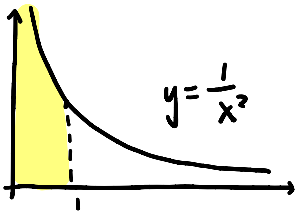
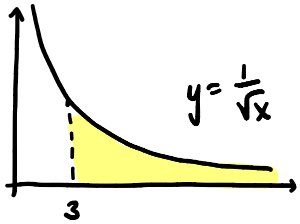
Methods in Calculus

Improper integrals

- One or both the limits are infinite
- The function is indefined within the limit





$\int_1^\infty \frac{1}{x^2}\,dx$ $\qquad$ $\int_0^3 \frac{1}{\sqrt{x}}\,dx$

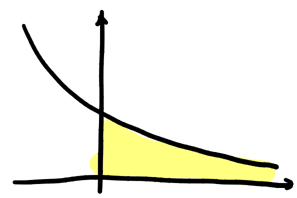
If an improper integral exists, it is convergent (the area converges on a value)

if not, it is divergent

$I = \int_0^\infty e^{-x}\,dx$ ← does this exist / is this convergent?

finite

consider $\int_0^t e^{-x}\,dx = \left[-e^{-x}\right]_0^t = -e^{-t} + 1$

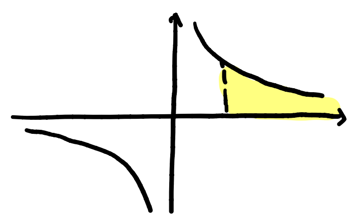


$\lim_{t\to\infty}\left(-e^{-t}+1\right) = \lim_{t\to\infty}\left(-\frac{1}{e^t}+1\right) = 1$

$I \to 1$ as $t \to \infty$

$\therefore I = \int_0^\infty e^{-x}\,dx = 1$ I is convergent

$\int_1^\infty \frac{1}{x}\,dx$ $\qquad$ consider $\int_1^t \frac{1}{x}\,dx = \left[\ln x\right]_1^t = \ln t - \ln 1 = \ln t$



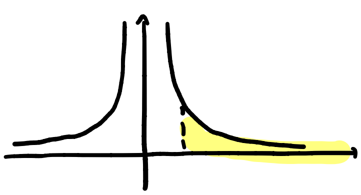
$\lim_{t\to\infty}(\ln t)$ $\qquad$ $I \to \infty$ as $t \to \infty$

$\therefore I = \int_0^\infty \frac{1}{x}\,dx$ is divergent / doesn't exist.

$\int_1^\infty \frac{1}{x^2}\,dx$ $\qquad$ consider $\int_1^t \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^t = -\frac{1}{t} + 1$

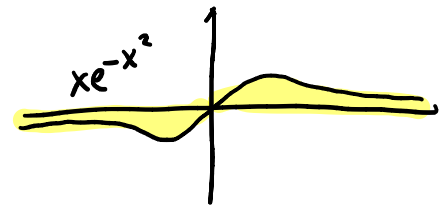
$\lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = -\lim_{t\to\infty}\left(\frac{1}{t}\right) + \lim_{t\to\infty}(1) = 1$ $\qquad$ $I \to 1$ as $t \to \infty$

$\therefore I = \int_1^\infty \frac{1}{x^2}\,dx = 1$ I is convergent.



If both limits are infinite, split the integral

$\int_{-\infty}^{\infty} xe^{-x^2}\,dx$ ← split into 2: $\int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx$



consider $\int_{-t}^{0} xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_{-t}^{0} = -\frac{1}{2} + \frac{1}{2}e^{-t^2}$ $\quad \lim\left(-\frac{1}{2}+\frac{1}{2}e^{-t^2}\right) = -\frac{1}{2}$

just reverse chain rule $\quad y = e^{-x^2} \quad \frac{dy}{dx} = -2xe^{-x^2} \quad \therefore I = -\frac{1}{2}e^{-x^2} + c$

consider $\int_{0}^{t} xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_{0}^{t} = -\frac{1}{2}e^{-t^2} + \frac{1}{2}$ $\quad \lim\left(-\frac{1}{2}e^{-t^2} + \frac{1}{2}\right) = \frac{1}{2}$

Both integrals converge, ∴ their sum converges

$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0$

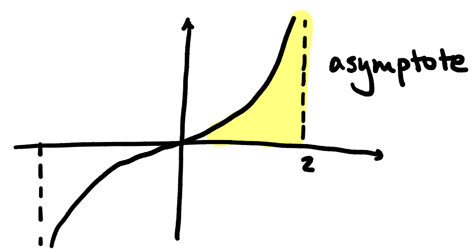
if either one doesn't converge, their sums diverge.

If the limits are <u>NOT</u> infinite but the function is undefined, do the same thing.

