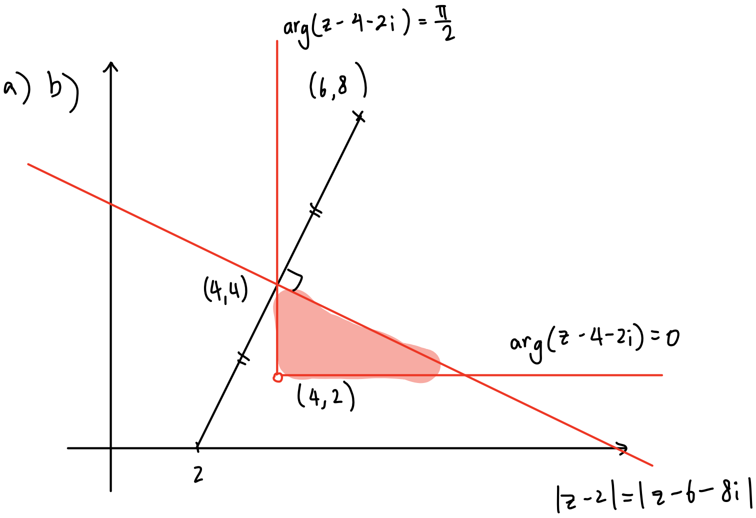


Ex 2F

3. $|z+1-i| \le 1 \qquad 0 \le \arg z \le \frac{3\pi}{4}$



5. a) b)



Mixed Ex 2 (Quiz Revision ö)

10. $z = \dfrac{a+3i}{2+ai}$

a) $a=4$ $\quad z = \dfrac{4+3i}{2+4i} = \dfrac{(4+3i)(2-4i)}{(2+4i)(2-4i)} = \dfrac{8-16i+6i+12}{4+16} = \dfrac{20-10i}{20} = 1-\frac{1}{2}i$

$|z| = \sqrt{1^2+\left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$ ✓

b) $z = \dfrac{a+3i}{2+ai} = \dfrac{(a+3i)(2-ai)}{(2+ai)(2-ai)} = \dfrac{2a-a^2i+6i+3a}{4+a^2} = \dfrac{5a}{4+a^2} + \dfrac{(-a^2+6)i}{4+a^2}$

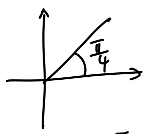
let $\arg(z) = \frac{\pi}{4}$ $\quad \tan\frac{\pi}{4} = \left(\dfrac{-a^2+6}{4+a^2}\right)\left(\dfrac{4+a^2}{5a}\right) = \dfrac{-a^2+6}{5a} = 1$

$-a^2+6 = 5a$

$a^2+5a-6=0$

$a = 1, -6$

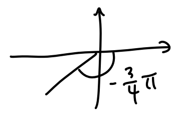
when $a=1$



$z = \frac{5}{5} + \frac{5}{5}i = 1+i$ $\quad \arg z = \frac{\pi}{4}$

$\therefore a = 1$ ✓

when $a=-6$



$z = -\frac{30}{40} - \frac{30}{40}i = -\frac{3}{4} - \frac{3}{4}i$ $\quad \arg z = -\frac{3}{4}\pi$

11. $z_1 = -1-i \quad z_2 = 1+i\sqrt{3}$

a) $|z_1| = \sqrt{2} \quad \arg(z_1) = -\frac{3}{4}\pi$

$|z_2| = \sqrt{1+3} = 2 \quad \arctan\sqrt{3} = \frac{1}{3}\pi = \arg(z_2)$

$\therefore z_1 = \sqrt{2}\left(\cos\left(-\frac{3}{4}\pi\right) + i\sin\left(-\frac{3}{4}\pi\right)\right)$ ✓

$z_2 = 2\left(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi\right)$ ✓

b) i) $|z_1 z_2| = |z_1||z_2| = 2\sqrt{2}$ ✓

ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} = \dfrac{\sqrt{2}}{2}$ ✓

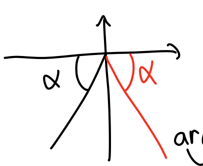
c) i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = -\frac{5}{12}\pi$ ✓

$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = -\frac{13}{12}\pi = \frac{11}{12}\pi$ ✓

