Left off last time having derived the velocity of a fluid element after it has traversed a distance l where

$$\left.\frac{\Delta\rho}{\rho}\right|_l = \text{density contrast at } l$$

$$v = (gl)^{1/2}\left[\left.\frac{\Delta\rho}{\rho}\right|_l\right]^{1/2}$$

When we put in a scale height

$$l = h = \frac{kT}{mg} = \frac{c_s^2}{g}$$

$$\boxed{\Rightarrow \quad v = c_s\left[\left.\frac{\Delta\rho}{\rho}\right|_H\right]^{1/2}}$$

when $\left.\frac{\Delta \rho}{\rho}\right|_{l=H} \ll 1$ then

the flux is roughly

$$F = V \,(\text{energy density})$$

Presume all in kinetic then

$$F = \rho\, v_c^3 \quad \text{or}$$

$$F = \rho\,(g l)^{3/2} \left(\left.\frac{\Delta \rho}{\rho}\right|_{l=H}\right)^{3/2}$$

put in $l = \frac{kT}{m_p g}$ = mixing length.

$$= \rho \frac{kT}{m_p} c_s \left(\left.\frac{\Delta \rho}{\rho}\right|_{l=H}\right)^{3/2}$$

This is a rough manifestation of what is called the mixing length theory.

$$\boxed{F \approx P c_s \left[\left.\frac{\Delta \rho}{\rho}\right|_{l=H}\right]^{3/2}}$$

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Now, what is the resulting heat transport? Here things get *very* fishy as we presume the bubble moves up some distance + then suddenly dissipates the energy. In this case, we find

crude measure of entropy ...

$$F_{conv} \approx V l \frac{d}{dx} n K T = \left(\frac{\Delta s}{s}\right) \frac{V l}{H} n K T$$

over one scale height, so we get

$$\boxed{F_{conv} = \left(\frac{\Delta s}{s}\right) v \, n k_B T \approx c_s \left(\frac{\Delta s}{s}\right)^{3/2} \frac{k_B T}{m_p} \rho}$$

The obvious measure of how much convection can or can't do for us is the ratio of this to the radiative heat transport

$$F_{rad} \approx \frac{1}{3} \frac{c}{\kappa \rho} \frac{d}{dr} a T^4 \approx \frac{a c T^4}{3 \kappa \rho H} \approx c P_{rad} \frac{1}{\tau}$$

so

$$\frac{F_{conv}}{F_{rad}} \approx \frac{c_s P \left(\frac{\Delta s}{s}\right)^{3/2}}{c \, P_{rad} \, (\kappa \rho H)^{-1}} \approx \frac{c_s}{c} \frac{P}{P_{rad}} \tau \frac{\Delta s}{s}$$

$$\approx \frac{c_s}{c} \left(\frac{P}{P_{rad}}\right) \tau \left(\frac{\Delta s}{s}\right)^{3/2}$$

So we have

$$\frac{F_{conv}}{F_{rad}} \sim \frac{C_s}{C} \frac{P}{P_{rad}} \tau \left(\frac{\Delta S}{S}\right)^{3/2}$$

as a local measure of just how "efficient" convection can be. In an atmosphere, say, where τ is small, what can happen?

Well, let's say $\frac{P}{P_{rad}} \simeq 10^4$, like I found for the sun. Then if we want

$$\frac{F_{conv}}{F_{rad}} \sim 1 \quad \text{we need}$$

$$\left(\frac{kT}{m_p c^2}\right)^{1/2} \frac{\cancel{C_s}}{\cancel{C}} (10^4)\, \tau \left(\frac{\Delta S}{S}\right)^{3/2} \simeq 1$$

$$\left(\frac{T}{10^4}\right)^{1/2} 0.3\, \tau \left(\frac{\Delta S}{S}\right)^{3/2} = 1 \qquad \tfrac{1}{2} \cdot 2$$

or

$$\left(\frac{\Delta S}{S}\right) \simeq 2 \left(\frac{10^4}{T}\right)^{1/3} \frac{1}{\tau^{2/3}}$$

$\Rightarrow$ Vigorous convection in the outer parts.

