Puzzler of Electric Fields.

Lecture 2

Imagine a pure ionized H atmosphere where we demand $n_e = n_p$, so

$$P = 2 n_p kT \Rightarrow \frac{dP}{dz} = -\rho g$$

$$\frac{dP}{dz} = -m_p n_p g \Rightarrow 2kT \frac{dn_p}{dz} = -m_p n_p g$$

so

$$h = \frac{2kT}{m_p g} \; ; \; n_e = n_p = \left(e^{-z/h}\right) n_0$$

Now why don't the protons & electrons gravitationally separate?

Well we need to consider hydrostatic balance separately for each species, then

$$\frac{1}{n_e}\frac{dP_e}{dz}\hat{z} = m_e \underset{\sim}{g} - e\underset{\sim}{E} \qquad \uparrow E \downarrow g.$$

$$\frac{1}{n_p}\frac{dP_p}{dz}\hat{z} = m_p \underset{\sim}{g} + e\underset{\sim}{E}$$

Sum:

$$\frac{dP}{dz} = -n_e(m_e + m_p) g \Rightarrow \frac{dP}{dz} = -\rho g$$

Diff.

$$0 = (m_e - m_p)\vec{g} - 2e\vec{E}$$

So we get that

$$2e\vec{E} = -(m_p - m_e)\vec{g} = -m_p\vec{g}$$

$$e\vec{E} = -\frac{1}{2}m_p\vec{g}$$

so

$$F_{\cancel{up,e}\ down,e} \approx \frac{1}{2}m_p g$$

$$F_{down,p} = \underset{\uparrow\ up}{\frac{1}{2}m_p g} - \underset{\uparrow\ down}{m_p g}$$

So, the $\vec{E}$ field does not have any direct relevance to HB, but does dramatically impact the relative diffusion of species in a WD.

[Add HW Problem on CF in WD interiors]

Self-Gravitating Objects

I left off last time with a discussion of the repercussion of hydrostatic balance for simple isothermal atmospheres. Now I want to discuss self-gravitating objects in the limit of no rotation.

i) Why Hydrostatic for a Star.

Well, start with the full eqn:

$$\rho \frac{d\underline{v}}{dt} = \rho \underline{g} - \underline{\nabla} P$$

and imagine that we turn off the pressure for a moment. Let's see how fast the star would respond, since

$$\rho \frac{d\underline{v}}{dt} = \rho \underline{g}$$

and

$$\underline{g} = -\frac{Gm(r)}{r^2}\hat{r}$$

define $m(r)$.

picture

$$\frac{dv_r}{dt} = -\frac{Gm(r)}{r^2}$$

Now, let's track the radius at which a fixed mass is inside, say m. This defines the 'coordinate' we are following, then

$$\frac{d v_r}{dt} = -\frac{Gm}{r^2}$$

and $v_r = 0$ at $t = 0$ and $r = r_0$
and

$$v_r = \frac{dr}{dt} \quad \text{so}$$

$$\boxed{\frac{d^2 r}{dt^2} = -\frac{Gm}{r^2}} \qquad (1)$$

I can also rewrite this as an energy equation as

$$\frac{1}{2} v^2 = Gm\left(\frac{1}{r} - \frac{1}{r_0}\right) \longrightarrow \text{ec next}$$

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yes? eqn well take $\frac{d}{dt}$ of E

$$\frac{d}{dt}\frac{1}{2}v^2 = Gm\frac{d}{dt}\frac{1}{r}$$

$$\cancel{v}\frac{dv}{dt} = -\frac{Gm}{r^2}\cancel{\frac{dr}{dt}} \Rightarrow \frac{dv}{dt} = -\frac{Gm}{r^2}$$

or could integrate once to get it as

$$\frac{dv}{dt} = -\frac{Gm}{r^2}$$

$$v\,dv = -\frac{Gm}{r^2} \cancel{dt} \frac{dr}{\cancel{dt}}$$

$$\Rightarrow \int_0^v v\,dv = -\int_{r_0}^{r} \frac{Gm}{r^2}\,dr$$

$$\frac{1}{2}v^2 = +\frac{Gm}{r}\Big|_{r_0}^{r} = Gm\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

$$\Rightarrow \boxed{\frac{1}{2}v^2 = Gm\left(\frac{1}{r} - \frac{1}{r_0}\right)}$$

