Begin lecture 7. 

but we can relate $T dS$ to $dE$ & $Pd$...

$$T\frac{ds}{dt} = \frac{dE}{dt} - \frac{P}{\rho^2}\frac{d}{dt}\rho = \cancel{-\varepsilon_{grav}}$$

$E$ = internal energy per gram

where we are now writing a Lagrangian formulation. This basically says how the deviation from adiabatic evolution is what generates luminosity.

Imagine an ideal gas:

$$E = \frac{3}{2}\frac{kT}{\mu m_p} = \frac{3}{2}\frac{P}{\rho}$$ so we get

$$\frac{3}{2}\frac{d}{dt}\frac{P}{\rho} = \frac{3}{2}\frac{1}{\rho}\frac{dP}{dt} - \frac{3}{2\rho^2}P\frac{d\rho}{dt}$$

so the full piece is

$$T\frac{ds}{dt} = \frac{3}{2}\frac{1}{\rho}\frac{dP}{dt} - \frac{5}{2}\frac{P}{\rho^2}\frac{d\rho}{dt}$$

$$= \frac{P}{\rho}\left[\frac{3}{2}\frac{d\ln P}{dt} - \frac{5}{2}\frac{d\ln\rho}{dt}\right] = \frac{3}{2}\frac{P}{\rho}\left[\frac{d\ln P}{dt} - \frac{5}{3}\frac{d\ln\rho}{dt}\right]$$

$$T\frac{ds}{dt} = \frac{3}{2}\frac{P}{\rho}\frac{d}{dt}\left[\ln\left(\frac{P}{\rho^{5/3}}\right)\right]$$

so

$$\boxed{T\frac{ds}{dt} = +\frac{3}{2}\frac{P}{\rho}\frac{d}{dt}\ln\left(\frac{P}{\rho^{5/3}}\right)}$$

Now we often call $T\frac{ds}{dt} = -\varepsilon_{grav}$ ← sign same as

so that $\frac{\partial L_r}{\partial M_r} = \varepsilon$ ...

This $\epsilon_{grav}$ should be a positive number for a contracting star, where (assume $\rho$ = const), then HB

$\rho \sim M/R^3$

$R \sim \left(\frac{M}{\rho}\right)^{1/3}$

$$\Rightarrow P \sim \frac{GM^2}{R^4} \sim \frac{GM^2}{M^{4/3}}\rho^{4/3}$$

so

$$\frac{P}{\rho^{5/3}} = C\rho^{-1/3}$$

then we get

$$\frac{d}{dt}\ln(C\rho^{-1/3}) = -\frac{1}{3}\frac{d\ln\rho}{dt}$$

or we get

$$\epsilon_{grav} = +\frac{1}{2}\frac{P}{\rho}\frac{d\ln\rho}{dt}$$

at the center which is obviously positive for collapse. Yet again, proof that ~~luminosity~~ energy is lost as the star contracts.

It is important to notice that the full eqn:

$$+\frac{3}{2}\frac{P}{\rho}\frac{d}{dt}\ln\left(\frac{P}{\rho^{5/3}}\right) = \epsilon_{nuc} - \frac{\partial L_r}{\partial M_r}$$

in the absence of $\epsilon_{nuc}$ defines a timescale

$$\frac{P}{\rho}\frac{1}{t} \sim \frac{L}{M} \rightarrow t \sim \left(\frac{kT}{m_p}\right)\frac{M}{L} \sim t_{KH}$$

When nothing is changing on this timescale we can neglect

$$T\frac{ds}{dt}$$

and write

$$\boxed{\varepsilon_{nuc} = \frac{\partial \mathcal{L}_r}{\partial m_r}}$$

as we will see this is basically OK for a "snapshot" of the star, but is typically the rare case. (i.e. its OK on the main sequence).

On the main sequence the burning timescale is so much longer than KH that this is an excellent approx.

