# Notes for PHYS 232: Lars Bildsten's Course on Stellar Structure and Evolution

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# Contents

# 1 Introduction

*Monday, September 30, 2013*

## 1.1 The Magnitude Scale

The magnitude is a logarithmic scale detailing the relative brightnesses of objects. It is based on the star Vega ($\approx 8$ pc away). It is defined such that its magnitude in the $U$, $B$, and $V$ magnitudes is zero. This corresponds to a flux definition as

$$3.5 \times 10^{-20} \ \mathrm{erg\,cm^{-2}\,s^{-1}\,Hz^{-1}} \tag{1.1}$$

for $M = 0$ (where $M$ is the magnitude). Or, in terms of a simple photon flux, we have

$$N_\lambda = 10^3 \ \gamma\,\mathrm{cm^{-2}\,s^{-1}\,\mathring{A}} \tag{1.2}$$

at the $V$ band for $V = 0$. A useful fact to know is that a magnitude difference of 5 corresponds to a brightness ratio of 100. The classic filters have centers and widths given approximately by Table 1 The **Absolute Magnitude** is the apparent magnitude of the star if it were located at a distance of 10 pc. A **parsec** (pc) is the distance at which a star appears to move one arcsecond in the sky over a half of a year due to the motion of the earth around the sun. In cgs, it's approximatione

$$1 \text{ pc} \approx 3 \times 10^{18} \text{ cm} \tag{1.3}$$

## 1.2 The HR Diagram

Most stars shine predominantly in the optical range of the electromagnetic spectrum. As a result, we get most of our information about stars by observing their optical output. It makes sense, then, that we might organize stars by there color, which is indicative of their surface temperature. When plotting a population of stars' luminosities against their surface temperature, we note a strong correlation between the two. As it turns out, the controlling parameter for these quantities is the mass of the star, at least while the star is on the **Main Sequence** (stars burning hydrogen to helium). The correlation between the mass of a main- sequence star and its luminosity is incredibly strong (see HR diagram examples).

Surveys of all nearby stars within 22 pc have found 2241 stars, which corresponds to a number density of stars as

$$n_* = \frac{2241}{\frac{4\pi}{3}(22 \text{ pc})^2} = 0.05 \text{ pc}^{-3} \tag{1.4}$$

| Band | $\lambda_0$ ($\mathring{A}$) | FWHM (Å) | Flux (# $\mathrm{cm^{-2}\,s^{-1}\,\mathring{A}^{-1}}$) |
|---|---|---|---|
| $U$ | 3650 | 680 | 780 |
| $B$ | 4400 | 980 | 1450 |
| $V$ | 5500 | 890 | 1000 |
| $R$ | 7000 | 2200 | 610 |
| $I$ | 9000 | 2400 | 380 |

Table 1: Basic filter band information used in optical astronomy.

which gives an average separation of $\langle r \rangle \approx 1.7$ pc

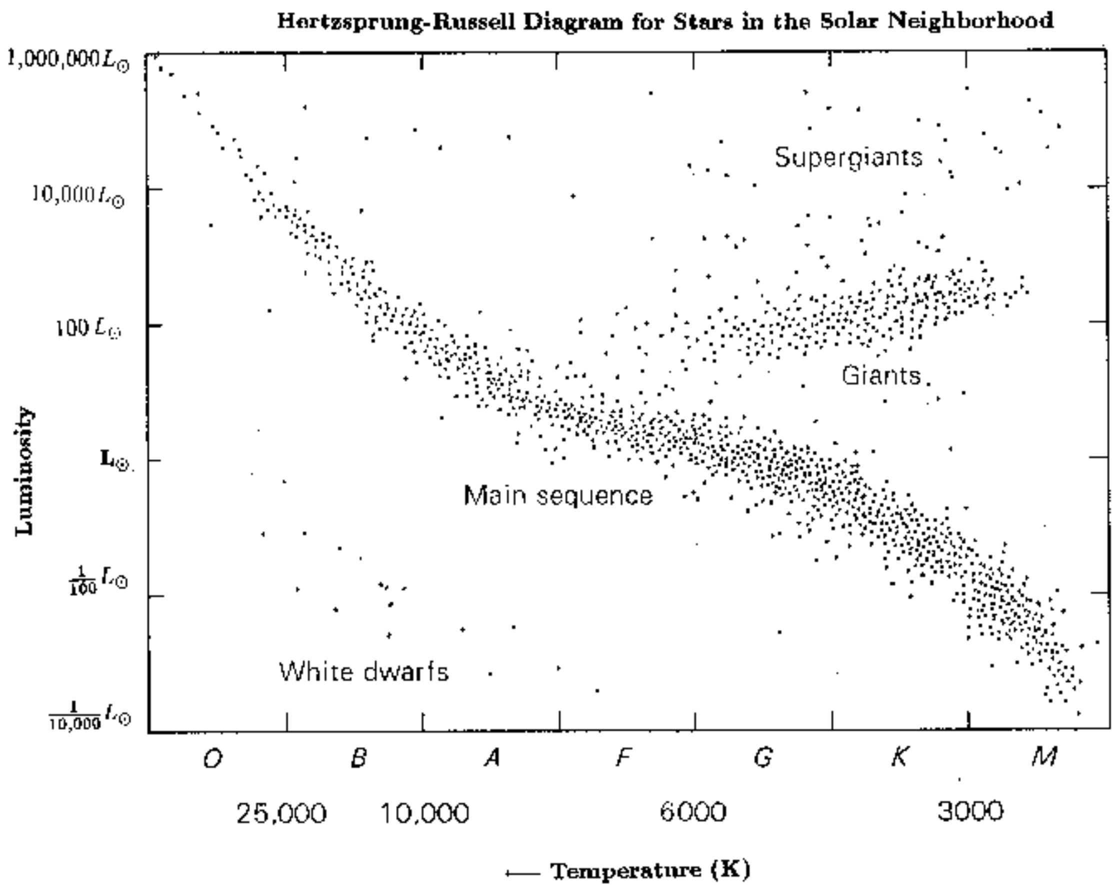


Figure 1: An HR diagram for stars in the local neighborhood (shamelessly stolen from Google Images)

## 1.3 Conditions for a Star on the HR Diagram

We are interested in knowing what defines the regime where a star can reside in a particular $L, T_{\text{eff}} = [L/(4\pi\sigma_{\text{SB}}R^2)]^{1/4}$. Why, for example, is there a dynamical range in the luminosity spanning over six orders of magnitude, while only a range of a factor of about 5 in the effective temperature? To gain some perspective, we might observe the number of stars as a function of brightness. We organize these stars by their **spectral type** (a rough measure of how big the star is) and find their approximate **mass density** (the amount of mass contained in these stars per unit volume):

| Spectral Type | $\rho$ ($M_\odot$/pc$^3$) |
|---|---|
| O-B | 0.4 |
| A-F | 4 |
| G-M | 40 |
| WD's | 20 |

Here we've used the standard labels for different spectral types, O, B, A, F, G, K, M, L, T, which are roughly in decreasing order of size and temperature (the reasoning for this scale is historical rather than logical, and the ordering is often remembered by the mnemonic, "Oh Be A Fine Gal/Guy, Kiss Me! Less Tongue!"). We see that the big, bright stars form an exceedingly small portion of the amount of stellar mass in our galaxy. We will find that this is because large stars exhaust their fuel much more quickly than smaller stars, and thus live and die much faster. We'll now observe another type of classification of stars used in our neighborhood, the Milky Way Galaxy.

