Notes for PHYS 232: Stellar Structure

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1 Introduction

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1.1 The HR Diagram

Most stars shine predominantly in the optical range of the electromagnetic spectrum. As a result, we get most of our information about stars by observing their optical output. It makes sense, then, that we might organize stars by there color, which is indicative of their surface temperature. When plotting a population of stars' luminosities against their surface temperature, we note a strong correlation between the two. As it turns out, the controlling parameter for these quantities is the mass of the star, at least while the star is on the **Main Sequence** (stars burning hydrogen to helium). The correlation between the mass of a main-sequence star and its luminosity is incredibly strong (see HR diagram examples).

1.2 Conditions for a Star on the HR Diagram

We are interested in knowing what defines the regime where a star can reside in a particular L, $T_{\text{eff}} = [L/(4\pi\sigma_{\text{SB}}R^2)]^{1/4}$. Why, for example, is there a dynamical range in the luminosity spanning over six orders of magnitude, while only a range of a factor of about 5 in the effective temperature? To gain some perspective, we might observe the number of stars as a function of brightness. We organize these stars by their **spectral type** (a rough measure of how big the star is) and find their approximate **mass density** (the amount of mass contained in these stars per unit volume):

Spectral Type	ρ ($M_\odot$/pc^3)
O-B	0.4
A-F	4
G-M	40
WD's	20

Here we've used the standard labels for different spectral types, O, B, A, F, G, K, M, L, T, which are roughly in decreasing order of size and temperature (the reasoning for this scale is historical rather than logical, and the ordering is often remembered by the mnemonic, "Oh Be A Fine Gal/Guy, Kiss Me! Less Talk!"). We see that the big, bright stars form an exceedingly small portion of the amount of stellar mass in our galaxy. We will find that this is because large stars exhaust their fuel much more quickly than smaller stars, and thus live and die much faster. We'll now observe another type of classification of stars used in our neighborhood, the Milky Way Galaxy.

1.2.1 Population I Stars

From Earth, the center of the galaxy is approximately 8.5 kpc away. Meanwhile, the disk is only about 100 pc wide. We've observed that stars in the thin disk (commonly known as **Population I Stars**) are orbiting at the orbital velocity with only a small amount of axial and radial motion. They are essentially dynamically cold and in nearly circular orbits. This is indeed where most of the **interstellar medium** (ISM) resides, causing much of the star formation in the galaxy. This region is also very metal rich. That is, compared to other parts of the universe, there is a much

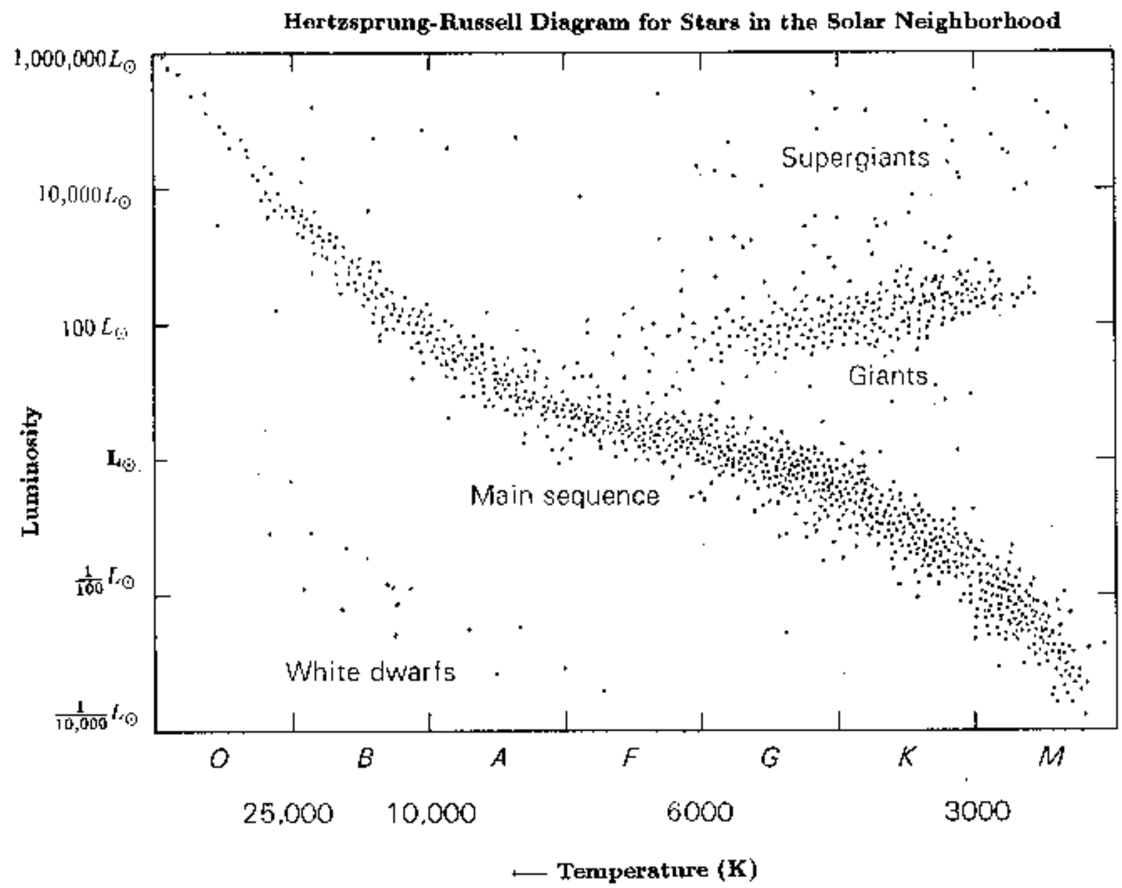

Figure 1: An HR diagram for stars in the local neighborhood (shamelessly stolen from Google Images)

higher concentration of elements heavier than helium present. We will denote the mass fraction of these "metals" with the letter Z, and in this region, we have $Z \sim 1-2\,\%$. These metals come from a previous generation of stars, who died in the past, giving off the metals we now have.

1.2.2 Population II Stars

Population II Stars reside mostly in the spheroid in the center of the galaxy. These are older stars in regions where star formation is largely shut down. Typically they are metal poor, with metallicities as low as $Z = 10^{-4} Z_{\odot}$. Kinematically, they are often on radial orbits (rather than their more azimuthal population II counterparts in the disk). We typically say that the globular clusters are part of this population. Sometimes these stars are seen passing through the disk at velocities comparable to the orbital velocities, and are easily identified due to their high velocities and unique spectra (due to the low metallicities).

2 The Isothermal, Plane Parallel Atmosphere

2.1 Scale Height and Column Depth

Before tackling the physics of stars, we first consider a simple toy model– the isothermal, plane parallel atmosphere. This model is somewhat applicable to the thin stellar atmosphere near the surface of a star, where curvature can be neglected and the acceleration due to gravity is nearly uniform.

Consider an atmosphere where the local acceleration due to gravity, $\mathbf{g}$ is constant in value and direction. The atmosphere is composed of an isothermal ideal gas with temperature T. We wish to find the distribution of particles in this atmosphere.

In a strictly statistical sense, we would expect the energy distribution to be comparable to $e^{-E/kT}$ (recall that the atmosphere is isothermal, so the average kinetic energy is uniform throughout). In our case, the energy of particles is linear in height, so we expect this probability to be proportional to $e^{-mgh/kT}$.

We will let $m_B \approx m_p$ be the baryon mass, μ be the mean molecular mass (measured in AMU), and ρ is the density in $\mathrm{g\,cm^{-3}}$. We suppose that the gas is in hydrostatic balance, so we have

$$\frac{dP}{dz} = -\rho g \tag{1}$$

Combining this with the ideal gas law,

$$P = nkT \tag{2}$$

We find that

$$kT\frac{dn}{dz} = -m_p \mu n g \tag{3}$$

which in turn gives us

$$\frac{d\ln n}{dz} = -\frac{m_p \mu g}{kT} \tag{4}$$

Solving this differential equation gives the expected result

$$n(z) = n(0)\exp\left(-\frac{m_p \mu g z}{kT}\right) = n(0)\exp\left(-\frac{z}{h}\right) \tag{5}$$

where we have defined the **scale height** $h \equiv kT/(\mu m_p g)$, which is the e-folding distance in number density. As it turns out, the scale height for earth's atmosphere is approximately 10 km. This model is really only valid in cases where $h \ll R$, (where R is the size of the object), so we now investigate the ratio of these two quantities:

$$\frac{h}{R} = \frac{kT}{\mu m_p \frac{GM}{R^2} R} = \frac{1}{\mu} \frac{kT/m_p}{GM/R} \sim \frac{v_{\rm th}^2}{v_{\rm esc}^2} \ll 1 \tag{6}$$

So if this approximation is valid, the thermal velocities of particles are typically much smaller than the escape velocity of the central body, so a star could retain its own atmosphere (thankfully, Earth does the same to its atmosphere!) For stars, we will find that $kT_c/m_p \sim GM/R$. Then, (6) tells us that

$$\frac{h}{R} \sim \frac{T_{\rm eff}}{T_c}. \tag{7}$$

This tells us that stars are quite sharp-edged (their scale heights are very small compared to their radii). We can also deduce a physical meaning for the scale height as being how far a particle needs to fall to gain an energy comparable to kT.

From the ideal gas law, we can easily see that the pressure will also fall off exponentially in this isothermal atmosphere. However, let's explore the pressure a bit more. First, we return to the ideal gas law,

$$P = \frac{\rho}{\mu m_p} kT = nkT \tag{8}$$

and the condition for hydrostatic equilibrium,

$$dP = -\rho g\, dz. \tag{9}$$

We now integrate (9) from $z = z$ to $z \to \infty$:

$$P(\infty) - P(z) = -\int_z^\infty \rho(z')g\, dz' \tag{10}$$

$$P(z) = g \int_z^\infty \rho(z')\, dz \tag{11}$$

Note that it is okay to take the integral to infinity so long as we are dealing with a constant $\mathbf{g}$. This result suggests the definition of the **column density**:

$$y(z) \equiv \int_z^\infty \rho(z)\, dz \tag{12}$$

On the surface of the earth, the column density is approximately $y = 1000\,\mathrm{g\,cm^{-3}}$. Think of it as the amount of mass sitting above you per unit area at a given altitude. The column density is an important number (for us) to determine the details of heat transport. For now though, we can write the pressure in this isothermal atmosphere in a compact form: $P(z) = gy(z)$.