Hayashi

# Contracting, Fully Conv Star

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At a given instant in a fully convective star

$$\frac{P}{\rho^{5/3}} = \text{constant}$$

( $n = 3/2$ polytrope )

but due to contraction it changes as for a given fluid element always knows about the central entropy so

$$\frac{d}{dt} \ln\left(\frac{P}{\rho^{5/3}}\right) = \frac{d \ln}{dt}\left(\frac{P_c}{\rho_c^{5/3}}\right)$$

go to full page

but $P_c = 0.77 \frac{GM^2}{R^4}$ ; $T_c = 0.54 \frac{GM\mu m_p}{K_B R}$

$$P = \frac{\rho k T}{\mu m_p} \qquad \rho_c = \frac{\mu m_p}{K_B} \frac{K_B R}{0.54\, G M \mu m_p} \frac{0.77\, GM^2}{R^4}$$

$$\rho_c = \left(\frac{0.77}{0.54}\right)\frac{M}{R^3}$$

so

$$\frac{P_c}{\rho_c^{5/3}} = \frac{0.77\, GM^2}{R^4} \times \frac{1}{0.77} \frac{R^5}{M^{5/3}} \propto R M^{1/3}$$

but $M$ is not changing

$$\frac{d}{dt} \ln R M^{1/3} = \frac{1}{R}\frac{dR}{dt}$$

Remember

$$\frac{3}{2}\frac{P}{\rho}\frac{d}{dt}\ln\left(\frac{P}{\rho^{5/3}}\right) = -\frac{\partial L}{\partial m_r}$$

$$\frac{3}{2}\frac{P}{\rho}\frac{1}{R}\frac{dR}{dt} = -\frac{dL}{dm_r}$$

We want

$$\frac{3}{2}\frac{P}{\rho}\frac{d}{dt}\ln\left(\frac{P}{\rho^{5/3}}\right) = -\frac{\partial L}{\partial m_r}$$

Well $\quad P \sim \frac{GM^2}{R^4} \qquad \rho \sim \frac{M}{R^3}$

so $\quad \frac{P}{\rho^{5/3}} \propto \frac{M^2}{R^4}\frac{R^5}{M^{5/3}} \sim M^{1/3} R$

So an adiabatic change in the star (say if you pull off a chunk of matter) follows

$$\boxed{R \propto \frac{1}{M^{1/3}}}$$

Putting this back in

$$\frac{d}{dt}\ln\left(\frac{P}{\rho^{5/3}}\right) = \frac{d}{dt}\ln\left(M^{1/3}R\right)$$

but $M$ = constant $\Rightarrow \frac{d}{dt}\ln R = \frac{1}{R}\frac{dR}{dt}$

so we get

$$\boxed{\frac{3}{2}\frac{P}{\rho}\frac{1}{R}\frac{dR}{dt} = -\frac{dL_r}{dm_r}}$$

so $$L = \frac{3}{2}\frac{1}{R}\left|\frac{dR}{dt}\right| \int \frac{P}{\rho}\, 4\pi r^2 dr \, \rho$$

but

$$L = \frac{3}{2}\frac{1}{R}\left|\frac{dR}{dt}\right| \int P\, 4\pi r^2 dr$$

But this we know from the Virial Thm is related to

$$\Omega = -\frac{3}{5-n}\frac{GM^2}{R} = \frac{-6}{7}\frac{GM^2}{R}$$

and $n = 3/2$

$$\Omega = \frac{-3}{\frac{10}{2}-\frac{3}{2}} = \frac{-6}{7}\frac{GM^2}{R}$$

$$3\int P\,dV = \frac{6}{7}\frac{GM^2}{R}$$

$$\Rightarrow \boxed{\int P\,dV = \frac{2}{7}\frac{GM^2}{R}}$$

so,

$$\boxed{L = \frac{3}{2}\frac{1}{R}\left|\frac{dR}{dt}\right|\frac{2}{7}\frac{GM^2}{R} = \frac{3}{7}\frac{GM^2}{R^2}\left|\frac{dR}{dt}\right|}$$

