COMP6216 Simulation Modelling Coursework 2: The Group Dynamics of Hard Workers and Lazy Workers

Brijee Rana
br1e21@soton.ac.uk
3338 5874

***Abstract*—In this report, we research the group dynamics of hard and lazy workers by using modelling techniques. We find that given certain parameters, the number of students that imitate the Lazy strategy increases as the number of courses increases, this reaches an equilibrium state where only lazy students exist. Changing parameters such as initial composition of the population, group size, cost of effort and the contribution of hard workers to group effort changes the composition of the group.**

I. Introduction

In this assignment I have been asked to solve a modelling problem using two different modelling techniques: differential equations based approach and agent-based modelling.

Firstly, I will develop a differential equation that models the problem. Then use analytical techniques to gain insight into the system's behaviour, based in the differential equation to find equilibrium points and their stability. To compare this to the analytical results I will use a numerical integration method on the differential equation. Finally, implement an agent-based model upon the same problem and compare its results with the other methods to answer the research questions.

The research method questions to answer are:

- Assuming we start with equal numbers of hard working and lazy workers, what is the composition of the group.
  1) After 4 years (i.e. 8 courses) if H=1 and L=0 and a=0.5.
  2) In the long run (after an "infinite" number of years)?
  3) How quickly is this equilibrium state reached?
- How do the following parameters influence results:
  1) Initial composition of the population
  2) group size (n)
  3) cost of effort, a
  4) contribution of hard workers to group effort (i.e. H and L)

II. The modelling problem

- There is a population of students working on a group project, following two strategies (S): work hard (H) or lazy (L).
- In every course, groups of size n are formed at random. Students use the strategy determined at the beginning of the course in their group work.
- Total group effort is determined by the composition of the group. In a group with h hard workers and l=n-h lazy workers total group effort is e=h*H+l*L.
- When group projects are marked, every student in a group gets the same mark. The lecturer determines this mark as m=e/n.
- Every student rethinks their strategy at the end of the semester. Each student following this process:
  1) Select another student from the population at random
  2) Compare between each other based on a measure π, where $\pi = m - aS$, a is a parameter and S is the strategy.
  3) If the randomly selected student has a better measure π that itself, then it adopts the students strategy with a probability proportional to the difference in π.
  4) If the randomly selected student does not have a better π than the student does not change strategy.
- Students take an infinite number of courses and follow the same procedure for every course they take.

III. Differential Equation Approach

The process of where the students decide if they should change strategies is based on replicator dynamics. Thus I can use the replicator equation which is defined as: $\dot{x}_i = x_i(\pi_i - \bar{\pi})$.

Using the replicator equation, we can apply it to our problem. To begin within, we can denote x to be the concentration of the students whose strategy is H, i.e hard work. This means that if we sum the concentration of H students and concentration of L students, it should sum to 1, i.e. concentration of Lazy students = 1- x. To find the average payoff $\bar{\pi}$, we need to find the average measure of π_H and π_L. Thus we have to look at the interaction with all other members of the population.

Interactions between a student with strategy H:
H-H has probability x

H-L has probability 1-x
Average measure of H, $\pi_H = x(m-aH)+(1-x)(m-aH)$

Interactions between a student with strategy L:
L-H has probability x
L-L has probability 1-x
Average measure of L, $\pi_L = x(m-aL)+(1-x)(m-aL)$.
Thus our average payoff is, $\bar{\pi} = x\pi_H + (1-x)\pi_L$.