That's one integral. But we then have

$$\boxed{\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = Gm\left(\frac{1}{r} - \frac{1}{r_0}\right)}$$

We could integrate it and get a characteristic time or just simplify to

$$\frac{d^2 r}{dt^2} = -\frac{Gm}{r^2}$$

$$\Downarrow$$

$$\frac{r}{t^2} \approx \frac{Gm}{r^2}$$

$$t \approx \sqrt{\frac{r^3}{Gm}}$$

$$\approx \sqrt{\frac{1}{G\langle\rho\rangle}}$$

$$\langle\rho\rangle_\odot \approx 1 \ \text{gr/cm}^3 \qquad \approx 1 \ \text{hr!}$$

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} \approx 1.39$$

So we find

$$\frac{dr}{dt} = -\sqrt{2Gm}\left(\frac{1}{r} - \frac{1}{r_0}\right)^{1/2}$$

$\uparrow$ – sign since we know which way it goes.

$$\Rightarrow \int_{r_0}^{0} \frac{dr}{\left(\frac{1}{r} - \frac{1}{r_0}\right)^{1/2}} = -\sqrt{2Gm}\int_0^{t_{ff}} dt$$

Call $X = r/r_0$, then

$$\int_1^0 \frac{dx}{\left(\frac{1}{x} - 1\right)^{1/2}} = -\frac{1}{r_0^{3/2}}\sqrt{2Gm}\; t_{ff}$$

$$\underbrace{\int_0^1 \frac{dx}{\left(\frac{1}{x} - 1\right)^{1/2}}}_{\text{"}\pi/2\text{"}} = \sqrt{\frac{2GM}{r_0^3}}\; t_{ff}$$

$$\Rightarrow t_{ff} = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2GM}}$$

$M \propto \rho r^3$

$= 0.5$ hr for $\odot$

which is the time it takes for a radius to fall to the same [illegible]

Which is ~~also~~ like

$$t_{ff} = \frac{\pi}{2}\sqrt{\frac{3}{2G\rho 4\pi}} \sim \frac{1}{\sqrt{G\rho}}$$

= dynamical time. So the region will come into HE on this sort of timescale. This is the shortest timescale in town for a star.

Another way to write this is

$$\omega^2 = \frac{GM}{R^3}$$

as the Kepler Law. (i.e. the dynamical time is comparable to the orbital period of a particle at that radius).

Virial: Relation between E_{gr} & E_{kin} 2:

Start with hydrostatic balance for the spherical star:

Mention that units are $\frac{erg}{cm^4}$

$$= \frac{dP}{dr} = -\rho \frac{G\, m(r)}{r^2}$$

mass inside of r

$\Downarrow$ Multiply both sides by $4\pi r^3 dr$ and integrate throughout the star. Well, we start with the RHS

$$= -\int \rho \frac{4\pi r^3 dr\; G\, m(r)}{r^2}$$

but call $dm = \rho\, 4\pi r^2 dr$ = mass in a shell at dr, then

$$E_{gr} = -\int \frac{dm\; G\, m(r)}{r} = -\int_0^M \left[\frac{G\, m(r)}{r}\right] dm$$

which is the gravitational energy of the sphere. If the density were uniform, $m(r) = \frac{4\pi r^3}{3}\rho$

o this w/ and

$$E_{gr} = -\int_0^R \frac{G \cdot 4\pi r^3 \rho}{3} * \rho\, 4\pi r^2$$

The LHS becomes

$$\int 4\pi r^3 dr \frac{dP}{dr} = 4\pi r^3 P\Big|_0^r - 3 * 4\pi \int P \cdot r^2 dr$$

and we take $P \Rightarrow 0$ at R

[Really it is $r^3 P \to 0$ as $R \to \infty$, which is also fine for an exponential atmosphere, more on that later]

so

$$= -3 \cdot 4\pi \int P \cdot r^2 dr$$

$$= -3 \int P \cdot 4\pi r^2 dr$$

$$\boxed{\simeq -3 \langle P \rangle \cdot V}$$

~~whereas~~ where

$$\langle P \rangle = \frac{1}{V} \int P \cdot 4\pi r^2 dr$$

and V = volume of the atms. Now the other side is really just the gravitational energy of the system, so we have

$$\langle P \rangle = -\frac{1}{3} \frac{E_{gr}}{V}$$

(watch for sign changes in the text.)

since $E_{gr} < 0$ by definition.

