Series (again!)

Method of differences

$r=1$: $a-b$
$r=2$: $b-c$
$r=3$: $c-d$
$\vdots \qquad \vdots$
$r=n$: $x-y$

If we add all the expressions, the terms cancel.

Just add the terms together to find terms that cancel.

Example ①

$$r^2(r+1)^2-(r-1)^2r^2 \equiv 4r^3$$

$$4\sum_{r=1}^{n} r^3 = \sum_{r=1}^{n}\left(r^2(r+1)^2-(r-1)^2r^2\right)$$

$r=1$: $1^2(2)^2-(0)^2 1^2$

$r=2$: $2^2(3)^2-(1)^2 2^2$ all terms cancel

$r=3$: $3^2(4)^2-(2)^2 3^2$

$\vdots \qquad \vdots$

$r=n$: $n^2(n+1)^2-(n-1)^2n^2$

Sum of terms $= n^2(n+1)^2 = 4\sum_{r=1}^{n} r^3$

$$\therefore \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Ex 2A

1 a) $\frac{1}{2}\left(r(r+1)-r(r-1)\right)=\frac{1}{2}\left(r^2+r-r^2+r\right)=r$

b) $\sum_{r=1}^{n} r = \frac{1}{2}\sum_{r=1}^{n}\left(r(r+1)-r(r-1)\right)$

$r=1:\ 1\cancel{(2)}-1(0)$

$r=2:\ \cancel{2(3)}-\cancel{2(1)}$

$r=3:\ \cancel{3(4)}-\cancel{3(2)}$

$\vdots \qquad \vdots$

$r=n:\ n(n+1)-\cancel{n(n-1)}$

Sum of terms $= n(n+1)$

$\therefore \sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

3 a) $\frac{1}{r(r+2)} \equiv \frac{A}{r}+\frac{Br+C}{r+2} = \frac{Ar+2A+Br^2+Cr}{r(r+2)} = \frac{Br^2+(A+C)r+2A}{r(r+2)}$

$2A=1 \qquad A+C=0$

$A=\frac{1}{2} \qquad C=-\frac{1}{2}$

$\therefore \frac{1}{r(r+2)}=\frac{1}{2r}-\frac{1}{2(r+2)}$

b) $\sum_{r=1}^{n}\frac{1}{r(r+2)} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+2}\right)$

$r=1:\ 1-\cancel{\frac{1}{3}}$

$r=2:\ \frac{1}{2}-\cancel{\frac{1}{4}}$

$r=3:\ \cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}$

$r=4:\ \cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}$

$\vdots$

$r=n-1:\ \cancel{\frac{1}{n-1}}-\frac{1}{n+1}$

$r=n:\ \cancel{\frac{1}{n}}-\frac{1}{n+2}$

Sum of series $=1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$

$=\frac{3}{2}-\frac{2n+3}{(n+1)(n+2)}=\frac{3n^2+9n+6-4n-6}{2(n+1)(n+2)}$

$=\frac{n(3n+5)}{2(n+1)(n+2)}$

$\therefore \sum_{r=1}^{n}\frac{1}{r(r+2)}=\frac{1}{2}\left(\frac{n(3n+5)}{2(n+1)(n+2)}\right)$

$=\frac{n(3n+5)}{4(n+1)(n+2)}$

5 a) $\frac{1}{r!} - \frac{1}{(r+1)!} = \frac{r+1}{(r+1)!} - \frac{1}{(r+1)!} = \frac{r}{(r+1)!}$

b) $\sum_{r=1}^{n} \frac{r}{(r+1)!} = \sum_{r=1}^{n} \left(\frac{1}{r!} - \frac{1}{(r+1)!} \right)$

$r=1: \frac{1}{1} - \cancel{\frac{1}{2!}}$

$r=2: \cancel{\frac{1}{2!}} - \cancel{\frac{1}{3!}}$

$r=3: \cancel{\frac{1}{3!}} - \cancel{\frac{1}{4!}}$

$\vdots$

$r=n: \cancel{\frac{1}{n!}} - \frac{1}{(n+1)!}$

Sum of terms: $1 - \frac{1}{(n+1)!}$

$\therefore \sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$

7 a) $\frac{1}{(2r+3)(2r+5)} = \frac{A}{2r+3} + \frac{B}{2r+5} = \frac{(2A+2B)r + 5A+3B}{(2r+3)(2r+5)}$

