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Lecture 4

I left off last time having "derived" the diffusion eqn for ~~radiative~~ heat transport:

$$F = \frac{1}{3} v \ell \frac{d}{dz} a T^4$$

where $v = c$, and $\ell = \frac{1}{n\sigma}$. The preference is to write this as an "opacity", where

$$F = \frac{1}{3} \frac{c}{n\sigma} \frac{d}{dz} a T^4 = \frac{4}{3} \frac{m_p a c T^3}{\rho \sigma} \frac{dT}{dz}$$

$$= \frac{4}{3} \frac{acT^3}{\rho} \frac{1}{\kappa} \frac{dT}{dz} \Rightarrow \kappa = \frac{\sigma}{m_p}.$$

so

$$\kappa = \text{Opacity} = \frac{cm^2}{gr}$$ are the units. For Thomson scattering in a pure H plasma, this is just

$$\kappa = \frac{\sigma_{Th}}{m_p} =$$ cross-section per gram of material, note it has units of [illegible], so we will see that κ_y [illegible]

Before going into even more details concerning the "opacity" lets ~~e~~ rewrite this in proper radial units

$$\underset{\sim}{F} = -\frac{1}{3} v \ell \underset{\sim}{\nabla} E$$

which, in spherical coordinates gives

$$\underset{\sim}{\nabla} E = \frac{\partial}{\partial r} E$$

$$\underset{\sim}{F} = F_r \hat{r} \quad \text{and} \quad F_r = \frac{L_r}{4\pi r^2}$$

so

$$\boxed{L_r = -(4\pi r^2)\,\frac{4}{3}\,\frac{acT^3}{\rho\kappa}\,\frac{dT}{dr}}$$

~~Now lets imagine a star with constant L_r in hydrostatic balance~~

$$\cancel{\frac{dP}{dr} = -\rho\,\frac{Gm(r)}{r^2}}$$

~~If constant luminosity,~~ then

$$L_r = -(4\pi r^2)\,\frac{c}{\rho\kappa}\left(\frac{d}{dr} P_{rad}\right) = -4\pi r^2\,\frac{c}{\kappa}\,\frac{d}{dy} P_{rad}$$

since $P_{rad} = \frac{1}{3} E_{rad}$.

where $dy = -\rho\, dr$: column density.
Then we have:

$$L(r) = 4\pi r^2 \frac{c}{\kappa} \frac{d}{dy} P_{rad}$$

$$\frac{d}{dr} P = -\frac{\rho}{r^2} G(m(r))$$

$$\Rightarrow \quad \frac{dP}{dy} = + \frac{G m(r)}{r^2}$$

so $y=0$ at the outer edge. ~~Lets now combine these~~ Lets talk some more about y as

$$dy = -\rho \, dr$$

$$\text{so} \quad \int dy = -\int \rho \, dr$$

and we will always take $y=0$ at the outside edge of the star. (More on Bdy Conditions in a moment). Our two eqns are thus

$$\frac{dP}{dy} = \frac{G m(r)}{r^2} \quad \& \quad \frac{dP_{rad}}{dy} = \frac{L(r)}{4\pi r^2} \frac{\kappa}{c}$$

$\rightarrow$ y increases as you go in

Combine these to get

$$\frac{dP}{dP_{rad}} = \frac{4\pi G m(r) c}{\kappa\, L(r)}$$

This has units of luminosity and is called

$$L_{Edd} = \frac{4\pi G M c}{\kappa}$$

(more later). Now imagine that:

$$\frac{dP}{dP_{rad}} = \left(\frac{4\pi G c}{\kappa}\frac{M}{L}\right)\underbrace{\left(\frac{m(r)}{M}\right)\left(\frac{L}{L(r)}\right)}_{\text{dimensionless functions which}}$$

$$= (\text{Const})\left(\quad\right)$$

or just write

$$\frac{dP_{rad}}{dP} = \frac{L\kappa}{4\pi G c M}\left[\cancel{\ } \eta(r)\right]$$

↑ #?