$\int_{0}^{2} \frac{x}{\sqrt{4-x^2}}\,dx$ $\quad$ consider $\int_{0}^{t} \frac{x}{\sqrt{4-x^2}}\,dx = \left[-\sqrt{4-x^2}\right]_{0}^{t} = -\sqrt{4-t^2} + 2$

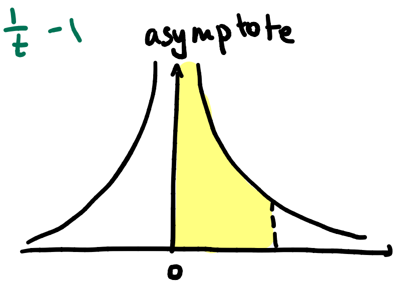
integration by substitution

$\lim_{t\to 2}\left(-\sqrt{4-t^2} + 2\right) = 2 \quad \therefore \int_{0}^{2} \frac{x}{\sqrt{4-x^2}}\,dx = 2$



when $x=2$, $\sqrt{4-x^2} = 0$ and we can't divide by 0
replace the limit with t and take the limit!

$\int_{0}^{1} \frac{1}{x^2}\,dx$ $\quad$ consider $\int_{t}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{t}^{1} = -1 + \frac{1}{t} = \frac{1}{t} - 1$



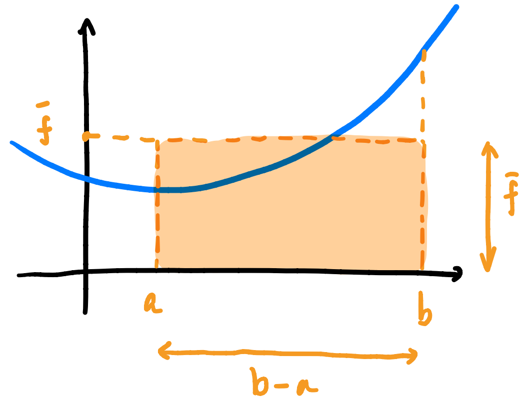
$\lim_{t\to 0}\left(\frac{1}{t} - 1\right)$ $\quad$ as $t \to 0$, $\frac{1}{t} - 1 \to \infty$

$\therefore \int_{0}^{1} \frac{1}{x^2}\,dx$ is divergent / doesn't exist

Mean Value of a F(unction) (over an interval)

$$\boxed{\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

The hardest part is just integrating the given function



how is this even part of the syllabus.

Differentiating Inverse Trigonometric Functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$ prove!

$$\frac{d}{dx}(\arccos) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

let $y = \arcsin x$

$$\sin y = \sin(\arcsin x) = x$$

$\sin y = x$

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$ implicit differentiation

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

Domain and range of inverse trig functions

f(x)	domain	range
arcsin x	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
arccos x	$[-1, 1]$	$[0, \pi]$
arctan x	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right), \quad |x| < a$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$$

(in formula book)

$$y = \arctan\left(\frac{1-x}{1+x}\right)$$

$$\tan y = \frac{1-x}{1+x}$$

$$\sec^2 y \frac{dy}{dx} = \frac{-(1+x)-(1-x)}{(1+x)^2} = -\frac{2}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}\left(-\frac{2}{(1+x)^2}\right) = -\frac{1}{1+\tan^2 y}\left(\frac{2}{(1+x)^2}\right) = -\frac{1}{1+\left(\frac{1-x}{1+x}\right)^2}\cdot\frac{2}{(1+x)^2} = -\frac{(1+x)^2}{(1+x)^2+(1-x)^2}\cdot\frac{2}{(1+x)^2}$$

$$= -\frac{2}{1+2x+x^2+1-2x+x^2} = -\frac{2}{2+2x^2} = -\frac{1}{1+x^2}$$

Mixed Ex3

1 a) $\int \frac{1}{e^x+e^{-x}}\,dx$ $\quad u=e^x$

$du = e^x dx = u\,dx$

$= \int \frac{1}{u+\frac{1}{u}}\,dx$

$= \int \frac{u}{u^2+1}\,dx = \int \frac{1}{u^2+1}\,du = \arctan u + c = \arctan e^x + c$ ✓

b) $\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx = \lim_{t\to\infty}\left(\int_{-t}^{0} \frac{1}{e^x+e^{-x}}\,dx + \int_{0}^{t} \frac{1}{e^x+e^{-x}}\,dx\right) = \lim_{t\to\infty}\left[\arctan e^x\right]_{-t}^{0} + \lim_{t\to 0}\left[\arctan e^x\right]_{0}^{t}$

$= \lim_{t\to\infty}\left(\arctan 1 - \arctan \frac{1}{e^t}\right) + \lim_{t\to\infty}\left(\arctan e^t - \arctan 1\right)$