We can insert this back into our replicator equation and we find that:
$\dot{x} = x(\pi_H - x\pi_H - (1-x)\pi_L)$
$\dot{x} = x(1-x)(\pi_H - \pi_L)$
$\dot{x} = x(1-x)(x(m-aH)+(1-x)(m-aH)-x(m-aL)-(1-x)(m-aL))$
$\dot{x} = x(1-x)((1-x)(m-aH-m+aL)+x(m-aH-m+aL))$
$\dot{x} = x(1-x)((1-x)(a(L-H))+x(a(L-H)))$
$\dot{x} = x(1-x)(a(L-H)-xa(L-H)+x(a(L-H))$
$\dot{x} = x(1-x)(1-x)(a(L-H))$
Thus, we have derived a 1D, non-linear, autonomous differential equation for our model is:
$\dot{x} = a(L-H)x(1-x)$

A. Analytical techniques

1) Equilibrium stability analysis: To gain insight into the system's behaviour, find the equilibrium and their stability. To find the equilibrium we set $\dot{x} = 0 = a(L-H)x(1-x)$
$x^* = 0$
$x^* = 1$

To investigate the linear stability of each equilibrium we can set $f(x) = a(L-H)x(1-x)$ and calculate the gradient of f(x) with respect to x,
$\frac{df}{dx} = a(L-H)\frac{d}{dx}[x(1-x)]$
u = x, v= 1-x, (uv)' = uv' + u'v = (x)(-1) + (1)(1-x) = 1-2x
$\frac{df}{dx} = a(L-H)(1-2x)$

Let us use the values stated in the research questions, a=0.5,L=0 and H=1.
$\frac{df}{dx}|_{x=0} = a(L-H)(1-2(0)) = a(L-H) = 0.5(0-1) = -0.5 < 0 \Rightarrow stable$ fixed-point
$\frac{df}{dx}|_{x=1} = a(L-H)(1-2(1)) = -a(L-H) = -0.5(0-1) = 0.5 > 0 \Rightarrow unstable$ fixed-point

2) Graphical analysis: We can also look at the system graphically to investigate its fixed points and their stability.

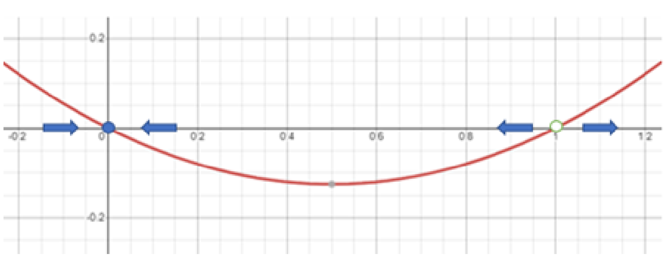


Fig. 1: Graphical analysis of fixed points

As you can see by figure 1, it matches with our equilibrium analysis above. That $x* = 0$ is a stable point in which the system moves towards and $x* = 1$ is an unstable fixed point that the system moves away from.

3) Integration: If a 1d ODE is of the form $\frac{d}{dx}x = f(t)$ with a r.h.s. independent of the unknown function x, it can be solved by integration.

The differential equation that I have derived $\frac{dx}{dt} = f(x,t) = a(L-H)x(1-x)$ is separable, therefore I can find the integral as a solution.
Assuming we start with equal numbers of hard working and lazy workers, thus initial condition x(0)=0.5.
$\frac{dx}{dt} = a(L-H)x(1-x), x_0 = 0.5$
$\frac{dx}{x(1-x)} = a(L-H)t \rightarrow \int_x^{x_0} \frac{dx}{x(1-x)}\, dx = a(L-H)\int_t^0 dt$

$\frac{1}{x(1-x)} = \frac{(1-x)+x}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$
$ln\frac{x}{x_0} - ln\frac{1-x}{1-x_0} = a(L-H)t$

$ln\frac{x}{1-x} + ln\frac{1-x_0}{x_0} = a(L-H)t$

$x = \frac{x_0}{1-x_0}e^{a(L-H)t} - x\frac{x_0}{1-x_0}e^{a(L-H)t}$

Subbing in $x_0 = 0.5, x = \frac{1}{1+e^{-a(L-H)}t}$

Let us use the conditions stated in the research questions, a=0.5,L=0 and H=1.

Thus, we get $x = \frac{1}{1+e^{0.5t}}$.
With this, we can calculate the composition of the group and answer some of the research questions.

