2.1: COMPLEX NUMBERS *again!*

$$\left.\begin{aligned}
\cos x &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+\frac{(-1)^r x^{2r}}{(2r)!}+\ldots \\
\sin x &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+\frac{(-1)^r x^{2r+1}}{(2r+1)!}+\ldots \\
e^x &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^r}{r!}+\ldots
\end{aligned}\right\} \text{MacLaurin expansions}$$

$$e^{i\theta} = 1+(i\theta)+\frac{(i\theta)^2}{2!}+\ldots$$

$$= 1+i\theta-\frac{\theta^2}{2!}-\frac{i\theta^3}{3!}+\frac{\theta^4}{4!}+\frac{i\theta^5}{5!}-\frac{\theta^6}{6!}-\ldots$$

$$= \left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\ldots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\ldots\right)$$

$e^{i\theta} = \cos\theta + i\sin\theta$ ← euler's relation

$z = a+bi$ — cartesian form

$z = r(\cos\theta + i\sin\theta)$ — modulus-argument form

$z = re^{i\theta}$ — exponential form

$r = |z|$ $\theta = \arg z$

when $\theta = \pi$, $\underline{e^{\pi i} + 1 = 0}$

euler's identity

⚠ $r(\cos\theta - i\sin\theta)$

$= r(\cos(-\theta) + i\sin(-\theta))$

mod-arg. form requires plus, NOT minus

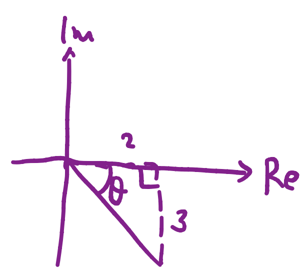
$z = a+bi$ eg. $2-3i$

↓

$z = r(\cos\theta + i\sin\theta)$

1. find modulus $r = |z| = \sqrt{2^2+3^2} = \sqrt{13}$
2. find argument $\theta = \arg z = -\arctan\left(\frac{3}{2}\right)$
3. write $z = \sqrt{13}\left(\cos(-0.983) + i\sin(-0.983)\right)$

$= \sqrt{13}e^{-0.983i}$



We know that $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$,

$\arg(z_1 \cdot z_2) = \arg z_1 + \arg z_2$,

$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

This proves the following, NOT the other way around.

Complex indices are complicated

Therefore, $z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$ $\quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

DeMoivre's Theorem

$$\boxed{(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)}$$

True $\forall n \in \mathbb{Z}^+$

Can be proven by induction $\forall n \in \mathbb{Z}^+$

Using this with trigonometric functions:

Showing $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

showing $\sin n\theta$ or $\cos n\theta$ equals

$\sin^n\theta + \sin^{n-2}\theta + \dots$

$\cos^n\theta + \cos^{n-2}\theta + \dots$

$\Downarrow$

$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$ (de Moivre's)

$= \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta$

(Binomial Expansion)

Compare real terms (since the answer is is in terms of cos)

$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$

$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$ (apply $\sin^2\theta = 1 - \cos^2\theta$ and simplify)

if $z = \cos\theta + i\sin\theta$, $\frac{1}{z} = z^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

$z + \frac{1}{z} = 2\cos\theta$	$z^n + \frac{1}{z^n} = 2\cos n\theta$
$z - \frac{1}{z} = 2i\sin\theta$	$z^n - \frac{1}{z^n} = 2i\sin n\theta$

$\underbrace{e^{\theta i} \pm e^{-\theta i}}_{\text{this form gives trig functions}}$

$$\boxed{\begin{aligned} \sin n\theta &= \frac{1}{2i}\left(e^{in\theta} - e^{-in\theta}\right) \\ \cos n\theta &= \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right) \end{aligned}}$$

Expressing $\cos^5\theta$ in the form $a\cos 5\theta + b\cos 3\theta + c\cos\theta$

if $z = \cos\theta + i\sin\theta$, $\left(z+\frac{1}{z}\right)^5 = (2\cos\theta)^5 = 32\cos^5\theta$ * use $z+\frac{1}{z} = 2\cos\theta$

This is different from the previous example. This example turns a power into θ coefficients.

