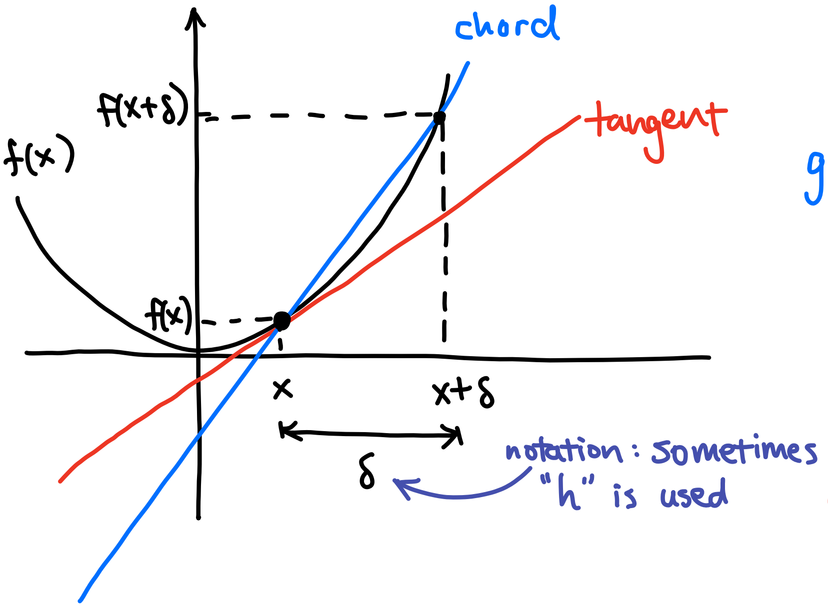



gradient $= \dfrac{f(x+\delta)-f(x)}{x+\delta-x}$

$= \dfrac{f(x+\delta)-f(x)}{\delta}$

gradient $= \underbrace{\lim\limits_{\delta\to 0} \dfrac{f(x+\delta)-f(x)}{\delta} = f'(x)}$

Differentiation by

FIRST PRINCIPLES

resulting change in f

NOTE: $f(x+\delta)-f(x) = \delta f$

$\delta = (x+\delta)-x = \delta x$

$f'(x) = \lim\limits_{\delta\to 0} \dfrac{\delta f}{\delta x}$

small change in x

## EX 12B

① $f(x) = x^2$

a) $f'(x) = \lim\limits_{\delta\to 0} \dfrac{f(x+\delta)-f(x)}{\delta} = \lim\limits_{\delta\to 0} \dfrac{x^2+2\delta x+\delta^2-x^2}{\delta} = \lim\limits_{\delta\to 0} \dfrac{2\delta x+\delta^2}{\delta}$

$= \lim\limits_{\delta\to 0} 2x+\delta = 2x \qquad \therefore f'(2) = 2\cdot 2 = 4$

③ $A(-2,8)$ on $y=x^3$ A has gradient $g$

a) let $y=f(x)$ $f'(x) = \lim\limits_{\delta\to 0} \dfrac{f(x+\delta)-f(x)}{\delta} = \lim\limits_{\delta\to 0} \dfrac{x^3+3x^2\delta+3x\delta^2+\delta^3-x^3}{\delta}$

$= \lim\limits_{\delta\to 0} 3x^2+3x\delta+\delta^2$

$f'(-2) = \lim\limits_{\delta\to 0}(12-6\delta+\delta^2) = g$

b) $\lim\limits_{\delta\to 0}(12-6\delta+\delta^2) = 12$

# POWER RULE (proof)

To differentiate $f(x) = x^n$, use binomial expansion

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^r b^{n-r}$$

$$f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$$

$$f(x+\delta) = (x+\delta)^n, \text{ where } (x+\delta)^n = \sum_{r=0}^{n} {}^nC_r x^r \delta^{n-r}$$

$$= \delta^n + nx\delta^{n-1} + {}^nC_2 x^2 \delta^{n-2} + \dots nx^{n-1}\delta + x^n$$

$$f'(x) = \lim_{\delta \to 0} \frac{\delta^n + nx\delta^{n-1} + \dots + nx^{n-1}\delta + x^n - x^n}{\delta}$$

$$= \lim_{\delta \to 0} \delta^{n-1} + nx\delta^{n-1} + \dots + nx^{n-1} = \boxed{nx^{n-1}}$$

