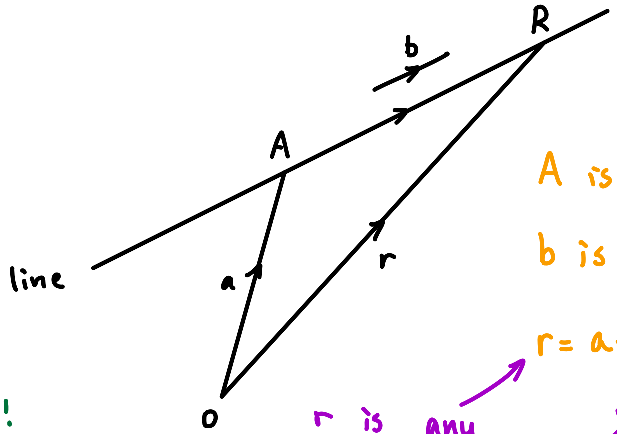
Vectors

Equation of a line



A is a point on the line

b is the direction of the line

$r = a + \lambda b$

r is any point on the line

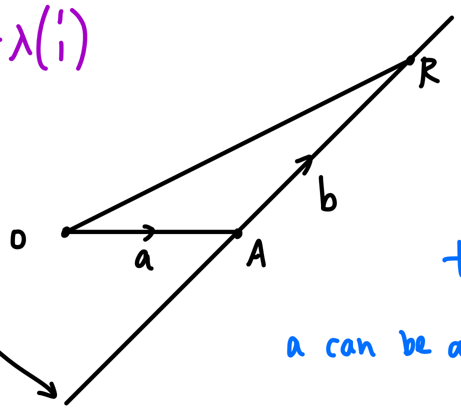
λ is a scalar

When λ varies from $-\infty$ to ∞, r takes every value on the line!

This equation works for both <u>2D AND 3D</u>

(with 3D vectors in 3D of course)

e.g. $r = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



a line in direction $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and passes through point $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

the equation is <u>NOT UNIQUE!</u>

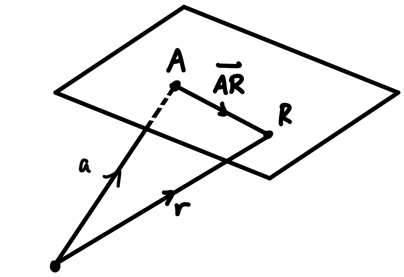
a can be any point the line passes through e.g. $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

b can be any vector in that or -ve direction e.g. $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$

Cartesian form of the line:

if $r = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, cartesian equation is $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3} = \lambda$

Equation of a plane (in 3 dimensions obviously)

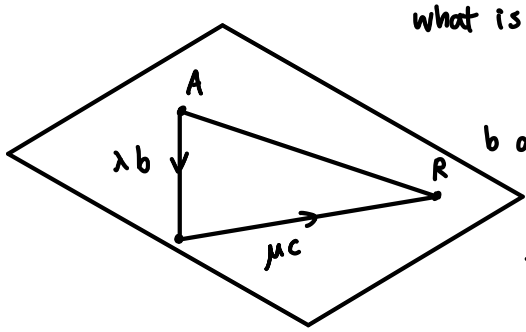


We need an expression $r = \boxed{\ ?\ ?\ ?\ }$ to express any point on the plane

$r = a + \overrightarrow{AR}$

$\overrightarrow{AR}$ lies on the plane.

what is $\overrightarrow{AR}$?



(and non-zero)

b and c are non-parallel AND in the plane

$\Rightarrow$ THEREFORE, $r = \underline{a} + \underline{\lambda} b + \underline{\mu} c$ (vector form of plane)

point on plane   variable scalar

Example 7

$A(2,2,-1)$ $B(3,2,-1)$ $C(4,3,5)$ all lie on the plane

$\overrightarrow{AB}$ is a vector in the plane $= \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}3\\2\\-1\end{pmatrix} - \begin{pmatrix}2\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$\overrightarrow{AC}$ is a vector in the plane $= \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}4\\3\\5\end{pmatrix} - \begin{pmatrix}2\\2\\-1\end{pmatrix} = \begin{pmatrix}2\\1\\6\end{pmatrix}$

$\therefore$ equation of plane is $r = \begin{pmatrix}2\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\0\end{pmatrix} + \mu\begin{pmatrix}2\\1\\6\end{pmatrix}$

point on plane   vectors on plane

Example 8

Plane: $r = \begin{pmatrix}3\\4\\-2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\2\end{pmatrix}$ $P = \begin{pmatrix}2\\2\\-1\end{pmatrix}$

Verify that P lies on the plane

if a point lies on the plane, it satisfies the plane equation (all 3 equations are consistent)

$r = \begin{pmatrix}3 + 2\lambda + \mu\\ 4 + \lambda - \mu\\ -2 + \lambda + 2\mu\end{pmatrix} = \begin{pmatrix}2\\2\\-1\end{pmatrix}$ if P lies on the plane

$$\begin{cases} 2\lambda + \mu = -1 & \text{— ①} \\ \lambda - \mu = -2 & \text{— ②} \\ \lambda + 2\mu = 1 & \text{— ③} \end{cases}$$

