Lect 3

Energy Transfer + T Grade

I left off last time with more details about how diff't stars behave under hydrostatic balance, especially those with ideal gas equations of state. For a constant density star, we know that

$$E_{gr} = \frac{-3}{5}\frac{GM^2}{R}$$

and HB tells us

$$\langle P \rangle = \frac{-1}{3}\frac{E_{gr}}{V} = \frac{2 \rho k_B T}{m_p}$$

so $\frac{1}{5}\frac{GM^2}{R} = 2(\rho V) k_B T \frac{1}{m_p}$

$\Downarrow$ $\rho V = M$

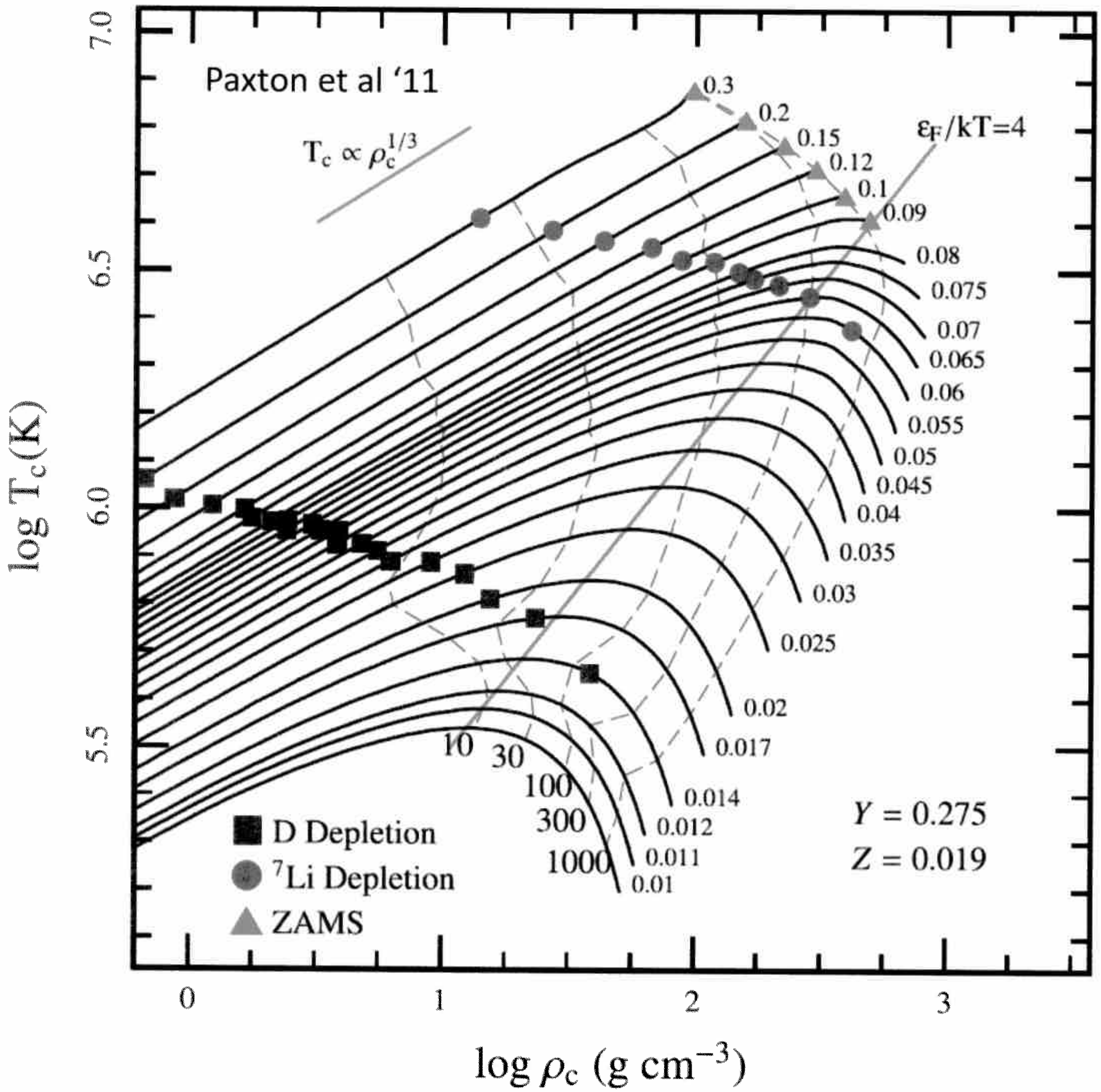
$$\Rightarrow \boxed{k_B T \approx \frac{1}{10}\frac{G M m_p}{R}}$$