After 8 courses, t=8, if H=1,L=0 and a=0.5, we can calculate that the concentration if hard workers in a group is 0.018 (2 s.f). This means that the concentration of lazy workers in a group is 0.982 (3 s.f).

As you can see in figure 2, equilibrium state is reached after 12 courses and the concentration of hard workers in a group is almost 0, which continues to be the case after in the long run, i.e. after an infinite number of years. In terms of parameters influencing the results. Changing the initial composition of the population only changes how long it will take for equilibrium to reach, i.e. if X(0) is ¿ 0.5 this time, then it will take longer or a greater number of courses for equilibrium to reach and vice versa. Group size, n is not a parameter in this differential equation so it cannot be tested. In terms of cost of effort a, as you can see in figure 6, when a=0 the number of lazy and hard students is the same but as it increases, the concentration of H decreases. In terms of the values of H and L, the greater the difference between them, the faster they converge to the equilibrium state. When they equal each other the concentration of lazy and hard workers is the same as seen in figure 7.

B. Numerical integration method

The integration method I decided to use to numerically integrate the differential equations is the Runge-kutta method. The Differential Equation that I derived earlier is not a

Fig. 2: The composition of hard and lazy students in a group over time

stiff system as it does not have rapidly varying components because it is a 1D ODE system. Therefore, I will not be using an implicit method such as the Backwards Euler. I should use an explicit method instead, I could use the Forwards Euler but it has a many sources of errors. These errors are that there is a truncation error, round-off error as it is a first-order scheme, and having a small step-size can be problematic. Thus I will be using the Runge-Kutta method which is a higher order approximation of the midpoints method. More specifically, as a trade-off between improving truncation error and computational cost, I will be using the 4th order Runge-Kutta method. As you can see in figure 3, the runge-kutta method of approximation reproduces the system correctly. The system converges to 0 as time (number of courses) increases and in the same time at around 10, i.e the system behaves similarly as the previous method.

As you can see by figure 4, the numerical integration of sample trajectories show the equilibrium and their stability. Since, $x* = 0$ is stable and $x* = 1$ is unstable, the system converges to 0, i.e the concentration of hard workers in a group converges to 0, given the parameters set in the research

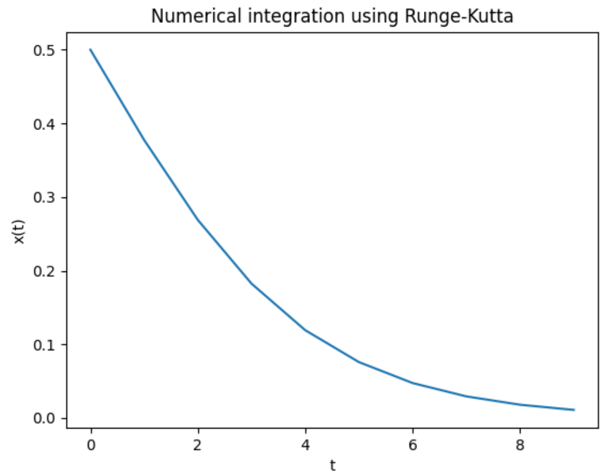


Fig. 3: Runge-Kutta method of approximation

questions.

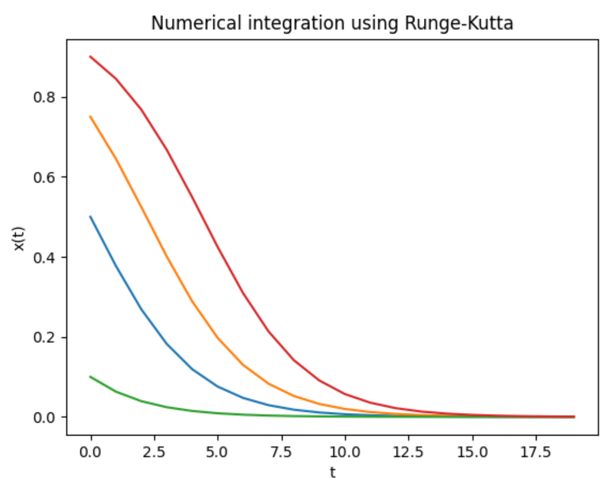


Fig. 4: Sample trajectories

In the runge-kutta method the step size, H=1. I did not need to choose a smaller step-size as the runge-kutta method is a higher-order of approximation and I did not choose a bigger step-size to avoid greater numerical integration error.