### 1.3.1 Population I Stars

From Earth, the center of the galaxy is approximately 8.5 kpc away. Meanwhile, the disk is only about 100 pc wide. We've observed that stars in the thin disk (commonly known as **Population I Stars**) are orbiting at the orbital velocity with only a small amount of axial and radial motion. They are essentially dynamically cold and in nearly circular orbits. This is indeed where most of the **interstellar medium** (ISM) resides, causing much of the star formation in the galaxy. This region is also very metal rich. That is, compared to other parts of the universe, there is a much higher concentration of elements heavier than helium present. We will denote the mass fraction of these "metals" with the letter $Z$, and in this region, we have $Z \sim 1 - 2\,\%$. These metals come from a previous generation of stars, who died in the past, giving off the metals we now have.

### 1.3.2 Population II Stars

**Population II Stars** reside mostly in the spheroid in the center of the galaxy. These are older stars in regions where star formation is largely shut down. Typically they are metal poor, with metallicities as low as $Z = 10^{-4} Z_\odot$. Kinematically, they are often on radial orbits (rather than their more azimuthal population I counterparts in the disk). We typically say that the globular clusters are part of this population. Sometimes these stars are seen passing through the disk at velocities comparable to the orbital velocities, and are easily identified due to their high velocities and unique spectra (due to the low metallicities).

# 2 A Simple Hydrostatic Atmosphere

## 2.1 Scale Height and Column Depth

Before tackling the physics of stars, we first consider a simple toy model– the isothermal, plane parallel atmosphere. This model is somewhat applicable to the thin stellar atmosphere near the surface of a star, where curvature can be neglected and the acceleration due to gravity is nearly uniform.

Consider an atmosphere where the local acceleration due to gravity, $\mathbf{g}$ is constant in value and direction. The atmosphere is composed of an isothermal ideal gas with temperature $T$. We wish to find the distribution of particles in this atmosphere.

In a strictly statistical sense, we would expect the energy distribution to be comparable to $e^{-E/kT}$ (recall that the atmosphere is isothermal, so the average kinetic energy is uniform throughout). In our case, the energy of particles is linear in height, so we expect this probability to be proportional to $e^{-mgh/kT}$.

We will let $m_B \approx m_p$ be the baryon mass, $\mu$ be the mean molecular mass (measured in AMU), and $\rho$ is the density in $\mathrm{g\,cm^{-3}}$. We suppose that the gas is in hydrostatic balance, so we have

$$\frac{dP}{dz} = -\rho g \tag{2.1}$$

Combining this with the ideal gas law,

$$P = nkT \tag{2.2}$$

We find that

$$kT\frac{dn}{dz} = -m_p \mu n g \tag{2.3}$$

which in turn gives us

$$\frac{d\ln n}{dz} = -\frac{m_p\mu g}{kT} \tag{2.4}$$

Solving this differential equation gives the expected result

$$n(z) = n(0)\exp\left(-\frac{m_p\mu g z}{kT}\right) = n(0)\exp\left(-\frac{z}{h}\right) \tag{2.5}$$

where we have defined the **scale height** $h \equiv kT/(\mu m_p g)$, which is the $e$-folding distance in number density. As it turns out, the scale height for earth's atmosphere is approximately 10 km. This model is really only valid in cases where $h \ll R$, (where $R$ is the size of the object), so we now investigate the ratio of these two quantities:

$$\frac{h}{R} = \frac{kT}{\mu m_p \frac{GM}{R^2} R} = \frac{1}{\mu}\frac{kT/m_p}{GM/R} \sim \frac{v_{\rm th}^2}{v_{\rm esc}^2} \ll 1 \tag{2.6}$$

So if this approximation is valid, the thermal velocities of particles are typically much smaller than the escape velocity of the central body, so a star could retain its own atmosphere (thankfully, Earth does the same to its atmosphere!) For stars, we will find that $kT_c/m_p \sim GM/R$. Then, (2.6) tells us that

$$\frac{h}{R} \sim \frac{T_{\rm eff}}{T_c}. \tag{2.7}$$

This tells us that stars are quite sharp-edged (their scale heights are very small compared to their radii). We can also deduce a physical meaning for the scale height as being how far a particle needs to fall to gain an energy comparable to $kT$.

From the ideal gas law, we can easily see that the pressure will also fall off exponentially in this isothermal atmosphere. However, let's explore the pressure a bit more. First, we return to the ideal gas law,

$$P = \frac{\rho}{\mu m_p}kT = nkT \tag{2.8}$$

and the condition for hydrostatic equilibrium,

$$dP = -\rho g \, dz. \tag{2.9}$$

We now integrate (2.9) from $z = z$ to $z \to \infty$:

$$P(\infty) - P(z) = -\int_z^\infty \rho(z') g \, dz' \tag{2.10}$$

$$P(z) = g \int_z^\infty \rho(z') \, dz \tag{2.11}$$

Note that it is okay to take the integral to infinity so long as we are dealing with a constant **g**. This result suggests the definition of the **column density**:

$$y(z) \equiv \int_z^\infty \rho(z) \, dz \tag{2.12}$$

On the surface of the earth, the column density is approximately $y = 1000\,\mathrm{g\,cm^{-2}}$. Think of it as the amount of mass sitting above you per unit area at a given altitude. The column density is an important number (for us) to determine the details of heat transport. For now though, we can write the pressure in this isothermal atmosphere in a compact form: $P(z) = gy(z)$.

