MODULE 4

Multiple Input Multiple Output Wireless Communications

Multiple Input Multiple Output Wireless Communications: Introduction to MIMO Communications, MIMO system Model, MIMO Zero Forcing Receiver, MIMO MMSE Receiver, Singular Value decomposition of MIMO Channel, SVD and MIMO capacity, Alamouti and Space-Time Codes, Nonlinear MIMO receiver: V-Blast, MIMO Beamforming. **[Text1:6.1,6.2, 6.3, 6.4, 6.5, 6.6, 6.8, 6.9, 6.10]**

Text 1: Aditya K Jagannatham, "Principles of Modern Wireless Communication systems, Theory and Practice ", Mc Graw Hill Education (India) Private Limited, 2017, ISBN 978-81- 265-4231-4.

4.1 Introduction to MIMO Wireless Communications

- Multiple-Input Multiple-Output (MIMO) wireless communications employ multiple antennas at the transmitter and the receiver.
- A schematic of a MIMO system with multiple antennas is shown in Figure 4.1. As MIMO systems have multiple antennas, they can be employed to increase the reliability of the signal through diversity combining. This leads to diversity gain and a net decrease in the bit-error rate of the wireless communication system.
- A unique aspect of MIMO wireless systems is that they enable a several-fold increase in the data rate of the wireless communication system by transmitting several information streams in parallel. This is termed **spatial multiplexing**.
- This can be thought of as transmitting multiple parallel streams in space through different spatial modes, i.e., multiplexing information streams in the space dimension as illustrated in Figure 4.1.

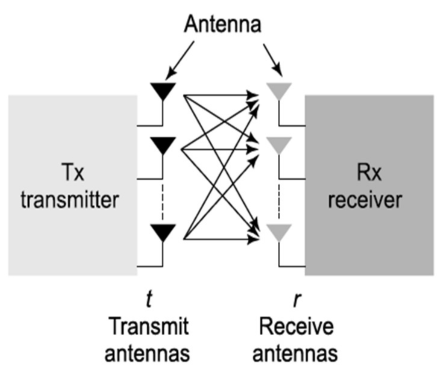

Figure 4.1 MIMO system schematic

4.2 MIMO System Model

- Consider a MIMO wireless system with t transmit antennas and r receive antennas. Such a MIMO system is also termed an r × t system.

- Let $x_1, x_2, \ldots, x_t$ denote the t symbols transmitted from the t transmit antennas in the MIMO system, i.e., x_i denotes the symbol transmitted from the i^{th} transmit antenna $1 \le i \le t$. These transmit symbols can be stacked to form the t-dimensional vector, also termed the transmit vector,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix}$$

- Corresponding to this transmission, let $y_1, y_2, \ldots, y_t$ denote the r received symbols across the r receive antennas in the MIMO systems, which can be stacked as the r-dimensional receive symbol vector,

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix}$$

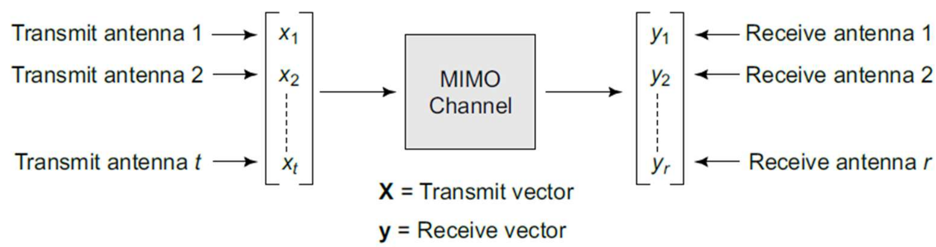

Figure 4.2 MIMO system input-output schematic

- This is shown schematically in Figure 4.2. Let the complex coefficient h_{ij} represent the fading channel coefficient between the i^{th} receive antenna and the j^{th} transmit antenna. Thus, there are a net of rt channel coefficients in this wireless scenario corresponding to all possible combinations of the r receive antennas and t transmit antennas. These can be arranged in a matrix form as

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & \ldots & h_{1t} \\ h_{21} & h_{22} & \ldots & h_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{r1} & h_{r2} & \ldots & h_{rt} \end{bmatrix}$$

where the $r \times t$ dimensional matrix H is termed the MIMO channel matrix.

- Let the additive noise at the receive antenna i be denoted by n_i, i.e., $n_1, n_2, \ldots, n_r$ denote the additive noise at the r receive antennas. Thus, the net MIMO input output system model can be represented in vector form as

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} h_{11} & h_{12} & \dots & h_{1t} \\ h_{21} & h_{22} & \dots & h_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{r1} & h_{r2} & \dots & h_{rt} \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_r \end{bmatrix}}_{\mathbf{n}}$$

This is succinctly represented using matrix notation as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

The receive symbol y_1 is given as

$$y_1 = h_{11}x_1 + h_{12}x_2 + \ldots + h_{1t}x_t + n_1$$

from which it can be seen that all the symbols $x_1, x_2, \ldots, x_t$ interfere at y_1 received at the receive antenna 1. Similarly, the receive symbol y_2 is given as

$$y_2 = h_{21}x_2 + h_{22}x_2 + \ldots + h_{2t}x_t + n_2$$

from which it can be once again seen that $x_1, x_2, \ldots, x_t$ interfere at y_2 received at the receive antenna 2.

- This is, in general, true for all the receive antennas, i.e., at each receive antenna i, the receive symbol y_i is a linear of all the transmit symbols $x_1, x_2, \ldots, x_t$ from the t transmit antennas, observed in additive noise n_i.
- For the special case of t = 1, i.e., single transmit antenna and multiple receive antennas, this is termed as **Single-Input Multiple-Output (SIMO) system** or the receive diversity system . This can be modelled as

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_r \end{bmatrix}}_{\mathbf{h}} x + \underbrace{\begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_r \end{bmatrix}}_{\mathbf{n}}$$

Similarly, for the case of one receive antenna, i.e., r = 1 and multiple transmit antennas, it is termed a **Multiple-Input Single-Output (MISO) system** model or a transmit diversity system. Its system model is given as

$$y = \underbrace{\begin{bmatrix} h_1 & h_1 & \ldots & h_t \end{bmatrix}}_{\mathbf{h}^T} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix}}_{\mathbf{x}} + n$$

Finally, for r = t = 1, i.e., a single receive and transmit antenna, it reduces to the single-input single-output (SISO) system, modelled as

$$y = hx + n$$

The above equation model the Rayleigh fading wireless -channel-based wireless communication. The covariance matrix of the noise R_n of the noise vector n defined as

$$\begin{aligned}
\mathbf{R_n} &= \mathrm{E}\left\{\mathbf{n}\mathbf{n}^H\right\} \\
&= \mathrm{E}\left\{\begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_L \end{bmatrix}\begin{bmatrix} n_1^* & n_2^* & \dots & n_L^* \end{bmatrix}\right\} \\
&= \begin{bmatrix} \mathrm{E}\left\{|n_1|^2\right\} & \mathrm{E}\left\{n_1 n_2^*\right\} & \dots & \mathrm{E}\left\{n_1 n_r^*\right\} \\ \mathrm{E}\left\{n_2 n_1^*\right\} & \mathrm{E}\left\{|n_2|^2\right\} & \dots & \mathrm{E}\left\{n_2 n_r^*\right\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{E}\left\{n_r n_1^*\right\} & \mathrm{E}\left\{n_r n_2^*\right\} & \dots & \mathrm{E}\left\{|n_r|^2\right\} \end{bmatrix} \\
&= \begin{bmatrix} \sigma_n^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_n^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix} \\
&= \sigma_n^2 \mathbf{I}_r
\end{aligned}$$

The noise vector n with the covariance structure above is termed spatially uncorrelated additive noise, since the noise samples at the different antennas i, j are independent, i.e., E { $n_i n_j^*$ } = 0 if $i \neq j$.

- Finally, to denote the transmission and reception across different time instants, one can add the time index k to the MIMO system model to frame the net model as

$$\underbrace{\begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_r(k) \end{bmatrix}}_{\mathbf{y}(k)} = \underbrace{\begin{bmatrix} h_{11} & h_{12} & \dots & h_{1t} \\ h_{21} & h_{22} & \dots & h_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{r1} & h_{r2} & \dots & h_{rt} \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_t(k) \end{bmatrix}}_{\mathbf{x}(k)} + \underbrace{\begin{bmatrix} n_1(k) \\ n_2(k) \\ \vdots \\ n_r(k) \end{bmatrix}}_{\mathbf{n}(k)}$$

Thus, the vectors y (k) , x (k) , n (k) define the receive, transmit, and noise vectors of the

MIMO wireless communication system at the time instant k.

- We have assumed the channel matrix H to be constant or, in other words, not dependent on the time instant k. This is also termed a **slow fading or quasi-static channel matrix**, indicating that the channel coefficients are constant over the block of MIMO vectors that are transmitted.
- We also assume that any two noise samples across two different time instants are uncorrelated, i.e., E { $n_i(k)\, n^*_j(l)$ } = 0 if k ≠ l. Hence, the noise covariance matrix is given as

$$\mathrm{E}\left\{\mathbf{n}(k)\,\mathbf{n}(l)^H\right\} = \sigma^2\delta(k-l)\,\mathbf{I}_r$$

where the delta function δ (k − l) = 1 if k = l and 0 otherwise.

This noise process, which is uncorrelated across different antennas and time instants is termed **spatio-temporally uncorrelated noise**.

4.3 MIMO Zero-Forcing (ZF) Receiver

- The process to recover the transmitted signal vector x from the received vector y at the MIMO receiver is described below. This can be considered as solving the system of linear equations,

$$\mathbf{y = Hx}$$

where $x_1, x_2, \ldots, x_t$ are the t unknowns and there are r equations corresponding to the r observations $y_1, y_2, \ldots, y_r$.

- Consider a simplistic scenario, where r = t, i.e., the number of receive antennas is equal to the number of transmit antennas. In this case, the matrix H is square.
- Further, if the matrix H is now invertible, the estimate ˆx of the transmit vector x is still given as

$$\hat{\mathbf{x}} = \mathbf{H}^{-1}\hat{\mathbf{y}}$$

However, frequently, one has more receive antennas than transmit antennas, i.e., r > t. In this scenario, the system y = Hx is given as

$$\underbrace{\begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ \vdots \\ \vdots \\ y_r(k) \end{bmatrix}}_{\mathbf{y}(k)} = \underbrace{\begin{bmatrix} h_{11} & h_{12} & \ldots & h_{1t} \\ h_{21} & h_{22} & \ldots & h_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ h_{r1} & h_{r2} & \ldots & h_{rt} \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_t(k) \end{bmatrix}}_{\mathbf{x}(k)}$$

from which it can be seen that the matrix H has more rows than columns. Such a matrix is popularly known as a **tall matrix** due to its structure. In this situation, one cannot exactly solve for x since there are more equations r than unknowns t. Hence, one can resort to choosing the vector x which minimizes the estimation error f (ˆx),

$$f(\mathbf{x}) = \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2$$

The above error function is also termed the **least-squares error function** and the resulting estimator is termed the **least-squares estimator**.

- We consider real vectors/matrices y, x, H. The above error function can be expanded as

$$\begin{aligned} f(\mathbf{x}) &= \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2, \\ &= (\mathbf{y} - \mathbf{H}\mathbf{x})^T(\mathbf{y} - \mathbf{H}\mathbf{x}) \\ &= \left(\mathbf{y}^T - \mathbf{x}^T\mathbf{H}^T\right)(\mathbf{y} - \mathbf{H}\mathbf{x}) \\ &= \mathbf{y}^T\mathbf{y}^T - \mathbf{x}^T\mathbf{H}^T\mathbf{y} - \mathbf{y}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{H}^T\mathbf{H}\mathbf{x} \\ &= \mathbf{y}^T\mathbf{y} - 2\mathbf{x}^T\mathbf{H}^T\mathbf{y} + \mathbf{x}^T\mathbf{H}^T\mathbf{H}\mathbf{x} \end{aligned} \qquad \text{--- 4.1}$$

where we have used the relation $y^T \mathbf{H}x = (y^T \mathbf{H}x)^T = \hat{}x^T \mathbf{H}^T y$ in the above simplification. This is due to the fact that $y^T \mathbf{H}x$ is a scalar and, hence, is equal to its transpose.

