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How two dimensional ants measure the curvature of their two dimensional universe

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Abstract

In this project, we will talk about two-dimensional ants detecting curvature in a two-dimensional universe. This is based on the mathematician Euclid's work. Firstly, we will cover the Euclidean distance, its historical background and Euclid's five postulates. The five postulates are mentioned within the section covering Euclid's geometry. Besides this, the section about Non-Euclidean geometry is linked to Euclid's geometry. An intuition of travelling from university to anyone's house will be explored. The theorem of $|x - y| = d(x, y)$ will also be proved.

In the following section, we will have a look at graphs of functions as matrices. Firstly, a function being bijective will be proved by using two random equations. We will have a look at two examples $f(x, y) = x^2 + y^2$ and $f(x, y) = \sin(x) + \cos(y)$ to illustrate the theory. This will be used as a template to consider general functions.

Finally, Riemannian Metrics on $\mathbb{R}^2$ will be looked at by considering definition of Riemannian metrics and tangent spaces. It is linked to curvature, where we will look at how a Riemannian curvature tensor can be used in real life. Christoffel symbols will be used to describe a metric connection.

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1 Two Dimensional Ants: An Overview

1.1 Background

1.1.1 Euclidean Distance

Euclidean Distance is the distance between any two points and it can be explored from the point of view of curves by using an intuitive description to show that it makes sense.

The general formula is

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

where

1. d - the Euclidean distance
2. (x_1, y_1) - the coordinates of the first point
3. (x_2, y_2) - the coordinates of the second point

Speed can be calculated by using the dot product as indicated in equation 2.

$$|\dot{\gamma}(t)| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t)} \tag{2}$$

The length of the total distance travelled is $\int (speed)dt$.

1.1.2 Euclid's Geometry

Around 325BC, Euclid, a greek mathematician used some known work and arranged them into elements, which were divided into thirteen chapters. These chapters were given the name 'book', which helped the whole world's understanding of geometry for the next generations [6].

Euclid based his geometry on five postulates as indicated below:

1. A straight line segment can be drawn joining any two points.

2. Any straight line segment can be extended indefinitely in a straight line.

3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as centre.

4. All right angles are congruent.

5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the parallel postulate [7].

However, the fifth postulate cannot be proved as a theorem so it is called the parallel postulate [7] as shown in Figure 1. This postulate doesn't apply on three-dimensional or curved geometries which is known as Non-Euclidean geometry [10]. From the diagram below, you will notice that there is a dash symbol next to the V. It is an alternative to V, which illustrates the fifth postulate.

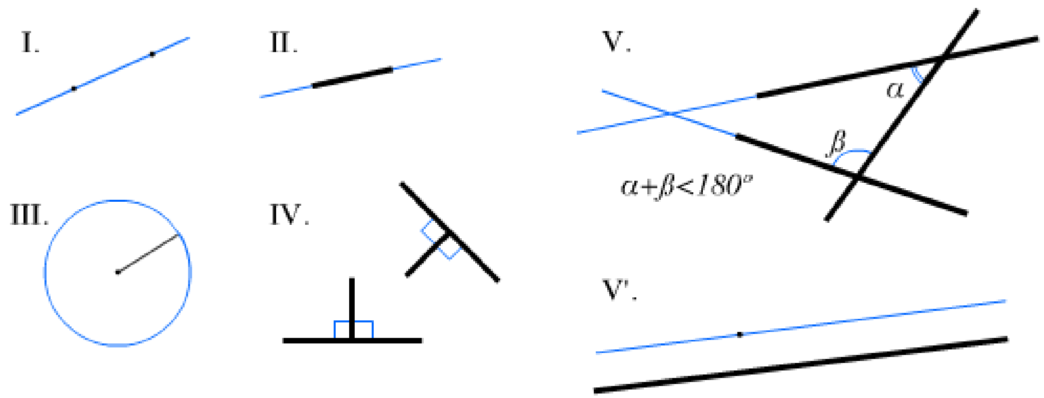

Figure 1: Euclid's Five Postulates

1.1.3 Non-Euclidean Geometry

This type of geometry is mainly focused on hyperbolic and spherical geometries. There are two historical threads that Euclid had developed such as understanding the movement of stars and planets in the sky that might be shaped like a hemisphere and the other one being associated with the fifth postulate (go to section 1.1.2) in Euclid's elements [3].

The main types of surfaces in Non-Euclidean geometry are conical, spherical, cylindrical and hyperbolic surfaces. A line does not intersect by itself in a Euclidean plane, but they can in other three-dimensional shapes such as a cone, a cylinder and a sphere [10].From Figure 2, the lines are intersecting in those shapes.

Circles are the straight lines of spherical geometry, which is 'a consequence of the properties of a sphere'. These kinds of curves are intrinsically straight which means that the curves have the same symmetry. As seen in Figure 3, there are three arcs that intersect with each other and a triangle is formed. In differential geometry, spherical geometry is known as 'the geometry of a surface with constant positive curvature'[3].

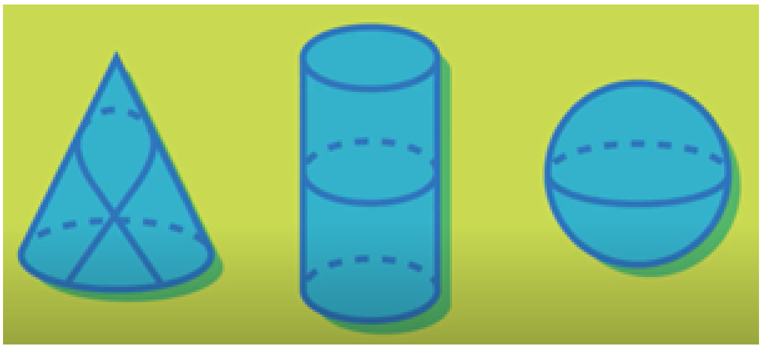

Figure 2: Non-Euclidean Shapes

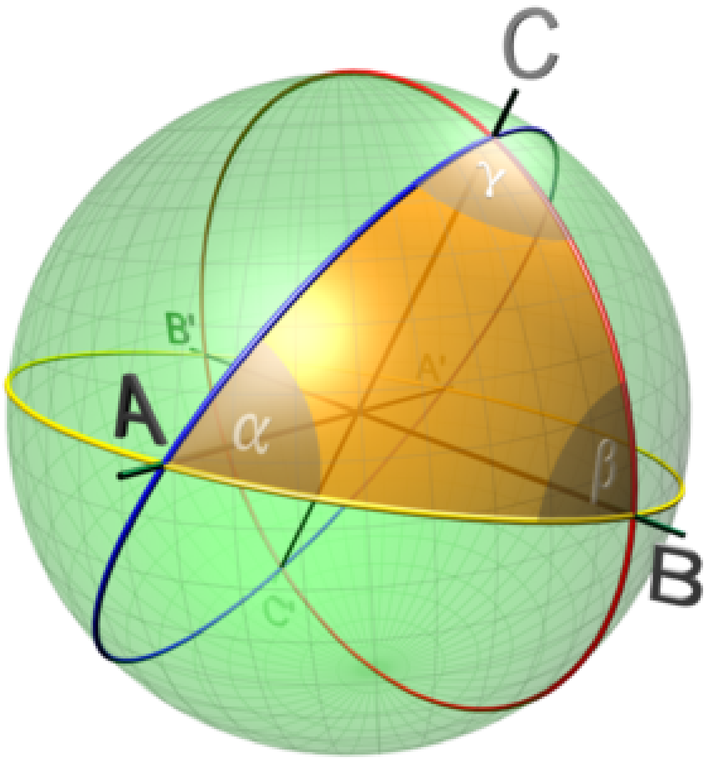

Figure 3: Spherical Geometry

Hyperbolic geometry has some involvement with the fifth postulate (go to section 1.1.2) and they are different by scale. In the mid-19th century, it was proved that hyperbolic surfaces have a constant negative curvature. However, some people were wondering whether a surface with hyperbolic geometry exists. From Figure 4, a triangle has been formed just like in Figure 3.

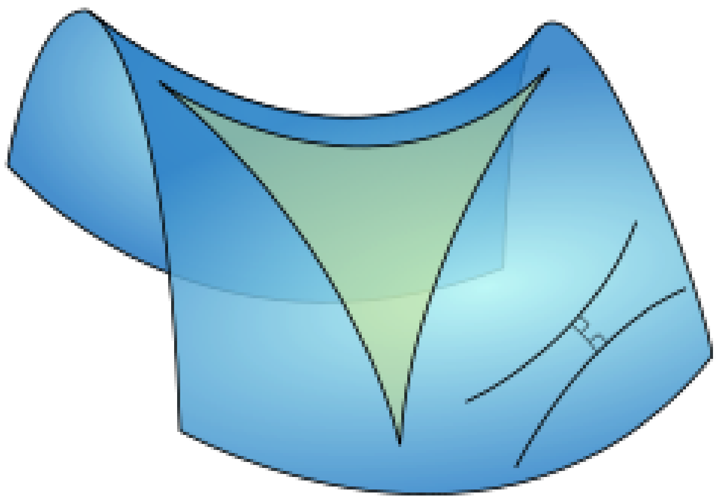

Figure 4: Hyperbolic Geometry

1.2 Intuition of Distance Travelled

For this, we are going to do an intuition of travelling from Brunel University to the author's house.

A person to go from Brunel University to get to the final destination:

1. From Brunel University, they go to the bus stop which is a 1 min walk.
2. Then they go on the bus and get off at a bus stop near Uxbridge station.
3. Now, they walk from the bus stop to Uxbridge station.
4. At Uxbridge station, they catch the train and get off at Eastcote station.
5. From Eastcote Station, it is an 18 min walk to get to the author's house.

The total length travelled from Brunel University to the house is 8.11(2 d.p)km, which is represented as the hypotenuse of the triangle on figure 5. On Google Maps, the 'Measure distance' function is used to measure the distance to get 7.72km. The quickest route to get to the house takes 40 minutes and we convert it to hours to get $\frac{2}{3}$. Now, we divide 8.11 by $\frac{2}{3}$ to get the speed of 11.58(2 d.p) km/hr.

