7: Algebraic Methods

$$\begin{pmatrix} 2x & -7 \\ x & +4 \end{pmatrix} \begin{pmatrix} 2x & -7 \\ x & -2 \end{pmatrix}$$

7.1: ALGEBRAIC FRACTIONS

Ex 7A (p.138)

(3) $\dfrac{6x^3+3x^2-84x}{6x^2-33x+42} = \dfrac{3x(2x^2+x-28)}{3(2x^2-11x+14)} = \dfrac{x(2x-7)(x+4)}{(2x-7)(x-2)} = \dfrac{x(x+4)}{x-2} = \dfrac{ax(x+b)}{x+c}$

$a=1,\ b=4,\ c=-2$

7.2: DIVIDING ALGEBRAIC EXPRESSIONS

Ex 7B (p.141)

(2) a)

$$\begin{array}{r l}
 & 6x^2+3x+2 \\
x+4 \,\big) & 6x^3+27x^2+14x+8 \\
 & \underline{6x^3+24x^2} \\
 & \quad 3x^2+14x+8 \\
 & \quad \underline{3x^2+12x} \\
 & \qquad 2x+8 \\
 & \qquad \underline{2x+8} \\
 & \qquad\quad 0
\end{array}$$

$\therefore (x+4)(6x^2+3x+2)$

quotient / $q(x)$

divisor / $d(x)$

remainder / $r(x)$

(6) c)

$$\begin{array}{r l}
 & -3x^2+5x+10 \\
x-2 \,\big) & -3x^3+11x^2+0x-20 \\
 & \underline{-3x^3+6x^2} \\
 & \quad 5x^2 \quad -20 \\
 & \quad \underline{5x^2-10x} \\
 & \qquad 10x-20 \\
 & \qquad \underline{10x-20} \\
 & \qquad\quad 0
\end{array}$$

$\therefore -(x-2)(3x^2-5x-10)$

Leave space here!

(14) $x^4-16 = (x^2+4)(x^2-4)$

$x^4-16 = (x^2+4)(x+2)(x-2)$

$\dfrac{x^4-16}{x+2} = (x^2+4)(x-2)$

Shortcut by recognizing the divisor!

(16) a)

$$\begin{array}{r l}
 & 3x^2-5x-62 \\
x-3 \,\big) & 3x^3-14x^2-47x-14 \\
 & \underline{3x^3-9x^2} \\
 & \quad -5x^2-47x \\
 & \quad \underline{-5x^2+15x} \\
 & \qquad -62x-14 \\
 & \qquad \underline{-62x+186} \\
 & \qquad\quad -200
\end{array}$$

$\therefore$ remainder $= -200$

remainder is ALWAYS lower degree than divisor!

part b) continued

b) $f(x)=3x^3-14x^2-47x-14=(x+2)(3x^2-20x-7)$ SKY-EYE-THROGH!

$3x^2$ -7 $-7x$ $-47x$ $-40x$

7.3: FACTOR THEOREM

if $f(p)=0 \iff f(p)$ is a factor of $f(x)$

GOES **BOTH** ways!

factorizing cubics using factor theorem:

1. sub values of p until $f(p)=0$
2. divide $f(x)$ by $(x-p)$
3. factorized!

factorizing higher degree polynomials using factor theorem:

1. sub values of p until $f(p)=0$
2. divide $f(x)$ by $(x-p)$
3. if the result is not cubic, repeat. else, use cubic factorization method.

Ex 7C (p.145)

(10) $f(1)=p+q-3-7=p+q-10=0$ — ①
$f(-1)=-p+q+3-7=-p+q-4=0$ — ②

because $(x+1)$ & $(x-1)$ are factors, $f(1)=f(-1)=0$

①-②: $2p-6=0$
$p=3$
$q=7$

b) $(x-1)(x+3)=x^2+2x-3$

You can divide by non-linear functions!