$\begin{cases} 2A+2B=0 & — ① \\ 5A+3B=1 & — ② \end{cases}$

$3\times①: 6A+6B=0 \quad — ①'$

$2\times②: 10A+6B=2 \quad — ②'$

$②' - ①': 4A=2$

$A=\frac{1}{2}$

$B=-A$

$=-\frac{1}{2}$

$\therefore \sum_{r=1}^{n} \frac{1}{(2r+3)(2r+5)} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r+3} - \frac{1}{2r+5} \right)$

$r=1: \frac{1}{5} - \cancel{\frac{1}{7}}$

$r=2: \cancel{\frac{1}{7}} - \cancel{\frac{1}{9}}$

$\vdots$

$r=n: \cancel{\frac{1}{2n+3}} - \frac{1}{2n+5}$

$\sum = \frac{1}{5} - \frac{1}{2n+5} = \frac{2n+5-5}{2n+5} = \frac{2n}{2n+5}$

$\sum_{r=1}^{n} \frac{1}{(2r+3)(2r+5)} = \frac{1}{2}\left(\frac{2n}{2n+5}\right) = \frac{n}{2n+5}$

9 $\sum_{r=1}^{n} (r+1)^2 - (r-1)^2$

$r=1: \cancel{2^2} - 0^2$

$r=2: \cancel{3^2} - 1^2$

$r=3: \cancel{4^2} - \cancel{2^2}$

$\vdots$

$r=n-1: n^2 - \cancel{(n-2)^2}$

$r=n: (n+1)^2 - \cancel{(n-1)^2}$

$\sum = n^2 + (n+1)^2 - 1 = n^2 + n^2 + 2n + 1 - 1$

$= 2n^2 + 2n = 2n(n+1)$

Higher derivatives (nothing new)

Some notation: $y=f(x)$ $\frac{dy}{dx}=f'(x)$ $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=f''(x)$ $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3}=f'''(x)$

at the 4th derivative, use roman numerals in brackets. (iv, v, vi, vii, viii, ix, x...) $\longrightarrow$ $\frac{d}{dx}\left(\frac{d^3y}{dx^3}\right)=\frac{d^4y}{dx^4}=f^{(iv)}(x)$ $\frac{d}{dx}\left(\frac{d^4y}{dx^4}\right)=\frac{d^5y}{dx^5}=f^{(v)}(x)$

general: $\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)=\frac{d^ny}{dx^n}=f^{(n)}(x)$

⚠ If you write $f^n(x)$ instead $f^{(n)}(x)$, that becomes a composite function, applying $f(x)$ n times.

example: $y=f(x)=\ln(1-x)$

$\frac{dy}{dx}=f'(x)=-\frac{1}{1-x}$ $\frac{d^2y}{dx^2}=f''(x)=-\frac{1}{(1-x)^2}$ I mean... what's new?

$\frac{d^3y}{dx^3}=f'''(x)=-\frac{2}{(1-x)^3}$ $\left.\frac{d^3y}{dx^3}\right|_{x=\frac{1}{2}}=f'''\left(\tfrac{1}{2}\right)=-\frac{2}{\left(\frac{1}{2}\right)^3}=-16$

McLaurin Series!

A way to write a function as an infinite sum of terms in the form ax^n (an ∞-degree polynomial)

let's say $y=f(x)$ and $f(x)$ can be differentiated infinitely times.

let $f(x)=a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_rx^r+\ldots=\sum_{r=0}^{\infty}a_rx^r$

since $x^r=0$ if $x=0$, $f(0)=a_0+0+0+\ldots=a_0$

$f'(x)=0+a_1+2a_2x+3a_3x^2+\ldots+ra_rx^{r-1}+\ldots=\sum_{r=0}^{\infty}ra_rx^{r-1}$

since $x^r=0$ if $x=0$, $f'(0)=a_1+0+0+\ldots=a_1$

$f''(x)=2a_2+6a_3x+\ldots+r(r-1)a_rx^{r-2}+\ldots$ $f''(0)=2a_2\longrightarrow a_2=\frac{f''(0)}{2}$

$f'''(x)=6a_3+24a_4x+\ldots+r(r-1)(r-2)a_rx^{r-3}+\ldots$ $f'''(0)=6a_3\longrightarrow a_3=\frac{f'''(0)}{6}$

And so on...

