Lect 12 $\boxed{\text{Intro}}$

Last time I showed that

$$\langle \sigma v \rangle = 2.6 \, \frac{E_G^{1/6}}{\sqrt{m_r}} \frac{S(E_0)}{(kT)^{2/3}} \exp\left[-3\left(\frac{E_G}{4kT}\right)^{1/3}\right]$$

where

$$E_G = (\pi \alpha Z_1 Z_2)^2 (2 m_r c^2)$$

This sets up an evident hierarchy of charge for all purely strong reaction and one can imagine that purely strong reactions will go in the charge hierarchy, whereas weak interaction might end up very delayed.

So today we ~~[illegible]~~ look first at light elements

D, T, ^{3}He, ^{4}He,

T ↓ decays

no [illegible] at mass 5

Now, lets do something with this rate. Call

$$S_s = 2000 \text{ barns} \cdot \text{keV}$$
$$= 3.2\times10^{-30} \text{ ergs} \cdot \text{cm}^2$$

Imagine some reaction that releases energy E_{nuc}, then we find

$$\mathcal{E} = \frac{n_1 n_2 \langle \sigma v \rangle E_{nuc}}{\rho} = \text{cm}^{-6} \frac{\text{cm}^3}{\text{se}} \text{ Erg}$$
$$= \frac{\text{cm}^{-3}\,\text{erg}}{\text{sec}} \frac{\text{cm}}{\text{gr}}$$

If $p+p$ then

$$\mathcal{E} = \frac{n_p}{m_p} E_{nuc} \langle \sigma v \rangle = \left(\frac{E_{nuc}}{m_p}\right) n_p \langle \sigma v \rangle$$

$$E_G = 0.988 \text{ MeV} \frac{m_r}{m_p} = 494 \text{ keV} \frac{m_r}{}$$

for $p+p$ so we find.

$$\mathcal{E} = \frac{E_{nuc}}{m_p^2} \rho \langle \sigma v \rangle \propto \frac{1}{T^{2/3}} \exp\left(\frac{-1}{T^{1/3}}\right)$$

And

$$\langle \sigma v \rangle_{pp} = 7\times10^{-13} \left(\frac{S}{S_s}\right) \frac{1}{T_7^{2/3}} \exp\left(\frac{-15.69}{T_7^{1/3}}\right)$$

pure strong

So, imagine that we have just one rate-limiting step, then we find

$$\varepsilon = 2.8 \times 10^{30} \left(\frac{S}{S_t}\right) \frac{\rho}{T_7^{2/3}} \exp\left(\frac{-15.69}{T_7^{1/3}}\right)$$

We know from before that

$$\frac{dL}{dm} = \varepsilon \Rightarrow L = \int \varepsilon \, dM$$

and clearly the Main Sequence is defined as the place where

$$L = \int \varepsilon \, dM = L_{\text{radiative losses}}.$$

You will do many of these integrals in ~~a few~~ the upcoming HW set. I just want to first ask for low mass stars when does the pp-cycle begin to halt contraction?

This is clearly when of order

$$\frac{GM^2}{R}$$ of energy is released in a contraction time.

$$L_{nuc} \approx \frac{GM^2}{R} \frac{1}{t_{cont}}$$

but $t_{cont} = \dfrac{GM^2/R}{L}$

then this is the same as

$$L_{nuc} = L_{rad}.$$

For fully convective stars on the lowest part of the main sequence we roughly have a fixed T_{eff}. I showed earlier that

$$T_{eff} \propto M^{7/51} L^{1/102}$$

Rescaling to the approp #'s

$$T_{eff} = 4000K \left(\frac{M}{M_\odot}\right)^{7/51}$$

so

$$L = 9 \times 10^{32} \frac{ergs}{sec} \left(\frac{M}{M_\odot}\right)^{28/51} \left(\frac{R}{R_\odot}\right)^4$$

Lets go to $M = 0.1 M_\odot$ $R = 0.1 R_\odot$ so

$$L = 2.5 \times 10^{28} \frac{ergs}{sec} \left(\frac{M}{0.1 M_\odot}\right)^{28/51} \left(\frac{R}{0.1 R_\odot}\right)^4$$

For the pp-reactions the $S = 4 \times 10^{-22}$ keV-Barn $= 2 \times 10^{-25}$ St.
so

$$\varepsilon = 5.6 \times 10^5 \frac{S}{T_7^{2/3}} \exp\left(\frac{-15.69}{T_7^{1/3}}\right)$$

Roughly, we can write a "nuclear lumin." as

$$L = M\mathcal{E}^{\int \mathcal{E}\, 4\pi r^2 \rho\, dr}$$

& use the polytropic relations we know & $\mu = 0.6$

$$T_c = 0.54 \frac{GM\mu m_p}{k_B R} \quad ; \quad \rho = \frac{3}{4\pi}\frac{M}{R^3}$$

