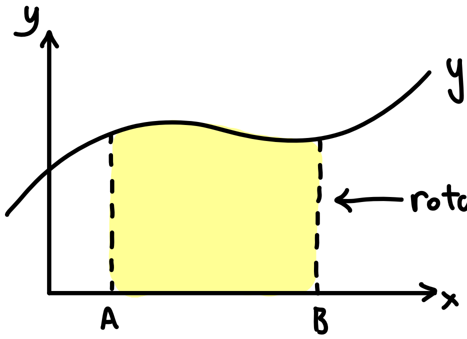
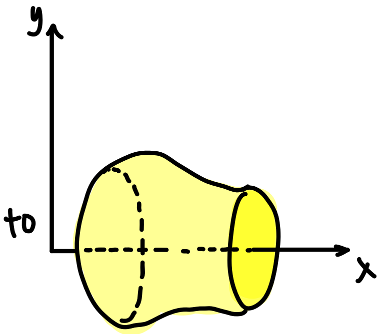
VOLUMES OF REVOLUTION

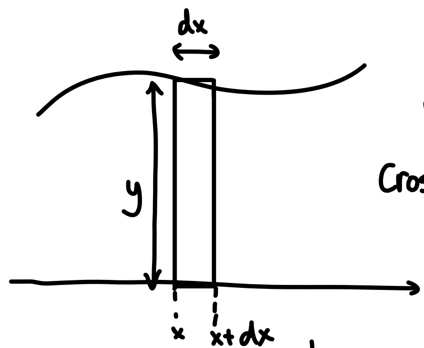
(ft. Integration)



← rotate area around x-axis to sweep out volume



Volume $= \boxed{\pi \int_a^b y^2 \, dx}$



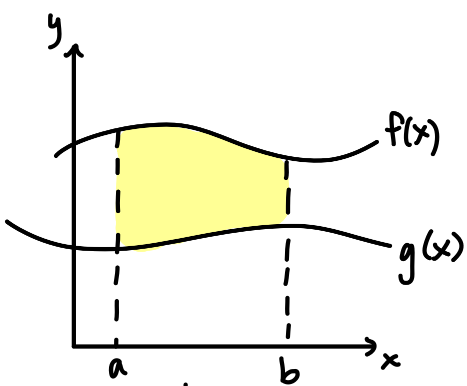
$r = y$

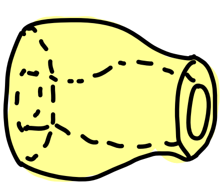
Cross-sectional area $= \pi r^2 = \pi y^2$

Volume of thin slice $= \pi y^2 h$

$= \pi y^2 dx$

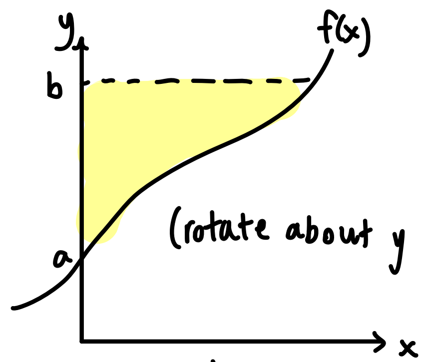
Volume of all slices $= \int_a^b \pi y^2 dx = \pi \int_a^b y^2 dx$



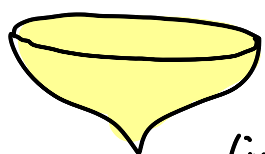


Volume $= \pi \int_a^b (f(x))^2 dx - \pi \int_a^b (g(x))^2 dx$

(hollow center)



(rotate about y axis)



Volume $= \pi \int_a^b x^2 dy$

(just flip x and y)

Exercise 5D (Modelling)

1. a) $k=10$ (10m tall tent at fair)

b) $0.01x^2 = -y^2+k^2$

$x^2 = -100y^2+100k^2$

$$\text{Volume} = \pi\int_a^b x^2\,dy = \pi\int_0^k(-100y^2+100k^2)\,dy = -100\pi\int_0^k(y^2-k^2)\,dy$$

$$= -100\pi\left[\frac{1}{3}y^3-k^2y\right]_0^k = -100\pi\left(\left(\frac{1}{3}k^3-k^3\right)-(0)\right) = -100\pi\left(-\frac{2}{3}k^3\right)$$

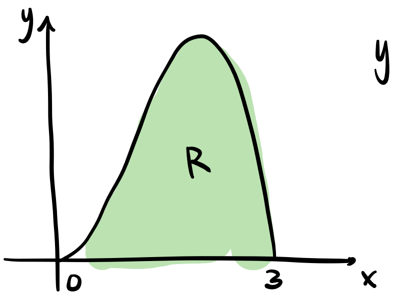
$$= \frac{200}{3}k^3\pi \qquad \text{with } k=100,\ \text{Volume} = \frac{2000000}{3}\pi\ m^3$$

$$\approx 2.09\times10^6\ m^3$$

c) Not exact shape

Mixed Exercise 5

1.