So the total energy of the system is

$$E_{tot} = E_{gr} + \underbrace{E_{KE}}_{= \sum e_{K,i} = N\langle K\rangle}$$

and for HB we know

$$E_{gr} = -3V\langle P\rangle,$$

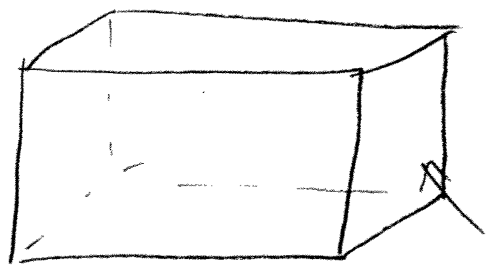
so

$$E_{tot} = -3V\langle P\rangle + E_{KE}$$

So we just need to relate the pressure density to the K.E. density.

Relationship between Pressure and internal K.E. density.

You should look at Reif or another similar text for a thorough def'n. I will only give here the simplest possible "derivation"



$\longrightarrow z.$

Area A

The particles have three degrees of freedom, x, y, z. Lets presume 1/3 move predominantly in the z direction, ~~½ going~~ at $\langle v \rangle$, with ½ in $+\hat{z}$ ½ in $-\hat{z}$. Then the wall is hit at the rate

$$r = A\left(\frac{n}{6}\right) v = \frac{\#}{\text{sec}} \text{ striking wall,}$$

but each one imparts

$$\Delta p = 2mv = 2p \text{ , so}$$

$$F = \frac{\Delta p}{\Delta t} = 2p * A \frac{n}{6} v$$

$$\text{Pressure} = \frac{F}{A} = \frac{1}{3} p * n \cdot v$$

but $p = mv$, so

$$P = \frac{1}{3} n \, m v^2 = \left(\frac{2}{3} n\right)\left(\frac{1}{2} m v^2\right)$$

$$\boxed{P = \frac{2}{3} n \langle K \rangle}$$

If the particles are relativistic, then

$$\boxed{P = \frac{1}{3} n \cancel{c} \frac{E}{\cancel{c}} = \frac{1}{3} n \langle K \rangle}$$

We could derive this more rigorously, but that is not the point of this course, as we want to show instead the macroscopic repercussions of this relation. It is crucial, because it implies that stars become almost unbound when they are supported by relativistic particles!

~~As remember~~

$$\cancel{\langle P \rangle = -\frac{1}{3} \frac{E_{gr}}{V} = \frac{1}{3} \frac{N}{V} E_{KE}}$$

$$\cancel{\Rightarrow}$$

So now we know that

$$P = \frac{2}{3} n \langle K \rangle \qquad NR$$

$$P = \frac{1}{3} n \langle K \rangle \qquad \text{Relat.}$$

Now lets go back as we know that

$$E_{tot} = -3V\langle P \rangle + E_K$$

~~and $E_K = N\langle K \rangle$~~ So that

$$E_K = N\langle K \rangle = N \frac{3P}{n} = 3V\langle P \rangle$$

i.e. $\Rightarrow$ $\underline{E_{tot} = 0}$ if relativistic,

However, if NR, then

$$E_K = \frac{3}{2} V \langle P \rangle \quad \text{so}$$

$$\boxed{E_{tot} = -\tfrac{3}{2} V \langle P \rangle}$$

or just

$$\boxed{E_{tot} = -\frac{1}{2} E_{GR}}$$

Stop Here.