Call $m = M/0.1 M_\odot \quad r = R/0.1 R_\odot$

$$T_7 = 0.74 \frac{m}{r} \quad \rho = 139 \frac{m}{r^3}$$

so

$$L_{nuc} \approx \frac{M(5.6\times10^5)\, 139}{\left(0.74 \frac{m}{r}\right)^{2/3}} \frac{m}{r^3} \exp\left(\frac{-15.69}{\left(0.74 \frac{m}{r}\right)^{1/3}}\right)$$

$$= 2\times10^{40} \frac{m^2 r^{2/3}}{r^3 m^{2/3}} \exp\left(\frac{-17.35\, r^{1/3}}{m^{1/3}}\right)$$

$6/3 - 2/3 \quad\quad 9/3 - 2/3 = 7/3$

$$\boxed{L = 2\times10^{40} \frac{m^{4/3}}{r^{7/3}} \exp\left(\frac{-17.35\, r^{1/3}}{m^{1/3}}\right)}$$

Now, equate this

$$2.5\times10^{28}\, m^{\frac{1}{2}}\, r^{4} = 2\times10^{40} \;—\; r^{..}$$

$$\Rightarrow \exp\left(\frac{17.35\, r^{1/3}}{m^{1/3}}\right) = 8\times10^{11}\,\frac{m^{5/6}}{r^{19/3}}$$

$$\Rightarrow \frac{17.35\, r^{1/3}}{m^{1/3}} = 27.41 + \frac{5}{6}\ln m - \frac{19}{3}\ln r$$

so

$$\boxed{r = m\left[1.58 + 4.8\times10^{-2}\ln m - 0.365\ln r\right]^{3}}$$

if $m = 1$

$$\Rightarrow r = \left(1.58 - 0.365\ln r\right)^{3}$$ we get

$r \approx 2.2$ so

$T_c \approx 3.4\times10^{6}$ K at $0.1 M_\odot$

Now this is obviously a crude version and it tells us the few imp't points about the main sequence

(1) as $M \propto R$ almost as the temperature is same. as the ... will do

more on this later, as we need to first see if the stars will nearly have the same T or not in the middle.

That was the first & crudest version of the main sequence. Your HW problem for next week will have even more such dimensional analysis.

An important trick is to expand

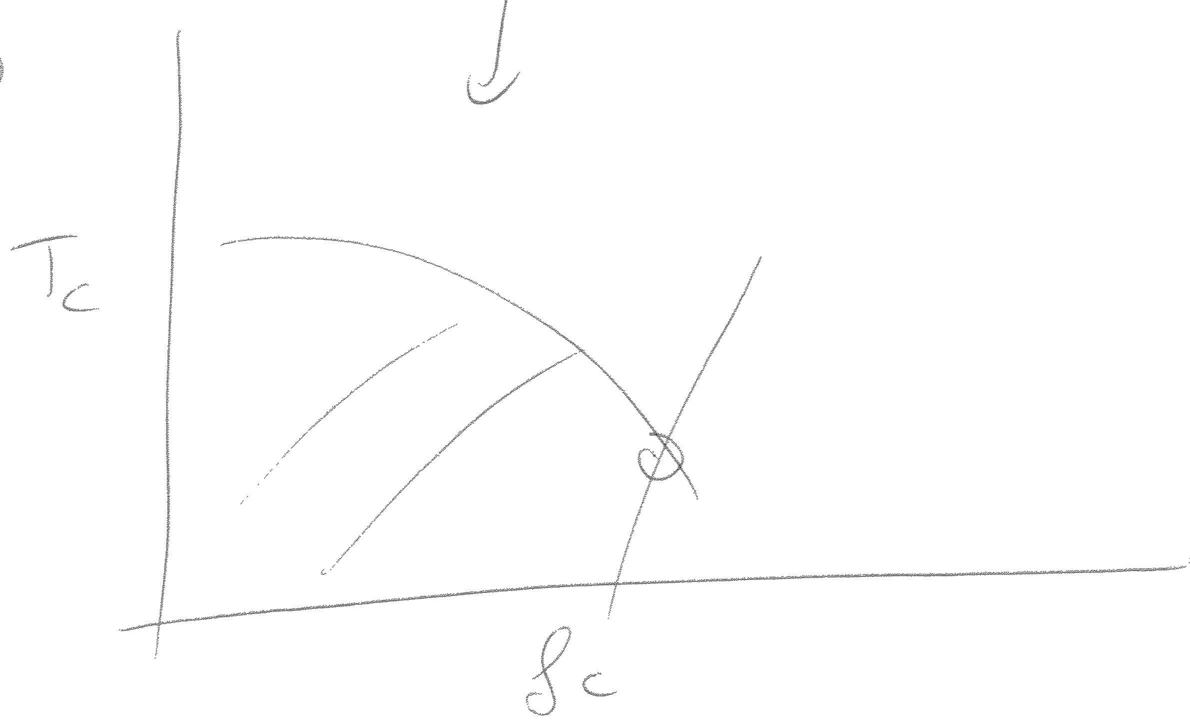
$$\exp\left(-\frac{a}{T_7^{1/3}}\right) \quad \text{as}$$

