Exc 2.6

$$p(x) = \mathcal{N}(x \mid b, B)$$
$$= (2\pi B)^{-1/2} e^{-(x-b)^2/2B} \qquad (G1)$$
$$= c\, e^{-\mathcal{J}(x)} \qquad (2)$$

i) $p = 0$ when $\mathcal{J}(x) = +\infty$

$\Leftrightarrow \quad x = \pm\infty$

ii) Note: log is monotonic

so $\frac{d\mathcal{J}}{dx} = 0 \Rightarrow \frac{dp}{dx} = 0$

$$\frac{d\mathcal{J}}{dx}(x) = \frac{x-b}{B} = 0 \text{ if } x = b \qquad (3)$$

iii) $\frac{dp}{dx} = -p \cdot \frac{d\mathcal{J}}{dx}$ by (2)

$$\frac{d^2p}{dx^2} = p \cdot \left(\frac{d\mathcal{J}}{dx}\right)^2 - p \cdot \frac{d^2\mathcal{J}}{dx^2} \quad \text{by "chain rule"}$$

$$= p \cdot \left(\left(\frac{x-b}{B}\right)^2 - 1/B\right) \qquad (4)$$

$$= 0 \text{ if } x = b + \sqrt{B}$$

iv) $$\int_{\mathbb{R}} e^{-x^2} dx = \left[\left(\int_{\mathbb{R}} e^{-u^2} du\right)\left(\int_{\mathbb{R}} e^{-v^2} dv\right)\right]^{1/2}$$

$$= \left(\int_{\mathbb{R}^2} e^{-(u^2+v^2)} du\, dv\right)^{1/2}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$$

$$= \left(\int_0^{2\pi} \int_{\mathbb{R}^+} e^{-r^2} r\, dr\, d\theta\right)^{1/2}$$

$$= \left(\int_0^{2\pi} \left[-\tfrac{1}{2} e^{-r^2}\right]_0^{+\infty} d\theta\right)^{1/2} = \left(\int_0^{2\pi} \tfrac{1}{2}\, d\theta\right)^{1/2} = \pi^{1/2}$$

b) By (4), $\frac{d^2p}{dx^2}(b) = -\frac{1}{B} p(b)$

$$= -\frac{1}{B} c$$

$$= -(2\pi)^{-1/2} B^{-3/2}$$

So, B is a simple funct of $\frac{d^2p}{dx^2}(b)$ i.e. we get an estimate of the variance (B) from $\frac{d^2p}{dx^2}(b)$.