## Differentiating Quadratics

$$f(x) = ax^2 + bx + c$$

$$f(x+\delta) = a(x+\delta)^2 + b(x+\delta) + c$$

$$= ax^2 + 2ax\delta + a\delta^2 + bx + b\delta + c$$

$$\frac{f(x+\delta) - f(x)}{\delta} = \frac{2ax\delta + a\delta^2 + b\delta}{\delta} = 2ax + a\delta + b$$

$$\lim_{\delta \to 0} (2ax + a\delta + b) = 2ax + b$$

EX 12E

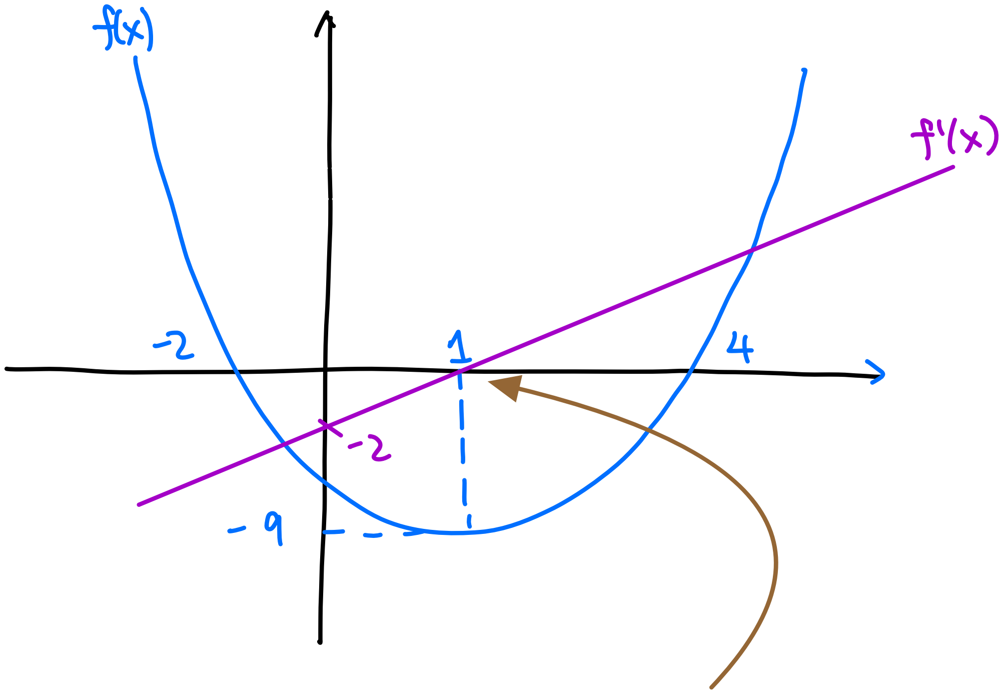
(7) $f(x) = x^2 - 2x - 8$

$= (x-4)(x+2)$

$= (x-1)^2 - 9$

$f'(x) = 2ax + b$

$= 2x - 2$




because at the turning point, rate of change $= 0$

# Derivative of functions with 2 or more terms

$$\underbrace{\text{if } y = f(x) \pm g(x),\ \frac{dy}{dx} = f'(x) \pm g'(x)}_{\text{easy!}}$$

Linear combinations

Set: $\{f_1(x), f_2(x), f_3(x), \dots, f_i(x)\}$

Set: $\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_i(x)\}$ where $\alpha \in \mathbb{R}$

let $L(x) = \sum_{i=1}^{n} \alpha_i f_i(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x) + \alpha_3 f_3(x) + \dots \alpha_n f_n(x)$

then $L'(x) = \sum_{i=1}^{n} \alpha_i f_i'(x)$ ie. what we said up there.

EX 12E

(6) $f(x) = \frac{12}{p\sqrt{x}} + x = \frac{12}{p} x^{-1/2} + x$ $\quad f'(x) = -\frac{6}{p} x^{-3/2} + 1$

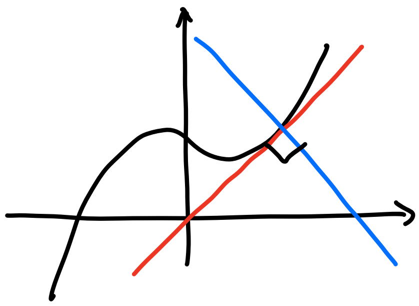
$f'(2) = -\frac{6}{p}\left(\frac{1}{2\sqrt{2}}\right) + 1 = 3$

$2p = -3\left(\frac{\sqrt{2}}{2}\right)$ $\quad \therefore p = -\frac{3}{4}\sqrt{2}$

# Gradient, Tangents & Normals

Gradient of tangent at $x=a$ $= \left.\frac{dy}{dx}\right|_{x=a} = f'(a)$

Gradient of normal at $x=a$ $= -\left(\left.\frac{dy}{dx}\right|_{x=a}\right)^{-1} = -(f'(a))^{-1}$