$$\ln(\quad) = \frac{-a}{T_7^{1/3}} \qquad \frac{d\ln}{dT_7} = \frac{+a}{3\,T_7^{4/3}}$$

or $$\boxed{\frac{d\ln}{d\ln T_7} = \frac{a}{3\,T_7^{1/3}}} = \nu$$

then $$\exp\left(-\frac{a}{T_7^{1/3}}\right) = \exp\left(-\frac{a}{T_0^{1/3}}\right) \cdot \left(\frac{T}{T}\right)^{\nu}$$

Added



+ A discussion of Brown Dwarfs and onset of degeneracy,

since $\rho_{MS} \sim \frac{M}{R^3} \propto \frac{1}{M^2}$ and $T_c \downarrow$

as $M \downarrow$ $\Rightarrow$ eventually become degenerate.

$\hookrightarrow$

(Emphasize hierarchy of charge + reactions) B

Deuterium & ~~Lithium~~

Burning

I earlier mentioned that D burning, via

$$p + D \rightarrow {}^3He + \gamma$$

was the first fuel in a star. The S-factor is 2.5×10^{-4} kev-barn and

$$m_r = \frac{1 \cdot 2}{3} m_p \qquad Z_1 = Z_2 = 1$$

8×10^{-20} $\Rightarrow$

$$E_G = (\pi \alpha Z_1 Z_2)^2 (2 m_r c^2)$$

$$= 655 \text{ keV}$$

so the reaction rate is

$$\langle \sigma v \rangle = 8 \times 10^{-20} \frac{1}{T_7^{2/3}} \exp\left(-\frac{17.24}{T_7^{1/3}}\right) \frac{cm^3}{sec}$$

~~The~~ Let's first ask if ~~can~~ when D ~~can supply any energy~~ burns as

$$\varepsilon = \frac{E}{\rho} n_p n_D \langle \sigma v \rangle = \frac{E_{nuc}}{m_p} n_D \langle \sigma v \rangle$$

$$cm^{-3}/sec \qquad \text{Skip.}$$

but $n_D = 10^{-5} n_p$ so

$$\varepsilon = \frac{E_{nuc} \times 10^{-5}}{m_p} n_p \langle \sigma v \rangle =$$

$$t_D = \frac{1}{n_p \langle \sigma v \rangle} = \frac{2\times10^{-5}}{\rho} T_7^{2/3} \exp\left(\frac{17.24}{T_7^{1/3}}\right)$$

Lets just ask for when

$$t_D = t_{collapse}$$

and since we want to burn ~~much~~ all of it, we will presume for now that it all occurs at the center where for an $n = 3/2$ polytrope (again fully conv.) we know that

$$\rho_c = 6 \langle \rho \rangle = \frac{6 \cdot 3 M}{4\pi R^3} = 8.3 \frac{M}{R^3} \ \mathrm{gr/cm^3}$$

$$T_c = 0.54 \frac{GM \mu m_p}{k_B R} = 7.43\times10^6 \frac{M}{R} \quad \text{in } \odot \text{ units}$$

Our earlier calcs gave us

$$t_{collapse} = \frac{R}{\left|\frac{dR}{dt}\right|} = \frac{\frac{3}{7}\frac{GM^2}{R}}{L}$$

$$T_{eff} = 4000\, M^{7/51}$$

$$L = 9\times10^{32} (R)^2 M^{28/51} \ \mathrm{erg/sec}$$

$$t_{collapse} = 1.8\times10^{15}\,\mathrm{s}\, \frac{M^2}{R\ R^2\ M^{28/51}}$$

$$\boxed{t_c = 1.8\times10^{15} \frac{M^{3/2}}{R^3}\ \mathrm{sec}}$$

(we earlier derived)

for T_7 use $T_7 = 0.74 \frac{M}{R}$, then

$t_{coll} = t_D$

$$\Rightarrow \frac{2\times10^{-5}}{8.3}\frac{R^3}{M}(0.74)^{2/3}\frac{M^{2/3}}{R^{2/3}}\exp\left[\frac{17.24 R^{1/3}}{(0.74)^{1/3}M^{1/3}}\right]$$

$$= 1.8\times10^{15}\frac{M^{3/2}}{R^3}$$

$$9.1\times10^{20}\frac{M^{3/2}}{R^3}\frac{MR^{2/3}}{M^{2/3}R^3} = \exp\left(\frac{19.06 R^{1/3}}{M^{1/3}}\right)$$

$$\Rightarrow 48.26 + \frac{11}{6}\ln M - \frac{16}{3}\ln R = 19.06\frac{R^{1/3}}{M^{1/3}}$$

$$R = M\left[2.53 + 0.096\ln M - 0.28\ln R\right]^3$$

M	R	T_c	$0.13R/M$	t_D yrs
0.03	0.430	5.16×10^5	1.86	7.5×10^6
0.1	1.17	6.3×10^5	1.52	1.7×10^6
0.3	2.86	7.7×10^5	1.24	5×10^5
1.0	7.58	9.76×10^5	0.98	1.3×10^5