In terms of parameters influencing the results. Fortunately, as shown in figures 8 and 9, my findings are exactly similar to my previous method as this is a numerical approximation to the previous method.

IV. Agent-Based Model Approach

I have implemented an agent-based model to address the same problem. As you can see in figure 5, the results are similar to the results obtained in previous methods. That is, the system converges to 0 at around 10 courses, meaning that the number of hard workers continuously decrease, given parameters set in the research question. The composition of the group after 4 years is that there are 2 hard workers which matches the previous methods estimation that at 4 years the concentration of hard workers is 0.018.

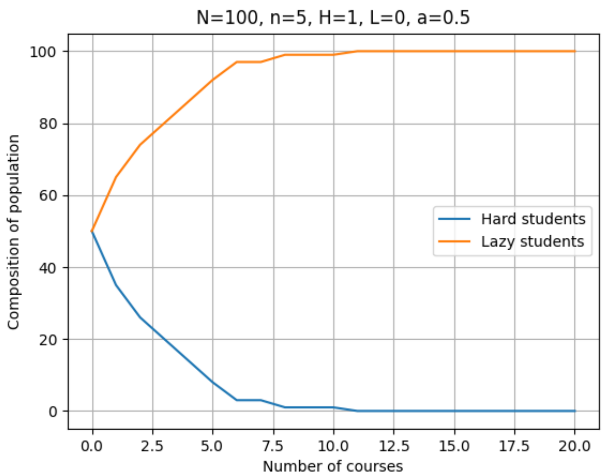


Fig. 5: Agent based method

In terms of parameters influencing the results. The initial composition of the population changes at what time the system converges to the equilibrium, if the number of hard workers are greater than number of lazy workers than it would take longer for the system to converge which is a similar finding to the previous methods. In terms of group size(n), as you can see by figure 11, changing the group size affects the results in an interesting way. When n=1, the number of hard workers increase as the number of courses increases until it converges to all students being hard workers as their performance measure is solely based on themselves and their contribution. When n=2, the results seem to show that as you there will be a greater number of lazy students as time progresses but there will not be a convergence such that only lazy students will exist. As you increase the group size, the convergence to the equilibrium occurs more ”smoothly” as more lazy students can coast of hard workers, thus the number of lazy students increase until there are only them. Figure 10, shows the effect of cost of effort, which the findings are similar to the previous methods in that when a is large, the number of hard workers in the population decrease until it converges to no students being hard workers. As the cost of effort a, gets significantly smaller, the number of lazy students decreases as the number of courses increases. However, in the agent based model when a is significantly small (e.g. a=0.01), the system until it converges to all students being hard workers which is different to the results from previous methods. Perhaps, as cost of effort decreases, there is less incentive to be a lazy worker. In terms of the values of H and L, the results are different to the previous methods such that, when H and L are equal, the number of hard workers rapidly increase and converge to where only hard workers exist instead of what is found in the other methods such that the proportion of hard and lazy workers are the same. This could be as being a lazy worker and hard worker has the same contribution so it does not matter if you are lazy or hard worker and the system randomly chooses to converge to one. Also, when the difference of H and L increases, convergence to the equilibrium state where the number of hard workers is 0 is not faster but slower.

V. Conclusion

Fortunately, the findings from my results for each method are similar. When we start with the same number of hard and lazy workers and parameters H=1, L=0 and a=0.5, after 4 years, the composition of the group is such that there concentration of 0.2 hard workers. In the long run the concentration of hard workers converges to 0, as the equilibrium state is reached in 10 courses, i.e. 5 years.

