Ch6 Circles

6.1: Midpoints & $\perp$ bisectors

The midpoint is the value halfway between 2 points

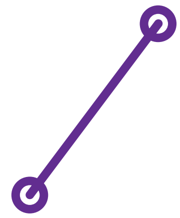
$M(x_m, y_m)$ is the midpoint between $P_1(x_1, y_1)$ & $P_2(x_2, y_2)$.

$$x_m = \frac{x_1 + x_2}{2} \qquad y_m = \frac{y_1 + y_2}{2}$$

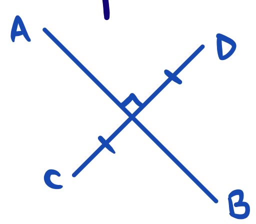
Average of x/y values

Line segment

finite part of a line with 2 distinct end points



The perpendicular bisector of a line segment AB will pass through the line AB at its midpoint and be at 90° to the line AB



AB is the perpendicular bisector of CD

Line PQ is the diameter of a circle

Midpoint of PQ is the center of the circle

Center = (2, −2)   P(8, −5)   what is Q?

$$\frac{8 + x_1}{2} = 2 \qquad \frac{-5 + y_1}{2} = -2 \qquad \therefore Q(-4, 1)$$

$$x_1 = -4 \qquad y_1 = 1$$

$\perp$ Bisector of PQ?

$$m_{PQ} = \frac{8-(-4)}{-5-1} = \frac{12}{-6} = -2 \qquad \therefore m = \frac{1}{2}$$

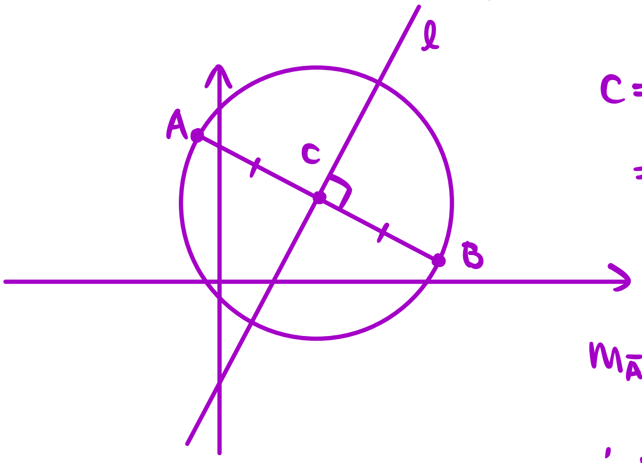
Sub $y = -2$, $x = 2$, $m = \frac{1}{2}$

$$-2 = 1 + c$$
$$c = 3$$

$\therefore$ the $\perp$ bisector of PQ is $y = \frac{1}{2}x + 3$

$\overline{AB}$ (line segment AB) is Ø of a circle of center C

A(-1,4), B(5,2) line ℓ passes through point C and is ⊥ to $\overline{AB}$



C = midpoint of $\overline{AB}$

$= \left(\frac{-1+5}{2}, \frac{4+2}{2}\right) = (2,3)$

$m_{\overline{AB}} = \frac{4-2}{-1-5} = \frac{2}{-6} = -\frac{1}{3}$

$\therefore m_\ell = 3$

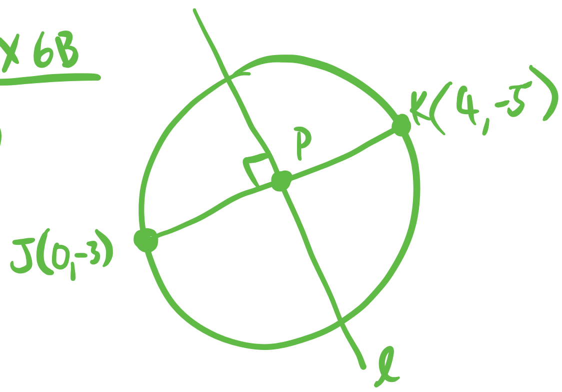
$\therefore$ ℓ passes through (2,3) & gradient 3