~~but~~

$$\frac{L}{L_\odot} \approx 0.034 \left(\frac{M}{M_\odot}\right)^{4/7}\left(\frac{R}{R_\odot}\right)^{103/49}$$

was what we found for the star using $T_{eff} = 2500 \left(M/M_\odot\right)^{[illegible]}\left(R/R_\odot\right)^{1/49}$

But

$$\frac{3}{7}\frac{GM^2}{R^2}\frac{dR}{dt} = -0.034 L_\odot \left(\frac{M}{M_\odot}\right)^{4/7}\left(\frac{R}{R_\odot}\right)^2$$

So we get $X = R/R_\odot$ then

$$\frac{dx}{dt} = -8.3\times10^{-17}\left(\frac{M}{M_\odot}\right)^{\frac{4}{7}-\frac{14}{7}}\left(\frac{R}{R_\odot}\right)^4$$

or $$\boxed{\frac{dx}{dt} = -\frac{X^4}{\tau}}$$

where $$\tau = 4\times10^8\,\mathrm{yr}\left(\frac{M}{M_\odot}\right)^{10/7}$$

Note already $\tau \approx 400\times10^6$ yr $\gg$ accretion time

so $$\int_{X_1}^{X_2}\frac{dx}{X^4} = -\int_0^t\frac{dt}{\tau} = -\frac{t}{\tau}$$

$t$ = current age, roughly!

$$-\frac{1}{3}\frac{1}{X_2^3} + \frac{1}{3X_1^3} = \frac{-t}{\tau}$$

star has contracted lots already.

lets presume $X_1 \gg X_2$ then

$$\frac{1}{X_2^3} = \frac{3t}{\tau} \Rightarrow R = R_\odot\left(\frac{\tau}{3t}\right)^{1/3}$$

So if t = time since ∞, then rather quickly we get to

$$\frac{R}{R_\odot} = \left(\frac{1.3\times10^{8}\,yr}{t}\right)^{1/3}\left(\frac{M}{M_\odot}\right)^{10/21}$$

---

insert: (reexplain why convective

Now, the stars become radiative when $L \sim L_{rad}$ which for $M_\odot$ stars is

$$\frac{L}{L_\odot} < \left(\frac{M}{M_\odot}\right)^{3}$$

or just

$$\left(\frac{M}{M_\odot}\right)^{4/7}\left(\frac{R}{R_\odot}\right)^{2}(0.034) < \left(\frac{M}{M_\odot}\right)^{3}$$

or $$\left(\frac{M}{M_\odot}\right)^{4/7}(0.034)\left(\frac{1.3\times10^{8}\,yr}{t}\right)^{2/3}\left(\frac{M}{M_\odot}\right) < \left(\frac{M}{M_\odot}\right)^{3}$$

$$\Rightarrow \boxed{t > 8\times10^{5}\,yrs\left(\frac{M_\odot}{M}\right)^{15/7}}$$ to become radiative.

So massive stars spend very little time on the Hayashi Track.

Things scale differently for lower mass stars where

$$L = L_\odot \left(\frac{M}{M_\odot}\right)^{5.5} \left(\frac{R_\odot}{R}\right)^{0.5}$$

$2+\frac{1}{2}-\frac{5}{2}$

Then our contraint is

$$0.034 \left(\frac{M}{M_\odot}\right)^{4/7} \left(\frac{R}{R_\odot}\right)^{2} < \left(\frac{M}{M_\odot}\right)^{\frac{11}{2}} \left(\frac{R_\odot}{R}\right)^{\frac{1}{2}}$$