So, ~~...~~ we will typically find that convection out/near low optical depths will be quite vigorous, often at/near sound speeds. This is definitely true for Sun.

Now lets go to the interior at $\tau \ggg 1$ and ask again.

$$\frac{F_{conv}}{F_{rad}} \sim \frac{c_s}{R}\left[\frac{P}{P_{rad}}\tau\frac{R}{c}\right]\left(\frac{\Delta\rho}{\rho}\right)^{3/2}$$

$$\sim \frac{1}{t_{dyn}}\left[\quad ? \quad\right]\left(\frac{\Delta\rho}{\rho}\right)^{3/2}$$

So what is this timescale? Well let's consider a random walk first. In N collisions a γ moves a distance $\ell \simeq \sqrt{N}\,\lambda$, but

$t = \frac{N\lambda}{c}$. Time to go $\ell \simeq R$ is then

$$\left(\frac{R}{\lambda}\right)^2 = \frac{ct}{\lambda} \Rightarrow t_R \simeq \frac{R^2}{\lambda c} = \frac{R}{\lambda}\frac{R}{c}$$

or

$$\boxed{t_R = \tau\frac{R}{c}}$$

So, that's how long it takes the radiation field to empty out. However, it must do that P/P_{rad}* times to empty the thermal content. So we get

$$\frac{F_{conv}}{F_{rad}} \simeq \frac{t_{KH}}{t_{dyn}}\left(\frac{\Delta\rho}{\rho}\right)^{3/2} = \frac{10^7\,yr}{1\,hr}\left(\frac{\Delta\rho}{\rho}\right)^{3/2}$$

This is what we mean by effic. convection. The conv. flux can easily carry ~~re~~ the required flux!

In addition, in the presence of convection the $\Delta\rho/\rho$ is $\ll 1 \Rightarrow$ background model is just the adiabat!

$$\frac{\Delta\rho}{\rho} \sim \left(\frac{t_s}{t_{KH}}\right)^{2/3} \simeq 5\times10^{-8}$$

Side Derivation of t_{KH}

$$L = R^2 \frac{ac}{\kappa\rho}\frac{1}{R}T^4 = \frac{R^2 c}{\tau} P_{rad}$$

$$t_{KH} = \frac{P_{gas} R^3}{L} \simeq \frac{P_{gas}}{P_{rad}}\frac{\tau R^3}{R^2 c}$$

$$\simeq \frac{P_{gas}}{P_{rad}}\tau\frac{R}{c}$$

$$\simeq \frac{P_{gas}}{P_{rad}} t_R.$$

Or the way to think of this is that the "valve" for convection is so efficient that an extremely slight superadiabatic gradient can easily push things around enough to carry heat out. The resulting speeds are then:

$$v \approx c_s \left(\frac{\Delta\rho}{\rho}\right)^{1/2} \approx c_s \left(\frac{t_s}{t_{KH}}\right)^{1/3} \lll c_s$$

For a typical massive MS core this gives:

$$t_{KH} = \frac{GM^2/R}{L_\odot \left(M/M_\odot\right)^{3.5}} = 10^7 \text{yrs} \left(\frac{M_\odot}{M}\right)^{2.5}$$

$$t_s \approx \frac{1}{\sqrt{G\rho}} \approx \frac{1}{\sqrt{G\,3M/4\pi R^3}}$$

$$R = R_\odot \left(M/M_\odot\right)^{\cancel{0.75}}$$

$$= \left(\frac{4\pi R_\odot^3}{3GM} \frac{M^3}{M_\odot^3}\right)^{1/2} = 3\times10^3 \text{s} \left(\frac{M}{M_\odot}\right)$$

so $$\frac{t_s}{t_{KH}} = 10^{-11}\left(\frac{M}{M_\odot}\right)\left(\frac{M}{M_\odot}\right)^{2.5} = 3\times10^{-9}\left(\frac{M}{10M_\odot}\right)^{3.5}$$

so

$$\boxed{\frac{v}{c_s} \approx 10^{-3}\left(\frac{M}{10M_\odot}\right)^{7/6}}$$

[Rotation? & B?]