2.2 Mean Molecular Weights

We'll now make a useful definition for calculating pressures and other useful quantities. For an ideal gas, the total pressure of a mixed gas is simply

$$P = \sum_{i=1}^{N} n_i kT \tag{13}$$

where n_i are just the number densities for each ion. The number density is computed via

$$n_i = \frac{X_i \rho}{A_i m_p}. \tag{14}$$

where X_i is the mass fraction of the i^{th} ion with mass number A_i and ρ is the overall mass density. Then the ion pressure is given by (assuming total ionization)

$$P_{\text{ion}} = kT \sum \frac{X_i \rho}{A_i m_p} = \frac{kT\rho}{m_p} \sum \frac{X_i}{A_i} = \frac{kT\rho}{\mu_{\text{ion}} m_p}. \tag{15}$$

Where we have defined the **mean molecular weight** of the ions to be

$$\mu_{\text{ion}} = \left[\sum \frac{X_i}{A_i}\right]^{-1} \tag{16}$$

For the electrons, we have (assuming total ionization)

$$P_e = n_e kT = kT \left(\sum Z_i n_i\right) = \frac{kT\rho}{m_p} \sum \frac{Z_i X_i}{A_i}. \tag{17}$$

(here Z_i is the atomic number, not the metallicity). Then the total pressure is just the sum of these two,

$$P = P_{\text{ion}} + P_e = \frac{\rho kT}{m_p} \left(\frac{1}{\mu_e} + \frac{1}{\mu_i}\right) \tag{18}$$

So we define the overall mean molecular weight via

$$\frac{1}{\mu} \equiv \frac{1}{\mu_e} + \frac{1}{\mu_i} \tag{19}$$

One might think of this as the average weight of a particle that supplies pressure within a gas. Later, we'll see that this quantity, and its evolution, plays a large and critical role in the the nature of stellar evolution. Since fusion tends to decrease the pressure support, the star must continuously readjust its structure so as to hold itself up.

Wednesday, January 11, 2012

2.3 Electric Fields in Stars

Imagine a pure, ionized hydrogen atmosphere which is, on the large scale, electrically neutral. We wish to find the scale height in a plasma of ionized hydrogen. In this plasma, we have $n_p = n_e$ due to electric neutrality. Then the overall pressure in this hydrogen plasma is

$$P = 2n_p kT \tag{20}$$

Using hydrostatic equilibrium, we get

$$2kT\frac{dn_p}{dz} = -m_p n_p g \tag{21}$$

which in turn gives us the differential equation

$$\frac{d\ln n_p}{dz} = -\frac{m_p g}{2kT} \tag{22}$$

Which gives us a scale height of

$$h = \frac{2kT}{m_p g} \tag{23}$$

We need to look at both plasmas separately while incorporating the electric field created between the protons and electrons. For electrons, we have

$$\frac{1}{n_e}\frac{dP_e}{dz} = -m_e g - eE. \tag{24}$$

Likewise for the protons,

$$\frac{1}{n_p}\frac{dP_p}{dz} = -m_p g + eE \tag{25}$$

where we've assumed that the electric field points up (the protons are heavier and would thus sink below the electrons). Now adding (24) and (25), we recover hydrostatic balance. However, subtracting the two equations will get us the electric field:

$$0 = -m_e g + m_p g - 2eE, \tag{26}$$

giving the result,

$$eE = \frac{1}{2}\left(m_p - m_e\right) g \quad \text{or} \quad e\mathbf{E} \approx -\frac{m_p \mathbf{g}}{2}. \tag{27}$$

So this field does not directly affect hydrostatic balance, but it does dramatically impact the relative difficulty of (?UNREADABLE TEXT IN LECTURE NOTES?) in a white dwarf.

3 Self-Gravitating Objects

So far we have only considered systems where the acceleration due to gravity is constant. In any self-gravitating object, this is obviously not true. We will, however, continue to assume that such objects do not rotate. Additionally, we will be ignoring mass loss. Essentially all we must write down are equations of mass conservation, momentum conservation, and energy conservation. We'll start with momentum conservation.

3.1 Momentum Conservation and the Free-Fall Timescale

The momentum equation for a fluid is just

$$\rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{g} - \boldsymbol{\nabla} P \tag{28}$$

This equation essentially states that a self-gravitating object is neither collapsing nor expanding. If we were to "shut off" gravity or the pressure gradient, the star would either explode or collapse, respectively. Such a collapse would occur on the **free-fall timescale**, which we will now derive. Taking the pressure gradient out of (28), we retrieve

$$\mathbf{g} = -\frac{Gm(r)}{r^2}\hat{r} \tag{29}$$

For this derivation, we will be using a **Lagrangian coordinate systems**. This is a system where the coordinates follow a particular fluid element. In essence, we are making the substitution

$$\frac{d}{dt} \to \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \tag{30}$$

Returning back to the derivation, (29) gives us

$$\frac{dv_r}{dt} = -\frac{Gm(r)}{r^2} \tag{31}$$

Initially, we have $t = 0$, $v_r = 0$, and $r = r_0$ with the radial velocity given by $v_r = dr/dt$. Then our differential equation is

$$\frac{d^2r}{dt^2} = -\frac{Gm(r)}{r^2} \tag{32}$$

As an order of magnitude estimate, this gives us

$$\frac{r}{t_{\text{ff}}^2} \sim \frac{Gm}{r^2} \quad \Rightarrow \quad t_{\text{ff}}^2 \sim \frac{1}{Gm/r^3} \tag{33}$$

So we define the free-fall timescale to be

$$t_{\text{ff}} = \frac{1}{\sqrt{G\left\langle \rho \right\rangle}} \tag{34}$$

This is also the same as the Keplerian orbital period, modulo some uninteresting constants. The punchline of this whole argument is that a star that is *not* in hydrostatic balance will respond on a timescale of the free-fall timescale. From this alone, we may conclude that the sun (and all other stars not currently exploding) is in hydrostatic balance. We will then assume that all stars are always in hydrostatic balance.

3.2 The Virial Theorem

Stars are held up by gas pressure, radiation pressure, or both. The pressure gradients are what will be the "restoring forces" against gravity for our cases. In spherical symmetry, hydrostatic balance tells us

$$\frac{dP}{dr} = -\rho\frac{Gm(r)}{r^2} \tag{35}$$

We will use this to derive the **Virial Theorem**, which relates the potential energy to the kinetic energy of a system. The equation of mass conservation states that

$$dm = 4\pi r^2 \rho(r)\, dr \tag{36}$$

Now we multiply both sides of (35) by $4\pi r^3\,dr$:

$$\int 4\pi r^3\,dP = -\int \rho\frac{G}{r^2}4\pi r^3\,drm(r) \tag{37}$$

$$= -\int \frac{Gm(r)dm}{r} = E_{\rm GR} \tag{38}$$

where $E_{\rm GR}$ is the gravitational binding energy. Performing a similar analysis to the left side of (37) gives

$$\int 4\pi r^3\,dr\frac{dP}{dr} = 4\pi r^2 P\Big|_0^R - 3\left[4\pi\int Pr^2\,dr\right] \tag{39}$$

$$= -3\int P4\pi r^2\,dr \tag{40}$$

$$= -3\left\langle P\right\rangle V \tag{41}$$

where we've defined the average pressure to be the pressure averaged over volume. Then the virial theorem tells us that

$$\boxed{\left\langle P\right\rangle = -\frac{1}{3}\frac{E_{\rm GR}}{V}} \tag{42}$$

Now we examine the total energy:

$$E_{\rm tot} = E_{\rm GR} + E_{\rm KE} = -3\left\langle P\right\rangle V + E_{\rm KE} \tag{43}$$

We need only relate the kinetic energy to the pressure to finish this equation off. For an ideal gas, we know that $P = NkT/V$, so the kinetic energy is $E_{\rm KE} = \frac{3}{2}NkT = \frac{3}{2}PV$. This gives a total energy of

$$E_{\rm tot} = -3\left\langle P\right\rangle V + \frac{3}{2}\left\langle P\right\rangle V = -E_{\rm KE} \tag{44}$$

Interestingly, this requires a negative heat capacity. That is, an increase in the temperature of the system causes a net *decrease* in total energy. However for radiation, pressure is given by $P = \frac{1}{3}aT^4$ and $E/V = aT^4$. Taking this to its conclusion gives us

$$E_{\rm tot} \to 0 \text{ as the particles become relativistic} \tag{45}$$

The origin of this result is in the momentum-energy relation of relativistic particles and non-relativistic particles. That is, $E = pc$ for ultra-relativistic particles and $E = p^2/2m$ for non-relativistic particles.

The limiting energy of ultra-relativistic stars puts an upper level on the mass of large stars, since a total energy of a star being zero means unbinding the star. In the "normal case" of an ideal gas star, the more traditional form of the virial theorem applies:

$$\frac{E_{\rm KE}}{\text{mass}} \sim \frac{GM}{R} \tag{46}$$

This is why stars typically behave with a negative heat capacity. That is, as a star radiates, $E_{\rm tot}$ is more negative, meaning that R must decrease and the temperature T (essentially the kinetic energy per particle) rises. This behavior would have to continue until a new energy source became available.

3.3 Applications of the Virial Theorm

The gravitational energy of an object is typically given by

$$E_{\rm GR} \approx -\frac{GM^2}{R} \tag{47}$$

Using the virial theorem, we have

$$E_{\rm GR} = 3\langle P\rangle V = 3Nk\langle T\rangle \tag{48}$$

Or,

$$\frac{GM}{R}(Nm_p) \sim 3NkT \tag{49}$$

So we have

$$\boxed{kT \sim \frac{GMm_p}{R}} \tag{50}$$

This temperature is the temperature of most of the material and is $T \sim T_c \sim$ core. For the sun, we then have $T \sim 10^7$ K. Interestingly, assuming hydrostatic equilibrium was all we needed to get a rough estimate of the sun's core temperature! One might note, though, that the surface temperature is significantly lower than the core temperature, so we must assume that there is heat loss taking place in the sun. Today the luminosity of the sun is

$$L_\odot = 4\times 10^{33}\ \text{erg/s} \tag{51}$$

If we assume there is no energy source for the sun other than gravitational energy, we can come up with a timescale (called the **Kelvin-Helmholtz timescale**)

$$t_{\rm KH} = \frac{E_{\rm GR}}{L} \approx 3\times 10^7\ \text{years} \tag{52}$$

for the sun. This has been known for awhile and since the Earth is known to have existed much longer than $t_{\rm KH}$, scientists deduced that another energy source within the sun was needed to explain its longevity. We now know that this energy source is, of course, fusion. Note that at the center of the sun, the temperature of 10^7 K corresponds to an energy per particle of about 1 keV. The binding energy of helium is approximately 7MeV, approximately 7000 times bigger than the thermal content. Thus, the sun could last approximately 7000 times longer, bringing the projected lifetime of the sun up to a more reasonable (but still wrong) number of about 200 billion years. We conclude that nuclear energy is a more promising form of energy for the sun than chemical energy.