$\frac{1}{2}$

so we get

$$0.034\, M^{4/7} \left(\frac{\tau}{t}\right)^{5/6} (M)^{5/4} < M^{11/2}$$

$$\Rightarrow\ 0.034\, M^{-103/28} < \left(\frac{t}{\tau}\right)^{5/6}$$

$$\Rightarrow \boxed{t > 2.4\times10^{6}\,\mathrm{yr}\left(\frac{M_\odot}{M}\right)^{4.4} \text{ to become radiative.}}$$

If we naively say $M \propto R$ on the MS then

$$t_{MS} \approx 1.3\times10^{8}\,\mathrm{yr}\left(\frac{M_\odot}{M}\right)^{1.5} = \text{time to}$$

reach main sequence, so if

$t_{ms} < t_{rad} \Rightarrow$

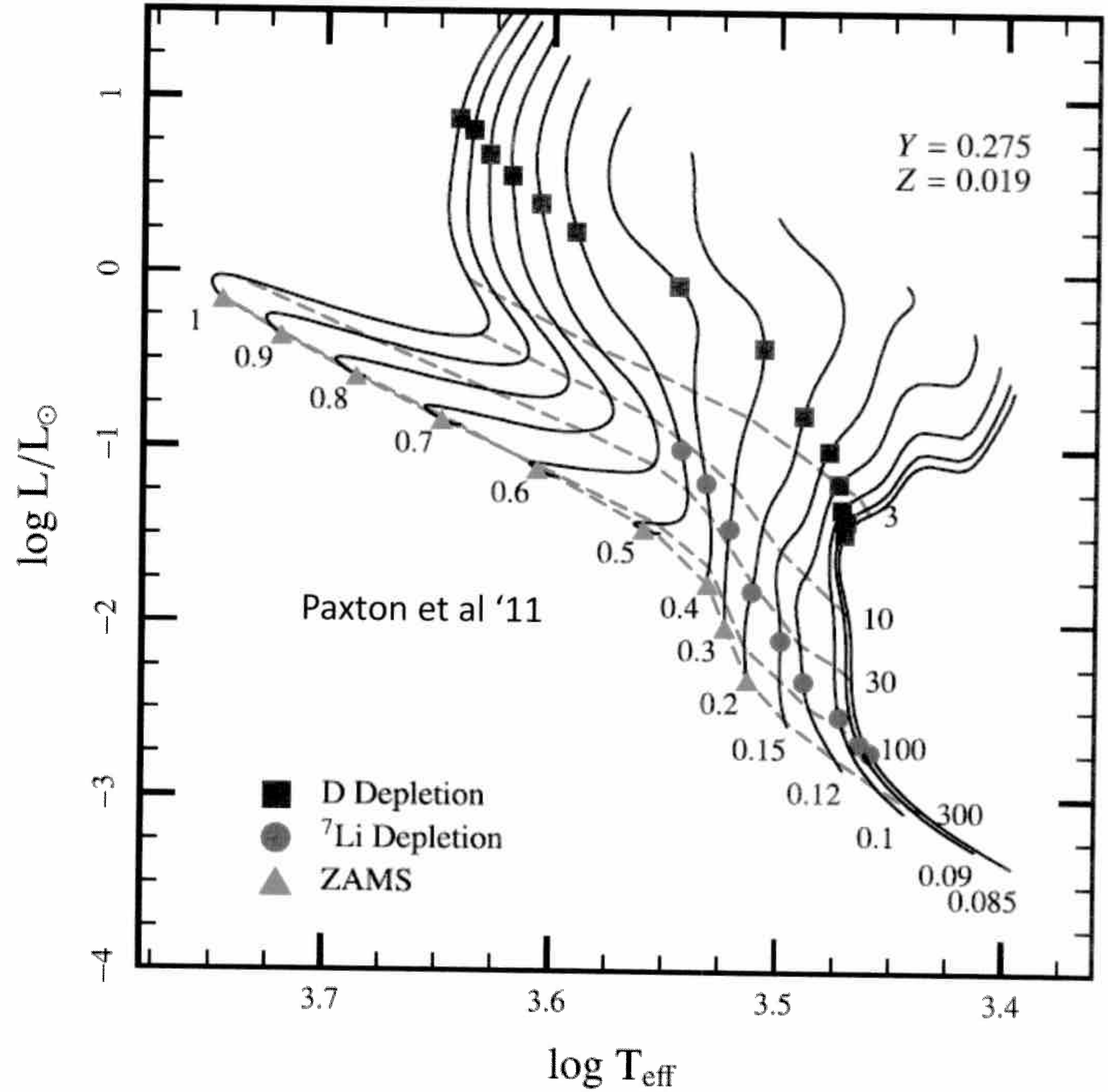


**Figure 15.** Location in the Hertzsprung-Russell (H-R) diagram for $0.085\, M_\odot < M < 1\, M_\odot$ stars as they arrive at the main sequence for $Y = 0.275$ and

So very low mass stars can reach the main sequence before becoming radiative. Our crude calcs give:

$$2.4\times10^{6} M^{-4.4} = 1.3\times10^{8} M^{-1.5}$$

$\Rightarrow$ $M < 0.25 M_\odot$ stars reach the main sequence & ignite H before having become ~~conve~~ radiative

$\Rrightarrow$ Discuss Stahlers plots.

~~Now we need to show why the stars convect at all. This is a bit of a chicken & egg problem as what I will actually show is that the convective star is loosing energy much too rapidly~~

$\Rightarrow$ Low mass ($M<0.3 M_\odot$) stars on the main sequence are fully convective!

$\Rightarrow$ Add Li depletion story later for Pleiades. This simple picture is really right.

# Where do we Stand at this point

(1) Hydrostatic Balance plus ideal gas $\Rightarrow$ $KT \sim \frac{GMm_p}{R}$

(2) Luminosities are known from mechanism of heat transfer

$\Rightarrow$ Radiation $\quad L \propto M^3$