$$= z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) + 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) + \frac{1}{z^5}$$ * Binomial

$$= z^5 + 5z^3 + 10z + 10\left(\frac{1}{z}\right) + 5\left(\frac{1}{z^3}\right) + \frac{1}{z^5}$$

$$= \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$$ * group terms

$$= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta = 32\cos^5\theta$$

Therefore, $\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$

Sums of Complex Geometric Series

First term $= w$ Common ratio $= z$ (both are allowed to be complex)

$$S_n = \sum_{r=0}^{n-1} wz^r = w + wz + wz^2 + wz^3 + \dots + wz^{n-1} = \frac{w(z^n - 1)}{z-1}$$

$$S_\infty = \sum_{r=0}^{\infty} wz^r = w + wz + wz^2 + wz^3 + \dots = \frac{w}{1-z} \quad (\text{for } |z|<1)$$

! exponential form is the easiest to deal with when working with powers

$z = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n} = e^{\frac{\pi i}{n}}$ show that $1+z+z^2+z^3+\ldots+z^{n-1} = 1+i\cot\left(\frac{\pi}{2n}\right)$

(example 13)

hint: $\dfrac{\sin\frac{\pi}{2n}}{\sin\frac{\pi}{2n}} + \dfrac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}$ ← 1 complex number ← $\frac{1}{2i}\left(e^{in\theta} - e^{-in\theta}\right)$

$$1+z+z^2+z^3+\ldots+z^{n-1} = \frac{z^n-1}{z-1} = \frac{\left(e^{\frac{\pi i}{n}}\right)^n - 1}{e^{\frac{\pi i}{n}} - 1}$$ * sum of geometric series

$$= \frac{e^{\pi i} - 1}{e^{\frac{\pi i}{n}} - 1} = \frac{-2}{e^{\frac{\pi i}{n}} - 1} = \frac{-2e^{-\frac{\pi i}{2n}}}{\left(e^{\frac{\pi i}{n}} - 1\right)e^{-\frac{\pi i}{2n}}}$$ * multiply top & bottom by $e^{-\frac{\pi i}{2n}}$

$$= \frac{-2e^{-\frac{\pi i}{2n}}}{e^{\frac{\pi i}{2n}} - e^{-\frac{\pi i}{2n}}} = \frac{-2e^{-\frac{\pi i}{2n}}}{2i\sin\frac{\pi}{2n}}$$ * $e^{\frac{\pi i}{2n}} - e^{-\frac{\pi i}{2n}} = 2i\sin\frac{\pi}{2n}$

$$= \frac{ie^{-\frac{\pi i}{2n}}}{\sin\frac{\pi}{2n}}$$ * $\frac{-1}{i} = i \implies \frac{-2}{2i} = i$

$$= \frac{i\left(\cos\left(-\frac{\pi}{2n}\right) + i\sin\left(-\frac{\pi}{2n}\right)\right)}{\sin\frac{\pi}{2n}} = \frac{i\cos\frac{\pi}{2n} + \sin\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}$$

$$= i\cot\frac{\pi}{2n} + 1 = 1 + i\cot\frac{\pi}{2n}$$

$$e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots + e^{ni\theta} = (\cos\theta + i\sin\theta) + (\cos 2\theta + i\sin 2\theta) + \ldots + (\cos n\theta + i\sin n\theta)$$

$$= (\cos\theta + \cos 2\theta + \ldots + \cos n\theta) + i(\sin\theta + \sin 2\theta + \ldots + \sin n\theta)$$

$$\Downarrow$$

$$\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \operatorname{Im}\left(\frac{e^{i\theta}\left(e^{ni\theta} - 1\right)}{e^{i\theta} - 1}\right)$$

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta = \operatorname{Re}\underbrace{\left(\frac{e^{i\theta}\left(e^{ni\theta} - 1\right)}{e^{i\theta} - 1}\right)}$$

$$e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots + e^{ni\theta} = \sum_{r=0}^{n-1} e^{i\theta}\left(e^{i\theta}\right)^r = \frac{e^{i\theta}\left(e^{ni\theta} - 1\right)}{e^{i\theta} - 1}$$

N^{th} roots of a complex number

What does that even mean?