## 2.2 Mean Molecular Weights

We'll now make a useful definition for calculating pressures and other useful quantities. For an ideal gas, the total pressure of a mixed gas is simply

$$P = \sum_{i=1}^{N} n_i kT \tag{2.13}$$

where $n_i$ are just the number densities for each ion. The number density is computed via

$$n_i = \frac{X_i \rho}{A_i m_p}. \tag{2.14}$$

where $X_i$ is the mass fraction of the $i^{\mathrm{th}}$ ion with mass number $A_i$ and $\rho$ is the overall mass density. Then the ion pressure is given by (assuming total ionization)

$$P_{\mathrm{ion}} = kT \sum \frac{X_i \rho}{A_i m_p} = \frac{kT\rho}{m_p} \sum \frac{X_i}{A_i} = \frac{kT\rho}{\mu_{\mathrm{ion}} m_p}. \tag{2.15}$$

Where we have defined the **mean molecular weight** of the ions to be

$$\mu_{\mathrm{ion}} = \left[ \sum \frac{X_i}{A_i} \right]^{-1} \tag{2.16}$$

For the electrons, we have (assuming total ionization)

$$P_e = n_e kT = kT \left( \sum Z_i n_i \right) = \frac{kT\rho}{m_p} \sum \frac{Z_i X_i}{A_i}. \tag{2.17}$$

(here $Z_i$ is the atomic number, not the metallicity). Then the total pressure is just the sum of these two,

$$P = P_{\text{ion}} + P_e = \frac{\rho kT}{m_p}\left(\frac{1}{\mu_e} + \frac{1}{\mu_i}\right) \tag{2.18}$$

So we define the overall mean molecular weight via

$$\frac{1}{\mu} \equiv \frac{1}{\mu_e} + \frac{1}{\mu_i} \tag{2.19}$$

For fully-ionized solar composition material, we have $\mu \approx 0.64$.

One might think of this as the average weight of a particle that supplies pressure within a gas. Later, we'll see that this quantity, and its evolution, plays a large and critical role in the the nature of stellar evolution. Since fusion tends to decrease the pressure support, the star must continuously readjust its structure so as to hold itself up.

*Wednesday, October 2, 2013*

## 2.3 Electric Fields in Stars

Imagine a pure, ionized hydrogen atmosphere which is, on the large scale, electrically neutral. We wish to find the scale height in a plasma of ionized hydrogen. In this plasma, we have $n_p = n_e$ due to electric neutrality. Then the overall pressure in this hydrogen plasma is

$$P = 2n_p kT \tag{2.20}$$

Using hydrostatic equilibrium, we get

$$2kT\frac{dn_p}{dz} = -m_p n_p g \tag{2.21}$$

which in turn gives us the differential equation

$$\frac{d\ln n_p}{dz} = -\frac{m_p g}{2kT} \tag{2.22}$$

Which gives us a scale height of

$$h = \frac{2kT}{m_p g} \tag{2.23}$$

We need to look at both plasmas separately while incorporating the electric field created between the protons and electrons. For electrons, we have

$$\frac{1}{n_e}\frac{dP_e}{dz} = -m_e g - eE. \tag{2.24}$$

Likewise for the protons,

$$\frac{1}{n_p}\frac{dP_p}{dz} = -m_p g + eE \tag{2.25}$$

where we've assumed that the electric field points up (the protons are heavier and would thus sink below the electrons). Now adding (2.24) and (2.25), we recover hydrostatic balance. However, subtracting the two equations will get us the electric field:

$$0 = -m_e g + m_p g - 2eE, \tag{2.26}$$

giving the result,

$$eE = \frac{1}{2}\left(m_p - m_e\right) g \quad \text{or} \quad e\mathbf{E} \approx -\frac{m_p \mathbf{g}}{2}. \tag{2.27}$$

So this field does not directly affect hydrostatic balance, but it does dramatically impact the relative surface abundances of elements since different isotopes feel different forces, causing diffusion and sedimentation.

# 3 Self-Gravitating Objects

So far we have only considered systems where the acceleration due to gravity is constant. In any self-gravitating object, this is obviously not true. We will, however, continue to assume that such objects do not rotate. Additionally, we will be ignoring mass loss. Essentially all we must write down are equations of mass conservation, momentum conservation, and energy conservation. We'll start with momentum conservation.

## 3.1 Momentum Conservation and the Free-Fall Timescale

The momentum equation for a fluid is just

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{g} - \boldsymbol{\nabla} P \tag{3.1}$$

This equation essentially states that a self-gravitating object is neither collapsing nor expanding. If we were to "shut off" gravity or the pressure gradient, the star would either explode or collapse, respectively. Such a collapse would occur on the **free-fall timescale**, which we will now derive. Taking the pressure gradient out of (3.1), we retrieve

$$\mathbf{g} = -\frac{Gm(r)}{r^2}\hat{r} \tag{3.2}$$

For this derivation, we will be using a **Lagrangian coordinate system**. This is a system where the coordinates follow a particular fluid element. In essence, we are making the substitution

$$\frac{d}{dt} \to \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \tag{3.3}$$

Returning back to the derivation, (3.2) gives us

$$\frac{dv_r}{dt} = -\frac{Gm(r)}{r^2} \tag{3.4}$$

Initially, we have $t = 0$, $v_r = 0$, and $r = r_0$ with the radial velocity given by $v_r = dr/dt$. Then our differential equation is

$$\frac{d^2 r}{dt^2} = -\frac{Gm(r)}{r^2} \tag{3.5}$$

As an order of magnitude estimate, this gives us

$$\frac{r}{t_{\text{ff}}^2} \sim \frac{Gm}{r^2} \quad \Rightarrow \quad t_{\text{ff}}^2 \sim \frac{1}{Gm/r^3} \tag{3.6}$$

So we define the free-fall timescale to be

$$t_{\text{ff}} = \frac{1}{\sqrt{G\langle\rho\rangle}} \tag{3.7}$$

This is also the same as the Keplerian orbital period, modulo some uninteresting constants. The punchline of this whole argument is that a star that is *not* in hydrostatic balance will respond on a timescale of the free-fall timescale. From this alone, we may conclude that the sun (and any other star not currently exploding) is in hydrostatic balance. We will then assume that all stars are always in hydrostatic balance.