Even though the long path is used to get back to the house, it is not the distance between Brunel and the house. Google Maps is used to show the path of the intuition.

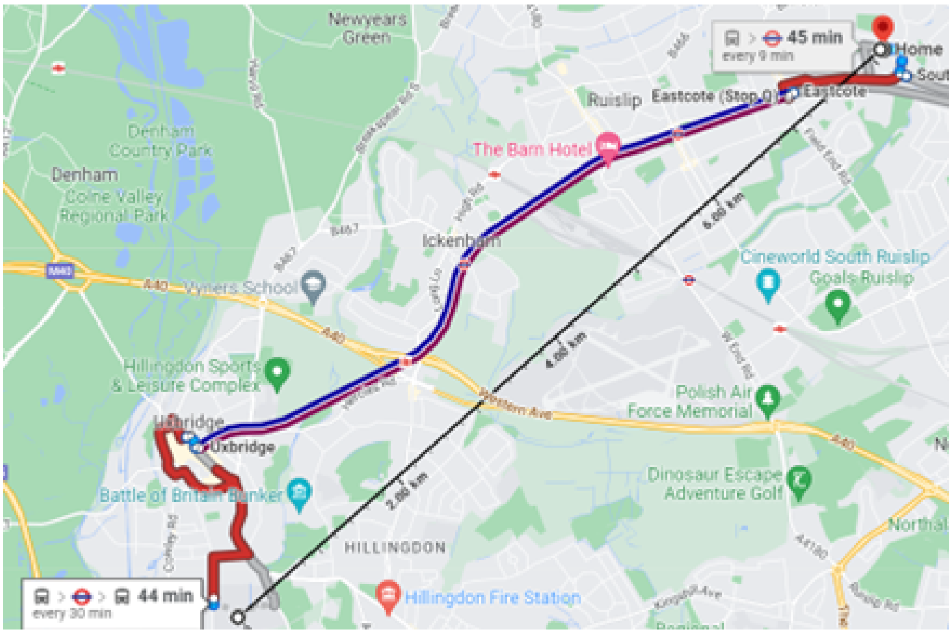

Figure 5: The distance of Brunel to home

The velocity is the speed of something in a given direction [4].

The length can be expressed as an integral of speed $\int v(t)dt$, where $v(t)$ is the speed over time.

1.3 Theorem of $|x - y| = d(x, y)$

If $x, y \in \mathbb{R}$, the distance is $|x - y|$ where $|x - y| = \sqrt{(x - y) \cdot (x - y)}$.

But intuitively, the distance between the two points $x, y \in \mathbb{R}$ should be the minimisation of the lengths of curves.

$C_{x,y} := \{\gamma \in C^1([0, 1], \mathbb{R}^2), \gamma(0) = x, \gamma(1) = y\}$ is the set of all C^1 curves between x and y. C^1 means that the functions are continuously differentiable.

We are trying to prove that $|x - y| = d(x, y)$ where

$$d(x, y) := inf\{len(\gamma) : \gamma \in \mathcal{C}_{x,y}\}.$$

$d(x, y)$ exists because if $len(\gamma) \geq 0$.

Theorem 1. $|x - y| = d(x, y)$

Proof. Let the function be $\sigma : [0, 1] \to \mathbb{R}^2$

Now there is a parametric equation, $\sigma(t) = (1-t)x + ty$.

The length of the function is

$$len(\sigma) = |x - y| \implies |x - y| \geq d(x, y)$$

We need to show that $len(\sigma) \geq |x - y|$ for curves γ.

$|x - y| = inf\{len(\gamma) : \gamma \in C^1([0, 1], \mathbb{R}^2), \gamma(0) = x, \gamma(1) = y\}$

$len(\sigma) = |x \cdot y| \to inf\{len(\gamma) : \gamma \in C^1([0, 1], \mathbb{R}^2), \gamma(0) = x, \gamma(1) = y\} \leq |x - y|$

This implies that $|x - y| \geq d(x, y)$.

Now it remains to show that

$$|x - y| \leq inf\{len(\gamma) : \gamma \in C^1([0, 1], \mathbb{R}^2), \gamma(0) = x, \gamma(1) = y\}$$

When $\dot{\gamma}(t)$ is a coordinate,

$$\int_0^1 \dot{\gamma}(t)dt = \int_0^1 (\dot{\gamma_1}(t), \dot{\gamma_2}(t))dt = \left(\int_0^1 \dot{\gamma_1}(t)dt, \int_0^1 \dot{\gamma_2}(t)dt\right).$$

Let $\gamma_i : [0, 1] \to \mathbb{R}$

Using the Fundamental Theorem of Calculus $\int_C df = f(u(b), v(b)) - f(u(a), v(a))$ (go to page 100 of [5]),

$\int_0^1 \dot{\gamma}_i(t)dt = \gamma_i(1) - \gamma_i(0)$.

$\int_0^1 \dot{\gamma}(t)dt = (\gamma_1(1) - \gamma_1(0), \gamma_2(1) - \gamma_2(0)) = \gamma(1) - \gamma(0) = y - x$

$y - x$ is going to be in modulus form.

For $|x - y| = |\int_0^1 \dot{\gamma}(t)dt| \leq \int_0^1 |\dot{\gamma}(t)|dt = len(\gamma)$, we obtain

$$|x - y| \leq inf\{len(\gamma) : \gamma(0) = x, \gamma(1) = y\},$$

and by linking this with $|x-y| \geq d(x,y)$, we get

$$|x-y| = inf\{len(\gamma) : \gamma(0) = x, \gamma(1) = y\}.$$

$\square$

2 Graphs of Functions as Matrices

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function.

Let $\Phi_f : \mathbb{R}^2 \to graph(f)$ and $\Phi_f(x, y) = (x, y, f(x, y))$.

2.1 Proof of Φ_f being bijective

$\Phi_f : \mathbb{R} \to graph(f)$ is bijective, where $\Phi_f(x, y) = (x, y, f(x, y))$ and $graph$ is represented as a graph for any function.

We need to show that:

1. $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $\Phi_f(x_1, y_1) = \Phi_f(x_2, y_2) \implies (x_1, y_1) = (x_2, y_2)$ is injective.
   Proof:
   Let $\Phi_f(x_1, y_1, f(x_1, y_1)) = \Phi_f(x_2, y_2, f(x_2, y_2))$ and using the definition of Φ_f's, we get
   $$(x_1, y_1, f(x_1, y_1)) = (x_2, y_2, f(x_2, y_2))$$
   This means that $x_1 = x_2$, $y_1 = y_2$ and $(x_1, y_1) = (x_2, y_2)$.
   $\therefore$ Φ_f is injective.

2. $\forall z \in graph(f)$, $\exists (x, y) \in \mathbb{R}^2$ such that $\Phi_f(x, y) = z$ is surjective.
   Let $z = (x, y, f(x, y)) \implies z = \Phi_f(x, y)$.
   If $z \longleftarrow graph(f) := \{(x, y, f(x, y)) : (x, y) \leftarrow \mathbb{R}^2\}$, it is equivalent to $z = (x, y, f(x, y)) = \Phi_f(x, y)$.
   $\therefore$ z is surjective.

It is now shown that $\Phi_f(x, y)$ is bijective, as it is both injective and surjective.

2.2 Examples

2.2.1 $f(x, y) = x^2 + y^2$

Here is one example of a function being a matrix, $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 + y^2$

The (x, y)-coordinate of the function is $(\gamma_1(t), \gamma_2(t))$ and it is substituted into equation and the coordinate of Φ_f is differentiated.

$$\dot{\sigma}(t) = \tfrac{d}{dt}(\gamma_1(t), \gamma_2(t), (\gamma_1(t))^2 + (\gamma_2(t))^2) = (\dot{\gamma_1}(t), \dot{\gamma_2}(t), 2\gamma_1(t)\dot{\gamma_1}(t) + 2\gamma_2(t)\dot{\gamma_2}(t))$$

The equation above is squared by doing a dot product of itself.

$$\begin{aligned}\dot{\sigma}(t)\cdot\dot{\sigma}(t) &= (\dot{\gamma_1}(t), \dot{\gamma_2}(t), 2\gamma_1(t)\dot{\gamma_1}(t) + 2\gamma_2(t)\dot{\gamma_2}(t)) \cdot (\dot{\gamma_1}(t), \dot{\gamma_2}(t), 2\gamma_1(t)\dot{\gamma_1}(t) + 2\gamma_2(t)\dot{\gamma_2}(t)) \\ &= \dot{\gamma_1}(t)^2 + \dot{\gamma_2}(t)^2 + (2\gamma_1(t)\dot{\gamma_1}(t) + 2\gamma_2(t)\dot{\gamma_2}(t))^2 = \dot{\gamma_1}(t)^2 + \dot{\gamma_2}(t)^2 + (2\gamma_1(t)\dot{\gamma_1}(t) + \\ &\quad 2\gamma_2(t)\dot{\gamma_2}(t))(2\gamma_1(t)\dot{\gamma_1}(t) + 2\gamma_2(t)\dot{\gamma_2}(t)) \\ &= \dot{\gamma_1}(t)^2 + \dot{\gamma_2}(t)^2 + 4\gamma_1(t)^2\dot{\gamma_1}(t)^2 + 4\gamma_1(t)\gamma_2(t)\dot{\gamma_1}(t)\dot{\gamma_2}(t) + 4\gamma_1(t)\gamma_2(t)\dot{\gamma_1}(t)\dot{\gamma_2}(t) + \\ &\quad 4\gamma_2(t)^2\dot{\gamma_2}(t)^2 \\ &= (1 + 4\gamma_1(t)^2)\dot{\gamma_1}(t)^2 + \dot{\gamma_1}(t)\dot{\gamma_2}(t)(4\gamma_1(t)\gamma_2(t) + 4\gamma_1(t)\gamma_2(t)) + (1 + 4\gamma_2(t)^2)\dot{\gamma_2}(t)^2\end{aligned}$$

