Proof by contradiction

1. Assume the opposite (assume the statement is false)
2. Contradiction
3. Statement of proof

e.g. $\sqrt{2}$ is an irrational #.

Assume $\sqrt{2}$ is rational :

$\sqrt{2} = \frac{a}{b}$ where it is the simplest form

$2 = \frac{a^2}{b^2}$

$2b^2 = a^2 \Rightarrow a^2$ is even $(a^2 = (2k)^2)$ (a is even)

$2b^2 = 4k^2$

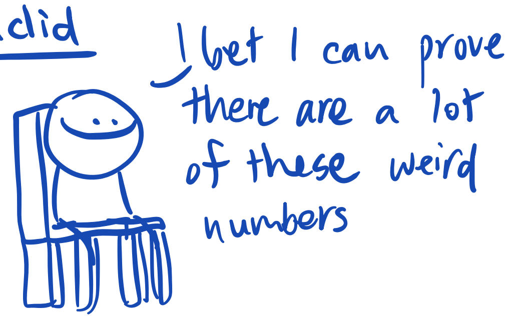
$b^2 = 2k^2 \Rightarrow b^2$ is even (b is even)

if a & b are even, $\frac{a}{b}$ is not the simplest form

$\therefore \sqrt{2}$ is irrational

e.g. There are infinite primes

Euclid



Assume there are a finite number of primes

$\{p_1, p_2, p_3, p_4 \ldots\}$ set of <u>all</u> prime numbers

Consider the number $N = p_1 \times p_2 \times p_3 \times p_4 \ldots + 1$

Dividing N by any prime number would yield a remainder of 1.

Therefore, N itself is a prime that isn't in the set of "all prime numbers"

$\therefore$ There are an infinite number of primes.

Algebraic fractions

Same as numeric fractions: cancel out common factors, then multiply numerators & denominators

That's it. Literally nothing else.

$$\text{e.g. } \underbrace{\frac{x+1}{2} \cdot \frac{3}{x^2-1} = \frac{x+1}{2} \cdot \frac{3}{(x+1)(x-1)}}_{\text{factorise to find factors}} = \underbrace{\frac{1}{2} \cdot \frac{3}{x-1}}_{\text{cancel common factors}} = \overbrace{\frac{3}{2x-2}}^{\text{multiply}}$$

Partial Fractions

split fraction into linear denominators

$$\text{from } \frac{n}{(x-a)(x-b)} \Rightarrow \overbrace{\frac{A}{(x-a)} + \frac{B}{(x-b)}}$$

Steps:

1. $$\frac{A(x-b)+B(x-a)}{(x-a)(x-b)} \equiv \frac{n}{(x-a)(x-b)}$$
$$\Downarrow$$
$$A(x-b)+B(x-a) \equiv n$$

2. Method 1: Substitution

sub $x=a$ or $x=b$ to eliminate B or A & solve

Method 2: Comparing Coefficients

$(A+B)x + ab - Ab - Ba \equiv n \Rightarrow$ compare coefficients

e.g. $$\frac{6x-2}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$$

$$6x-2 \equiv A(x+1) + B(x-3)$$

Substitution:

Sub $x=3$ to find A

Sub $x=-1$ to find B

Coefficients:

$$6x-2 \equiv (A+B)x + A - 3B - 3$$

Repeated factors $(x-a)^2$ on the denominator can be split into even more partial fractions.

Improper fractions : degree of numerator $\geq$ degree of denominator

e.g. $\dfrac{x^2+2x+3}{x+4}$ $\begin{array}{l}\leftarrow 2^{nd} \text{ degree}\\ \leftarrow 1^{st} \text{ degree}\end{array}$ or $\dfrac{x^2+2x+3}{x^2+5x-4}$ $\begin{array}{l}\leftarrow 2^{nd} \text{ degree}\\ \leftarrow 2^{nd} \text{ degree}\end{array}$

Simplifying : $\dfrac{x^2+5x+8}{x-2}$

① long division

$$\begin{array}{r}
x+7 \\
x-2 \overline{\big) x^2+5x+8} \\
\underline{x^2-2x} \\
7x+8 \\
\underline{7x-14} \\
22
\end{array}$$

$$\therefore\ x^2+5x+8 = \underbrace{(x-2)}_{\text{divisor}}\underbrace{(x+7)}_{\text{quotient}} + 22 \leftarrow \text{remainder}$$

~or~

$$\frac{x^2+5x+8}{x-2} = \underbrace{x+7+\frac{22}{x+2}}_{\text{rewritten as mixed fraction!}}$$

Ex 1F (p.17)