①+②: $3\lambda = -3$

$\lambda = -1$

$\mu = -1 - 2\lambda = -1 + 2 = 1$

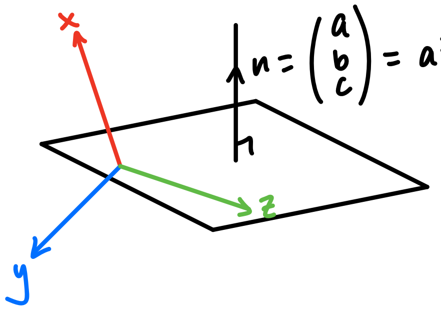
① and ② are consistent

③ LHS $= \lambda + 2\mu = -1 + 2 = 1 =$ RHS

③ is consistent with solutions

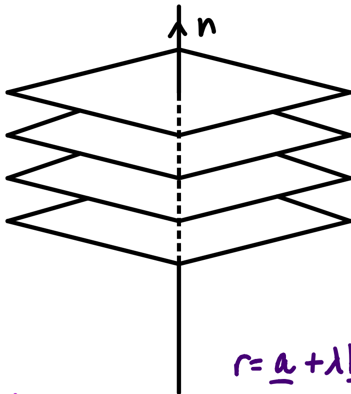
$\therefore$ P lies on the plane

Normal Vector of a Plane



$n = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ai + bj + ck$

a normal vector n describes infinite number of planes



$\downarrow$

choose a point the plane passes through to pinpoint the correct plane

n describes the direction of the plane

cartesian form of a plane : $ax+by+cz=d$

$\hookrightarrow n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$r = \underline{a} + \lambda \underline{b} + \mu \underline{c}$

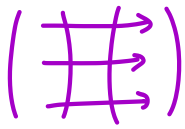
$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1\lambda \\ b_2\lambda \\ b_3\lambda \end{pmatrix} + \begin{pmatrix} c_1\mu \\ c_2\mu \\ c_3\mu \end{pmatrix} \begin{matrix} — ① \\ — ② \\ — ③ \end{matrix} \right.$$

Solve for λ and μ in terms of x, y and z $\rightarrow$ Solve for μ and λ using only 2 eqns, then substitute into 3rd eqn to find cartesian form

Scalar Product (dot product)

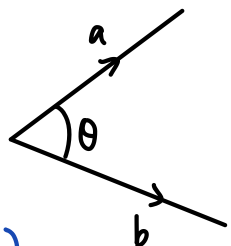
$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = ad + be + cf$

vector · vector = scalar



multiply across and add

$a \cdot b = |a||b| \cos\theta$



scalar product is used to find the angle between two vectors.

If $a \perp b$, $a \cdot b = 0$ $(\cos 90° = 0)$ $\rightarrow$ non-zero vectors a and b are $\perp$ <u>if and only if</u> $a \cdot b = 0$

If $a // b$, $a \cdot b = |a||b|$ $(\cos 0° = 1)$

$\Downarrow$

$a \cdot a = |a|^2$

points A, B, C lie on the plane

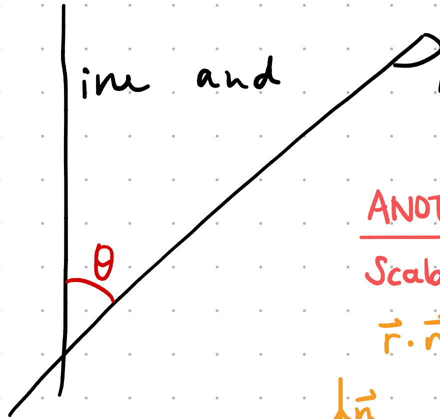
$ax+by+cz=1$

$\Downarrow$

$$\begin{cases} A_1 a + A_2 b + A_3 c = 1 \\ B_1 a + B_2 b + B_3 c = 1 \\ C_1 a + C_2 b + C_3 c = 1 \end{cases}$$

Simultaneous to find a, b and c and normal vector.

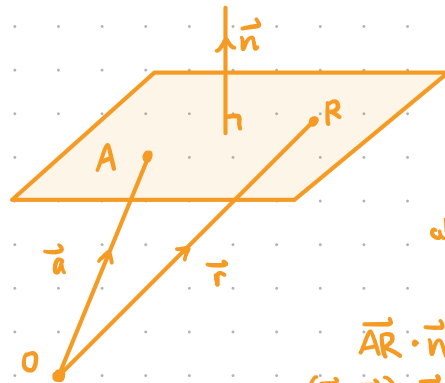
Angle between a line and plane



ANOTHER EQUATION OF PLANES

Scalar product form

$\vec{r}\cdot\vec{n} = k$, where $k = a\cdot\vec{n}$



$\vec{AR} = \vec{r} - \vec{a}$

along the plane, ⊥ to normal

$\vec{AR}\cdot\vec{n} = 0$

$(\vec{r}-\vec{a})\cdot\vec{n} = 0$

$\vec{r}\cdot\vec{n} - \vec{a}\cdot\vec{n} = 0$

scalar product is distributive

$\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

A and $\vec{a}$ are constant

$\therefore\ \vec{a}\cdot\vec{n}$ is constant

$\vec{r}\cdot\vec{n} = \vec{k}$

Scalar product form example

Example 19

Plane Π passes through A, ⊥ to $\underline{n}$

Capital Π used to name planes

underline means a vector

$\vec{OA} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ $\quad n = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$

a) Scalar product form: $\underline{a}\cdot\underline{n} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}\cdot\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 6+3+5 = 14$

scalar product form is cartesian form in a way

$\therefore\ \Pi : \underline{r}\cdot\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 14$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}\cdot\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 3x+y-z$

b) Cartesian form: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}\cdot\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 14$ $\quad\therefore\ \Pi : 3x+y-z = 14$