- To find the minimum of the error function f (x) with respect to x, we have to set the derivative with respect to x equal to 0.
  Consider a multidimensional function g (x). The vector derivative of g (x) with respect to x is defined as

$$\frac{\partial g(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial g(\mathbf{x})}{\partial x_1} \\ \frac{\partial g(\mathbf{x})}{\partial x_2} \\ \vdots \\ \frac{\partial g(\mathbf{x})}{\partial x_t} \end{bmatrix}$$

  which is basically a t-dimensional vector with the i^{th} component equal to the derivative of f (x) with respect to x_i.

  Consider any vector $c = [c_1, c_2, \ldots, c_t]^T$. Let the function g (x) be defined as

$$g(\mathbf{x}) = \mathbf{c}^T\mathbf{x} = c_1x_1 + c_2x_2 + \ldots + c_tx_t$$

  Hence, it can be seen that, $\partial g(x)/\partial x_i = c_i$. Therefore, it can be readily deduced that in this case

$$\frac{\partial g(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial g(\mathbf{x})}{\partial x_1} \\ \frac{\partial g(\mathbf{x})}{\partial x_2} \\ \vdots \\ \frac{\partial g(\mathbf{x})}{\partial x_t} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_t \end{bmatrix} = \mathbf{c}$$

In fact, it can also be seen that since $c^T x = x^T c$, we have

$$\frac{\partial \mathbf{c}^T\mathbf{x}}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}^T\mathbf{c}}{\partial \mathbf{x}} = \mathbf{c}$$

Going back to the expansion of the error function f (x) from Eq. (4.1), it can be seen that the derivative of each component with respect to x can be computed as follows.
Consider the quantity $y^T y = \| y \|^2$
Observe that this does not depend on x. Hence, we have $\partial y^T y/\partial x = 0$.
Consider the component $2x^T \mathbf{H}^T y$. This is in the form of $x^T c$, where $c = 2\mathbf{H}^T y$.
Hence, the derivative of this component with respect to x is given as $\partial(2x^T \mathbf{H}^T y)/\partial x = 2H^T y$.
Now, consider the last component $x^T H^T Hx$. This can be differentiated employing the product rule as

$$\begin{aligned} \frac{\partial\left(\mathbf{x}^T\mathbf{H}^T\mathbf{H}\mathbf{x}\right)}{\partial \mathbf{x}} &= \mathbf{H}^T\mathbf{H}\mathbf{x} + \left(\mathbf{x}^T\mathbf{H}^T\mathbf{H}\right)^T \\ &= 2\mathbf{H}^T\mathbf{H}\mathbf{x} \end{aligned}$$

Hence, employing the above results, the derivative for the error function f (x) can be simplified As

$$\frac{\partial f(\mathbf{x})}{\mathbf{x}} = -2\mathbf{H}^T\mathbf{y} + 2\mathbf{H}^T\mathbf{H}\mathbf{x}$$

At the optimal estimate of the transmit vector ˆx where the above error is minimized, we must have the derivative equal to 0. Using this condition,

$$\left.\frac{\partial f(\mathbf{x})}{\mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}} = 0$$

$$-2\mathbf{H}^T\mathbf{y} + 2\mathbf{H}^T\mathbf{H}\hat{\mathbf{x}} = 0$$

$$\mathbf{H}^T\mathbf{H}\hat{\mathbf{x}} = \mathbf{H}^T\mathbf{y}$$

$$\hat{\mathbf{x}} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{y}$$

Finally, for the case of complex vectors/matrices x, y, H, the transpose in the above expression can be replaced by the Hermitian operator to yield

$$\hat{\mathbf{x}} = \left(\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{H}^H\mathbf{y}$$

The above decoder for the MIMO wireless system to decode the transmitted symbol vector x from the received symbol vector y is termed the **zero-forcing receiver** or simply the **ZF receiver**. Hence, the zero-forcing decoder can be expressed as

$$\hat{\mathbf{x}}_{\mathrm{ZF}} = \underbrace{\left(\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{H}^H}_{\mathbf{F}_{\mathrm{ZF}}}\mathbf{y} \qquad \text{-------- 4.2}$$

The matrix $\mathbf{F}_{\mathrm{ZF}} = \left(\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{H}^H$ is also termed the zero-forcing receiver matrix and the estimate ˆx_{ZF} is, therefore, given as

$$\hat{\mathbf{x}}_{\mathrm{ZF}} = \mathbf{F}_{\mathrm{ZF}}\mathbf{y}$$

4.3.1 Properties of the Zero-Forcing Receiver Matrix F_{ZF}

Consider the matrix product $\mathbf{F}_{ZF}\mathbf{H}$**,** which can be simplified as

$$\begin{aligned}\mathbf{F}_{\mathrm{ZF}}\mathbf{H} &= \left(\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{H}^H\mathbf{H}\\ &= \left(\mathbf{H}^H\mathbf{H}\right)^{-1}\left(\mathbf{H}^H\mathbf{H}\right) = \mathbf{I}_t \qquad \text{-------- 4.3}\end{aligned}$$

where I_t denotes the identity matrix of dimension t. Thus, multiplying the matrix F_{ZF} with the channel matrix indeed produces the identity matrix.

- The matrix F_{ZF} acts as an inverse of the channel matrix H. However, observe that for r > t, the matrix H is rectangular and strictly speaking does NOT have a matrix inverse.
- Hence, this matrix F_{ZF} is termed the **pseudo-inverse of H**.
- For instance, if F_{ZF} to be the inverse of H, it must also satisfy the property that HF_{ZF} = I. However, we have,

$$\mathbf{H}\mathbf{F}_{\mathrm{ZF}} = \mathbf{H}\left(\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{H}^H \neq \mathbf{I}_t$$

  Thus, the matrix product HF_{ZF} is not equal to the identity matrix in general. Further, the matrix inverse is a unique matrix. However, it can be shown that the left or pseudo-inverse of the matrix is H when r > t is not unique. Thus, in general, F_{ZF} is not the inverse of the channel matrix H. However, if r = t and the matrix H is invertible then the pseudo-inverse is actually equal to the inverse. This can be seen as follows.

$$\begin{aligned}\mathbf{F}_{\mathrm{ZF}} &= \mathbf{H}\left(\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{H}^H\\ &= \mathbf{H}^{-1}\left(\mathbf{H}^H\right)^{-1}\mathbf{H}^H\\ &= \mathbf{H}^{-1}\end{aligned}$$

  Thus, for this particular case, the pseudo-inverse reduces to the matrix inverse and is unique.

4.3.2 Principle of Orthogonality Interpretation of ZF Receiver

- We present an intuitive-reasoning-based approach to derive the zero-forcing decoder above. Consider once again the system of equations at the receiver for r > t.

$$\underbrace{\begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ \vdots \\ \vdots \\ y_r(k) \end{bmatrix}}_{\mathbf{y}(k)} = \underbrace{\begin{bmatrix} h_{11} & h_{12} & \dots & h_{1t} \\ h_{21} & h_{22} & \dots & h_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \underbrace{h_{r1}}_{\mathbf{h}_1} & \underbrace{h_{r2}}_{\mathbf{h}_2} & \dots & \underbrace{h_{rt}}_{\mathbf{h}_t} \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_t(k) \end{bmatrix}}_{\mathbf{x}(k)}$$

Denoting the column $h_i = [h_{1i}, h_{2i}, \dots, h_{ri}]^T$ we denote the i^{th} column of the channel matrix H. Then, the system of equations above can be succinctly represented as

$$\mathbf{y} = \underbrace{x_1\mathbf{h}_1 + x_2\mathbf{h}_2 + \dots + x_t\mathbf{h}_t}_{\hat{\mathbf{y}}}$$

where $\hat{y}$ is the approximation of y and we are interested in minimizing $\| y - \hat{y} \|^2$. Observe that there are t columns of the channel matrix H, which are $h_1, h_2, \dots, h_t$. Hence, they represent a t-dimensional subspace. However, the vector y can lie anywhere in the r-dimensional space, and is unlikely to lie exclusively in the t-dimensional subspace represented by the columns of the channel. This is shown schematically in Figure 4.3, where approximation error $e = y - \hat{y}$ the error is shown for different choices of $\hat{y}$.

- The approximation is minimum, when it is orthogonal to the space spanned by the columns of H. This is termed the principle of orthogonality, which is extremely helpful in understanding the intuition behind complex estimation problems. Observe that the error vector e is orthogonal to h_i if $h^H_i e = 0$.

Therefore, since e is orthogonal to the subspace spanned by the columns of H, it follows that it is orthogonal to each of the columns of H.

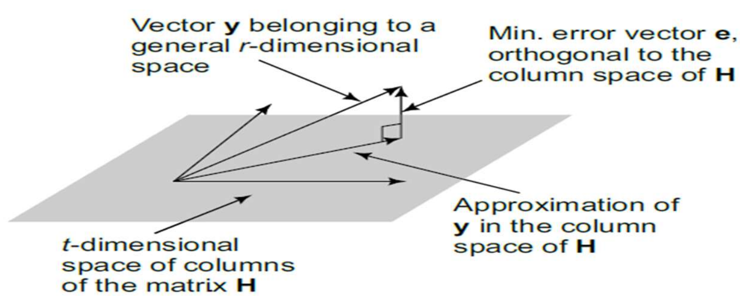

Figure 4.3 Zero-forcing: principle of orthogonality

$$\mathbf{h}_1^H \mathbf{e} = 0$$
$$\mathbf{h}_2^H \mathbf{e} = 0$$
$$\vdots$$
$$\mathbf{h}_t^H \mathbf{e} = 0,$$

$$\mathbf{H}^H\mathbf{e} = \mathbf{0}_{t\times 1}$$

We now employ the above principle of orthogonality to derive the expression for the zero forcing estimate $\hat{x}_{ZF}$ of the transmitted symbol vector x. This can be derived by substituting the expression $e = y - H\hat{x}_{ZF}$ in the above result as

$$\mathbf{H}^H \underbrace{(\mathbf{y} - \mathbf{H}\hat{\mathbf{x}}_{\mathrm{ZF}})}_{\mathbf{e}} = 0$$

$$\mathbf{H}^H\mathbf{y} - \mathbf{H}^H\mathbf{H}\hat{\mathbf{x}}_{\mathrm{ZF}} = 0$$

$$\mathbf{H}^H\mathbf{y} = \mathbf{H}^H\mathbf{H}\hat{\mathbf{x}}_{\mathrm{ZF}}$$

$$\hat{\mathbf{x}}_{\mathrm{ZF}} = \left(\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{H}^H\mathbf{y}$$

which is exactly identical to the expression for the zero-forcing MIMO decoder derived earlier in Eq. (4.2).

Example 1: Compute the MIMO zero-forcing receiver for the channel matrix H given as

$$\mathbf{H} = \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 4 & 2 \end{bmatrix} \qquad \text{----- 4.4}$$

Solution: The MIMO channel matrix above is of size 3 × 2, implying that the number of receive antennas is r = 3, while the number of transmit antennas is t = 2. Thus, the MIMO system model can be described as

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 4 & 2 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}}_{\mathbf{n}}$$

The transmit vector x is of dimension 2 × 1, while the receive and noise vectors y, n respectively are of dimension 3 × 1. The above MIMO system model can also be explicitly written to describe the signal received at each receive antenna as

$$y_1 = 2x_1 + 3x_2 + n_1$$
$$y_2 = x_1 + 3x_2 + n_2$$
$$y_3 = 4x_1 + 2x_2 + n_3$$

which basically represents a system of r = 3 equations for t = 2 unknowns x_1, x_2. To compute the zero-forcing decoder F_{ZF} , we first compute the matrix $(H^H H)^{-1} H^H$.