## 3.2 The Virial Theorem

Stars are held up by gas pressure, radiation pressure, or both. The pressure gradients are what will be the "restoring forces" against gravity for our cases. In spherical symmetry, hydrostatic balance tells us

$$\frac{dP}{dr} = -\rho\frac{Gm(r)}{r^2} \tag{3.8}$$

We will use this to derive the **Virial Theorem**, which relates the potential energy to the kinetic energy of a system. The equation of mass conservation states that

$$dm = 4\pi r^2 \rho(r)\, dr \tag{3.9}$$

Now we multiply both sides of (3.8) by $4\pi r^3\, dr$:

$$\int 4\pi r^3\, dP = -\int \rho\frac{Gm(r)}{r^2}4\pi r^3\, dr \tag{3.10}$$

$$= -\int \frac{Gm(r)dm}{r} = E_{\text{GR}} \tag{3.11}$$

where $E_{\text{GR}}$ is the gravitational binding energy. Performing a similar analysis to the left side of (3.10) gives

$$\int 4\pi r^3\, dr\frac{dP}{dr} = 4\pi r^3 P\Big|_0^R - 3\left[4\pi\int Pr^2\, dr\right] \tag{3.12}$$

$$= -3\int P\, 4\pi r^2\, dr \tag{3.13}$$

$$= -3\langle P\rangle V \tag{3.14}$$

where we've defined the average pressure to be the pressure averaged over volume. Also, in (3.13) we've noted that the boundary terms must vanish since $P(R) = 0$ (not necessarily the case during star formation in dark clouds, though). Then the virial theorem tells us that

$$\boxed{\langle P\rangle = -\frac{1}{3}\frac{E_{\text{GR}}}{V}} \tag{3.15}$$

Now we examine the total energy:

$$E_{\text{tot}} = E_{\text{GR}} + E_{\text{KE}} = -3\langle P\rangle V + E_{\text{KE}} \tag{3.16}$$

We need only relate the kinetic energy to the pressure to finish this equation off. For an ideal gas, we know that $P = NkT/V$, so the kinetic energy is $E_{\text{KE}} = \frac{3}{2}NkT = \frac{3}{2}PV$. This gives a total energy of

$$E_{\text{tot}} = -3\langle P\rangle V + \frac{3}{2}\langle P\rangle V = -E_{\text{KE}} \tag{3.17}$$

Interestingly, this requires a negative heat capacity. That is, an increase in the temperature of the system causes a net *decrease* in total energy. However for radiation, pressure is given by $P = \frac{1}{3}aT^4$ and $E/V = aT^4$. Taking this to its conclusion gives us

$$E_{\text{tot}} \to 0 \text{ as the particles become relativistic} \tag{3.18}$$

The origin of this result is in the momentum-energy relation of relativistic particles and non-relativistic particles. That is, $E = pc$ for ultra-relativistic particles and $E = p^2/2m$ for non-relativistic particles.

The limiting energy of ultra-relativistic stars puts an upper level on the mass of large stars, since a total energy of a star being zero means unbinding the star. In the "normal case" of an ideal gas star, the more traditional form of the virial theorem applies:

$$\frac{E_{\text{KE}}}{\text{mass}} \sim \frac{GM}{R} \tag{3.19}$$

This is why stars typically behave with a negative heat capacity. That is, as a star radiates, $E_{\text{tot}}$ is more negative, meaning that $R$ must decrease and the temperature $T$ (essentially the kinetic energy per particle) rises. This behavior would have to continue until a new energy source became available.

## 3.3 Applications of the Virial Theorm

The gravitational energy of an object is typically given by

$$E_{\text{GR}} \approx -\frac{GM^2}{R} \tag{3.20}$$

Using the virial theorem, we have

$$-E_{\text{GR}} = 3\langle P\rangle V = 3Nk\langle T\rangle \tag{3.21}$$

Or,

$$\frac{GM}{R}(Nm_p) \sim 3NkT \tag{3.22}$$

So we have

$$\boxed{kT \sim \frac{GMm_p}{R}} \tag{3.23}$$

This temperature is the temperature of most of the material and is $T \sim T_c$ (temperature in the core). For the sun, we then have $T \sim 10^7$ K. Interestingly, assuming hydrostatic equilibrium was all we needed to get a rough estimate of the sun's core temperature! One might note, though, that the surface temperature is significantly lower than the core temperature, so we must assume that there is heat loss taking place in the sun. Today the luminosity of the sun is

$$L_\odot = 4 \times 10^{33} \text{ erg/s} \tag{3.24}$$

If we assume there is no energy source for the sun other than gravitational energy, we can come up with a timescale (called the **Kelvin-Helmholtz timescale**)

$$t_{\text{KH}} = \left| \frac{E_{\text{GR}}}{L} \right| \approx 3 \times 10^7 \text{ years} \tag{3.25}$$

for the sun. This has been known for awhile and since the Earth is known to have existed much longer than $t_{\text{KH}}$, scientists deduced that another energy source within the sun was needed to explain its longevity. We now know that this energy source is, of course, fusion. Note that at the center of the sun, the temperature of $10^7$ K corresponds to an energy per particle of about 1 keV. The binding energy of helium is approximately 7MeV, approximately 7000 times bigger than the thermal content. Thus, the sun could last approximately 7000 times longer, bringing the projected lifetime of the sun up to a more reasonable (but still wrong) number of about 200 billion years. We conclude that nuclear energy is a more promising form of energy for the sun than chemical energy.

*Monday, October 7, 2013 (No updates)*

## 3.4 Gas Pressure and Radiation Pressure

Recall from the case of the constant density star that the gravitational energy is given by

$$E_{\text{GR}} = -\frac{3}{5}\frac{GM^2}{R} \tag{3.26}$$

And the average pressure is given by the virial theorem to be

$$\langle P \rangle = -\frac{1}{3}\frac{E_{\text{GR}}}{V} = (n_e + n_p)kT = 2n_p kT = \frac{2\rho kT}{m_p} \tag{3.27}$$

Here we've sort of assumed that the star is isothermal. This tells us that the average thermal energy is given by

$$kT = \frac{1}{10}\frac{GMm_p}{R} \tag{3.28}$$

Where the mass is given by

$$M = \rho\frac{4\pi}{3}R^3 \tag{3.29}$$

and the central temperature is given approximately by (scaled by solar units)

$$T_c \approx 2 \times 10^6\,\text{K} \left(\frac{\rho_c}{1\,\text{g}\,\text{cm}^{-3}}\right)^{1/3} \left(\frac{M}{M_\odot}\right)^{2/3} \tag{3.30}$$

These scalings are actually recovered in simulations (see Figure 2).