We have the matrices $A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t)\end{pmatrix}$ and $\dot{\gamma}(t) = \begin{pmatrix}\dot{\gamma_1}(t) \\ \dot{\gamma_2}(t)\end{pmatrix}$ and they are multiplied.

$$A(t)\dot{\gamma}(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t)\end{pmatrix}\begin{pmatrix}\dot{\gamma_1}(t) \\ \dot{\gamma_2}(t)\end{pmatrix} = \begin{pmatrix} A_{11}(t)\dot{\gamma_1}(t) + A_{12}(t)\dot{\gamma_2}(t) \\ A_{21}(t)\dot{\gamma_1}(t) + A_{22}(t)\dot{\gamma_2}(t)\end{pmatrix}$$

$A(t)\dot{\gamma}(t)$ is multiplied by $\dot{\gamma}(t)$

$$\begin{aligned}\dot{\gamma}(t)^T \cdot A(t)\dot{\gamma}(t) = \begin{pmatrix}\dot{\gamma_1}(t) & \dot{\gamma_2}(t)\end{pmatrix} \cdot \begin{pmatrix} A_{11}(t)\dot{\gamma_1}(t) + A_{12}(t)\dot{\gamma_2}(t) \\ A_{21}(t)\dot{\gamma_1}(t) + A_{22}(t)\dot{\gamma_2}(t)\end{pmatrix} &= \dot{\gamma_1}(t)(A_{11}(t)\dot{\gamma_1}(t) + \\ &\quad A_{12}(t)\dot{\gamma_2}(t)) + \\ &\quad \dot{\gamma_2}(t)(A_{21}(t)\dot{\gamma_1}(t) + \\ &\quad A_{22}(t)\dot{\gamma_2}(t)) \\ &= A_{11}(t)\dot{\gamma_1}(t)^2 + \\ &\quad A_{12}(t)\dot{\gamma_1}(t)\dot{\gamma_2}(t) + \\ &\quad A_{21}(t)\dot{\gamma_1}(t)\dot{\gamma_2}(t) + \\ &\quad A_{22}(t)\dot{\gamma_2}(t)^2 \\ &= A_{11}(t)\dot{\gamma_1}(t)^2 + \\ &\quad \dot{\gamma_1}(t)\dot{\gamma_2}(t)(A_{12}(t) + \\ &\quad A_{21}(t)) + A_{22}(t)\dot{\gamma_2}(t)^2\end{aligned}$$

$\dot{\sigma}(t) \cdot \dot{\sigma}(t) = \dot{\gamma}(t)^T \cdot A(t)\dot{\gamma}(t)$

$$(1+4\gamma_1(t)^2)\dot{\gamma_1}(t)^2 + \dot{\gamma_1}(t)\dot{\gamma_2}(t)(4\gamma_1(t)\gamma_2(t) + 4\gamma_1(t)\gamma_2(t)) + (1+4\gamma_2(t)^2)\dot{\gamma_2}(t)^2 = A_{11}(t)\dot{\gamma_1}(t)^2 + \dot{\gamma_1}(t)\dot{\gamma_2}(t)(A_{12}(t) + A_{21}(t)) + A_{22}(t)\dot{\gamma_2}(t)^2$$

The elements are $A_{11}(t) = 1+4\gamma_1(t)^2$, $A_{12}(t) = A_{21}(t) = 4\gamma_1(t)\gamma_2(t)$ and $A_{22}(t) = 1+4\gamma_2(t)^2$.

$\therefore A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} = \begin{pmatrix} 1+4\gamma_1(t)^2 & 4\gamma_1(t)\gamma_2(t) \\ 4\gamma_1(t)\gamma_2(t) & 1+4\gamma_2(t)^2 \end{pmatrix}.$

2.2.2 $f(x, y) = \sin(x) + \cos(y)$

Here is another example, $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \sin(x) + \cos(y)$.

$(x, y) = (\gamma_1(t), \gamma_2(t))$ is going to be substituted into the equation in section 2.2.2 and it well get differentiated.

$\dot{\sigma}(t) = \frac{d}{dt}(\gamma_1(t), \gamma_2(t), \sin(\gamma_1(t)) + \cos(\gamma_2(t))) = (\dot{\gamma_1}(t), \dot{\gamma_2}(t), \cos(\gamma_1(t))\dot{\gamma_1}(t) - \sin(\gamma_2(t))\dot{\gamma_2}(t))$

$\dot{\sigma}(t)$ is going to be multiplied by a dot product of itself where each point of the coordinate will be multiplied by itself.

$$\begin{aligned}
\dot{\sigma}(t) \cdot \dot{\sigma}(t) &= (\dot{\gamma_1}(t), \dot{\gamma_2}(t), \cos(\gamma_1(t))\dot{\gamma_1}(t) - \sin(\gamma_2(t))\dot{\gamma_2}(t)) \cdot \\
&\qquad (\dot{\gamma_1}(t), \dot{\gamma_2}(t), \cos(\gamma_1(t))\dot{\gamma_1}(t) - \sin(\gamma_2(t))\dot{\gamma_2}(t)) \\
&= \dot{\gamma_1}(t)^2 + \dot{\gamma_2}(t)^2 + (\cos(\gamma_1(t))\dot{\gamma_1}(t) - \sin(\gamma_2(t))\dot{\gamma_2}(t))^2 \\
&= \dot{\gamma_1}(t)^2 + \dot{\gamma_2}(t)^2 + (\cos(\gamma_1(t))\dot{\gamma_1}(t) - \\
&\qquad \sin(\gamma_2(t))\dot{\gamma_2}(t))(\cos(\gamma_1(t))\dot{\gamma_1}(t) - \sin(\gamma_2(t))\dot{\gamma_2}(t)) \\
&= \dot{\gamma_1}(t)^2 + \dot{\gamma_2}(t)^2 + \cos(\gamma_1(t))^2\dot{\gamma_1}(t)^2 - \sin(\gamma_2(t))\cos(\gamma_1(t))\dot{\gamma_1}(t)\dot{\gamma_2}(t) \\
&\qquad - \sin(\gamma_2(t))\cos(\gamma_1(t))\dot{\gamma_1}(t)\dot{\gamma_2}(t) + \sin(\gamma_2(t))^2\dot{\gamma_2}(t)^2 \\
&= (1+\cos(\gamma_1(t))^2)\dot{\gamma_1}(t)^2 - \sin(\gamma_2(t))\cos(\gamma_1(t))\dot{\gamma_1}(t)\dot{\gamma_2}(t) - \\
&\qquad \sin(\gamma_2(t))\cos(\gamma_1(t))\dot{\gamma_1}(t)\dot{\gamma_2}(t) + (1+\sin(\gamma_2(t))^2)\dot{\gamma_2}(t)^2
\end{aligned}$$

$\dot{\gamma}(t)$ is going to be multiplied by $A(t)$ and then multiplied by the transpose of $\dot{\gamma}(t)$.

$$A(t)\dot{\gamma}(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} \begin{pmatrix} \dot{\gamma_1}(t) \\ \dot{\gamma_2}(t) \end{pmatrix} = \begin{pmatrix} A_{11}(t)\dot{\gamma_1}(t) + A_{12}(t)\dot{\gamma_2}(t) \\ A_{21}(t)\dot{\gamma_1}(t) + A_{22}(t)\dot{\gamma_2}(t) \end{pmatrix}$$

$$\begin{aligned}
\dot{\gamma}(t)^T \cdot A(t)\dot{\gamma}(t) &= \begin{pmatrix} \dot{\gamma_1}(t) & \dot{\gamma_2}(t) \end{pmatrix} \cdot \begin{pmatrix} A_{11}(t)\dot{\gamma_1}(t) + A_{12}(t)\dot{\gamma_2}(t) \\ A_{21}(t)\dot{\gamma_1}(t) + A_{22}(t)\dot{\gamma_2}(t) \end{pmatrix} \\
&= \dot{\gamma_1}(t)(A_{11}(t)\dot{\gamma_1}(t) + A_{12}(t)\dot{\gamma_2}(t)) + \dot{\gamma_2}(t)(A_{21}(t)\dot{\gamma_1}(t) + A_{22}(t)\dot{\gamma_2}(t)) \\
&= A_{11}(t)\dot{\gamma_1}(t)^2 + A_{12}(t)\dot{\gamma_1}(t)\dot{\gamma_2}(t) + A_{21}(t)\dot{\gamma_1}(t)\dot{\gamma_2}(t) + A_{22}(t)\dot{\gamma_2}(t)^2 \\
&= A_{11}(t)\dot{\gamma_1}(t)^2 + \dot{\gamma_1}(t)\dot{\gamma_2}(t)(A_{12}(t) + A_{21}(t)) + A_{22}(t)\dot{\gamma_2}(t)^2
\end{aligned}$$

Let $\dot{\sigma}(t) \cdot \dot{\sigma}(t) = \dot{\gamma}(t)^T \cdot A(t)\dot{\gamma}(t)$

$$\begin{aligned}
&(1 + \cos(\gamma_1(t))^2)\dot{\gamma_1}(t)^2 - \sin(\gamma_2(t))\cos(\gamma_1(t))\dot{\gamma_1}(t)\dot{\gamma_2}(t) \\
&\sin(\gamma_2(t))\cos(\gamma_1(t))\dot{\gamma_1}(t)\dot{\gamma_2}(t) + (1 + \sin(\gamma_2(t))^2)\dot{\gamma_2}(t)^2 = A_{11}(t)\dot{\gamma_1}(t)^2 + \dot{\gamma_1}(t)\dot{\gamma_2}(t)(A_{12}(t) \\
&\qquad + A_{21}(t)) + A_{22}(t)\dot{\gamma_2}(t)^2
\end{aligned}$$

The elements are $A_{11}(t) = 1 + (\cos \circ \gamma_1)^2$, $A_{12}(t) = A_{21}(t) = -(\cos \circ \gamma_1)(\sin \circ \gamma_2)$ and $A_{22}(t) = 1 + (\sin \circ \gamma_2)^2$.

$$\therefore A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} = \begin{pmatrix} 1 + (\cos \circ \gamma_1)^2 & -(\cos \circ \gamma_1)(\sin \circ \gamma_2) \\ -(\cos \circ \gamma_1)(\sin \circ \gamma_2) & 1 + (\sin \circ \gamma_2)^2 \end{pmatrix}.$$

2.3 General function Φ_f

Let the function Φ_f be $f : \mathbb{R}^2 \to \mathbb{R}$, $\Phi_f = (x, y, f(x, y))$.