$z^n = w$ where $z, w \in \mathbb{C}$

and we are trying to solve for z

(nth roots of w)

write numbers in modulus-argument form to apply de Moivre's theorem

write the argument of w as $\theta + 2k\pi$ to account for different arguments resulting in the same complex number.

example: $z^4 = 2 + 2i\sqrt{3}$

$|2+2i\sqrt{3}| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4$

$\arg(2+2i\sqrt{3}) = \arctan\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$ therefore, $z^4 = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

write z in mod-arg form and add $2k\pi$ to argument of w

$$\underbrace{(r(\cos\theta + i\sin\theta))^4 = r^4(\cos 4\theta + i\sin 4\theta)}_{\text{apply de Moivre's}} = 4\left(\cos\left(\frac{\pi}{3} + \underbrace{2k\pi}\right) + i\sin\left(\frac{\pi}{3} + \underbrace{2k\pi}\right)\right)$$

these account for the weird arguments.

Comparing their modulus, $r^4 = 4 \rightarrow r = \sqrt{2}$ $(r \geq 0)$

Comparing their arguments, $4\theta = \frac{\pi}{3} + 2k\pi$

Substitute <u>consecutive</u> values of k (here we use $-2, -1, 0, 1$)

$$\left.\begin{array}{ll} k=0:\ \theta = \frac{\pi}{12} & k=-1:\ \theta = -\frac{5}{12}\pi \\ k=1:\ \theta = \frac{7}{12}\pi & k=-2:\ \theta = -\frac{11}{12}\pi \end{array}\right\}$$

The 4 argument form 4 roots (all of them are 4^{th} roots of $2+2i\sqrt{3}$)

Answer (given in exponential form): $\sqrt{2}e^{\frac{\pi i}{12}}, \sqrt{2}e^{\frac{7\pi i}{12}}, \sqrt{2}e^{-\frac{5\pi i}{12}}, \sqrt{2}e^{-\frac{11\pi i}{12}}$

MAKE SURE ALL ARGUMENTS ARE BETWEEN $-\pi$ and π !

Ex (1F)

(2) a) $z^7 = 1$ 7^{th} roots of $\cos 0 + i\sin 0$

$$\cos 7\theta + i\sin 7\theta = \cos(2k\pi) + i\sin(2k\pi)$$

$7\theta = 2k\pi$ $\quad k = -3, -2, -1, 0, 1, 2, 3$

$\theta = \frac{2k\pi}{7}$ $\quad \theta = -\frac{6\pi}{7}, -\frac{4\pi}{7}, -\frac{2\pi}{7}, 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}$

$\therefore z = \text{cis}\left(\pm\frac{6\pi}{7}\right), \text{cis}\left(\pm\frac{4\pi}{7}\right), \text{cis}\left(\pm\frac{2\pi}{7}\right), \text{cis}0$

c) $z^5 = -32$ 5^{th} roots of $32(\cos\pi + i\sin\pi)$

$$r^5(\cos 5\theta + i\sin 5\theta) = 2^5(\cos(2k\pi + \pi) + i\sin(2k\pi + \pi))$$

$r = 2$ $\quad \theta = \frac{2k\pi + \pi}{5} = \frac{(2k+1)\pi}{5}$ $\quad k = -2, -1, 0, 1, 2$

$\theta = -\frac{3\pi}{5}, -\frac{\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}, \pi$

$\therefore z = 2\text{cis}\left(\pm\frac{3\pi}{5}\right), 2\text{cis}\left(\pm\frac{\pi}{5}\right), 2\text{cis}\pi$

e) $z^4 = 2 - 2i\sqrt{3}$ $\quad \arg(2 - 2i\sqrt{3}) = \arctan\left(\frac{-2\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

$|2 - 2i\sqrt{3}| = \sqrt{4 + 12} = 4$

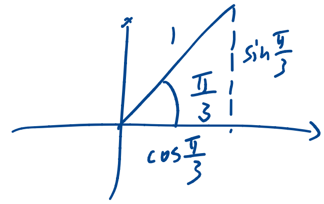
$$r^4(\cos 4\theta + i\sin 4\theta) = 4\left(\cos\left(2k\pi - \frac{\pi}{3}\right) + i\sin\left(2k\pi - \frac{\pi}{3}\right)\right)$$

$r = \sqrt{2}$ $\quad \theta = \frac{k\pi}{2} - \frac{\pi}{12} = \frac{(6k-1)\pi}{12}$ $\quad k = -1, 0, 1, 2$