Observe that $H^H H$ can be simplified as

$$\mathbf{H}^H\mathbf{H} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 17 \\ 17 & 22 \end{bmatrix}$$

$$\left(\mathbf{H}^H\mathbf{H}\right)^{-1} = \frac{1}{21 \times 22 - 17 \times 17} \begin{bmatrix} 22 & -17 \\ -17 & 21 \end{bmatrix}$$

$$= \frac{1}{173} \begin{bmatrix} 22 & -17 \\ -17 & 21 \end{bmatrix}$$

Thus, the zero-forcing receiver matrix $F_{ZF} = (H^H H)^{-1} H^H$ can be expressed as

$$\mathbf{F}_{ZF} = \left(\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{H}^H$$

$$= \frac{1}{173} \begin{bmatrix} 22 & -17 \\ -17 & 21 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 3 & 3 & 2 \end{bmatrix}$$

$$= \frac{1}{173} \begin{bmatrix} -7 & -29 & 54 \\ 29 & 46 & -26 \end{bmatrix}$$

$$= \begin{bmatrix} -0.04 & -0.17 & 0.31 \\ 0.17 & 0.27 & -0.15 \end{bmatrix} \quad \text{---- 4.5}$$

Thus, the zero-forcing estimate of the transmit vector is given as

$$\hat{\mathbf{x}}_{\mathrm{ZF}} = \mathbf{F}_{ZF}\mathbf{y}$$

$$= \left(\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{H}^H\mathbf{y}$$

$$= \begin{bmatrix} -0.04 & -0.17 & 0.31 \\ 0.17 & 0.27 & -0.15 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

The estimate of each of the transmit symbols ˆx1, ˆx2 as

$$\hat{x}_1 = -0.04y_1 - 0.17y_2 + 0.31y_3$$

$$\hat{x}_2 = +0.17y_1 + 0.27y_2 - 0.15y_3$$

The zero-forcing receiver matrix matrix F_{ZF} is of dimension 2 × 3, i.e., t × r.

One of the chief **disadvantages** of the zero-forcing receivers is noise amplification. This can be understood as follows. Consider the SISO wireless system for r = t = 1. We have the system model given as

$$y = hx + n$$

Hence, the zero-forcing receiver is given as $f_{ZF} = (h*h)^{-1} h^* = h^{-1} = 1/h$. Hence, the zeroforcing estimate of the transmitted symbol x is given as

$$\hat{x}_{ZF} = f_{ZF} \times y$$

$$= \frac{1}{h}(hx + n)$$

$$= x + \frac{n}{h} \qquad \text{---- 4.6}$$

If the fading coefficient h is close to zero then the factor n/h is very high. This is termed **noise amplification**, which significantly distorts the performance of the receiver.

4.4 MIMO MMSE Receiver

- We develop the Minimum Mean-Squared Error (MMSE) receiver for the MIMO wireless communication system. The MMSE receiver is based on a Bayesian approach, meaning that the transmit vector x is assumed to be random in nature. Thus, if $\hat{x}_{MMSE}$ denotes the estimated symbol vector, the MMSE receiver minimizes the average or mean of the squared error

$$\mathrm{E}\left\{\|\hat{\mathbf{x}}_{\mathrm{MMSE}} - \mathbf{x}\|^2\right\}$$

Thus, it is aptly named **minimum mean-squared error estimator**. To illustrate the development of the MMSE receiver, we consider a single-input multiple-output (SIMO) wireless system, i.e., t = 1, and generalize the result to the case of a MIMO system. Hence, consider the SIMO system model given as

$$y = hx + n$$

where x is now a scalar transmitted symbol. Thus, the basic problem can be interpreted as estimating the symbol x given the vector $y = [y_1, y_2, \ldots, y_r]^T$. Let $c = [c_1, c_2, \ldots, c_r]^T$. One can now define a linear estimator of x as

$$\hat{x} = c^T y \qquad \text{-------------4.7}$$

$$= \begin{bmatrix} c_1 & c_2 & \dots & c_r \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix}$$

$$= c_1 y_1 + c_2 y_2 + \ldots + c_r y_r$$

Such an estimator as above is termed as a linear estimator, since the estimate is a linear function of y. We now wish to find the best or optimal linear estimator $\hat{x}_{MMSE}$, which minimizes the mean squared error and hence is also termed the **Linear Minimum Mean Squared Estimator** (LMMSE).
The average mean squared error is defined as

$$\mathrm{E}\left\{ \|\hat{\mathbf{x}} - \mathbf{x}\|^2 \right\}$$

Employing the form in Eq. (4.7), the resulting equation can be simplified as

$$\begin{aligned}
\mathrm{E}\left\{ \|\hat{\mathbf{x}} - \mathbf{x}\|^2 \right\} &= \mathrm{E}\left\{ \left(\mathbf{c}^T\mathbf{y} - \mathbf{x}\right)\left(\mathbf{c}^T\mathbf{y} - \mathbf{x}\right)^T \right\} \\
&= \mathrm{E}\left\{ \left(\mathbf{c}^T\mathbf{y} - \mathbf{x}\right)\left(\mathbf{y}^T\mathbf{c} - \mathbf{x}^T\right) \right\} \\
&= \mathrm{E}\left\{ \mathbf{c}^T\mathbf{y}\mathbf{y}^T\mathbf{c} - x\mathbf{y}^T\mathbf{c} - \mathbf{c}^T\mathbf{y}x + xx^T \right\} \\
&= \mathbf{c}^T \underbrace{\mathrm{E}\left\{\mathbf{y}\mathbf{y}^T\right\}}_{\mathbf{R}_{yy}} \mathbf{c} - \underbrace{\mathrm{E}\left\{x\mathbf{y}^T\right\}}_{\mathbf{R}_{xy}} \mathbf{c} - \mathbf{c}^T \underbrace{\mathrm{E}\left\{\mathbf{y}x\right\}}_{\mathbf{R}_{yx}} + \mathrm{E}\left\{x^2\right\}_{\mathbf{R}_{xx}} \\
&= \mathbf{c}^T\mathbf{R}_{yy}\mathbf{c} - 2\mathbf{c}^T\mathbf{R}_{yx} + \mathbf{R}_{xx}
\end{aligned}$$

where the covariance matrix R_{yy} is defined as $R_{yy} = E\{ yy^H \}$. . Similarly, $R_{yx} = E\{yx\} = R^T_{xy}$ and $R_{xx} = E\{ x^2 \}$. Also note that we have used the fact $c^T R_{yx} = (c^T R_{yx})^T = R_{xy}c$. Hence, the average MSE as a function of the receive beamformer c, denoted by MSE (c), is given as

$$\overline{\mathrm{MSE}}\left(\mathbf{c}\right) = \mathbf{c}^T\mathbf{R}_{yy}\mathbf{c} - 2\mathbf{c}^T\mathbf{R}_{yx} + \mathbf{R}_{xx}$$

Thus, the optimal beamformer c which minimizes the average or mean squared error can be obtained by differentiating MSE (c) with respect to c and setting equal to zero as

$$\frac{\partial \overline{\mathrm{MSE}}\left(\mathbf{c}\right)}{\partial \mathbf{c}} = 0$$

$$\frac{\partial}{\partial \mathbf{c}}\left(\mathbf{c}^T\mathbf{R}_{yy}\mathbf{c} - 2\mathbf{c}^T\mathbf{R}_{yx} + \mathbf{R}_{xx}\right) = 0$$

$$2\mathbf{R}_{yy}\mathbf{c} - 2\mathbf{R}_{yx} = 0$$

$$\mathbf{c} = \mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}$$

Thus, the optimal LMMSE beamforming vector c is given as $c = R^{-1}_{yy} R_{yx}$. This is also termed in signal processing as the optimal Wiener filter. The above can be generalized in the case of complex vectors by replacing the transpose by the Hermitian operation.

Hence, the MMSE estimate of x is given as

$$\begin{aligned}\hat{\mathbf{x}}_{\text{MMSE}} &= \mathbf{c}^H \mathbf{y} \\ &= \left(\mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}\right)^H \mathbf{y} \\ &= \mathbf{R}_{xy}\mathbf{R}_{yy}^{-1}\mathbf{y}\end{aligned}$$

We now compute the MMSE receiver for the MIMO wireless system. Consider again the MIMO system model given as

$$y = Hx + n$$

Let the transmit symbols x_i, $1 \leq i \leq t$ be such that each is of power Pd, i.e., $E\{|x_i|^2\} = P_d$, with elements on different transmit antennas being uncorrelated, i.e., $E\{x_i x^*_j\} = 0$ when $i \neq j$. Hence, the covariance R_{xx} of the transmit symbols is given as

$$\begin{aligned}\mathbf{R}_{xx} &= \mathrm{E}\left\{\mathbf{x}\mathbf{x}^H\right\} \\ &= \mathrm{E}\left\{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix}\begin{bmatrix} x_1^* & x_2^* & \dots & x_t^* \end{bmatrix}\right\} \\ &= \begin{bmatrix} \mathrm{E}\left\{|x_1|^2\right\} & \mathrm{E}\left\{x_1 x_2^*\right\} & \dots & \mathrm{E}\left\{x_1 x_t^*\right\} \\ \mathrm{E}\left\{x_2 x_1^*\right\} & \mathrm{E}\left\{|x_2|^2\right\} & \dots & \mathrm{E}\left\{x_2 x_t^*\right\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{E}\left\{x_t x_1^*\right\} & \mathrm{E}\left\{x_t x_2^*\right\} & \dots & \mathrm{E}\left\{|x_t|^2\right\} \end{bmatrix} \\ &= \begin{bmatrix} P_d & 0 & 0 & \dots & 0 \\ 0 & P_d & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & P_d \end{bmatrix} \\ &= P_d \mathbf{I}_t\end{aligned}$$

Hence, R_{yy} , the covariance of the receive vector y can be simplified as

$$\begin{aligned}
\mathbf{R}_{yy} &= \mathrm{E}\left\{\mathbf{y}\mathbf{y}^H\right\} \\
&= \mathrm{E}\left\{(\mathbf{Hx}+\mathbf{n})(\mathbf{Hx}+\mathbf{n})^H\right\} \\
&= \mathrm{E}\left\{\mathbf{Hxx}^H\mathbf{H} + \mathbf{nx}^H\mathbf{H}^H + \mathbf{Hxn}^H + \mathbf{nn}^H\right\} \\
&= \mathbf{H}\underbrace{\mathrm{E}\left\{\mathbf{xx}^H\right\}}_{\mathbf{R}_{xx}}\mathbf{H} + \underbrace{\mathrm{E}\left\{\mathbf{nx}^H\right\}}_{0}\mathbf{H}^H + \mathbf{H}\underbrace{\mathrm{E}\left\{\mathbf{xn}^H\right\}}_{0} + \underbrace{\mathrm{E}\left\{\mathbf{nn}^H\right\}}_{\mathbf{R}_{nn}} \\
&= \underbrace{P_d\mathbf{HH}^H + \sigma_n^2\mathbf{I}_r}_{\mathbf{R}_{yy}}
\end{aligned}$$

where we have used the fact that E { nx ^{H} } = E {xn ^{H} } = 0, since the noise at the receiver and transmit symbols are uncorrelated, in the above simplification. Further, the cross-covariance matrix R_{yx} can be simplified as

$$\begin{aligned}
\mathbf{R}_{yx} &= \mathrm{E}\left\{\mathbf{yx}^H\right\} \\
&= \mathrm{E}\left\{(\mathbf{Hx}+\mathbf{n})\mathbf{x}^H\right\} \\
&= \mathrm{E}\left\{\mathbf{Hxx}^H + \mathbf{nx}^H\right\} \\
&= \mathbf{H}\underbrace{\mathrm{E}\left\{\mathbf{xx}^H\right\}}_{\mathbf{R}_{xx}} + \underbrace{\mathrm{E}\left\{\mathbf{nx}^H\right\}}_{0} \\
&= P_d\mathbf{H}
\end{aligned}$$