Group size does affect the results as when it is small it is unlikely converge and as the size increases the number of hard workers decrease. The initial composition of the population changes how quickly the equilibrium state is reached, the greater the initial number of hard workers, the longer it takes.As the cost of effort increases, the rate of students imitating the lazy strategy increases. When H and L are the same, the number of hard and lazy students should be the same, as the difference between H and L increases the rate at which students imitate the lazy strategy should increase.

There are some factors to consider when thinking about these results, such that doing hard work, does not determine the quality of the effort or work done on the project. Thus when determining marks as e/n does not accurately show the quality of the work. e.g. the students could be working hard on wrong approaches to the projects. It is also unlikely that all students use the strategy and perhaps some students are naturally lazy while others are naturally hard working.

References

[1] M. T. Heath, "Scientific Computing: An Introductory Survey", McGraw-Hill, New York, 2002.

[2] S. F. Railsback and V. Grimm, "Agent-Based and Individual-Based Modelling", Princeton University Press, Princeton, NJ (2012).

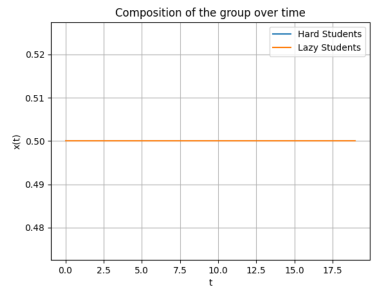
[3] S. H. Strogatz, "Nonlinear Dynamics and Chaos", Westview Press (1994).

VI. Parameter Graphs

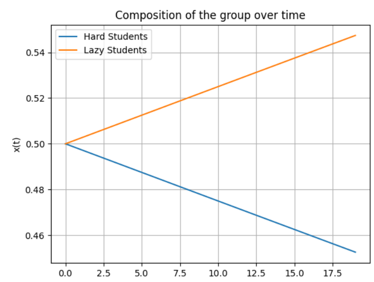
A. DE integrated solution

B. Runge-Kutta method

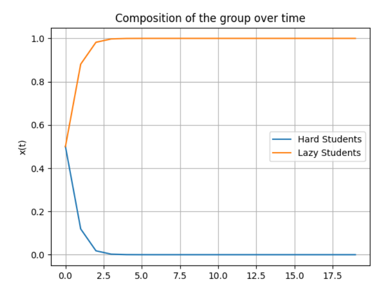
C. Agent-Based Model



(a) DE: a=0

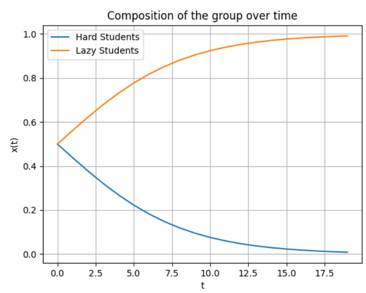


(b) DE: a=0.01

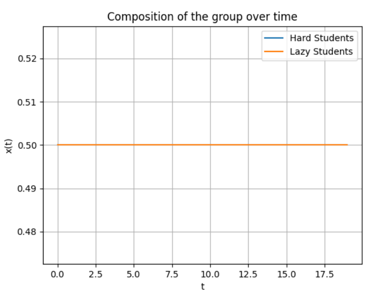


(c) DE: a=2

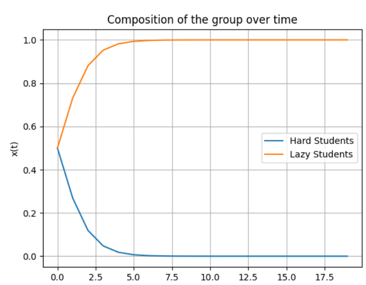
Fig. 6: DE integrated solution: parameter a



