CHAPTER 6: MATRICES

A grid of numbers = matrix

$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

Rows × Columns

2 × 2 matrix

$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \end{pmatrix}$

Rows × Columns

2 × 3 matrix

In general: n×m

Special Matrices

Square $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

rows = columns

Zero $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

all elements are zero

numbers in a matrix = elements

Identity $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

1's diagonally down, everything else is 0 (only square matrices)

Referring to matrices

CAPITAL LETTERS!

e.g.

$A = \begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}$ $M = \begin{pmatrix} 3 & 2 & 3 \\ 4 & 3 & 1 \end{pmatrix}$

$B = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$

n×n identity matrix = I_n

$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Adding & Subtracting

Requirements: 2 matrices of the same size/dimensions

$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 2 & 5 \\ 4 & 3 & 1 \end{pmatrix}$ illegal

$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$ legal

$= \begin{pmatrix} 1+3 & 1+2 \\ 2+5 & 3+4 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 7 & 7 \end{pmatrix}$ ADD numbers element-wise

Do the same thing with subtraction!

Multiplication

Scalar multiplication

$3 \times \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 3 \\ 15 & 6 \end{pmatrix}$

multiply element-wise.

the "EASY" one

Matrix multiplication the BIGGIE

Requirements: columns of A = rows of B (in A×B)

$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \times \begin{pmatrix} g \\ h \\ i \end{pmatrix} = \begin{pmatrix} ag + bh + ci \\ dg + eh + fi \end{pmatrix}$! AB ≠ BA

2×1 matrix

WHAT THE HELL JUST HAPPENED?

$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \times \begin{pmatrix} g \\ h \\ i \end{pmatrix}$ multiply element-wise, then add them together

then repeat for each row in A

then repeat for each column in B

sizes $\underbrace{2 \times \underbrace{3 \quad \times \quad 3}_{\text{same \#}} \times 1}_{\text{result size}}$

Determinants

Inverse: opposite of something

e.g. $3 \times 3^{-1} = 1$ but ONLY IF $3 \neq 0$

$a^{-1} = \frac{1}{a} \quad (a \neq 0)$

if $a = 0$, a^{-1} doesn't exist

Inverse of a matrix:

$\underbrace{A^{-1} \times A = I}_{\text{All are squares}} = A \times A^{-1}$

Inverse of a matrix

Does the inverse exist?

Det(A) determines if A^{-1} exists

If $\mathrm{Det}(A) = 0$, no inverse exists, A is singular

If $\mathrm{Det}(A) \neq 0$, inverse exists, A is non-singular

2×2 Matrices

alternative notation

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\det M = |M| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \underbrace{ad-bc}$

the origin of this will be explained later.

e.g. $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\det M = (1)(4) - (2)(3) = -2 \neq 0 \quad \therefore M^{-1}$ exists

e.g. $A = \begin{pmatrix} 4 & p+2 \\ -1 & 3-p \end{pmatrix}$ Given that A is singular, $\det A = 12 - 4p - (-p-2) = 14 - 3p = 0$

$p = \frac{14}{3}$

3×3 Matrices

minors of a, b and c

Let $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then $\det M = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

I'm sorry WHAT?

$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ go through a, b and c. delete rows & columns

$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ the remaining elements form a minor of a, b and c

$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ multiply with minors, add with alternating signs

$\underbrace{\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}}$

signs for each row

you don't need to choose abc

any row/column will work

(if you follow the signs)

e.g. $\begin{vmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} + 4\begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix} = 1(6-4) - 2(9+1) + 4(12+2) = 38$

$n \times n$ matrices

e.g. 4x4 $\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$ $\qquad \underbrace{\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}}_{\text{signs to follow}}$

A level: up to 3x3 determinants (thank god)

$$\det = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$ same algorithm $\therefore$

Inverse Matrices

2x2: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ if $\det A = 0$, $\frac{1}{\det A}$ undefined, A^{-1} undefined

3x3: $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ $\quad$ Matrix of minors, $M = \begin{pmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & \begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & \begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$

minors of each element

Apply $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ pattern to M to get C

Matrix of cofactors, $C = \begin{pmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix} = \begin{pmatrix} M_a & -M_b & M_c \\ -M_d & M_e & -M_f \\ M_g & -M_h & M_i \end{pmatrix}$

transpose C

(swap rows & columns)