$3 = 2(3) + c$

$c = -3$ $\quad \therefore y = 3x - 3$

Ex 6B

3



$m_{JK} = \frac{-3-(-5)}{0-4} = -\frac{2}{4} = -\frac{1}{2}$

$m_\ell = 2$

$P\left(\frac{4+0}{2}, \frac{-5-3}{2}\right) = P(2,-4)$

$-4 = 4 + c$

$c = -8$ $\quad \therefore \ell : y = 2x - 8$

5 X(7,-2), Y(4,q)

$(y = 4x + b) \perp \overline{XY}$

$m_{xy} = \frac{q+2}{-3} = -\frac{1}{3}q - \frac{2}{3} = -\frac{1}{4}$

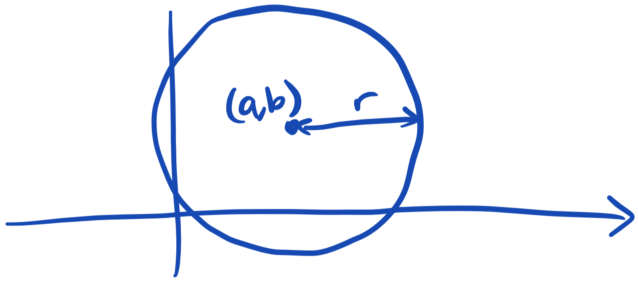
$-4q - 8 = -3$

$4q = -5$

$q = -\frac{5}{4}$

Equation of a circle

Important equations in this color



$(x-a)^2+(y-b)^2=r^2$ Eqn of circle with center (a,b) & radius r.

$x^2-2ax+a^2+y^2-2by+b^2-r^2=0$

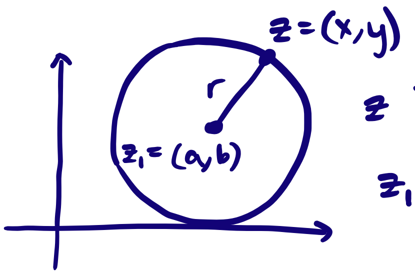
let $f=-a$, $g=-b$, $c^2=a^2+b^2-r^2$

$x^2+y^2+2fx+2gy+c^2=0$

and $c^2=a^2+b^2-r^2$

$c^2=f^2+g^2-r^2$

$r=\sqrt{f^2+g^2-c^2}$



z is general (on circle)

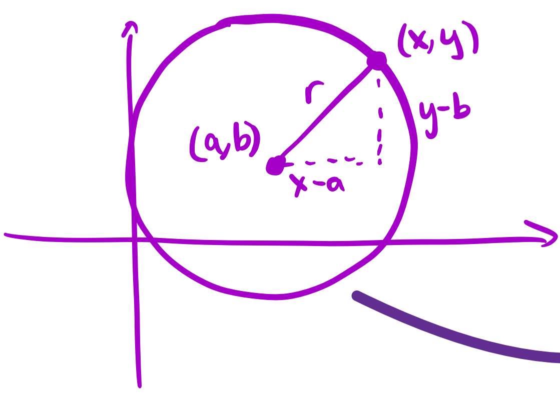
z_1 is specific

Consider $|z-z_1|$ = distance between (x,y) and (a,b)

Then, $\underbrace{|z-z_1|}=r$, $r\in\mathbb{R}$

when z is always r units away from z_1, the shape is a circle

$$r=\sqrt{(x-a)^2+(y-b)^2}$$



EX 6C (p. 119)

9 a) $x^2+y^2-10x+4y-20=0$

$(x-5)^2-25+(y+2)^2-4-20=0$

$(x-5)^2+(y+2)^2=49$

b) centre $=(5,-2)$ $\quad r=\sqrt{49}=7$

10 c) $x^2+y^2-6y-22x+40=0$

$(x-3)^2-9+(y-11)^2-121+40=0$

$(x-3)^2+(y-11)^2=90$

centre $=(3,11)$ $\quad r=\sqrt{90}=3\sqrt{10}$

EX 6D (p. 122)