Can Deuterium ignition hold the star up? Well yes, as the question is but phrased as is

$$E_{nuc} = (5.5\,\text{MeV})\,n_p\times(2\times10^{-5}) = \frac{M}{m_p}\cdot(2\times10^{-5})(5.5\,\text{MeV})$$

relative to $E_{grav} = \frac{3}{7}\frac{GM}{R}$

Well

$$t_{D,MS} = t_{cont} \frac{(2\times10^{-5})(5.5\,\mathrm{MeV})}{\frac{3}{7} GMm_p/R}$$

$$\frac{D}{H} = 2\times10^{-5}$$

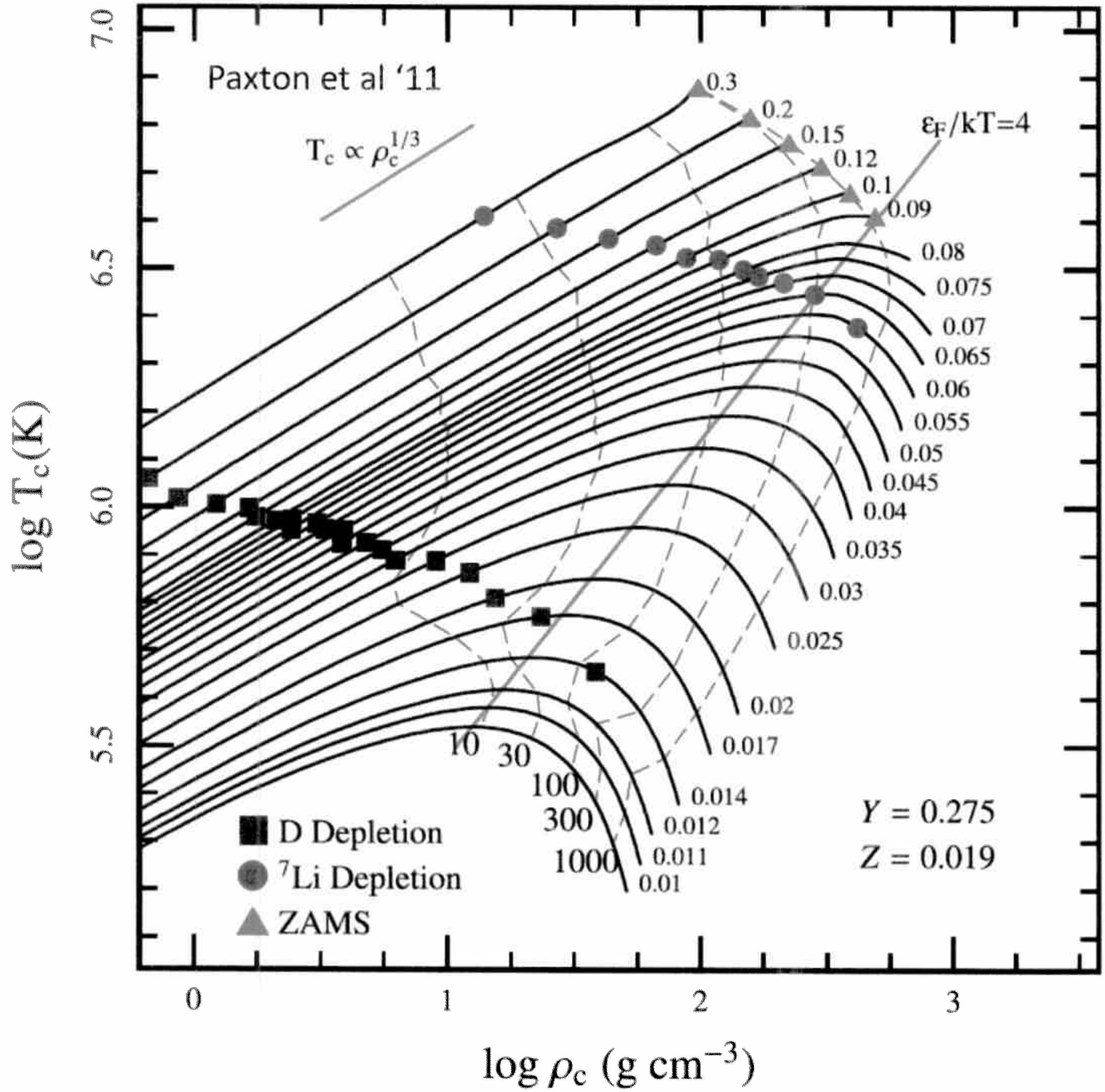
$$= t_{cont}\left[0.13\left(\frac{R}{R_\odot}\right)\left(\frac{M_\odot}{M}\right)\right]$$