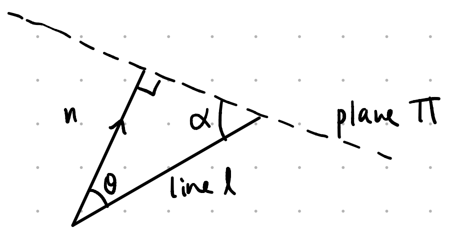
Acute ∠ between line & plane example

Example 20

line l: $\mathbf{r} = 2\mathbf{i}+\mathbf{j}-5\mathbf{k}+\lambda(3\mathbf{i}+4\mathbf{j}-12\mathbf{k}) = \begin{pmatrix}2\\1\\-5\end{pmatrix} + \lambda\begin{pmatrix}3\\4\\-12\end{pmatrix}$

plane Π: $\mathbf{r}\cdot(2\mathbf{i}-2\mathbf{j}-\mathbf{k}) = \mathbf{r}\cdot\begin{pmatrix}2\\-2\\-1\end{pmatrix} = 2$

IGNORE POSITION



normal $= \begin{pmatrix}2\\-2\\-1\end{pmatrix}$ line l direction $= \begin{pmatrix}3\\4\\-12\end{pmatrix}$

$$\begin{pmatrix}2\\-2\\-1\end{pmatrix}\cdot\begin{pmatrix}3\\4\\-12\end{pmatrix} = 6-8+12 = 10 = |n||l|\cos\theta$$

$|n| = \sqrt{4+4+1} = 3$ $|l| = \sqrt{9+16+144} = 13$

$\cos\theta = \dfrac{10}{|n||l|} = \dfrac{10}{39}$

$\theta = \arccos\left(\frac{10}{39}\right)$

$\alpha = 90° - \theta = 90° - \arccos\left(\frac{10}{39}\right) = 14.9°$ (3sf)

Alternative: $\cos\theta = \sin\alpha$

since $\alpha = 90° - \theta$ (and phase difference of sin & cos $= 90°$)

$\sin\alpha = \frac{10}{39}$

$\alpha = \arcsin\left(\frac{10}{39}\right) = 14.9°$

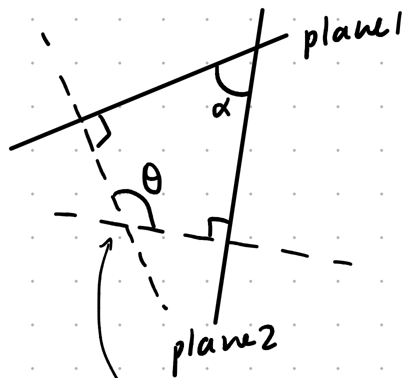
General form of acute ∠ between line and equation:

$$\sin\theta = \left|\frac{\underline{b}\cdot\underline{n}}{|\underline{b}||\underline{n}|}\right|$$

The example is not a proof, just a demonstration.

Acute angle between 2 planes

Example 21



$\underline{n}_1 = \begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix}$ $\underline{n}_2 = \begin{pmatrix} 7 \\ -4 \\ 4 \end{pmatrix}$

$\underline{n}_1 \cdot \underline{n}_2 = 28 - 16 - 28 = -16$

$|n_1| = |n_2| = \sqrt{16+16+49} = 9$

$\cos\theta = \dfrac{\underline{n}_1 \cdot \underline{n}_2}{|n_1||n_2|} = \dfrac{-16}{81}$

obtuse angle,

$\theta = 101.4°$ NOT acute!

($\angle$ sum of quadilateral = 360°)

Angle between planes, $\alpha = 180° - 101.4° = 78.6°$

Acute angle between 2 planes:

$\cos\theta = \left| \dfrac{n_1 \cdot n_2}{|n_1||n_2|} \right|$

Not proof, just demonstration

Points of Intersection (of 2 lines)

$\begin{cases} \underline{r} = \underline{a} + \lambda \underline{b} \\ \underline{r} = \underline{c} + \mu \underline{d} \end{cases}$

2 3D lines ⟶ form simultaneous equations to solve for λ and μ

remember to make sure λ and μ satisfy all 3 equations

If 2 lines are not parallel but do not intersect,
they are skew to show 2 lines are skew, show they do not intersect,
then show that they don't have the same direction.
direction vectors not multiples of each other

Example 22

$r = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ $r = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 1 \\ 4 \end{pmatrix}$