5 $x-y-10=0$

$y=x-10$

$x^2-4x+x^2-20x+100=21$

$2x^2-24x+79=0$

$\frac{\Delta}{4}=144-158<0$

$\therefore$ Not intersecting

10 a) $x^2-10x+k^2x^2-12kx+57$

$=(k^2+1)x^2-(12k+10)x+57=0$

$\Delta=144k^2+240k+100-228k^2-228>0$

$84k^2-240k+128>0$

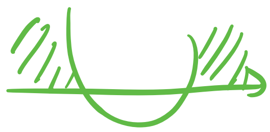
$21k^2-60k+32>0$

b) $21k^2-60k+32=0$

$k=\frac{60\pm\sqrt{3600-2688}}{42}$

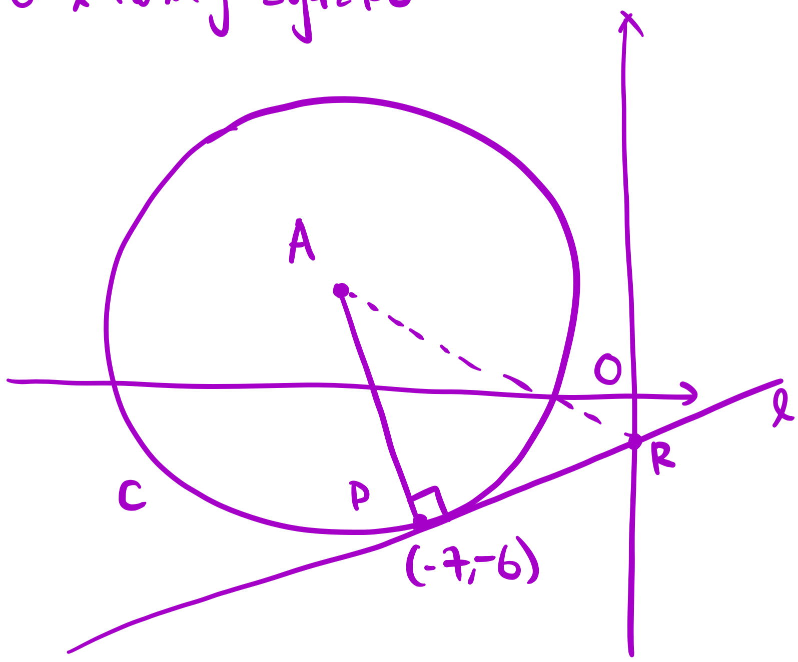
$=2.15,\ 0.71$

$\therefore k<0.71,\ k>2.15$



EX 6E (p.126)

(5) Circle C $x^2+18x+y^2-2y+29=0$



a) $49-126+36+12+29=0$

b) $2f=18$ $\quad 2g=-2$ $\quad \therefore A(-9,1)$

$a=-9$ $\quad b=1$

$m_{\overline{AP}}=\frac{7}{-2}=-\frac{7}{2}$ $\quad m_l=\frac{2}{7}$ $\quad -6=-2+c$

$c=-4$

$\therefore l: y=\frac{2}{7}x-4$

c) $R=C=-4$

d) $\frac{1}{2}\left|\begin{matrix}-9 & -7 & 0 & -9 \\ 1 & -6 & -4 & 1\end{matrix}\right|$

$=\frac{1}{2}\left|(54+28+0)-(-7+0+36)\right|$

$=\frac{1}{2}(53)=26.5$

(8) a) $x^2-6x+9+y^2-2py+p^2=5$ $\quad y=-2x+5$

$x^2-6x+9+4x^2-20x+25+4px-10p+p^2=5$

$5x^2+(4p-26)x+p^2-10p+39=0$

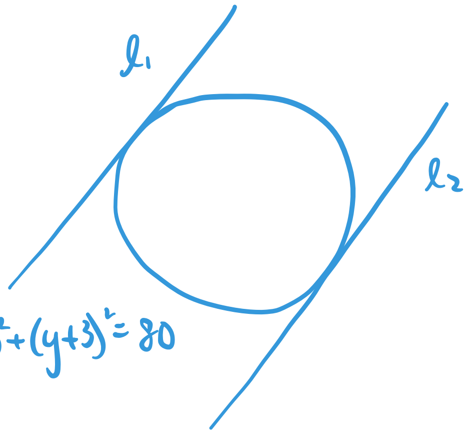
$\Delta=16p^2+208p+676-4p^2+40p-156=0$

$12p^2+248p+520=0$

$3p^2+62p+130=0$

EX 6E (p.126)