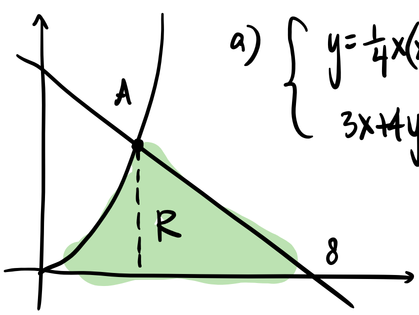


$y = x^2\sqrt{9-x^2} \qquad y^2 = x^4(9-x^2) = 9x^4-x^6$

$$\text{Volume about x-axis} = \pi\int_0^3(9x^4-x^6)\,dx = \pi\left[\frac{9}{5}x^5-\frac{1}{7}x^7\right]_0^3$$

$$= \pi\left(\left(\frac{2187}{5}-\frac{2187}{7}\right)-(0)\right) = \frac{4374}{35}\pi$$

5.



a) $\begin{cases} y = \frac{1}{4}x(x+1)^2 & \text{—①} \\ 3x+4y=24 & \text{—②} \end{cases}$

② : $y = -\frac{3}{4}x+6 \rightarrow$ ① : $-\frac{3}{4}x+6 = \frac{1}{4}x^3+\frac{1}{2}x^2+\frac{1}{4}x$

$\frac{1}{4}x^3+\frac{1}{2}x^2+x-6=0$

$x^3+2x^2+4x-24=0$

$x=2$

$y = -\frac{3}{2}+6 = 4.5 \quad \therefore A(2, 4.5)$

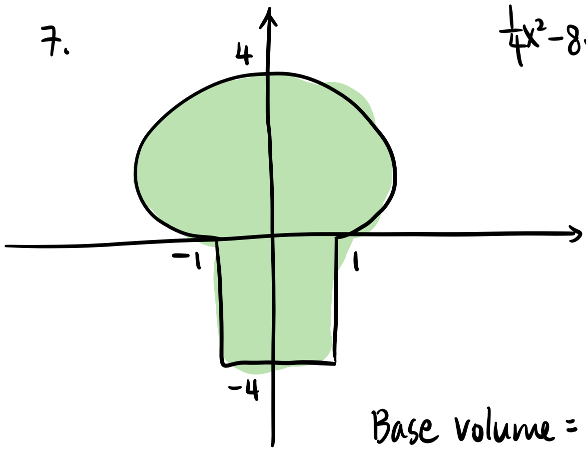
b) $$V_1 = \pi\int_0^2\left(\frac{1}{16}x^2(x^4+4x^3+6x^2+4x+1)\right)dx = \frac{1}{16}\pi\int_0^2(x^6+4x^5+6x^4+4x^3+x^2)\,dx$$

$$= \frac{1}{16}\pi\left[\frac{1}{7}x^7+\frac{2}{3}x^6+\frac{6}{5}x^5+x^4+\frac{1}{3}x^3\right]_0^2 = \frac{1}{16}\pi\left(\frac{128}{7}+\frac{128}{3}+\frac{192}{5}+16+\frac{8}{3}\right) = \frac{1549}{210}\pi$$

$V_2 = \text{cone} = \frac{1}{3}\pi(4.5)^2\times6 = \frac{81}{2}\pi$

$V = \frac{1549}{210}\pi+\frac{81}{2}\pi = \frac{5027}{105}\pi$

7.



$\frac{1}{4}x^2 - 8\sqrt{y} + 4y = 0$

$x^2 = 32\sqrt{y} - 16y$

$\text{Volume} = \pi\int_0^4 x^2\,dy = \pi\int_0^4 (32\sqrt{y} - 16y)\,dy$

$= \pi\left[\frac{64}{3}y^{\frac{3}{2}} - 8y^2\right]_0^4 dy = \pi\left(\left(\frac{512}{3} - 128\right) - (0)\right) = \frac{128}{3}\pi$

Base volume $= 4 \times \pi \times (1)^2 = 4\pi$

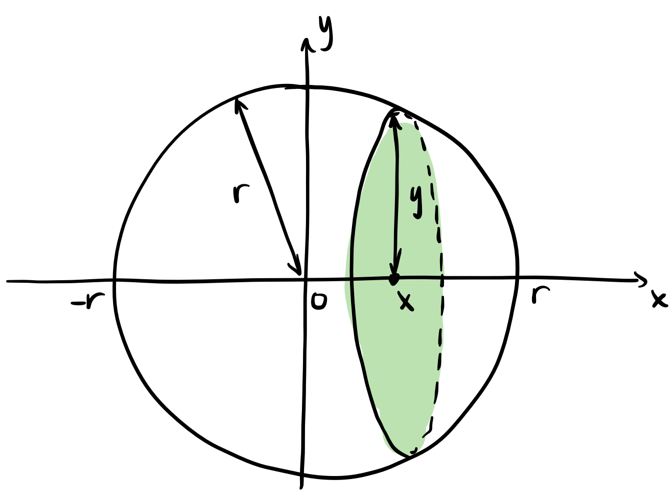
total $= \frac{128}{3}\pi + \frac{12}{3}\pi = \frac{140}{3}\pi \text{ cm}^3$