$\sigma : [0, 1] \to \mathbb{R}^2$, $\sigma(t) = (\sigma_1(t), \sigma_2(t))$

$\sigma(t) = (\sigma_1(t), \sigma_2(t))$ is substituted into Φ_f to get $\Phi_f(\sigma(t)) = (\sigma_1(t), \sigma_2(t), f(\sigma_1(t), \sigma_2(t)))$.

Now differentiate $\Phi_f(\sigma(t))$ to get $(\Phi \circ \sigma)'(t)$ and do the dot product squared of the function.

$(\Phi_f \circ \sigma)'(t) = (\dot{\sigma_1}(t), \dot{\sigma_2}(t), \dot{\sigma_1}(t) \cdot f_{\sigma_1(t)}(\sigma_1(t), \sigma_2(t)) + \dot{\sigma_2}(t) \cdot f_{\sigma_2(t)}(\sigma_1(t), \sigma_2(t)))$

The function is expanded.

$$
\begin{aligned}
(\Phi_f \circ \sigma)'(t) \cdot (\Phi_f \circ \sigma)'(t) &= (\dot{\sigma_1}(t), \dot{\sigma_2}(t), \dot{\sigma_1}(t) \cdot f_{\sigma_1(t)}(\sigma(t)) + \dot{\sigma_2}(t) \cdot f_{\sigma_2(t)}(\sigma(t))) \cdot (\dot{\sigma_1}(t), \\
&\quad \dot{\sigma_2}(t), \dot{\sigma_1}(t) \cdot f_{\sigma_1(t)}(\sigma(t)) + \dot{\sigma_2}(t) \cdot f_{\sigma_2(t)}(\sigma(t))) \\
&= \dot{\sigma_1}(t)^2 + \dot{\sigma_2}(t)^2 + (\dot{\sigma_1}(t) \cdot f_{\sigma_1(t)}(\sigma(t)) + \dot{\sigma_2}(t) \cdot f_{\sigma_2(t)}(\sigma(t)))^2 \\
&= \dot{\sigma_1}(t)^2 + \dot{\sigma_2}(t)^2 + (\dot{\sigma_1}(t) \cdot f_{\sigma_1(t)}(\sigma(t)) + \dot{\sigma_2}(t) \cdot f_{\sigma_2(t)}(\sigma(t))) \\
&\quad (\dot{\sigma_1}(t) \cdot f_{\sigma_1(t)}(\sigma(t)) + \dot{\sigma_2}(t) \cdot f_{\sigma_2(t)}(\sigma(t))) \\
&= \dot{\sigma_1}(t)^2 + \dot{\sigma_2}(t)^2 + \dot{\sigma_1}(t)^2 \cdot f_{\sigma_1(t)}(\sigma(t))^2 + \\
&\quad \dot{\sigma_1}(t)\dot{\sigma_2}(t) f_{\sigma_1(t)}(\sigma(t)) f_{\sigma_2(t)}(\sigma(t)) + \\
&\quad \dot{\sigma_1}(t)\dot{\sigma_2}(t) f_{\sigma_1(t)}(\sigma(t)) f_{\sigma_2(t)}(\sigma(t)) + \dot{\sigma_2}(t)^2 \cdot f_{\sigma_2(t)}(\sigma(t))^2 \\
&= (1 + f_{\sigma_1(t)}(\sigma(t))^2)\dot{\sigma_1}(t)^2 + (1 + f_{\sigma_2(t)}(\sigma(t))^2)\dot{\sigma_2}(t)^2 + \dot{\sigma_1}(t)\dot{\sigma_2}(t) \\
&\quad f_{\sigma_1(t)}(\sigma(t)) f_{\sigma_2(t)}(\sigma(t)) + \dot{\sigma_1}(t)\dot{\sigma_2}(t) f_{\sigma_1(t)}(\sigma(t)) f_{\sigma_2(t)}(\sigma(t))
\end{aligned}
$$

Here, we have matrices $A_f(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}$ and $\dot{\sigma}(t) = \begin{pmatrix} \dot{\sigma_1}(t) \\ \dot{\sigma_2}(t) \end{pmatrix}$.

Now $A_f(t)$ is multiplied by $\dot{\sigma}(t)$ to get

$$
A_f(t)\dot{\sigma}(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} \begin{pmatrix} \dot{\sigma_1}(t) \\ \dot{\sigma_2}(t) \end{pmatrix} = \begin{pmatrix} A_{11}(t)\dot{\sigma_1}(t) + A_{12}(t)\dot{\sigma_2}(t) \\ A_{21}(t)\dot{\sigma_1}(t) + A_{22}(t)\dot{\sigma_2}(t) \end{pmatrix}
$$

$A_f(t)\dot{\sigma}(t)$ is multiplied by the transpose of $\dot{\sigma}(t)$ to get

$$
\begin{aligned}
\dot{\sigma}(t)^T A_f(t)\dot{\sigma}(t) &= \begin{pmatrix} \dot{\sigma_1}(t) & \dot{\sigma_2}(t) \end{pmatrix} \begin{pmatrix} A_{11}(t)\dot{\sigma_1}(t) + A_{12}(t)\dot{\sigma_2}(t) \\ A_{21}(t)\dot{\sigma_1}(t) + A_{22}(t)\dot{\sigma_2}(t) \end{pmatrix} \\
&= \dot{\sigma_1}(t)(A_{11}(t)\dot{\sigma_1}(t) + A_{12}(t)\dot{\sigma_2}(t)) + \\
&\quad \dot{\sigma_2}(t)(A_{21}(t)\dot{\sigma_1}(t) + A_{22}(t)\dot{\sigma_2}(t)) \\
&= A_{11}(t)\dot{\sigma_1}(t)^2 + A_{12}(t)\dot{\sigma_1}(t)\dot{\sigma_2}(t) + \\
&\quad A_{21}(t)\dot{\sigma_1}(t)\dot{\sigma_2}(t) + A_{22}(t)\dot{\sigma_2}(t)^2
\end{aligned}
$$

Now let $(\Phi_f \circ \sigma)'(t) \cdot (\Phi_f \circ \sigma)'(t) = \dot{\sigma}(t)^T A_f(t)\dot{\sigma}(t)$

$$
\begin{aligned}
(1 + f_{\sigma_1(t)}(\sigma(t))^2)\dot{\sigma_1}(t)^2 + (1 + f_{\sigma_2(t)}(\sigma(t))^2)\dot{\sigma_2}(t)^2 &+ \dot{\sigma_1}(t)\dot{\sigma_2}(t) f_{\sigma_1(t)}(\sigma(t)) f_{\sigma_2(t)}(\sigma(t)) + \\
&\quad \dot{\sigma_1}(t)\dot{\sigma_2}(t) f_{\sigma_1(t)}(\sigma(t)) f_{\sigma_2(t)}(\sigma(t)) \\
&= A_{11}(t)\dot{\sigma_1}(t)^2 + A_{12}(t)\dot{\sigma_1}(t)\dot{\sigma_2}(t) + \\
&\quad A_{21}(t)\dot{\sigma_1}(t)\dot{\sigma_2}(t) + A_{22}(t)\dot{\sigma_2}(t)^2
\end{aligned}
$$

The elements of $A_f(t)$ are $A_{11}(t) = (1 + f_{\sigma_1(t)}(\sigma(t)))^2$, $A_{22}(t) = (1 + f_{\sigma_2(t)}(\sigma(t)))^2$ and $A_{12}(t) = A_{21}(t) = f_{\sigma_1(t)}(\sigma(t))f_{\sigma_2(t)}(\sigma(t))$.

$\therefore A_f(t) = \begin{pmatrix} 1 + f_{\sigma_1(t)}(\sigma(t))^2 & f_{\sigma_1(t)}(\sigma(t))f_{\sigma_2(t)}(\sigma(t)) \\ f_{\sigma_1(t)}(\sigma(t))f_{\sigma_2(t)}(\sigma(t)) & 1 + f_{\sigma_2(t)}(\sigma(t))^2 \end{pmatrix}$.

Let the function $B : \mathbb{R}^2 \to \mathbb{R}$, $B_f(x, y) = \begin{pmatrix} 1 + (\delta_x f)^2(x, y) & (\delta_x f)(\delta_y f)(x, y) \\ (\delta_x f)(\delta_y f)(x, y) & 1 + (\delta_y f)^2(x, y) \end{pmatrix}$

$\dot{\sigma}(t) = (\dot{\sigma_1}(t), \dot{\sigma_2}(t))$ is substituted into $B_f(x, y)$ to get

$$(B_f \circ \sigma)\dot{\sigma}(t) = \begin{pmatrix} 1 + (\delta_x f)^2(\dot{\sigma}(t)) & (\delta_x f)(\delta_y f)(\dot{\sigma}(t)) \\ (\delta_x f)(\delta_y f)(\dot{\sigma}(t)) & 1 + (\delta_y f)^2(\dot{\sigma}(t)) \end{pmatrix}$$

.