7



$(x+5)^2+(y+3)^2=80$

$m_{l_1}=m_{l_2}=2$

Sub $y=2x+c$ into eqn of circle

$(x+5)^2+(2x+(3+c))^2=80$

$x^2+10x+25+4x^2+12x+4c+9+6c+c^2=80$

$5x^2+(22+4c)x+c^2+6c-46=0$

$\Delta=(22+4c)^2-20(c^2+6c-46)$

$=484+176c+16c^2-20c^2-120c+920$

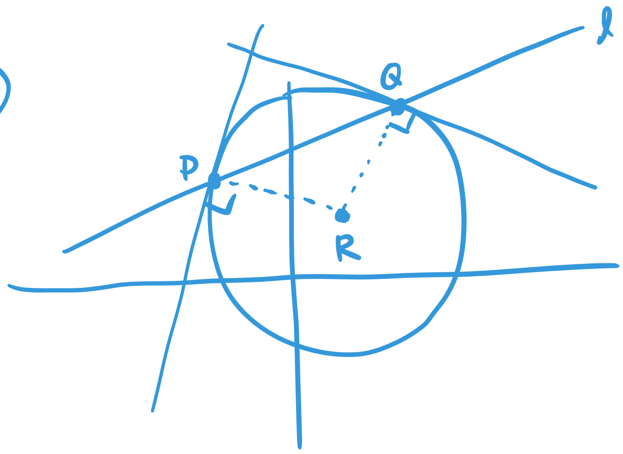
$=-4c^2+56c+1404=0$

$c=27,-13$

$\therefore l_1: y=2x+27$

$l_2: y=2x-13$

11



Circle has eqn $x^2-4x+y^2-6y=7$

l has eqn $x-3y+17=0$

a) Find P & Q $\quad x=3y-17$

$(3y-17)^2-4(3y-17)+y^2-6y=7$

$9y^2-102y+289-12y+68+y^2-6y=7$

$10y^2-120y+350=0$

$5y^2-60y+175=0$

$y=7,5$

$x=4,-2$

$\therefore P(-2,5)$ & $Q(4,7)$

b) find the eqn of the tangents the the circle at P & Q

centre of circle $(x-2)^2+(y-3)^2=7+4+9$

$(x-2)^2+(y-3)^2=20$

$\therefore R(2,3)$

$m_{RP}=\frac{2}{-4}=-\frac{1}{2}$ $\quad m_{RQ}=\frac{4}{2}=2$

$m_P=-(-\frac{1}{2})^{-1}=2$ $\quad m_Q=-(2)^{-1}=-\frac{1}{2}$

tan at P

$5=-4+c$

$c=9$

$\therefore y=2x+9$

tan at Q

$7=-2+c$

$c=9$

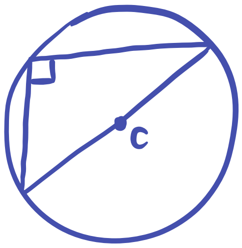
$y=-\frac{1}{2}x+9$

c) find $\perp$ bisector of $\overline{PQ}$

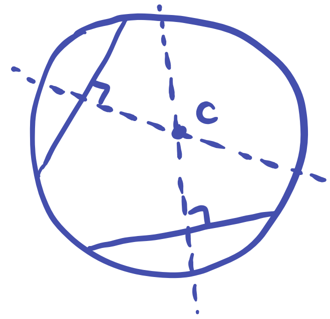
$m_{PQ}=\frac{2}{6}=\frac{1}{3}$ $\quad m_\perp=-3$ $\quad$ passes through $(2,3)$

$3=-6+c$

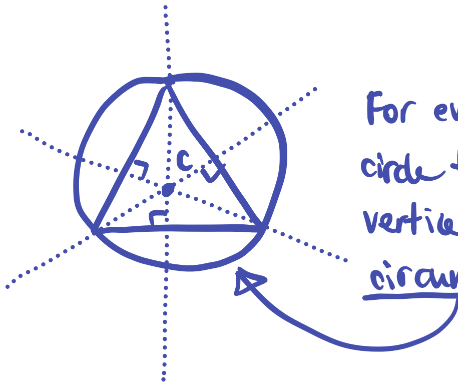
$c=9$ $\quad \therefore y=-3x+9$



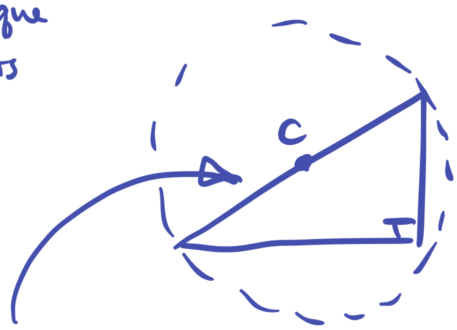
2 lines extended from diameter meeting on circumfrence form a 90° angle