12. $z = 2-2i\sqrt{3}$

a) $|z| = \sqrt{4+12} = 4$ ✓

b)



$\alpha = \arctan\left(\frac{2\sqrt{3}}{2}\right) = \frac{1}{3}\pi$

$\arg z = -\pi + \alpha = -\frac{2}{3}\pi$

$\arg z = -\alpha = -\frac{1}{3}\pi$

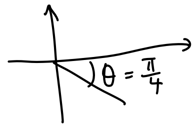
c) $w = 4\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

$\left|\dfrac{w}{z}\right| = \dfrac{|w|}{|z|} = \dfrac{4}{4} = 1$ ✓

d) $\arg\left(\dfrac{w}{z}\right) = \arg w - \arg z = -\frac{\pi}{4} - \frac{2}{3}\pi = -\frac{11}{12}\pi$

$-\frac{\pi}{4} - \left(-\frac{1}{3}\right)\pi = \frac{1}{12}\pi$

13. $4-4i$



$\arg = -\frac{\pi}{4}$ $\quad \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ $\quad \cos\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

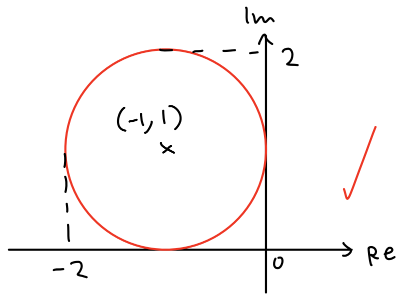
$4\left(\frac{2}{\sqrt{2}}\right)\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$ ✓

14. $P: |z+1-i| = 1$

a) centre $(-1, 1)$ radius 1

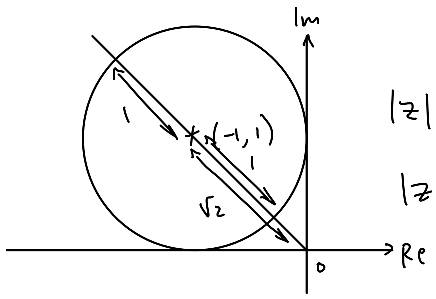
$(x+1)^2 + (y-1)^2 = 1$ ✓

b)



✓

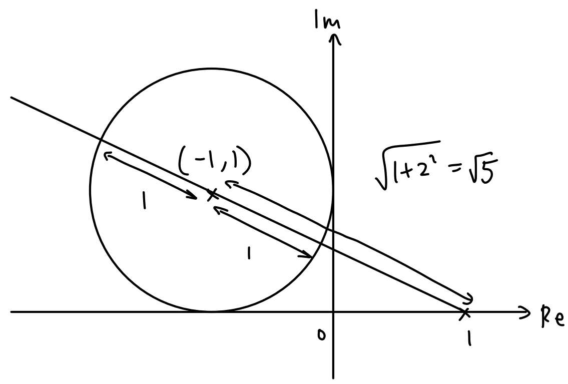
c)



$|z|_{max} = \sqrt{2} + 1$

$|z|_{min} = \sqrt{2} - 1$ ✓

d)



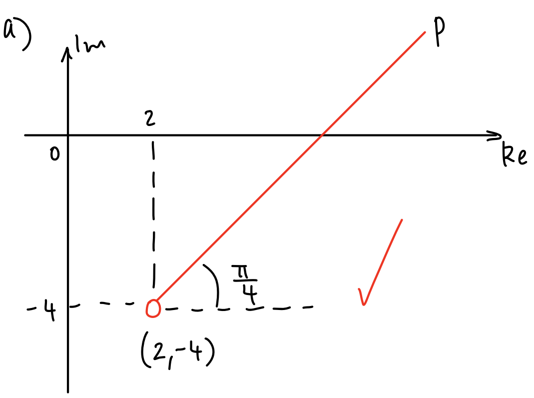
$\sqrt{1+2^2} = \sqrt{5}$

$|z-1|_{min} = \sqrt{5} - 1$

$|z-1|_{max} = \sqrt{5} + 1$ ✓

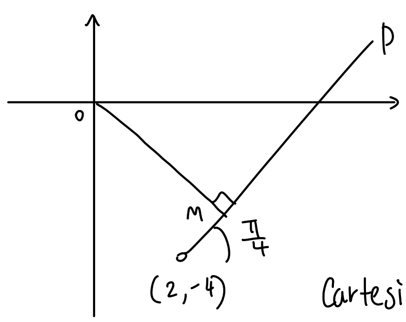
15. $\arg(z-2+4i) = \frac{\pi}{4}$

a)