$\begin{cases} 3 + \lambda = 0 - 5\mu & \text{— (1)} \\ 1 - 2\lambda = -2 + \mu & \text{— (2)} \\ 1 - \lambda = 3 + 4\mu & \text{— (3)} \end{cases}$

3 dimensions ⟶ 3 equations
2 lines ⟶ 2 unknowns

ONLY SUBSTITUTE INTO THE EQUATIONS USED TO FIND THE SOLUTION ((1) or (3))

(1)+(3): $4 = 3 - \mu$

$\mu = -1 \longrightarrow$ (1): $\lambda = -5\mu - 3 = 5 - 3 = 2$

(2): LHS $= 1 - 2(2) = -3 = -2 + (-1) =$ RHS

$\therefore$ Consistent, lines intersect.

solutions work for (2), the lines intersect.

sub into line equations

Coordinates $= r = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix} = (5, -3, -1)$

Points of Intersection (of lines and planes)

$\underline{r} = \underbrace{\underline{a} + \lambda \underline{b}}$ $\quad \underline{r} \cdot \underline{n} = k$ $\quad (\underline{a} + \lambda \underline{b}) \cdot \underline{n} = k$

substitute

$\downarrow$

only 1 equation in λ, solve for λ

$\downarrow$

substitute into line to get intersection

Example 23

line l: $r = \begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $\quad$ plane Π: $r \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 4$

$$\begin{pmatrix} -1+\lambda \\ 1+\lambda \\ -5+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -1+\lambda+2+2\lambda-15+6\lambda = 9\lambda - 14 = 4 \Rightarrow \lambda = 2$$

$$r = \begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

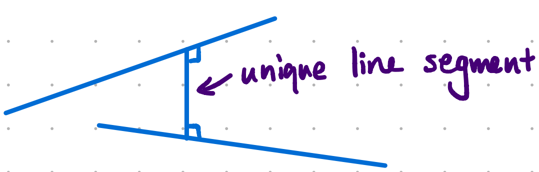
$\therefore$ coordinates $= (1, 3, -1)$

if line is parallel to plane it would have no solutions (λ's cancel, $0=2$ etc.)

if line is along the plane, it would have infinitely many solutions (λ's cancel, $2=2$ etc.)

Perpendicular / Shortest Distance

Between 2 lines:



Define A and B as general points on the lines, then find $\vec{AB}$ (from B−A)

$\vec{AB} \perp l_1$ and l_2, forming 2 equations and 2 unknowns (λ and μ, dot product $=0$)

Solve and enjoy.

Example 25 $\quad l_1$: $r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\quad l_2$: $r = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

let A be on l_1, B be on l_2

$$A = \begin{pmatrix} 1 \\ \lambda \\ \lambda \end{pmatrix} \quad B = \begin{pmatrix} -1+2\mu \\ 3-\mu \\ -1-\mu \end{pmatrix} \quad \vec{AB} = B - A = \begin{pmatrix} -1+2\mu-1 \\ 3-\mu-\lambda \\ -1-\mu-\lambda \end{pmatrix} = \begin{pmatrix} -2+2\mu \\ 3-\mu-\lambda \\ -1-\mu-\lambda \end{pmatrix}$$

get an equation for $\vec{AB}$

simultaneous

$\vec{AB} \perp$ to l_1, $\therefore \begin{pmatrix} -2+2\mu \\ 3-\mu-\lambda \\ -1-\mu-\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 3-\mu-\lambda-1-\mu-\lambda = 2-2\mu-2\lambda = 0$ ①

$\vec{AB} \perp$ to l_2, $\therefore \begin{pmatrix} -2+2\mu \\ 3-\mu-\lambda \\ -1-\mu-\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = -4+4\mu-3+\mu+\lambda+1+\mu+\lambda = -6+6\mu+2\lambda = 0$ ②

①+②: $-4+4\mu = 0$, $\mu = 1$ $\Rightarrow \lambda = 1-\mu = 0$ $\quad$ find λ and μ

$\Downarrow$

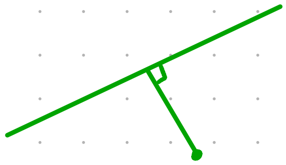
get $\vec{AB}$

$$\vec{AB} = \begin{pmatrix} -2+2 \\ 3-1 \\ -1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$$

find $\perp$ distance

$\perp$ distance $= \sqrt{0^2+2^2+(-2)^2} = 2\sqrt{2}$

Between line and point:



Similar method to 2 lines,
Set B to be a general point,
find $\overrightarrow{AB}$, dot $\overrightarrow{AB}$ and
line direction $= 0$.
Solve and enjoy.