Let $A_f(t) = (B_f \circ \sigma)\dot{\sigma}(t)$.

$$\begin{pmatrix} 1 + f_{\sigma_1(t)}(\sigma(t))^2 & f_{\sigma_1(t)}(\sigma(t))f_{\sigma_2(t)}(\sigma(t)) \\ f_{\sigma_1(t)}(\sigma(t))f_{\sigma_2(t)}(\sigma(t)) & 1 + f_{\sigma_2(t)}(\sigma(t))^2 \end{pmatrix} = \begin{pmatrix} 1 + (\delta_x f)^2(\dot{\sigma}(t)) & (\delta_x f)(\delta_y f)(\dot{\sigma}(t)) \\ (\delta_x f)(\delta_y f)(\dot{\sigma}(t)) & 1 + (\delta_y f)^2(\dot{\sigma}(t)) \end{pmatrix}$$

Given any $\sigma : I \to \mathbb{R}^2$

Let $(\Phi_f \circ \sigma)'(t) \cdot (\Phi_f \circ \sigma)'(t) = \dot{\sigma}(t) \cdot (B_f \circ \sigma)(t)\dot{\sigma}(t)$

$$\begin{aligned} (B_f \circ \sigma)(t)\dot{\sigma}(t) &= \begin{pmatrix} 1 + (\delta x f)^2(x, y) & (\delta_x f)(\delta_y f)(x, y) \\ (\delta_x f)(\delta_y f)(x, y) & 1 + (\delta_y f)^2(x, y) \end{pmatrix} \begin{pmatrix} \dot{\sigma_1}(t) \\ \dot{\sigma_2}(t) \end{pmatrix} \\ &= \begin{pmatrix} \dot{\sigma_1}(t)(1 + (\delta_x f)^2(x, y)) + \\ \quad \dot{\sigma_2}(t)((\delta_x f)(\delta_y f)(x, y)) \\ \dot{\sigma_1}(t)((\delta_x f)(\delta_y f)(x, y)) + \\ \quad \dot{\sigma_2}(t)(1 + (\delta_y f)^2(x, y)) \end{pmatrix} \end{aligned}$$

$\sigma(t)$ is differentiated to get $\dot{\sigma}(t)$ and multiplied by $(B_f \circ \sigma)(t)\dot{\sigma}(t)$.

$$\begin{aligned}
\dot{\sigma}(t)^T \cdot (B_f \circ \sigma)(t)\dot{\sigma}(t) &= \begin{pmatrix}\dot{\sigma_1}(t) & \dot{\sigma_2}(t)\end{pmatrix} \begin{pmatrix} \dot{\sigma_1}(t)(1+(\delta_x f)^2(x,y))+ \\ \quad \dot{\sigma_2}(t)((\delta_x f)(\delta_y f)(x,y)) \\ \dot{\sigma_1}(t)((\delta_x f)(\delta_y f)(x,y))+ \\ \quad \dot{\sigma_2}(t)(1+(\delta_y f)^2(x,y)) \end{pmatrix} \\
&= \dot{\sigma_1}(t)(\dot{\sigma_1}(t)(1+(\delta_x f)^2(x,y)) + \dot{\sigma_2}(t)((\delta_x f)(\delta_y f)(x,y)))+ \\
&\quad \dot{\sigma_2}(t)(\dot{\sigma_1}(t)((\delta_x f)(\delta_y f)(x,y)) + \dot{\sigma_2}(t)(1+(\delta_y f)^2(x,y))) \\
&= \dot{\sigma_1}(t)(\dot{\sigma_1}(t) + \dot{\sigma_1}(t)(\delta_x f)^2(x,y) + \dot{\sigma_2}(t)(\delta x f)(\delta_y f)(x,y))+ \\
&\quad \dot{\sigma_2}(t)(\dot{\sigma_1}(t)(\delta_x f)(\delta_y f)(x,y) + \dot{\sigma_2}(t) + \dot{\sigma_2}(t))(\delta_y f)^2(x,y)) \\
&= \dot{\sigma_1}(t)^2 + \dot{\sigma_1}(t)^2(\delta_x f)^2(x,y) + \dot{\sigma_1}(t)\dot{\sigma_2}(t)(\delta_x f)(\delta_y f)(x,y)+ \\
&\quad \dot{\sigma_1}(t)\dot{\sigma_2}(t)(\delta_x f)(\delta_y f)(x,y) + \dot{\sigma_2}(t)^2 + \dot{\sigma_2}(t)^2(\delta_y f)^2(x,y) \\
&= \dot{\sigma_1}(t)^2(1+(\delta_x f)^2(x,y)) + \dot{\sigma_1}(t)\dot{\sigma_2}(t)((\delta_x f)(\delta_y f)(x,y)+ \\
&\quad (\delta_x f)(\delta_y f)(x,y)) + \dot{\sigma_2}(t)^2(1+(\delta_y f)^2(x,y))
\end{aligned}$$

Let $(\Phi_f \circ \sigma)(t) \cdot (\Phi_f \circ \sigma)(t) = \dot{\sigma}(t)^T \cdot (B_f \circ \sigma)(t)\dot{\sigma}(t)$

$$\begin{aligned}
(1+f_{\sigma_1(t)}(\sigma(t))^2)\dot{\sigma_1}(t)^2 + (1+f_{\sigma_2(t)}(\sigma(t))^2)\dot{\sigma_2}(t)^2 + \dot{\sigma_1}(t)\dot{\sigma_2}(t)f_{\sigma_1(t)}(\sigma(t))f_{\sigma_2(t)}(\sigma(t)) \\
+ \dot{\sigma_1}(t)\dot{\sigma_2}(t)f_{\sigma_1(t)}(\sigma(t))f_{\sigma_2(t)}(\sigma(t)) \\
= \dot{\sigma_1}(t)^2(1+(\delta_x f)^2(x,y)) + \dot{\sigma_1}(t)\dot{\sigma_2}(t)((\delta_x f)(\delta_y f)(x,y)+ \\
(\delta_x f)(\delta_y f)(x,y)) + \dot{\sigma_2}(t)^2(1+(\delta_y f)^2(x,y))
\end{aligned}$$

The properties of a $B_f(x,y)$ matrix:

1. To find the determinant of $B_f(x,y)$,

$$\begin{aligned}
det(B_f(x,y)) &= det\left(\begin{pmatrix} 1+(\delta_x f)^2(x,y) & (\delta_x f)(\delta_y f)(x,y) \\ (\delta_x f)(\delta_y f)(x,y) & 1+(\delta_y f)^2(x,y) \end{pmatrix}\right) \\
&= (1+(\delta_x f)^2(x,y))(1+(\delta_y f)^2(x,y))- \\
&\quad ((\delta_x f)(\delta_y f)(x,y))((\delta_x f)(\delta_y f)(x,y)) \\
&= 1+(\delta_y f)^2(x,y) + (\delta_x f)^2(x,y) + (\delta_x f)^2(x,y)(\delta_y f)^2(x,y)- \\
&\quad (\delta_x f)^2(x,y)(\delta_y f)^2(x,y) \\
&= 1+(\delta_x f)^2(x,y) + (\delta_y f)^2(x,y)
\end{aligned}$$

$\therefore det(B_f(x,y)) > 0.$

2. To find the transpose of $B_f(x, y)$,

$$
\begin{aligned}
B_f(x,y)^T &= \begin{pmatrix} 1 + (\delta_x f)^2(x,y) & (\delta_x f)(\delta_y f)(x,y) \\ (\delta_x f)(\delta_y f)(x,y) & 1 + (\delta_y f)^2(x,y) \end{pmatrix}^T \\
&= \begin{pmatrix} 1 + (\delta_x f)^2(x,y) & (\delta_x f)(\delta_y f)(x,y) \\ (\delta_x f)(\delta_y f)(x,y) & 1 + (\delta_y f)^2(x,y) \end{pmatrix}
\end{aligned}
$$

$\therefore B_f(x, y)$ is symmetric.

3 Riemannian Metrics on $\mathbb{R}^2$

3.1 Definition

A Riemannian Metric $B(x, y)$ on a smooth manifold M is a smoothly chosen inner product on each of the tangent spaces [1].

$B : \mathbb{R}^2 \to Matrices(\mathbb{R}^2)$, i.e. for $(x, y) \leftarrow \mathbb{R}^2$, $B(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ matrix.

$(x, y) \mapsto B(x, y)$ is smooth such that $\forall (x, y) \in \mathbb{R}^2$,

1. the determinant of $B(x, y)$ is greater than 0, which is positively definite.
2. $B(x, y) = B(x, y)^T$ which is symmetric.

The vectors with a base point at (x, y) are $(x, y) \in \mathbb{R}^2$, $u, v \in T_{(x,y)}\mathbb{R}^2$ where $\{((x, y), u) : u \in \mathbb{R}^2\}$.

$g_B(x, y)[u, v] := u \cdot B(x, y)v$

g_B is the Riemannian metric corresponding to B.

3.2 Tangent space

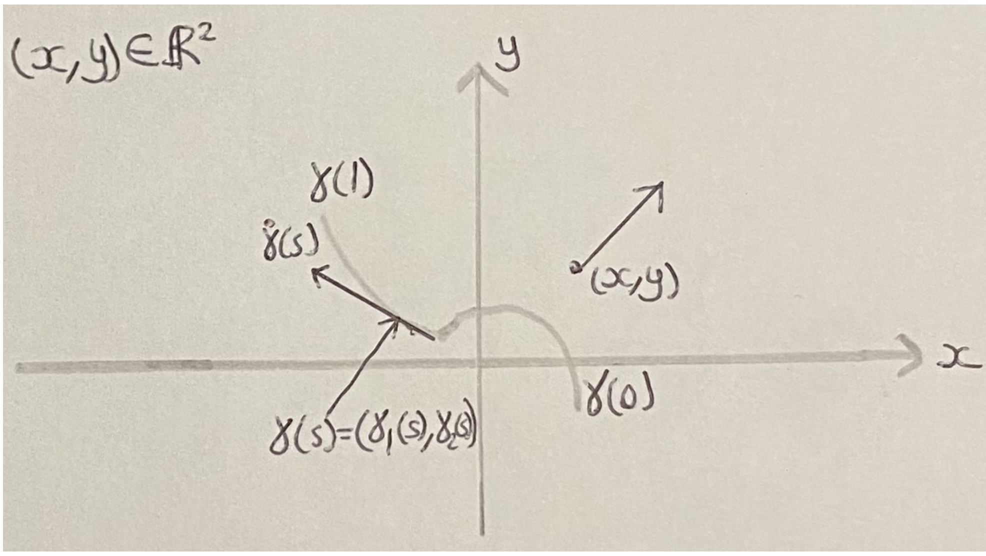

Figure 6: The graph of $\dot{\gamma}(s)$

$T_{(x,y)}\mathbb{R}^2$ is a tangent space at (x, y), the set of all 2-vectors with a base point (x, y).