$C^T = \begin{pmatrix} M_a & -M_d & M_g \\ -M_b & M_e & M_h \\ M_c & -M_f & M_i \end{pmatrix}$

mirror along diagonal

$A^{-1} = \frac{1}{\det A} C^T$ if $\det A = 0$, $\frac{1}{\det A}$ undefined, A^{-1} undefined

Systems of Equations

$$\begin{aligned} x + 2y &= 3 \\ 5x - 3y &= 4 \end{aligned} \Rightarrow \begin{pmatrix} x+2y \\ 5x-3y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

matrix product of 2 matrizes!

let $\begin{pmatrix} 1 & 2 \\ 5 & -3 \end{pmatrix} = A$

$\begin{pmatrix} x \\ y \end{pmatrix} = X \quad \begin{pmatrix} 3 \\ 4 \end{pmatrix} = B$

$\Rightarrow AX = B$

$A^{-1}AX = A^{-1}B$

⚠ order matters!

$X = \underbrace{A^{-1}B}_{\text{solution!}}$

$\det A = -3 - 10 = -13$

$$A^{-1} = -\frac{1}{13}\begin{pmatrix} -3 & -2 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{13} & \frac{2}{13} \\ \frac{5}{13} & -\frac{1}{13} \end{pmatrix} \qquad A^{-1}B = \begin{pmatrix} \frac{3}{13} & \frac{2}{13} \\ \frac{5}{13} & -\frac{1}{13} \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{9}{13} + \frac{8}{13} \\ \frac{15}{13} - \frac{4}{13} \end{pmatrix} = \begin{pmatrix} \frac{17}{13} \\ \frac{11}{13} \end{pmatrix}$$

Therefore, $X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{17}{13} \\ \frac{11}{13} \end{pmatrix} \Rightarrow x = \frac{17}{13},\ y = \frac{11}{13}$

If we were trying to solve a system of $eq^{\underline{n}}$ with 3 unknowns, we use 3×3 matrices!

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} j \\ k \\ l \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1}\begin{pmatrix} j \\ k \\ l \end{pmatrix}$$

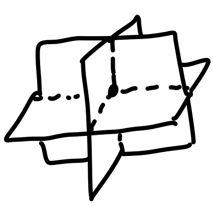
let the system of equations be

$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = v$

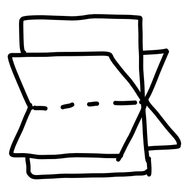
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}v$

A system of $eq^{\underline{n}}s$ is consistent if there is exactly 1 solution

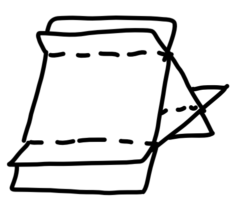
each $eq^{\underline{n}}$ represents a plane in 3D space.



1 pt. of intersection
A^{-1} exists
consistent



1 line of intersection
A^{-1} doesn't exist
consistent



No common intersections
A^{-1} doesn't exist
inconsistent

Exercise 6F

a) $\begin{pmatrix} 2 & -6 & 4 \\ 3 & 2 & -9 \\ -2 & 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 32 \\ -49 \\ -3 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -6 & 4 \\ 3 & 2 & -9 \\ -2 & 4 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 32 \\ -49 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{19}{25} & \frac{11}{25} & \frac{23}{25} \\ \frac{3}{10} & \frac{1}{5} & \frac{3}{5} \\ \frac{8}{25} & \frac{2}{25} & \frac{11}{25} \end{pmatrix}\begin{pmatrix} 32 \\ -49 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix}$$

$$\det\begin{vmatrix} 2 & -6 & 4 \\ 3 & 2 & -9 \\ -2 & 4 & 1 \end{vmatrix} = 50$$

$$M = \begin{pmatrix} \begin{vmatrix} 2 & -9 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -9 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix} \\ \begin{vmatrix} -6 & 4 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -6 \\ -2 & 4 \end{vmatrix} \\ \begin{vmatrix} -6 & 4 \\ 2 & -9 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 3 & -9 \end{vmatrix} & \begin{vmatrix} 2 & -6 \\ 3 & 2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 38 & -15 & 16 \\ -22 & 10 & -4 \\ 46 & -30 & 22 \end{pmatrix}$$

$$C = \begin{pmatrix} 38 & 15 & 16 \\ 22 & 10 & 4 \\ 46 & 30 & 22 \end{pmatrix} \qquad C^T = \begin{pmatrix} 38 & 22 & 46 \\ 15 & 10 & 30 \\ 16 & 4 & 22 \end{pmatrix}$$

$$\frac{1}{50}C^T = \begin{pmatrix} \frac{19}{25} & \frac{11}{25} & \frac{23}{25} \\ \frac{3}{10} & \frac{1}{5} & \frac{3}{5} \\ \frac{8}{25} & \frac{2}{25} & \frac{11}{25} \end{pmatrix}$$

