Differentiating $\frac{dy}{dx}$ (Trigonometric Functions)

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$

Prove using first principles and small angle approximations.

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Prove by Quotient Rule and $\tan x = \frac{\sin x}{\cos x}$

Proof of $\frac{d}{dx}(\sin x) = \cos x$

$$f(x) = \sin(x)$$

$$f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} = \lim_{\delta \to 0} \frac{\sin(x+\delta) - \sin(x)}{\delta}$$

$$= \lim_{\delta \to 0} \frac{\sin x \cos\delta + \cos x \sin\delta - \sin x}{\delta}$$

$$= \lim_{\delta \to 0} \left(\sin x \left(\frac{\cos\delta - 1}{\delta} \right) + \cos x \left(\frac{\sin\delta}{\delta} \right) \right) = \sin x \cdot \lim_{\delta \to 0} \left(\frac{\cos\delta - 1}{\delta} \right) + \cos x \cdot \lim_{\delta \to 0} \left(\frac{\sin\delta}{\delta} \right)$$

$$\because \lim_{\delta \to 0} \sin\delta \approx \delta, \quad \therefore \lim_{\delta \to 0} \frac{\sin\delta}{\delta} \approx \lim_{\delta \to 0} \frac{\delta}{\delta} = 1$$

$$\because \lim_{\delta \to 0} \cos\delta \approx 1 - \frac{\delta^2}{2}, \quad \therefore \lim_{\delta \to 0} \left(\frac{\cos\delta - 1}{\delta} \right) \approx \lim_{\delta \to 0} \frac{1 - \frac{\delta^2}{2} - 1}{\delta} = \lim_{\delta \to 0} \left(-\frac{\delta}{2} \right) = 0$$

Small Angle Approximations

$$\therefore f'(x) = \sin x \cdot 0 + \cos x \cdot 1 = \cos x$$

$$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$$ (Chain rule)

This applies for all 3 trigonometric functions.

Other Trig Functions : Differentiated

$$\frac{dy}{dx}(\sec x) = \sec x \tan x$$

$$\frac{dy}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{dy}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

Differentiating

$\frac{dy}{dx}$(Exponential Functions)

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\ln(ax)) = \frac{a}{ax}$$

From chain rule

Differentiating (Chain Rule)

$\frac{dy}{dx}$ (Composite Functions)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

ie. let $u = g(x)$

$$\frac{d}{dx}\left(f(g(x))\right) = \frac{d}{dx}\left(f(u)\right) = f'(u) \cdot g'(x)$$

Example: $y = e^{3x}$ $\quad u = 3x$

$= e^{u}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du}(e^u) \cdot \frac{d}{dx}(3x)$$

$$= e^u \cdot 3 = 3e^u = 3e^{3x}$$

Differentiating (Product Rule)

$\frac{dy}{dx}$ (Product of Functions)

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Example: $f(x) = \sin x \quad g(x) = \cos x$

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$= \cos x \cdot \cos x + \sin x \cdot (-\sin x) = \cos^2 x - \sin^2 x$$

Quotient Rule

minus, not plus.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) \cdot g(x) - f(x) \cdot g(x)}{(g(x))^2}$$

Example: $\frac{d}{dx}\left(\frac{3x}{2x+1}\right) = \frac{\frac{d}{dx}(3x) \cdot (2x+1) - 3x \cdot \frac{d}{dx}(2x+1)}{4x^2+4x+1}$

$$= \frac{(6x+3) - (6x)}{4x^2+4x+1} = \frac{3}{4x^2+4x+1}$$

Sometimes it's better to simplify the fraction instead of using this rule.

e.g. $\frac{d}{dx}\left(\frac{x}{\sqrt{x}}\right) = \frac{d}{dx}(\sqrt{x})$

Parametric Differentiation

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

These cancel out!

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

$$\frac{dx}{dt} = -\sin t \qquad \frac{dy}{dt} = \cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \cos t \cdot \left(-\frac{1}{\sin t}\right)$$

$$= -\frac{\cos t}{\sin t} = -\cot(t)$$

Implicit Differentiation

If we can't rearrange something into the form $y = f(x)$, use implicit differentiation!