Thus, the optimal MMSE receiver is given as

$$\begin{aligned}
\mathbf{C} &= \mathbf{R}_{yy}^{-1}\mathbf{R}_{yx} \\
&= \left(P_d\mathbf{HH}^H + \sigma_n^2\mathbf{I}\right)^{-1}P_d\mathbf{H} \\
&= P_d\left(P_d\mathbf{HH}^H + \sigma_n^2\mathbf{I}\right)^{-1}\mathbf{H}
\end{aligned}$$

Hence, the MMSE estimate ˆx $_{MMSE}$ of the transmit vector x is given as

$$\begin{aligned}
\hat{\mathbf{x}}_{\text{MMSE}} &= \mathbf{C}^H\mathbf{y} \\
&= P_d\mathbf{H}^H\left(P_d\mathbf{HH}^H + \sigma_n^2\mathbf{I}\right)^{-1}\mathbf{y} \quad \text{----- 4.8}
\end{aligned}$$

We now derive an alternative structure for the MIMO MMSE receiver. Observe that, we have,

$$P_d\mathbf{H}^H\mathbf{HH}^H + \sigma_n^2\mathbf{H}^H = P_d\mathbf{H}^H\mathbf{HH}^H + \sigma_n^2\mathbf{H}^H$$

$$\Rightarrow \left(P_d\mathbf{H}^H\mathbf{H} + \sigma_n^2\mathbf{I}\right)\mathbf{H}^H = \mathbf{H}^H\left(P_d\mathbf{HH}^H + \sigma_n^2\mathbf{I}\right)$$

$$\Rightarrow \mathbf{H}^H \left(P_d \mathbf{H}\mathbf{H}^H + \sigma_n^2 \mathbf{I}\right)^{-1} = \left(P_d \mathbf{H}^H \mathbf{H} + \sigma_n^2 \mathbf{I}\right)^{-1} \mathbf{H}^H$$

Thus, the MIMO MMSE receiver in Eq. (4.8) can also be expressed as

$$\hat{\mathbf{x}}_{\text{MMSE}} = P_d \mathbf{H}^H \left(P_d \mathbf{H}\mathbf{H}^H + \sigma_n^2 \mathbf{I}\right)^{-1} \mathbf{y}$$

$$= P_d \left(P_d \mathbf{H}^H \mathbf{H} + \sigma_n^2 \mathbf{I}\right)^{-1} \mathbf{H}^H \mathbf{y} \qquad \text{------ 4.9}$$

Thus, the above expression is an alternative form of implementation of the **MIMO MMSE receiver**. Observe that the matrix $HH^H + \sigma^2_n I$ is of dimension r × r, while $P_d H^H H + \sigma^2_n I$ is t × t dimensional. Thus, if r > t, inversion of the latter matrix is of a lower computation complexity. Since this is frequently the case in MIMO wireless systems, the alternative MIMO MMSE receiver version is more popular for implementation.

4.4.1 Robustness of MMSE to Noise Amplification

The MMSE receiver does not lead to noise amplification, as is the case with the ZF receiver, which was seen in Eq. (4.6). Consider the SISO case for which H = h. The system model is given as

$$y = hx + n$$

The MMSE estimate of x is given from Eq. (4.8) as

$$\hat{\mathbf{x}}_{\text{MMSE}} = P_d \frac{h^*}{P_d h h^* + \sigma_n^2} y$$

$$= P_d \frac{h^*}{P_d |h|^2 + \sigma_n^2} y$$

Thus, it can be seen that for $|h| \approx 0$, the MMSE receiver becomes

$$\hat{\mathbf{x}}_{\text{MMSE}} \approx P_d \frac{h^*}{\sigma_n^2} \mathbf{y}$$

Thus, since it does not lead to division by a quantity close to 0, unlike the ZF, it does not lead to noise enhancement.

4.4.2 Low and High SNR Properties of the MMSE Receiver

We explore the low and high SNR behaviour of the MMSE receiver, which provides valuable insights into its nature. Consider the expression for the MMSE receiver in Eq. (4.9). Observe that at high SNR, i.e., $P_d / \sigma^2_n \rightarrow \infty$, the term $P_d H^H H$ dominates over $\sigma^2_n I_t$. Hence, we have

$$P_d \mathbf{H}^H \mathbf{H} + \sigma_n^2 \mathbf{I} \approx P_d \mathbf{H}^H \mathbf{H}$$

Employing this approximation in the MMSE receiver in Eq. (4.9), we have

$$\hat{\mathbf{x}}_{\text{MMSE}} = P_d \left(P_d \mathbf{H}^H \mathbf{H} + \sigma_n^2 \mathbf{I}\right)^{-1} \mathbf{H}^H \mathbf{y}$$

$$\approx P_d \left(P_d \mathbf{H}^H \mathbf{H}\right)^{-1} \mathbf{H}^H \mathbf{y}$$

$$= \left(\mathbf{H}^H \mathbf{H}\right)^{-1} \mathbf{H}^H \mathbf{y}$$

which is identical to the MIMO zero-forcing receiver from Eq. (4.2). Thus, at high SNR, the MMSE receiver reduces to the MIMO zero-forcing receiver. On the other hand, at low SNR, i.e., as $P_d / \sigma^2_n \rightarrow 0$, the term $P_d H^H H$ negligible compared to $\sigma^2_n I_t$. Thus, we have

$$P_d \mathbf{H}^H \mathbf{H} + \sigma_n^2 \mathbf{I}_t \approx \sigma_n^2 \mathbf{I}_t$$

Employing the above approximation at low SNR, the MMSE receiver can be simplified as

$$\hat{\mathbf{x}}_{\text{MMSE}} = P_d \left(P_d \mathbf{H}^H \mathbf{H} + \sigma_n^2 \mathbf{I}\right)^{-1} \mathbf{H}^H \mathbf{y}$$

$$\approx P_d \left(\sigma_n^2 \mathbf{I}\right)^{-1} \mathbf{H}^H \mathbf{y}$$

$$= \frac{P_d}{\sigma_n^2} \mathbf{H}^H \mathbf{y}$$

which reduces to the matched filter, i.e., proportional to H^H . Thus, the optimal MMSE receiver can be approximated as the zero-forcing receiver at high SNR, while at low SNR, it behaves similar to the matched filter. This behaviour of the MMSE decoder is schematically represented in Figure 4.4.

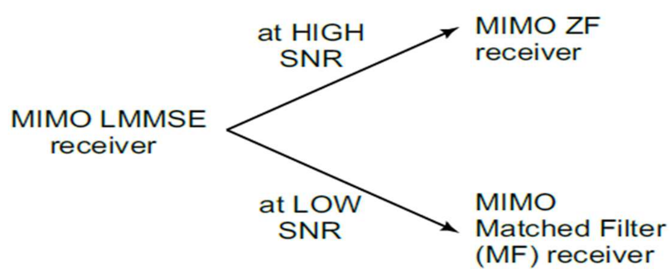

Figure 4.4 MIMO MMSE receiver: asymptotic behaviour

4.5 Singular Value Decomposition (SVD) of the MIMO Channel

Consider an r × t MIMO channel H with r ≥ t, i.e., number of receive antennas greater than or equal to the number of transmit antennas. The SVD of the channel matrix H is given as

$$\mathbf{H} = \underbrace{\left[\begin{array}{cccc} u_{11} & u_{12} & \cdots & u_{1t} \\ u_{21} & u_{22} & \cdots & u_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ \underbrace{u_{r1}}_{\mathbf{u}_1} & \underbrace{u_{r2}}_{\mathbf{u}_2} & \cdots & \underbrace{u_{rt}}_{\mathbf{u}_t} \end{array}\right]}_{\mathbf{U}} \underbrace{\left[\begin{array}{cccc} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_t \end{array}\right]}_{\boldsymbol{\Sigma}}$$

$$\underbrace{\left[\begin{array}{cccc} v_{11}^* & v_{21}^* & \cdots & v_{t1}^* \\ v_{12}^* & v_{22}^* & \cdots & v_{t2}^* \\ \vdots & \vdots & \ddots & \vdots \\ v_{1t}^* & v_{2t}^* & \cdots & v_{tt}^* \end{array}\begin{array}{l} \}\,\mathbf{v}_1^H \\ \}\,\mathbf{v}_2^H \\ \\ \}\,\mathbf{v}_t^H \end{array}\right]}_{\mathbf{V}^H},$$

$= U\Sigma V^H$ --------------- 4.10

where the matrices U, Σ, V, which are r × t, t × t and t × t dimensional respectively, satisfy important properties, which we examine next. The columns of the matrix U and V are unit norm, i.e., we have

$$\|\mathbf{u}_i\|^2 = \|\mathbf{v}_i\|^2 = 1,\ 1 \leq i \leq t$$

Further, the columns of matrix U and V are orthogonal, i.e.,

$$\mathbf{u}_i^H \mathbf{u}_j = \mathbf{v}_i^H \mathbf{v}_j = 0,\ i \neq j,\ 1 \leq i, j \leq t$$

Thus, the columns of matrices U and V are orthonormal. As a result, the t × t dimensional square matrix V is unitary, i.e.,

$$\mathbf{V}^H \mathbf{V} = \mathbf{V}\mathbf{V}^H = \mathbf{I}_t$$

Further, since if r = t, the matrix U is also a unitary matrix. Otherwise, U simply satisfies the relation $U^H U = It$. Further, the quantities $\sigma_1, \sigma_2, \ldots, \sigma_t$ are known as the singular values of the matrix Σ. These singular values are non-negative and ordered, i.e., each $\sigma_i \geq 0$ and

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_t$$

Finally, an important property of the singular values is that the number of nonzero singular values is equal to the rank of the matrix H.

4.5.1 Examples of Singular Value Decomposition

Example 2: Consider a 2 × 1 SIMO wireless system with channel matrix H given as

$$\mathbf{H} = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

------------ 6.11

$$\mathbf{H} = \left[\begin{array}{c} 1 \\ 1 \end{array}\right], = \left[\begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array}\right]\sqrt{2}$$

$$= \underbrace{\left[\begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array}\right]}_{\mathbf{U}} \underbrace{\left[\begin{array}{c} \sqrt{2} \end{array}\right]}_{\mathbf{\Sigma}} \underbrace{\left[\begin{array}{c} 1 \end{array}\right]}_{\mathbf{V}^H}$$

This is a simple example of the SVD and yet illustrates several important properties.

- For instance, $U = [\,1/\sqrt{2}\,,\ 1/\sqrt{2}\,]^T$, which is of dimension $r \times t$, i.e., 2×1.
- Further, U only has a single column $u1 = [\,1/\sqrt{2}\,,\ 1/\sqrt{2}]^T$, which is unit-norm, i.e., $\| u_1 \|^2 = 1$.
- Further, the singular value $\sigma_1 = \sqrt{2}$, which is greater than 0.
- Also, the number of nonzero singular values is 1, which is equal to the rank of the matrix.
- The rank of the matrix is easily seen to be equal to 1 in this case since it is simply a column vector. Also observe that $V = [1]$. Hence, we trivially have

$$\mathbf{V}^H\mathbf{V} = \mathbf{V}\mathbf{V}^H = 1 = \mathbf{I}_1$$

Thus, the above decomposition satisfies all the properties of the SVD.