$T_{(x,y)}\mathbb{R}^2 = \{((x, y), u) : u \in \mathbb{R}^2\}$ where (x, y) is the base point of U.

A tangent bundle is collection of tangent vectors with points that are tangent and a type of case for a vector bundle that has a bundle rank [9]. A vector bundle is a a group of vector spaces that get parametrized by another space to every x of the vector space [2].

In a tangent bundle, $T\mathbb{R}^2 = \bigcup\limits_{(x,y)\in\mathbb{R}^2} T_{(x,y)}\mathbb{R}^2$.

$u \in T_{(x,y)}\mathbb{R}^2$ is another way of representing $((x, y), u)$.

On Figure 6, $\dot{\gamma}(s) \in T_{\gamma(s)}\mathbb{R}^2$. $\gamma(0)$ has a higher y-value than $\gamma(1)$. This shows that there is some steepness between the points $\gamma(0)$ and $\gamma(1)$.

As we have seen, $len_B(\gamma) = \int_0^1 \sqrt{g_B(\gamma_t)[\dot{\gamma}(t), \dot{\gamma}(t)]}\, dt$.

4 Curvature

4.1 Riemannian curvature tensor

A Riemannian curvature tensor is a four-index tensor(a generalisation of a matrix) that is useful in general relativity and it has other tensors such as the Ricci curvature tensor and scalar curvature and the only tensor that is constructed from the metric tensor [8].

Christoffel symbols are needed to compute a Riemannian curvature tensor. Ants in our 2-D universe would measure curvature via Riemannian curvature tensor.

4.2 Christoffel symbols

4.2.1 Formula for Christoffel symbols

By using this formula,

$$\begin{aligned}\Gamma^i_{kl} &= \frac{1}{2}\sum_{m=1}^{2} g^{im}\left(\partial_l g_{ml} + \partial_k g_{ml} - \partial_m g_{kl}\right)\\ &= \frac{1}{2}g^{i1}\left(\partial_l g_{1l} + \partial_k g_{1l} - \partial_1 g_{kl}\right) + \frac{1}{2}g^{i2}\left(\partial_l g_{2l} + \partial_k g_{2l} - \partial_2 g_{kl}\right),\end{aligned}$$

we can find the Christoffel symbols of the second kind for two functions that are going to be covered in this section.

g^{im} is the inverse of the matrix and the g's are the coefficients in the matrix.

∂_1 is the differential of a function with respect to x.

∂_2 is the differential of a function with respect to y.

Let $A = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$

The inverse of the coefficients of matrix A are $g^{11} = \frac{1}{det(A)}g_{22}$, $g^{22} = \frac{1}{det(A)}g_{11}$, $g^{12} = \frac{1}{det(A)}g_{21}$ and $g^{21} = \frac{1}{det(A)}g_{12}$.

4.2.2 $1 + x^2$

Using the matrix $A = \begin{pmatrix} 1+x^2 & 0 \\ 0 & 1 \end{pmatrix}$,

To find A^{-1},

$$A^{-1} = \begin{pmatrix} 1+x^2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{1+x^2}\begin{pmatrix} 1 & 0 \\ 0 & 1+x^2 \end{pmatrix}$$

The differentials of this equation are:

1. $\partial_1 = \frac{\partial}{\partial x}\left(1+x^2\right) = 2x$
2. $\partial_2 = \frac{\partial}{\partial y}\left(1+x^2\right) = 0$

The inverse of the coefficients of matrix A are

1. $g^{11} = \frac{1}{det(A)} g_{22} = \frac{1}{1+x^2} \cdot 1 = \frac{1}{1+x^2}$
2. $g^{22} = \frac{1}{det(A)} g_{11} = \frac{1}{1+x^2} \cdot (1+x^2) = \frac{1+x^2}{1+x^2} = 1$
3. $g^{12} = \frac{1}{det(A)} g_{21} = \frac{1}{1+x^2} \cdot 0 = 0$
4. $g^{21} = \frac{1}{det(A)} g_{12} = \frac{1}{1+x^2} \cdot 0 = 0$

When $i = 1$, $k = 1$ and $l = 1$,

$$\begin{aligned}\Gamma^1_{11} &= \frac{1}{2} g^{11}\left(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}\right) + \frac{1}{2} g^{12}\left(\partial_1 g_{21} + \partial_1 g_{21} - \partial_2 g_{11}\right) \\ &= \frac{1}{2}\left(\frac{1}{1+x^2}(2x + 2x - 2x)\right) + \frac{1}{2}(0\,(0+0-0)) = \frac{1}{2}\left(\frac{1}{1+x^2}(2x)\right) + \frac{1}{2}(0\,(0)) \\ &= \frac{1}{2}\left(\frac{2x}{1+x^2}\right) = \frac{x}{1+x^2}\end{aligned}$$

When $i = 1$, $k = 1$ and $l = 2$,

$$\begin{aligned}\Gamma^1_{21} &= \frac{1}{2}g^{11}\left(\partial_2 g_{12} + \partial_1 g_{12} - \partial_1 g_{12}\right) + \frac{1}{2}g^{12}\left(\partial_1 g_{22} + \partial_1 g_{22} - \partial_2 g_{12}\right) \\ &= \frac{1}{2}\left(\frac{1}{1+x^2}(0+0-0)\right) + \frac{1}{2}(0(0+0-0)) \\ &= \frac{1}{2}\left(\frac{1}{1+x^2}(0)\right) + \frac{1}{2}(0(0)) = \frac{1}{2}(0) + \frac{1}{2}(0) = 0\end{aligned}$$

When $i = 1$, $k = 2$ and $l = 1$,

$$\begin{aligned}\Gamma^1_{12} &= \frac{1}{2}g^{11}\left(\partial_1 g_{12} + \partial_2 g_{12} - \partial_1 g_{12}\right) + \frac{1}{2}g^{12}\left(\partial_1 g_{22} + \partial_2 g_{22} - \partial_2 g_{12}\right) \\ &= \frac{1}{2}\left(\frac{1}{1+x^2}(0+0-0)\right) + \frac{1}{2}(0(0+0-0)) \\ &= \frac{1}{2}\left(\frac{1}{1+x^2}(0)\right) + \frac{1}{2}(0) = \frac{1}{2}(0) + \frac{1}{2}(0) = 0\end{aligned}$$

When $i = 1$, $k = 2$ and $l = 2$,

$$\begin{aligned}\Gamma^1_{22} &= \frac{1}{2}g^{11}\left(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}\right) + \frac{1}{2}g^{12}\left(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}\right) \\ &= \frac{1}{2}\left(\frac{1}{1+x^2}(0+0-0)\right) + \frac{1}{2}(0(0+0-0)) \\ &= \frac{1}{2}\left(\frac{1}{1+x^2}(0)\right) + \frac{1}{2}(0) = \frac{1}{2}(0) + \frac{1}{2}(0) = 0\end{aligned}$$

When $i = 2$, $k = 1$ and $l = 1$,

$$\begin{aligned}\Gamma^2_{11} &= \frac{1}{2}g^{21}\left(\partial_1 g_{11} + \partial_1 g_{12} - \partial_1 g_{11}\right) + \frac{1}{2}g^{22}\left(\partial_1 g_{21} + \partial_1 g_{21} - \partial_2 g_{11}\right) \\ &= \frac{1}{2}(0(2x+0-2x)) + \frac{1}{2}(1(0+0-0)) \\ &= \frac{1}{2}(0(0)) + \frac{1}{2}(1(0)) = 0\end{aligned}$$

When $i = 2$, $k = 1$ and $l = 2$,

$$\begin{aligned}\Gamma^2_{12} &= \frac{1}{2}g^{21}\left(\partial_2 g_{12} + \partial_1 g_{12} - \partial_1 g_{12}\right) + \frac{1}{2}g^{22}\left(\partial_2 g_{22} + \partial_1 g_{22} - \partial_2 g_{12}\right) \\ &= \frac{1}{2}(0(0+0-0)) + \frac{1}{2}(1(0+0-0)) \\ &= \frac{1}{2}(0(0)) + \frac{1}{2}(1(0)) = 0\end{aligned}$$

When $i = 2$, $k = 2$ and $l = 1$,

$$\begin{aligned}
\Gamma^2_{21} &= \frac{1}{2} g^{21} \left(\partial_1 g_{11} + \partial_2 g_{11} - \partial_1 g_{21} \right) + \frac{1}{2} g^{22} \left(\partial_1 g_{21} + \partial_2 g_{21} - \partial_2 g_{21} \right) \\
&= \frac{1}{2}(0(2x + 0 - 0)) + \frac{1}{2}(1(0 + 0 - 0)) \\
&= \frac{1}{2}(0) + \frac{1}{2}(0) = 0
\end{aligned}$$

When $i = 2$, $k = 2$ and $l = 2$,

$$\begin{aligned}
\Gamma^2_{22} &= \frac{1}{2} g^{21} \left(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22} \right) + \frac{1}{2} g^{22} \left(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22} \right) \\
&= \frac{1}{2}(0(0 + 0 - 1)) + \frac{1}{2}(1(0 + 0 - 0)) \\
&= \frac{1}{2}(0(-1)) + \frac{1}{2}(1(0)) = \frac{1}{2}(0) + \frac{1}{2}(0) = 0
\end{aligned}$$

For all Γ's, the values are 0 except for Γ^1_{11} which is $\frac{x}{1+x^2}$.