EX12F

⑥ $P: x = \frac{1}{2}$ on $y = 2x^2 = f(x)$

Normal at $P$ intersects $f(x)$ at $P$ & $Q$

$f'(x) = 4x \quad f'(\frac{1}{2}) = 2 \quad f(\frac{1}{2}) = \frac{1}{2}$

gradient of normal $= -\frac{1}{2}$

eq$^n$ $= (y - \frac{1}{2}) = -\frac{1}{2}(x - \frac{1}{2})$

$y - \frac{1}{2} = -\frac{1}{2}x + \frac{1}{4}$

$y = -\frac{1}{2}x + \frac{3}{4}$

$-\frac{1}{2}x + \frac{3}{4} = 2x^2$

$-2x + 3 = 8x^2$

$8x^2 + 2x - 3 = 0$

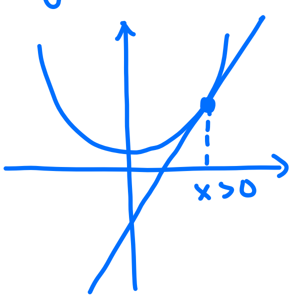
$\begin{matrix} 4x & +3 \\ 2x & -1 \end{matrix}$

$x = \frac{1}{2}, -\frac{3}{4}$

$f(-\frac{3}{4}) = \frac{9}{8} \quad \therefore Q(-\frac{3}{4}, \frac{9}{8})$

CHALLENGE: $L$ is a tangent to $f(x) = 4x^2 + 1$ $m_L > 0$ & $L$ crosses $(0, -8)$

gradient $= \frac{-8 - y}{0 - x} = \frac{-8 - 4x^2 - 1}{0 - x} = f'(x) = 8x$




$-4x^2 - 9 = -8x^2$

$4x^2 = 9$

$x = \pm\frac{3}{2} = \frac{3}{2} \quad f(\frac{3}{2}) = 10 \quad m_L = \frac{10 - (-8)}{\frac{3}{2} - 0}$

$= 18 \cdot \frac{2}{3} = 12$

$\therefore y = 12x - 8$

Some random questions:

① $P(p^2, 2p)$ $Q(q^2, 2q)$ curve $C: y^2 = 4x$ $R =$ intersection of tangents at $P$ & $Q$

$y = 2\sqrt{x} = f(x)$

$f'(x) = \frac{1}{\sqrt{x}}$ $f'(p^2) = \frac{1}{p}$ $y - 2p = \frac{1}{p}(x - p^2)$

$y = \frac{1}{p}x + p$ and $y = \frac{1}{q}x + q$

intersection: $\frac{1}{p}x + p = \frac{1}{q}x + q$

$qx + p^2q = px + pq^2$ $\therefore R(pq, p+q)$

$x(p-q) = pq(p-q)$

$x = pq$ $y = p+q$

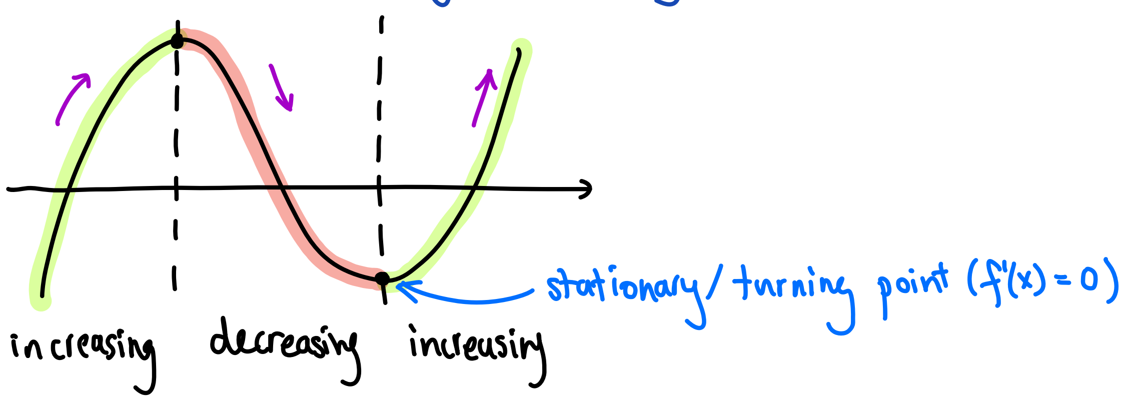
# Increasing and decreasing functions

Increasing function: $f'(x) \geq 0$ for all values of $x$

Decreasing function: $f'(x) \leq 0$ for all values of $x$

(strict increasing/decreasing function: $f'(x) > / < 0$ for all values of $x$)

## Functions with increasing/decreasing intervals



# Second Derivatives

| | FIRST | | SECOND |
|---|---|---|---|
| leibniz | $\frac{dy}{dx} = 4x^3$ | $\longrightarrow$ | $\frac{d^2y}{dx^2} = 12x^2$ |
| $y = x^4$ | $y' = 4x^3$ | $\longrightarrow$ | $y'' = 12x^2$ |
| newton | $\dot{y} = 4x^3$ | $\longrightarrow$ | $\ddot{y} = 12x^2$ |
| legrange | $f'(x) = 4x^3$ | $\longrightarrow$ | $f''(x) = 12x^2$ |

You don't need third in A-Level.