Virial Thm for an Orbit

$$E_{gr} = \frac{-GmM}{r}$$

$$E_{KE} = \frac{1}{2} m v^2, \quad \text{but} \quad \frac{v^2 m}{r} = \frac{GmM}{r^2}$$

$$\Rightarrow v^2 = \frac{GM}{r}$$

so

$$E_{KE} = +\frac{1}{2} m \frac{GM}{r}$$

$$E_{tot} = E_{gr} + E_{KE}$$

$$= \frac{-GmM}{r} + \frac{1}{2} \frac{GMm}{r} = -\frac{1}{2} \frac{GmM}{r}$$

Now, what happens if the star radiates away some energy? Well we can imagine that

$$E_{tot} = + E_K + E_{gr} = - E_K .$$

and if it radiates the E_{tot} gets more negative which implies E_K gets bigger! $\Rightarrow$ The star looses total energy in a contraction but does it by getting hotter.

$\Rightarrow$ As noted by Philips a 1% decrease in total energy comes about from a 2% decrease in grav and a 1% increase in KE.

Even stranger, imagine that we torch up a star with an extra bit of energy ΔE. This will lead to a less bound object of lower temperature.

The star, when heated on a timescale > t_{ff} (so the hydrostatic balance survives) actually cools $\Rightarrow$ $C < 0$. It does this because the work needed to change the radius ($P\,dV$) is larger than the heat we put in so that the expansion to a new radius required some of the internal K.E.

Last time I went through all the manipulation of the Virial thm to explain how $T \uparrow$ as star loses energy.

Even stranger is the following:

"Why a Star Doesn't Explode"

$\Rightarrow C_V < 0$. $\rightarrow$ As long as heating is on a timescale $> t_{ff}$.

It does this because the work needed to change the radius ($p dV$) is larger than that pdV in so that the expansion to a new radius used up some internal K.E.

Now that we have these rules, we can say what happens as a star contracts. Well

$$E_{gr} = - \int_0^M \frac{G\,m(r)}{r}\,dm$$

$$\approx - \frac{GM^2}{R}$$

so if a star loses energy it must contract to smaller R. Now we can also use this to make some estimates. In particular

$$- E_{GR} = -3 \langle P \rangle V = -\cancel{3}\,\frac{2}{\cancel{3}}\,n\overset{N}{\overset{\shortparallel}{V}}\langle K \rangle$$

$$= -2N \langle K \rangle$$

gives

$$\frac{GM^2}{R} \approx +\cancel{3}\,\frac{2}{\cancel{3}}\,k_B T * N$$

but $N = \frac{M}{m_p}$, so

$$\boxed{2\,k_B T = \frac{G M m_p}{R}} \approx 10^7$$

Emphasize that HB $\Rightarrow T$!

now this should look familiar, as it says $k_B T \sim GMm_p/R$ in which case the scale height is roughly the radius.

We can also crudely get this from hydrostatic balance, where

$$\frac{dP}{dr} = -\rho \frac{Gm(r)}{r^2}$$

$$\Rightarrow \quad \frac{P}{R} \approx + \frac{M}{R^3} \frac{GM}{R^2}$$

$$\Rightarrow \quad P \approx \left(\frac{GM}{R^2}\right)\left(\frac{M}{R^2}\right) = \frac{\rho k_B T}{m}$$

$$\Rightarrow \quad \frac{GM}{R} m = k_B T \quad \checkmark \quad \left(\begin{matrix}\text{same as}\\ \text{earlier} + \\ \text{mechanics}\end{matrix}\right)$$

Now, the sun at the present time has $L = 4 \times 10^{33}$ ergs/sec, and

$$E_{gr} \approx \frac{-GM^2}{R} = 3.8 \times 10^{48} \text{ ergs}$$

so that if there was no heat source we would expect the radius to change in

$$t_{KH} = \frac{GM^2/R}{L} \approx 3 \times 10^7 \text{ years!}$$

Now, it ends up that we know that the Earth has been around for 4.5×10^9 yrs so we need something for much longer than this so we conclude that something must be supplying energy at the center.