$f^{(n)}(0)=n!\,a_n\longrightarrow a_n=\frac{f^{(n)}(0)}{n!}$ with this, we can find all the coefficients.

$$\boxed{\text{Therefore, } f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(r)}(0)}{r!}x^r+\ldots=\sum_{r=0}^{\infty}\frac{f^{(r)}(0)}{r!}x^r}$$

Ex 2C

1. a) $f(x)=(1-x)^{-1}$ $f'(x)=(1-x)^{-2}$ $f''(x)=2(1-x)^{-3}$ $f'''(x)=6(1-x)^{-4}$ $f^{(r)}(x)=r!(1-x)^{-r-1}$

$f(0)=1$ $f'(0)=1$ $f''(0)=2$ $f'''(0)=6$ $f^{(r)}(0)=r!$

$\therefore f(x)=1+x+\frac{2!}{2!}x^2+\frac{3!}{3!}x^3+\ldots+\frac{r!}{r!}x^r=1+x+x^2+\ldots+x^r+\ldots$

b) $f(x)=(1+x)^{\frac{1}{2}}$ $f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}}$ $f''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}}$ $f'''(x)=\frac{3}{8}(1+x)^{-\frac{5}{2}}$

$f(0)=1$ $f'(0)=\frac{1}{2}$ $f''(0)=-\frac{1}{4}$ $f'''(0)=\frac{3}{8}$

$\therefore f(x)=1+\frac{x}{2}-\frac{x^2}{4\times 2!}+\frac{3x^3}{8\times 3!}+\ldots=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}+\ldots$

2. $f(x)=e^{\sin x}$ $f'(x)=\cos x\, e^{\sin x}$ $f''(x)=\cos^2 x\, e^{\sin x}$

$f(0)=1$ $f'(0)=1$ $f''(0)=1$ $\therefore f(x)=1+x+\frac{x^2}{2}+\ldots$

3. a) $f(x)=\cos x$ $f'(x)=-\sin x$ $f''(x)=-\cos x$ $f'''(0)=\sin x$

$f(0)=1$ $f'(0)=0$ $f''(0)=-1$ $f'''(0)=0$

$\therefore f(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+(-1)^r\frac{x^{2r}}{(2r)!}$

b) $f\left(\frac{\pi}{6}\right)=1-\frac{\pi^2}{36\times 2}+\frac{\pi^4}{1296\times 24}=0.86605388354\ldots$

$\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}=0.8660254038\ldots$ $\therefore$ correct to 3dp.

4. $f(x)=e^x$ $f'(x)=f''(x)=f'''(x)=f^{(n)}(x)=e^x$ $f^{(n)}(0)=1$

$\therefore e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^r}{r!}+\ldots$ a) $e=e^1=1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\ldots\simeq 2.718$

$f(x)=\ln(1-x)$ $f'(x)=-\frac{1}{1-x}$ $f''(x)=-\frac{1}{(1-x)^2}$ $f'''(x)=-\frac{2}{(1-x)^3}$ $f^{(iv)}(x)=-\frac{6}{(1-x)^4}$ $f^{(n)}(x)=-\frac{(n-1)!}{(1-x)^n}$

$f(0)=0$ $f'(0)=-1$ $f''(0)=-1$ $f'''(0)=-2$ $f^{(iv)}(0)=-6$ $f^{(n)}(0)=-(n-1)!$

$\therefore \ln(1-x)=-x-\frac{x^2}{2}-\frac{2x^3}{6}-\ldots-\frac{(r-1)!\,x^r}{r!}=-x-\frac{x^2}{2}-\frac{x^3}{3}-\ldots-\frac{x^r}{r}-\ldots$

b) $\ln\left(\frac{6}{5}\right)=\ln(1-(-0.2))=-0.2-\frac{(0.2)^2}{2}+\frac{(0.2)^3}{3}-\ldots\simeq -0.182$

⑤ a) $f(x)=e^{3x}$ $f'(x)=3e^{3x}$ $f''(x)=9e^{3x}$ $f^{(n)}(x)=3^n e^{3x}$

$f(0)=1$ $f'(0)=3$ $f''(0)=9$ $f^{(n)}(0)=3^n$

$$e^{3x}=1+3x+\frac{9x^2}{2}+\frac{27x^3}{6}+\frac{81x^4}{24}+\ldots=1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3+\frac{27}{8}x^4+\ldots$$

b) $f(x)=\ln(1+2x)$ $f'(x)=\frac{2}{(1+2x)}$ $f''(x)=-\frac{4}{(1+2x)^2}$ $f'''(x)=\frac{16}{(1+2x)^3}$ $f^{(iv)}(x)=-\frac{96}{(1+2x)^4}$

$f(0)=0$ $f'(0)=2$ $f''(0)=-4$ $f'''(x)=16$

$$\ln(1+2x)=2x-\frac{4x^2}{2}+\frac{16x^3}{6}-\frac{96x^4}{24}=2x-2x^2+\frac{8}{3}x^3-4x^4+\ldots$$

c) $f(x)=\sin^2 x$ $f'(x)=2\sin x\cos x=\sin 2x$ $f''(x)=2\cos 2x$ $f'''(x)=-4\sin 2x$

