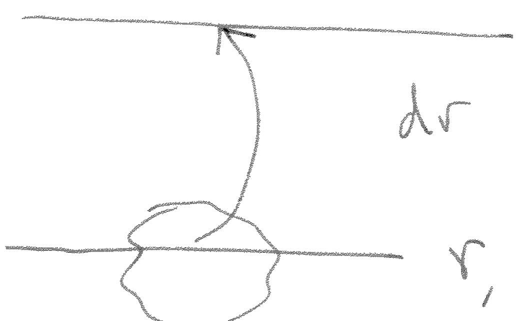
Convection

We have only discussed heat transport via ~~e~~ e^- conduction or radiation. However there are times when the required T gradient is <u>so</u> steep that convection occurs.

Your intuition might be that <u>hot</u> air rises, however I will show that 'a mild gradient can be accomodated, but eventually the T gradient' can become too large.

Draw a balloon and a fluid element and push it up to $r + dr$.



Do it adiabatically, so that in <u>the</u> bubble

$$TdS = dE + pdV = 0$$

In which case moving it up reduces the pressure and so the ~~bubble~~ balloon expands.

Presume that the motion is much slower than

$$t_{cr} = \frac{h}{c_s} = \text{time for sound to cross a scale height}$$

$\Rightarrow$ Bubble is always in press. equilibrium.

Adiabatic

—— 2

—— 1

$$\Rightarrow P V^{\gamma} = \text{constant}$$

or

$$\frac{P_1}{\rho_1^{\gamma}} = \frac{P_2}{\rho_2^{\gamma}}$$

ρ's are for Bubble

so density is

$$\rho_{2,b}^{\gamma} = \rho_1^{\gamma} \frac{P_2}{P_1}$$

$< 1 \Rightarrow$ always less dense.

The bubble clearly reduces it's density, but is it less dense than surroundings or more dense.

$\boxed{\text{Stability}}$ only when

$$\rho_{2,b} > \rho_{2,*} \quad \text{or} \quad \left(\frac{P_2}{P_1}\right)^{\frac{1}{\gamma}} \rho_1 > \rho_{2,*}$$

So for stability we want

$$\left(\frac{P_2}{P_1}\right)^{\frac{1}{\gamma}} \rho_1 > \rho_2$$

where now everything refers to star.

Write

$$P_2 = P_1 + \frac{dP}{dr}\Delta r \qquad [\Delta r > 0]$$

$$\rho_2 = \rho_1 + \frac{d\rho}{dr}\Delta r \text{ , then}$$

$$\left[1 + \frac{1}{\gamma}\Delta r\frac{d\ln P}{dr}\right] > 1 + \frac{d\ln\rho}{dr}\Delta r$$

comment here about Entropy

or that
$$\boxed{\frac{1}{\gamma}\frac{d\ln P}{dr} > \frac{d\ln\rho}{dr}}$$

For an ideal gas we write

$$P = \frac{\rho k T}{\mu m_p} \quad \text{so} \quad d\ln\rho = d\ln P - d\ln T$$

and so we get

$$\frac{1}{\gamma}\frac{d\ln P}{dr} > \frac{d\ln P}{dr} - \frac{d\ln T}{dr}$$

or

$$\left(\frac{1}{\gamma}-1\right)\frac{d\ln P}{dr} > -\frac{d\ln T}{dr}$$

for <u>stability</u>. Now we know that $\gamma = \frac{5}{3}$

$$\frac{1}{\gamma}-1 \text{ is } < 0$$

and $$\frac{d\ln P}{dr} < 0$$

so LHS is a positive #.

The RHS has $-\frac{1}{T}\frac{dT}{dr}$ which is also a positive # so we say

$$\left|\frac{d\ln T}{dr}\right| < \left(1-\frac{1}{\gamma}\right)\left|\frac{d\ln P}{dr}\right|$$

or

$$\boxed{\frac{d\ln T}{d\ln P} < 1-\frac{1}{\gamma} = 1-\frac{3}{5} = \frac{2}{5} \ (\text{stable})}$$

so when

$$\frac{d\ln T}{d\ln P} < \frac{2}{5}$$ the model is stable.

We can also include a gradient in μ if we wanted.