(a) DE: H=0.5, L=0

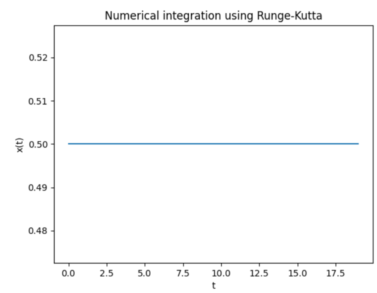


(b) DE: H=1, L=1

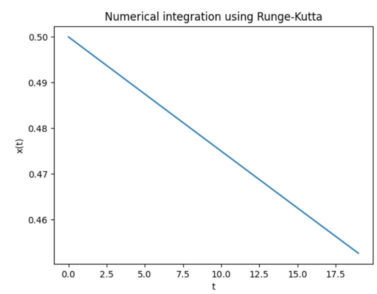


(c) DE: H=3, L=1

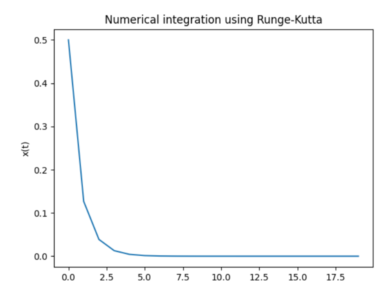
Fig. 7: DE integrated solution: parameters H and L



(a) R-K: a=0

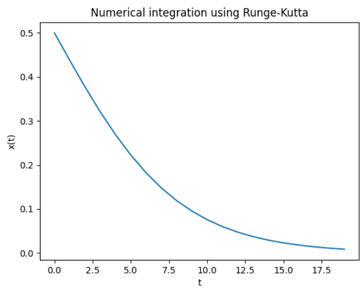


(b) R-K: a=0.01

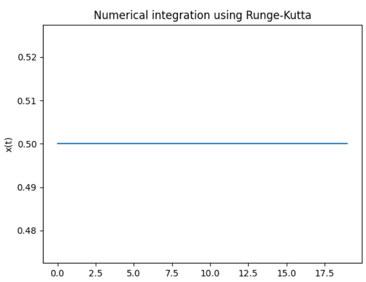


(c) R-K: a=2

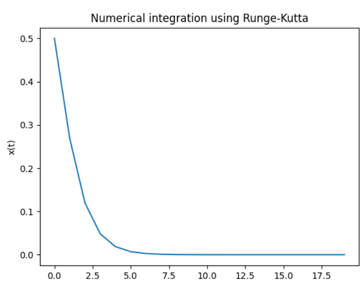
Fig. 8: Runge-Kutta method: parameter a



(a) R-K: H=0.5, L=0

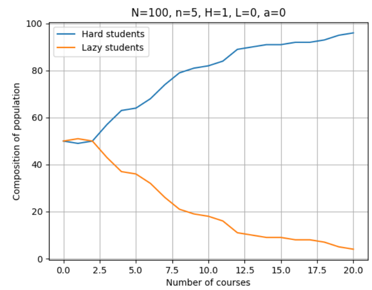


(b) R-K: H=1, L=1

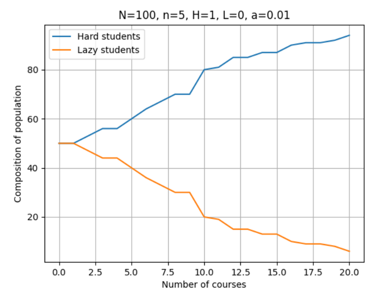


(c) R-K: H=3, L=1

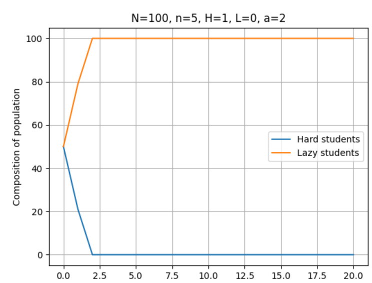
Fig. 9: Runge-Kutta method: parameters H and L