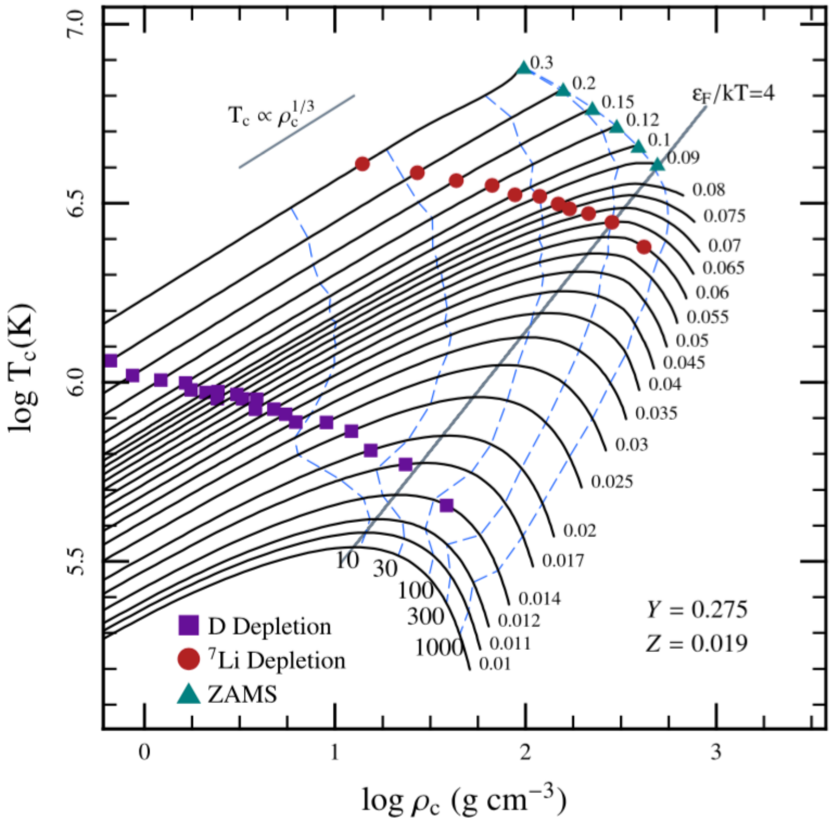


**Figure 16.** Trajectories of central conditions for fully convective $M < 0.3\,M_\odot$ stars as they approach the main sequence ($M > 0.08\,M_\odot$) or become brown dwarfs for $Y = 0.275$ and $Z = 0.019$. Each solid line shows $T_c$ and $\rho_c$ for a fixed mass, $M$, noted at the end of the line (when the age is 3 Gyr). The dashed blue lines are isochrones for ages of 10, 30,100, 300 Myr and 1 Gyr. The purple squares and red circles show where D and $^7$Li is depleted by a factor of 100. The green triangles show the ZAMS.

Figure 2: From Paxton et al. 2011 (MESA I)

Here we've only considered the case of the pressure due to an ideal gas, thus far ignoring the contributions from radiation pressure. We then want to know when radiation pressure becomes comparable to gas pressure. That is,

$$P_{\rm rad} = \frac{1}{3}aT^4 \geq P_{\rm gas} \tag{3.31}$$

The temperature in the star is approximately

$$kT \sim \frac{GMm_p}{R} \tag{3.32}$$

and the pressure gradient is, (again, very approximately)

$$\frac{dP}{dr} = -\rho g \approx \frac{P_c}{R} \sim \frac{M}{R^3}\frac{GM}{R^2} \tag{3.33}$$

Then the condition we are seeking is

$$\frac{1}{3}a\left(\frac{GMm_p}{Rk}\right)^4 \gtrsim \frac{GM^2}{R^4} \tag{3.34}$$

Interestingly, $R$ cancels in (3.34), so this condition is dependent only on the mass of the star. Thus, we can get a hard limit that is independent of any other properties of the star. Dropping tons more constants, this gives

$$M^2 > \frac{k^4}{aG^3m_p^4} \tag{3.35}$$

Recall that the radiation constant is given by $a = \frac{\pi^2}{15}\frac{k^4}{(\hbar c)^3}$. Putting this in (3.35), we have

$$\frac{M^2}{m_p^2} > \frac{k^4(\hbar c)^3}{G^3m_p^6k^4} \sim \left(\frac{\hbar c}{Gm_p^2}\right)^3 \tag{3.36}$$

Then the limit on the mass is then

$$\boxed{M > m_p\left(\frac{\hbar c}{Gm_p^2}\right)^{3/2}} \tag{3.37}$$

stars above this mass (approximately) have significant radiation pressure. Recall the fine structure constant

$$\alpha = \frac{1}{137} = \frac{e^2}{\hbar c} \tag{3.38}$$

Noting that the Coulomb energy is

$$E_{\text{Coulomb}} = \frac{e^2}{r} \tag{3.39}$$

Analagous to (3.39), we define a dimensionless measure of the strength of gravity, which appears in (3.37):

$$\alpha_{\text{G}} = \frac{Gm_p^2}{\hbar c} \approx 6\times 10^{-39} \tag{3.40}$$

Then the fundamental stellar mass given in (3.37) is

$$M > m_p\frac{1}{\alpha_{\text{G}}^{3/2}} \approx 2M_\odot \tag{3.41}$$

After all that work... it turns out that the mass where radiation pressure *actually* starts to matter is closer to $60-90\ M_\odot$.

Of interest in this case is that as $P_{\text{rad}} \gg P_{\text{gas}}$, then $E_{\text{tot}} \to 0$ from the virial theorem. In this state, the star has nearly enough energy to unbind itself, so radiation pressure sets an upper limit on the mass of a star as other instabilities will act to unbind it when the pressure is radiation dominated.

### 3.5 Summary

Note that here we have used hydrostatic balance to find the central temperature as a function of mass and radius. Additionally we have realized that energy losses from the surface require the radius of a star to decrease and the core temperature to increase (at least until another energy source is present). We will later show that the main sequence is just that place where the power generated by nuclear reactions is equal to that released by the star so that the star need not contract. What we have *not* done yet is to derive the rate of heat loss from the star.

# 4 Heat Transfer in Stars

In studying the ways in which heat moves outward through a star, we will first be ignoring convection, though it is a powerful mechanism when it is available to the star. To move heat through a star, there are electrons, ions, and photons available. Recall (or perhaps you don't) **Fick's Law**, which tells us that the heat flux can be determined via

$$F = \mathrm{ergs\,cm^{-2}\,s^{-1}} = -\frac{1}{3}v\ell\frac{d}{dx}(E) \tag{4.1}$$

where here $E$ is the energy density. We'll start first with the energy density of an ideal gas of electrons, but before we do that, let's derive (4.1).