Then $d\ln\rho = d\ln P - d\ln T + d\ln\mu$ and we get

$$\frac{1}{\gamma}\frac{d\ln P}{dr} > \frac{d\ln P}{dr} - \frac{d\ln T}{dr} + \frac{d\ln\mu}{dr}$$

or

$$\left(\frac{1}{\gamma}-1\right)\frac{d\ln P}{dr} - \frac{d\ln\mu}{dr} > -\frac{d\ln T}{dr}$$

So if μ increases with height then

$$\frac{1}{\mu}\frac{d\mu}{dr} > 0$$

so the condition on T is weakened.
Imagine isothermal. Then just must have

$$\left(\frac{1}{\gamma}-1\right)\frac{d\ln P}{dr} > \frac{d\ln\mu}{dr} \quad \text{(Stable)}$$

Conv. with mean mol. weight gradient in the problem.

$$P = \frac{\rho k T}{\mu m_p}$$

$$\rho = \frac{\mu m_p P}{k T}$$

$$\frac{d\rho}{dr} = \frac{m_p}{k}\left[\frac{P}{T}\frac{d\mu}{dr} + \frac{\mu}{T}\frac{dP}{dr} - \frac{\mu P}{T^2}\frac{dT}{dr}\right]$$

and we want.

$$\frac{1}{\rho}\frac{d\rho}{dr} = \frac{1}{\mu}\frac{d\mu}{dr} + \frac{1}{P}\frac{dP}{dr} - \frac{1}{T}\frac{dT}{dr}$$

so:

$$\frac{d\ln\mu}{dr} - \frac{1}{T}\frac{dT}{dr} < -\frac{\rho g}{P\gamma} + \frac{\rho g}{P}$$

so

$$\boxed{-\frac{d\ln T}{dr} < \frac{\rho g}{P}\left(1 - \frac{1}{\gamma}\right) - \frac{d\ln\mu}{dr}}$$

Now lets first note that $dT/dr < 0$ nearly always and so we get

The models we previously discussed had $P \propto T^4$

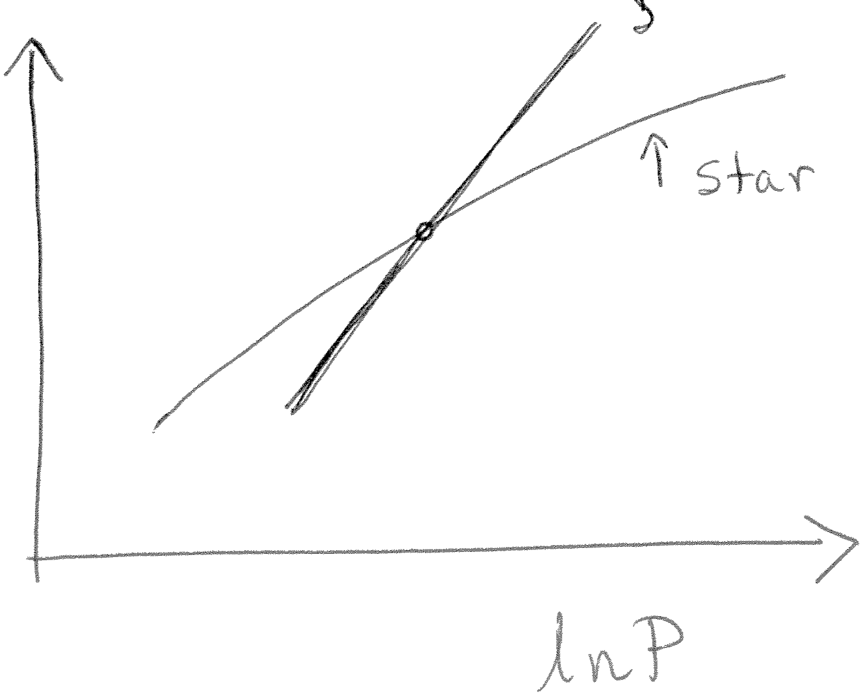
or $T \propto P^{1/4}$

so $\frac{d\ln T}{d\ln P} = \frac{1}{4} < \frac{2}{5}$ so

the model is stable.