For Grossman + Graboske, the ~~Read~~ table is

M	R
0.03	0.54
0.1	1.17

So the star sits on the D main sequence for a time of about $< 10^7$ yrs. The lowest mass star which can do this is about $0.015\,M_\odot$, those of lower mass collapse further and become degenerate before igniting D.

It is a bit of a coincidence that the D/H ratio is ~~fast~~ large enough so that this burning can temp. hold the star up. Obviously D rich environments can even lead to longer lifetimes.


Paxton et al '11
$T_c \propto \rho_c^{1/3}$
$\epsilon_F/kT=4$
0.3
0.2
0.15
0.12
0.1
0.09
0.08
0.075
0.07
0.065
0.06
0.055
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.017
0.014
0.012
0.011
0.01
10
30
100
300
1000
D Depletion
^{7}Li Depletion
ZAMS
$Y = 0.275$
$Z = 0.019$
log T_c(K)
log ρ_c (g cm^{-3})
7.0
6.5
6.0
5.5
0
1
2
3

Burrows et al. 97
ApJ 491, 856

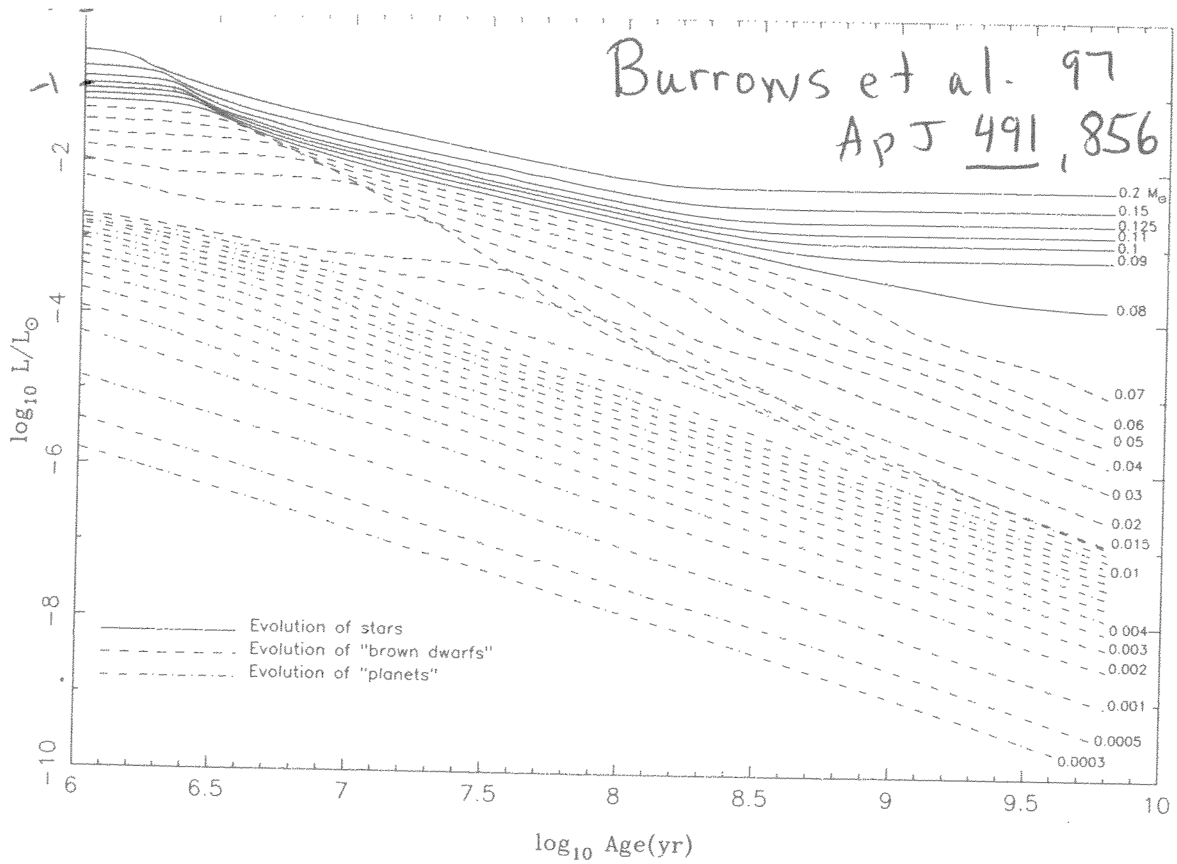


FIG. 7.—Evolution of the luminosity (in $L_\odot$) of solar-metallicity M dwarfs and substellar objects vs. time (in yr) after formation. The stars, "brown dwarfs" and "planets" are shown as solid, dashed, and dot-dashed curves, respectively. In this figure, we arbitrarily designate as "brown dwarfs" those objects that burn deuterium, while we designate those that do not as "planets." The masses (in $M_\odot$) label most of the curves, with the lowest three corresponding to the mass of Saturn, half the mass of Jupiter, and the mass of Jupiter.