$\theta = -\frac{7\pi}{12}, -\frac{\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}$

$\therefore z = \sqrt{2}\text{cis}\left(-\frac{7\pi}{12}\right), \sqrt{2}\text{cis}\left(-\frac{\pi}{12}\right), \sqrt{2}\text{cis}\left(\frac{5\pi}{12}\right), \sqrt{2}\text{cis}\left(\frac{11\pi}{12}\right)$

4\. a) $(z+1)^3 = -1 \qquad -1 = \cos\pi + i\sin\pi$



$$\cos 3\theta + i\sin 3\theta = \cos(2k\pi+\pi) + i\sin(2k\pi+\pi)$$

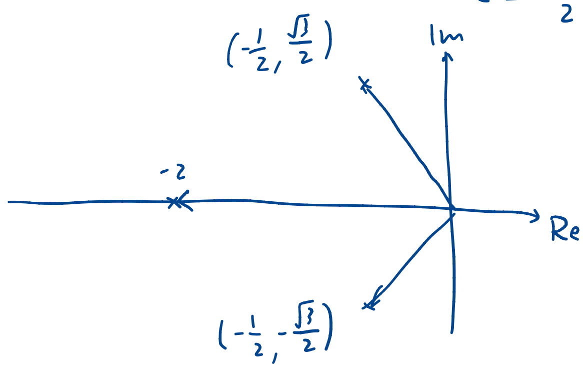
$$\theta = \frac{(2k+1)\pi}{3} \qquad k = -1, 0, 1$$

$$\theta = -\frac{\pi}{3}, \frac{\pi}{3}, \pi \qquad \therefore\ z+1 = \text{cis}\left(-\frac{\pi}{3}\right), \text{cis}\frac{\pi}{3}, \text{cis}\,\pi$$

$$z+1 = \frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -1$$

$$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -2$$

b)



c) centre $= (-1, 0)$ radius $= 1$

6\. a) $|-2-2i\sqrt{3}| = \sqrt{4+12} = 4 \qquad \arctan\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3} \qquad \arg(-2-2i\sqrt{3}) = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$

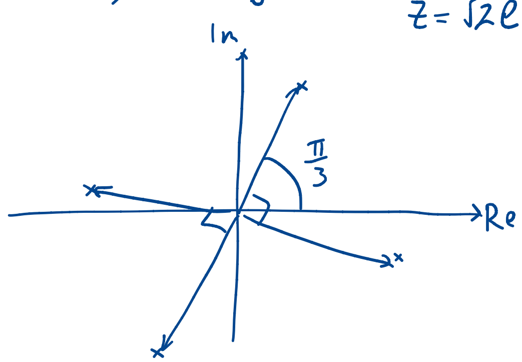
b) $z^4 = -2-2i\sqrt{3} = 4\,\text{cis}\left(-\frac{2\pi}{3}\right)$

$$r^4 \text{cis}\, 4\theta = 4\,\text{cis}\left(2k\pi - \frac{2\pi}{3}\right) = 4\,\text{cis}\left(\frac{6k\pi - 2\pi}{3}\right) = 4\,\text{cis}\left(\frac{2\pi(3k-1)}{3}\right)$$

$$r = \sqrt{2} \qquad \theta = \frac{\pi(3k-1)}{6} \qquad k = -1, 0, 1, 2$$

$$\theta = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$$

$$z = \sqrt{2}e^{-\frac{2\pi}{3}i}, \sqrt{2}e^{-\frac{\pi i}{6}}, \sqrt{2}e^{\frac{\pi}{3}i}\ \sqrt{2}e^{\frac{5\pi}{6}i}$$



N^{th} roots of Unity (ie. N^{th} roots of 1)