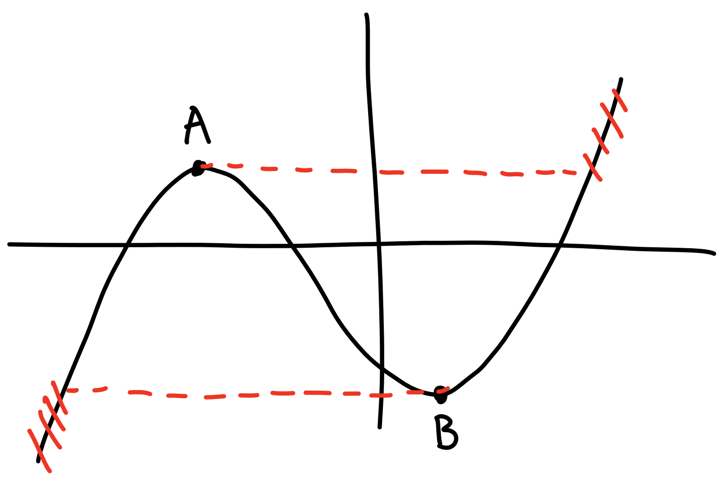
These are used to classify Stationary Points

# Stationary points

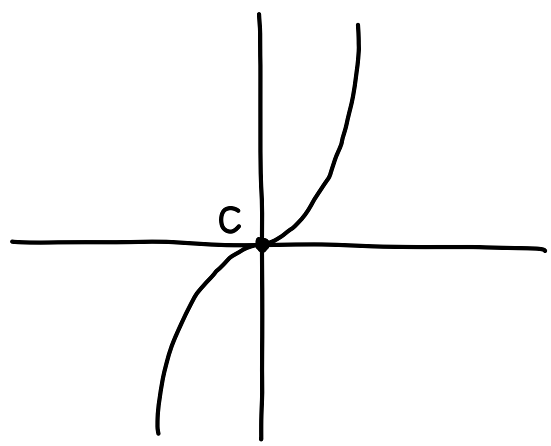
A stationary point is any point $(x_0, y_0)$ on $y = f(x)$ where $\left.\frac{dy}{dx}\right|_{x=x_0} = 0$

Maxima, minima and points of inflection are all stationary points.

Only maxima and minima are turning points (not points of inflection)

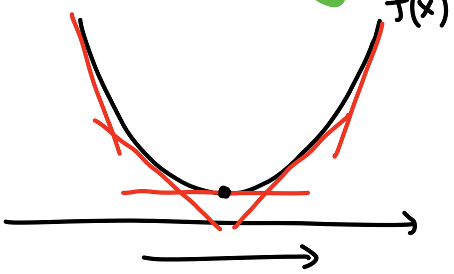



A & B are LOCAL maximum/minimum




C is a point of inflection

# Classifying Stationary Points




A function $g(x)$ is increasing over some interval $I$ provided that $g'(x)>0$ for $x \in I$

$\Downarrow$

$f'(x)$ is increasing. Therefore, $\frac{d}{dx}(f'(x)) > 0$

ie. $f''(x) > 0$ at a minimum point.

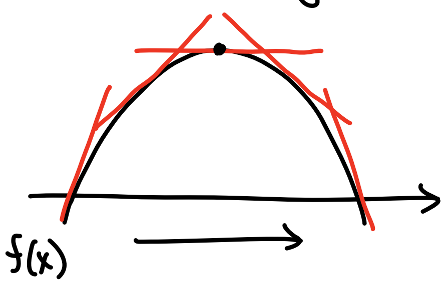
gradient at x is given by $f'(x)$

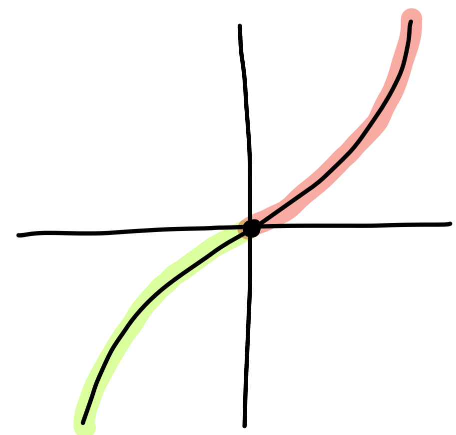
$f'(x)$ is decreasing. Therefore, $\frac{d}{dx}(f'(x)) < 0$

ie. $f''(x) < 0$ at a minimum point.