<u>Example 26</u>

line: $\frac{x-1}{2} = \frac{y-1}{-2} = \frac{z+3}{-1}$ and point: $A(1,2,-1)$

convert to vector form → $r = \begin{pmatrix}1\\1\\-3\end{pmatrix} + \lambda\begin{pmatrix}2\\-2\\-1\end{pmatrix}$ $\quad A = \begin{pmatrix}1\\2\\-1\end{pmatrix}$

general point → B is on line, $B = \begin{pmatrix}1+2\lambda\\1-2\lambda\\-3-\lambda\end{pmatrix}$ $\quad \overrightarrow{AB} = \begin{pmatrix}2\lambda\\-1-2\lambda\\-2-\lambda\end{pmatrix}$ ← find $\overrightarrow{AB}$

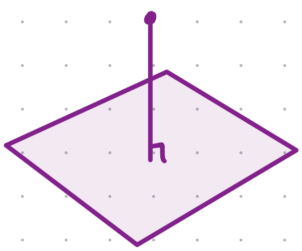
$\overrightarrow{AB} \perp$ line, $\overrightarrow{AB} \cdot$ line dir $= 0$ → $\overrightarrow{AB} \cdot \begin{pmatrix}2\\-2\\-1\end{pmatrix} = \begin{pmatrix}2\lambda\\-1-2\lambda\\-2-\lambda\end{pmatrix} \cdot \begin{pmatrix}2\\-2\\-1\end{pmatrix} = 4\lambda+2+4\lambda+2+\lambda = 9\lambda+4=0$

find λ → $\lambda = -\frac{4}{9}$

$\overrightarrow{AB} = \begin{pmatrix}-8/9\\-1/9\\-14/9\end{pmatrix}$ ← substitute

$\perp$ dist $= |\overrightarrow{AB}| = \frac{\sqrt{29}}{3} \approx 1.80$ ← find magnitude

Between point and plane



$\Pi : ax+by+cz=d$

point: (α, β, γ)

$$\Rightarrow \perp \text{ distance} = \underbrace{\frac{|a\alpha + b\beta + c\gamma - d|}{\sqrt{a^2+b^2+c^2}}}_{\text{this is in the formula booklet.}}$$

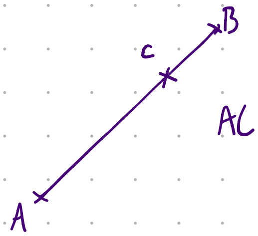
If a plane is in the form $r.\hat{n} = k$, $k = \perp$ distance from <u>origin</u>

hat means <u>unit</u> normal vector (length = 1)

Mixed Ex 9

1. a) $\overrightarrow{AB} = \begin{pmatrix}1\\2\\2\end{pmatrix} - \begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}0\\3\\-1\end{pmatrix}$ $\quad l: r = \begin{pmatrix}1\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\3\\-1\end{pmatrix}$ ✓

b)



$\overrightarrow{AC} = \frac{2}{3}\overrightarrow{AB} = \frac{2}{3}\begin{pmatrix}0\\3\\-1\end{pmatrix} = \begin{pmatrix}0\\2\\-2/3\end{pmatrix}$

$AC : CB = 2:1$

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \begin{pmatrix}1\\-1\\3\end{pmatrix} + \begin{pmatrix}0\\2\\-2/3\end{pmatrix} = \begin{pmatrix}1\\1\\7/3\end{pmatrix}$ ✓

3. $l: r = \begin{pmatrix}2\\3\\-4\end{pmatrix} + \lambda\begin{pmatrix}0\\2\\3\end{pmatrix}$ ✓

5. $l: r = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\-2\end{pmatrix}$ $\quad 3\times\begin{pmatrix}3\\1\\-2\end{pmatrix} = \begin{pmatrix}9\\3\\-6\end{pmatrix}$ ✓

$\begin{pmatrix}1\\2\\-1\end{pmatrix} + 2\begin{pmatrix}3\\1\\-2\end{pmatrix} = \begin{pmatrix}7\\4\\-5\end{pmatrix}$ ✓ $\quad \therefore\ r = \begin{pmatrix}7\\4\\-5\end{pmatrix} + \lambda\begin{pmatrix}9\\3\\-6\end{pmatrix}$

7. a) $A = \begin{pmatrix}2\\-1\\2\end{pmatrix}$ $\quad B = \begin{pmatrix}1\\3\\-1\end{pmatrix}$ $\quad C = \begin{pmatrix}4\\2\\5\end{pmatrix}$ $\quad \overrightarrow{AB} = \begin{pmatrix}-1\\4\\-3\end{pmatrix}$ ✓ $\quad \overrightarrow{BC} = \begin{pmatrix}3\\-1\\6\end{pmatrix}$ ✓

$\Pi: r = \begin{pmatrix}2\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}-1\\4\\-3\end{pmatrix} + \mu\begin{pmatrix}3\\-1\\6\end{pmatrix}$ ✓

b) $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\-1\\2\end{pmatrix} + \begin{pmatrix}-\lambda\\4\lambda\\-3\lambda\end{pmatrix} + \begin{pmatrix}3\mu\\-\mu\\6\mu\end{pmatrix}$

$$\begin{cases} x = 2 - \lambda + 3\mu & \text{— (1)} \\ y = -1 + 4\lambda - \mu & \text{— (2)} \\ z = 2 - 3\lambda + 6\mu & \text{— (3)} \end{cases}$$

(3) + 6(2): $6y + z = -4 + 21\lambda$

$\lambda = \frac{6y+z+4}{21}$

$\mu = -1 + 4\lambda - y = -\frac{21}{21} + \frac{24y+4z+16}{21} - \frac{21y}{21}$