roughly, so we get

$\frac{4\pi}{3}\rho R^3 = M$, so $R = \left(\frac{3M}{4\pi\rho}\right)^{1/3}$

which then gives us:

$$\boxed{T_c \approx 2 \times 10^6 \left(\frac{\rho_c}{1\ \mathrm{g/cm^3}}\right)^{1/3}\left(\frac{M}{M_\odot}\right)^{2/3} K}$$


Paxton et al '11
$T_c \propto \rho_c^{1/3}$
$\varepsilon_F/kT=4$
0.3
0.2
0.15
0.12
0.1
0.09
0.08
0.075
0.07
0.065
0.06
0.055
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.017
0.014
0.012
0.011
0.01
10
30
100
300
1000
D Depletion
^{7}Li Depletion
ZAMS
$Y = 0.275$
$Z = 0.019$
log T_c(K)
7.0
6.5
6.0
5.5
0
1
2
3
log ρ_c (g cm^{-3})

Rad'n Pressure

When does

$$P_r = \frac{1}{3} a T^4 \gtrsim P_{gas}$$

I start by using the known relation for low mass, which are

$$P_{gas} \simeq P_c \approx \frac{GM^2}{R^4}$$

and

$$KT \sim \frac{GMm_p}{R}$$

then

$$\frac{1}{3} a \left(\frac{GMm_p}{KR} \right)^4 \gtrsim \frac{GM^2}{R^4}$$

notice that R cancels and we have:

$$M^2 > \frac{k_B^4}{a G^3 m_p^4}$$

but

$$a = \frac{\pi^2}{15} \frac{k_B^4}{(\hbar c)^3}$$

$$\Rightarrow \quad \frac{M^2}{m_p^2} \gtrsim \frac{\cancel{k_B^4} (\hbar c)^3}{G^3 m_p^6 \cancel{k_B^4}} \gtrsim \left(\frac{\hbar c}{G m_p^2} \right)^3$$

or just

$$M > m_p \left(\frac{\hbar c}{G m_p^2}\right)^{3/2}$$

Roughly speaking, this is where a $M_\odot$ comes from. Call

$$\alpha_G = \frac{G m_p^2}{\hbar c} = \frac{e^2}{\hbar c}\,\frac{G m_p^2}{e^2}$$

$$\alpha_G = \alpha\,(8.1 \times 10^{-37}) = 6 \times 10^{-39}$$

$$M > m_p \left(\frac{1}{\alpha_G}\right)^{3/2} \simeq 1.85\, M_\odot .$$

The current, more careful calc's get #'s more like 60-90 $M_\odot$ as we will discuss later. But the important point here is

So this sets the most massive star possible and we will go into more detail on this later.

Where We Stand Today at Understanding MS.

① HB implies T is high and as ~~I show~~ this can be held up with ideal gas pressure for most stars.