$\Uparrow$

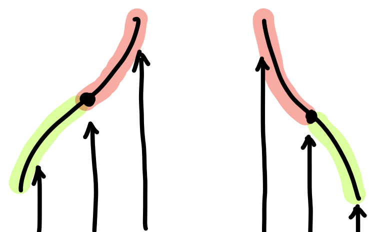
A function $g(x)$ is decreasing over some interval $I$ provided that $g'(x)<0$ for $x \in I$




A point of inflection is a point where the curve changes concavity. All inflection points have $f''(x)=0$, but some points have $f''(x)=0$ that aren't inflection points

Point of inflection:



| | | | | | | |
|---|---|---|---|---|---|---|
| $\frac{d^2y}{dx^2}$ | $-$ | $0$ | $+$ | $+$ | $0$ | $-$ |
| $\frac{dy}{dx}$ | $+$ | $0$ | $+$ | $-$ | $0$ | $-$ |

Ex12I

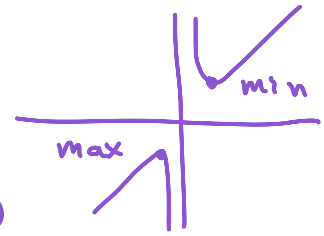
(3) e) $y = x + x^{-1}$ or $x + \frac{1}{x}$

$$\frac{dy}{dx} = 1 - x^{-2} = 0$$
$$1 = \frac{1}{x^2}$$
$$x = \pm 1$$

$$\frac{d^2y}{dx^2} = 2x^{-3} = \frac{2}{x^3}$$

$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 2 > 0$ min at $x=1$, $y = 1 + \frac{1}{1} = 2$ ie $(1,2)$

$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = -2 < 0$ max at $x=-1$, $y = -1 - \frac{1}{1} = -2$ ie $(-1,-2)$




h) $y = x^{\frac{1}{2}}(x-6) = x^{\frac{3}{2}} - 6x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - \frac{3}{x^{\frac{1}{2}}} = 0$$
$$3x = 6$$
$$x = 2$$

$$\frac{d^2y}{dx^2} = \frac{3}{4x^{\frac{1}{2}}} + \frac{3}{2x^{\frac{3}{2}}}$$

$\left.\frac{d^2y}{dx^2}\right|_{x=2} = \frac{3}{4\sqrt{2}} + \frac{3}{4\sqrt{2}} = \frac{3}{2\sqrt{2}} > 0$ min at $x=2$, $y = \sqrt{2}(2-6) = -4\sqrt{2}$ ie $(2, -4\sqrt{2})$

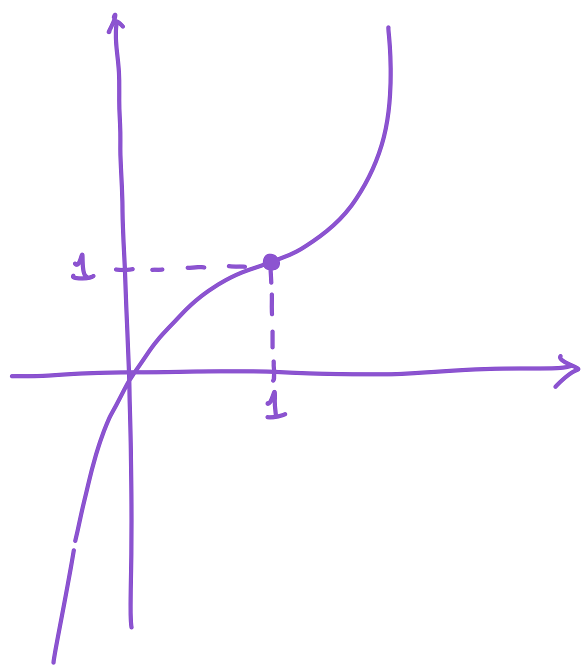
(5) $y = x^3 - 3x^2 + 3x$

$$\frac{dy}{dx} = 3x^2 - 6x + 3 = 0$$
$$x^2 - 2x + 1 = 0$$
$$x = 1$$

$$\frac{d^2y}{dx^2} = 6x - 6$$

$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 0$

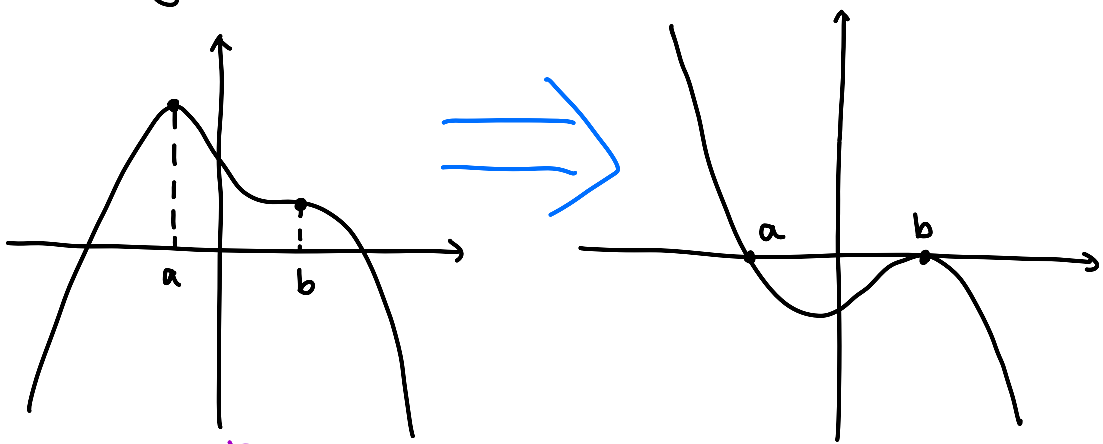
| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $\frac{dy}{dx}$ | 3 | 0 | 3 |
| | ↗ | · | ↗ |