Example 3: Let the 2 × 2 channel matrix H be given as

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

$$\mathbf{H} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{bmatrix}}_{\mathbf{\Sigma}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{V}^H}$$

and claim that $\sigma_1 = 1$, $\sigma_2 = \sqrt{5}$ are the singular values of H.
However, this is incorrect since $\sigma_1 < \sigma_2$, meaning that the above is not a valid SVD. However, one can recast H as follows.

$$\mathbf{H} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} \sqrt{5} & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{\Sigma}} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{V}^H}$$

where the matrices U, V are 2 × 2 permutation matrices which flip the rows and columns of the inner diagonal matrix respectively. It can also be observed that $VV^H = V^HV = I_t$. Also, in this case, since $r = t$, we also have $UU^H = U^HU = I_t$. Further, $\sigma_1 = \sqrt{5} > \sigma_2 = 1 > 0$. Thus, the singular values are positive and ordered. Hence, this is a valid SVD of the channel matrix H.

Example 4: We now look at another example of a channel matrix H and compute its SVD. Let H be given as

$$\mathbf{H} = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$$

The columns of H, i.e., $h_1 = [1, 1]^T$ and $h_2 = [2, -2]^T$ are orthogonal since $h^H{}_1 h_2 = 1 \times 2 + 1 \times (-2) = 0$. This fact can be employed to compute the SVD. Thus, the matrix H can be decomposed in steps as

$$\mathbf{H} = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The above decomposition has a structure that looks very close to the SVD. It now remains to normalize the columns of the left and right matrices to generate orthonormal vectors. This can be done as follows.

$$\mathbf{H} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}}_{\mathbf{\Sigma}} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{V}^H}$$

It can be seen that the singular values are $\sigma_1 = 2\sqrt{2} > \sigma_2 = \sqrt{2} > 0$. Further, the number of nonzero singular values and, hence, the rank is 2. Since r = t = 2, the matrices U, V are both square 2 × 2, unitary and contain orthonormal columns.

4.6 Singular Value Decomposition and MIMO Capacity

The singular value decomposition is central to understanding the spatial multiplexing properties of the MIMO channel and deriving the fundamental limit on the capacity of the MIMO channel. Consider the r × t MIMO wireless system,

$$\mathbf{y = Hx + n}$$

Let the SVD of the channel matrix H be given as $H = U\Sigma V^H$. Thus, replacing H with its SVD, the above MIMO system model is given as

$$\mathbf{y} = \underbrace{\mathbf{U\Sigma V}^H}_{\mathbf{H}}\mathbf{x} + \mathbf{n}$$

At the receiver, multiplying by U^H , we have

$$\underbrace{\mathbf{U}^H\mathbf{y}}_{\tilde{\mathbf{y}}} = \mathbf{U}^H\left(\mathbf{U\Sigma V}^H\mathbf{x} + \mathbf{n}\right)$$

$$\tilde{\mathbf{y}} = \underbrace{\mathbf{U}^H\mathbf{U}}_{\mathbf{I}_t}\mathbf{\Sigma V}^H\mathbf{x} + \underbrace{\mathbf{U}^H\mathbf{n}}_{\tilde{\mathbf{n}}}$$

$$= \mathbf{\Sigma V}^H\mathbf{x} + \tilde{\mathbf{n}}$$

This operation of multiplying by U^H at the receiver is a part of the signal processing at the receiver or receive processing. Further, prior to transmission, let the transmit vector x be generated as x = V˜x, where the vector ˜x contains the transmit symbols. This operation is termed as transmit precoding. Thus, substituting this expression for x above, we have

$$\begin{aligned}\tilde{\mathbf{y}} &= \mathbf{\Sigma}\mathbf{V}^H\mathbf{x} + \tilde{\mathbf{n}}\\ &= \mathbf{\Sigma}\mathbf{V}^H\mathbf{V}\tilde{\mathbf{x}} + \tilde{\mathbf{n}}\\ &= \mathbf{\Sigma}\tilde{\mathbf{x}} + \tilde{\mathbf{n}}\end{aligned}$$

The above equivalent system model of the MIMO system after the receive and transmit processing operations can be explicity written as

$$\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\\ \vdots\\ \tilde{y}_t\end{bmatrix} = \begin{bmatrix}\sigma_1 & 0 & \dots & 0\\ 0 & \sigma_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \sigma_t\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\\ \vdots\\ \tilde{x}_t\end{bmatrix} + \begin{bmatrix}\tilde{n}_1\\ \tilde{n}_2\\ \vdots\\ \tilde{n}_t\end{bmatrix}$$

The above system can, in fact, be written in a much simpler decoupled form as

$$\begin{aligned}\tilde{y}_1 &= \sigma_1\tilde{x}_1 + \tilde{n}_1\\ \tilde{y}_2 &= \sigma_2\tilde{x}_2 + \tilde{n}_2\\ &\vdots\\ \tilde{y}_t &= \sigma_t\tilde{x}_t + \tilde{n}_t\end{aligned} \quad \text{------- 4.12}$$

Thus, the above equivalent system represents the parallelization of the MIMO channel with t information streams being transmitted in parallel. There is no interference between the t information streams carrying symbols $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_t$. This is termed **spatial multiplexing**, where the t independent information streams are being multiplexed over the multiple spatial dimensions, arising due to the presence of multiple transmit and receive antennas in the system. Also, the covariance of the modified noise ˜n can be derived as

$$\begin{aligned}\mathbf{R}_{\tilde{\mathbf{n}}} &= \mathrm{E}\left\{\tilde{\mathbf{n}}\tilde{\mathbf{n}}^H\right\}\\ &= \mathrm{E}\left\{\mathbf{U}^H\mathbf{n}\left(\mathbf{U}^H\mathbf{n}\right)^H\right\}\\ &= \mathrm{E}\left\{\mathbf{U}^H\mathbf{n}\mathbf{n}^H\mathbf{U}\right\}\\ &= \mathbf{U}^H\underbrace{\mathrm{E}\left\{\mathbf{n}\mathbf{n}^H\right\}}_{\sigma_n^2\mathbf{I}}\mathbf{U}\\ &= \mathbf{U}^H\sigma_n^2\mathbf{I}\mathbf{U} = \sigma_n^2\underbrace{\mathbf{U}^H\mathbf{U}}_{\mathbf{I}_t}\\ &= \sigma_n^2\mathbf{I}_t\end{aligned} \quad \text{---------- 4.13}$$

Thus, once again, the noise $\tilde{n}$ has a covariance proportional to the identity matrix, indicating equal variance, uncorrelated noise components. Further, the variance of each noise component $\tilde{n}_i$, $1 \leq i \leq t$ is equal to σ^2_n . Consider now the i^{th} parallel MIMO channel above. This is given as

$$\tilde{y}_i = \sigma_i \tilde{x}_i + \tilde{n}_i.$$

Hence, the SNR of the system is given as $\sigma_i^2 \frac{E\{|x_i|^2\}}{\sigma_n^2} = \sigma_i^2 \frac{P_i}{\sigma_n^2}$, where Pi is the power of the i^{th} data stream x_i. Thus, the MIMO system can be viewed as a collection of t parallel channels each with noise power σ^2_n and power gain σ^2_i . This is schematically shown in Figure 4.5. From the above expression for the SNR of the i^{th} channel, the Shannon capacity Ci of the channel

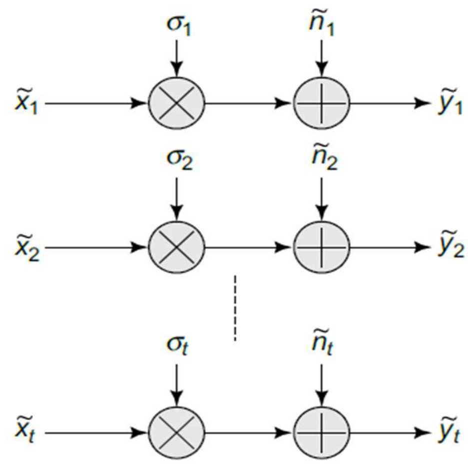

Figure 4.5 MIMO SVD parallel channels

$$C_i = \log_2 \left(1 + \frac{P_i \sigma_i^2}{\sigma_n^2}\right)$$

Thus, the MIMO system can be thought of as a collection of t parallel data pipes, with Capacities

$$C_1 = \log_2 \left(1 + \frac{P_1 \sigma_1^2}{\sigma_n^2}\right)$$

$$C_2 = \log_2 \left(1 + \frac{P_2 \sigma_2^2}{\sigma_n^2}\right)$$

$$\vdots$$

$$C_t = \log_2 \left(1 + \frac{P_t \sigma_t^2}{\sigma_n^2}\right)$$

Thus, the net MIMO C capacity is given as the sum of the individual capacities

$$C = \sum_{i=1}^{t} \log_2 \left(1 + \frac{P_i \sigma_i^2}{\sigma_i^2}\right)$$

The total power P at the transmitter can be allocated to the individual streams to maximize the net capacity. Thus, one can maximize the above sum capacity subject to the power constraint

$$P_1 + P_2 + \ldots + P_t \leq P$$

4.6.1 Optimal MIMO Capacity

Thus, the optimal MIMO power allocation problem can be formulated as

$$\max. \quad \sum_{i=1}^{t} \log_2\left(1 + \frac{P_i \sigma_i^2}{\sigma_i^2}\right)$$

$$\text{s.t.} \quad \sum_{i=1}^{t} P_i \leq P$$

where s.t. in the above problem statement stands for subject to and denotes the optimization constraint. We employ the standard Lagrange-multiplier-based technique for the above constrained optimization problem. Denoting the Lagrange multiplier by λ, the Lagrangian cost function f (P , λ) for the above optimization problem can be formulated as

$$f\left(\overline{P}, \lambda\right) = \sum_{i=1}^{t} \log_2\left(1 + \frac{P_i \sigma_i^2}{\sigma_i^2}\right) + \lambda\left(P - \sum_{i=1}^{t} P_i\right)$$

Where $\overline{P} = [P_1,\ P_2,\ \ldots,\ P_t]^T$. Differentiating f (P , λ) with respect to Pi and setting equal to 0, we obtain

$$\frac{\partial}{\partial P_i} f\left(\overline{P}, \lambda\right) = 0$$

$$\frac{\frac{\sigma_i^2}{\sigma_n^2}}{1 + \frac{P_i \sigma_i^2}{\sigma_n^2}} - \lambda = 0$$

Solving the above equation yields

$$\frac{\sigma_i^2}{\sigma_n^2}\frac{1}{\lambda} = 1 + \frac{P_i \sigma_i^2}{\sigma_n^2}$$

$$P_i = \left(\frac{1}{\lambda} - \frac{\sigma_n^2}{\sigma_i^2}\right)^+$$

where the function $x^+ = x$ if $x \geq 0$ and 0 otherwise. This is because of the fact that each power $P_i \geq 0$, i.e., power cannot be negative. It now remains to find the Lagrange multiplier λ, which can be found from the constraint equation as

$$\sum_{i=1}^{t} P_i = P$$

$$\sum_{i=1}^{t}\left(\frac{1}{\lambda}-\frac{\sigma_n^2}{\sigma_i^2}\right)^+ = P$$

The above optimal power allocation is also termed water filling. This can be seen as follows. Consider a vessel with t bars and the height of the i^{th} bar equal to σ^2_n / σ^2_i . If water is poured into this vessel to the level $1/\lambda$ then the level of water at the i^{th} bar is $\left(\frac{1}{\lambda}-\frac{\sigma_n^2}{\sigma_i^2}\right)^+$. This is shown schematically in Figure 4.6. Observe that the power allocated is proportional to the singular value, i.e., larger the σ_i, larger is the power allocated. Further, also observe that due to the nature of water filling, weak channels with low σ_i are not allocated any power.