Let's try to find z such that $z^3 = 1$

$z^3 = \cos 0 + \sin 0$

$r^3(\cos 3\theta + i\sin 3\theta) = \cos 2k\pi + i\sin 2k\pi$

Comparing their modulus, $r^3 = 1$
$r = 1$

Comparing their arguments, $3\theta = 2k\pi$

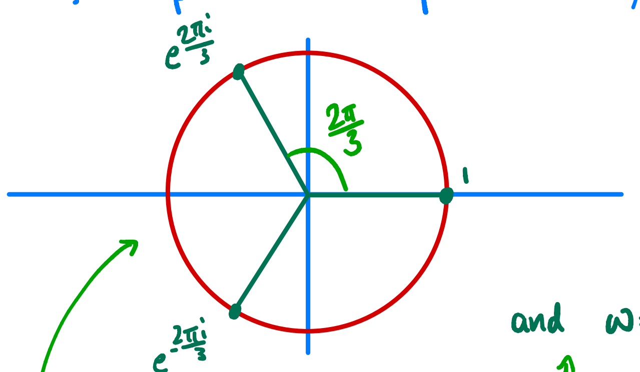
$k=-1: \quad \theta = -\frac{2}{3}\pi$

$k=0: \quad \theta = 0$

$k=1: \quad \theta = \frac{2}{3}\pi$

Answer: $\sqrt[3]{1} = 1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}$

If we plot these complex numbers,



if we do $\left(e^{\frac{2\pi i}{3}}\right)^2$,

we get $e^{\frac{4\pi i}{3}} = e^{-\frac{2\pi i}{3}}$

if $\omega = e^{\frac{2\pi i}{3}}$, the 3^{rd} roots of unity (the 3^{rd} roots of 1) are $1, \omega$ and ω^2

and $\omega = e^{\frac{2\pi i}{n}}$

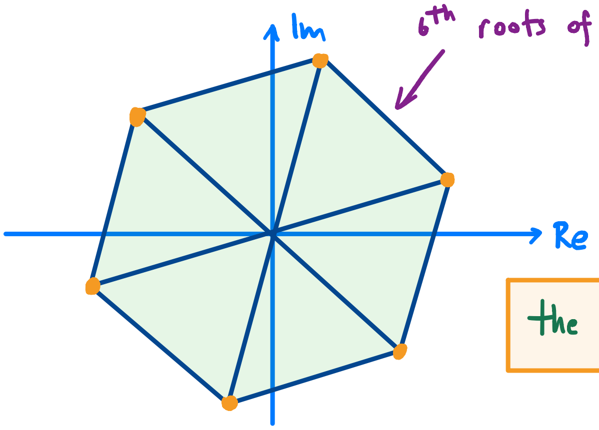
The n^{th} roots of unity form a regular polygon, so the first root would have an argument $\frac{2\pi}{n}$, and the rest of the roots are just powers of the first root.

> Verdict: solutions to $z^n = 1$ are called "n^{th} roots of unity"
>
> the n^{th} roots of unity are $1, \omega, \omega^2, \dots, \omega^{n-1}$ where $\omega = e^{\frac{2\pi i}{n}}$
>
> the sum of the n roots equal 0 (think of vectors)
>
> the n^{th} roots of unity form a regular n-gon in the complex plane

Use these to work out all kinds of stuff related to regular n-gons!

Geometric Problems

When plot on an argand diagram, the n^{th} roots of any complex number form an n-gon.



6^{th} roots of $7+24i$ form a hexagon

the n^{th} roots of unity are $1, \omega, \omega^2, \dots \omega^{n-1}$

if we know 1 of the roots (e.g. z_1),

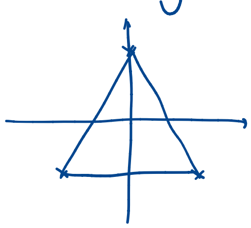
$$\boxed{\text{the } n^{th} \text{ roots of a number are } z_1, z_1\omega, z_1\omega^2, \dots z_1\omega^{n-1}}$$

Use this to find roots faster, and also to solve geometrical problems

(all points are same distance from origin, angles from each other are constant.)

Ex 1G

1 a) Equilateral triangle, one vertex at $(0, 4)$



$$0+4i = 4\left(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}\right)$$

3^{rd} roots of number $= z, z\omega, z\omega^2$

$$z = 4\left(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}\right) \qquad \omega = \cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3}$$

$$z\omega = 4\left(\cos\tfrac{7}{6}\pi + i\sin\tfrac{7}{6}\pi\right) = 4\left(\cos\left(-\tfrac{5}{6}\pi\right) + i\sin\left(-\tfrac{5}{6}\pi\right)\right) = -2\sqrt{3} - 2i$$

$$z\omega^2 = 4\left(\cos\left(-\tfrac{1}{6}\pi\right) + i\sin\left(-\tfrac{1}{6}\pi\right)\right) = 2\sqrt{3} - 2i$$

$$\therefore (0,4), (-2\sqrt{3}, -2), (2\sqrt{3}, -2)$$

c) Regular pentagon, one vertex at $(-1, \sqrt{3})$ $\quad z = -1+\sqrt{3}i = 2\left(\cos\tfrac{2}{3}\pi + i\sin\tfrac{2}{3}\pi\right)$