$\perp$ bisector of chords intersect at centor



For every Δ there exists a unique circle that passes through all its vertices. This is called a circumcircle



hypotenuse of ∟Δ is the diameter of its circumcircle

EX 6E

(6) $A(3,15)$ $B(-13,3)$ $C(-7,-5)$ $D(8,0)$

a) Show ΔABC form a ∟Δ

$|AB| = \sqrt{16^2 + 12^2} = 20$

$|BC| = \sqrt{6^2 + 8^2} = 10$

$|AC| = \sqrt{10^2 + 20^2} = 10\sqrt{5}$

$|AB|^2 + |BC|^2 = 400 + 100$

$= 500 = |AC|^2$

$\therefore \Delta ABC$ is a ∟Δ

b) eqn of its circumcircle

$\overline{AC}$ is the diameter of this circle

Midpoint ie center: $\left(\frac{3-7}{2}, \frac{15-5}{2}\right) = (-2, 5)$ radius $= \frac{\text{diameter}}{2} = \frac{10\sqrt{5}}{2} = 5\sqrt{5}$

$(x+2)^2 + (y-5)^2 = 125$

c) Show A, B, C and D lie on circle

A, B, C lie on circle $\because$ circumcircle of ΔABC

$100 + 25 = 125$ $\therefore$ LHS = RHS

$\therefore$ D lies on circle

Mixed Ex 6 (p. 132)

1. a) $C =$ midpoint of $\overline{QR} = \left(\frac{11-5}{2}, \frac{12+0}{2}\right) = (3,6)$

   b) $|\overline{QR}| = \sqrt{16^2+12^2} = 20 \quad \therefore$ radius $= 20/2 = 10$

   c) $(x-3)^2+(y-6)^2 = 100$

   d) $(13-3)^2+(6-6)^2 = 100+0 = 100$

   $\because$ LHS = RHS $\therefore$ P lies on circle

2. circle center $= (5,-2)$ radius $= \sqrt{30}$

   distance from $(0,0)$ to $(5,-2) = \sqrt{5^2+2^2} = \sqrt{29} < \sqrt{30}$

   $\because$ distance < radius $\therefore (0,0)$ lies in circle

3. a) $x^2+y^2+8y=-7$ $\quad\therefore$ centre $= (0,-4)$

   $(x+0)^2+(y+4)^2 = 9$ $\quad$ radius $= 3$

   b) sub $x=0$ $\quad y^2+6y = -2y-7$

   $y^2+8y+7=0$

   $y = -1, -7$ $\quad\therefore (0,-1), (0,-7)$

   c) sub $y=0$ $\quad x^2+3x-3x+7=0$

   $x^2+7=0$

   $\Delta = -28 < 0$ $\therefore$ No intersections

4. a) $P=(8,8)$ $\quad (8+1)^2+(8-3)^2 = 81+25 = 106$ $\quad\because$ LHS=RHS $\therefore$ P lies on circle

   b) $Q=(-1,3)$ $\quad |PQ| = \sqrt{9^2+5^2} = \sqrt{106}$

5. a) $\underbrace{A: 1+0=1 \qquad B: \tfrac{1}{4}+\tfrac{3}{4}=1 \qquad C: \tfrac{1}{4}+\tfrac{3}{4}=1}$

   $\because$ LHS=RHS $\therefore$ Points lie on circle

   b) $|AB| = \sqrt{(3/2)^2+(\sqrt{3}/2)^2} = \sqrt{3}$

   $|BC| = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

   $|AC| = \sqrt{(3/2)^2+(\sqrt{3}/2)^2} = \sqrt{3}$

   $\because |AB| = |BC| = |AC|$

   $\therefore \Delta ABC$ is equilateral

6 a) $(3-k)^2+(0-3k)^2 = 9-6k+k^2+9k^2 = 10k^2-6k+9=13$ $\begin{pmatrix}5k & +2\\ k & -1\end{pmatrix}$