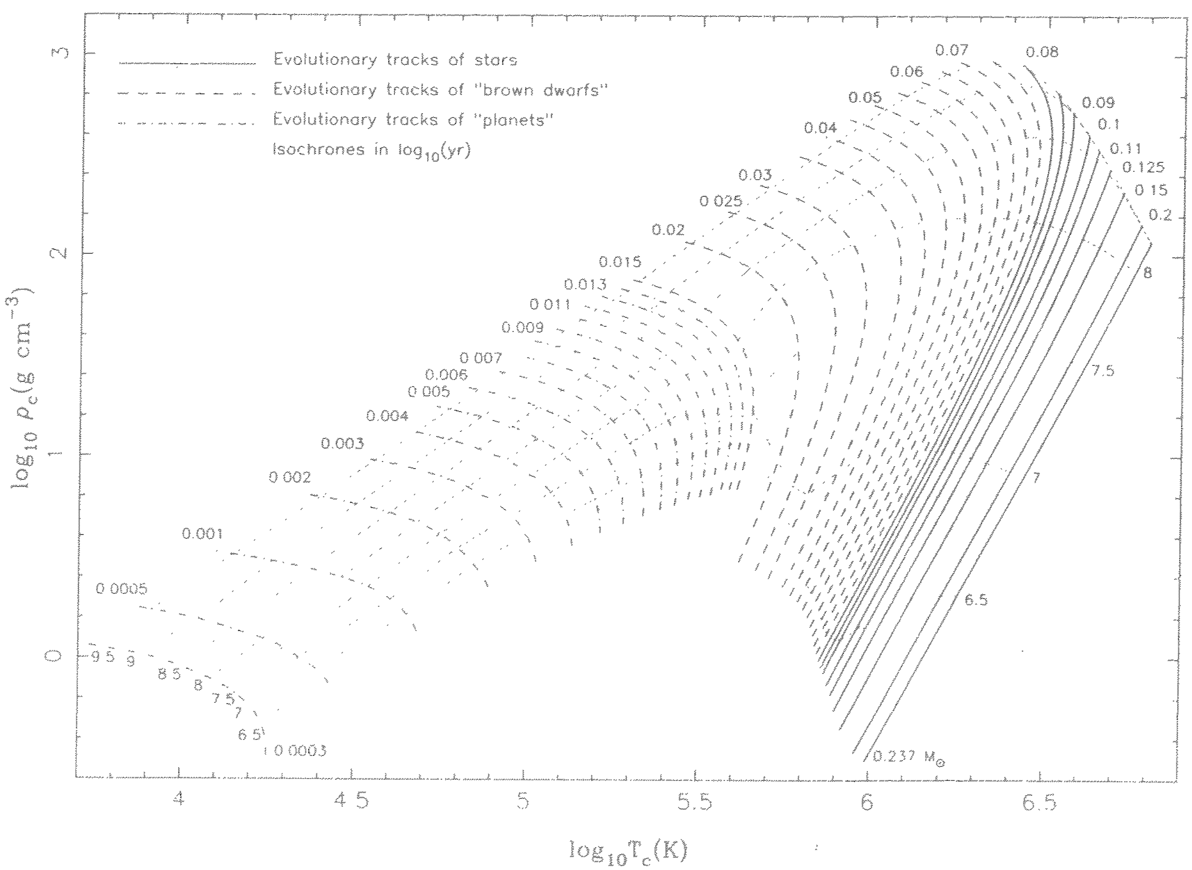


FIG. 8.—Evolutionary tracks of central density (in g cm^{-3}) vs. central temperature (in K) for stars (*solid lines*), "brown dwarfs" (*dashed lines*), and "giant planets" (*dot-dashed lines*), as in Fig. 7. The isochrones are drawn as gray curves and are labeled in $\log_{10}$ yr. The pronounced wave in the isochrones between about $\log_{10} T_c = 5.5$ and 6 is due to deuterium burning. A given mass defines a unique relationship between central temperature and density that is independent of metallicity. The only effect of the metallicity is to change the rate at which the central temperature and density evolve and the positions of the isochrones.