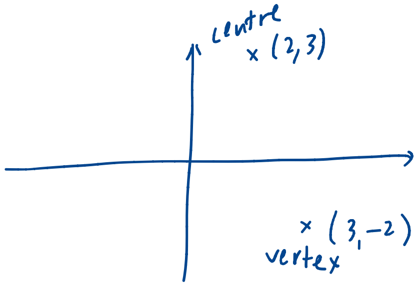
$$z, z\omega, z\omega^2, z\omega^3, z\omega^4 \qquad \omega = \cos\tfrac{2\pi}{5} + i\sin\tfrac{2\pi}{5}$$

$$z\omega = 2\left(\cos\left(-\tfrac{14}{15}\pi\right) + i\sin\left(-\tfrac{14}{15}\pi\right)\right) \qquad z\omega^2 = 2\left(\cos\left(-\tfrac{8}{15}\pi\right) + i\sin\left(-\tfrac{8}{15}\pi\right)\right)$$

$$z\omega^3 = 2\left(\cos\left(-\tfrac{2}{15}\pi\right) + i\sin\left(-\tfrac{2}{15}\pi\right)\right) \qquad z\omega^4 = 2\left(\cos\tfrac{4}{15}\pi + i\sin\tfrac{4}{15}\pi\right)$$

$$\left(2\cos\tfrac{2}{3}\pi, 2\sin\tfrac{2}{3}\pi\right), \left(2\cos\left(-\tfrac{14}{15}\pi\right), 2\sin\left(-\tfrac{14}{15}\pi\right)\right), \left(2\cos\left(-\tfrac{8}{15}\pi\right), 2\sin\left(-\tfrac{8}{15}\pi\right)\right), \left(2\cos\left(-\tfrac{2}{15}\pi\right), 2\sin\left(-\tfrac{2}{15}\pi\right)\right), \left(2\cos\tfrac{4}{15}\pi, 2\sin\tfrac{4}{15}\pi\right)$$

② equilateral triangle at (2, 3) and vertex at (3, -2)



vertex relative to centre $= \begin{pmatrix}3\\-2\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}1\\-5\end{pmatrix}$ or $1-5i = z$

$$w = \operatorname{cis}\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

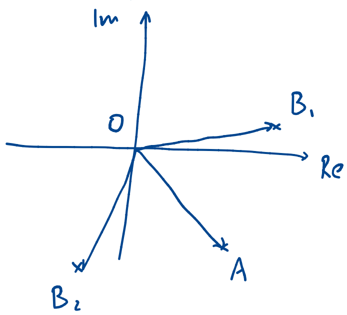
$$zw = (1-5i)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -\frac{1}{2} + \frac{5\sqrt{3}}{2} + \frac{5}{2}i + \frac{\sqrt{3}}{2}i$$

$$= \frac{5\sqrt{3}-1}{2} + \left(\frac{\sqrt{3}+5}{2}\right)i$$

$$zw^2 = \left(\frac{5\sqrt{3}-1}{2} + \left(\frac{\sqrt{3}+5}{2}\right)i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -\frac{5\sqrt{3}-1}{4} - \frac{3+5\sqrt{3}}{4} - \frac{\sqrt{3}+5}{4}i + \frac{5(3)-\sqrt{3}}{4}i$$

$$= -\frac{2+10\sqrt{3}}{4} + \frac{10}{4}i = -\frac{1+5\sqrt{3}}{2} + \frac{5}{2}i$$

③ $A = \sqrt{3}(1-i) = \sqrt{3} - \sqrt{3}i = \sqrt{6}e^{-\frac{1}{4}\pi i}$



B_1, A and B_2 form 60° between each other.