$\boxed{\text{Energetics : What can it be?}}$

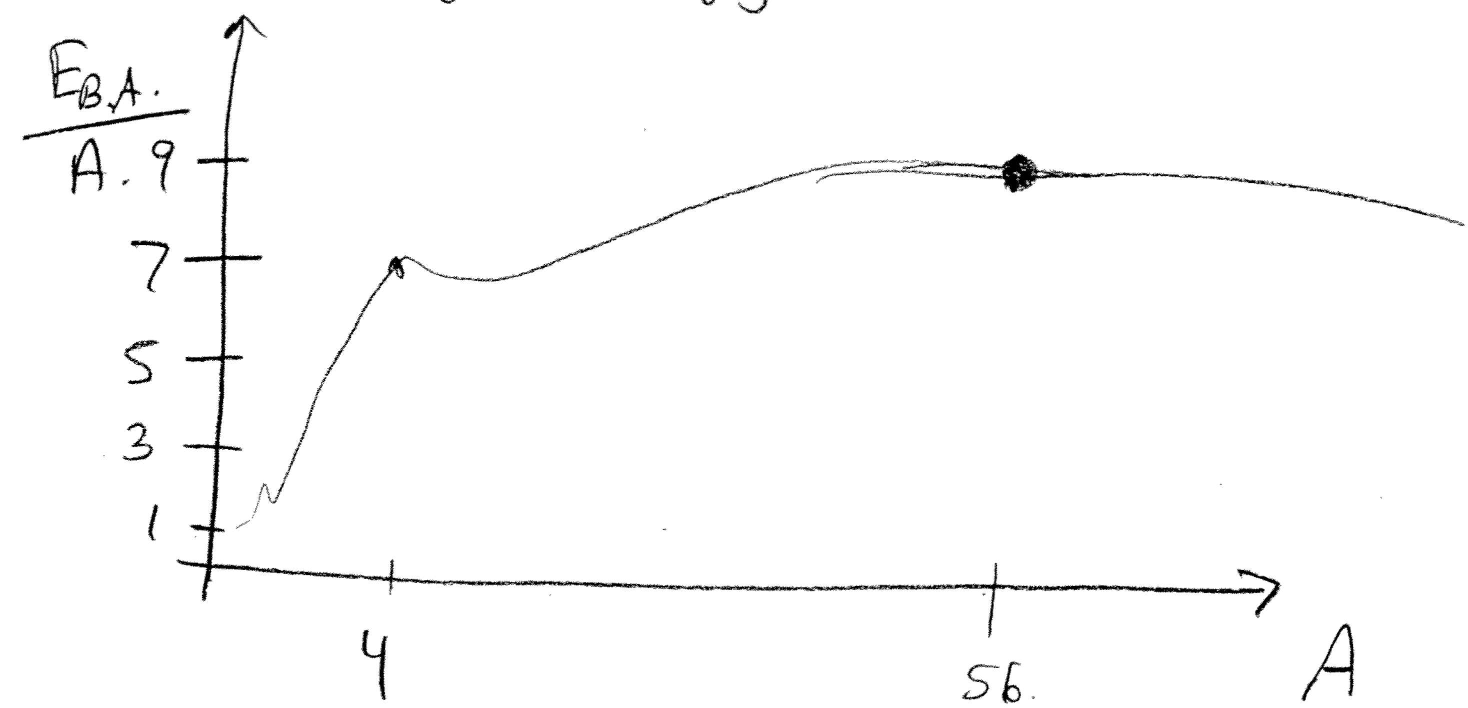
$$\frac{E_{gr}}{M} = \frac{GM}{R} \approx 10^{15} \frac{\text{ergs}}{\text{gr}}$$

$$\frac{5\,\text{eV}}{\text{atom}} = \text{Chemistry} = 5 \times 10^{12} \frac{\text{ergs}}{\text{gr}}$$

$$\frac{7\,\text{MeV}}{m_p} = \text{Nuclear} = 7 \times 10^{18} \frac{\text{ergs}}{\text{gr.}}$$