STOP

How do we think slightly more physically about this term.
Well, for that it is easier to draw a picture:



We decided that the perturbation was adiabatic, so ~~P~~

$PV^{\gamma} =$ constant

(drop μ term).

$$-\frac{1}{T}\frac{dT}{dr} < \frac{\rho g}{P}\left(1-\frac{1}{\gamma}\right).$$

or just:

$$-\frac{d\ln T}{dr} < -\frac{d\ln P}{dr}\left(1-\frac{1}{\gamma}\right)$$

$$\Rightarrow \frac{d\ln T}{d\ln P} < \left(1-\frac{1}{\gamma}\right) = 1-\frac{3}{5} = \frac{2}{5}$$

So we can say this as if

T is steeper than $P^{2/5}$

then convection will occur. What about our standard model, which gave us

$$\rho \propto T^3 \Rightarrow P = \frac{\rho k T}{\mu m_p}$$

$$\Rightarrow P \propto T^4 \Rightarrow T \propto P^{1/4}$$

so the Standard model <u>is</u> stable to convection, as $1/4 < 2/5$!

or since $d\ln\rho = d\ln P - d\ln T$

$$1 - \frac{d\ln T}{d\ln P} > \frac{1}{\gamma}$$

$$\left(1 - \frac{1}{\gamma}\right) > \frac{d\ln T}{d\ln P} \quad \text{stable} \ \checkmark.$$

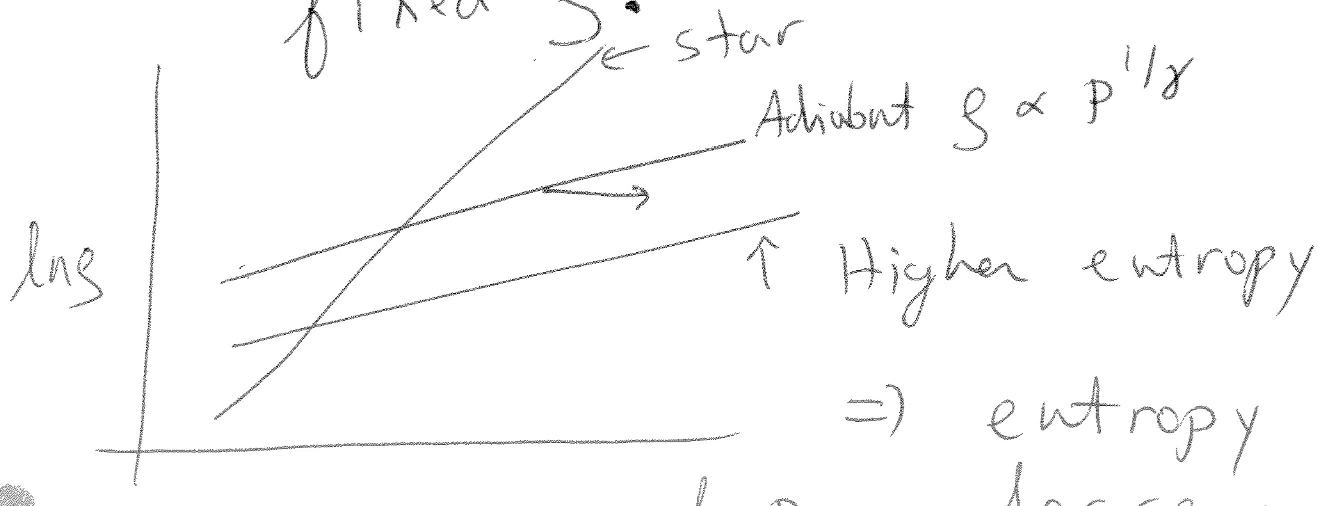
What is the sign of the entropy gradient? Well

$\cancel{Tds = dE + pdV}$

$$S \propto \ln \frac{T^{3/2}}{\rho}$$ as I kind of showed. So a ~~att~~ ~~write~~

$\cancel{T^{3/2} = T_0^{3/2} \frac{\rho}{\rho_0}}$ at fixed ρ

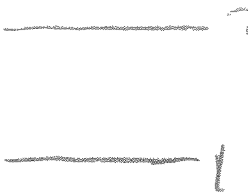
if you want S to increase you increase $T \Rightarrow P\uparrow$ at fixed ρ.



$\Rightarrow$ entropy decreases inward for a <u>stable</u> model!

Now imagine the temp. gradient is steeper than the adiabat, what happens?