$f(0)=0$ $f'(0)=0$ $f''(0)=2$ $f'''(0)=0$

$f^{(iv)}(x)=-8\cos 2x$ $f^{(v)}(x)=16\sin 2x$ $f^{(vi)}(x)=32\cos 2x$

$f^{(iv)}(0)=-8$ $f^{(v)}(0)=0$ $f^{(vi)}(x)=32$

$$\sin^2 x=\frac{2x^2}{2!}-\frac{8x^4}{4!}+\ldots=x^2-\frac{1}{3}x^4+\ldots$$

⑥ $$\cos\left(x-\frac{\pi}{4}\right)=\cos x\cos\frac{\pi}{4}+\sin x\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}\cos x+\frac{\sqrt{2}}{2}\sin x=\frac{\sqrt{2}}{2}(\cos x+\sin x)$$

$$=\frac{\sqrt{2}}{2}\left[\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)+\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right)\right]$$

$$=\frac{1}{\sqrt{2}}\left(1+x-\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)=\frac{1}{\sqrt{2}}\left(1+x-\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}+\ldots\right)$$

⑦ $f(x)=(1-x)^2\ln(1-x)$

a) $$f'(x)=\frac{(1-x)^2}{-(1-x)}+\left(-2(1-x)\ln(1-x)\right)=-(1-x)-2(1-x)\ln(1-x)$$

$$f''(x)=\frac{d}{dx}(-1+x)-2\frac{d}{dx}\left((1-x)\ln(1-x)\right)=1-2\left(-\frac{1-x}{1-x}-\ln(1-x)\right)$$

$$=1+2(1+\ln(1-x))=3+2\ln(1-x)$$

b) $f(0)=\ln 1=0$ $f'(0)=-1+x-2(1-x)\ln(1-x)\Big|_{x=0}=-1-2\ln 1=-1$

$f''(0)=3+2\ln 1=3$ $f'''(x)=-\frac{2}{1-x}$ $f'''(0)=-2$

c) $$(1-x)^2\ln(1-x)=-x+\frac{3x^2}{2}-\frac{2x^3}{6}+\ldots=-x+\frac{3}{2}x^2-\frac{1}{3}x^3+\ldots$$

⑧ a) $3\sin x - 4x\cos x + x = 3\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) - 4x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + x$

$= \left(3x - \frac{x^3}{2} + \frac{x^5}{40} - \dots\right) + \left(-4x + 2x^3 - \frac{x^5}{6} + \dots\right) + x = -\frac{x^3}{2} + 2x^3 + \frac{x^5}{40} - \frac{x^5}{6} + \dots$

$= \frac{3}{2}x^3 - \frac{17}{120}x^5 + \dots$

b) $\frac{3\sin x - 4x\cos x + x}{x^3} = \frac{\frac{3}{2}x^3 - \frac{17}{120}x^5 + \dots}{x^3} = \frac{3}{2} - \frac{17}{120}x^2 + \dots$

$\therefore \lim_{x\to 0}\left(\frac{3\sin x - 4x\cos x + x}{x^3}\right) = \lim_{x\to 0}\left(\frac{3}{2} - \frac{17}{120}x^2 + \dots\right) = \frac{3}{2}$

⑨ $f(x) = \ln(\cos x)$

a) $f'(x) = \frac{1}{\cos x}\cdot(-\sin x) = -\tan x$

b) $f''(x) = -\sec^2 x \quad f'''(x) = -2\sec x(\sec x \tan x) = -2\sec^2 x \tan x$

$f^{(iv)}(x) = \frac{d}{dx}(-2\tan x)\cdot\sec^2 x + (-2\tan x)\cdot\frac{d}{dx}(\sec^2 x)$

$= -2\sec^2 x \cdot \sec^2 x - 2\tan x \cdot 2\sec^2 x \tan x = -2\sec^4 x - 4\tan^2 x \sec^2 x$

$f'(0) = 0 \quad f''(0) = -\frac{1}{\cos^2 0} = -1 \qquad f'''(0) = -\frac{2}{\cos^2 0}\cdot\tan 0 = 0 \qquad f^{(iv)}(0) = -\frac{2}{\cos^4 0} - 0 = -2$

c) $f(0) = \ln 1 = 0 \quad \ln(\cos x) = -\frac{x^2}{2} - \frac{2x^4}{24} = -\frac{1}{2}x^2 - \frac{1}{12}x^4 + \dots$

d) $\ln\cos\frac{\pi}{4} = \ln\frac{\sqrt{2}}{2} = \ln\frac{1}{\sqrt{2}} = -\frac{1}{2}\ln 2$