Wednesday, January 18, 2012

3.4 Gas Pressure and Radiation Pressure

Recall from the case of the constant density star that the gravitational energy is given by

$$E_{\rm GR} = -\frac{3}{5}\frac{GM^2}{R} \tag{53}$$

And the average pressure is given by the virial theorem to be

$$\langle P\rangle = -\frac{1}{3}\frac{E_{\rm GR}}{V} = (n_e + n_p)kT = 2n_p kT = \frac{2\rho kT}{m_p} \tag{54}$$

Here we've sort of assumed that the star is isothermal. This tells us that the average thermal energy is given by

$$kT = \frac{1}{10}\frac{GMm_p}{R} \tag{55}$$

Where the mass is given by

$$M = \rho\frac{4\pi}{3}R^3 \tag{56}$$

and the central temperature is given approximately by (scaled by solar units)

$$T_c \approx 2\times 10^6\,{\rm K}\left(\frac{\rho_c}{1\,{\rm g\,cm^{-3}}}\right)^{1/3}\left(\frac{M}{M_\odot}\right)^{2/3} \tag{57}$$

These scalings are actually recovered in simulations (see MESA plot from class). Here we've only considered the case of the pressure due to an ideal gas, thus far ignoring the contributions to radiation pressure. We then want to know when radiation pressure becomes comparable to gas pressure. That is,

$$P_{\rm rad} = \frac{1}{3}aT^4 \geq P_{\rm gas} \tag{58}$$

The temperature in the star is approximately

$$kT \sim \frac{GMm_p}{R} \tag{59}$$

and the pressure gradient is, (again, very approximately)

$$\frac{dP}{dr} = -\rho g \approx \frac{P_c}{R} \sim \frac{M}{R^3}\frac{GM}{R^2} \tag{60}$$

Then the condition we are seeking is

$$\frac{1}{3}a\left(\frac{GMm_p}{Rk}\right)^4 \gtrsim \frac{GM^2}{R^4} \tag{61}$$

Interestingly, R cancels in (61), so this condition is dependent only on the mass of the star. Thus, we can get a hard limit that is independent of any other properties of the star. Dropping tons more constants, this gives

$$M^2 > \frac{k_b^4}{aG^3m_p^4} \tag{62}$$

Recall that the radiation constant is given by $a = \frac{\pi^2}{15}\frac{k^4}{(\hbar c)^3}$. Putting this in (62), we have

$$\frac{M^2}{m_p^2} > \frac{k^4(\hbar c)^3}{G^3m_p^6k^4} \sim \left(\frac{\hbar c}{Gm_p^2}\right)^3 \tag{63}$$

Then the limit on the mass is then

$$\boxed{M > m_p \left(\frac{\hbar c}{G m_p^2}\right)^{3/2}} \tag{64}$$

stars above this mass (approximately) have significant radiation pressure. Recall the fine structure constant

$$\alpha = \frac{1}{137} = \frac{e^2}{\hbar c} \tag{65}$$

Noting that the Coulomb energy is

$$E_{\text{Coulomb}} = \frac{e^2}{r} \tag{66}$$

Analagous to (66), we define a unit less measure of the strength of gravity, which appears in (64):

$$\alpha_{\text{G}} = \frac{G m_p^2}{\hbar c} \approx 6 \times 10^{-39} \tag{67}$$

Then the fundamental stellar mass given in (64) is

$$M > m_p \frac{1}{\alpha_{\text{G}}^{3/2}} \approx 2 M_\odot \tag{68}$$

After all that work... it turns out that the mass where radiation pressure *actually* starts to matter is closer to $60 - 90\ M_\odot$.

Of interest in this case is that as $P_{\text{rad}} \gg P_{\text{gas}}$, then $E_{\text{tot}} \to 0$ from the virial theorem. In this state, the star has enough energy to unbind itself, so radiation pressure sets an upper limit on the mass of a star.

3.5 Summary

Note that here we have used hydrostatic balance to find the central temperature as a function of mass and radius. Additionally we have realized that energy losses from the surface require the radius of a star to decrease and the core temperature to increase (at least until another energy source is present). We will later show that the main sequence is just that place where the power generated by nuclear reactions is equal to that released by the star so that the star need not contract. What we have *not* done yet is to derive the rate of heat loss from the star.

4 Heat Transfer in Stars

In studying the ways in which heat moves outward through a star, we will first be ignoring convection, though it is a powerful mechanism when it is available to the star. To move heat through a star, there are electrons, ions, and photons available. Recall (or perhaps you don't) **Fick's Law**, which tells us that the heat flux can be determined via

$$F = \text{ergs}\,\text{cm}^{-2}\,\text{s}^{-1} = -\frac{1}{3} v \ell \frac{d}{dx}(E) \tag{69}$$

where here E is the energy density. We'll start first with the energy density of an ideal gas of electrons, but before we do that, let's derive (69).

4.1 Heat Flux Derivation

Imagine a medium with a gradient in temperature across a surface membrane. Let's say "above" the membrane, the temperature is T_1 and "below" the membrane, the temperature is T_2, with $T_2 > T_1$. Particles from region 2 then transport heat when they travel from region 2 to region 1. Let's call E the internal energy per unit volume as defined before, and then let's see what happens at the surface. Particles, on average, will be coming from a distance $x + \ell$ above the membrane, where $\ell = (\sigma n)^{-1}$ is the mean free path of the particles. Then the downward flow of energy is

$$F_{\text{down}} \approx \frac{1}{6} v E(x + \ell) \tag{70}$$

whereas particles from beneath move upward and carry heat from below at

$$F_{\text{up}} \approx \frac{1}{6} v E(x - \ell) \tag{71}$$

So the net flux in the positive $\hat{x}$ direction is

$$F_x = -\frac{1}{6} v E(x + \ell) + \frac{1}{6} v E(x - \ell) \tag{72}$$

Writing the energy densities as linear functions,

$$E(x + \ell) = E(x) + \ell \frac{dE}{dx} \tag{73}$$

$$E(x - \ell) = E(x) - \ell \frac{dE}{dx} \tag{74}$$

so

$$F_x = -\frac{1}{3} v \ell \frac{dE}{dx} \tag{75}$$

4.2 Heat Transport by Electrons

For electrons, the energy density is

$$E = \frac{3}{2} k T n_e \tag{76}$$

Then the energy density gradient is

$$\frac{dE}{dx} = \frac{dE}{dT} \frac{dT}{dx} \frac{dE}{dT} = \frac{3}{2} n k \frac{dT}{dx} \tag{77}$$

Then from Fick's Law, we have

$$F = -\frac{1}{3} v \ell \frac{3}{2} n k \frac{dT}{dx} = -\frac{1}{2} v \ell n k \frac{dT}{dx} \tag{78}$$

Where the mean free path is

$$\ell = \frac{1}{n\sigma} \tag{79}$$

for the scattering cross section σ. Then (78) becomes

$$F = -\left[\frac{1}{2} v \frac{k}{\sigma}\right] \boldsymbol{\nabla} T \tag{80}$$

From the theory of Coulomb scattering, the cross section would be

$$\sigma_{\text{Coulomb}} \sim b^2 \sim \frac{e^4}{(kT)^2} \tag{81}$$

Which gives a flux of

$$L = 4\pi R^2 F = 4\pi R \frac{(kT)^{7/2}}{m_e^{1/2} e^4} \tag{82}$$

For the sun, this would give

$$L \approx 5 \times 10^{31}\,\mathrm{erg\,s^{-1}} \tag{83}$$

which is two orders of magnitude too small, so we conclude that the sun does not transmit its heat via conduction through electrons. Now we'll move on to photons

4.3 Heat Transport by Photons

For photons, the main scatterer will be electrons, so the cross section in question of the mean free path is the Thomson cross section. Additionally, the energy density is now $E = aT^4$. Additionally the speed of photons is obviously the speed of light. Then the ratio of the fluxes due to photons and electrons is

$$\frac{F_\gamma}{F_e} = \frac{c}{(kT/m_e)^{1/2}} \frac{e^4/(kT)^2}{\sigma_{\text{Th}}} \frac{E_{\text{rad}}}{E_{\text{gas}}} \tag{84}$$

Remember that the Thomson cross section is given by

$$\sigma_{\text{Th}} = \frac{8\pi}{3} \frac{e^4}{(m_e c^2)^2} \tag{85}$$

Then (84) becomes

$$\frac{F_\gamma}{F_e} = \left(\frac{m_e c^2}{kT}\right)^{1/2} \left(\frac{m_e c^2}{kT}\right)^2 \frac{P_{\text{rad}}}{P_{\text{gas}}} \tag{86}$$

Comparing the pressures gives

$$\frac{P_{\text{rad}}}{P_{\text{gas}}} \approx 10^{-4} \left(\frac{M}{M_\odot}\right)^2 \tag{87}$$

Then plugging (87) into (86), we see that heat transport by photons dominates heat transport by electrons whenever

$$M > 0.03\, M_\odot \left(\frac{T}{10^7\ \mathrm{K}}\right)^{5/4} \tag{88}$$

So if conduction ever dominates, it is in very low mass stars (also white dwarfs in their late lives). For our cases, photons are always going to be the dominant transport mechanism. We still haven't found out what the actual luminosity will be in the case of radiative diffusion. We do so now.

$$L = 4\pi R^2 F = 4\pi R^2 \frac{1}{3} c\ell \frac{d}{dr}\left(aT^4\right) \approx R^2 c \frac{1}{n_e \sigma_{\text{Th}}} \frac{1}{R} aT^4 \approx R^2 \frac{c m_p}{\rho \sigma_{\text{Th}}} \frac{1}{R} a \left(\frac{GMm_p}{Rk}\right)^4 \tag{89}$$

where we have noted that $kT = \frac{GMm_p}{R}$. Continuing on,

$$L \approx \frac{cm_p a(GMm_p)^4}{M\sigma_{\rm Th}k^4} \propto M^3 \tag{90}$$

This relation is surprisingly accurate for stars with masses greater than a solar mass. Note that we have derived the stellar luminosity with *no knowledge* of the source of energy. The luminosity is set by the modes of heat transport available to the star.