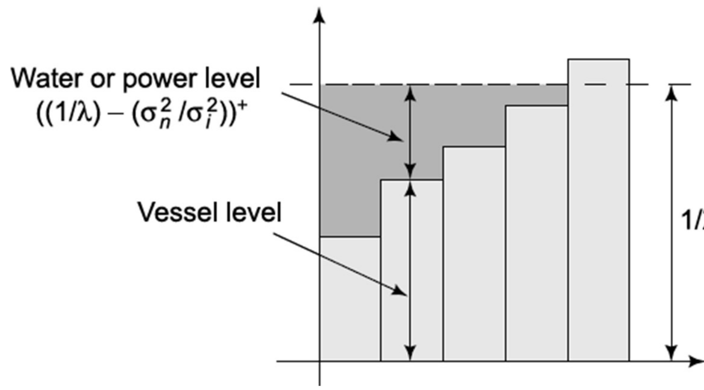

Figure 4.6 MIMO water-filling capacity

Observe that the water-filling equation in Eq. (4.14) above is nonlinear due to the x^+ function. It can be solved iteratively as follows. Set N = t initially. Assume $\frac{1}{\lambda}-\frac{\sigma_n^2}{\sigma_i^2}$ for $1 \le i \le N$. Solve the equation

$$\sum_{i=1}^{N}\left(\frac{1}{\lambda}-\frac{\sigma_n^2}{\sigma_i^2}\right) = P$$

Now, check if $\frac{1}{\lambda}-\frac{\sigma_n^2}{\sigma_N^2} \geq 0.$ If this is the case then the λ computed yields the desired power allocation. However, if $\frac{1}{\lambda}-\frac{\sigma_n^2}{\sigma_N^2}$ **< 0,** then set $P_N = 0$ and $N = t - 1$ and repeat the process as above. Example clarifies this concept of MIMO capacity and optimal power allocation.

Example 5: Consider the MIMO channel matrix H given as

$$\mathbf{H} = \begin{bmatrix} 2 & -6 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Considering a transmit power of $P = -1.25$ dB and noise power $\sigma^2_n = 3$ dB, compute the MIMO capacity and optimal power allocation**.**

Solution: The above MIMO channel matrix is 3 × 3, i.e., a MIMO system with r = t = 3 antennas. We also have, $\sigma^2_n = 3$ dB = 2 and P = −1.25 dB = 0.75. Observe that the columns of the channel matrix H are orthogonal. For instance, if c1, c2 denote the first two columns of H, we have

$$\mathbf{c}_1^H \mathbf{c}_2 = 2 \times (-6) + 3 \times (4)$$

$$= 0$$

Hence, the SVD of the channel matrix H is given as

$$\mathbf{H} = \begin{bmatrix} 2 & -6 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{13}} & -\frac{6}{\sqrt{52}} & 0 \\ \frac{3}{\sqrt{13}} & \frac{4}{\sqrt{52}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -\frac{6}{\sqrt{52}} & \frac{2}{\sqrt{13}} & 0 \\ \frac{4}{\sqrt{52}} & \frac{3}{\sqrt{13}} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} \sqrt{52} & 0 & 0 \\ 0 & \sqrt{13} & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{\mathbf{\Sigma}} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{V}^H}$$

Thus, the singular values σ1, σ2, σ3 for the above channel matrix are given as

$$\sigma_1 = \sqrt{52} \Rightarrow \sigma_1^2 = 52$$

$$\sigma_2 = \sqrt{13} \Rightarrow \sigma_2^2 = 13$$

$$\sigma_3 = 2 \Rightarrow \sigma_3^2 = 4$$

It can be seen that the channel matrix H has 3 nonzero singular values. Hence, the rank of the channel matrix is 3. The channel capacity C in terms of the powers P_1, P_2, P_3 assigned to the different channel singular modes is given as

$$C = \log_2\left(1 + \frac{P_1\sigma_1^2}{\sigma_n^2}\right) + \log_2\left(1 + \frac{P_2\sigma_2^2}{\sigma_n^2}\right) + \log_2\left(1 + \frac{P_3\sigma_3^2}{\sigma_n^2}\right)$$

$$= \log_2\left(1 + \frac{P_1 \times 52}{2}\right) + \log_2\left(1 + \frac{P_2 \times 13}{2}\right) + \log_2\left(1 + \frac{P_3 \times 4}{2}\right)$$

The above expression for capacity has to be maximized for total power $P_1 + P_2 + P_3 = 0.75$. We set N = t = 3 and solve the Lagrangian in Eq. (4.14) as

$$\left(\frac{1}{\lambda} - \frac{1}{26}\right) + \left(\frac{1}{\lambda} - \frac{2}{13}\right) + \left(\frac{1}{\lambda} - \frac{1}{2}\right) = 0.75$$

$$\frac{1}{\lambda} = \frac{0.75 + \frac{1}{26} + \frac{2}{13} + \frac{1}{2}}{3}$$

$$\frac{1}{\lambda} = 0.48$$

Let us now compute the power allocation to the channel 3, i.e., P_3, which is given as, $P_3 = 1/\lambda - 1/2 = 0.48 - 0.5 = -0.02 \leq 0$. Thus, since the power P_3 is coming out to be negative, this is not a possible allocation. This implies that the power bar corresponding to the channel 3 lies above the water level $1/\lambda$, as per the schematic shown in Figure 4.6. Thus, the singular mode 3 is allotted 0 power in the optimal allocation. Hence, we now set N = 2 and resolve the equation for the Lagrangian variable λ as

$$\left(\frac{1}{\lambda} - \frac{1}{26}\right) + \left(\frac{1}{\lambda} - \frac{2}{13}\right) = 0.75$$

$$\frac{1}{\lambda} = \frac{0.75 + \frac{1}{26} + \frac{2}{13}}{2} = 0.4712$$

Recomputing the powers P1, P2, we have

$$P_1 = 0.4712 - \frac{1}{26} = 0.4327 > 0$$

$$P_2 = 0.4712 - \frac{2}{13} = 0.3174 > 0$$

Thus, since $P_1, P_2 > 0$, this is a feasible power allocation. Hence, we have, $P_1 = 0.4327 = -3.63$ dB and $P_2 = 0.3174 = -4.98$ dB. As already described, $P_3 = 0$. Therefore, the capacity C is given as

$$C_{\max} = \log_2\left(1 + \frac{52 \times 0.4327}{2}\right) + \log_2\left(1 + \frac{13 \times 0.3174}{2}\right) = 5.23 \text{ b/s/Hz}$$

where the units b/s/Hz is read as bits per second per hertz. The optimal MIMO transmission scheme can now be derived as follows. Observe from the SVD of $H = U\Sigma V^H$ computed above that the matrix V is given as

$$\underbrace{\begin{bmatrix} \underbrace{\begin{matrix}0\\1\\0\end{matrix}}_{\mathbf{v}_1} & \underbrace{\begin{matrix}1\\0\\0\end{matrix}}_{\mathbf{v}_2} & \underbrace{\begin{matrix}0\\0\\1\end{matrix}}_{\mathbf{v}_3} \end{bmatrix}}_{\mathbf{V}}$$

The transmit vector x is given as

$$\mathbf{x} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}$$

$$= \quad \mathbf{v}_1\tilde{x}_1 + \mathbf{v}_2\tilde{x}_2$$

where ˜x3 = 0 since $P_3 = 0$. Also, we have $\sqrt{P_1} = 0.66$ and $\sqrt{P_2} = 0.56$. Therefore, the net transmit vector x is given as

$$\mathbf{x} \quad = \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \underbrace{\tilde{x}_1}_{\sqrt{P_1}\, b_1} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \underbrace{\tilde{x}_2}_{\sqrt{P_2}\, b_2}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} 0.66\, b_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} 0.56\, b_2$$

where b_1, b_2 are unit power-transmit symbols belonging to an appropriate transmit constellation such as BPSK, QPSK, etc. Thus, one can compute the optimal power allocation corresponding to the capacity of the MIMO channel to derive the optimal precoded transmit vectors for the MIMO system.

4.7 Alamouti and Space-Time Codes

- We describe a power set of codes which are termed space-time codes for error protection coding in MIMO communication systems.
- In a traditional error-control coding framework, the block code is applied only over the time dimension or basically over a block of concatenated symbols.
- However, due to the nature of the MIMO system, one can exploit the spatial dimension as well. That is to say, the one can additionally encode the symbols over the spatial dimension or across the multiple antennas, in addition to coding over the time dimension. This gives rise to the paradigm of space-time encoding, which leads to a significantly superior performance in MIMO systems and multiple-antenna systems in general.
- We begin with a basic introduction to the Alamouti code which is described for a 1 × 2 system, i.e., for a system with r = 1 receive antenna and t = 2 transmit antennas. This is an example of a MISO system, which is a special case of the MIMO system.
- Let the 1 × 2 channel matrix be denoted by [h1, h2], where h1, h2 denote the channel coefficients between transmit antennas 1, 2 and the single receive antenna respectively. Hence, the system model can be represented as

$$y = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n \qquad \text{-----4.15}$$

  where x_1, x_2 are the symbols transmitted from the two transmit antennas and n denotes the additive white Gaussian noise at the receiver.

- Consider now a symbol x which is transmitted as follows. Let x_1 be generated as $\frac{h_1^*}{\|\mathbf{h}\|}x$ and similarly, x_2 be generated as $x_2 = \frac{h_2^*}{\|\mathbf{h}\|}x.$. Here,‖ h ‖ denotes the norm of vector h and is defined as

$\|\mathbf{h}\| = \sqrt{|h_1|^2 + |h_2|^2}$. Thus, the transmit vector is now given as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{h_1^*}{\|\mathbf{h}\|} \\ \frac{h_2^*}{\|\mathbf{h}\|} \end{bmatrix} x$$

This is termed transmit beamforming, i.e., transmitting the symbol x in the direction given by the vector

$$\begin{bmatrix} \frac{h_1^*}{\|\mathbf{h}\|} \\ \frac{h_2^*}{\|\mathbf{h}\|} \end{bmatrix}$$

Substituting the above expression for the beamformed symbol in Eq. (4.15), the output y is given as

$$\begin{aligned} y &= \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} \frac{h_1^*}{\|\mathbf{h}\|} \\ \frac{h_2^*}{\|\mathbf{h}\|} \end{bmatrix} x + n, \\ &= \left(\frac{|h_1|^2}{\|\mathbf{h}\|} + \frac{|h_2|^2}{\|\mathbf{h}\|} \right) x + n, \\ &= \|\mathbf{h}\|\, x + n \end{aligned}$$

Observe that the SNR for the above system is, therefore, given as

$$\mathrm{SNR} = \frac{\|\mathbf{h}\|^2 P}{\sigma_n^2} \quad \text{-------- 4.16}$$

which is identical to that of the receive diversity with Maximum Ratio Combining (MRC). Thus, the above example seems to indicate that the performance that can be achieved through multiple antennas at the transmitter is equivalent to that achieved with multiple antennas at the receiver. However, a close examination reveals the following point. Consider again the transmit vector given as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{h_1^*}{\|\mathbf{h}\|} \\ \frac{h_2^*}{\|\mathbf{h}\|} \end{bmatrix} x$$

- To perform this at the transmitter requires knowledge of the channel coefficients h_1, h_2 at the transmitter, which is termed Channel State Information (CSI) in the context of wireless communications. However, the channel coefficients h_1, h_2 are estimated at the receiver.
- To implement beamforming at the transmitter, information about these channel coefficients has to be fed back to the transmitter. This is a challenging task. Therefore, one cannot always count on possessing the knowledge of the channel at the transmitter. The Alamouti

space-time code is an ingenuous scheme which overcomes this constraint through a novel transmission procedure and is described next.