↑ could change with position

First off how big is the #, well lets stick to massive stars, where:

$$L \approx L_{\odot} \left(\frac{M}{M_{\odot}}\right)^3$$

and

$$L_{edd} = \frac{4\pi G M c m_p}{\sigma_{Th}}$$

$$= 1.2\times10^{38} \frac{ergs}{sec} \left(\frac{M}{M_{\odot}}\right)$$

$$= 3.13\times10^{4} L_{\odot} (M/M_{\odot})$$

so

$$\frac{L}{L_{edd}} \approx 3.2\times10^{-5} \left(\frac{M}{M_{\odot}}\right)^2$$

so $M \gtrsim 100$'s of $M_{\odot}$. What is interesting then for almost all stars is that we can proceed with the assumption that $P_{rad} \ll P$, but we:

$$\frac{dP_{rad}}{dP} = \frac{L}{L_E}\, \eta(r)$$

Skip when short lecture

so integrate:

$$\int_R^r dP_{rad} = \int_R^r \frac{L}{L_E} \eta(r)\, dP$$

$$\Rightarrow \quad P_{rad}(r) = \frac{L}{L_E}\int_R^r \eta(r)\, dP$$

so

$$P_{rad}(r) = \frac{L}{4\pi c G M}\int_R^r \kappa\, \eta(r)\, dP$$

$$= \frac{L\, P(r) \langle \kappa \eta(r) \rangle}{4\pi c G M}$$

where

$$\langle \kappa \eta(r) \rangle = \frac{1}{P(r)}\int_0^{P(r)} \kappa \eta\, dP$$

is some averaage. Now, we will come back to this later in the course, but one can imagine it to be a remonably good approximation to take

$$\langle \kappa \eta(r) \rangle = \text{constant}.$$

Then we simply get.

$$\frac{dP_{rad}}{dP} \approx \frac{L \langle \kappa \eta \rangle}{4\pi G c M}$$ set $\langle \eta \rangle \sim 1$

so

$$P_{rad}(r) = \frac{L}{L_E} P(r)$$ integrat from outside in!

now, the typical trick is to define

$$\beta = \frac{P_{gas}}{P_{tot}}$$ delay a page

the $P_{gas} = \beta P_{tot} = \beta (P_g + P_r)$

so $P_g(1-\beta) = \beta P_r \Rightarrow$

$$P_r = P_g \left(\frac{1-\beta}{\beta} \right)$$

so we then get

$$\frac{1}{3} a T^4 = \frac{L}{L_E} \frac{\rho k T}{\mu m_p}$$ start

so

$$T^3 = \frac{L}{L_E} \frac{3 k \rho}{a \mu m_p} = \left(\frac{L}{L_E} \frac{3k}{a \mu m_p} \right) \rho$$

~~But~~

$$\cancel{P = \frac{\rho k T}{\mu m_p} \propto \rho^{4/3}}$$

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So we have

$$T^3 = \frac{L}{L_E} \frac{3k}{a\mu m_p} \rho$$

within the star, even though we have not done so much. Now since we know that

$$P = \frac{\rho k T}{\mu m_p}$$

and $T^3 \propto \rho$ as given above,

then

$$P \propto \rho^{4/3}$$

$\Rightarrow$ By considering the heat transport we have obtained a relation between P, ρ & T that can, (still with <u>some</u> approximations) allow us to construct a real star!

so we can say that the **run** of density & pressure in such a star is that of a polytrope of index 3

$$P = \rho^{1+\frac{1}{n}} \qquad n=3$$

$$\Rightarrow P = \rho^{4/3} K$$

$\Rightarrow$ Allows for complete soln. Show $\frac{dP}{dr} \Rightarrow$ & $\frac{dm}{dr}$.

In reality, we will define a constant $\beta = P_{gas} / P_{tot}$ in which case

$$P_g = \beta P_t = \beta (P_g + P_r), \quad \text{so}$$

we get:

$$P_{rad} = \frac{1-\beta}{\beta} P_{gas} = \frac{1}{3} a T^4$$

(we can later relate β to the Edd. Luminosity, but no biggy right now).