Monday, January 23, 2012

4.3.1 The Eddington Limit and the Eddington Standard Model

Continuing with heat transfer via radiation with electron scattering being the primary source of opacity, the flux is given via Fick's law as

$$F = \frac{1}{3}v\ell\frac{d}{dz}(aT^4) \tag{91}$$

where now $v = c$ and $\ell = (n_e\sigma_{\rm Th})^{-1}$. Plugging these in to (91), the flux becomes

$$F = \frac{1}{3}c\frac{1}{n_e\sigma_{\rm Th}}\frac{d}{dz}(aT^4) = \frac{4}{3}\frac{acT^3}{\rho\kappa}\frac{dT}{dz} \tag{92}$$

where we've defined the **opacity** via

$$\kappa_{\rm es} \equiv \frac{\sigma_{\rm Th}}{m_p} \tag{93}$$

The opacity is measured in units of area per unit mass, indicating it is the cross-sectional area per unit mass. For electron scattering, it is simply a constant, but when other processes are relevant, it may depend on temperature and density. With the flux available, we can write the luminosity as

$$L(r) = F(r)4\pi r^2 = -4\pi r^2\frac{4}{3}\frac{acT^3}{\rho\kappa}\frac{dT}{dr} = -4\pi r^2\frac{c}{\kappa\rho}\frac{d}{dr}P_{\rm rad} \tag{94}$$

Noting the product $\rho\,dr$ showing up in the denominator of (94), we are reminded of hydrostatic equilibrium:

$$\frac{dP}{dr} = -\rho(r)g(r) \quad\Rightarrow\quad dP = -\rho(r)g(r)\,dr \tag{95}$$

Putting in $g(r)$ explicitly,

$$\frac{dP}{\rho(r)dr} = -\frac{Gm(r)}{r^2} \tag{96}$$

Bringing the column depth back in ($y = \int\rho(r)\,dr$), this becomes

$$\frac{dP}{dy} = \frac{Gm(r)}{r^2} \tag{97}$$

where we've used the fact that $y = 0$ at the surface and increases *inwards* (hence the sign change). Assuming g is a constant, this would give our old result

$$\boxed{P = gy} \tag{98}$$

Now returning to (94):

$$\frac{dP_{\text{rad}}}{dy} = \frac{L(r)}{4\pi r^2}\frac{\kappa}{c} \tag{99}$$

and the result we just derived

$$\frac{Gm(r)}{r^2} \tag{100}$$

Taking the ratio of (99) and (100) gives us

$$\frac{dP}{dP_{\text{rad}}} = \frac{4\pi Gm(r)c}{\kappa L(r)} \tag{101}$$

Suppose $M = m(R) =$ total mass of star and $L = L(R) =$ luminosity of star

$$\frac{dP}{dP_{\text{rad}}} = \left[\frac{4\pi GcM}{\kappa L}\right]\frac{m(r)}{M}\frac{L}{L(r)} \tag{102}$$

We have now introduced the **Eddington Luminosity**,

$$L_{\text{Edd}} = \frac{4\pi GcM}{\kappa} \tag{103}$$

In general, the Eddington Luminosity is much larger than the luminosity, since it is the luminosity where the only pressure gradient that matters is the radiation pressure (where the star becomes unstable). Evaluating the Eddington Luminosity in solar units and with electron scattering,

$$L_{\text{Edd}} = 3.1 \times 10^4\ L_\odot \left(\frac{M}{M_\odot}\right) \tag{104}$$

Also recall how luminosity scales with mass:

$$L = L_\odot \left(\frac{M}{M_\odot}\right) \tag{105}$$

So we see that for low-mass stars, their luminosities are indeed much smaller than the Eddington Luminosity. Now let's investigate the ratio of the luminosity to the Eddington Luminosity:

$$\frac{L}{L_{\text{Edd}}} \approx 3 \times 10^{-5} \left(\frac{M}{M_\odot}\right)^2 \tag{106}$$

So until $M \geq 100\, M_\odot$, it will be the case that $L \ll L_{\text{Edd}}$ and thus $P_{\text{rad}} \ll P$. Now returning to (102), we define a dimensionless quantity, $\eta(r)$ by

$$\eta(r) \equiv \frac{L(r)}{L} M m(r) \tag{107}$$

And (102) looks like

$$\frac{dP_{\text{rad}}}{dP} = \frac{L}{L_{\text{Edd}}}\eta(r) \tag{108}$$

Aside: this model of stars is the called the "Eddington Standard Model" and was used to describe stars before their source of energy was known.

4.3.2 Polytropic Relations

Now integrating (108), we have

$$\int_R^r dP_{\rm rad} = \frac{L}{L_{\rm Edd}} \int_R^r \eta(r)\, dP \tag{109}$$

We can integrate the left side, leaving us with

$$P_{\rm rad}(r) = \frac{L}{L_{\rm Edd}} \int_R^r \eta(r)\, dP \tag{110}$$

Mathematically speaking, we are now stuck because we do not have any knowledge of $\eta(r)$. The formal approach of solving this problem is to define spatial averages of $\eta(r)$ for different choices of mass and luminosity profiles. We will, however, just let $\eta(r) \approx 1$, which gives us

$$P_{\rm rad} = \frac{L}{L_{\rm Edd}} P_{\rm gas}(r) \tag{111}$$

And now substituting in our equations of state for the radiation pressure and assuming that the radiation pressure is negligible compared to the gas pressure:

$$\frac{1}{3} a T^4 = \frac{L}{L_{\rm Edd}} \frac{\rho k T}{\mu m_p} \tag{112}$$

Solving this for T^3, we have

$$\boxed{T(r)^3 = \frac{L}{L_{\rm Edd}} \frac{d\kappa_R}{a \mu m_p} \rho(r)} \tag{113}$$

So we have gone from a situation where we had only typical or central values to an actual equation that let's us find $T(r)$, $\rho(r)$, and $P(r)$. It appears then, that in our low radiation pressure limit, $\rho \propto T^3$. Additionally, for the ideal gas law, we see that $P \propto \rho^{4/3}$. For some given $L/L_{\rm Edd}$, we have a relation between pressure and density via

$$\frac{dP}{dr} = -\rho(r) \frac{Gm(r)}{r^2} \tag{114}$$

$$dm(r) = \rho 4\pi r^2\, dr \tag{115}$$

Let us briefly consider an adiabatic change in an ideal gas. We recall from Freshman physics that $PV^\gamma = \text{const}$. For a monatomic, ideal gas, $\gamma = 5/3$, so we have $P \propto \rho^{5/3}$ and $\rho T \propto P$, so $T \propto \rho^{2/3}$. Then we'll call the "entropy" $T/\rho^{2/3}$. For our star, we have $T^3 \propto \rho \Rightarrow T \propto \rho^{1/3}$. In our case, the "entropy" is then $a/\rho^{1/3}$. Thus the entropy is highest in the outer layers and smallest in the center.

4.3.3 Heat Transfer in the Outer Atmosphere

Near the surface of the star, photons naturally leave (which is why we see them!) We want to know what the place or condition is like. Near the surface, the density must decrease exponentially since $T \ll T_c$ and thus g is a constant. Our previous arguments for the scale height and the plane-parallel isothermal atmosphere is largely applicable here, so the length scale of relevance is the scale height:

$$h = \frac{kT}{\mu m_p g} \tag{116}$$

It then makes sense that the condition for photons to escape would be for the mean free path to be comparable to the scale height:

$$\ell \sim h \quad \Rightarrow \quad \frac{m_p}{\rho\sigma_{\mathrm{Th}}} \approx \frac{kT}{mg} \tag{117}$$

or

$$g\frac{m_p}{\sigma_{\mathrm{Th}}} \sim \frac{\rho kT}{m_p} \sim P_{\mathrm{gas}} \tag{118}$$

Again, this can be rewritten as

$$g\kappa^{-1} \sim P_{\mathrm{gas}} \quad \text{where} \quad \kappa = \frac{\sigma_{\mathrm{Th}}}{m_p} \tag{119}$$

Then we can say the condition where photons can escape is

$$P_{\mathrm{gas}} \leq \frac{g}{\kappa} \tag{120}$$

This argument assumed a constant opacity. In general, we must do a line integral through the depth of the outer atmosphere. Imagine we ask the probably for a photon to reach some depth in the star. We define the **optical depth** to be

$$d\tau = \frac{dr}{\ell} = dr\ \kappa(r)\rho(r) \tag{121}$$

Then the probability would be

$$P(\tau) \propto e^{-\tau} \tag{122}$$

Then the optical depth at a certain radius would be given by

$$\tau = \int_R^r \kappa(r)\rho(r)\,dr \tag{123}$$

or, invoking hydrostatic equilibrium,

$$\tau = \int \kappa(r)\frac{dP(r)}{g} = \frac{1}{g}\int \kappa(r)\,dP(r) \tag{124}$$

Then the condition for photons for leave becomes when the optical depth is unity, or

$$\tau \sim 1 = \frac{1}{g}\kappa P(r) \quad \Rightarrow \quad P(\tau = 1) = \frac{g}{\kappa} \tag{125}$$

Additionally, we can redefine the flux equation in terms of optical depth into a cute form:

$$F = \frac{c}{3}\frac{d}{d\tau}\left(aT^4\right) \tag{126}$$

4.3.4 The Effective Temperature

At the surface, the flux is given by

$$F = \frac{L}{4\pi R^2} \equiv \sigma_{\rm SB} T_{\rm eff}^4 \tag{127}$$

Assuming that the flux is constant at or near the surface, we may use the fact that $a = 4\sigma_{\rm SB}/c$ and (126) to get

$$\frac{d}{d\tau} T^4 = \frac{3}{4} T_{\rm eff}^4 \tag{128}$$

Recall that τ is dimensionless, so there is no problem here. For $\tau \gg 1$, we have

$$T^4 = \frac{3}{4} T_{\rm eff}^4 \tau \tag{129}$$

This implies that as τ increases (going deeper and deeper into the star), the radiation field becomes more and more isotropic. Naively, we might assume that the flux would be given by the energy density multiplied by the speed of light, but our result in (129) essentially tells us that $F = acT^4/\tau$.

Wednesday, January 25, 2012

4.4 Convection

Another important form of heat transport in stars is that of convection, which is where an instability causes the bulk movement of material (rather than the diffusion of photons or electrons) to transport heat throughout the star.

The origin of the instability is that a fluid element that rises adiabatically (faster than the heat transfer time scale) is **lighter** than the surrounding fluid and so it "runs away". This is a linear instability, in that a displacement exponentially grows in time.