$= \frac{3y+4z-5}{21}$

$x = 2 - \frac{6y+z+4}{21} + \frac{9y+12z-15}{21}$

$21x = 42 - 6y - z - 4 + 9y + 12z - 15$

$21x - 3y - 11z = 23$ ✓

9. $L=\begin{pmatrix}4\\7\\7\end{pmatrix}$ $M=\begin{pmatrix}1\\3\\2\end{pmatrix}$ $N=\begin{pmatrix}2\\4\\6\end{pmatrix}$

a) $\overrightarrow{ML} = L-M = \begin{pmatrix}3\\4\\5\end{pmatrix}$ ✓ $\overrightarrow{MN} = \vec{N}-\vec{M} = \begin{pmatrix}1\\1\\4\end{pmatrix}$ ✓

b) $\overrightarrow{ML}\cdot\overrightarrow{MN} = |\overrightarrow{ML}||\overrightarrow{MN}|\cos\angle LMN$ ✓

$$\cos\angle MN = \frac{\begin{pmatrix}3\\4\\5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\4\end{pmatrix}}{\sqrt{3^2+4^2+5^2}\sqrt{1+1+4^2}} = \frac{3+4+20}{\sqrt{50}\sqrt{18}} = \frac{27}{30} = \frac{9}{10}$$ ✓

11. $\overrightarrow{OA} = \begin{pmatrix}1\\2\\-3\end{pmatrix}$ $\overrightarrow{OB} = \begin{pmatrix}5\\0\\-3\end{pmatrix}$

a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}4\\-2\\0\end{pmatrix}$ $l: r = \begin{pmatrix}1\\2\\-3\end{pmatrix} + \lambda\begin{pmatrix}4\\-2\\0\end{pmatrix}$ ✓

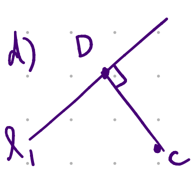
b) $\overrightarrow{OA} = \begin{pmatrix}1\\2\\-3\end{pmatrix} = \begin{pmatrix}4\\-4\\3\end{pmatrix} + (-3)\begin{pmatrix}1\\-2\\2\end{pmatrix}$ $\mu=3$ satisfies, $\therefore$ A lies on l_2 ✓

c) $\begin{pmatrix}4\\-2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix} = \left|\begin{matrix}4\\-2\\0\end{matrix}\right|\left|\begin{matrix}1\\-2\\2\end{matrix}\right|\cos\theta$

$$\cos\theta = \frac{4+4+0}{\sqrt{16+4+0}\sqrt{1+4+4}} = \frac{8}{6\sqrt{5}} = \frac{4}{3\sqrt{5}}$$

$$\theta = \cos^{-1}\left(\frac{4}{3\sqrt{5}}\right) = 53.4^\circ$$ ✓

d)



D is an arbitrary point on $l_1 \longrightarrow \overrightarrow{OD} = \begin{pmatrix}1+4\lambda\\2-2\lambda\\-3\end{pmatrix}$

$\overrightarrow{OC} = \begin{pmatrix}0\\4\\-5\end{pmatrix}$ $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \begin{pmatrix}1+4\lambda\\-2-2\lambda\\2\end{pmatrix}$

$\begin{pmatrix}1+4\lambda\\-2-2\lambda\\2\end{pmatrix}\cdot\begin{pmatrix}4\\-2\\0\end{pmatrix} = 4+16\lambda+4+4\lambda = 0$

$20\lambda = -8$

$\lambda = -2/5$

$\overrightarrow{CD} = \begin{pmatrix}1-8/5\\-2+4/5\\2\end{pmatrix} = \begin{pmatrix}-3/5\\-6/5\\2\end{pmatrix}$

$$|\overrightarrow{CD}| = \sqrt{\frac{9}{25}+\frac{36}{25}+4} = \frac{\sqrt{145}}{5} = 2.408$$ ✓

13. $l_1: r=\begin{pmatrix}3\\0\\-1\end{pmatrix}+s\begin{pmatrix}0\\1\\0\end{pmatrix}$ $l_2: r=\begin{pmatrix}9\\-2\\-1\end{pmatrix}+t\begin{pmatrix}1\\-2\\1\end{pmatrix}$ A and B are arbitrary points on l_1 and l_2 respectively.

$=\begin{pmatrix}3\\s\\-1\end{pmatrix}=\vec{OA}$ $=\begin{pmatrix}9+t\\-2-2t\\-1+t\end{pmatrix}=\vec{OB}$

$\vec{AB}=\vec{OB}-\vec{OA}=\begin{pmatrix}6+t\\-2-2t-s\\t\end{pmatrix}$ $\begin{pmatrix}6+t\\-2-2t-s\\t\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}=-2-2t-s=0$

$s+2t+2=0$ — ①

$\begin{pmatrix}6+t\\-2-2t-s\\t\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}=6+t+4+4t+2s+t=2s+6t+10=0$ — ②

② − 2①: $2t+6=0$
$t=-3$ $s=-2-2t=-2+6=4$

$\vec{AB}=\begin{pmatrix}6-3\\-2+6-4\\-3\end{pmatrix}=\begin{pmatrix}3\\0\\-3\end{pmatrix}$ $|\vec{AB}|=\sqrt{9+9}=3\sqrt{2}$