✓

b)



Cartesian of P:

$y + 4 = x - 2$

$y = x - 6$

Cartesian of OM:

$y = -x$

$\begin{cases} y = x - 6 & — ① \\ y = -x & — ② \end{cases}$

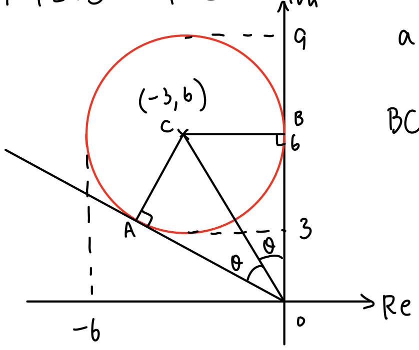
① + ②: $2y = -6$

$y = -3 \quad x = 3$

$M(3, -3)$

$|z|_{min} = |OM| = \sqrt{9+9} = 3\sqrt{2}$ ✓

16. $|z+3-6i|=3$



$\arg A = (\arg z)_{max}$

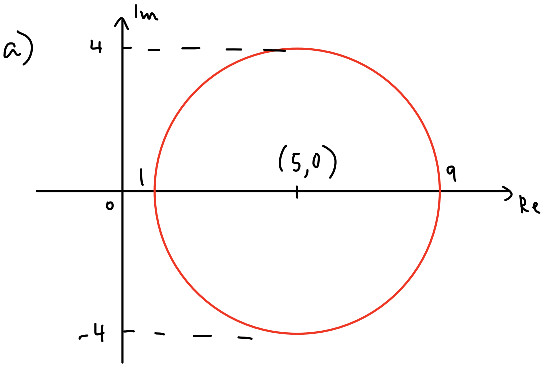
$BC=3 \quad OC=\sqrt{9+36}=3\sqrt{5}$

$\theta = \arcsin\left(\frac{3}{3\sqrt{5}}\right) = \arcsin\left(\frac{1}{\sqrt{5}}\right)$

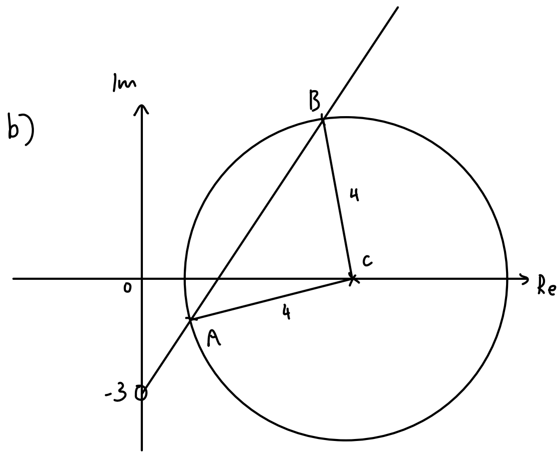
$(\arg z)_{max} = \frac{\pi}{2} + 2\arcsin\left(\frac{1}{\sqrt{5}}\right)$ ✓

17. $p: |z-5|=4$

a)



b)



gradient $= \tan\frac{\pi}{3} = \sqrt{3}$ line: $y=\sqrt{3}x-3$

circle: $(x-5)^2+y^2=4^2$

$x^2-10x+25+3x^2-6\sqrt{3}x+9=16$

$4x^2-(10+6\sqrt{3})x+18=0$

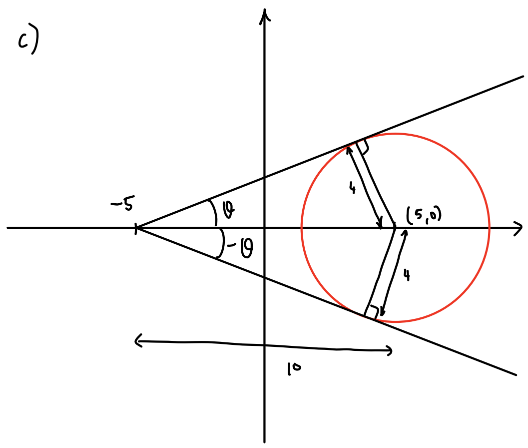
$2x^2-(5+3\sqrt{3})x+9=0$

$x=\frac{5+3\sqrt{3}\pm\sqrt{52+30\sqrt{3}-72}}{4}$

$\approx 3.96,\ 1.14$

$y \approx 3.86,\ -1.03$ $\therefore$

c)