Suppose we ahve a fluid element in hydrostatic balance. We imagine displacing this fluid element from r to $r+dr$. So we assume that the displaced element responds adiabatically. Thermodynamcis tells us that

$$T\,dS = dE + P\,dV = 0 \tag{130}$$

for an adiabatic process. This is equivalent to the requirement that the timescale of response is much less than that for heat to enter or leave the fluid element. We presume that hte timescale is longer than the time for pressure equilibrium as well:

$$t = \frac{h}{c_s} = \text{time for the pressure to equilibrate} \quad \Rightarrow \quad v \ll c_s \tag{131}$$

where h is agin the scale height. Suppose the element starts in location 1 and travels to location 2. This adiabatic process requires that PV^γ be a constant within the bubble, which means that

$$\frac{P_1}{\rho_1^\gamma} = \frac{P_2}{\rho_2^\gamma} \tag{132}$$

where the subscripts refer to where the bubble is located. Rearranging (132) gives us the new density

$$\rho_{2,b}^{\gamma} = \rho_1^{\gamma}\frac{P_2}{P_1} \tag{133}$$

So we require $\rho_{2,b} > \rho_{2,*}$, where the * indicates the density of the star's "background" conditions. For the star, we require

$$\rho_1\left(\frac{P_2}{P_1}\right)^{1/\gamma} > \rho_2 \tag{134}$$

For an infinitesimal change, this gives

$$P_2 = P_1 + \Delta r\frac{dP}{dr} \tag{135}$$

and

$$\rho_2 = \rho_1 + \frac{d\rho}{dr}\Delta r \tag{136}$$

Then (134) requires

$$\frac{1}{\gamma}\frac{d\ln P}{d\ln r} > \frac{d\ln\rho}{dr} \tag{137}$$

Now, we recall that

$$P = \frac{\rho kT}{\mu m_p} \quad\Rightarrow\quad d\ln\rho = d\ln P - d\ln T \tag{138}$$

and thus (137) can be written as

$$\frac{1}{\gamma}\frac{d\ln P}{dr} > \frac{d\ln P}{dr} - \frac{d\ln T}{dr} \tag{139}$$

For an ideal, monatomic gas, this becomes

$$\frac{d\ln T}{d\ln P} < 1 - \frac{1}{\gamma} = \frac{2}{5} \tag{140}$$

Recall that earlier we found that for radiative heat transport in the Eddington standard model, $\rho \propto T^3$, and thus $P \propto \rho T \propto T^4$, so $T \propto P^{1/4}$. Then computing the logarithmic derivative in (140),

$$\frac{d\ln T}{d\ln P} = \frac{1}{4} < \frac{2}{5} \tag{141}$$

So stars folloiwng this model are stable and do not transport heat by convection. A model that is stable to convection has the entropy increasing with radius. Note that when we say "entropy", we mean the **specific entropy**, or the entropy per unit mass.

4.4.1 The Unstable Case

Now we wish to understand what happens when the background model is unstable. The density in the bubble is

$$\rho_{2,b} = \rho_2\left(\frac{P_2}{P_1}\right)^{1/\gamma} \tag{142}$$

and for the star,

$$\rho_{2,*} = \rho_1 + \Delta r \left.\frac{d\rho}{dr}\right|_* \tag{143}$$

The density contrast is then

$$\Delta\rho = \rho_{2,*} - \rho_{2,b} = \Delta r \left[\left.\frac{d\rho}{dr}\right|_* - \frac{1}{\gamma}\frac{\rho}{P}\left.\frac{dP}{dr}\right|_* \right] \tag{144}$$

or

$$\Delta\rho = \rho\Delta r \left[\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma} \right] \tag{145}$$

where we have invoked hydrostatic equilibrium. Stability requires $\Delta < 0$, or

$$\frac{d\ln\rho}{dr} < -\frac{\rho g}{\gamma P} \tag{146}$$

In the unstable case, though, we require $\Delta\rho > 0$. the acceleration, a, or the displaced element will be given by

$$a = \frac{\Delta\rho}{\rho} g = g\Delta r \left[\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma} \right] \tag{147}$$

This is the equation of mothion for the fluid element. Written in terms of the displacement coordinate x, (147) is

$$\ddot{x} = gx \left(\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma} \right) \tag{148}$$

Strictly speaking this equation holds for unstable or stable cases. Clearly this is a harmonic oscillator, so the physics should be familiar. In any case, we have

$$-\omega^2 = g \left(\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma} \right) \tag{149}$$

If $-\omega^2 < 0$, then the solution is stable, and the element oscillates at the Brunt Vüisülü frequency is

$$N^2 - g \left[\frac{d\ln\rho}{dr} + \frac{\rho g}{\gamma P} \right] \tag{150}$$

This is the local frequency of oscillation of a fluid element in a convectively stable atmosphere. In the Earth's atmosphere, this frequency is around five or ten minutes.

Now if this coefficient is negative, the solution is unstable (i.e., groes exponentially). Note that $N^2 \sim g/h$, so

$$N^2 \sim \frac{g m_p g}{kT} \sim g^2 \frac{1}{v_{\rm th}^2} \quad \Rightarrow N \sim \frac{g}{v_{\rm th}} \sim \frac{g}{c_s} \tag{151}$$

So our displacement solution looks like

$$x = x_0 e^{t/\tau} \tag{152}$$

where $1/\tau^2 = -N^2$. So the speed is

$$\dot{x} = \frac{x_0}{\tau} e^{t/\tau} = \ell/\tau = \text{speed after moving a distance } \ell \tag{153}$$

Generally, the speed is given by

$$v = \ell\sqrt{-N^2} \tag{154}$$

Suppose that the stellar interior is strongly unstable, so that $N^2 \sim -g/h$. Then the velocity after traveling some length ℓ is

$$v = \ell\left(\frac{g}{h}\right)^{1/2} = \ell\left(\frac{gm_pg}{kT}\right)^{1/2} = \ell\frac{g}{v_{\rm th}} =\approx \frac{g}{v_{\rm th}}h = \frac{g}{v_{\rm th}}\frac{v_{\rm th}^2}{g} = v_{n\rm th} \tag{155}$$

So if $N^2 \sim -g/h$, i.e., the star is strongly unstable, then a runaway fluid element will reach the sound speed after traversing one scale height! Recall, though that the change in density is

$$\left.\frac{\Delta\rho}{\rho}\right|_{\text{wt }\ell} \approx \ell\left(\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma}\right) \tag{156}$$

and thus the velocity is

$$v = \ell\sqrt{\frac{g}{\ell}\left.\left(\frac{\Delta\rho}{\rho}\right)\right|_\ell} = (g\ell)^{1/2}\left(\left.\frac{\Delta\rho}{\rho}\right|_\ell\right)^{1/2} \tag{157}$$

Monday, January 30, 2012

4.4.2 Convective Efficiency

We left off last time with a rough relation between the velocity of an upwardly rising fluid element and the density contrast between the fluid element and the background material at some height ℓ above the element's original location. In the unstable case, we found that the element was less dense than the surrounding fluid, by examining the quantity $\Delta\rho/\rho|_\ell$. The velocity at which the element traveled was given by $v = (g\ell)^{1/2}(\Delta\rho/\rho|_\ell)^{1/2}$.

Inserting the scale height in for ℓ,

$$\ell = h = \frac{kT}{mg} = \frac{c_s^2}{g} \tag{158}$$

Then the velocity would be

$$v = c_s\left[\left.\frac{\Delta\rho}{\rho}\right|_h\right]61/2 \tag{159}$$

Now we consider the flux due to convection. The flux is just the velocity times the energy density, and so we can write

$$F = v\rho v^2 = \rho v^3 \tag{160}$$

Assuming $\Delta\rho/\rho \ll 1$, we may assume that the element is moving subsonically.

The business of heat flux via convection is covered in **Mixing Length Theory**. In this "theory", we imagine a convective element traveling up a **mixing height**, ℓ and then allowing the fluid element to equilibrate with the surroundings. Since the preferred lengthscale in a star is the scale height, we typically choose $\ell_{\rm MT} = \alpha h$ for some dimensionless constant α. Applying this to (160),

$$F_{\rm conv} = \rho\frac{kT}{m_p}c_s\left(\left.\frac{\Delta\rho}{\rho}\right|_{\ell=h}\right)^{3/2} = Pc_s\left(\left.\frac{\Delta\rho}{\rho}\right|_{\ell=h}\right)^{3/2} \tag{161}$$

Now recall our result for the radiative flux:

$$F_{\mathrm{rad}} = \frac{1}{3}\frac{c}{\kappa\rho}\frac{d}{dr}\left(aT^4\right) \approx \frac{cP_{\mathrm{rad}}}{\tau} \tag{162}$$

where in the second equality, we're let $d/dr \to 1/h$. Now we find the ratio in these two fluxes:

$$\frac{F_{\mathrm{conv}}}{F_{\mathrm{rad}}} \sim \frac{\tau P c_s \left(\Delta\rho/\rho|_h\right)^{3/2}}{cP_{\mathrm{rad}}} \sim \frac{c_s}{c}\frac{P}{P_{\mathrm{rad}}}\tau\left(\left.\frac{\Delta\rho}{\rho}\right|_h\right)3/2 \tag{163}$$

For typical, low-mass stars, we assume that $P/P_{\mathrm{rad}} \gg 1$, let's find when both fluxes give equal contributions (also plugging in some solar values):

$$\left(\frac{kT}{m_p c^2}\right)^{1/2} 10^4\tau\left(\frac{\Delta\rho}{\rho}\right)^{1/2} \sim 1 \tag{164}$$

The requirement becomes

$$\frac{\Delta\rho}{\rho} \approx 2\left(\frac{10^4}{T}\right)^{1/3}\frac{1}{\tau^{2/3}} \tag{165}$$

Where τ and T get smaller, the simple theory implies convection at near the sound speed, c_s. In truth, you can't do this problem correctly in one dimension, so this result isn't all that good. What does this imply about the surface, though? Stellar convection at or near the surface becomes sonic, which we say is **inefficient** *and* $\Delta\rho/\rho \to 1$.