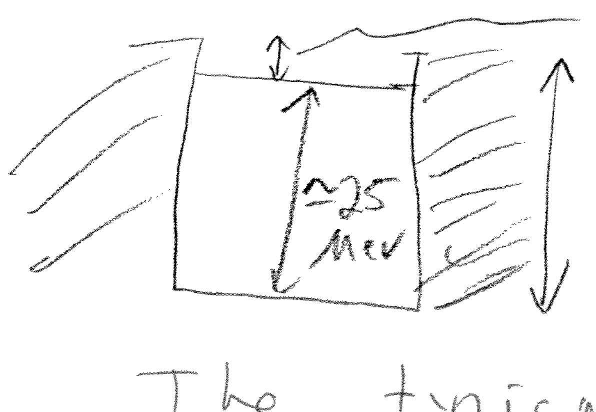
Nuclear energy release looks the most promising the only trouble is that the guess at the interior temp. is much too low for a naive guess at nuclear physics. Barring that, if we just claim we can do it then the sun lives longer by $\sim 7 \times 10^3$ or so, ~~getting~~ comparable ~~to~~ a Hubble time!

Binding energy curve.



One can crudely view a nucleus as a collection of particles confined by the strong reaction ~~to~~ be in a finite radius.

Binding Energy $\simeq$ 5 MeV

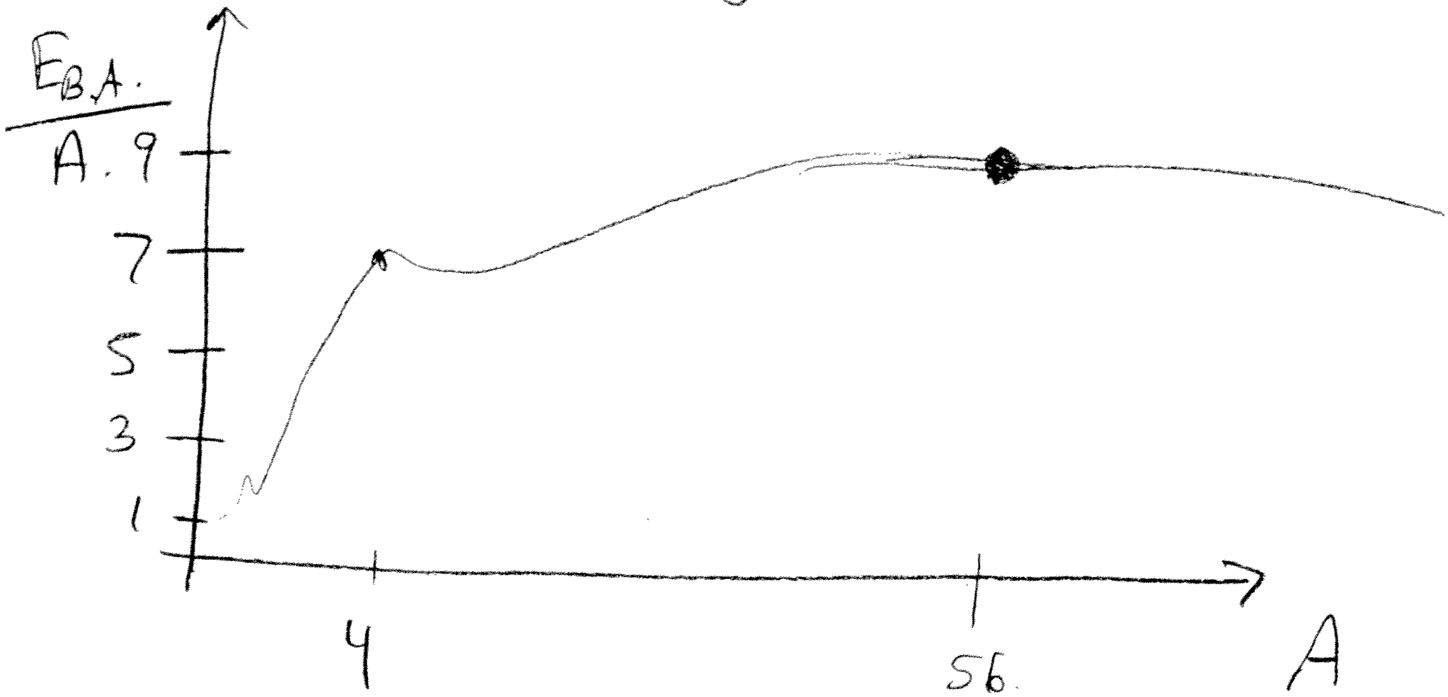


$\simeq$ 30 MeV deep.