$10k^2-6k-4=0$

$5k^2-3k-2=0$

$k=-5/2,\ 1$

b) $k=1$ $(x-1)^2+(y-3)^2=13$

7 $y=3x-9$ $x^2+px+9x^2-54x+81+12x-36=20$

$10x^2-(42-p)x+25=0$

$\Delta = 1764-84p+p^2-1000 = p^2-84p+764<0$

if $p^2-84p+764=0$, $\therefore 42-10\sqrt{10} < p < 42+10\sqrt{10}$

$p=42\pm10\sqrt{10}$

8 x intercept $=(4,0)$ y intercept $=(0,-8)$

$A(4,0)$ $B(0,-8)$

$|AB| = \sqrt{4^2+8^2} = 4\sqrt{5}$ $r=\frac{4\sqrt{5}}{2}=2\sqrt{5}$ $r^2=20$

center $=(4/2, -8/2)=(2,-4)$ $\therefore (x-2)^2+(y+4)^2=20$

9 a) $(x-8)^2+(y-10)^2=r^2$

$(4-8)^2+(0-10)^2 = 16+100 = r^2$

$r=\sqrt{116}=2\sqrt{29}$

b) $(a-8)^2+(0-10)^2=(a-8)^2+100=116$

$a-8=\pm\sqrt{16}$

$a=8\pm4=12,4$ $\therefore a=12$

10 $(x-5)^2+0=36$ $\therefore P(11,0),\ Q(-1,0)$

$x-5=\pm\sqrt{36}$

$x=5\pm6 = 11, -1$

11) $(0+4)^2+(y-7)^2=121$

$y-7=\pm\sqrt{105}$

$y=7\pm\sqrt{105}$ $\quad m=7+\sqrt{105}\,,\ n=7-\sqrt{105}$

12) a) $(a+5)^2+2^2=125$

$a+5=\pm\sqrt{121}$

$a=5\pm 11=16,-6=16$

$(0+5)^2+(b+2)^2=125$

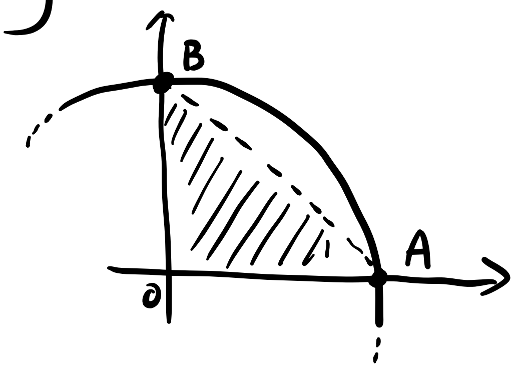
$b+2=\pm\sqrt{100}$

$b=2\pm 10=-8,12=12$

Rej. −ve #s

b) $A(16,0)$ $\quad B(0,12)$ $\quad m_{AB}=-\frac{12}{16}=-\frac{3}{4}$

$\therefore y=-\frac{3}{4}x+12$



c) Area $=\frac{1}{2}bh=\frac{1}{2}(16)(12)=96$ units squared

13) a) $(x-p)^2+(y-q)^2=625$

$(-7,0):\ (-7-p)^2+(-q)^2=625$

$p^2+14p+49+q^2=625$

$p^2+14p-576=-q^2$

$(7,0):\ (7-p)^2+(-q)^2=625$

$p^2-14p+49+q^2=625$

$p^2-14p-576=-q^2$

$p^2+14p-576=p^2-14p-576$

$28p=0$

$p=0$

$\therefore q=\sqrt{576}=24$ (rej. −ve)

b) $x^2+(y-24)^2=625$

$y-24=\pm 25$

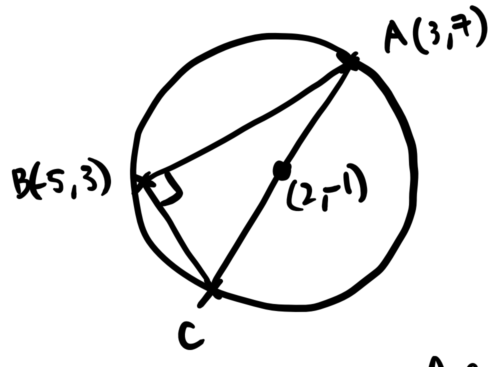
$y=49,-1$ $\quad \therefore (0,49),(0,-1)$

(14) $C(5,1)$ $\quad m_{AC} = \frac{-8}{-8} = 1 \qquad m_{\perp} = -1$

$-7 = 3 + c$

$c = 10 \qquad \therefore y = -x + 10$

(15)