Example: eqn of circle

$x^2 + y^2 = 16$ ← since y is a function of x,

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(16)$ we can apply the chain rule!

$2x + 2y\frac{dy}{dx} = 0$ (e.g. $y^2 \rightarrow \underbrace{2y}_{\text{Derivative of "outside function" (ie } x^2 \rightarrow 2x)} \frac{dy}{dx}$ } Derivative of "inside function" (ie y)

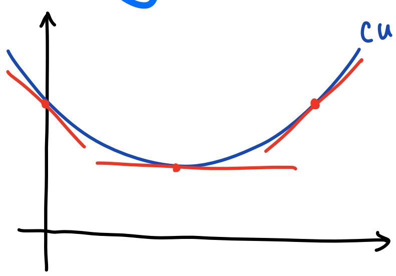
$\frac{dy}{dx} = -\frac{2x}{2y}$ Solve for $\frac{dy}{dx}$

$= -\frac{x}{y}$

In general: $y^n \rightarrow ny^{n-1}\frac{dy}{dx}$

$$f(x) \cdot h(y) = \overbrace{f'(x)h(y) + f(x)h'(y)\underbrace{\frac{dy}{dx}}_{\text{Chain Rule}}}^{\text{Product Rule}}$$

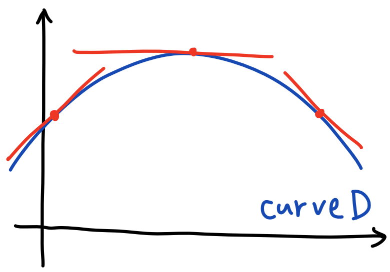
Using 2nd Derivatives



curve C is <u>convex</u> within an interval

$\underbrace{\frac{d^2y}{dx^2} > 0}_{\downarrow}$ for values of x within this interval

The slope of the tangents are increasing



curve D is <u>concave</u> within an interval

$\underbrace{\frac{d^2y}{dx^2} < 0}_{\downarrow}$ for values of x within this interval

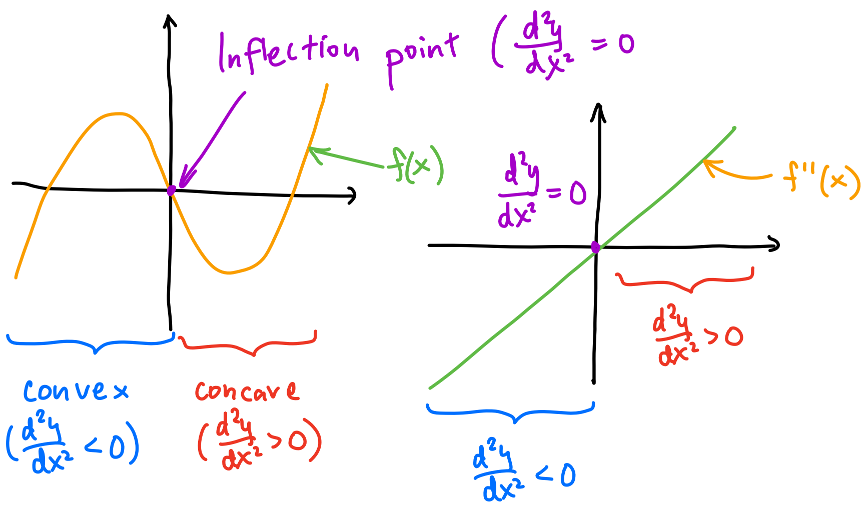
The slope of the tangents are decreasing

"Curvature" of a graph / function

Points of Inflection

When $\frac{d^2y}{dx^2}$ changes sign (curvature changes sign)

e.g.



!! $\frac{d^2y}{dx^2} = 0$ ALONE is INSUFFICIENT EVIDENCE of an inflection point!

(The CHANGE IN SIGN is crucial!)