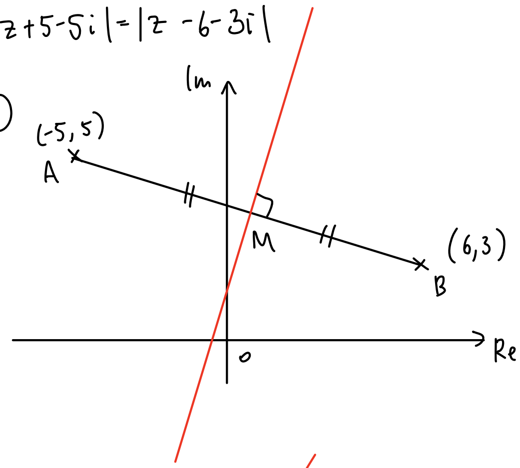
$\theta = \arcsin\frac{4}{10} = 0.412$

If $\arg(z+5)=\theta$ has no solutions with circle,

$0.412<\theta<\pi$ ✓, $-\pi<\theta<-0.412$ ✓

18. $|z+5-5i| = |z-6-3i|$

a)



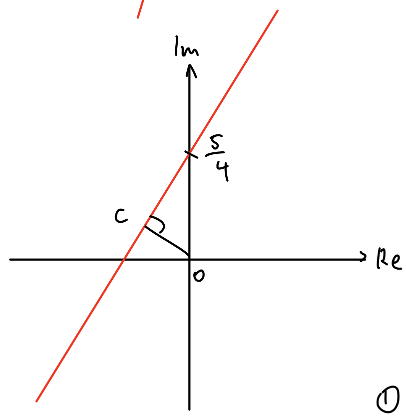
b) $M\left(\frac{6-5}{2}, \frac{5+3}{2}\right) = M\left(\frac{1}{2}, 4\right)$

$m_{AB} = \frac{-2}{11}$ $\quad m_{\perp} = \frac{11}{2}$

$y - 4 = \frac{11}{2}\left(x - \frac{1}{2}\right)$

$y = \frac{11}{2}x + \frac{5}{4}$

c)



$m_{OC} = -\frac{2}{11}$ $\quad y = -\frac{2}{11}x$

$\begin{cases} y = \frac{11}{2}x + \frac{5}{4} & — ① \\ y = -\frac{2}{11}x & — ② \end{cases}$

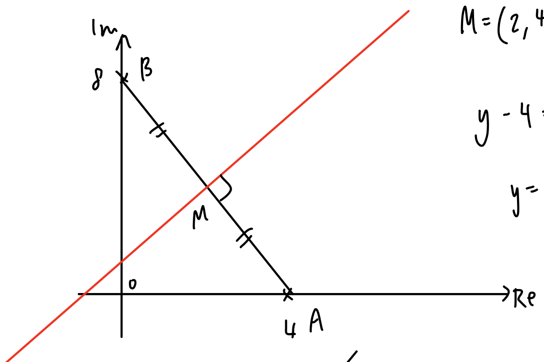
① - ②: $\frac{125}{22}x + \frac{5}{4} = 0$

$x = -\frac{11}{50}$ $\quad y = \frac{1}{25}$

$\therefore C\left(-\frac{11}{50}, \frac{1}{25}\right)$

$|z|_{min} = |OC| = \sqrt{\left(\frac{11}{50}\right)^2 + \left(\frac{1}{25}\right)^2} = \frac{\sqrt{5}}{10}$ ✓