$= \lim_{t\to\infty}\left(\frac{\pi}{4} - \arctan\frac{1}{e^t}\right) + \lim_{t\to\infty}\left(\arctan e^t - \frac{\pi}{4}\right) = \left(\frac{\pi}{4} - \arctan 0\right) + \left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{2}$ ✓

2. $f(x) = \frac{1-\cos x}{\sin^2 x} = \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}$ $\quad \int\left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right)dx = \int \frac{1}{\sin^2 x}\,dx - \int \frac{\cos x}{\sin^2 x}\,dx$

$\int \frac{1}{\sin^2 x}\,dx = \int \operatorname{cosec}^2 x\,dx = -\cot x + c$

$\int \frac{\cos x}{\sin^2 x}\,dx$ $\quad$ let $u=\sin x$ $\quad du = \cos x\,dx$

$= \int \frac{1}{u^2}\,du = -\frac{1}{u} + c = -\frac{1}{\sin x} + c$ $\quad \therefore \int f(x)\,dx = -\cot x + \frac{1}{\sin x} + c$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)\,dx = \left[-\cot x + \frac{1}{\sin x}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \left(-\frac{1}{\tan\frac{\pi}{3}} + \frac{1}{\sin\frac{\pi}{3}}\right) - \left(-\frac{1}{\tan\frac{\pi}{6}} + \frac{1}{\sin\frac{\pi}{6}}\right)$

$= -\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{3}{\sqrt{3}} - 2 = \frac{4}{\sqrt{3}} - \frac{2\sqrt{3}}{\sqrt{3}} = \frac{4-2\sqrt{3}}{\sqrt{3}}$ or $\frac{4}{\sqrt{3}} - 2$

$\bar{f} = \left(\frac{4-2\sqrt{3}}{\sqrt{3}}\right) \div \left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \left(\frac{4-2\sqrt{3}}{\sqrt{3}}\right) \times \left(\frac{6}{\pi}\right) = \frac{24-12\sqrt{3}}{\pi\sqrt{3}}$ ✓ $\frac{6}{\pi}\left(\frac{4}{\sqrt{3}} - 2\right)$

3. $f(x) = x\sin 2x$ $\quad \int x\sin 2x\,dx = -\frac{1}{2}x\cos 2x - \int\left(-\frac{1}{2}\cos 2x\right)dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$

$\int_0^{\frac{\pi}{2}} f(x)\,dx = \left[-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}} = \left(-\frac{\pi}{4}\cos\pi + \frac{1}{4}\sin\pi\right) - \left(-\frac{1}{4}\sin 0\right) = \frac{\pi}{4}$

$\bar{f} = \frac{\pi}{4} \div \left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4} \times \frac{2}{\pi} = \frac{1}{2}$ ✓

4. a) $\frac{d}{dx}\left(\arccos x^2\right) = -\frac{1}{\sqrt{1-x^4}}(2x) = -\frac{2x}{\sqrt{1-x^4}}$

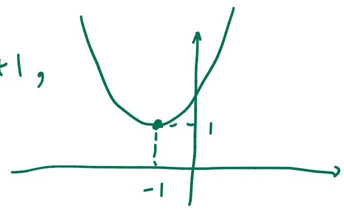
b) $\int \frac{3x}{\sqrt{16-x^4}}\,dx = \int \frac{3x}{4\sqrt{1-\frac{1}{16}x^4}}\,dx = \frac{3}{8}\int \frac{2x}{\sqrt{1-\frac{1}{16}x^4}}\,dx = \frac{3}{8}\int \frac{2x}{\sqrt{1-\left(\frac{1}{2}x\right)^4}}\,dx = -\frac{3}{4}\arccos\left(\frac{1}{2}x\right)^2 \cdot 2 + c$

$= -\frac{3}{2}\arccos \frac{1}{4}x^2 + c$

5. a) $f(x) = \arctan\left(\frac{2x+3}{x-1}\right)$ $\quad f'(x) = \frac{1}{1+\left(\frac{2x+3}{x-1}\right)^2} \cdot \left(\frac{2(x-1)-(2x+3)}{(x-1)^2}\right) = \frac{1}{\frac{x^2-2x+1+4x^2+12x+9}{(x-1)^2}} \cdot \left(-\frac{5}{(x-1)^2}\right)$

$= \frac{(x-1)^2}{5x^2+10x+10} \cdot \left(-\frac{5}{(x-1)^2}\right) = -\frac{1}{x^2+2x+2}$

b) Considering $g(x) = x^2+2x+2 = (x+1)^2+1$,



$g(2) = 4+4+2 = 10$

$g(-2) = 4-4+2 = 2$

in domain $-2 \le x \le 2$, range: $1 \le g(x) \le 10$

$\therefore f'(x) = -\frac{1}{x^2+2x+2} = -\frac{1}{g(x)}$ range: $-1 \le f'(x) \le -\frac{1}{10}$ $\quad \therefore |f'(x)| \le 1$, $-2 \le x \le 2$

