# An UPGRADE to the Binomial Expansion!

The old binomial formula: Works only if $n$ is a natural # ($n=1,2,3,4,5\ldots$ etc)

$$(a+b)^n = \sum_{r=0}^{n} \left({}^nC_r\, a^{n-r} b^r\right)$$

The new binomial formula: Works only if $|x|<1$

$$(1+x)^n = \sum_{r=0}^{\infty} \left(\frac{n(n-1)(n-2)\ldots(n-(r-1))x^r}{r!}\right)$$

or

$$(1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\ldots+\overbrace{\frac{n(n-1)(n-2)\ldots(n-(r-1))x^r}{r!}}^{\text{general term}}$$

Infinite series

## Ex 4A (p. 96)

1 (b) (i) $(1+x)^{-6} = 1-6x+\frac{(-6)(-7)x^2}{2}+\frac{(-6)(-7)(-8)x^3}{6}+\ldots = 1-6x+21x^2-56x^3+\ldots$

(ii) $-1<x<1$

(f) (i) $(1+x)^{-\frac{2}{3}} = 1-\frac{3}{2}x+\frac{(-\frac{3}{2})(-\frac{5}{2})x^2}{2}+\frac{(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})x^3}{6}+\ldots = 1-\frac{2}{3}x+\frac{15}{8}x^2-\frac{35}{16}x^3+\ldots$

(ii) $-1<x<1$

3 (b) (i) $\frac{1}{(1+3x)^4} = (1+3x)^{-4} = 1-12x+\frac{(-4)(-5)(3x)^2}{2}+\frac{(-4)(-5)(-6)(3x)^3}{6}+\ldots = 1-12x+90x^2-540x^3+\ldots$

(ii) $-1<3x<1$ or $-\frac{1}{3}<x<\frac{1}{3}$ or $|x|<\frac{1}{3}$

(f) (i) $\frac{\sqrt[3]{1-2x}}{1-2x} = (1-2x)^{-\frac{2}{3}} = 1-\frac{2}{3}(2x)+\frac{(-\frac{2}{3})(-\frac{5}{3})(2x)^2}{2}+\frac{(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})(2x)^3}{6}+\ldots$

$= 1+\frac{4}{3}x+\frac{20}{9}x^2+\frac{320}{81}x^3+\ldots$

(ii) $-1<-2x<1$ or $-\frac{1}{2}<x<\frac{1}{2}$ or $|x|<\frac{1}{2}$

Deriving the new binomial formula.

① A power series: $\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

(an infinite polynomial)

where $a_i \in \mathbb{R}$, $i = 0, 1, 2 \ldots$

Suppose we want to represent some $f(x)$ as a power series. what must $a_1, a_2, a_3 \ldots$ be? (where $f(x)$ is continuous)

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

if we set $x=0$,

$f(0) = a_0$

if we differentiate both sides,

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$

if we set $x=0$,

$f'(0) = a_1$

if we differentiate both sides,

$f''(x) = 2a_2 + 6a_3 x + \ldots$

if we set $x=0$,

$f''(0) = 2a_2$

Repeat to find all coefficients!

From this pattern,

$$a_k = \frac{f^{(k)}(0)}{k!}$$

differentiate k times

Therefore,

$$f(x) = \frac{f^{(0)}(0)}{0!} x^0 + \frac{f^{(1)}(0)}{1!} x^1 + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Now let $f(x) = (1+x)^n$

$f(x) = (1-x)^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$f'(x) = n(1+x)^{n-1}$

$f''(x) = n(n-1)(1+x)^{n-2}$

$f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$

$$a_k = \frac{f^{(k)}(0)}{k!}$$

$k=0$, $a_0 = \frac{f^{(0)}(0)}{0!} = f(0) = (1+0)^n = 1$

$k=1$, $a_1 = \frac{f^{(1)}(0)}{1!} = f'(0) = n$

$k=2$, $a_2 = \frac{f^{(2)}(0)}{2!} = \frac{f''(0)}{2!} = \frac{n(n-1)}{2!}$

$k=3$, $a_3 = \frac{f^{(3)}(0)}{3!} = \frac{f'''(0)}{3!} = \frac{n(n-1)(n-2)}{3!}$

Why is the formula valid only if $|x| < 1$?

d'Alembert's ratio test:

A series $\sum_{k=0}^{\infty} a_k$ converges to a finite value

provided $\lim_{k \to 0} \left| \frac{a_{k+1}}{a_k} \right| < 1$ (ie as $k$ increases, $|a_{k+1}| < |a_k|$)

For $(1-x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots$,

the general term $= \frac{n(n-1)\ldots(n-(r-1))x^k}{k!} = a_k$

then, $\left| \frac{a_{k+1}}{a_k} \right| = \left| \left( \frac{n(n-1)\ldots(n-(k-1))(n-k)\, x^{k+1}}{(k+1)!} \right) \left( \frac{k!}{n(n-1)\ldots(n-(k-1))\, x^k} \right) \right|$

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{(n-k)x}{k+1} \right| = \lim_{k \to \infty} \frac{|n-k|}{k+1} |x| = \lim_{k \to \infty} \frac{\left| \frac{n}{k} - 1 \right|}{1 + \frac{1}{k}} |x|$$

$$= \frac{|0-1|}{1+0} |x| = |x|$$

Therefore if we want the expression to not explode, $|x| < 1$

# Expanding $(a+bx)^n$

$$(a+bx)^n = \left[a\left(1+\frac{b}{a}x\right)\right]^n = a^n \overbrace{\left(1+\frac{b}{a}x\right)^n}^{\text{expand with new formula}}$$

$$(4+2x)^{\frac{1}{2}} = \sqrt{4}\left(1+\frac{1}{2}x\right)^{\frac{1}{2}} = 2\left[1+\left(\frac{1}{2}\right)\left(\frac{1}{2}x\right)+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}x\right)^2\left(\frac{1}{2!}\right)+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{2}x\right)^3\left(\frac{1}{3!}\right)+\dots\right]$$

For $|x| < \frac{1}{2}$

$$(2+x)^{\frac{1}{2}} = \sqrt{2}\left(1+\frac{x}{2}\right)^{\frac{1}{2}} = \sqrt{2}\left[1+\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{x}{2}\right)^2\left(\frac{1}{2!}\right)+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{2}\right)^3\left(\frac{1}{3!}\right)+\dots\right]$$

For $|x|<1$

$$\sqrt{\frac{2+x}{1-x}} = (2+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}} = \sqrt{2}\left(1+\frac{x}{2}\right)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$=\sqrt{2}\left[1+\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{x}{2}\right)^2\left(\frac{1}{2!}\right)+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{x}{2}\right)^3\left(\frac{1}{3!}\right)+\dots\right]\left[1-\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)+\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{x}{2}\right)^2\left(\frac{1}{2!}\right)-\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{x}{2}\right)^3\left(\frac{1}{3!}\right)+\dots\right]$$