> What I'm trying to do here is to write a density model in 1-D that doesn't make me blush. *–Lars Bildsten*

To find a situation where this *does* work is deep in the interior of the star, where $\tau \ggg 1$ (yes, three greater than signs). The relevant lengthscale here is no longer the scale height, which begins to approach the radial coordinate:

$$h = \frac{P}{\rho g} \sim \frac{GM^2}{R^4 M}\frac{R^3 R^2}{GM} \sim R \tag{166}$$

so we'll just use the radius of the star as our characteristic length scale. The ratio of fluxes is now

$$\frac{F_{\mathrm{conv}}}{F_{\mathrm{rad}}} \sim \frac{c_s}{R}\left[\frac{P}{P_{\mathrm{rad}}}\tau\frac{R}{c}\right]\left(\left.\frac{\Delta\rho}{\rho}\right|_R\right)^{3/2} \tag{167}$$

Note that the time it takes for a photon to diffuse from a star is $t_{\mathrm{rw}} = \tau R/c$. Thus, that is the time it takes to evacuate the radiation field of energy, so multiplying by the ratio in pressures is actually the Kelvin-Helmholtz time. Using this fact, (167) can be written as

$$\frac{F_{\mathrm{conv}}}{F_{\mathrm{rad}}} = \frac{t_{\mathrm{KH}}}{t_{\mathrm{dyn}}}\left(\left.\frac{\Delta\rho}{\rho}\right|_R\right)^{3/2} \tag{168}$$

Deep in the star, $t_{\mathrm{KH}} \ggg t_{\mathrm{dyn}}$, so convection is efficient when

$$\left.\frac{\Delta\rho}{\rho}\right|_R \sim \left(\frac{t_{\mathrm{dyn}}}{t_{\mathrm{KH}}}\right) \lll 1 \tag{169}$$

When convection is efficient, then, the stellar model nearly follows the adiabatic relation! Deep interior convection, in this limit, only implies that we need to know the adiabatic relation very well.

Note that the ratio of times scales like

$$\frac{t_{\rm dyn}}{t_{\rm KH}} \approx 3 \times 10^{-9} \left(\frac{M}{10\ M_\odot}\right)^{3.5} \tag{170}$$

Noting that $L \propto M^{3.5}$ and $R = R_\odot(M/M_\odot)$ on the main sequence, this gives us

$$\frac{v}{c_s} \approx 10^{-3} \left(\frac{M}{10\ M_\odot}\right)^{7/6} \tag{171}$$

Now remember the adiabatic condition for convection derived earlier:

$$\left.\frac{d\ln T}{d\ln P}\right|_* = \left.\frac{d\ln T}{d\ln P}\right|_{\rm adiabatic} = \frac{2}{5} \tag{172}$$

So in the convective zone, we must have $T \propto P^{2/5} \propto (\rho T)^{2/5}$, or more directly, $T \propto \rho^{2/3}$. So far, we have motivated *two* distinct **polytropic relations** (models where $P \propto \rho^{(n+1)/n}$ for some n):

1. **Fully Convective and Efficient**: $P \propto \rho^{5/3}$, where the prefactor is the specific entropy of the star, which is spatially constant. Thus if we have the mass of a star and the entropy, we can calculate everything we want to know about this star (in this simple model).

2. **Constant L/M** star with constant (electron scattering) opacity, where $P \propto \rho^{4/3}$.

4.5 Fully Convective Star

Imagine a star of ideal gas *and* from the core to the photosphere is fully convective *and* efficient (this will likely be a bad approximation near the surface). Somewhere near the surface, photons must be made to allow the energy to escape the star. Near the photosphere, the pressure is

$$P_{\rm ph} \approx \frac{g}{\kappa_{\rm ph}} \tag{173}$$

If we were to plot $\ln T$ against $\ln P$ (as we did in lecture), for this star, it would have a slope of 2/5. Getting out towards the surface, though, the curve would have to be a bit shallower. We hope this only happens out towards one scale height in from the surface. Incorporating this physics will change R, but only on the order of $h|_{\rm location}$. Noting that

$$\frac{h}{R} \sim \frac{T}{T_c} \tag{174}$$

we see this is an extremely small error, since T/T_c is very small out towards the surface. We assume $P \propto \rho^{5/3}$ (an $n = 3/2$ polytrope). In this model, the central pressure is

$$P_c = 0.77\frac{GM^2}{R^4} \tag{175}$$

The central temperature is

$$T_c = 0.54 \frac{GM\mu m_p}{kR} \tag{176}$$

Requiring the entropy be the same everywhere requires

$$\frac{T_c}{P_c^{2/5}} = \frac{T_{\rm ph}}{P_{\rm ph}^{2/5}} = \frac{T_{\rm eff}}{P_{\rm ph}^{2/5}} \tag{177}$$

Since we've assumed the perfect adiabat (admittedly a lie... also, μ is changing). The surface thermal energy is

$$kT_{\rm eff} = 0.6 \left(\frac{GM\mu m_p}{R}\right)\left(\frac{R^2}{M\kappa_{\rm ph}}\right)^{2/5} \tag{178}$$

Putting in $\kappa_{\rm ph} = \kappa_{\rm es}$ gives us

$$T_{\rm eff} = 200 \text{ K} \left(\frac{M}{M_\odot}\right)^{3/5}\left(\frac{R_\odot}{R}\right)^{1/5} \tag{179}$$

So we have a big problem with assumign electron scattering (or convection).

Wednesday, February 1, 2012

4.6 Fully Convective Stars and the Hayashi Track

MISSING DUE TO MY INABILITY TO WAKE UP BEFORE NOON

5 Star Formation and the Jean's Mass

5.1 The Jean's Mass

In the interstellar medium, there are large masses of cold dust and gas that can give rise to isolated regions of stellar formation. Often these manifestations are called **cold cores**. Suppose one of these cold cores is in the ISM with some density ρ and temperature T (likely around 10-20 K). The gravitational energy is something like

$$E_{\rm GR} \approx -\frac{GM^2}{R} \tag{180}$$

And the thermal content is something like

$$E_{\rm th} = \frac{3}{2}\frac{M}{m_p}kT \tag{181}$$

In order for this cloud to collapse, we must have the total energy being less than zero. Thus, we must have

$$\frac{GM^2}{R} > \frac{3}{2}\frac{M}{m_p}kT \tag{182}$$

If we assume a constant density profile, this tells us that the mass of the object must be

$$M > 500\ M_\odot \left(\frac{T}{10\ \mathrm{K}}\right)^{3/2} \left(\frac{1\ \mathrm{cm}^{-3}}{n}\right)^{1/2} \tag{183}$$

This mass is called the **Jean's Mass**. If a cloud is more massive than this, it is possible to collapse. Now, it may stay as one large blob, or it may fragment... we're not sure yet. The question is then, "How does the Jean's mass change in a collapsing cloud?" We know that the Jeans mass scales as

$$M_\mathrm{J} \propto T^{3/2} \rho^{-1/2} \tag{184}$$

As the star collapses the density must increase. *If* the material maintains its entropy, then we must have $T \propto \rho^{2/3}$. Then the Jean's mass scales as

$$M_\mathrm{J} \propto T^{3/2} \rho^{-1/2} \propto \rho^{1/2} \tag{185}$$

So if the collapse is adiabatic, the collapse does not lead to fragmentation. The cloud will reach some critical density, at which point the Jean's mass is higher than the mass of the cloud, halting collapse.

Cooling of the gas (i.e., the temperature remains constant) gives the Jean's mass trivially scaling as

$$M_\mathrm{J} \propto \frac{1}{\rho^{1/2}} \tag{186}$$

This allows for the possibility of fragmentation since the mass will remain above the Jean's mass. More likely, this will be the case at earlier times, and then later the collapse becomes more adiabatic until a freeze-out. What ends the fragmentation is that at high density, the optical depths are increasing and the gas cannot radiate on the contraction timescale. If we can analyze the microphysics governing these processes, we could determine a minimum Jean's mass, which would explain the fragmentation. At low metal content, the minimum masses are much higher. This is thought to be the reason why the first stars were so large.

When cooling is efficient, then the collapse and fragmentation occurs on the dynamical timescale:

$$t_\mathrm{dyn} \approx \frac{1}{\sqrt{G\rho}} = \frac{10^7\ \mathrm{years}}{(n/100\ \mathrm{cm}^{-3})^{1/2}} \tag{187}$$

Then it typically takes $1-10$ Myrs for the collapse to occur. At the center, an object at hydrostatic balance forms. This causes an **accretion luminosity**

$$L \approx \dot{M}\frac{GM_c}{R_c} \tag{188}$$

which is powered by a loss of gravitational potential energy. The shocks on the core surface typically have energies of

$$kT_\mathrm{shock} \approx \frac{GM_c m_p}{R_c} \approx 0.1\ \mathrm{keV} \tag{189}$$

5.2 Pre-Main Sequence Stars

Recall the main equations of stellar structure:

$$\frac{dP}{dr} = -\rho(r)g(r) \tag{190}$$

$$dm(r) = 4\pi r^2 \rho(r)\, dr \tag{191}$$

Fluxes defined by temperature gradient

We need to be able to have an equation describing energy (the next moment, as it were). Looking at the second law of thermodynamics, we have

$$dQ = TdS = dE + PdV \tag{192}$$

The corresponding time-dependent equation in the Lagrangian is

$$T\frac{dS}{dt} = \frac{dE}{dt} + P\frac{dV}{dt} \tag{193}$$

where E is the internal energy per unit mass. In terms of the specific units (per unit mass),

$$dQ = dE + pd\left(\frac{1}{\rho}\right) = dE - \frac{P}{\rho^2}d\rho \tag{194}$$

The Lagrangian equation becomes

$$T\frac{ds}{dt} = \frac{dQ}{dt} \equiv \text{loss or gain} \tag{195}$$

First we have heat gained via neuclear reactions, ε. Secondly, there is heat gained or lost due to a gradient in $\mathbf{F}$. Then (195) becomes our last equation for stellar evolution:

$$\boxed{T\frac{ds}{dt} = \varepsilon_{\text{nuc}} - \frac{\boldsymbol{\nabla}\cdot\mathbf{F}}{\rho}} \tag{196}$$

Here we ahve $F\mathbf{F} = F_r\hat{r}$. Then

$$\frac{\boldsymbol{\nabla}\cdot\mathbf{F}}{\rho} = \frac{1}{\rho}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) \tag{197}$$

and the luminoisty is

$$L(r) = 4\pi r^2 F_r \tag{198}$$

Then we can rewrite (197) as

$$\frac{\boldsymbol{\nabla}\cdot\mathbf{F}}{\rho} = \frac{1}{\rho 4\pi r^2}\frac{\partial}{\partial r}L(r) = \frac{dL(r)}{dm(r)} \tag{199}$$