Mixed Exercise

1. $A=\begin{pmatrix}1 & -3\\ 2 & 1\end{pmatrix}$ $AB=\begin{pmatrix}4 & 1 & 9\\ 1 & 9 & 4\end{pmatrix}$

$A\times B=AB$

$A^{-1}\times A\times B=A^{-1}\times AB=B$

$A^{-1}=\frac{1}{\det A}\begin{pmatrix}1 & 3\\ -2 & 1\end{pmatrix}=\frac{1}{1+6}\begin{pmatrix}1 & 3\\ -2 & 1\end{pmatrix}=\frac{1}{7}\begin{pmatrix}1 & 3\\ -2 & 1\end{pmatrix}$

$A^{-1}\times AB=\frac{1}{7}\begin{pmatrix}1 & 3\\ -2 & 1\end{pmatrix}\times\begin{pmatrix}4 & 1 & 9\\ 1 & 9 & 4\end{pmatrix}=\frac{1}{7}\begin{pmatrix}4+3 & 1+27 & 9+12\\ -8+1 & -2+9 & -18+4\end{pmatrix}$

$=\frac{1}{7}\begin{pmatrix}7 & 28 & 21\\ -7 & 7 & -14\end{pmatrix}=\begin{pmatrix}1 & 4 & 3\\ -1 & 1 & -2\end{pmatrix}$ ✓

3. a) $XB=BA$

$X=BAB^{-1}$

b) $A=\begin{pmatrix}5 & 3\\ 0 & -2\end{pmatrix}$ $B=\begin{pmatrix}2 & 1\\ -1 & -1\end{pmatrix}$ $B^{-1}=-1\times\begin{pmatrix}-1 & -1\\ 1 & 2\end{pmatrix}=\begin{pmatrix}1 & 1\\ -1 & -2\end{pmatrix}$

$X=B\times A\times B^{-1}=\begin{pmatrix}2 & 1\\ -1 & -1\end{pmatrix}\times\begin{pmatrix}5 & 3\\ 0 & -2\end{pmatrix}\times\begin{pmatrix}1 & 1\\ -1 & -2\end{pmatrix}$

$=\begin{pmatrix}2 & 1\\ -1 & -1\end{pmatrix}\times\begin{pmatrix}2 & -1\\ 2 & 4\end{pmatrix}=\begin{pmatrix}6 & 2\\ -4 & -3\end{pmatrix}$ ✓

5. $A=\begin{pmatrix}1 & 0 & 2\\ t & 3 & 1\\ -2 & -1 & 1\end{pmatrix}$ $\det A=1\begin{vmatrix}3 & 1\\ -1 & 1\end{vmatrix}-0\begin{vmatrix}t & 1\\ -2 & 1\end{vmatrix}+2\begin{vmatrix}t & 3\\ -2 & -1\end{vmatrix}$

$=4+2(-t+6)=-2t+16=0$

$t=8$ ✓

7. a) $A=\begin{pmatrix}k & -2\\ -4 & k\end{pmatrix}$ $\det A=k^2-8\neq 0$

$k^2\neq 8$

$k\neq\pm 2\sqrt{2}$ ✓

b) $A^{-1}=\frac{1}{k^2-8}\begin{pmatrix}k & 2\\ 4 & k\end{pmatrix}$ ✓

9. a) $M=\begin{pmatrix}2 & -m\\ m & -1\end{pmatrix}$

$\det M=-2+m^2=0$

$m=\pm\sqrt{2}$ ✓

b) $M^{-1}=\frac{1}{m^2-2}\begin{pmatrix}-1 & m\\ -m & 2\end{pmatrix}$ ✓

11. $\begin{pmatrix}1&1&1\\1&-4&2\\2&1&-3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6\\-2\\0\end{pmatrix}$ let $D=\begin{pmatrix}1&1&1\\1&-4&2\\2&1&-3\end{pmatrix}$