Lithium ^{6} & ^{7} are the next to go via

$$p + {}^7Li \rightarrow 2\,{}^4He$$

$S_0 = 120$ keV · barn $\quad m_r = \frac{7}{8} m_p$

so $E_G = (\pi \alpha Z_1 Z_2)^2 (2 m_r c^2)$

$= 7.736$ MeV and

$$\langle \sigma v \rangle = \frac{5.1 \times 10^{-14}}{T_7^{2/3}} \exp\left(\frac{-39.27}{T_7^{1/3}} \right)$$

Again, everything is convective, but it is clear that the higher Gamow peak means we must go to higher T's.

Play same game as with D

$$t_{cont} = 1.8 \times 10^{15} \frac{M^{3/2}}{R^3} \quad ; \quad \boxed{\begin{array}{l} T_7 = 0.74 \frac{M}{R} \\ \rho = 8.3\ M/R^3 \end{array}}$$

$$t_{Li} = \frac{1}{n_p \langle \sigma v \rangle} = \frac{m_p}{\rho \langle \sigma v \rangle}$$

$$= \left(4 \times 10^{-12} \frac{R^3}{M} \right) (0.74)^{2/3} \frac{M^{2/3}}{R^{2/3}} \exp\left(\frac{43.4}{M^{1/3}} R^{1/3} \right)$$

$$5.5\times10^{26}\,\frac{M^{3/2}R^{2/3}}{K^3R^3}\,\frac{M}{M^{2/3}} = \exp\left(\frac{43.4\,R^{1/3}}{M^{1/3}}\right)$$

$$61.57 + \frac{11}{6}\ln M - \frac{16}{3}\ln R = 43.4\,\frac{R^{1/3}}{M^{1/3}}$$

$$R = M\left[1.419 + 4.2\times10^{-2}\ln M - 0.123\ln R\right]^3$$

M	R	T	t_{cont}
0.08	0.26	2.27×10^6 K	74 Myr
0.20	0.57	2.6×10^6 K	27 Myr
1.0	2.286	3.2×10^6 K	

Clearly still not on the main sequence but close, as the low mass main sequence stars have $T_c = 4\times10^6$ and hence all ^7Li depleted. The ^7Li abundance is not large enough to supply the star with any energy at this time to halt collapse, so ~~yes~~ there is no Lithium main sequence.

~~You will do much more on this important problem in the next HW set.~~

What is critical about Li is that it acts as a thermometer, since the burning is so T sensitive. If Li is gone, ~~then~~

$\boxed{\text{PP vs CNO}}$

I showed last time that there is a cycle, the simplest version being:

$$p + p \rightarrow D + e^+ + \nu_e$$

$$p + D \rightarrow {}^3He + \gamma$$

$${}^3He + {}^3He \longrightarrow {}^4He + 2p$$

which goes via the p-p rate limiting step. If we apply this to the Sun we find that

$$L_\odot = \varepsilon M_\odot$$

with

$$\varepsilon_{pp} = 3\times 10^{30} \frac{S}{S_t} \frac{S}{T_7^{2/3}} \exp\left(\frac{-15.7}{T_7^{1/3}}\right)$$

$$L_\odot = 6\times 10^{32} \frac{1}{T_7^{2/3}} \exp\left(\frac{-15.7}{T_7^{1/3}}\right)$$

$$\Rightarrow \quad +11.9 = \frac{15.7}{T_7^{1/3}} \qquad T_7^{1/3} = \frac{15.7}{11.9}$$

$$\Rightarrow \quad T_c \approx 2\times 10^7\ K \qquad \left(1.5\times 10^7 \text{ is real value}\right)$$

This raises the question as to whether some other reaction might do it, as Bethe originally raised the question as to whether

there could be a catalytic ~~cycle~~ reaction
cycle involving the element s?

Well, lets ck.

$$L_n \propto \frac{S}{S_t} \exp\left(-3\left(\frac{E_G}{4kT}\right)^{1/3}\right)$$

$$E_G = (\pi\alpha Z_1 Z_2)^2 (2 m_r c^2)$$

$\boxed{pp}$ $\boxed{\text{vs } p + ?\ \text{Heavy}}$

$$10^{-25} \exp\left(-3\left(\frac{E_{G,pp}}{4kT}\right)^{1/3}\right) = \exp\left(-3\left(\frac{E_{G?}}{4kT}\right)^{1/3}\right)$$

$$+57.6 + 3\left(\frac{E_{G,pp}}{4kT}\right)^{1/3} = +3\left(\frac{E_{G?}}{4kT}\right)^{1/3}$$

so $$19.2(4kT)^{1/3} + E_{G,pp}^{1/3} = E_{G,?}^{1/3}$$

$$E_{G,?} = \left[E_{G,pp}^{1/3} + 19.2(4kT)^{1/3}\right]^3$$

$$= \left[29 T_7^{1/3} + 7.9\right]^3 \text{ keV}$$

$$= 50\ \text{MeV} = (\pi\alpha Z)^2 (2 m_p c^2)$$

$$\Rightarrow \boxed{Z = 7!}$$

So this says that the competition between weak + strong interaction gives us a big factor in the exponent as we can ~~go~~ afford the extra Coulomb tunneling