$= -\frac{1}{2}\left(\frac{\pi^2}{16}\right) - \frac{1}{12}\left(\frac{\pi^4}{256}\right) = \frac{\pi^2}{16}\left(-\frac{1}{2} - \frac{1}{12}\left(\frac{\pi^2}{16}\right)\right) = -\frac{1}{2}\ln 2$

$\ln 2 = \frac{\pi^2}{16}\left(1 + \frac{1}{6}\left(\frac{\pi^2}{16}\right)\right) = \frac{\pi^2}{16}\left(1 + \frac{\pi^2}{96}\right)$

⑩ $f(x) = \tan x \quad f'(x) = \sec^2 x \quad f''(x) = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x$

$f'''(x) = 4\sec^2 x \tan x \cdot \tan x + 2\sec^2 x \cdot \sec^2 x = 4\sec^2 x \tan^2 x + 2\sec^4 x$

$f^{(iv)}(x) = 8\sec^2 x \tan x \cdot \tan^2 x + 8\tan x \sec^2 x \cdot \sec^2 x + 8\sec^3 x \cdot \sec x \tan x$

$= 8\sec^2 x \tan^3 x + 8\tan x \sec^4 x + 8\sec^4 x \tan x = 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$

$f^{(v)}(x) = 16\sec^2 x \tan x \cdot \tan^3 x + 24\tan^2 x \cdot \sec^2 x \cdot \sec^2 x + 64\sec^4 x \tan^2 x + 16\sec^6 x$

$= 16\sec^2 x \tan^4 x + 88\tan^2 x \sec^4 x + 16\sec^6 x$

$f(0) = 0 \quad f'(0) = 1 \quad f''(0) = 0 \quad f'''(0) = 2 \quad f^{(iv)}(0) = 0 \quad f^{(v)}(0) = 16 \qquad \tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5$

Series expansion of Compound(Functions( ))

function	expansion		in formula book
e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^r}{r!}+\dots$	$\forall x$	✓
$\ln(1+x)$	$x-\frac{x^2}{2}+\frac{x^3}{3}-\dots+(-1)^{r+1}\frac{x^r}{r}+\dots$	$-1<x\le 1$	✓
$\sin x$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots+(-1)^r\frac{x^{2r+1}}{(2r+1)!}+\dots$	$\forall x$	✓
$\cos x$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots+(-1)^r\frac{x^{2r}}{(2r)!}+\dots$	$\forall x$	✓
$\arctan x$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\dots+(-1)^r\frac{x^{2r+1}}{2r+1}+\dots$	$-1\le x\le 1$	✓
$\tan x$	$x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\dots$		✗

No general term anyways

Compound function

$$\cos(2x^2) = 1-\frac{(2x^2)^2}{2!}+\frac{(2x^2)^4}{4!}-\frac{(2x^2)^6}{6!}+\dots = 1-\frac{4x^4}{2}+\frac{16x^8}{24}-\frac{64x^{12}}{720}+\dots$$

$$= 1-2x^4+\frac{2}{3}x^8-\frac{4}{45}x^{12}+\dots$$

just sub $2x^2$ into $\cos x$ expansion

$$\ln\left(\frac{\sqrt{1+2x}}{1-3x}\right) = \ln\left(\sqrt{1+2x}\right)-\ln(1-3x) = \frac{1}{2}\ln(1+2x)-\ln(1-3x)$$

$$= \frac{1}{2}\underbrace{\left(2x-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}-\dots\right)}-\underbrace{\left(-3x-\frac{(-3x)^2}{2}+\frac{(-3x)^3}{3}-\dots\right)}$$

$$= \frac{1}{2}\left(2x-2x^2+\frac{8}{3}x^3-\dots\right)-\left(-3x-\frac{9}{2}x^2-9x^3-\dots\right)$$

$$= x-x^2+\frac{4}{3}x^3+3x+\frac{9}{2}x^2+9x^3+\dots = 4x+\frac{7}{2}x^2+\frac{31}{3}x^3+\dots$$

valid for $-1<2x\le 1$ valid for $-1<-3x\le 1$

$-\frac{1}{2}<x\le\frac{1}{2}$ ← intersect → $-\frac{1}{3}\le x<\frac{1}{3}$ ← Both inequalities to be satisfied.

Therefore, $\ln\left(\frac{\sqrt{1+2x}}{1-3x}\right) = 4x+\frac{7}{2}x^2+\frac{31}{3}x^3+\dots$, $-\frac{1}{3}\le x<\frac{1}{3}$