This small velocity is thus consistent with our earlier assumptions, in particular that the bubble has time for pressure equilibrium.

Now, the other particularly convenient outcome is that we can assume the fluid is very nearly following the adiabatic relation, so that

$$\left.\frac{d\ln T}{d\ln P}\right|_* = \left.\frac{d\ln T}{d\ln P}\right|_{ad} = \frac{2}{5}$$

So the star which is fully convective would have

$$T \propto P^{2/5}.$$

or if fully convective, then

$T \propto P^{2/5}$ and $P \propto \rho T$

$$\Rightarrow \quad \frac{P}{\rho} \propto P^{2/5} \qquad \rho \propto P^{3/5}$$

or $\boxed{P \propto \rho^{5/3}}$

or an equation of state that follows

$$P \propto \rho^{1+\frac{1}{n}} \quad \Rightarrow n = \frac{3}{2} \text{ polytrope.}$$

Again, for these objects we can construct the whole star.

So we have found two polytropes

1. Fully convective star

$$P \propto \rho^{5/3}$$

2. Constant L/M star with Thomson scattering + radiative diffusion

$$P \propto \rho^{4/3}.$$

A Fully Convective Star

So, imagine a fully conv. star. As noted earlier the flux it carries is very sens. to the presumed superadiab. in which case it is not well determined. What sets it?

Well, somewhere the photons must come from. Lets again say, and imagine, that the convection just barely reaches the photosphere, so that the star stops at

$$P_{ph} = \frac{g}{\kappa}$$

whereas inside of this point

$$T \propto P^{2/5}$$

so $$P \propto \rho T \propto \rho P^{2/5} \Rightarrow P^{3/5} \propto \rho$$

$$P \propto \rho^{5/3} = \rho^{1+\frac{1}{n}} \qquad n = \frac{3}{2} \text{ polytrope}$$

Now, we know P & T roughly at the center of the star, as

For an $n = 3/2$ polytrope, we find:

$$P_c = 0.77 \frac{GM^2}{R^4} \; ; \; T_c = 0.54 \frac{GM\mu m_p}{k_B R}$$

so we must have:

$$\frac{T_c}{P_c^{2/5}} = \frac{T_p}{P_p^{2/5}}$$

(or the other way to say it is that we have K fixed).

$$\frac{0.54 \dfrac{GM\mu m_p}{k_B R}}{0.9 \dfrac{G^{2/5} M^{4/5}}{R^{8/5}}} = \frac{T_{eff} \, \kappa^{2/5}}{\left(\dfrac{GM}{R^2}\right)^{2/5}}$$

so

$$k_B T_{eff} = \frac{0.6 \, GM\mu m_p \, R^{8/5}}{R \, M^{4/5}} \frac{M^{2/5}}{R^{4/5}} \frac{1}{\kappa^{2/5}}$$

$$\boxed{k_B T_{eff} = 0.6 \left(\frac{GM\mu m_p}{R}\right) \left[\frac{R^2}{M \kappa_{ph}}\right]^{2/5}}$$

If this is the case and κ = constant, then we just get.

$$L = \sigma T^4 4\pi R^2$$

$$T_{eff} \simeq 200K \left(\frac{M}{M_\odot}\right)^{3/5} \left(\frac{R_\odot}{R}\right)^{1/5}$$

which is still not good, if we want electron scattering to be the major source of opacity.

It ends up that, in this case, the outer bdy sets the temperature.

And H^- which is a very loosely bound ion (0.75eV) and contributes opacity as

$$\gamma + H^- \rightarrow e^- + H$$