Challenge

a) $x^2 + y^2 = r^2$

height at given x is $\sqrt{r^2 - x^2} = y$

area of disc $= \pi y^2 = \pi(r^2 - x^2) = A$



b) $A = \pi(r^2 - x^2)$

$\text{Volume} = \int_{-r}^{r} \pi(r^2 - x^2)\,dx$

$= \pi\int_{-r}^{r}(r^2 - x^2)\,dx$

$= \pi\left[r^2x - \frac{1}{3}x^3\right]_{-r}^{r}$

$= \pi\left(\left(r^3 - \frac{1}{3}r^3\right) - \left(-r^3 + \frac{1}{3}r^3\right)\right)$

$= \pi\left(2r^3 - \frac{2}{3}r^3\right) = \frac{4}{3}\pi r^3$

Mixed Exercise

1. $\text{Volume} = \pi\int y^2\,dx = \pi\int_0^3 x^4(9-x^2)\,dx = \pi\int_0^3 (9x^4-x^6)\,dx = \pi\left[\frac{9}{5}x^5 - \frac{1}{7}x^7\right]_0^3$

$$= \pi\left(\left(\frac{2187}{5} - \frac{2187}{7}\right) - (0)\right) = \frac{4374}{35}\pi \approx 125$$

3. a) $f(x) = x^2+4x+4 = y$

$(x+2)^2 = y$

$x+2 = \sqrt{y}$

$x = \sqrt{y}-2$

$x^2 = 4-4\sqrt{y}+y$

b) $\text{Volume} = \pi\int x^2\,dy = \pi\int_4^9 (4-4\sqrt{y}+y)\,dy$

$= \pi\left[4y - \frac{8}{3}y^{\frac{3}{2}} + \frac{1}{2}y^2\right]_4^9$

$= \pi\left(\left(36-72+\frac{81}{2}\right) - \left(16 - \frac{64}{3} + 8\right)\right)$

$= \frac{11}{6}\pi$

5. a) $3x + x(x+1)^2 = 24$

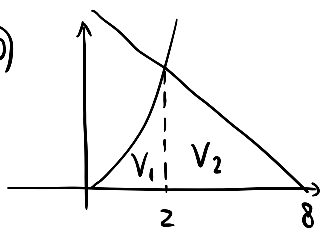
$x^3+2x^2+x+3x = 24$

$x^3+2x^2+4x-24 = 0 = f(x)$

$f(2) = 0 \quad \therefore$ 2 is a root.

$6+4y = 24$

$y = 18 \div 4 = 4.5 \quad \therefore (2, 4.5) = A$

b) 

$V_1 = \pi\int_0^2 \frac{1}{16}x^2(x+1)^4\,dx = \frac{\pi}{16}\int_0^2 x^2(x^4+4x^3+6x^2+4x+1)\,dx$

$= \frac{\pi}{16}\int_0^2 (x^6+4x^5+6x^4+4x^3+x^2)\,dx$

$= \frac{\pi}{16}\left[\frac{1}{7}x^7 + \frac{2}{3}x^6 + \frac{6}{5}x^5 + x^4 + \frac{1}{3}x^3\right]_0^2$

$= \frac{\pi}{16}\left(\frac{128}{7} + \frac{128}{3} + \frac{192}{5} + 16 + \frac{8}{3}\right) = \frac{1549}{210}\pi$

$V_2 = \pi\int_2^8 y^2\,dx = \pi\int_2^8 \left(-\frac{3}{4}x+6\right)^2 dx = \pi\int_2^8 \left(\frac{9}{16}x^2 - 9x + 36\right)dx = \pi\left[\frac{3}{16}x^3 - \frac{9}{2}x^2 + 36x\right]_2^8$

$= \pi\left((96-288+288) - \left(\frac{3}{2} - 18 + 72\right)\right) = \frac{81}{2}\pi$

$V_1 + V_2 = \frac{5027}{105}\pi \approx 150$

7. $\frac{1}{4}x^2 - 8\sqrt{y} + 4y = 0$

$x^2 = 32\sqrt{y} - 16y$

Volume of cap $= \pi \int_0^4 x^2 \, dy = \pi \int_0^4 (32y^{\frac{1}{2}} - 16y) \, dy$

$= \pi \left[\frac{64}{3} y^{\frac{3}{2}} - 8y^2 \right]_0^4 = \pi \left(\frac{512}{3} - 128 \right) = \frac{128}{3}\pi$

Volume of stem $= \pi r^2 \times h = 4\pi$

Total volume $= \frac{140}{3}\pi$ cm^3