15. a) $\lambda=0: r=\begin{pmatrix}1\\1\\-2\end{pmatrix}$ $\lambda=1: r=\begin{pmatrix}1\\1\\-2\end{pmatrix}+\begin{pmatrix}2\\0\\-1\end{pmatrix}=\begin{pmatrix}3\\1\\-3\end{pmatrix}$ $\vec{OA}=\begin{pmatrix}4\\3\\1\end{pmatrix}$

$ax+by+cz=1$

$\begin{cases}a+b-2c=1\\3a+b-3c=1\\4a+3b+c=1\end{cases}$ ✓ $\begin{cases}a=-\frac{2}{15}\\b=\frac{3}{5}\\c=-\frac{4}{15}\end{cases}$ $\therefore -2x+9y-4z=15$

$\Pi: r\cdot\begin{pmatrix}-2\\9\\-4\end{pmatrix}=15$ ✓

b) $\lambda=0: r=\begin{pmatrix}1\\2\\2\end{pmatrix}$ $\lambda=1: r=\begin{pmatrix}1\\2\\2\end{pmatrix}+\begin{pmatrix}2\\1\\-3\end{pmatrix}=\begin{pmatrix}3\\3\\-1\end{pmatrix}$ $\vec{OA}=\begin{pmatrix}3\\5\\1\end{pmatrix}$

$\begin{cases}a+2b+2c=1\\3a+3b-c=1\\3a+5b+c=1\end{cases}$ ✓ $\rightarrow \begin{cases}a=1\\b=-\frac{1}{2}\\c=\frac{1}{2}\end{cases}$ $\therefore 2x-y+z=2$

$\Pi: r\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=2$ ✓

c) $\lambda=0: r=\begin{pmatrix}2\\-1\\1\end{pmatrix}$ $\lambda=1: r=\begin{pmatrix}2\\-1\\1\end{pmatrix}+\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}3\\1\\3\end{pmatrix}$ $\vec{OA}=\begin{pmatrix}7\\8\\6\end{pmatrix}$

$\begin{cases}2a-b+c=1\\3a+b+3c=1\\7a+8b+6c=1\end{cases}$ ✓ $\rightarrow \begin{cases}a=\frac{4}{11}\\b=-\frac{5}{22}\\c=\frac{1}{22}\end{cases}$ $\therefore 8x-5y+z=22$

$\Pi: r\cdot\begin{pmatrix}8\\-5\\1\end{pmatrix}=22$ ✓

17. a) $\vec{OA} = \begin{pmatrix}1\\3\\3\end{pmatrix}$ $\vec{OB} = \begin{pmatrix}3\\1\\4\end{pmatrix}$ $\vec{OC} = \begin{pmatrix}2\\4\\1\end{pmatrix}$ $\begin{cases} a+3b+3c=1 \\ 3a+b+4c=1 \\ 2a+4b+c=1 \end{cases} \longrightarrow \begin{matrix} a=\frac{1}{10} \\ b=\frac{1}{6} \\ c=\frac{2}{15} \end{matrix}$

$\frac{1}{10}x+\frac{1}{6}y+\frac{2}{15}z=1$

$3x+5y+4z=30$ $\underline{n}=\begin{pmatrix}3\\5\\4\end{pmatrix}$ $|\underline{n}|=5\sqrt{2}$ $\hat{\underline{n}}=\frac{1}{5\sqrt{2}}\begin{pmatrix}3\\5\\4\end{pmatrix}=\begin{pmatrix}3/5\sqrt{2}\\1/\sqrt{2}\\4/5\sqrt{2}\end{pmatrix}$ ✓

b) $3x+5y+4z=30$ ✓

c) let D be an arbitrary point on Π.

$\vec{OD}=\lambda\begin{pmatrix}3\\5\\4\end{pmatrix}$ $\begin{pmatrix}3\lambda\\5\lambda\\4\lambda\end{pmatrix}\cdot\begin{pmatrix}3\\5\\4\end{pmatrix}=9\lambda+25\lambda+16\lambda=50\lambda=30$

$\lambda=\frac{3}{5}$

$\therefore \vec{OD}=\begin{pmatrix}9/5\\3\\12/5\end{pmatrix}$ $|\vec{OD}|=\sqrt{\frac{81}{25}+9+\frac{144}{25}}=3\sqrt{2}$ ✓

19. a) $A=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $B=\begin{pmatrix}5\\-2\\1\end{pmatrix}$ $C=\begin{pmatrix}3\\2\\6\end{pmatrix}$ $\begin{cases} a+b+c=1 \\ 5a-2b+c=1 \\ 3a+2a+6c=1 \end{cases} \rightarrow \begin{cases} a=\frac{3}{5} \\ b=\frac{4}{5} \\ c=-\frac{2}{5} \end{cases}$ $3x+4y-2z=5$

normal vector $=\begin{pmatrix}3\\4\\-2\end{pmatrix}$ ✓

b) $3x+4y-2z-5=0$ ✓

21. $l_1: r=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}$ $l_2: r=\begin{pmatrix}1\\4\\-4\end{pmatrix}+\mu\begin{pmatrix}-3\\0\\1\end{pmatrix}$

a) $\begin{cases} 2\lambda+3\mu=0 \\ \lambda=3 \\ -2\lambda-\mu=-4 \end{cases}$ $\lambda=3$ $\mu=-\frac{2}{3}\mu=-2$