They are part of a regular hexagon with centre O.

$$B_1 = A \times e^{\frac{2\pi}{6}i} = \sqrt{6}e^{-\frac{1}{4}\pi i} \times e^{\frac{1}{3}\pi i} = \sqrt{6}e^{\frac{1}{12}\pi i}$$ ✓

$$B_2 = A \times e^{5\left(\frac{2\pi}{6}\right)i} = \sqrt{6}e^{-\frac{1}{4}\pi i} \times e^{\frac{5}{3}\pi i} = \sqrt{6}e^{\frac{17}{12}\pi i} = \sqrt{6}e^{-\frac{7}{12}\pi i}$$ ✓

④ a) $z^4 = [r(\cos\theta + i\sin\theta)]^4 = r^4(\cos 4\theta + i\sin 4\theta) = -12i = 12\left(\cos\left(-\frac{1}{2}\pi\right) + i\sin\left(-\frac{1}{2}\pi\right)\right)$

$$= 12\left(\cos\left(-\frac{1}{2}\pi + 2k\pi\right) + i\sin\left(-\frac{1}{2}\pi + 2k\pi\right)\right)$$

$r^4 = 12$

$r = \sqrt[4]{12}$

$4\theta = -\frac{1}{2}\pi + 2k\pi$

$\theta = -\frac{1}{8}\pi + \frac{1}{2}k\pi$

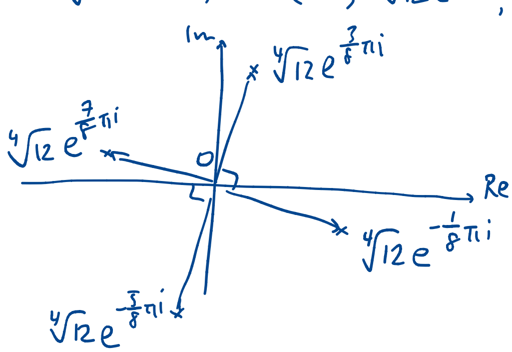
$k = -1$: $\theta = -\frac{1}{8}\pi - \frac{1}{2}\pi = -\frac{5}{8}\pi$

$k = 0$: $\theta = -\frac{1}{8}\pi$

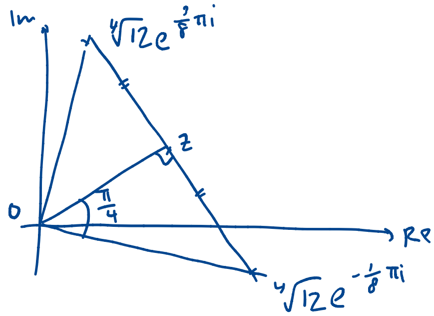
$k = 1$: $\theta = -\frac{1}{8}\pi + \frac{1}{2}\pi = \frac{3}{8}\pi$

$k = 2$: $\theta = -\frac{1}{8}\pi + \pi = \frac{7}{8}\pi$

$$z = \sqrt[4]{12}e^{-\frac{5}{8}\pi i},\ \sqrt[4]{12}e^{-\frac{1}{8}\pi i},\ \sqrt[4]{12}e^{\frac{3}{8}\pi i},\ \sqrt[4]{12}e^{\frac{7}{8}\pi i},$$



b) Consider $\sqrt[4]{12}\,e^{\frac{3}{8}\pi i}$ and $\sqrt[4]{12}\,e^{-\frac{1}{8}\pi i}$ and their midpoint, z



$$\cos\frac{\pi}{4} = \frac{|z|}{\sqrt[4]{12}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$|z| = \frac{\sqrt{2} \times \sqrt[4]{3}}{\sqrt{2}} = \sqrt[4]{3}$$

$$\arg z = \frac{3}{8}\pi - \frac{1}{4}\pi = \frac{1}{8}\pi$$

$z = \sqrt[4]{3}\,e^{\frac{1}{8}\pi i}$ is one of the 4^{th} roots of w

$$\sqrt[4]{3}\,e^{\frac{1}{8}\pi i} = \sqrt[4]{3}\left(\cos\frac{1}{8}\pi + i\sin\frac{1}{8}\pi\right)$$

$$w = z^4 = 3\left(\cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi\right) \quad \text{or} \quad 3e^{\frac{1}{2}\pi i} \quad \text{or} \quad 3i$$

5 a) Regular hexagon with centre at origin, vertex $8+8i$