19. a) $|z-4| = |z-8i|$

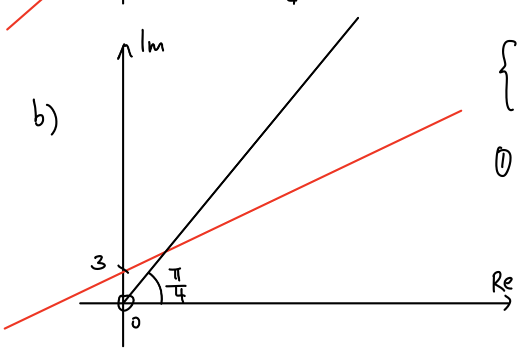


$M = (2,4)$ $\quad m_{AB} = -\frac{8}{4} = -2$ $\quad m_{\perp} = \frac{1}{2}$

$y - 4 = \frac{1}{2}(x-2)$

$y = \frac{1}{2}x + 3$ ✓

b)



$\begin{cases} y = x & — ① \\ y = \frac{1}{2}x + 3 & — ② \end{cases}$

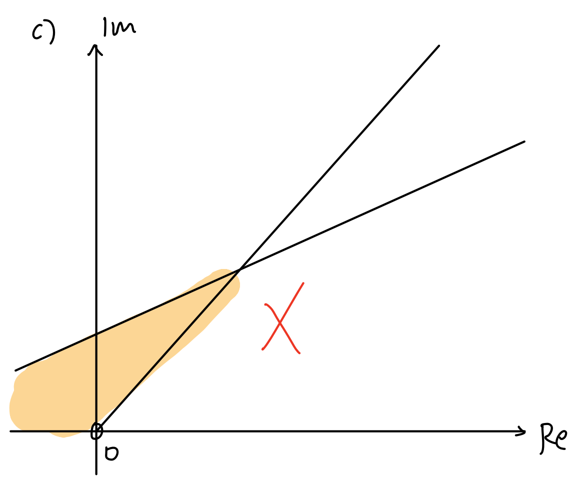
① - ②: $\frac{1}{2}x - 3 = 0$

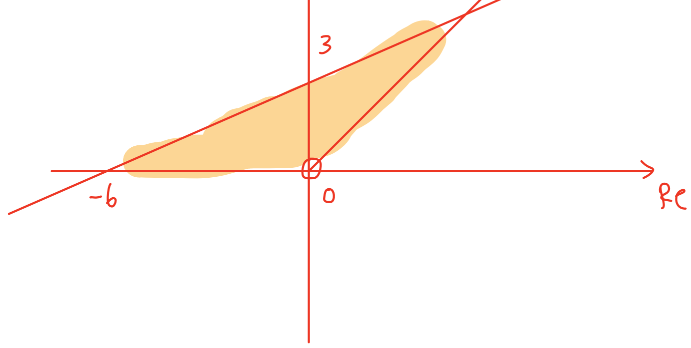
$x = 6$

$y = 6$

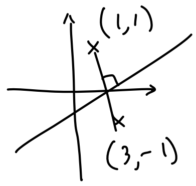
$\therefore z = 6 + 6i$ ✓

c)





20. a) i) $|z-3+i| = |z-1-i|$



midpoint $M = \left(\frac{3+1}{2}, 0\right) = M(2,0)$

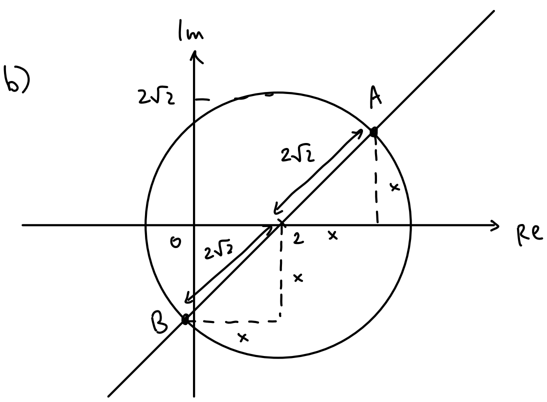
gradient $= \frac{2}{-2} = -1$ $\perp$ gradient $= 1$

$\therefore y = x - 2$ ✓

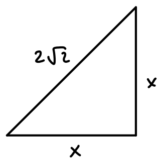
ii) $|z-2| = 2\sqrt{2}$ centre $(2,0)$ radius $2\sqrt{2}$

$(x-2)^2 + y^2 = 8$

b)



gradient $= 1$ $\therefore$



$2x^2 = 8$

$x = 2$

$\therefore A(4,2)$

$B(0,-2)$

$\therefore z = 4+2i, -2i$ ✓

c)