Problem 2

Part a.

$$L(\beta \mid X, \vec{y}) = \prod_{i=1}^{n} p_i^{y_i}(1-p_i)^{1-y_i} \quad , \; y_i = 1, 0$$

$$= \prod_{i=1}^{n} \left(\frac{e^{X_i\beta}}{1+e^{X_i\beta}}\right)^{y_i} \cdot \left(1 - \frac{e^{X_i\beta}}{1+e^{X_i\beta}}\right)^{1-y_i} \quad , \; y_i = 1, 0$$

$$= \prod_{i=1}^{n} \left(\frac{e^{X_i\beta}}{1+e^{X_i\beta}}\right)^{y_i} \cdot \prod_{i=1}^{n} \left(1 - \frac{e^{X_i\beta}}{1+e^{X_i\beta}}\right)^{1-y_i}$$

$$= \prod_{i=1}^{n} \left(\frac{e^{X_i\beta}}{1+e^{X_i\beta}}\right)^{y_i} \cdot \prod_{i=1}^{n} \left(\frac{1}{1+e^{X_i\beta}}\right)^{1-y_i}$$

Part b.

$$\ln\left(L(\beta \mid X, \vec{y})\right) = \ln\left[\prod_{i=1}^{n} \left(\frac{e^{X_i\beta}}{1+e^{X_i\beta}}\right)^{y_i} \cdot \prod_{i=1}^{n} \left(\frac{1}{1+e^{X_i\beta}}\right)^{1-y_i}\right]$$

Prop

$\ln(x \cdot y) = \ln(x) + \ln(y)$

$$= \ln\left(\prod_{i=1}^{n} \left(\frac{e^{X_i\beta}}{1+e^{X_i\beta}}\right)^{y_i}\right) + \ln\left(\prod_{i=1}^{n} \left(\frac{1}{1+e^{X_i\beta}}\right)^{1-y_i}\right)$$

$$= \sum_{i=1}^{n} \ln\left(\left(\frac{e^{X_i\beta}}{1+e^{X_i\beta}}\right)^{y_i}\right) + \sum_{i=1}^{n} \ln\left(\left(\frac{1}{1+e^{X_i\beta}}\right)^{1-y_i}\right)$$

$$= \sum_{i=1}^{n} y_i \cdot \ln\left(\frac{e^{X_i\beta}}{1+e^{X_i\beta}}\right) + \sum^{n} (1-y_i) \cdot \ln\left(\frac{1}{1+e^{X_i\beta}}\right)$$

Prop

$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

$$= \sum_{i=1}^{n} y_i \cdot \underbrace{\ln\left(e^{X_i\beta}\right)}_{\text{evaluates}} - y_i\left(1+e^{X_i\beta}\right) + \sum_{i=1}^{n} (1-y_i) \cdot \underbrace{\ln(1)}_{0} - (1-y_i) \cdot \ln\left(1+e^{X_i\beta}\right)$$

$$= \sum_{i=1}^{n} y_i \cdot X_i\beta + \sum_{i=1}^{n} -(1-y_i) \cdot \ln\left(1+e^{X_i\beta}\right) + \sum_{i=1}^{n} -y_i\left(1+e^{X_i\beta}\right)$$

...

$$= \sum_{i=1}^{n} y_i \cdot X_i\beta + \sum_{i=1}^{n} (y_i - 1)\ln\left(1 + e^{X_i\beta}\right) + \sum_{i=1}^{n} -y_i\left(1 + e^{X_i\beta}\right)$$

$$= \sum_{i=1}^{n} y_i \cdot X_i\beta + \sum_{i=1}^{n} (-1)\cdot\ln\left(1 + e^{X_i\beta}\right) + \sum_{i=1}^{n} (y_i)\cdot\ln\left(1 + e^{X_i\beta}\right) - \sum_{i=1}^{n} (y_i)\left(1 + e^{X_i\beta}\right)$$

$$\ell\left(\beta \mid X, \vec{y}\right) = \sum_{i=1}^{n} y_i \cdot X_i\beta - \sum_{i=1}^{n} \ln\left(1 + e^{X_i\beta}\right)$$

Problem 2

Part. c.

Find $\nabla \ell(\beta|X,\vec{y})$, $X_i\beta = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}$

$$\ell(\beta|X,\vec{y}) = \sum_{i=1}^{n} y_i \cdot (\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}) - \sum_{i=1}^{n} \ln\left(1 + e^{\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}}\right)$$

$$\nabla \ell(\beta|X,\vec{y}) = \begin{bmatrix} \dfrac{\partial \ell}{\partial \beta_0} = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \dfrac{e^{\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}}}{1 + e^{\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}}} \\ \dfrac{\partial \ell}{\partial \beta_1} = \sum_{i=1}^{n} y_i x_{i1} - \sum_{i=1}^{n} \dfrac{x_{i1} \cdot e^{\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}}}{1 + e^{\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}}} \\ \dfrac{\partial \ell}{\partial \beta_2} = \sum_{i=1}^{n} y_i x_{i2} - \sum_{i=1}^{n} \dfrac{x_{i2} \cdot e^{\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}}}{1 + e^{\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}}} \end{bmatrix}$$