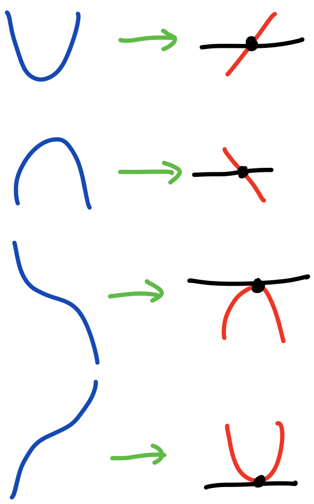

therefore, point of inflection

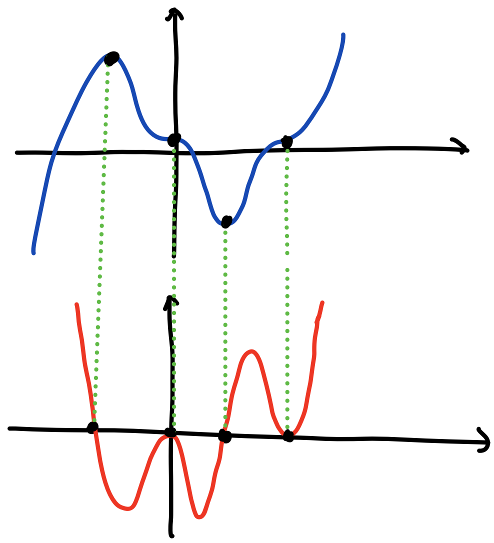
# Sketching gradient functions

plotting $f'(x)$. Remember: stationary point on $f(x) \Rightarrow$ x-intercept on $f'(x)$

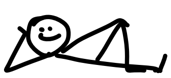



## General Rules





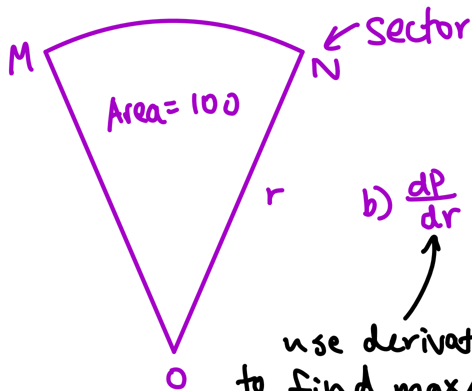
# Modelling



Use math to "summarize" a real-life scenario, then solve problems.

Ex 12K (p.281)

8




a) Area $= \frac{1}{2} r^2 \theta = 100$

$\theta = \frac{200}{r^2}$

Perimeter $= P = 2r + r\theta$

$= 2r + \frac{200}{r}$

$\left(r > \sqrt{\frac{100}{\pi}}\right)$

b) $\frac{dP}{dr} = 2 - \frac{200}{r^2} = 0$

use derivative to find max/min

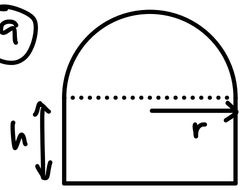
$2r^2 = 200$

$r = \sqrt{100} = 10$, $\underline{P = 20 + 20 = 40}$

# More modelling:

Ex 12K (p. 281)

(9)




a) Perimeter $= \pi r + 2h + 2r = 40$

$$h = \frac{40 - 2r - \pi r}{2}$$

$$= 20 - r - \frac{\pi}{2} r$$

Area $= A = \frac{\pi}{2} r^2 + 2rh$

$$= \frac{\pi}{2} r^2 + 40r - 2r^2 - \pi r^2$$

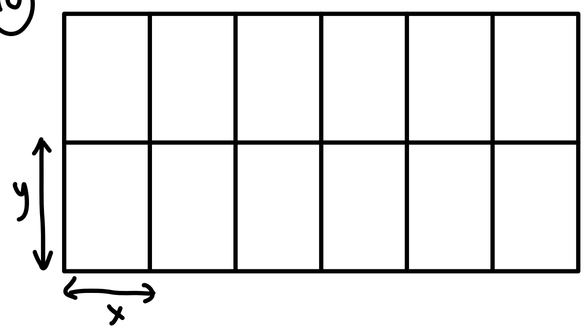
$$= 40r - 2r^2 - \frac{\pi r^2}{2}$$

b) $\frac{dA}{dr} = 40 - 4r - \pi r = 0$

$$r = \frac{40}{\pi + 4}, \quad A = 40\left(\frac{40}{\pi+4}\right) - 2\left(\frac{40}{\pi+4}\right)^2 - \frac{\pi\left(\frac{40}{\pi+4}\right)^2}{2}$$

$$= 112 \text{ units}^2$$

(10)




a)