The typical size of the nucleus is a fermi $\approx 10^{-13}$ cm and the typical energies are then just calculable from Q.M. Things roughly grow at constant density so if

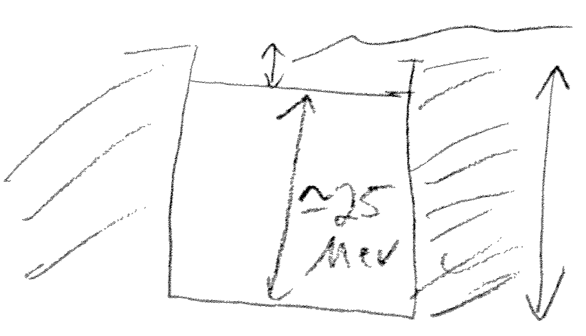
A = # of nucleons $(n + p)$

Binding energy curve.



One can crudely view a nucleus as a collection of particles confined by the strong reaction ~~to~~ be in a finite radius.

Binding Energy $\simeq 5$ MeV



$\simeq 30$ MeV deep.

The typical size of the nucleus is a fermi $\approx 10^{-13}$ cm and the typical energies are then just calculable from Q.M. Things roughly grow at constant density so if

$A = \#$ of nucleons ([illegible])

we have: $r_n = 1.3\,A^{1/3}\,\text{fm}$, so

$$\rho_N = \frac{A\,m_p}{\frac{4\pi}{3} r^3} = \frac{m_p}{\frac{4\pi}{3}(1.3)^3\ \text{fm}^3}$$

$\rho_N = 2 \times 10^{14}\ \text{gr/cm}^3$, we'll not see a star this dense until we get ~~to~~ a neutron star.

So, what we want to do is find a way to fuse the nuclei. However, ~~the~~ the temperature at the center of the Sun is

$$P = 2\rho\, k_B T \approx \frac{GM}{R^2}\frac{M}{R^2}$$

$$\Rightarrow T_c \approx 1.5 \times 10^7 \text{ K} \Rightarrow k_B T \approx \text{keV}$$

is much lower than the Coulomb barrier associated with penetrating to nuclear distances

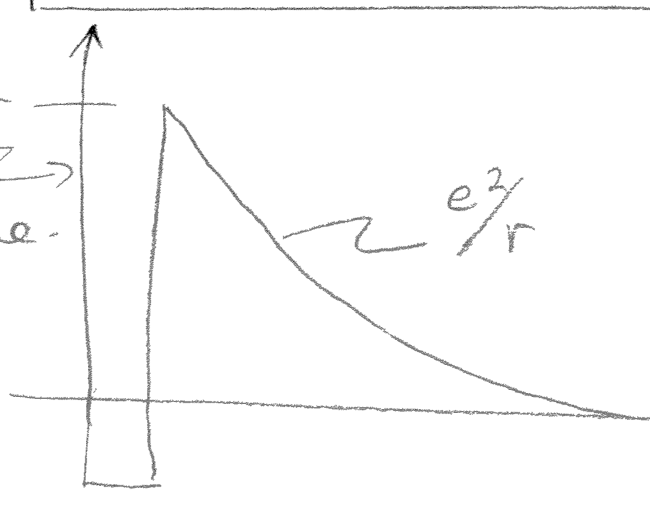
Units:

$V = \frac{e^2}{r}$ $\quad e = 4.8 \times 10^{-10}$, or

$$\frac{e^2}{\hbar c} = \frac{1}{137}$$

$\hbar c = 200 \text{ MeV} \cdot \text{fm} =$

$\text{MeV} = 10^6 \text{ eV} = 10^6 \cdot 1.6 \times 10^{-12} \text{ ergs} \approx 10^{-6}$



$$\frac{e^2}{1\text{fm}} \approx \frac{(4.8\times10^{-10})^2}{10^{-13}} = \frac{25\times10^{-20}}{10^{-13}} = 25\times10^{-7} \sim \text{MeV}$$