$C = (3 - 2(1), 7 - 2(7+1))$

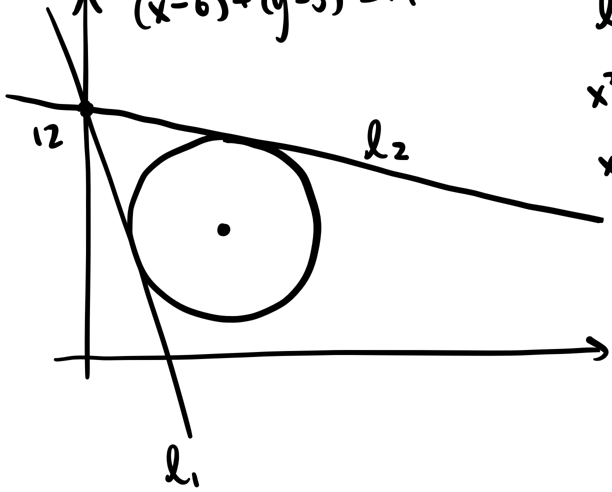
$= (1, -9)$

$|AB| = \sqrt{8^2 + 4^2} = 4\sqrt{5}$

$|BC| = \sqrt{6^2 + 12^2} = 6\sqrt{5}$

Area $= \frac{1}{2}bh = \frac{1}{2}(4\sqrt{5})(6\sqrt{5}) = 60$ units squared

(16) $(x-6)^2 + (y-5)^2 = 17$



$l_1 \;\&\; l_2 : y = mx + 12$

$x^2 - 12x + 36 + y^2 - 10y + 25 = 17$

$x^2 - 12x + 36 + m^2x^2 + 24mx + 144 - 10mx - 120 + 25 = 17$

$m^2x^2 + x^2 + 14mx - 12x + 68 = 0$

$(m^2+1)x^2 + (14m - 12)x + 68 = 0$

$\Delta = 196m^2 - 336m + 144 - 272m^2 - 272$

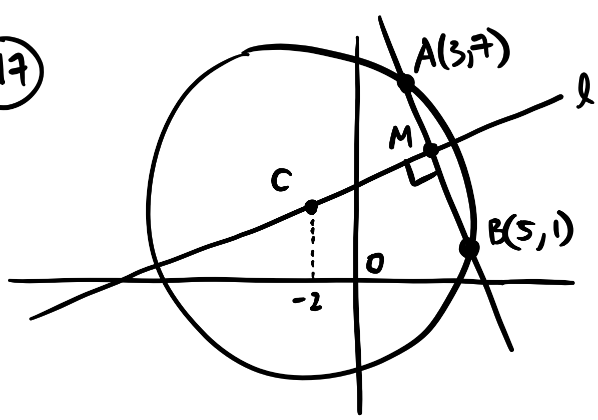
$= -76m^2 - 336m - 128$

$= -19m^2 - 84m - 32 = 0$

$m = -\frac{8}{19}, -4$

$\therefore l_1 : y = -4x + 12 \qquad l_2 : y = -\frac{8}{19}x + 12$

(17)



a) $M = \left(\frac{5+3}{2}, \frac{7+1}{2}\right) = (4, 4)$

$m_{AB} = \frac{6}{-2} = -3 \qquad 4 = \frac{4}{3} + c$

$m_l = \frac{1}{3} \qquad c = \frac{8}{3} \qquad \therefore y = \frac{1}{3}x + \frac{8}{3}$

b) $(x+2)^2 + (y-a)^2 = r^2$

$(3,7): 25 + a^2 - 14a + 49 = r^2$

$(5,1): 49 + a^2 - 2a + 1 = r^2$

$25 + a^2 - 14a + 49 = 49 + a^2 - 2a + 1$

$12a = 24$

$a = 2 \qquad r = \sqrt{50} \qquad \therefore (x+2)^2 + (y-2)^2 = 50$

part c) on next page ☺

c) Shoelace formula $\frac{1}{2}\left|\begin{matrix}3\\7\end{matrix}\times\begin{matrix}5\\-1\end{matrix}\times\begin{matrix}-2\\2\end{matrix}\times\begin{matrix}3\\7\end{matrix}\right| = \frac{1}{2}\left|(-3+10-14)-(35+2+6)\right|$

$= 25$ units squared