$$(\pi\alpha)^2(2m_pc^2)Z^2 = \left[29T_7^{1/3}+7.9\right]^3 \text{ keV}$$

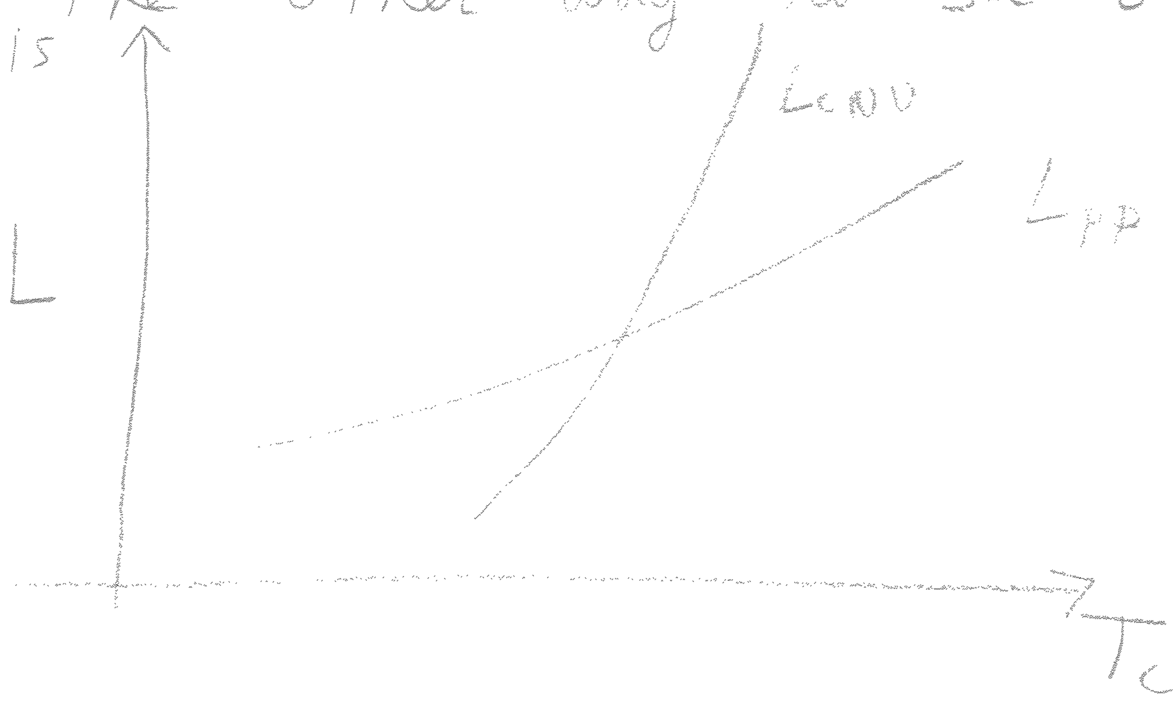
$$Z^2 = 25\left(T_7^{1/3}+0.27\right)^3$$

$$\boxed{Z_c = 5\left(T_7^{1/3}+0.27\right)^{3/2}}$$

T_7	Z
1	7.15
0.5	5.5

But the lower-mass main sequence is much colder than this so ~~then~~ the L_s are so much lower.

~~The~~ The other way to see this is



Where these things are very T sensitive since

$$L \propto \exp\left(-3\left(\frac{E_G}{4kT}\right)^{1/3}\right)$$

$$\frac{d\ln L}{d\ln T} = \left(\frac{E_G}{4kT}\right)^{1/3}$$

, which for pp at $T = 10^7$ is 5.23 but for $p + {}^{14}N$ is $E_G = 48.1$ MeV and then we get 24! so it is much steeper which means that ~~[illegible]~~ eventually it wins.

$\boxed{\text{CN Cycle}}$

Hans Bethe + von Weizsäcker both in 1938 showed that there is a CN cycle that works this way using all CN in the star as catalysts:

S_0 (keV-Barn)	Reaction
1.4	$p + {}^{12}C \rightarrow {}^{13}N + \gamma$
$\tau = 870$ sec	${}^{13}N \rightarrow {}^{13}C + e^+ + \nu_e$
5.5	$p + {}^{13}C \rightarrow {}^{14}N + \gamma$
$\boxed{2.75}$	$\boxed{p + {}^{14}N \rightarrow {}^{15}O + \gamma}$
$\tau = 178$ sec	${}^{15}O \rightarrow {}^{15}N + e^+ + \nu_e$
5.3×10^4	$p + {}^{15}N \rightarrow {}^{12}C + \alpha$

||| non-resonant → over.

rate limiting step.