4.7.1 Alamouti Code: Procedure

- The Alamouti code is a space-time code proposed for a 1 × 2 MISO system.
- The interesting aspect of the Alamouti-code is that it achieves a diversity order of 2 without CSI at the transmitter.
- Consider two symbols x_1, x_2. In an Alamouti-coded system, in the first transmit instant, the symbol x_1 is transmitted from the transmit antenna 1, while x_2 is transmitted from the transmit antenna 2. Therefore, the transmit symbol vector in the first time instant is given as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Further, the received symbol y(1) at the receiver corresponding to this transmission is given as

$$y(1) = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n(1) \qquad \text{------- 4.17}$$

In the second time instant, the symbol $-x^*_2$ is transmitted from the first transmit antenna, while x^*_1 is transmitted from the second transmit antenna. As we will see later, this is the unique aspect of the Alamouti code which enables it to achieve diversity gain of the order 2 at the receiver. Therefore, the transmit symbol vector in the second time instant is given as

$$\begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix}$$

Further, the received symbol y(2) can be expressed as

$$y(2) = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix} + n(2) \qquad \text{------ 4.18}$$

Consider the conjugate of y(2) at the receiver, the above equation can be simplified as

$$\begin{aligned} y^*(2) &= \begin{bmatrix} h_1^* & h_2^* \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + n^*(2) \\ &= \begin{bmatrix} -h_1^* & h_2^* \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + n^*(2) \\ &= \begin{bmatrix} h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n^*(2) \end{aligned} \qquad \text{----- 4.19}$$

Now, the received symbols y(1) and $y^*(2)$ from equations (4.17) and (4.19) above can be stacked to write the combined system model for the first and second time instants in the Alamouti code as

$$\underbrace{\begin{bmatrix} y(1) \\ y^*(2) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} n(1) \\ n^*(2) \end{bmatrix}}_{\mathbf{n}}$$
------ 4.20

Thus, both the symbols have been stacked which effectively converts the Alamouti coded system into a 2 × 2 MIMO system, with the channel matrix

$$\begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix}$$
----------- 4.21

Further, the noise $n*(2)$ is statistically identical to n(2), i.e., $n^*(2)$ is zero mean circularly symmetric Gaussian noise with variance σ^2_n . Moreover, observe a very important property of the Alamouti channel matrix. Consider the columns c1, c2 of the channel matrix given as

$$\mathbf{c}_1 = \begin{bmatrix} h_1 \\ h_2^* \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} h_2 \\ -h_1^* \end{bmatrix}$$

Then, we have, $c^H_1 c_2$ given as

$$\mathbf{c}_1^H \mathbf{c}_2 = \begin{bmatrix} h_1^* & h_2 \end{bmatrix} \begin{bmatrix} h_2 \\ -h_1^* \end{bmatrix}$$

$$= h_1^* h_2 - h_2 h_1^*$$

$$= 0$$

It can, therefore, be seen that the columns c1, c2 are orthogonal. This tremendously simplifies the receive processing of the Alamouti code. Consider now beamforming using the vector w1 defined in terms of c1 as

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{c}_1\|}\mathbf{c}_1$$

$$= \frac{1}{\|\mathbf{h}\|} \begin{bmatrix} h_1 \\ h_2^* \end{bmatrix}$$

One can now employ this as a receive beamformer to derive the processed symbol as

$$\mathbf{w}_1^H \mathbf{y} = \begin{bmatrix} \frac{h_1^*}{\|\mathbf{h}\|} & \frac{h_2}{\|\mathbf{h}\|} \end{bmatrix} \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathbf{w}_1^H \mathbf{n}$$

$$= \begin{bmatrix} \|\mathbf{h}\| & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \tilde{n}_1$$

$$= \|\mathbf{h}\| \, x_1 + \tilde{n}_1$$

Further, since w1 is a unit-norm vector, $\tilde{n}1 = w^H_1$ n is Gaussian noise with variance σ^2_n .
Therefore, the SNR at the receiver is given as

$$\mathrm{SNR} = \frac{\|\mathbf{h}\|^2}{\sigma_n^2} \mathrm{E}\left\{|x_1|^2\right\}$$

$$= \frac{\|\mathbf{h}^2\|}{\sigma_n^2} P_1,$$

where $\| h \|^2 = |h1|^2 + |h2|^2$. Therefore, the diversity gain or diversity order of BER at the receiver is 2, since it is analogous to the maximum ratio combiner with 2 antennas shown in Eq. (4.16). Similarly, to decode x2, the beamformer w2 is given as

$$\mathbf{w}_2 = \frac{\mathbf{c}_2}{\|\mathbf{c}_2\|} = \frac{1}{\|\mathbf{h}\|} \begin{bmatrix} h_2 \\ -h_1^* \end{bmatrix}$$

Thus, the SNR of the decoded streams of the Alamouti code is $\frac{\|\mathbf{h}^2\|}{\sigma_n^2} P_1, \; \frac{\|\mathbf{h}^2\|}{\sigma_n^2} P_2,$, where P1, P2 are the power allocated to x1, x2 respectively.

- The total transmit power P is fixed. This has to be allocated to the individual streams. Therefore, we have P1 = P2 = P/2 . Hence, the net output SNR of each stream is

$$\mathrm{SNR} = \frac{P}{2} \frac{\|\mathbf{h}^2\|}{\sigma_n^2} = \frac{1}{2} \frac{\|\mathbf{h}^2\|}{\sigma_n^2} P$$

or in other words, equal to half that with CSI at the transmitter as can be seen from Eq. (4.16).

- Thus, the absence of CSI results in a loss of 3 dB in output SNR corresponding to this factor of 1/2 .
- Also, the orthogonality of the columns c1, c2 of the effective channel matrix is a key property of the Alamouti code. Hence, the Alamouti code is also termed an **Orthogonal Space-Time Block Code (OSTBC**).
- The term space-time refers to the fact that the Alamouti code involves two symbols x1, x2 which are transmitted over two antennas over two instants of time. Therefore, the symbols are coded across both the space and time dimensions, leading to the name "space-time" code. This is schematically shown in Figure 4.7.
- The Alamouti code transmits a net of two symbols x1, x2 in time instants. Therefore, on an average it transmits one symbol per time instant. Hence, the net rate of the code is 1 symbol per time instant, i.e., the rate R = 1. Such a code is termed a **full rate code**. Therefore, the Alamouti code is a full-rate code.

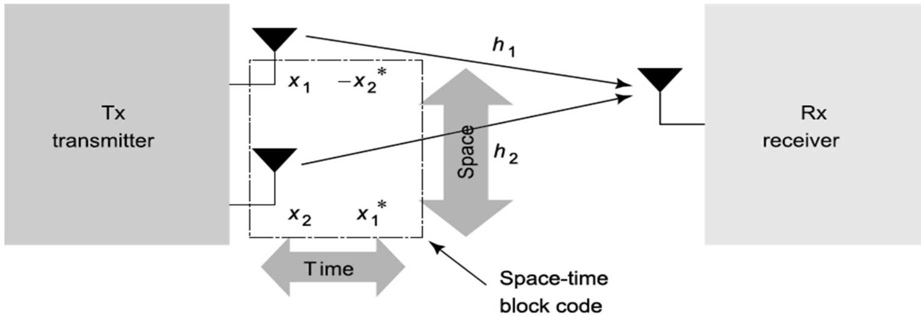

Figure 4.7 Alamouti orthogonal space-time block code

Example 6: Consider the 1 × 2 wireless system given as

$$y = \begin{bmatrix} 1+j & 3+4j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n$$

Clearly, indicate the processing at the transmitter and the receiver for the above system with the Alamouti code.

Solution: The channel coefficients h1, h2 are 1 + j, 3 + 4j, corresponding to the channels of transmit antennas 1, 2 respectively. Therefore, corresponding to the transmission of symbols x1, x2 from the first and second transmit antennas in the first time instant, we have the received symbol y(1) given as

$$y(1) = \begin{bmatrix} 1+j & 3+4j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n(1)$$

Further, corresponding to the transmission of $-x_2^*$, x_1^* from the first and second transmit antennas in the second time instant, we have

$$y(2) = \begin{bmatrix} 1+j & 3+4j \end{bmatrix} \begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix} + n(2)$$

which can be simplified by considering the conjugate $y^*(2)$ of y(2) as,

$$\begin{aligned} y^*(2) &= \begin{bmatrix} 1-j & 3-4j \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + n^*(2) \\ &= \begin{bmatrix} -1+j & 3-4j \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + n^*(2) \\ &= \begin{bmatrix} 3-4j & -1+j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n^*(2) \end{aligned}$$

Therefore, stacking now y(1), $y^*(2)$, we have

$$\underbrace{\begin{bmatrix} y(1) \\ y^*(2) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1+j & 3+4j \\ 3-4j & -1+j \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathbf{n}$$

where H above indicates the effective Alamouti channel matrix and y is the effective received vector. Therefore, the columns c1, c2 are given as

$$\mathbf{c}_1 = \begin{bmatrix} 1+j \\ 3-4j \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 3+4j \\ -1+j \end{bmatrix}$$

It can now be easily verified that the columns c1, c2 are orthogonal. Consider $c1^H c2$, which can be simplified as

$$\begin{aligned} \mathbf{c}_1^H \mathbf{c}_2 &= (1+j)^* (3+4j) + (3-4j)^* (-1+j) \\ &= (1-j)(3+4j) + (3+4j)(-(1-j)) \\ &= (1-j)(3+4j) - (3+4j)(1-j) \\ &= 0 \end{aligned}$$

Beamformer to detect x1 is given as

$$\mathbf{w}_1 = \frac{\mathbf{c}_1}{\|\mathbf{c}_1\|} = \frac{1}{\sqrt{27}} \begin{bmatrix} 1+j \\ 3-4j \end{bmatrix}$$

To detect x1, one has to perform the receive beamforming operation $w_1^H y$. Similarly, w2, the beamformer to detect x2 is given as

$$\mathbf{w}_2 = \frac{\mathbf{c}_2}{\|\mathbf{c}_2\|} = \frac{1}{\sqrt{27}} \begin{bmatrix} 3+4j \\ -1+j \end{bmatrix}$$

Therefore, x2 can be detected by performing the operation $w_2^H y$ at the receiver.

4.8 Nonlinear MIMO Receiver: V-BLAST

- We have seen the zero-forcing (ZF) and minimum mean squared error (MMSE) receivers, which are linear MIMO receivers.
- We now look at the first nonlinear MIMO receiver, termed V-BLAST, short for Vertical Bell Labs Layered Space-Time receiver.
- V-BLAST employs Successive Interference Cancellation (SIC) in which the impact of each estimated symbol is cancelled prior to the detection of the next symbol.

- This SIC principle on which V-BLAST is based leads to its nonlinear nature. Consider the MIMO system model given as

$$y = Hx + n$$

$$= \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \dots & \mathbf{h}_t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix} + \mathbf{n} \quad \text{----- 4.22}$$

The vectors h1, h2, . . . , ht correspond to the t columns of the channel matrix H.