$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&1&1\\1&-4&2\\2&1&-3\end{pmatrix}^{-1}\begin{pmatrix}6\\-2\\0\end{pmatrix}$

$\det D = 1\begin{vmatrix}-4&2\\1&-3\end{vmatrix} - 1\begin{vmatrix}1&2\\2&-3\end{vmatrix} + 1\begin{vmatrix}1&-4\\2&1\end{vmatrix}$

$= 10+7+9 = 26$

$$M=\begin{pmatrix}\begin{vmatrix}-4&2\\1&-3\end{vmatrix} & \begin{vmatrix}1&2\\2&-3\end{vmatrix} & \begin{vmatrix}1&-4\\2&1\end{vmatrix}\\ \begin{vmatrix}1&1\\1&-3\end{vmatrix} & \begin{vmatrix}1&1\\2&-3\end{vmatrix} & \begin{vmatrix}1&1\\2&1\end{vmatrix}\\ \begin{vmatrix}1&1\\-4&2\end{vmatrix} & \begin{vmatrix}1&1\\1&2\end{vmatrix} & \begin{vmatrix}1&1\\1&-4\end{vmatrix}\end{pmatrix}=\begin{pmatrix}10&-7&9\\-4&-5&-1\\6&1&-5\end{pmatrix}$$

$C=\begin{pmatrix}10&7&9\\4&-5&1\\6&-1&-5\end{pmatrix}$ $C^T=\begin{pmatrix}10&4&6\\7&-5&-1\\9&1&-5\end{pmatrix}$ $D^{-1}=\frac{1}{26}\begin{pmatrix}10&4&6\\7&-5&-1\\9&1&-5\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{26}\begin{pmatrix}10&4&6\\7&-5&-1\\9&1&-5\end{pmatrix}\begin{pmatrix}6\\-2\\0\end{pmatrix}=\frac{1}{26}\begin{pmatrix}60-8\\42+10\\54-2\end{pmatrix}=\frac{1}{26}\begin{pmatrix}52\\52\\52\end{pmatrix}=\begin{pmatrix}2\\2\\2\end{pmatrix}$$ ✓

13. a) $\begin{pmatrix}2&3&1\\-1&1&2\\a&1&4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6\\7\\b\end{pmatrix}$ $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2&3&1\\-1&1&2\\a&1&4\end{pmatrix}^{-1}\begin{pmatrix}6\\7\\b\end{pmatrix}$

$$\begin{vmatrix}2&3&1\\-1&1&2\\a&1&4\end{vmatrix}=2\begin{vmatrix}1&2\\1&4\end{vmatrix}-3\begin{vmatrix}-1&2\\a&4\end{vmatrix}+1\begin{vmatrix}-1&1\\a&1\end{vmatrix}$$

$= 4-3(-4-2a)+(-1-a) = 4+12+6a-1-a = 5a+15 = 0$

$a=-3$ ✓

$2x+3y+z=6$

$-x+y+2z=7 \rightarrow -3x+3y+6z=21$

$ax+y+4z=b \quad 3ax+3y+12z=3b$

$5x-5z=-15$

$(3a+3)x+6z=3b-21$

$-6x+6z=3b-21$

$6x-6z=21-3b$

$\frac{21-3b}{6}=-\frac{15}{5}$

$105-15b=-90$

$15b=195$

$b=13$ ✓

2. $A = \begin{pmatrix} a & b \\ 2a & 3b \end{pmatrix}$ a) $A^{-1} = \frac{1}{3ab-2ab}\begin{pmatrix} 3b & -b \\ -2a & a \end{pmatrix} = \frac{1}{ab}\begin{pmatrix} 3b & -b \\ -2a & a \end{pmatrix} = \begin{pmatrix} 3/a & -1/a \\ -2/b & 1/b \end{pmatrix}$

b) $Y = \begin{pmatrix} a & 2b \\ 2a & b \end{pmatrix}$ $X = YA^{-1} = \begin{pmatrix} a & 2b \\ 2a & b \end{pmatrix} \times \begin{pmatrix} 3/a & -1/a \\ -2/b & 1/b \end{pmatrix} = \begin{pmatrix} 3-4 & -1+2 \\ 6-2 & -2+1 \end{pmatrix}$