In this case we get

$$\frac{1-\beta}{\beta} P_{gas} = \frac{1-\beta}{\beta} \left(\frac{\rho k_B T}{\mu m_p} \right) = \frac{1}{3} a T^4$$

so

$$T(r) = \left[\frac{3 k_B}{a \mu m_p} \frac{1-\beta}{\beta}\right]^{1/3} \rho^{1/3}$$

or

$$P = \frac{P_{gas}}{\beta} = \frac{\rho k T}{\mu m_p} \frac{1}{\beta}$$

$$\Rightarrow \cancel{P(r) = \rho^{4/3}(r) \left[\left(\frac{k}{m_p \mu a}\right)^4\right]}$$

$$P(r) = \frac{1}{\beta} \frac{k}{\mu m_p} \left[\frac{3 k_B}{a \mu m_p} \frac{1-\beta}{\beta}\right]^{1/3} \rho(r)^{4/3}$$

$$= \left(\left(\frac{k}{\mu m_p}\right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4}\right)^{1/3} \rho^{4/3}(r)$$

$$= K \rho^{4/3}(r)$$

Solving the same for $P = (K)\rho^{4/3}$ yields a condition in terms of the mass; which we equate with the prefactor to get

$$\frac{1-\beta}{\beta^4} = 3\times10^{-3}\mu^4\left(\frac{M}{M_\odot}\right)^2$$

$$\Rightarrow \quad T(r) = 4.6\times10^6\,K\,(\beta\mu)\left(\frac{M}{M_\odot}\right)^{2/3}\rho^{1/3}(r)$$

$\mu^2 \frac{M}{M_\odot}$	β
1	0.997
2	0.9885
5	0.9412
10	0.8463
50	0.501

This just gives the dependencies. Fully solving the problem is more complicated

or if $\mu = 0.6$

		P_{rad}/P_{Gas}
2.8	0.997	3×10^{-3}
13.9	0.9412	6.2×10^{-2}
27.8	0.8463	0.18
138	0.501	1.00

$$\beta = \frac{P_G}{P_{Tot}}$$

So again $M = 100\,M_\odot$!

$$P_{rad} = \frac{1-\beta}{\beta} P_{Gas}$$

We had presumed that the radiation field was reasonably isotropic deep in the star. A measure of that is

$$\frac{\text{Flux}}{cE} \approx \frac{\sigma_{SB}\, T_{eff}^4}{\frac{1}{3} a c T^4}$$

$$\approx \frac{T_{eff}^4}{T^4}$$

So what is this? Well, in outer layers we have.

$$F = -\frac{1}{3} c \frac{1}{\rho \sigma n_e} \frac{d}{dr} a T^4$$

$$= +\frac{1}{3} c \frac{m_p}{\sigma} \frac{d}{dy} a T^4$$

So if $F = \text{constant}$ gives.

$$F \approx \frac{1}{3} c a \frac{d}{d\tau} T^4 \Rightarrow \frac{T_{eff}^4}{T^4} \approx \frac{1}{\tau}$$

Boundary / Photosphere

Now, what happens at the surface? Photons begin free-streaming when $\tau \simeq 1$, where yet again

$$\tau = \int \kappa \rho \, dz, \text{ so}$$

then

"Photosphere" = $\tau \sim 1$ so

if κ = const. this is where:

$$\tau \simeq 1 = \kappa \int \rho \, dz$$

or better yet $\tau = 1 = \kappa y$

But $P = gy = \frac{1}{\mu}\rho \ldots$

+ Then Eddington Limit at the photosphere.

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The Eddington Limit

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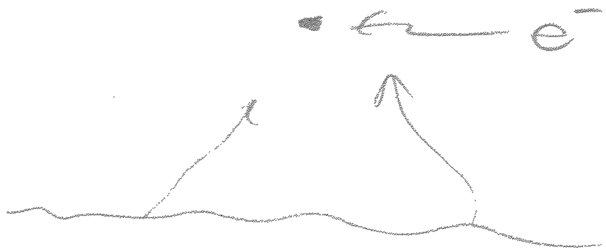

An electron sitting above the photosphere is struck every:

$$\phi = \frac{\#}{cm^2 \cdot sec}$$

$$t_{coll} = \frac{1}{\sigma_{es}(n_\gamma c)}$$

$$\frac{\#}{cm^2\, sec} = n_\gamma c$$

and gets $\Delta p = E_\gamma / c$ so

$$\text{Force} = \frac{\Delta p}{t} = \sigma_{es}\, n_\gamma\, c\, \frac{E_\gamma}{c}$$

$$\text{Force} = \sigma_{es} \frac{F}{c} > m_p g$$ or just

$$\frac{\sigma_{es}}{c} \frac{L}{4\pi \cancel{R^2}} > m_p \frac{GM}{\cancel{R^2}}$$

$$\Rightarrow \boxed{L > \frac{4\pi m_p G M c}{\sigma_{es}}}$$