$$
\begin{array}{r}
2x^2-9x-18 \\
x^2+2x-3 \overline{)\, 2x^4-5x^3-42x^2-9x+54} \\
\underline{2x^4+4x^3-6x^2} \\
9x^3-36x^2-9x+54 \\
\underline{9x^3-18x^2+27x} \\
-18x^2-36x+54 \\
\underline{-18x^2-36x+54} \\
0
\end{array}
$$

$$\begin{pmatrix} 2x & +3 \\ x & -6 \end{pmatrix}$$

$\therefore f(x)=(x-1)(x+3)(2x+3)(x-6)$

Challenge

a) $f(1)=2-5-42-9+54=0$
$f(-3)=162+135-378+27+54=0$

7.4: MATHEMATICAL PROOF

All proofs must: state assumptions, show logical steps & write a statement

① Proof by deduction

KNOWN FACT ⟹ LOGICAL STEPS ⟹ STATEMENT OF PROOF

e.g. statement: product of 2 odd #s is odd.

demonstration: $7 \times 5 = 35$

odd × odd = odd

proof: $p, q \in \mathbb{Z}$, $\therefore 2p+1$ & $2q+1$ = odd numbers

$(2p+1)(2q+1) = 4pq + 2p + 2q + 1$

$= 2(2pq + p + q) + 1$

Since $p, q \in \mathbb{Z}$, $(2pq + p + q) \in \mathbb{Z}$

$\therefore 2(2pq+p+q)+1$ is 1 more than an even number

$\therefore$ product of 2 odd #s is odd.

Another Example: prove $(2x+3)(x-4)(x+6) \equiv 2x^3 + 7x^2 - 42x - 72$

LHS = *Expand left hand side and gather terms* = RHS

$\therefore (2x+3)(x-4)(x+6) \equiv 2x^3 + 7x^2 - 42x - 72$

statement of proof

MUST have the identity symbol ($\equiv$)

Proof of factor theorem:

If $(ax-b)$ is factor of $f(x)$, $f(x) = (ax-b)\,g(x)$

and it follows that $f\left(\frac{b}{a}\right) = \left(a\frac{b}{a} - b\right) g\left(\frac{b}{a}\right) = (b-b)\,g\left(\frac{b}{a}\right) = 0 \cdot g\left(\frac{b}{a}\right) = 0$

therefore when $(ax-b)$ is a factor of $f(x)$, $f\left(\frac{b}{a}\right) = 0$

Prove that for consecutive integers, the difference of their squares is equal to their sum.

Let $n \in \mathbb{Z}$, then n and $n+1$ are consecutive integers.

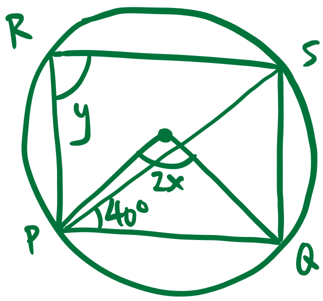
Difference between their squares is $(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n+1$

Which is equal to $n + (n+1)$

which is the sum of the 2 integers n & $n+1$

therefore <u>difference between squares of 2 consecutive integers is equal to their sum</u>.

An angle $2x$ is subtended at the circle



PQRS is a cyclic quadilateral

<u>Show that $y = x + 40°$</u>

$\angle PSQ = \frac{2x}{2} = x$ $\because \angle$ subtended at center is twice as $\angle$ subtended at circumfrence

$\angle PQS + y = 180°$ $\because$ Opposite $\angle$s of a cyclic quadilateral sum to $180°$

$\angle PQS = 180° - y$

$40° + x + \angle PQS = 180°$

$40° + x + 180° - y = 180°$

therefore, $y = x + 40°$

EX7E

(7) Prove $(x+6)^2 \geq 2x+11 \ \forall x \in \mathbb{R}$ (for all real values of x)

Consider $(x+5)^2 = x^2+10x+25$

$(x+6)^2 = x^2+12x+36$

Then, $(x+6)^2 = (x+5)^2 + 2x+11$

Since $(x+5)^2 \geq 0 \quad \forall x \in \mathbb{R}$,

then it follows that $(x+6)^2 \geq 2x+11$

(8) $a \in \mathbb{R}^+$ (positive real number), prove $a + \frac{1}{a} \geq 2$

$(a-1)^2 \geq 0$

$a^2 - 2a + 1 \geq 0$

$a - 2 + \frac{1}{a} \geq 0$

$a + \frac{1}{a} \geq 2$ for any $a \in \mathbb{R}^+$

Take the square of a value

(9) a) $p, q \in \mathbb{R}^+$, prove that $p+q \geq \sqrt{4pq}$

We have that $(p-q)^2 \geq 0$

and $(p+q)^2 = (p-q)^2 + 4pq$

then $(p+q)^2 \geq 4pq$

Since $p, q > 0$,

$p+q \geq \sqrt{4pq} \quad p, q \in \mathbb{R}^+$

b) let $p = q = -1$, then

$p+q = -2 \quad \sqrt{4pq} = 2$

$-2 \not\geq 2$

② Proof By Exhaustion

$\downarrow$

To have examined/considered all possible cases

If a statement is proved by exhaustion, then the statement is proved to be true for all possible cases which can be considered.