6. a) One or both the limits are infinity OR function undefined within limits. domain

b) $\int_0^\infty \frac{1}{(x+1)\sqrt{x}}\,dx$ upper limit $= \infty$ AND $\frac{1}{(x+1)\sqrt{x}}$ undefined at $x=0$. lower limit

c) $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$ $\quad \int \frac{1}{(x+1)\sqrt{x}}\,dx$ $\quad$ let $u = \sqrt{x}$

$du = -\frac{1}{\sqrt{x}}\,dx$ $\quad \frac{1}{2\sqrt{x}}\,dx$

$= \int -\frac{1}{u^2+1}\,du \times 2$

$= +2\arctan u + c = +2\arctan\sqrt{x} + c$ $\quad 2\arctan\sqrt{x} + c$

$\int_0^\infty \frac{1}{(x+1)\sqrt{x}}\,dx = \lim_{s\to 0}\int_s^1 \frac{1}{(x+1)\sqrt{x}}\,dx + \lim_{t\to\infty}\int_1^t \frac{1}{(x+1)\sqrt{x}}\,dx$ $\quad \arctan\sqrt{t} \to \frac{\pi}{2}$ as $t \to \infty$

$= \lim_{s\to 0}\left[-\arctan\sqrt{x}\right]_s^1 + \lim_{t\to\infty}\left[-\arctan\sqrt{x}\right]_1^t = \lim_{s\to 0}\left(-\arctan 1 + \arctan\sqrt{s}\right) + \lim_{t\to\infty}\left(-\arctan\sqrt{t} + \arctan 1\right)$

$= -\frac{\pi}{4} + 0 - \frac{\pi}{2} + \frac{\pi}{4} = -\frac{\pi}{2}$ $\quad \int_0^\infty \frac{1}{(x+1)\sqrt{x}}\,dx = \pi$

7. $f(x)=\frac{1+5x}{\sqrt{1-5x^2}}=\frac{1}{\sqrt{1-5x^2}}+\frac{5x}{\sqrt{1-5x^2}}$

$\int f(x)\,dx=\int\frac{1}{\sqrt{1-5x^2}}\,dx+\int\frac{5x}{\sqrt{1-5x^2}}\,dx$ $\qquad \int\frac{1}{\sqrt{1-5x^2}}\,dx=\frac{1}{\sqrt5}\arcsin\sqrt5 x+c$

$\int\frac{5x}{\sqrt{1-5x^2}}\,dx$ $\qquad u=5x^2$

$du=10x\,dx$

$=\frac12\int\frac{10x}{\sqrt{1-5x^2}}\,dx=\frac12\int\frac{1}{\sqrt{1-u}}\,du=-\frac12\sqrt{1-u}\cdot 2+c=-\sqrt{1-u}+c=-\sqrt{1-5x^2}$

$\therefore \int f(x)\,dx=-\sqrt{1-5x^2}+\frac{1}{\sqrt5}\arcsin\sqrt5 x+c$ $\qquad A=-1\quad B=\frac{1}{\sqrt5}$ ✓

8. a) $\int_0^t\frac{1}{x^2+1}\,dx=\left[\arctan x\right]_0^t=\arctan t$

$x=\tan\theta \quad dx=\sec^2\theta\,d\theta$

$1+x^2=1+\tan^2\theta+\sec^2\theta$ $\Rightarrow \int\frac{1}{x^2+1}\,dx=\int\frac{\sec^2\theta}{\sec^2\theta}\,d\theta=\theta+c$

$=\arctan x+c$

b) i) $\int_0^\infty\frac{1}{x^2+1}\,dx=\lim_{t\to\infty}\int_0^t\frac{1}{x^2+1}\,dx=\lim_{t\to\infty}(\arctan t)=\frac{\pi}{2}$ ✓ $\left(\arctan t\to\frac{\pi}{2} \text{ as } t\to\infty\right)$

ii) $\int_{-\infty}^\infty\frac{1}{x^2+1}\,dx=\int_{-\infty}^0\frac{1}{x^2+1}\,dx+\int_0^\infty\frac{1}{x^2+1}\,dx=\lim_{t\to\infty}\left[\arctan x\right]_{-t}^0+\lim_{t\to\infty}\left[\arctan x\right]_0^t$

$=\lim_{t\to\infty}(-\arctan t)+\lim_{t\to\infty}(\arctan t)=-\frac{\pi}{2}+\frac{\pi}{2}=0$ $\qquad \frac{\pi}{2}+\frac{\pi}{2}=\pi$

$\lim_{t\to\infty}(-\arctan(-t))$ $\qquad \arctan(-t)\to-\frac{\pi}{2}$ as $t\to\infty$