### 4.1 Heat Flux Derivation (not done in class)

Imagine a medium with a gradient in temperature across a surface membrane. Let's say "above" the membrane, the temperature is $T_1$ and "below" the membrane, the temperature is $T_2$, with $T_2 > T_1$. Particles from region 2 then transport heat when they travel from region 2 to region 1. Let's call $E$ the internal energy per unit volume as defined before, and then let's see what happens at the surface. Particles, on average, will be coming from a distance $x + \ell$ above the membrane, where $\ell = (\sigma n)^{-1}$ is the mean free path of the particles. Then the downward flow of energy is

$$F_{\rm down} \approx \frac{1}{6}vE(x+\ell) \tag{4.2}$$

whereas particles from beneath move upward and carry heat from below at

$$F_{\rm up} \approx \frac{1}{6}vE(x-\ell) \tag{4.3}$$

Think of the factor of 1/6 as being the portion of the flux moving through a particular face of a cube, in this case, the face pointing up or down. So the net flux in the positive $\hat{x}$ direction is

$$F_x = -\frac{1}{6}vE(x+\ell) + \frac{1}{6}vE(x-\ell) \tag{4.4}$$

Writing the energy densities as linear functions,

$$E(x+\ell) = E(x) + \ell\frac{dE}{dx} \tag{4.5}$$

$$E(x-\ell) = E(x) - \ell\frac{dE}{dx} \tag{4.6}$$

so

$$F_x = -\frac{1}{3}v\ell\frac{dE}{dx} \tag{4.7}$$

## 4.2 Heat Transport by Electrons

For electrons, the energy density is

$$E = \frac{3}{2}kTn_e \tag{4.8}$$

Then the energy density gradient is

$$\frac{dE}{dx} = \frac{dE}{dT}\frac{dT}{dx} = \frac{3}{2}nk\frac{dT}{dx} \tag{4.9}$$

Then from Fick's Law, we have

$$F = -\frac{1}{3}v\ell\frac{3}{2}nk\frac{dT}{dx} = -\frac{1}{2}v\ell nk\frac{dT}{dx} \tag{4.10}$$

Where the mean free path is

$$\ell = \frac{1}{n\sigma} \tag{4.11}$$

for the scattering cross section $\sigma$. Then (4.10) becomes

$$F = -\left[\frac{1}{2}v\frac{k}{\sigma}\right]\boldsymbol{\nabla}T \tag{4.12}$$

From the theory of Coulomb scattering, the cross section would be

$$\sigma_{\text{Coulomb}} \sim b^2 \sim \frac{e^4}{(kT)^2} \tag{4.13}$$

Which gives a flux of

$$L = 4\pi R^2 F = 4\pi R\frac{(kT)^{7/2}}{m_e^{1/2}e^4} \tag{4.14}$$

For the sun, this would give

$$L \approx 5 \times 10^{31}\,\mathrm{erg\,s^{-1}} \tag{4.15}$$

which is two orders of magnitude too small, so we conclude that the sun does not transmit its heat via conduction through electrons. Now we'll move on to photons

## 4.3 Radiative Diffusion

For photons, the main scatterer will be electrons, so the cross section in question of the mean free path is the Thomson cross section. Additionally, the energy density is now $E = aT^4$. Additionally the speed of photons is obviously the speed of light. Then the ratio of the fluxes due to photons and electrons is

$$\frac{F_\gamma}{F_e} = \frac{c}{(kT/m_e)^{1/2}}\frac{e^4/(kT)^2}{\sigma_{\text{Th}}}\frac{E_{\text{rad}}}{E_{\text{gas}}} \tag{4.16}$$

Remember that the Thomson cross section is given by

$$\sigma_{\text{Th}} = \frac{8\pi}{3}\frac{e^4}{(m_ec^2)^2} \tag{4.17}$$

Then (4.16) becomes

$$\frac{F_\gamma}{F_e} = \left(\frac{m_e c^2}{kT}\right)^{1/2} \left(\frac{m_e c^2}{kT}\right)^2 \frac{P_{\rm rad}}{P_{\rm gas}} \tag{4.18}$$

Comparing the pressures gives

$$\frac{P_{\rm rad}}{P_{\rm gas}} \approx 10^{-4} \left(\frac{M}{M_\odot}\right)^2 \tag{4.19}$$

(this is essentially a more accurate estimate of what we determined in (3.41).) Then plugging (4.19) into (4.18), we see that heat transport by photons dominates heat transport by electrons whenever

$$M > 0.03\, M_\odot \left(\frac{T}{10^7\ \rm K}\right)^{5/4} \tag{4.20}$$

So if conduction ever dominates, it is in very low mass stars (also white dwarfs in their late lives). For our cases, photons are always going to be the dominant transport mechanism. We still haven't found out what the actual luminosity will be in the case of radiative diffusion. We do so now.

$$L = 4\pi R^2 F = 4\pi R^2 \frac{1}{3} c\ell \frac{d}{dr}\left(aT^4\right) \approx R^2 c \frac{1}{n_e \sigma_{\rm Th}} \frac{1}{R} aT^4 \approx R^2 \frac{c m_p}{\rho \sigma_{\rm Th}} \frac{1}{R} a \left(\frac{GMm_p}{Rk}\right)^4 \tag{4.21}$$

where we have noted that $kT = \frac{GMm_p}{R}$. Continuing on,

$$L \approx \frac{c m_p a (GMm_p)^4}{M \sigma_{\rm Th} k^4} \propto M^3 \tag{4.22}$$

This relation is surprisingly accurate for stars with masses greater than a solar mass. Note that we have derived the stellar luminosity with *no knowledge* of the source of energy. The luminosity is set by the modes of heat transport available to the star.

*Wednesday, October 9, 2013*

### 4.3.1 The Eddington Limit and the Eddington Standard Model

Continuing with heat transfer via radiation with electron scattering being the primary source of opacity, the flux is given via Fick's law as

$$F = \frac{1}{3} v\ell \frac{d}{dz}(aT^4) \tag{4.23}$$

where now $v = c$ and $\ell = (n_e \sigma_{\rm Th})^{-1}$. Plugging these in to (4.23), the flux becomes

$$F = \frac{1}{3} c \frac{1}{n_e \sigma_{\rm Th}} \frac{d}{dz}(aT^4) = \frac{4}{3} \frac{acT^3}{\rho\kappa} \frac{dT}{dz} \tag{4.24}$$

where we've defined the **opacity** by generalizing the relationship between the density and mean free path via

$$\ell = \frac{1}{n\sigma} \equiv \frac{1}{\rho\kappa} \qquad \Rightarrow \qquad \kappa_{\rm es} \equiv \frac{n_e \sigma_{\rm Th}}{\rho} = \frac{\sigma_{\rm Th}}{\mu_e m_p} \approx 0.4\ {\rm cm^2\ g^{-1}} \tag{4.25}$$

The opacity is measured in units of area per unit mass, indicating it is the cross-sectional area per unit mass. For electron scattering, it is essentially constant, but when other processes are relevant, it may depend on temperature and density. With the flux available, we can write the luminosity as

$$L(r) = F(r)4\pi r^2 = -4\pi r^2 \frac{4}{3}\frac{acT^3}{\rho\kappa}\frac{dT}{dr} = -4\pi r^2 \frac{c}{\kappa\rho}\frac{d}{dr}P_{\text{rad}} \tag{4.26}$$