or $L \propto M^{5.5} R^{-0.5}$

(3) Stars can live for a time

$$t_{KH} \sim \frac{GM^2/R}{L} \ll 10^9 \text{ yrs}$$

so we need one more piece of physics

$\Downarrow$

Nuclear Energy

Lect 10

# Thermonuclear Fusion

After the big bang, we are left with (by ~~number~~ mass).

~75% protons

~25% Helium.

As we will see later, the first few generations of massive stars adds heavier elements into the mix, and by today at the solar radius $Z = 0.02$, so, roughly we have $^{12}C$, $^{14}N$, $^{16}O$ at a level of a part in $10^3$ by number.

We want to take advantage of the nuclear fuel as it releases $\simeq \frac{7\,MeV}{m_p}$ in fusion to heavier elements. The first, and obvious, problem to overcome is the need to make neutrons. For a pure H gas it ends up that the first reaction is

$$p + p \rightarrow d + e^+ + \nu_e \leftarrow \text{Weak Interact.}$$

followed rapidly by

$$p + d \rightarrow {}^3He + \gamma$$

$\uparrow$ 5.49 MeV

and typically closed via:

$$^{3}He + {}^{3}He \rightarrow {}^{4}He + 2p$$

(again, no weak).

The Coulomb potential is obviously crucial in terms of tunnelling, ~~but is irrelevant (almost) to the nuclear physics,~~ ~~cap~~

~~$\frac{e^2}{1fm} = 1\,MeV \ll E_{nuc}$ in the well as I will now describe.~~

## Liquid Drop

We know from experiment that a large nucleus is basically at constant density, as $r_n \sim A^{1/3}$, actually

$$r_0 \approx 1.3\,fm\,A^{1/3}$$

so

$$\rho = \frac{3Am_p}{4\pi(1.3fm)^3 A} \approx 2\times 10^{14}\,\frac{gr}{cm^3}$$

Now, lets describe the binding energy of a nucleus with

$Z$ protons, $N$ neutrons.

$A = N + Z$ nucleons.