So in the case of no nuclear burning (a pre-main sequence star), we have

$$T\frac{ds}{dt} = -\frac{dL(r)}{dm(r)} \tag{200}$$

We assume that the lumonisty gradient is positive due to the temperature gradient, so under the absence of an internal energy source, the entropy will decrease. It will decrease at the rate set by the heat loss to infinity. Adding on the rest to (200),

$$T\frac{ds}{dt} = -\frac{dL(r)}{dm(r)} = \frac{dE}{dt} - \frac{P}{\rho^2}\frac{d\rho}{dt} \tag{201}$$

Note that the internal energy is

$$E = \frac{3}{2}\frac{kT}{\mu m_p} = \frac{3}{2}\frac{P}{\rho} \tag{202}$$

Inserting this into (201) gives us

$$T\frac{ds}{dt} = \frac{3}{2}\frac{P}{\rho}\frac{d}{dt}\left(\ln P/\rho^{5/3}\right) = -\varepsilon_{\rm grav} \tag{203}$$

Textbooks will often refer to this as $-\varepsilon_{\rm grav}$. This is done so that

$$\frac{\partial L(r)}{\partial m(r)} = \varepsilon_{\rm nuc} + \varepsilon_{\rm grav} \tag{204}$$

Monday, February 6, 2012
Let's do some dimensional analysis for a contracting star. The pressure is

$$P \sim \frac{GM^2}{R^4} \sim \frac{GM^2}{M^{4/3}}\rho^{4/3} \tag{205}$$

Then the quantity in the log in (204) would be

$$\frac{P}{\rho^{5/3}} \approx C\frac{1}{\rho^{1/3}} \tag{206}$$

If the star is fully convective, then $P/\rho^{5/3}$ must be spatially constant, and we must have

$$T\frac{ds}{dt} = \frac{3}{2}\frac{P}{\rho}\frac{d}{dt}\left[\ln\left(C\rho_c^{-1/3}\right)\right] = -\frac{1}{2}\frac{P}{\rho}\frac{d\ln\rho_c}{dt} \tag{207}$$

Assuming that $\varepsilon_{\rm nuc} = 0$, (203) becomes

$$-\frac{1}{2}\frac{P}{\rho}\frac{d\ln\rho_c}{dt} = -\frac{dL(r)}{dm(r)} \quad \Rightarrow \quad \frac{dL(r)}{dm(r)} = \frac{1}{2}\frac{P}{\rho}\frac{d\ln\rho_c}{dt} \tag{208}$$

where $\rho_c \propto M/R^3$. Now we want to get (208) in terms of more physical variables. Assuming M is constant in time,

$$\frac{dL(r)}{dm(r)} = \frac{1}{2}\frac{P}{\rho}\frac{d}{dt}\left[\ln\left(\frac{1}{R^3}\right)\right] \quad \Rightarrow \quad \frac{dL(r)}{dm(r)} = -\frac{3}{2}\frac{P}{\rho}\frac{d}{dt}\ln R \tag{209}$$

So, assuming a fully-convective star with no mass loss, we have

$$\boxed{\frac{dL(r)}{dm(r)} = -\frac{3}{2}\frac{P(r)}{\rho(r)}\frac{1}{R}\frac{dR}{dt}} \tag{210}$$

Backtracking a bit, earlier we showed that

$$\text{entropy} \propto \frac{P}{\rho^{5/3}} \sim \frac{GM^2}{R^4}\frac{R^5}{M^{5/3}} \propto RM^{1/3} \tag{211}$$

for a sta in hydrostatic balance. An adiabatic adjustment to this star must require $R \propto 1/M^{1/3}$ (entropy must remain constant). Thus if the mass decreases, the radius must increase (like in white dwarfs).

Now we want to integrate (210). Multiplying both sides by dm/dr and integrating from the center to the surface gives

$$L = -\frac{3}{2}\frac{1}{R}\frac{dR}{dt}\int_0^R \frac{P(r)}{\rho(r)}\rho(r)4\pi r^2\, dr = -\frac{3}{2}\frac{1}{R}\frac{dR}{dt}\int_0^R P(r)4\pi r^2\, dr \tag{212}$$

We know the last integral from the Virial theorem's relation to the gravitational energy. The short answer is that

$$\int d^3r\, P = \frac{2}{7}\frac{GM^2}{R} \tag{213}$$

Then the luminosity is given by

$$L = -\frac{3}{7}\frac{GM^2}{R^2}\frac{dR}{dt} \tag{214}$$

This is essentially Kelving-Helmholtz contraction, but for a star that is fully convective, we can derive the "real equation" (as opposed to our more bogus results earlier).

5.3 Low Mass Stars

Earlier we derived that as stars are on the Hayashi track,

$$\frac{L}{L_\odot} \approx 0.03\ L_\odot \left(\frac{M}{M_\odot}\right)^{4/7}\left(\frac{R}{R_\odot}\right)^2 \tag{215}$$

Plugging this result in to our previous result can give us the radius as a function of time:

$$\frac{3}{7}\frac{GM^2}{R^2}\frac{dR}{dt} = -0.03\left(\frac{M}{M_\odot}\right)^{4/7}\left(\frac{R}{R_\odot}\right)^2 \tag{216}$$

We can now read off scalings: $\dot{R} \propto R^4$, $1/t \propto R^3$, and so $R \propto t^{-1/3}$. Explicitly, the radius for a fully-convective star coming down the Hayashi track

$$\frac{R}{R_\odot} = \left(\frac{M}{M_\odot}\right)^{10/21}\left(\frac{130\text{ Myr}}{t}\right)^{1/3} \tag{217}$$

For masses greater than about a half of a solar mass, convection will dominate heat transport until the luminosity reaches the level that was predicted by earlier opacity arguments that assumed radiative diffusion. The onset of a radiative core is then roughly when $L \lesssim L_{\rm rad}$. The time it takes for this to happen, we have

$$t > 10^6\text{ years}\ \left(\frac{M_\odot}{M}\right)^2 \tag{218}$$

So massive stars become radiative at a very young age. For low-mass stars, though,

$$t > 2.6 \times 10^6 \text{ years } \left(\frac{M_\odot}{M}\right)^{4.4} \tag{219}$$

For very low mass stars, this time approaches the Hubble time. However, very low mass stars will remain convective on the main sequence.

6 Nuclear Reactions in Stars

The big bang only mad (by mass) 75% protons and 25% Helium. In stars, we want to fuse protons into helium and release about 7 MeV per baryon, or 7×10^18 erg g^{-1}, which is *far* higher than the thermal energy content of matter in stars. As it turns out, the thermal energy is too low to bring two protons close enough together to fuse, so tunneling must take place. Additionally, there is no stable nucleus with just two (or more) protons, so weak interactions are required to have any stable reactions.

6.1 Liquid Drop Model

The "size" of a nucleus is approximately

$$r_0 = 1.3 \text{ fm } A^{1/3} \tag{220}$$

where A is the mass number ($A = N + Z$). Then the nuclear density is approximately

$$\rho = \frac{Am_p}{\frac{4\pi}{3}R^3} \approx 2 \times 10^{14} \text{ g cm}^{-3} \tag{221}$$

There is an energy well with

$$E_{\text{volume}} = -14 \text{ MeV } A \tag{222}$$

There must also be a surface term, dictating that surface particles are less bound than their counterparts deeper in

$$N_{\text{surf}} = A^{2/3} \sim \frac{\text{surface}}{\text{volume}} \tag{223}$$

So then we have

$$E_{\text{surf}} = 13 \text{ MeV } A^{2/3} \tag{224}$$

The Coulomb energy to assemble the nucleus is necessarily

$$E_{\text{Coulomb}} \sim \frac{e^2Z^2}{r} = \frac{3}{5}\frac{e^2Z^2}{R} \approx 0.56 \text{ MeV}\frac{Z^2}{A^{1/3}} \tag{225}$$

Nuclei typically prefer to have roughly equal numbers of neutrons and protons in order to minimize degeneracy pressure. Thus, often we will have $N = Z$ since then the kinetic energy is lower. This gives rise to a "symmetry energy",

$$E_{\text{sym}} = 18 \text{ MeV}\frac{(N-Z)^2}{A} \tag{226}$$

Wednesday, February 8, 2012

6.2 Tunneling through the Coulomb Barrier

In order for two positively charged nuclei to fuse, they must get close enough so that the strong force can overpower the Coulomb potential:

$$V(r) = \frac{Z_1 Z_2 e^2}{r} = 1.4 \text{ MeV} \frac{Z_1 Z_2}{(r/1 \text{ fm})} \tag{227}$$

At the center of the sun, where $T_c \sim 10^7$ K, the thermal energy is only about $kT \sim 8.6\,\text{keV}\left(T/10^8 \text{ K}\right)$. Then clearly the thermal energy in the sun is nowhere near high enough to overcome the Coulomb barrier. If there was no tunneling, we would need temperatures of order

$$T \sim \frac{Z_1 Z_2 e^2}{kR} \sim 1.6 \times 10^{10} \text{ K } \frac{Z_1 Z_2}{(R/1 \text{ fm})} \tag{228}$$

Stars pretty much never reach this temperature, so clearly tunneling is how fusion happens in stars.

There are two important ingredients for calculating reaction rates:

1. Tunneling
2. Maxwell-Boltzmann distribution of particle energies

The liklihood of a particle being able to tunnel through the Coulomb barrier is proportional to its energy, which in turn is determined by the Maxwell-Boltzmann statistics of the ensemble of particles. To get a better handle on what's going on here, consider a simple reaction:

$$\mathrm{a} + \mathrm{X} \rightarrow \mathrm{Y} + \mathrm{b} \tag{229}$$

Sometimes this is written as X(a,b)Y in astrophysics (no clue why). We define the **cross section** as

$$\sigma \equiv \frac{\#\text{reactions/X nucleus/unit time}}{\#\text{incident particles/cm}^2\text{/unit time}} \tag{230}$$

which clearly has the dimension of length squared (area). Now suppose the number density of a particles is given by n_a and the typical relative velocity is given by v_rel. Then the incoming flux of particles is just $n_\mathrm{a} v_\mathrm{rel}$. From this logic, we may decuce that

$$n_\mathrm{a} \sigma v_\mathrm{rel} = \frac{\#\text{reactions}}{\text{X nucleus} \times \text{unit time}} \tag{231}$$

Usually we write the reaction rate in the number of reactions per unit time per unit volume, or

$$r_\mathrm{aX} = n_\mathrm{X} n_\mathrm{a} \sigma v_\mathrm{rel} \tag{232}$$

Really we need to integrate over a relative velocity distribution to get an understanding of the actual reaction rate, so

$$r_\mathrm{aX} = \int_0^\infty dv \; n_\mathrm{a} n_\mathrm{X} \sigma(v) v phi(v) \tag{233}$$

where $\phi(v)$ is the probability density of a pair of particles having a relative velocity v. More often this is written as

$$r_\mathrm{aX} = n_\mathrm{a} n_\mathrm{X} \left\langle \sigma v \right\rangle \tag{234}$$

with

$$\langle \sigma v \rangle = \int_0^\infty dv \ \sigma(v) v \phi(v) = 4\pi \left(\frac{\mu}{2\pi kT} \right)^{3/2} int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2kT} \right) dv \tag{235}$$

where μ is now the reduced mass, $\mu = m_1 m_2 / (m_1 + m2)$.Note that if both reactants are the same species, (232) must be divided by a factor of 2 since we cannot imagine having a "target" and a "scatterer" (this screws up relative velocities and other things, see Clayton for the details). Similarly, if you have N like reactants, we must divide by $N!$ to adjust the reaction rate appropriately.