Noting the product $\rho\, dr$ showing up in the denominator of (4.26), we are reminded of hydrostatic equilibrium:

$$\frac{dP}{dr} = -\rho(r)g(r) \quad \Rightarrow \quad dP = -\rho(r)g(r)\, dr \tag{4.27}$$

Putting in $g(r)$ explicitly,

$$\frac{dP}{\rho(r)dr} = -\frac{Gm(r)}{r^2} \tag{4.28}$$

Bringing the column depth back in ($y = \int \rho(r)\, dr$), this becomes

$$\frac{dP}{dy} = \frac{Gm(r)}{r^2} \tag{4.29}$$

where we've used the fact that $y = 0$ at the surface and increases *inwards* (hence the sign change). Assuming $g$ is a constant, this would give our old result

$$\boxed{P = gy} \tag{4.30}$$

Now returning to (4.26):

$$\frac{dP_{\text{rad}}}{dy} = \frac{L(r)}{4\pi r^2}\frac{\kappa}{c} \tag{4.31}$$

and the result we just derived

$$\frac{dP}{dy} = \frac{Gm(r)}{r^2} \tag{4.32}$$

Taking the ratio of (4.31) and (4.32) gives us

$$\frac{dP}{dP_{\text{rad}}} = \frac{4\pi Gm(r)c}{\kappa L(r)} \tag{4.33}$$

Suppose $M = m(R) =$ total mass of star and $L = L(R) =$ luminosity of star

$$\frac{dP}{dP_{\text{rad}}} = \left[\frac{4\pi GcM}{\kappa L}\right]\frac{m(r)}{M}\frac{L}{L(r)} \tag{4.34}$$

We have now introduced the **Eddington Luminosity**,

$$L_{\text{Edd}} = \frac{4\pi GcM}{\kappa} \tag{4.35}$$

In general, the Eddington Luminosity is much larger than the luminosity, since it is the luminosity where the only pressure gradient that matters is the radiation pressure (where the star becomes unstable). Evaluating the Eddington Luminosity in solar units and with electron scattering,

$$L_{\text{Edd}} = 3.1 \times 10^4\ L_\odot \left(\frac{M}{M_\odot}\right) \tag{4.36}$$

Also recall how luminosity scales with mass:

$$L = L_\odot \left(\frac{M}{M_\odot}\right)^3 \tag{4.37}$$

So we see that for low-mass stars, their luminosities are indeed much smaller than the Eddington Luminosity. Now let's investigate the ratio of the luminosity to the Eddington Luminosity:

$$\frac{L}{L_{\mathrm{Edd}}} \approx 3 \times 10^{-5} \left(\frac{M}{M_\odot}\right)^2 \tag{4.38}$$

So until $M \geq 100\, M_\odot$, it will be the case that $L \ll L_{\mathrm{Edd}}$ and thus $P_{\mathrm{rad}} \ll P$. Now returning to (4.34), we define a dimensionless quantity, $\eta(r)$ by

$$\eta(r) \equiv \frac{L(r)}{L} \frac{M}{m(r)} \tag{4.39}$$

And (4.34) looks like

$$\frac{dP_{\mathrm{rad}}}{dP} = \frac{L}{L_{\mathrm{Edd}}} \eta(r) \tag{4.40}$$

Aside: this model of stars is the called the "Eddington Standard Model" and was used to describe stars before their source of energy was known.

### 4.3.2 Aside: Alternate Derivation of the Eddington Luminosity

Cosnider a situaton where photons are streaming through a medium of protons and electrons with a number density of photons $n_\gamma$ with the Thomson cross section (they interact primarily with electrons), $\sigma_{\mathrm{es}}$. Then the collision time (the average time between interactions with electrons) is

$$t_{\mathrm{coll}} = \frac{1}{\sigma_{\mathrm{es}} n_\gamma c} \tag{4.41}$$

And the momentum change from such collisions is given simply by

$$\Delta p = \frac{E_\gamma}{c} = \frac{h\nu}{c} \tag{4.42}$$

With the impulse and the time between interactions we can define a force:

$$F = \frac{\Delta p}{\Delta t} = \sigma_{\mathrm{es}} n_\gamma E_\gamma = \sigma_{\mathrm{es}} \frac{F}{c} \tag{4.43}$$

where the second $F$ is now the flux. Now when this force is greater than local gravity, $\sigma_{\mathrm{es}} F/c > m_p g$ (now we use the proton mass since we assume the electrons "pull" the protons along for the ride), we get radiative levitation with the condition

$$\frac{\sigma_{\mathrm{es}} L}{4\pi r^2 c} > m_p \frac{GM}{r^2} \qquad \Rightarrow \qquad L > L_{\mathrm{Edd}} = \frac{4\pi G M c}{\kappa} \tag{4.44}$$

So this is the luminosity at which the radiation field drives a mass loss from the star. The details of what actually happens in these conditions is still an active area of research.

### 4.3.3 Polytropic Relations

Returning now to the Eddinton standard model, we integrate (4.40) and get

$$\int_R^r dP_{\rm rad} = \frac{L}{L_{\rm Edd}} \int_R^r \eta(r)\, dP \tag{4.45}$$

We can integrate the left side, leaving us with

$$P_{\rm rad}(r) = \frac{L}{L_{\rm Edd}} \int_R^r \eta(r)\, dP \tag{4.46}$$

Mathematically speaking, we are now stuck because we do not have any knowledge of $\eta(r)$. The formal approach of solving this problem is to define spatial averages of $\eta(r)$ for different choices of mass and luminosity profiles. We will, however, just let $\eta(r) \approx 1$, which gives us

$$P_{\rm rad} = \frac{L}{L_{\rm Edd}} P_{\rm gas}(r) \tag{4.47}$$

And now substituting in our equations of state for the radiation pressure and assuming that the radiation pressure is negligible compared to the gas pressure:

$$\frac{1}{3} a T^4 = \frac{L}{L_{\rm Edd}} \frac{\rho k T}{\mu m_p} \tag{4.48}$$

Solving this for $T^3$, we have

$$\boxed{T(r)^3 = \frac{L}{L_{\rm Edd}} \frac{3k}{a\mu m_p} \rho(r)} \tag{4.49}$$