The main term in the binding energy is a volume term that is

$$E_{vol} = -14\,\text{MeV} * A$$

The next term is a "surface" term, ~~now~~ due to the phen. that a nucleon at the surface of the nucleus is not as "bound" since it interacts with fewer nucleons. Since the volume term is known, the best way to calculate this is to say we are overestimating the volume piece as those at the surface are <u>less</u> bound. The nuclear surface area is

$$A_{rea} = 4\pi R^2$$

~~Each nucleon occupies a "volume" V = 1.3 fm~~

$$\frac{1}{1.3\,\text{fm}}$$

$$\cancel{N_{surf} = \frac{Area}{\pi}}$$

$$N_{surf} = \frac{4\pi R_0^2 A^{2/3}}{\pi R_0^2} = \#\text{ on surface}$$

$$N_{surf} \approx 4A^{2/3} \propto A^{2/3}$$

and the energy is roughly

$$\boxed{E_{surf} = 13.1\,\text{MeV}\,A^{2/3}}$$

and is positive, since we presume all are bound.

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~~#~~6

## Nuclear Structure &
## The Semi-empirical Mass Formula

We know from experiment that a nucleus is basically at constant density.

$$\rho = \frac{3 A m_p}{4\pi (1.3\,fm)^3 A} = 2\times 10^{14} \frac{gr}{cm^3}$$

Now, lets discuss the binding energy of a nucleus ~~ell~~ with $Z$ protons, $N$ neutrons and $A = N+Z$.

### i) Coulomb Physics

Imagine a nucleus of radius $R = R_0 A^{1/3}$ that has charge

$$n_Q = \frac{Z}{V} \qquad V = \frac{4\pi}{3} R^3$$

spread uniformly throughout. The total Coulomb energy is then

$$E_{coul} = \int \frac{e^2}{r} q(r)\, dq$$

$$q(r) = n_Q * \frac{4\pi}{3} r^3 \qquad dq = n_Q * 4\pi r^2 dr,$$

or just write

$$\left(\frac{3q}{4\pi n_Q}\right)^{1/3} = r$$

, so

$$E_{coul} \approx e^2 \int_0^Z \frac{q\, dq}{q^{1/3}} \left(\frac{3}{4\pi n_Q}\right)^{-1/3} = e^2 \left(\frac{3}{4\pi n_Q}\right)^{-1/3} \ldots$$

So we get

$$E_{coul} = e^2 \left(\frac{3}{4\pi n_Q}\right)^{-1/3} \frac{3}{5} Q^{5/3}$$

$$= e^2 \left(\frac{3 \frac{4\pi}{3} R}{4\pi Z}\right)^{-1/3} \frac{3}{5} Q^{5/3} \approx \frac{3}{5} \frac{e^2 Q^2}{R}$$

$$\boxed{E_{coul} \approx \frac{3}{5} \frac{e^2 Z^2}{R}}$$

[Now, in reality there are a few problems with this, namely that (1) protons do know about each other and thus correlations enter in and reduce the probabilities and (2) protons might prefer to just live at the edge of the nucleus rather than stay in the middle].

But, ok if it were this we get

$$\boxed{E_{coul} = 0.66\,\text{MeV} \frac{Z^2}{A^{1/3}}}$$

[All the above caveats does no more than bring 0.66 down to 0.56] ~~most~~. So

$$\boxed{E_{coul} = 0.56\,\text{MeV} \frac{Z^2}{A^{1/3}}}$$

## The Symmetry Energy Term

It is observed that the radius of a nucleus increases with A as

$$r \approx 1.3\,\mathrm{fm}\,A^{1/3}$$

so that the # density of nucleons is

$$\rho = \frac{3 A m_p}{4\pi A\,(1.5\times10^{-13})^3} = C = 2\times10^{14}\ \frac{\mathrm{gr}}{\mathrm{cm}^3}.$$