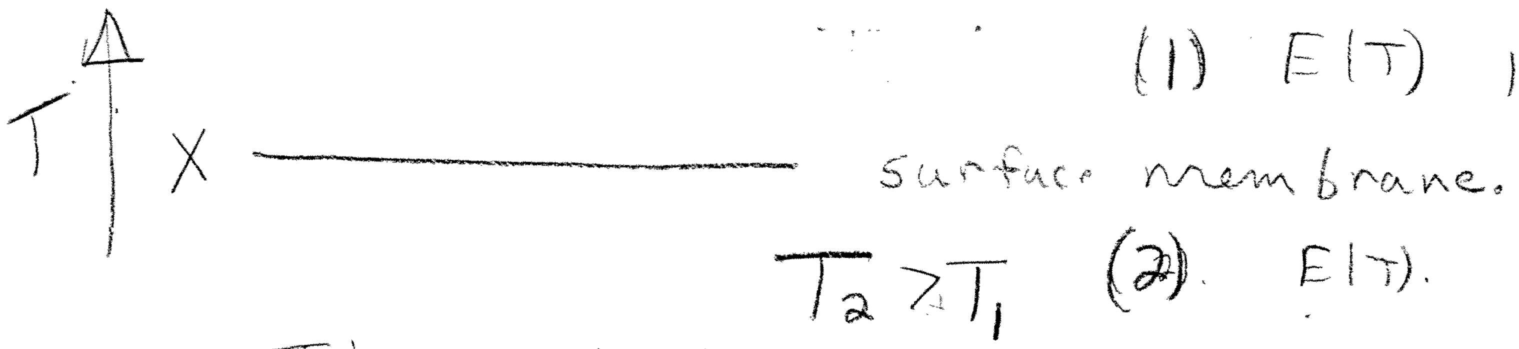
② Since T is high ⇒ star loses heat ⇒ (rate of loss I do today) and if there is no energy source it would contract ⇒ Problem: for the sun we know it has lived longer than the implied value

⇒ energy source.

③ We will show that the MS is just that place where the equilibrium sets in between ~~the~~ L and $\int \varepsilon \, dM$

Heat Transport

Imagine a medium with a gradient in temperature present.



The particles in region (2) are slightly hotter than region (1) and therefore transport heat when they go from (2) → (1). Lets call
E = internal energy per unit volume (as I defined before), then lets see what happens at the surface. Particles, on average, will be coming from a distance X+ℓ above the membrane, where

$$\ell = \text{mean free path} = \frac{1}{\sigma n v}$$

so Downward heat flow

$$F_{down} = \frac{\text{erg}}{\text{cm}^2 \cdot \text{sec}} \approx \frac{1}{6} v \, E(X+\ell)$$

whereas particles from beneath move upward and carry heat from below ~~at~~

$$F_{up} \approx \frac{1}{6} v \; E(X-\ell)$$

so the net flux in the positive $\hat{x}$ direction is

$$F_X = -\frac{1}{6} v \; E(X+\ell) + \frac{1}{6} v \; E(X-\ell).$$

which gives:

$$F_X = \frac{1}{6} v \left[-E(X+\ell) + E(X-\ell) \right]$$

write $E(X+\ell) = E(X) + \ell \frac{dE}{dX}$

$$E(X-\ell) = E(X) - \ell \frac{dE}{dX}$$

so

$$\boxed{F_X = -\frac{1}{3} v \ell \frac{dE}{dX}}$$

which we can (for a gas) rewrite as

$$\frac{dE}{dX} = \frac{dE}{dT} \frac{dT}{dX}$$

where for an ideal gas gives:

$$\frac{dE}{dT} = \frac{3}{2} n k_B$$

and the flux is

$$F_X = -\frac{1}{3} v \ell \frac{3}{2} n k_B \frac{dT}{dX}$$

or roughly:

$$\boxed{F \approx -\frac{1}{2} v \ell n k_B \frac{dT}{dx}}$$

Now, lets presume we have an ionized gas and

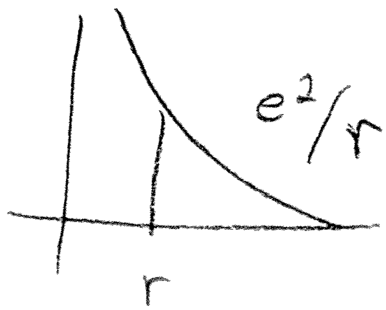
$$\ell \approx \frac{1}{n\sigma_s}, \quad \text{then}$$

$$F = -\frac{1}{2} v \frac{k_B}{\sigma} \frac{dT}{dx}$$

Units $= \frac{\cancel{cm}}{sec} \frac{ergs}{* cm^2 \, ^\circ C \cancel{cm}} = \checkmark$.