6.2.1 Barrier Penetration

Recall the Schrödinger equation,

$$\left[-\frac{\hbar^2}{2\mu} \nabla^2 + V \right] \psi = E\psi \tag{236}$$

For simplicity, we will only do a one-dimensional example to model barrier penetration. We supose a free particle encounters a finite-width and finite-height barrier. Outside the barrier, then, (236) reduces to

$$-\frac{\hbar^2}{2\mu} \nabla^2 \psi = E\psi \quad \Rightarrow \quad \psi \propto e^{-\pm ikx} \tag{237}$$

and the energy is given by $E = \hbar^2 k^2 / (2\mu)$. Inside the barrier, though, we have decaying modes:

$$-\frac{\hbar^2}{2\mu} \nabla^2 \psi = (E_V)\psi \quad \Rightarrow \quad \psi \propto e^{\pm \kappa x} \tag{238}$$

Where the "energy" is $E - V = -\hbar^2 \kappa^2 / (2\mu)$. Now we use this solution in solving for the actual Coulomb potential,

$$V(r) = \frac{Z_1 Z_2 e^2}{r} \tag{239}$$

by using the WKB approximation. In this regime, we have

$$\frac{\hbar^2 \kappa^2}{2\mu} = \frac{Z_1 Z_2 e^2}{r} - E \tag{240}$$

Using the WKB approxiamtion to calculate the barrier penetration, which assumes that the length-scale of changes in $V(r)$ is much larger than the DeBroglie wavelength of the particle, we get the probability density being

$$\psi^2 \propto \exp\left(-2 \int \kappa \ dr \right) \tag{241}$$

where the integral is between the two classical turning points. In this case, there is really only one turning point,

$$r_c = \frac{Z_1 Z_2 e^2}{E} \tag{242}$$

Rewriting κ in terms of the turning point,

$$\kappa = \left(\frac{2\mu E}{\hbar^2} \right)^{1/2} \left[\frac{r_c}{r} - 1 \right]^{1/2} \tag{243}$$

Then the integral in (241) becomes

$$\int_{r_c}^{r_{\rm in}} \kappa dr = \int_{r_c}^{0} \frac{\pi\alpha}{2} Z_1 Z_2 \left(\frac{2\mu c^2}{E}\right)^{1/2} \tag{244}$$

Here we've extended the upper limit to zero to make the integral doable, but this is actually a pretty good approximation (see Clayton for the details). This gives the probability of tunneling to be

$$P_{\rm tunnel} \propto \exp\left[-\pi\alpha Z_1 Z_2 \left(\frac{2\mu c^2}{E}\right)^{1/2}\right] = \exp\left[-\sqrt{E_{\rm G}/E}\right] \tag{245}$$

Where we have defined the **Gamow** energy via

$$E_{\rm G} \equiv (\pi\alpha Z_1 Z_2)^2\, 2\mu c^2 \approx 0.98 \text{ MeV } Z_1^2 Z_2^2 \left(\frac{\mu}{m_p}\right) \tag{246}$$

For a proton-proton interaction, $E_{\rm G} \sim 0.5$ MeV, and for a carbon-proton interaction, $E_{\rm G} \sim 33$ MeV. These values are significantly lower than that required for straight-up thermal energy to do the work, but still pretty high compared to kT_c. The probabiliy for a 1 keV proton to interact with another proton is then

$$P \propto \exp\left[-\left(\frac{500 \text{ keV}}{1 \text{ keV}}\right)^{1/2}\right] \sim 2 \times 10^{-10} \tag{247}$$

So we really need to be out on the Boltzmann tail for this to occur.

6.2.2 Nuclear Reaction Rates

The De Broglie waelength of a particle is

$$\lambda = \frac{h}{p} = \frac{h}{(kT\mu)^{1/2}} \approx 10^{-10} \text{ cm in solar core} \tag{248}$$

While the "size" of the nucleus is

$$r_{\rm nuc} \approx 1.3 \times 10^{-13} \text{ cm } A^{1/3} \tag{249}$$

Clearly, classical scattering just isn't going to cut it since the De Broglie wavelength is much longer than the target size. Partial wave analysis can get an effective cross-section given by

$$\pi\lambda^2 \,(\text{dimensionless stuff}) \exp\left[-\left(\frac{E_{\rm G}}{E}\right)^{1/2}\right] \tag{250}$$

Where the "dimensionless stuff" comes from nasty nuclear physics. Often this is written in terms of the reduced wavelenth, $\lambda/2\pi$.

$$\pi\lambda^2 = 4\pi^3\bar{\lambda}^2 = \frac{4\pi^3}{k^2} \tag{251}$$

So the energy is

$$E \approx \frac{\hbar^2 k^2}{2\mu} \quad \Rightarrow \quad 4\pi^3\bar{\lambda}^2 = \frac{2\pi^3\hbar^2}{\mu E} = 2000 \text{ barns} \left(\frac{\text{keV}}{\text{E}}\right) \tag{252}$$

Then the cross section can be written as

$$\sigma(E) = \frac{S(E)}{E} \exp\left[-\left(\frac{E_{\mathrm{G}}}{E}\right)^{1/2}\right] \tag{253}$$

Where the "stupid factor", $S(E)$ is a slowly changing function of the energy that takes into account the details of the nuclear reaction. Typical values of $S(E)$ are around 2000 keV barns. One important exception is the Deuterium synthesis reaction:

$$\mathrm{p} + \mathrm{p} \rightarrow \mathrm{D} + \mathrm{e}^{+} + \nu_{\mathrm{e}} \tag{254}$$

where $S \sim 4 \times 10^{-22}$ kev barns. This value is so small because a weak interaction is involved. Now we return back to calculating $\langle \sigma v \rangle$, using our newly found cross-section and writing the energy as $E = \frac{1}{2}\mu v^2$ and $dE = \mu v\ dv$. (Note that the non-relativistic energies are fine for our purposes.) Then we have

$$\langle \sigma v \rangle = \frac{1}{(kT)^{3/2}} \left(\frac{8}{\pi\mu}\right)^{1/2} \int_0^\infty dE\ S(E) \exp\left[-\frac{E}{kT} - \left(\frac{E_{\mathrm{G}}}{E}\right)^{1/2}\right] \tag{255}$$

Before evaluating this integral, we'll need to know how to use the method of steppest descent. Suppose we have an integral with

$$I = \int_{-1}^{\infty} g(x) e^{-f(x)} dx \tag{256}$$

where $g(x)$ is slowly varying and $f(x)$ is sharply peaked. (In our case, the Maxwell-Boltzmann distribution multiplied by the probability of tunneling is sharply peaked. Particles far out on the Maxwell-Boltzmann tail are very likely to tunnel, but incredibly unlikely to exist. Likewise, particles with low energies are in ample supply, but they aren't tunneling anytime soon, so there is some magic window in the middle that is most relevant.) We Taylor expand $f(x)$ about the peak (where $f'(x_0) = 0$):

$$f(x) \approx f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 \tag{257}$$

Then (256) can be approximated by

$$I = g(x_0) e^{-f(x_0)} \int_{-\infty}^{\infty} \exp\left[-\frac{f''(x_0)}{2}(x_0)^2\right] = g(x_0) e^{-f(x_0)} \sqrt{\frac{2\pi}{f''(x_0)}} \tag{258}$$

Then for our uses, we have

$$f(E) = \frac{E}{kT} + \left(\frac{E_{\mathrm{G}}}{E}\right)^{1/2} \tag{259}$$

Solving for E_0 gives $E_0^3 = \frac{1}{4} E_{\mathrm{G}} (kT)^2$. Then the relevant functions and derivatives are

$$f(E_0) = 3\left(\frac{E_{\mathrm{G}}}{4kT}\right)^{1/3} \tag{260}$$

$$f''(E_0) = 3\left[2E_{\mathrm{G}}(kT)^5\right]^{1/6} \tag{261}$$

Using this result in (255), we get

$$\langle\sigma v\rangle \approx \frac{1}{(kT)^{3/2}}\left(\frac{8}{\pi\mu}\right)^{1/2} S(E_0)\exp\left[-3\left(\frac{E_{\mathrm{G}}}{4kT}\right)^{1/3}\right]\int_0^\infty \exp\left[-\frac{(E-E_0)^2}{(\Delta/2)^2}\right] \tag{262}$$

$$\Rightarrow \boxed{\Delta = \frac{4}{2^{1/3}\sqrt{3}}E_{\mathrm{G}}^{1/5}(kT)^{5/6}} \tag{263}$$

$$= \boxed{2.6\frac{E_{\mathrm{G}}^{1/6}}{\sqrt{\mu}}\frac{S(E_0)}{(kT)^{2/3}}\exp\left[-3\left(\frac{E_{\mathrm{G}}}{4kT}\right)^{1/3}\right] = \langle\sigma v\rangle} \tag{264}$$

Now for some numbers. For proton-proton reactions, $E_{\mathrm{G}} = 494$ keV, $E_0 = 4.5$ keV $\left(T/10^7\ \mathrm{K}\right)^{2/3}$, and $\Delta/E_0 = 1\cdot(T/10^7\ \mathrm{K})^{1/6}$. These reactions are quite prevalent in the solar core. However, for proton-carbon reactions, $E_{\mathrm{G}} = 36$ MeV, $E_0 = 18$ keV$(T/10^7\ \mathrm{K})^{2/3}$, and $\Delta/E_0 = 0.5(T/10^7\ \mathrm{K})^{1/6}$. These do happen in the sun, but not very much. They are more prevalent in larger stars who derive their energy from the CNO cycle (more on this later).