(10) a) $x^4-1 = (x^2+1)(x+1)(x-1)$

b) $\dfrac{x^4-1}{x+1} = \dfrac{(x^2+1)(x+1)(x-1)}{x+1} = (x-1)(x^2+1)$

$\Downarrow$

$(ax+b)(cx^2+dx+e)$ $a=1,\ b=-1,\ c=1,\ d=0,\ e=1$

MIXED EX

(2) Assume q^2 is irrational but q is rational

q can be expressed as $\frac{a}{b}$

q^2 would be expressed as $\frac{a^2}{b^2}$

but q^2 is irrational

$\begin{pmatrix} x & -4 \\ x & +1 \end{pmatrix}$ $\begin{pmatrix} x & +5 \\ x & +1 \end{pmatrix}$

$\therefore$ If q^2 is irrational q is irrational

(4) a) $$\frac{4x^2-8x}{x^2-3x-4} \cdot \frac{x^2+6x+5}{2x^2+10x} = \frac{4x(x-2)}{(x+1)(x-4)} \cdot \frac{(x+5)(x+1)}{2x(x+5)} = \frac{2(x-2)}{(x-4)}$$

b) $$\ln((4x^2-8x)(x^2+6x+5)) = 6 + \ln((x^2-3x-4)(2x^2+10x))$$

$$\ln\left(\frac{(4x^2-8x)(x^2+6x+5)}{(x^2-3x-4)(2x^2+10x)}\right) = \ln e^6$$

$$\frac{(x-2)}{(x-4)} = \frac{1}{2}e^6$$

$$x-2 = \frac{1}{2}e^6 x - 2e^6$$

$$x\left(1-\frac{1}{2}e^6\right) = 2-2e^6$$

$$x = \frac{2-2e^6}{1-\frac{1}{2}e^6} = \frac{4e^6-4}{e^6-2}$$

(6) $$\frac{5x+3}{x^2-3x-10} = \frac{5x+3}{(x-5)(x+2)} = \frac{A(x+2)+B(x-5)}{(x-5)(x+2)}$$

$\begin{pmatrix} x & -5 \\ x & +2 \end{pmatrix}$

$$5x+3 = A(x+2)+B(x-5)$$

$\therefore \frac{5x+3}{(x-5)(x+2)} = \frac{4}{x-5} + \frac{1}{x+2}$

$\boxed{x=-2}$

$-10+3 = -7B$

$B=1$

$\boxed{x=5}$

$25+3 = 7A$

$A=4$

$$\therefore \frac{6x+1}{x-5} + \frac{4}{x-5} + \frac{1}{x+2} = \frac{6x+5}{x-5} + \frac{1}{x+2}$$

$$= \frac{(6x+5)(x+2)+x-5}{(x-5)(x+2)}$$

$$= \frac{6x^2+17x+10+x-5}{x^2-3x-10}$$

$$= \frac{6x^2+18x+5}{x^2-3x-10}$$

(8) $f(x) = \frac{x-3}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x(x-1)}$

$x-3 = A(x-1) + Bx$

$\boxed{x=1}$

$B = 1-3 = -2$

$\boxed{x=0}$

$-A = -1-3$

$A = 4$

$\therefore f(x) = \frac{4}{x} - \frac{2}{x-1}$

(10) $f(x) = \frac{16x-1}{(3x+2)(2x-1)} = \frac{D}{3x+2} + \frac{E}{2x-1} = \frac{D(2x-1)+E(3x+2)}{(3x+2)(2x-1)}$

$16x-1 = D(2x-1) + E(3x+2)$

$\boxed{x=-\frac{2}{3}}$

$D\left(-\frac{4}{3}-1\right) = -\frac{32}{3} - 1$

$D = 5$

$\boxed{x=\frac{1}{2}}$

$E\left(\frac{3}{2}+2\right) = 8-1$

$E = 2$

$\therefore f(x) = \frac{5}{3x+2} + \frac{2}{2x-1}$

(12) $h(x) = \frac{21x^2-13}{(x+5)(3x-1)^2} = \frac{D}{x+5} + \frac{E}{3x-1} + \frac{F}{(3x-1)^2}$

$= \frac{D}{x+5} + \frac{E(3x-1)+F}{(3x-1)^2}$

$= \frac{D(3x-1)^2 + (E(3x-1)+F)(x+5)}{(x+5)(3x-1)^2}$

$21x^2-13 = D(3x-1)^2 + (E(3x-1)+F)(x+5)$

$\boxed{x=\frac{1}{3}}$

$\frac{21}{9} - 13 = F\left(\frac{1}{3}+5\right)$

$F = -2$

$\boxed{x=-5}$

$525-13 = 256D + 0$

$D = 2$

$2(9x^2-6x+1) + (3Ex-E-2)(x+5)$

$= 18x^2-12x+2-3Ex^2+15Ex-Ex-2x-5E-10$

$= (18-3E)x^2 + \ldots = 21x^2-13$

$\therefore 18-3E = 21$

$E = -1$

$\therefore h(x) = \frac{2}{x+5} - \frac{1}{3x-1} - \frac{2}{(3x-1)^2}$