(a) ABM: a=0

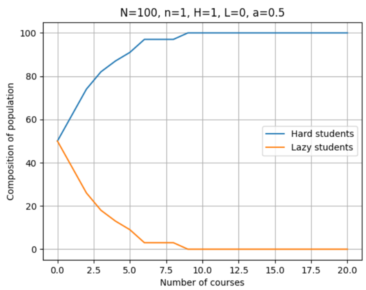


(b) ABM: a=0.01

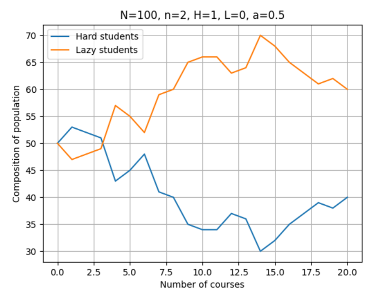


(c) ABM: a=2

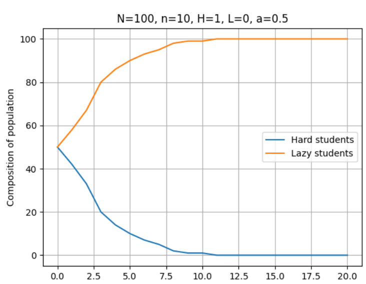
Fig. 10: Agent-based method: parameter a



(a) ABM: n=1

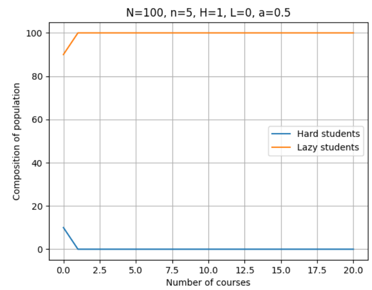


(b) ABM: n=2

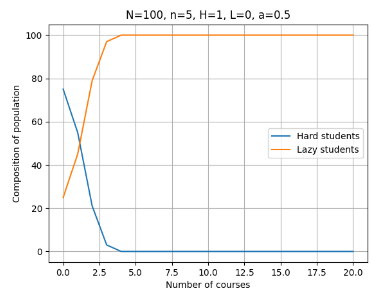


(c) ABM: n=10

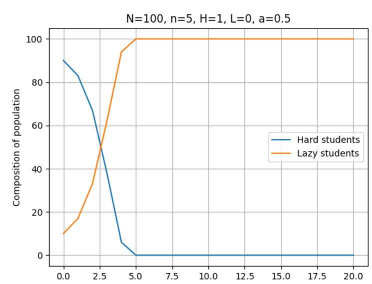
Fig. 11: Agent-based method: parameter n



(a) ABM: x(0)=0.1

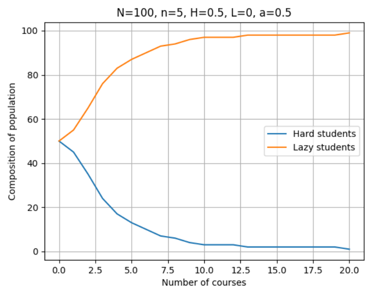


(b) ABM: x(0)=0.75

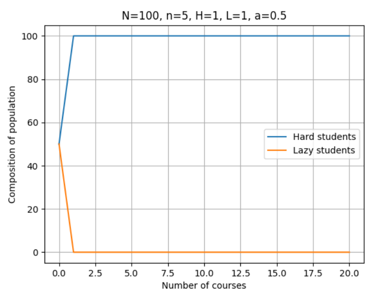


(c) ABM: x(0)=0.9

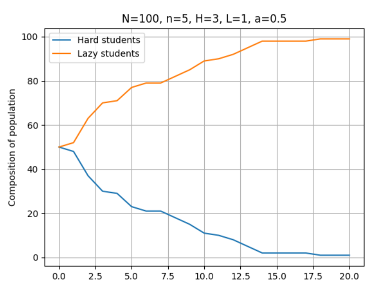
Fig. 12: Agent-based method: parameter x(0)



(a) ABM: H=0.5, L=0



(b) ABM: H=1, L=1



(c) ABM: H=3, L=1

Fig. 13: Agent-based method: parameters H and L