Total wire $= 14y + 18x = 1512$

$$y = \frac{1512 - 18x}{14} = \frac{756 - 9x}{7}$$

Area $= A = (6x)(2y) = 12xy = 1296x - \frac{108x^2}{7}$

b) $\frac{dA}{dx} = 1296 - \frac{216x}{7} = 0$

$$216x = 9072$$

$$x = 42, \quad A = 1296(42) - \frac{108(42)^2}{7} = 27216 \text{ units}^2$$

# Mixed Ex 12 (p.283)

① $f(x)=10x^2 \quad f'(x)=\lim_{\delta\to 0}\frac{f(x+\delta)-f(x)}{\delta} = \lim_{\delta\to 0}\frac{10x^2+20\delta x+10\delta^2-10x^2}{\delta} = \lim_{\delta\to 0} 20x+10\delta$

$= 20x$

③ $y=3x^2+3+\frac{1}{x^2} \quad \frac{dy}{dx}=6x-\frac{2}{x^3}$

A: $\left.\frac{dy}{dx}\right|_{x=1}=6-2=4$ B: $\left.\frac{dy}{dx}\right|_{x=2}=12-\frac{1}{4}=\frac{47}{4}$ C: $\left.\frac{dy}{dx}\right|_{x=3}=18-\frac{2}{27}=\frac{484}{27}$

⑤ $y=x^3-11x+1 \quad \frac{dy}{dx}=3x^2-11=1$

$x=\pm\sqrt{4}=\pm 2 \quad y=-13, 15$

⑦ $y=3\sqrt{x}-\frac{4}{\sqrt{x}}=3x^{\frac{1}{2}}-4x^{-\frac{1}{2}} \quad \frac{dy}{dx}=\frac{3}{2x^{\frac{1}{2}}}+\frac{2}{x^{\frac{3}{2}}}=\frac{3}{2\sqrt{x}}+\frac{2}{x\sqrt{x}}$

⑨ a) $(x^{\frac{3}{2}}-1)(x^{-\frac{1}{2}}+1)=x+x^{\frac{3}{2}}-x^{-\frac{1}{2}}-1$

b) $\frac{dy}{dx}=1+\frac{3}{2}x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}-1=\frac{3}{2\sqrt{x}}+\frac{1}{2x\sqrt{x}}$

c) $\left.\frac{dy}{dx}\right|_{x=4}=\frac{3}{4}+\frac{1}{16}=\frac{13}{16}$

⑪ $f(x)=ax^2+bx+c$ passes through $(1,2)$ & $(2,1)$ $\quad f'(2)=1$

$f'(x)=2ax+b$

$\therefore \begin{cases} f'(2)=4a+b=1 & \text{— i} \\ f(1)=a+b+c=2 & \text{— ii} \\ f(2)=4a+2b+c=1 & \text{— iii} \end{cases}$

iii−ii: $3a+b=-1$ — iv

i−iv: $a=2$

$\therefore b=-7$

$\therefore c=7$

⑬ $y=\frac{8}{x}-x+3x^2,\ x>0 \quad \frac{dy}{dx}=-\frac{8}{x^2}-1+6x$

$\left.\frac{dy}{dx}\right|_{x=2}=-2-1+12=9$ = tangent gradient

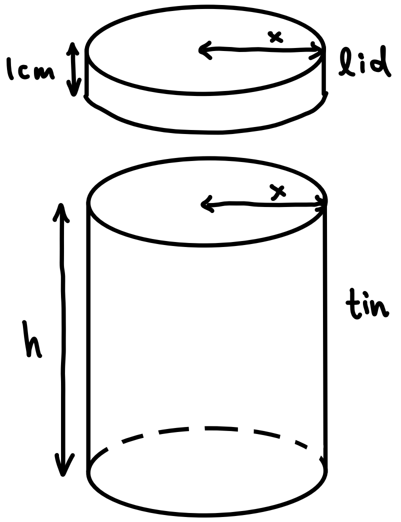
$y|_{x=2}=4-2+18=20$

normal gradient $=m=-\frac{1}{9}$

$(y-20)=-\frac{1}{9}(x-2)$

$y=-\frac{1}{9}x+\frac{2}{9}+20=-\frac{1}{9}x+\frac{182}{9}$

(28)