Can this work in the sun? Well what is $\sigma = ?$

(I gave an expanded discussion and discussed $\ln \Lambda$)



$$e^2/r \approx KT \Rightarrow \sigma \approx \pi r^2$$

$$\approx \pi \frac{e^4}{(kT)^2}$$

fill in details.

$$\sigma_{90^\circ} \approx \frac{e^4}{(KT)^2} \qquad v \approx \left(\frac{K_B T}{m_e}\right)^{1/2}$$

so $F \approx \frac{k_B T}{R} \left(\frac{k_B T}{m_e}\right)^{1/2} \frac{(k_B T)^2}{e^4}$ ✓

in which case

$$\boxed{L \approx 4\pi R^2 F \approx 4\pi R \frac{(k_B T)^{7/2}}{m_e^{1/2} e^4} \approx 5\times10^{31} \frac{erg}{sec}\left(\frac{R}{7\times10^{10}}\right) T_7^{7/2}}$$

$\leftarrow$ Too small by a few orders of magnitude

An alternative is that the radiation diffuses nearly as well. Two things to consider:

1) Relative energy densities
2) Relative mean free paths.

For radiation we just translate

$$l \approx \frac{1}{n_e \sigma} = \text{mean free path}$$

$$\frac{dE}{d}$$

$$E = aT^4 = \text{energy density, so:}$$

$$F_X = -\frac{1}{3} c \frac{1}{n\sigma_{sc}} \frac{d}{dX} aT^4$$

$$\approx \frac{-c}{n\sigma_{sc}} \frac{4}{3} a T^3 \frac{dT}{dX}$$

so $$F_X \approx \frac{-ac\frac{4}{3}T^3}{n\sigma_{sc}} \frac{dT}{dX}$$

So we want to compare the two conductivities

$$K_r \approx \frac{a c T^3}{n\, \sigma_{\gamma\text{-}e}} \approx \left(\frac{c\, k_B}{\sigma_{\gamma\text{-}e}}\right)\frac{a T^4}{n k_B T}$$

$$K_g \approx v \frac{k_B}{\sigma_{e\text{-}i}}$$

so $$\frac{K_r}{K_g} \approx \left(\frac{a T^4}{n k_B T}\right)\left(\frac{\sigma_{e\text{-}i}}{\sigma_{\gamma\text{-}e}}\right)\frac{c}{v}$$

$$\approx \left(\frac{P_r}{P_{gas}}\right)\left(\frac{\sigma_{e\text{-}i}}{\sigma_{\gamma\text{-}e}}\right)\left(\frac{c}{v_e}\right)$$

Now, we know that to build a stable star, we must have $P_r \leq P_{gas}$, how do we compare

$\sigma_{e\text{-}i}$ to $\sigma_{\gamma\text{-}e}$.

⇒ ~~Go into e-ion collisions in some detail followed by Thomson scattering, ...~~

$$\cancel{\approx \underbrace{\frac{P_r}{P_{gas}}}_{\ll 1}\;\underbrace{\left(\frac{m_e c^2}{kT}\right)^2\left(\frac{m_e c^2}{kT}\right)^{1/2}}_{\gg 1}}$$

$\Rightarrow K_R \gg K_G.$

So, what are the relative cross-sections?

$$\sigma_{e-i} \simeq \frac{e^4}{(kT)^2}$$

and if, for now we presume that Thomson scattering is the relevant microphysics we get

$$\sigma_{Th} = \frac{8\pi}{3} \frac{e^4}{(m_e c^2)^2} = \sigma_{\gamma-e}$$

$$\Rightarrow \boxed{\frac{K_r}{K_G} = \frac{P_r}{P_G} \left(\frac{m_e c^2}{kT}\right)^2 \left(\frac{m_e c^2}{kT}\right)^{1/2}}$$

But we already showed that

$\frac{P_r}{P_G} \ll 1$, so what gives?