Prove that $(n+1)^3 \geq 3^n$, $n \in \mathbb{N}$, $n \leq 4$

Exhaust ALL the possibilities

$n=1$, $(1+1)^3 = 3 = 3^1 = 3$

$n=2$, $(2+1)^3 = 27 \geq 3^2 = 9$

$n=3$, $(3+1)^3 = 64 \geq 3^3 = 27$

$n=4$, $(4+1)^3 = 125 \geq 3^4 = 81$

Hence, $(n+1)^3 \geq 3^n$ for $n \in \mathbb{N}$, $n \leq 4$

Prove that every perfect cub is a multiple of 9, 1 more than a multiple of 9, or 1 less than a multiple of 9

Every integer is either: i) a multiple of 3,
ii) 1 more than a multiple of 3,
or iii) 1 less than a multiple of 3

CASE ① let $m = 3n$, then

$$m^3 = (3n)^3 = 27n^3 = 9(3n^3)$$

CASE ② let $m = 3n-1$, then

$$m^3 = (3n-1)^3 = 27n^3 - 27n^2 + 9n - 1 = 9(3n^3 - 3n^2 + n) - 1$$

CASE ③ let $m = 3n+1$, then

$$m^3 = (3n+1)^3 = 27n^3 + 27n^2 + 9n + 1 = 9(3n^3 + 3n^2 + n) + 1$$

Hence, every perfect cube can be written as a multiple of 9, 1 more than a multiple of 9, or 1 less than a multiple of 9

Prove that for $x \in \mathbb{Z}$, $f(x) = x^3+x+1$ is odd

consider x even, $x = 2n$ for $n \in \mathbb{Z}$

then, $f(x) = (2n)^3 + 2n + 1$

$= 8n^3 + 2n + 1$

$= 2(4n^3 + n) + 1$

consider x odd, $x = 2n+1$ for $n \in \mathbb{Z}$

then, $f(x) = 8n^3 + 12n^2 + 6n + 1 + 2n + 1 + 1$

$= 8n^3 + 12n^2 + 8n + 3$

$= 2(4n^3 + 6n^2 + 4n + 1) + 1$

Exhausted all the possibilities (x is even, x is odd)

therefore, $f(x) = x^3+x+1$ is odd for $x \in \mathbb{Z}$

Prove that $n^7 - n$ is divisible by 7 for $n \in \mathbb{Z}$

we can factorize $n^7 - n$ as $n(n^6-1) = n(n^3+1)(n^3-1)$

$= n(n-1)(n^2+n+1)(n+1)(n^2-n+1)$

Every integer n can be written as

$7m, 7m+1, 7m+2, \ldots, 7m+6$ for some $m \in \mathbb{Z}$

CASE 1 $n = 7m$, n is multiple of 7, then $n^7 - n$ is a multiple of 7

CASE 2 $n = 7m+1$, $n-1$ is multiple of 7, then n^7-n is a multiple of 7

CASE 3 $n = 7m+2$, $n^2+n+1 = 49m^2+28m+4+7m+3$

$= 7(7m^2+5m+1)$, then n^7-n is a multiple of 7

CASE 4 $n = 7m+3$, $n^2-n+1 = 49m^2+42m+9-7m-2$

$= 7(7m^2+5m+1)$, then n^7-n is a multiple of 7

CASE 5 $n = 7m+4$, $n^2+n+1 = 49m^2+56m+16+7m+5$

$= 7(7m^2+9m+3)$, then n^7-n is a multiple of 7

CASE 6 $n = 7m+5$, $n^2-n+1 = 49m^2+70m+25-7m-4$

$= 7(7m^2+9m+3)$, then n^7-n is a multiple of 7

CASE 7 $n = 7m+6$, $n+1 = 7(m+1)$, then n^7-n is a multiple of 7

therefore, n^7-n is a multiple of n^7-n for $n \in \mathbb{Z}$