Proof by Induction

$n = 1, \ 2, \ 3, \ 4, \ \ldots, \ k, \ k+1, \ldots$

step 1: show that statement is true for $n=1$

step 2: assume something is true for $n=k$

step 3: show that if statement is true for $n=k$, it is true for $n=k+1$

step 4: summarize that statement is true for all $n \in \mathbb{Z}, n \geq 1$ $(\forall n \in \mathbb{Z}^+)$

EXAMPLE prove by induction that for all $n \in \mathbb{Z}^+$, $\sum_{r=1}^{n}(2r-1) = n^2$

step 1: $n=1 \rightarrow LHS = \sum_{r=1}^{1}(2r-1) = 2-1=1 = 1^2 = RHS$

step 2: assume true for $n=k \rightarrow LHS = \sum_{r=1}^{k}(2r-1) = k^2 = RHS$

step3: consider $n=k+1$: keyword need to show that $\sum_{r=1}^{k+1}(2r-1) = (k+1)^2$

separate from summation

$$LHS = \sum_{r=1}^{k+1}(2r-1) = \sum_{r=1}^{k}(2r-1) + (2(k+1)-1) = k^2 + 2k+1 = (k+1)^2 = RHS$$

substitute from step 2

step 4: since true for $n=1$, and true for $n=k+1$ if true for $n=k$, statement true for all $n \in \mathbb{Z}^+$

Exercise 8A

1. Prove by induction $\forall n \in \mathbb{Z}^+$, $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

for $n=1$: $LHS = 1 = \frac{1}{2}(1)(2) = 1 = RHS$

assume result is true for $n=k$: $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$

consider $n=k+1$,

need to show that $\sum_{r=1}^{k+1} r = \frac{1}{2}(k+1)(k+2)$

$$LHS = \sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1) = \frac{1}{2}k(k+1) + (k+1) = \frac{1}{2}k^2 + \frac{3}{2}k + 1 = \frac{1}{2}(k^2+3k+2) = \frac{1}{2}(k+1)(k+2) = RHS$$

since true for $n=1$ and true for $n=k+1$ if true for $n=k$, true $\forall n \in \mathbb{Z}^+$

Exercise 1B

1. a) Show that $\forall n \in \mathbb{Z}^+$ $8^n - 1$ is divisible by 7

$n=1$: $8-1=7$ true

assume $n=k$ is true $8^k - 1$ is divisible by 7 $f(k) = 8^k - 1$

$n=k+1$: $8^{k+1} - 1 = 8(8^k) - 1$ $f(k+1) = 8(8^k) - 1$

$f(k+1) - f(k) = 8(8^k) - 8^k - 1 + 1 = 7(8^k)$ divisible by 7

Therefore, $8^n - 1$ is divisible by 7 $\forall n \in \mathbb{Z}^+$

2. $f(n) = 13^n - 6^n$

a) $f(k) = 13^k - 6^k$ $f(k+1) = 13(13^k) - 6(6^k) = (6+7)13^k - 6(6^k) = 6(13^k - 6^k) + 7(13^k)$

$= 6f(k) + 7(13^k)$

b) prove that $\forall n \in \mathbb{Z}^+$, $f(n)$ divisible by 7

$f(1) = 13 - 6 = 7$ true

assume $f(k) = 13^k - 6^k$ divisible by 7

$f(k+1) = 6f(k) + 7(13^k)$ since $f(k)$ is divisible by 7, $6f(k)$ is divisible by 7.

since $7(13^k)$ is divisible by 7, $6f(k) + 7(13^k)$ is divisible by 7

$\therefore f(n)$ divisible by 7 $\forall n \in \mathbb{Z}^+$