Radiation will carry out the heat transport when

$$\frac{P_r}{P_G} > \left(\frac{kT}{m_e c^2}\right)^{5/2} = 10^{-7} \left(\frac{T}{10^7 K}\right)^{5/2}$$

So we do need to get some more accurate sense of P_r / P_G

Well, let's use a constant density star.

$$\frac{P_r}{P_G} = \frac{\frac{1}{3} a \left[\frac{1}{10} \frac{G M m_p}{R k_B} \right]^4}{\frac{1}{3} \frac{1}{\quad} \frac{3}{5} \frac{G M^2}{R} \frac{3}{4\pi R^3}}$$

$$= \frac{\frac{a}{10^4} \frac{G^3 M^2 m_p^4}{k_B^4}}{\frac{9}{20\pi}} = \frac{20\pi}{9} \frac{a}{10^4} \frac{G^3 M^2 m_p^4}{k_B^4}$$

$$= 10^{-4} \left(\frac{M}{M_\odot} \right)^2$$

Real ^{ZAMS} Sun has $\rho_c = 82.2$ $T_c = 1.44 \times 10^7$

$$\frac{P_r}{P_G} = \frac{\frac{1}{3} a T^4}{2\rho k T / m_p} = 5.5 \times 10^{-4},$$

so not so bad.

Go back and get.

$$10^{-4} \left(\frac{M}{M_\odot} \right)^2 > 10^{-7} \left(\frac{T}{10^7} \right)^{5/2}$$

$$M > 0.03 M_\odot \left[\frac{T}{10^7} \right]^{5/4}$$

So if all went as I said, then conduction might only be an issue for low-mass objects. However, we will show that convection plays a role there <u>and</u> the opacities are not just Thomson.

So now we are in a position to find what the luminosity is for a star of mass M & radius R when Thomson scattering is all that is important.

$$L = 4\pi R^2 F = (4\pi R^2)\left(\frac{4}{3}\frac{caT^3 m_p}{\rho\,\sigma_{Th}}\frac{T}{R}\right)$$

so

$$L = 4\pi R^2 \cdot \frac{4}{3}\frac{acm_p}{\sigma_{Th}}\frac{T^4}{\rho R}$$

but remember that

$$k_B T = \frac{GMm_p}{2R} \quad \text{and} \quad \rho = \frac{3M}{4\pi R^3}$$

so

$$L = \frac{16\pi}{3}\frac{Racm_p}{\sigma_{Th}}\frac{4\pi R^3}{3M}\left(\frac{GMm_p}{Rk_B}\right)^4$$

The radius cancels! and we get

$$L = \frac{64\pi^2}{9}\frac{acm_p}{\sigma_{Th}}\frac{G^4 m_p^4}{k_B^4} M^3 \frac{1}{2^4}$$

$\cancel{\approx}\ \propto M^3$ and depends on our modeling to get the prefactor right

Note that the [illegible] dependence says that we need to know quite well T_e to get $L = L_\odot$

Much more detailed calcs and fitting to the upper main sequence gives

$$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^{3.5} \quad \text{for } M \geq M_\odot$$

Lower mass stars have a more complicated opacity, which I will derive later.

The important point is that we can, rather simply, derive the solar luminosity without any ~~pr~~ knowledge of what is supplying the luminosity.

Just as a reminder, this gives

$$t_{KH} \approx \frac{3/5\; GM^2/R}{L} \approx 10^7\,\text{yrs}\left(\frac{R_\odot}{R}\right)\left(\frac{M_\odot}{M}\right)^{1.5}$$

as $L_\odot \approx 4 \times 10^{33}$ ergs s^{-1}

And, yet again, we must burn fuel at this rate to stay fixed for some time. If there is no energy supplied to the system, it continues to contract on this kind of timescale. This is how a star forms as we will see.