4.2.3 $\sin(x) + \cos(y)$

Using the matrix, $A = \begin{pmatrix} 1 + \cos^2(x) & -\cos(x)\sin(y) \\ -\cos(x)\sin(y) & 1 + \sin^2(y) \end{pmatrix}$

To find A^{-1},

$$\begin{aligned}
A^{-1} &= \begin{pmatrix} 1+\cos^2(x) & -\cos(x)\sin(y) \\ -\cos(x)\sin(y) & 1+\sin^2(y) \end{pmatrix}^{-1} \\
&= \frac{1}{(1+\cos^2(x))(1+\sin^2(y))-(-\cos(x)\sin(y))(-\cos(x)\sin(y))} \\
&\quad \begin{pmatrix} 1+\sin^2(y) & \cos(x)\sin(y) \\ \cos(x)\sin(y) & 1+\cos^2(x) \end{pmatrix} \\
&= \frac{1}{1+\sin^2(y)+\cos^2(x)+\cos^2(x)\sin^2(y)-\cos^2(x)\sin^2(y)} \\
&\quad \begin{pmatrix} 1+\sin^2(y) & \cos(x)\sin(y) \\ \cos(x)\sin(y) & 1+\cos^2(x) \end{pmatrix} \\
&= \frac{1}{1+\sin^2(y)+\cos^2(x)} \begin{pmatrix} 1+\sin^2(y) & \cos(x)\sin(y) \\ \cos(x)\sin(y) & 1+\cos^2(x) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1+\sin^2(y)}{1+\sin^2(y)+\cos^2(x)} & \frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)} \\ \frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)} & \frac{1+\cos^2(x)}{1+\sin^2(y)+\cos^2(x)} \end{pmatrix}
\end{aligned}$$

The inverse of the coefficients of matrix A are

1. $g^{11} = \frac{1}{det(A)} g_{22} = \frac{1}{1+\sin^2(y)+\cos^2(x)} \cdot (1+\sin^2(y)) = \frac{1+\sin^2(y)}{1+\sin^2(y)+\cos^2(x)}$
2. $g^{22} = \frac{1}{det(A)} g_{11} = \frac{1}{1+\sin^2(y)+\cos^2(x)} \cdot (1+\cos^2(x)) = \frac{1+\cos^2(x)}{1+\sin^2(y)+\cos^2(x)}$
3. $g^{12} = \frac{1}{det(A)} g_{21} = \frac{1}{1+\sin^2(y)+\cos^2(x)} \cdot (-\cos(x)\sin(y)) = -\frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)}$
4. $g^{21} = \frac{1}{det(A)} g_{12} = \frac{1}{1+\sin^2(y)+\cos^2(x)} \cdot (-\cos(x)\sin(y)) = -\frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)}$

When $i = 1$, $k = 1$ and $l = 1$,

$$
\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2} g^{11} \left(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11} \right) + \frac{1}{2} g^{12} \left(\partial_1 g_{21} + \partial_1 g_{21} - \partial_2 g_{11} \right) \\
&= \frac{1}{2} \left(\frac{1+\sin^2(y)}{1+\sin^2(y)+\cos^2(x)} \right) \left(-2\cos(x)\sin(x) - 2\cos(x)\sin(x) + 2\cos(x)\sin(x) \right) + \\
&\quad \frac{1}{2} \left(-\frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)} \right) \left(\sin(x)\sin(y) + \sin(x)\sin(y) - 0 \right) \\
&= \frac{1}{2} \left(\frac{1+\sin^2(y)}{1+\sin^2(y)+\cos^2(x)} \right) \left(-2\cos(x)\sin(x) \right) - \\
&\quad \frac{1}{2} \left(\frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)} \right) \left(2\sin(x)\sin(y) \right) \\
&= -\frac{(1+\sin^2(y))(\cos(x)\sin(y))}{1+\sin^2(y)+\cos^2(x)} - \frac{(\cos(x)\sin(y))(\sin(x)\sin(y))}{1+\sin^2(y)+\cos^2(x)} \\
&= -\frac{(\cos(x)\sin(y) + \cos(x)\sin^3(y))}{1+\sin^2(y)+\cos^2(x)} - \frac{\sin(x)\sin^2(y)\cos(x)}{1+\sin^2(y)+\cos^2(x)} \\
&= -\frac{(\cos(x)\sin(y) + \cos(x)\sin^3(y) + \sin(x)\sin^2(y)\cos(x))}{1+\sin^2(y)+\cos^2(x)}
\end{aligned}
$$

When $i = 1$, $k = 1$ and $l = 2$,

$$
\begin{aligned}
\Gamma^1_{21} &= \frac{1}{2} g^{11} \left(\partial_2 g_{12} + \partial_1 g_{12} - \partial_1 g_{12} \right) + \frac{1}{2} g^{12} \left(\partial_1 g_{22} + \partial_1 g_{22} - \partial_2 g_{12} \right) \\
&= \frac{1}{2} \left(\frac{1+\sin^2(y)}{1+\sin^2(y)+\cos^2(x)} \right) \left(-\cos(x)\cos(y) + \sin(x)\sin(y) - \sin(x)\sin(y) \right) + \\
&\quad \frac{1}{2} \left(-\frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)} \right) \left(0 + 0 + \cos(x)\cos(y) \right) \\
&= \frac{1}{2} \left(\frac{1+\sin^2(y)}{1+\sin^2(y)+\cos^2(x)} \right) \left(-\cos(x)\cos(y) \right) \\
&\quad - \frac{1}{2} \left(\frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)} \right) \left(\cos(x)\cos(y) \right) \\
&= \frac{1}{2} \left(\frac{-\cos(x)\cos(y) - \cos(x)\cos(y)\sin^2(y)}{1+\sin^2(y)+\cos^2(x)} \right) - \frac{1}{2} \left(\frac{\cos^2(x)\cos(y)\sin(y)}{1+\sin^2(y)+\cos^2(x)} \right) \\
&= -\frac{(\cos(x)\cos(y) + \cos(x)\cos(y)\sin^2(y) + \cos^2(x)\cos(y)\sin(y))}{2(1+\sin^2(y)+\cos^2(x))}
\end{aligned}
$$

When $i = 1$, $k = 2$ and $l = 1$,

$$\begin{aligned}
\Gamma^1_{12} &= \frac{1}{2}g^{11}\left(\partial_1 g_{12} + \partial_2 g_{12} - \partial_1 g_{12}\right) + \frac{1}{2}g^{12}\left(\partial_1 g_{22} + \partial_2 g_{22} - \partial_2 g_{12}\right) \\
&= \frac{1}{2}\left(\frac{1+\sin^2(y)}{1+\sin^2(y)+\cos^2(x)}\right)\left(\sin(x)\sin(y) - \cos(x)\cos(y) - \sin(x)\sin(y)\right) + \\
&\quad \frac{1}{2}\left(-\frac{cos(x)sin(y)}{1+sin^2(y)+cos^2(x)}\right)\left(0+0+cos(x)cos(y)\right) \\
&= \frac{1}{2}\left(\frac{1+\sin^2(y)}{1+\sin^2(y)+\cos^2(x)}\right)\left(-\cos(x)\cos(y)\right) \\
&\quad - \frac{1}{2}\left(\frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)}\right)\left(\cos(x)\cos(y)\right) \\
&= -\frac{\left(\cos(x)\cos(y) + \cos(x)\cos(y)\sin^2(y) + \cos^2(x)\cos(y)\sin(y)\right)}{2(1+\sin^2(y)+\cos^2(x))}
\end{aligned}$$

When $i = 1$, $k = 2$ and $l = 2$,

$$\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2}g^{11}\left(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}\right) + \frac{1}{2}g^{12}\left(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}\right) \\
&= \frac{1}{2}\left(\frac{1+\sin^2(y)}{1+\sin^2(y)+\cos^2(x)}\right)\left(-\cos(x)\cos(y) - \cos(x)\cos(y) - 0\right) + \\
&\quad \frac{1}{2}\left(-\frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)}\right)\left(2\sin(y)\cos(y) + 2\sin(y)\cos(y) - 2\sin(y)\cos(y)\right) \\
&= \frac{1}{2}\left(\frac{1+\sin^2(y)}{1+\sin^2(y)+\cos^2(x)}\right)\left(-2\cos(x)\cos(y)\right) - \\
&\quad \frac{1}{2}\left(\frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)}\right)\left(2\sin(y)\cos(y)\right) \\
&= \frac{(1+\sin^2(y))(-\cos(x)\cos(y))}{1+\sin^2(y)+\cos^2(x)} - \frac{(\cos(x)\sin(y))(\sin(y)\cos(y))}{1+\sin^2(y)+\cos^2(x)} \\
&= -\frac{\left(\cos(x)\cos(y) + \cos(x)\cos(y)\sin^2(y) + \sin(y)\cos(y)\cos^2(x)\right)}{1+\sin^2(y)+\cos^2(x)}
\end{aligned}$$

When $i = 2$, $k = 1$ and $l = 1$,

$$\begin{aligned}
\Gamma^2_{11} &= \frac{1}{2}g^{21}\left(\partial_1 g_{11} + \partial_1 g_{12} - \partial_1 g_{11}\right) + \frac{1}{2}g^{22}\left(\partial_1 g_{21} + \partial_1 g_{21} - \partial_2 g_{11}\right) \\
&= \frac{1}{2}\left(-\frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)}\right)\left(-2\cos(x)\sin(x) + \sin(x)\sin(y) + 2\cos(x)\sin(x)\right) \\
&\quad + \frac{1}{2}\left(\frac{1+\cos^2(x)}{1+\sin^2(y)+\cos^2(x)}\right)\left(\sin(x)\sin(y) + \sin(x)\sin(y) - 0\right) \\
&= \frac{1}{2}\left(-\frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)}\right)\left(\sin(x)\sin(y)\right) + \\
&\quad \frac{1}{2}\left(\frac{1+\cos^2(x)}{1+\sin^2(y)+\cos^2(x)}\right)\left(2\sin(x)\sin(y)\right) \\
&= -\frac{\frac{1}{2}(\cos(x)\sin(x))(\sin(x)\sin(y))}{1+\sin^2(y)+\cos^2(x)} + \frac{(\sin(x)\sin(y))(1+\cos^2(x))}{1+\sin^2(y)+\cos^2(x)} \\
&= \frac{\sin(x)\sin(y) + \sin(x)\sin(y)\cos^2(x) - \frac{1}{2}\cos(x)\sin^2(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)}
\end{aligned}$$