$= \begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix}$

4. $A = \begin{pmatrix} a & 2 & -1 \\ -1 & 1 & -1 \\ b & 2 & 1 \end{pmatrix}$ $A^2 = A \times A = \begin{pmatrix} a & 2 & -1 \\ -1 & 1 & -1 \\ b & 2 & 1 \end{pmatrix}\begin{pmatrix} a & 2 & -1 \\ -1 & 1 & -1 \\ b & 2 & 1 \end{pmatrix}$

$= \begin{pmatrix} a^2-2-b & 2a+2-2 & \cdots \\ \vdots & \ddots & \\ \cdots & & \cdots \end{pmatrix} = \begin{pmatrix} -4 & 2 & \cdots \\ \vdots & & \vdots \\ \cdots & & \cdots \end{pmatrix}$

$2a+2-2 = 2a = 2$
$a = 1$ $a^2-2-b = -b-1 = -4$
$b = 3$

6. $M = \begin{pmatrix} 1 & 0 & 0 \\ x & 2 & 0 \\ 3 & 1 & 1 \end{pmatrix}$ $\det M = 1\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 1(2-0) = 2$

Matrix of minors $= \begin{pmatrix} 2 & x & x-6 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

$C = \begin{pmatrix} 2 & -x & x-6 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ $C^T = \begin{pmatrix} 2 & 0 & 0 \\ -x & 1 & 0 \\ x-6 & -1 & 2 \end{pmatrix}$

$M^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ -x & 1 & 0 \\ x-6 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -x/2 & 1/2 & 0 \\ x/2-3 & -1/2 & 1 \end{pmatrix}$

8. $B=\begin{pmatrix} k & 6 \\ -1 & k-2 \end{pmatrix}$

a) $\det B = k^2-2k-(-6) = k^2-2k+6$

b) $k^2-2k+6 = (k-1)^2+5$ minimum value of $5>0$

$\therefore \det B > 0 \ \forall k \in \mathbb{R} \ \therefore B$ is non singular $\forall k \in \mathbb{R}$

c) $B^{-1} = \frac{1}{k^2-2k+6}\begin{pmatrix} k-2 & -6 \\ 1 & k \end{pmatrix}$

$21B^{-1} = \frac{21}{k^2-2k+6}\begin{pmatrix} k-2 & -6 \\ 1 & k \end{pmatrix}$

$21B^{-1}+B = \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$

$\frac{21}{k^2-2k+6}(k) + k-2 = -8$

$21k + k^3-2k^2+6k = -6k^2+12k-36$

$k^3+4k^2+15k+36=0$

$x=-3$

$\frac{21}{k^2-2k+6} - 1 = 0$

$21-k^2+2k-6=0$

$k^2-2k+15=0$

$k=-3,5$

$k \quad +3$

$k \quad -5$

$\therefore k=-3$

10. $A=\begin{pmatrix} 3 & 4 & 5 \\ 1 & a & -1 \\ -2 & 1 & 1 \end{pmatrix}$

a) $\det A = 3(a+1)-4(1-2)+5(1+2a)$

$= 3a+3+4+5+10a = 13a+12=0$

$a=-\frac{12}{13}$

b) $\det A = 13a+12$

$M=\begin{pmatrix} a+1 & -1 & 1+2a \\ -1 & 13 & 11 \\ -5a-4 & -8 & 3a-4 \end{pmatrix}$ $C=\begin{pmatrix} a+1 & 1 & 1+2a \\ 1 & 13 & -11 \\ -5a-4 & 8 & 3a-4 \end{pmatrix}$

$C^T=\begin{pmatrix} a+1 & 1 & -5a-4 \\ 1 & 13 & 8 \\ 1+2a & -11 & 3a-4 \end{pmatrix}$

$A^{-1}=\frac{1}{13a+12}\begin{pmatrix} a+1 & 1 & -5a-4 \\ 1 & 13 & 8 \\ 1+2a & -11 & 3a-4 \end{pmatrix}$

12. $x =$ # of hampshire $\quad y =$ # of dorset horn $\quad z =$ # of wiltshire horn

$$x + y + z = 2500$$
$$x + 0y - z = 300$$
$$1.06x + 1.04y + 1.03z = 2610$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.06 & 1.04 & 1.03 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2500 \\ 300 \\ 2610 \end{pmatrix}$$

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = v \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} v$$