Well the bubble is buoyant as it rises again if it is adiabatic, then:



$$\rho_2^b = \rho_1 \left(\frac{P_2}{P_1}\right)^{1/\gamma} \qquad P_2 = P_1 + \Delta r \frac{dP}{dr}$$

whereas

$$\rho_2^* = \rho_1 + \Delta r \frac{d\rho}{dr}\Big|_*$$

so the density contrast between the rising bubble & the star is

$$\Delta\rho = \rho_2^* - \rho_2^b = \rho_1 + \Delta r \frac{d\rho}{dr}\Big|_* - \rho_1\left(1+\frac{1}{\gamma}\frac{[dP/dr]\,\Delta r}{P}\right)$$

$$= \Delta r\left[\frac{d\rho}{dr}\Big|_* - \frac{1}{\gamma}\frac{\rho}{P}\frac{dP}{dr}\right]$$

or just

$$\Delta\rho = \Delta r\left[\frac{d\rho}{dr} + \frac{1}{\gamma}\frac{\rho}{P}\rho g\right]$$

$$\Delta\rho = \rho\Delta r\left[\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma}\right]$$

Again, for stability we want

$$\Delta\rho < 0 \qquad \rho_2^b > \rho_2^a$$

so

$$\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma} < 0 \qquad \text{or}$$

$$\boxed{\frac{d\ln\rho}{dr} < -\frac{\rho g}{P\gamma}}$$

same as before

Whereas the whole piece is >0 for instability. We can forge ahead and see right away that when >0 the lifted bubble is buoyant and is accelerated at

$$a = \frac{\Delta\rho}{\rho_b} g \qquad \text{so} \qquad (\rho = \rho_b)$$

$$a = \ddot{X} = g\,\Delta r\left[\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma}\right]$$

or just $(X = \Delta r)$

$$\ddot{X} = gX\left(\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma}\right)$$

in which case there is a characteristic frequency in the problem

$$N^2 = -g\left[\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma}\right]$$

as if $N^2 > 0$ then we have stable, buoyant oscillations as

$$\ddot{X} = -N^2 X$$

at a frequency

$$N^2 = -g\left[\frac{1}{\rho}\frac{d\rho}{dr} + \frac{\rho g}{P\gamma}\right] \approx \frac{g}{H}$$

or since $H \approx \dfrac{kT}{m_p g}$ it is

Amplify a bit on g-modes & buoyant osillations

$$N \sim \sqrt{\frac{g m_p g}{kT}} \sim \frac{g}{V_{th}} \sim$$

or if $H \sim R$ $\quad N^2 \approx \dfrac{GM}{R^3} \approx G\rho$

So the characteristic frequency is the time to cross a scale-height or the dynamical time.

If unstable, then

$$\ddot{X} = X \left(+\omega^2\right) \Rightarrow X = X_0 e^{t/\tau}$$

i.e. exponential growth in the displacement. The velocity of the fluid element is

$$\dot{X} = V = X_0 \frac{1}{\tau} e^{t/\tau} = \frac{X_0}{\tau} e^{t/\tau}$$

Unstable modes hw

$$\ddot{X} = X g\left[\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma}\right] = -N^2 X$$

and $N^2 < 0$ is unstable and we find

$$\ddot{X} = (-N^2)X = \frac{1}{\tau^2}X$$

Set $\frac{1}{\tau^2} = -N^2 \Rightarrow$ put in $X = X_0 e^{t/\tau}$

$$\dot{X} = V = \frac{X_0}{\tau}e^{t/\tau} \qquad \ddot{X} = \frac{X_0}{\tau^2}e^{t/\tau}$$

so that is an excellent soln.

Now after moving a length $\ell = X_0 e^{t/\tau}$ the velocity would be

$$v = \frac{X_0}{\tau}e^{t/\tau} = \frac{\ell}{\tau}$$

or

$$v = \ell\sqrt{-N^2} = \ell\sqrt{g\left[\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma}\right]}$$

but

$$\left.\frac{\Delta\rho}{\rho}\right|_{at\ \ell} \approx \ell\left(\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma}\right)$$

$$v = \ell\sqrt{\frac{g}{\ell}\left.\frac{\Delta\rho}{\rho}\right|_{at\ \ell}} = (g\ell)^{1/2}\left(\frac{\Delta\rho}{\rho}\right)^{1/2}$$

after moving a [illegible]. (over)

So if we put in $H = \ell =$

$$v = \left(g \frac{2kT}{m_p g}\right)^{1/2} \left(\frac{\Delta \rho}{\rho}\bigg|_{\ell = H}\right)^{1/2}$$

$$\simeq C_s \left(\frac{\Delta \rho}{\rho}\bigg|_{\ell = H}\right)^{1/2}$$

So if $\frac{\Delta \rho}{\rho}\bigg|_{\ell = H}$ is big then

$v = C_s \Rightarrow$ goes back to the old statements about falling a scale ht. to get the kT.