Now the n + p are degenerate particles at these type of densities & if we just do that we find that

$$n_n = \frac{8\pi}{3h^3} P_f^3 = \frac{\rho_N}{2m_p}$$

so

$$P_f = \left(\frac{3h^3\rho_N}{16\pi m_p}\right)^{1/3} = 0.26\, m_n c = \hbar k_F = \frac{h}{2\pi}\frac{2\pi}{\lambda}$$

so

$$\boxed{E_F = 32\,\mathrm{MeV}\ ;\ \lambda \approx 5\,\mathrm{fm}}$$

$$E_F = \frac{p^2}{2m} = \frac{h^2}{\lambda^2\, 2m}$$

The avg K.E. is $\frac{3}{5}E_F = 20\,\mathrm{MeV}$

so

$$\langle T\rangle \approx 20\,\mathrm{MeV} \Rightarrow U = -40\,\mathrm{MeV}$$

($\Rightarrow$ Avg VW [illegible] $\approx$ [illegible] MeV)

Pauli Principle requires that we move filled to unfilled. Lets see ~~to~~ how this changes the K.E. content in the nucleus relative to $Z = N = A/2$?

Imagine we go to

$$Z' = \frac{A}{2}(1-f) \qquad N' = \frac{A}{2}(1+f)$$

f = fraction of protons changed into neutrons.

For a n.r gas we know that $E_f \propto p_f^2 \rightarrow p_f^3 \propto n \Rightarrow E_f \propto n^{2/3}$.

So, lets write

$$\langle T \rangle_n = \frac{3}{5} E_{F,n} N$$

$$\langle T \rangle_p = \frac{3}{5} E_{F,p} Z$$

Initially $E_{F,n} = E_{F,p} = E_F \quad N = Z$

$$\boxed{\langle T \rangle = \frac{3}{5} E_F A}$$

After.

$$\langle T \rangle = \frac{3}{5} \frac{A}{2} (1+f) E_{F,n} + \frac{3}{5} \frac{A}{2} (1-f) E_{F,p}$$

$$= \frac{3}{5}\frac{A}{2}\left[E_F\left(\frac{n_n}{n_0}\right)^{2/3}(1+f) + E_F\left(\frac{n_p}{n_0}\right)^{2/3}(1-f)\right]$$

$$= \frac{3}{5}\frac{A}{2}E_F\left[(1+f)^{5/3} + (1-f)^{5/3}\right]$$

$$= \frac{3}{5}\frac{A}{2}E_F\left\{1 + \frac{5}{3}f + \frac{\frac{5}{3}\frac{2}{3}}{2}f^2 + 1 - \frac{5}{3}f + \frac{\frac{5}{3}\frac{2}{3}}{2}f^2\right]$$

$$= \frac{3}{5}\frac{A}{2}E_F\left[2 + \frac{10}{9}f^2\right]$$

So $\boxed{\Delta E = \frac{1}{3} A E_f f^2}$

But
$$f = \frac{2N'}{A} - 1 = \frac{2N' - A}{A}$$

$$= \frac{2N' - N' - Z'}{A} = \frac{N' - Z'}{A}$$

So
$$\boxed{\Delta E = \frac{1}{3} E_F \frac{(N' - Z')^2}{A}}$$

$$\simeq 10\,\text{MeV}\left(\frac{(N'-Z')^2}{A}\right)$$

[about 1/2 of what is most ~~important~~ needed! The other bits

The full symmetry piece is

$$E_{sym} = 18.1\,\text{MeV}\,\frac{(N-Z)^2}{A}$$

so we get

$$E = -14\,A + 18.1\,\frac{(N-Z)^2}{A} + 0.56\,\frac{Z^2}{A^{1/3}} + 13.1\,A^{2/3} \quad \text{in MeV.}$$

Now, this tells us much that we need to know as we can calculate the curve of B.E. & all other relevant quantities. What we must first sort out is how fusion can possibly occur at these very low energies.