When $i = 2$, $k = 1$ and $l = 2$,

$$\begin{aligned}
\Gamma^2_{12} &= \frac{1}{2}g^{21}\left(\partial_2 g_{12} + \partial_1 g_{12} - \partial_1 g_{12}\right) + \frac{1}{2}g^{22}\left(\partial_2 g_{22} + \partial_1 g_{22} - \partial_2 g_{12}\right) \\
&= \frac{1}{2}\left(-\frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)}\right)\left(-\cos(x)\cos(y) + \sin(x)\sin(y) - \sin(x)\sin(y)\right) + \\
&\quad \frac{1}{2}\left(\frac{1+\cos^2(x)}{1+\sin^2(y)+\cos^2(x)}\right)\left(2\sin(y)\cos(y) + 0 + \cos(x)\cos(y)\right) \\
&= \frac{1}{2}\left(-\frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)}\right)\left(-\cos(x)\cos(y)\right) + \\
&\quad \frac{1}{2}\left(\frac{1+\cos^2(x)}{1+\sin^2(y)+\cos^2(x)}\right)\left(2\sin(y)\cos(y) + \cos(x)\cos(y)\right) \\
&= \frac{1}{2}\left(\frac{\cos^2(x)\sin(y)\cos(y)}{1+\sin^2(y)+\cos^2(x)}\right) + \frac{1}{2}\left(\frac{(1+\cos^2(x))(2\sin(y)\cos(y) + \cos(x)\cos(y))}{1+\sin^2(y)+\cos^2(x)}\right) \\
&= \frac{\frac{1}{2}\cos^2(x)\sin(y)\cos(y)}{1+\sin^2(y)+\cos^2(x)} + \\
&\quad \frac{1}{2}\left(\frac{2\sin(y)\cos(y) + \cos(x)\cos(y) + 2\sin(y)\cos^2(x)\cos(y) + \cos^3(x)\cos(y)}{1+\sin^2(y)+\cos^2(x)}\right) \\
&= \frac{\frac{1}{2}\cos^2(x)\sin(y)\cos(y)}{1+\sin^2(y)+\cos^2(x)} + \\
&\quad \frac{\sin(y)\cos(y) + \frac{1}{2}\cos(x)\cos(y) + \sin(y)\cos^2(x)\cos(y) + \frac{1}{2}\cos^3(x)\cos(y)}{1+\sin^2(y)+\cos^2(x)} \\
&= \frac{\frac{3}{2}\cos^2(x)\sin(y)\cos(y) + \sin(y)\cos(y) + \frac{1}{2}\cos(x)\cos(y) + \frac{1}{2}\cos^3(x)\cos(y)}{1+\sin^2(y)+\cos^2(x)}
\end{aligned}$$

When $i = 2$, $k = 2$ and $l = 1$,

$$
\begin{aligned}
\Gamma^2_{21} &= \frac{1}{2} g^{21} \left(\partial_1 g_{11} + \partial_2 g_{11} - \partial_1 g_{21} \right) + \frac{1}{2} g^{22} \left(\partial_1 g_{21} + \partial_2 g_{21} - \partial_2 g_{21} \right) \\
&= \frac{1}{2} \left(-\frac{\cos(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)} \right) \left(-2\cos(x)\sin(x) + 0 - \sin(x)\sin(y) \right) + \\
&\quad \frac{1}{2} \left(\frac{1+\cos^2(x)}{1+\sin^2(y)+\cos^2(x)} \right) \left(\sin(x)\sin(y) - \cos(x)\cos(y) + \cos(x)\cos(y) \right) \\
&= \frac{1}{2} \left(\frac{\cos(x)\sin(y)(2\cos(x)\sin(x) + \sin(x)\sin(y))}{1+\sin^2(y)+\cos^2(x)} \right) + \\
&\quad \frac{1}{2} \left(\frac{(1+\cos^2(x))(\sin(x)\sin(y))}{1+\sin^2(y)+\cos^2(x)} \right) \\
&= \frac{1}{2} \left(\frac{2\cos^2(x)\sin(x)\sin(y) + \sin(x)\cos(x)\sin^2(y)}{1+\sin^2(y)+\cos^2(x)} \right) + \\
&\quad \frac{1}{2} \left(\frac{\sin(x)\sin(y) + \cos^2(x)\sin(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)} \right) \\
&= \frac{\frac{1}{2}(2\cos^2(x)\sin(x)\sin(y) + \sin(x)\cos(x)\sin^2(y))}{1+\sin^2(y)+\cos^2(x)} + \\
&\quad \frac{\frac{1}{2}(\sin(x)\sin(y) + \cos^2(x)\sin(x)\sin(y))}{1+\sin^2(y)+\cos^2(x)} \\
&= \frac{\cos^2(x)\sin(x)\sin(y) + \frac{1}{2}\sin(x)\cos(x)\sin^2(y)}{1+\sin^2(y)+\cos^2(x)} + \\
&\quad \frac{\frac{1}{2}\sin(x)\sin(y) + \frac{1}{2}cos^2(x)sin(x)sin(y)}{1+sin^2(y)+cos^2(x)} \\
&= \frac{\frac{3}{2}\cos^2(x)\sin(x)\sin(y) + \frac{1}{2}\sin(x)\cos(x)\sin^2(y) + \frac{1}{2}\sin(x)\sin(y)}{1+\sin^2(y)+\cos^2(x)}
\end{aligned}
$$

When $i = 2$, $k = 2$ and $l = 2$,

$$\begin{aligned}
\Gamma^2_{22} &= \frac{1}{2} g^{21} \left(\delta_2 g_{12} + \delta_2 g_{12} - \delta_1 g_{22}\right) + \frac{1}{2} g^{22} \left(\delta_2 g_{22} + \delta_2 g_{22} - \delta_2 g_{22}\right) \\
&= \frac{1}{2} \left(- \frac{\cos(x)\sin(y)}{1 + \sin^2(y) + \cos^2(x)} \right) \left(-\cos(x)\cos(y) - \cos(x)\cos(y) - 0\right) \\
&\quad + \frac{1}{2} \left(\frac{1 + \cos^2(x)}{1 + \sin^2(y) + \cos^2(x)} \right) \left(2\sin(y)\cos(y) + 2\sin(y)\cos(y) - 2\sin(y)\cos(y)\right) \\
&= \frac{1}{2} \left(- \frac{\cos(x)\sin(y)}{1 + \sin^2(y) + \cos^2(x)} \right) \left(-2\cos(x)\cos(y)\right) + \\
&\quad \frac{1}{2} \left(\frac{1 + \cos^2(x)}{1 + \sin^2(y) + \cos^2(x)} \right) \left(2\sin(y)\cos(y)\right) \\
&= \frac{1}{2} \left(\frac{2\cos^2(x)\sin(y)\cos(y)}{1 + \sin^2(y) + \cos^2(x)} \right) + \frac{1}{2} \left(\frac{(1 + \cos^2(x))(2\sin(y)\cos(y))}{1 + \sin^2(y) + \cos^2(x)} \right) \\
&= \frac{\cos^2(x)\sin(y)\cos(y)}{1 + \sin^2(y) + \cos^2(x)} + \frac{1}{2} \left(\frac{2\sin(y)\cos(y) + 2\cos^2(x)\sin(y)\cos(y)}{1 + \sin^2(y) + \cos^2(x)} \right) \\
&= \frac{\cos^2(x)\sin(y)\cos(y)}{1 + \sin^2(y) + \cos^2(x)} + \frac{\sin(y)\cos(y) + \cos^2(x)\sin(y)\cos(y)}{1 + \sin^2(y) + \cos^2(x)} \\
&= \frac{\sin(y)\cos(y) + 2\cos^2(x)\sin(y)\cos(y)}{1 + \sin^2(y) + \cos^2(x)}
\end{aligned}$$

After doing these calculations for this function, the values of the Christoffel Symbols are different except for $\Gamma^1_{12} = \Gamma^1_{22=} = -\frac{(\cos(x)\cos(y) + \cos(x)\cos(y)\sin^2(y) + \cos^2(x)\cos(y)\sin(y))}{2(1 + \sin^2(y) + \cos^2(x))}$.

5 Conclusion and Recommendations

In the dissertation background plan, there were a number of goals that we wanted to achieve. These were to:

1. Find out about Euclidean distance from the point of view of curves.
2. Reformulate the distance by taking the infimum over the lengths of curves that are a subset of $\mathbb{R}^2$
3. Detect geometry intrinsically for any three-dimensional shape.
4. Look for some information on the curvature of the examples that are computed.

The original goal was to detect 2-dimensional ants that can detect curvature in a 2-dimensional universe. We talked about the graph of a 2-D function by considering a way of measuring distance through deforming a 2-plane via matrix-valued functions and realise that matrices have a positive-definite and symmetric structure.

In this report, the relationship between consider positive-definite and symmetric functions and distance were considered. The length of a curve can be calculated by computing it with respect to the partial derivatives of the matrix. A 2-D universe is where the ants live. The aim of this project is to see how ants measure curvature by using the Riemannian Curvature via Christoffel symbols.

If there was more time given, it would have been possible to do more curvature calculations as they are difficult to be computed by hand. By overcoming this problem, we could have learned to use the computational systems such as SageMath and SageManifolds to solve the problems straight away.

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