- Consider now the left-inverse or pseudo-inverse of the channel matrix Q described in Eq. (4.3).
- Let this matrix be denoted by the t × r matrix Q, i.e., QH = Ir. Further, let the matrix Q be written as

$$\mathbf{Q} = \begin{bmatrix} \mathbf{q}_1^H \\ \mathbf{q}_2^H \\ \vdots \\ \mathbf{q}_t^H \end{bmatrix}$$

where q_1^H , q_2^H, . . . , q_t^H denote the t rows of the matrix Q. Therefore, QH = It can be written as

$$\begin{bmatrix} \mathbf{q}_1^H \\ \mathbf{q}_2^H \\ \vdots \\ \mathbf{q}_t^H \end{bmatrix} \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \dots & \mathbf{h}_t \end{bmatrix} = \mathbf{I}_t$$

$$\Rightarrow \begin{bmatrix} \mathbf{q}_1^H\mathbf{h}_1 & \mathbf{q}_1^H\mathbf{h}_2 & \dots & \mathbf{q}_1^H\mathbf{h}_t \\ \mathbf{q}_2^H\mathbf{h}_1 & \mathbf{q}_2^H\mathbf{h}_2 & \dots & \mathbf{q}_2^H\mathbf{h}_t \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{q}_t^H\mathbf{h}_1 & \mathbf{q}_t^H\mathbf{h}_2 & \dots & \mathbf{q}_t^H\mathbf{h}_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \quad \text{----- 4.23}$$

Therefore we have

$$\mathbf{q}_1^H\mathbf{h}_1 = \mathbf{q}_2^H\mathbf{h}_2 = \dots = \mathbf{q}_t^H\mathbf{h}_t = 0$$

$$\mathbf{q}_1^H\mathbf{h}_2 = \mathbf{q}_1^H\mathbf{h}_3 = \dots = \mathbf{q}_{t-1}^H\mathbf{h}_t = 0$$

$$\mathbf{q}_i^H\mathbf{h}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{------ 4.24}$$

From Eq. (4.22), observe that the system model can be explicitly given in terms of the columns of the channel matrix H as

$$\mathbf{y} = \mathbf{h}_1 x_1 + \mathbf{h}_2 x_2 + \dots + \mathbf{h}_t x_t + \mathbf{n}$$

Employing now the fact that $q_1^H h_1$ and $q_1^H h_2$, $q_1^H h_3$, etc., is zero, one can use q_1 as a receive beamformer. Therefore, performing $q_1^H y$ at the receiver, we have

$$\begin{aligned}
\tilde{y}_1 &= \mathbf{q}_1^H \mathbf{y} \\
&= \mathbf{q}_1^H \left(\mathbf{h}_1 x_1 + \mathbf{h}_2 x_2 + \ldots + \mathbf{h}_t x_t + \mathbf{n} \right) \\
&= \underbrace{\mathbf{q}_1^H \mathbf{h}_1}_{1} x_1 + \underbrace{\mathbf{q}_1^H \mathbf{h}_2}_{0} x_2 + \ldots + \underbrace{\mathbf{q}_1^H \mathbf{h}_t}_{0} x_t + \underbrace{\mathbf{q}_1^H \mathbf{n}}_{\tilde{n}_1} \\
&= x_1 + \tilde{n}_1
\end{aligned}$$

Thus, ˜y1 can now be employed to decode x1. Now, the interference caused by x1 is removed from y to form ˇy2 as

$$\begin{aligned}
\check{\mathbf{y}}_2 &= \mathbf{y} - \mathbf{h}_1 x_1 \\
&= \mathbf{h}_1 x_1 + \mathbf{h}_2 x_2 + \ldots + \mathbf{h}_t x_t + \mathbf{n} \mathbf{h}_1 x_1 \\
&= \mathbf{h}_2 x_2 + \ldots + \mathbf{h}_t x_t + \mathbf{n} \\
&= \underbrace{\begin{bmatrix} \mathbf{h}_2 & \mathbf{h}_3 & \ldots & \mathbf{h}_t \end{bmatrix}}_{\mathbf{H}^{(2)}} \underbrace{\begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_t \end{bmatrix}}_{\mathbf{x}^{(2)}} + \mathbf{n} \\
&= \mathbf{H}^{(2)} \mathbf{x}^{(2)} + \mathbf{n}
\end{aligned}$$

- The above system model can now be seen to correspond to a reduced MIMO system with channel matrix $H^{(2)}$ of r rows and t − 1 columns. Thus, it represents an $r \times (t-1)$ MIMO system, with $x_2, x_3, \ldots, x_t$ denoting the t − 1 transmit symbols.
- Now, consider $Q^{(2)}$ as the zero-forcing receiver for $H^{(2)}$ and repeat the above process by decoding x_2, and so on.
- The advantage of this scheme is that the diversity order and the associated diversity gain progressively increases as we proceed through the scheme for decoding the different transmit symbols x_1 through x_t. In fact, in the last stage, when only one symbol x_t is left for decoding, the effective system model is given as

$$\check{\mathbf{y}}_t = \mathbf{H}^{(t)} x_t + \mathbf{n}$$

Notice that the effective channel $H^{(t)}$ is the column vector h_t. This can be decoded by receive beamforming along h_t, in other words, maximum ratio combining.
In V-BLAST, streams that are decoded later experience progressively higher diversity.

Example 7: Consider the 2 × 2 MIMO system given below and describe the various stages of the V-BLAST receiver.

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}}_{\mathbf{n}} \quad \text{--- 4.25}$$

Solution: The channel matrix H is given as

$$\mathbf{H} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Therefore, the left inverse of H, i.e., Q = H† is given as $Q = H^{-1}$ since the matrix H is square and invertible. Therefore, we have

$$\mathbf{Q} = \mathbf{H}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

It can be clearly seen that the rows of the matrix Q are $q_1^H = [\ \mathbf{3}\ \mathbf{-2}\ ]$, $q_2^H = [\ \mathbf{-1}\ \mathbf{1}\ ]$. Further, it can be readily seen that $q_1^H h_1$ is given as

$$\mathbf{q}_1^H \mathbf{h}_1 = \begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$$

and $q_1^H h_2$ can be simplified as

$$\mathbf{q}_1^H \mathbf{h}_2 = \begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0$$

Therefore, the first row q_1^H is orthogonal to h_2, i.e., the second column of the channel matrix H. Therefore, decoding in q_1^H in the first stage of V-BLAST, we have

$$\begin{aligned} \tilde{y}_1 &= \mathbf{q}_1^H \mathbf{y} \\ &= \begin{bmatrix} 3 & -2 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\mathbf{q}_1^H \mathbf{n}}_{\tilde{n}_1} \\ &= x_1 + \tilde{n}_1 \end{aligned}$$

Therefore, the symbol x1 can be detected from ˜y1. The interference caused by this can now be canceled from the received signal y to form ˇy2 as

$$\check{\mathbf{y}}_2 = \mathbf{y} - \mathbf{h}_1 x_1$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} x_2 + \mathbf{n}$$

The above system corresponds effectively to a receive diversity system with 2 receive antennas and a single transmit antenna. In fact, the optimum receiver scheme is to perform maximum ratio combining with the receive beamformer w = [2 3]^T . The diversity order of decoding this stream is, therefore, equal to 2.

4.9 MIMO Beamforming

- Beamforming basically implies that only spatial dimension is used for transmission amongst the many dimensions that are available. Consider the MIMO system given as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

$$= \mathbf{U}\Sigma\mathbf{V}^H\mathbf{x} + \mathbf{n}$$

$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_t \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_t \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^H \\ \mathbf{v}_2^H \\ \vdots \\ \mathbf{v}_t^H \end{bmatrix} \mathbf{x} + \mathbf{n}$$

  where H = UΣV^H is the singular value decomposition (SVD) of the channel matrix H.
- The right and left singular vectors v_1, u_1 respectively, which correspond to the largest singular value σ_1, represent the dominant transmit and receive modes of the MIMO system. Hence, one can transmit a single symbol ˜x1 from the transmitter by beamforming along the vector v_1 as

$$\mathbf{x} = \mathbf{v}_1 \tilde{x}_1$$

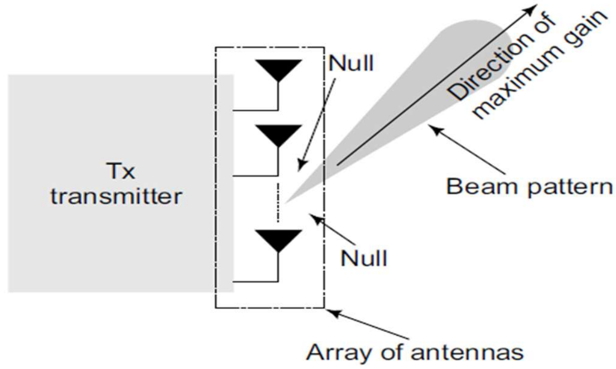

Figure 4.8 MIMO beamforming

Thus, the symbol ˜x1 is being transmitted along the abstract direction represented by the vector v_1 in t dimensional space as shown in Figure 6.8. Substituting this in the MIMO system model, we have

$$\mathbf{y} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_t \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_t \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^H \\ \mathbf{v}_2^H \\ \vdots \\ \mathbf{v}_t^H \end{bmatrix} \mathbf{v}_1 \tilde{x}_1 + \mathbf{n}$$

$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_t \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_t \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^H \mathbf{v}_1 \\ \mathbf{v}_2^H \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_t^H \mathbf{v}_1 \end{bmatrix} \tilde{x}_1 + \mathbf{n}$$

It can now be seen that $v_1^H v_1 = 1$, while $v_2^H v_1 = \ldots = v_t^H v_1 = 0$, since v_i, $2 \le i \le t$ are orthogonal to v_1. Therefore, the received signal y can be further simplified as

$$\mathbf{y} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_t \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tilde{x}_1 + \mathbf{n}$$

$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_t \end{bmatrix} \begin{bmatrix} \sigma_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tilde{x}_1 + \mathbf{n}$$

$$= \sigma_1 \mathbf{u}_1 \tilde{x}_1 + \mathbf{n}$$

The received vector y effectively corresponds to a multiple receive antenna system with the effective channel vector u_1.
One can now perform MRC at the receiver using the beamforming vector u_1 as,

$$\begin{aligned}\tilde{y}_1 &= \mathbf{u}_1^H \mathbf{y}\\ &= \mathbf{u}_1^H (\sigma_1 \mathbf{u}_1 \tilde{x}_1 + \mathbf{n})\\ &= \sigma_1 \mathbf{u}_1^H \mathbf{u}_1 \tilde{x}_1 + \underbrace{\mathbf{u}_1^H \mathbf{n}}_{\tilde{n}_1}\\ &= \sigma_1 \tilde{x}_1 + \tilde{n}_1\end{aligned}$$

Further, it can be seen from Eq. (4.13) that the variance of the noise ˜n1 = $u_1^H n$ is σ_n^2 . Thus, the SNR at the receiver is given as

$$\mathrm{SNR} = \sigma_1^2 \frac{\mathrm{E}\left\{|\tilde{x}_1|^2\right\}}{\sigma_n^2} = \sigma_1^2 \frac{P}{\sigma_n^2}$$

where P = E { |˜x1| ^{2} } is the transmit power corresponding to the symbol ˜x1.

- It can be seen from the above expression that the net transmit power P is amplified by a factor of σ_n^2 corresponding to the largest singular value σ_1. This MIMO beamforming scheme is termed **Maximum Ratio Transmission (MRT).**
- Since only one dimension is being used in this scheme, it results in a simplistic transmission and reception scheme compared to spatial multiplexing MIMO schemes such as MIMO-ZF, MIMO-MMSE, MIMO V-BLAST, etc.
- However, since it is transmitting along the MIMO spatial dimension with channel gain σ_1 corresponding to the largest singular value, it leads to a high SNR at the receiver. Further, two additional points are worth of being noted about the maximum ration transmission scheme.
- Firstly, it is capacity optimal at low SNR. Secondly, it achieves the full diversity order of MIMO communication, i.e., rt, where r, t denote the number of receive and transmit antennas respectively.

Question Bank

1. Briefly explain Multiple-input multiple output systems. Or With neat diagram explain MIMO system model.
2. Explain Pre-coding and Beam forming. Or Write a short note on MIMO Beamforming.
3. Explain singular value decomposition of MIMO channel.
4. Explain SVD and MIMO capacity.
5. Define Beam forming and briefly explain MIMO diversity gain.
6. Discuss transmit diversity with channel known at transmitter.
7. Discuss transmit diversity with channel unknown at transmitter – The Alamouti scheme.
8. Explain receiver diversity in detail.

9. Discuss receiver diversity with selection combining and threshold combining.
10. Briefly discuss Maximal-Ratio combining and Equal-Gain combining.
11. Discuss the capacity of time-varying frequency-selective fading channels with respect to time.