Made of $80\pi \text{cm}^2$ sheet metal

a) Area $= 2\pi xh + \pi x^2 + 2\pi x + \pi x^2 = 80\pi$

$$2xh + x^2 + +2x + x^2 = 80$$

$$h = \frac{80 - 2x^2 - 2x}{2x} = \frac{40 - x - x^2}{x}$$

Volume $= V = \pi x^2 h = \dfrac{40\pi x^2 - \pi x^3 - \pi x^4}{x} = \pi(40x - x^2 - x^3)$

b) $\dfrac{dV}{dx} = \pi \dfrac{d}{dx}(40x - x^2 - x^3) = \pi(40 - 2x - 3x^2) = 0$

$$3x^2 + 2x - 40 = 0$$

rej.

$$x = \frac{10}{3} \text{ or } -4 = \frac{10}{3}$$

c) $\dfrac{d^2V}{dx^2} = \pi \dfrac{d}{dx}(40 - 2x - 3x^2) = \pi(-2 - 6x)$ $\quad \left.\dfrac{d^2V}{dx^2}\right|_{x=\frac{10}{3}} = \pi(-2 - 20) = -22\pi < 0$

$\therefore$ this is a maximum

d) $V|_{x=\frac{10}{3}} = \pi\left(\dfrac{400}{3} - \dfrac{100}{9} - \dfrac{1000}{27}\right) = 268 \text{ cm}^2$

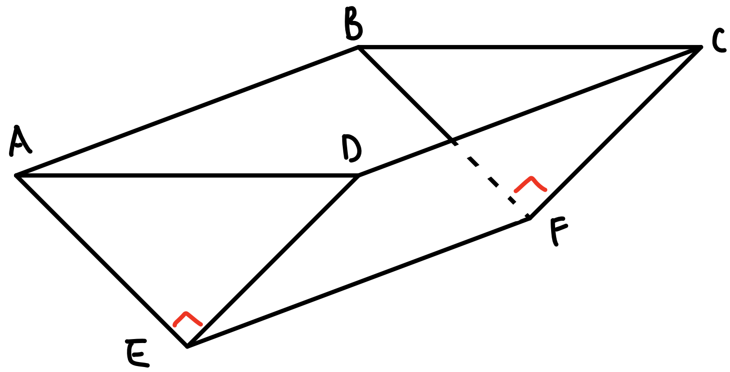
e) when $x = \frac{10}{3}$, area used in lid $= \pi x^2 + 2\pi x = \dfrac{100}{9}\pi + \dfrac{20}{3}\pi = \dfrac{160}{9}\pi$

percentage $= \dfrac{\frac{160}{9}\pi}{80\pi} = 22.2\%$

(29) Tank for storing water

$AD = x$ m

$\Delta ADE$ & $\Delta BCF$ are isosceles




a) $(AD)^2 = 2(AE)^2$

$AE = \frac{AD}{\sqrt{2}} = \frac{AD\sqrt{2}}{2}$ Area of $\Delta ADE = \frac{1}{2}(AE)^2 = \frac{1}{2}\left(\frac{2(AD)^2}{4}\right) = \frac{1}{4}x^2$ $m^2$

Capacity (ie volume) $= 4000 m^3$ total area of 2 triangles & 2 rectangles $= S$ $m^2$

b) let $EF = DC = AB = l$

$S = \frac{1}{2}x^2 + xl\sqrt{2} = \frac{x^2}{2} + \frac{16000\sqrt{2}}{x}$

Volume $= \frac{1}{4}x^2 l = 4000$

$l = \frac{16000}{x^2}$

c) $\frac{dS}{dx} = x - \frac{16000\sqrt{2}}{x^2} = 0$

$x^3 = 16000\sqrt{2}$

$x = \sqrt[3]{16000\sqrt{2}}$

$S\big|_{x=\sqrt[3]{16000\sqrt{2}}} = \frac{(16000\sqrt{2})^{\frac{2}{3}}}{2} + \frac{16000\sqrt{2}}{\sqrt[3]{16000\sqrt{2}}} = \frac{(16000\sqrt{2})^{\frac{2}{3}}}{2} + (16000\sqrt{2})^{\frac{2}{3}}$

$= \frac{3}{2}(16000\sqrt{2})^{\frac{2}{3}} = 1200$ $m^2$

d) $\frac{d^2S}{dx^2} = 1 + \frac{32000\sqrt{2}}{x^3}$ $\frac{d^2S}{dx^2}\Big|_{x=\sqrt[3]{16000\sqrt{2}}} = 1 + \frac{32000\sqrt{2}}{16000\sqrt{2}} = 1+2 = 3 > 0$

since $\frac{d^2S}{dx^2} > 0$, this is a minimum