$\downarrow$ into ③

LHS $=-6+2=-4=$ RHS $\therefore$ lines intersect. ✓

b) $r=\begin{pmatrix}1\\1\\0\end{pmatrix}+3\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}7\\4\\-6\end{pmatrix}$ ✓

c) $\begin{pmatrix}2\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}-3\\0\\1\end{pmatrix}=\left|\begin{matrix}2\\1\\-2\end{matrix}\right|\left|\begin{matrix}-3\\0\\1\end{matrix}\right|\cos\theta$

$\cos\theta=\left|\frac{-6-2}{\sqrt{4+1+4}\sqrt{9+1}}\right|=\frac{8}{3\sqrt{10}}=\frac{8\sqrt{10}}{30}=\frac{4\sqrt{10}}{15}$ ✓

23. a) $\vec{OA} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$ $\quad \vec{OB} = \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix}$ $\quad \vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ ✓

b) $l: r = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ ✓

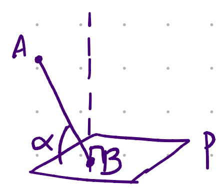
c) $\vec{OC} = \begin{pmatrix} 4 \\ 10 \\ 2 \end{pmatrix}$ $\quad \vec{OP} = \begin{pmatrix} 6-\lambda \\ 3-\lambda \\ 4+2\lambda \end{pmatrix}$ $\quad \vec{CP} = \begin{pmatrix} 2-\lambda \\ -7-\lambda \\ 2+2\lambda \end{pmatrix}$

$$\begin{pmatrix} 2-\lambda \\ -7-\lambda \\ 2+2\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = -2+\lambda+7+\lambda+4+4\lambda = 6\lambda+9=0$$

$$\lambda = -\frac{3}{2}$$

$$\vec{OP} = \begin{pmatrix} 6+3/2 \\ 3+3/2 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 15/2 \\ 9/2 \\ 1 \end{pmatrix}$$ ✓

25. a) $P: r \cdot \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = 1$ $\quad \vec{OA} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ $\quad \vec{OB} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$



$$\vec{BA} = \vec{OA} - \vec{OB} = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$$

$$\left| \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \right| = \left| \begin{matrix} -3 \\ 0 \\ 3 \end{matrix} \right| \left| \begin{matrix} -2 \\ -2 \\ 3 \end{matrix} \right| \cos(90° - \alpha)$$

$$\cos(90° - \alpha) = \frac{-6+9}{\sqrt{9+9}\sqrt{4+4+9}} = \frac{3}{3\sqrt{34}} = \frac{1}{\sqrt{34}}$$

$$90° - \alpha = 80.125°$$

$\alpha = 10°$ (nearest degree) ✓

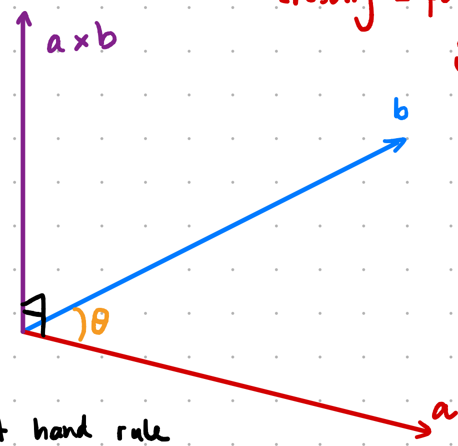
b) $$\frac{|2-4+6-1|}{\sqrt{4+4+9}} = \frac{3}{\sqrt{17}} = \frac{3\sqrt{17}}{17}$$ ✓

Vector/Cross Product (out of syllabus)

$$\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin\theta \, \hat{\underline{n}}$$

normal vector $\perp$ to a and b

axb is $\perp$ to a AND b



crossing 2 parallel vectors gives the 0 vector

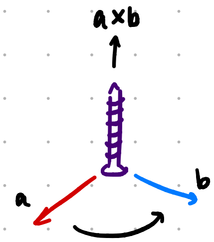
$\underline{i} \times \underline{i} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\underline{a \times b \neq a \times b}$

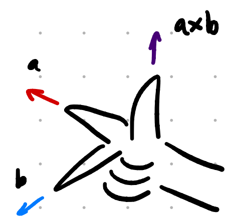
Turning a screw from a to b

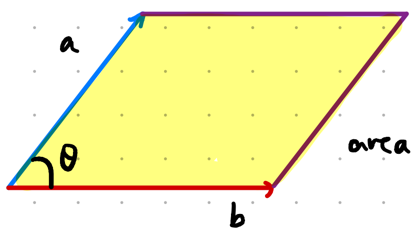
↓

screw moves in direction of axb



right hand rule





area of parallelogram = $|a||b|\sin\theta$ = magnitude of axb

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

← matrix determinants!

$$= (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$$

$$= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

← there's a proof, but its complicated.