So we have gone from a situation where we had only typical or central values to an actual equation that lets us find $T(r)$, $\rho(r)$, and $P(r)$. It appears then, that in our low radiation pressure limit, $\rho \propto T^3$. Additionally, for the ideal gas law, we see that $P \propto \rho^{4/3}$. This is an example of the more general idea of a **polytrope**, which is a gas whose pressue and density are related simply by

$$P \propto \rho^{1+\frac{1}{n}} \tag{4.50}$$

Here $n$ is the **polytropic index**. These will be important later, especially for white dwarf stars. For our $n = 3$ polytrope we can take a given $L/L_{\rm Edd}$ and recover a relation between pressure and density via

$$\frac{dP}{dr} = -\rho(r) \frac{Gm(r)}{r^2} \tag{4.51}$$

$$dm(r) = \rho 4\pi r^2\, dr \tag{4.52}$$

Let us briefly consider an adiabatic change in an ideal gas. We recall from Freshman physics that $PV^\gamma = \text{const}$. For a monatomic, ideal gas, $\gamma = 5/3$, so we have $P \propto \rho^{5/3}$ and $\rho T \propto P$, so $T \propto \rho^{2/3}$. Then we'll call the "entropy" $T/\rho^{2/3}$. For our star, we have $T^3 \propto \rho \Rightarrow T \propto \rho^{1/3}$. In our case, the "entropy" is then $a/\rho^{1/3}$. Thus the entropy is highest in the outer layers and smallest in the center.

At this time we should also introduce another parameter often used in stellar astrophysics, $\beta$. Typically we designate

$$\beta \equiv \frac{P_{\text{gas}}}{P_{\text{tot}}} \tag{4.53}$$

This parameter gives us a sense of how radiation dominated the gas is at a given location. We can write the radiation pressure in terms of $\beta$ and $P_{\text{gas}}$ via

$$P_{\text{rad}} = \left(\frac{1-\beta}{\beta}\right) P_{\text{gas}} \tag{4.54}$$

And so the pressure run in the star is given by

$$P(r) = \frac{1}{\beta}\frac{k}{\mu m_p}\left[\frac{3k}{a\mu m_p}\frac{1-\beta}{\beta}\right]^{1/3} \rho(r)^{4/3} \tag{4.55}$$

where (4.55) comes from combining the definition of $\beta$ with hydrostatic equilibrium and radiative diffusion.

### 4.3.4 Heat Transfer in the Outer Atmosphere

Near the surface of the star, photons naturally leave (which is why we see them!) We want to know what the place or condition is like. Near the surface, the density must decrease exponentially since $T \ll T_c$ and thus $g$ is a constant. Our previous arguments for the scale height and the plane-parallel isothermal atmosphere is largely applicable here, so the length scale of relevance is the scale height:

$$h = \frac{kT}{\mu m_p g} \tag{4.56}$$

It then makes sense that the condition for photons to escape would be for the mean free path to be comparable to the scale height:

$$\ell \sim h \quad \Rightarrow \quad \frac{m_p}{\rho \sigma_{\text{Th}}} \approx \frac{kT}{mg} \tag{4.57}$$

or

$$g\frac{m_p}{\sigma_{\text{Th}}} \sim \frac{\rho k T}{m_p} \sim P_{\text{gas}} \tag{4.58}$$

Again, this can be rewritten as

$$g\kappa^{-1} \sim P_{\text{gas}} \quad \text{where} \quad \kappa = \frac{\sigma_{\text{Th}}}{m_p} \tag{4.59}$$

Then we can say the condition where photons can escape is

$$P_{\text{gas}} \leq \frac{g}{\kappa} \tag{4.60}$$

Below this pressure, our diffusion approximation starts to fail as the mean free path length is exceeding the local scale height. This argument assumed a constant opacity. In general, we must do a line integral through the depth of the outer atmosphere. Imagine we ask the probability for a photon to reach some depth in the star. We define the **optical depth** to be

$$d\tau = \frac{dr}{\ell} = dr\ \kappa(r)\rho(r) \tag{4.61}$$

Then the probability would be

$$P(\tau) \propto e^{-\tau} \tag{4.62}$$

Then the optical depth at a certain radius would be given by

$$\tau = \int_R^r \kappa(r)\rho(r)\,dr \tag{4.63}$$

or, invoking hydrostatic equilibrium,

$$\tau = \int \kappa(r)\frac{dP(r)}{g} = \frac{1}{g}\int \kappa(r)\,dP(r) \tag{4.64}$$

Then the condition for photons for leave becomes when the optical depth is unity, or

$$\tau \sim 1 = \frac{1}{g}\kappa P(r) \quad \Rightarrow \quad P(\tau = 1) = \frac{g}{\kappa} \tag{4.65}$$

Additionally, we can redefine the flux equation in terms of optical depth into a cute form:

$$F = \frac{c}{3}\frac{d}{d\tau}\left(aT^4\right) \tag{4.66}$$

### 4.3.5 The Effective Temperature

At the surface, the flux is given by

$$F = \frac{L}{4\pi R^2} \equiv \sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4 \tag{4.67}$$

Assuming that the flux is constant at or near the surface, we may use the fact that $a = 4\sigma_{\mathrm{SB}}/c$ and (4.66) to get

$$\frac{d}{d\tau}T^4 = \frac{3}{4}T_{\mathrm{eff}}^4 \tag{4.68}$$

Recall that $\tau$ is dimensionless, so there is no problem here. For $\tau \gg 1$, we have

$$T^4 = \frac{3}{4}T_{\mathrm{eff}}^4\tau \tag{4.69}$$

This implies that as $\tau$ increases (going deeper and deeper into the star), the radiation field becomes more and more isotropic. Naively, we might assume that the flux would be given by the energy density multiplied by the speed of light, but our result in (4.69) essentially tells us that $F = acT^4/\tau$.

*Monday, October 14, 2013*

## 4.4 Convection

Another important form of heat transport in stars is that of convection, which is where an instability causes the bulk movement of material (rather than the diffusion of photons or electrons) to transport heat throughout the star. Convection occurs only when the temperature gradient is very steep. Otherwise, radiative diffusion dominates heat transport.

The origin of the instability is that a fluid element that rises adiabatically (faster than the heat transfer time scale) is **lighter** than the surrounding fluid and so it "runs away". This is a linear instability, in that a displacement exponentially grows in time.

Suppose we have a fluid element in hydrostatic balance. We imagine displacing this fluid element from $r$ to $r + dr$. So we assume that the displaced element responds adiabatically. Thermodynamcis tells us that

$$T\,dS = dE + P\,dV = 0 \qquad (4.70)$$

for an adiabatic process (here these quantities are related to the inside of the fluid element). This is equivalent to the requirement that the timescale of response is much less than that for heat to enter or leave the fluid element (i.e., the local thermal timescale. We presume that the timescale is longer than the time for pressure equilibrium as well:

$$t = \frac{h}{c_s} = \text{time for the pressure to equilibrate}$$
$$\Rightarrow \quad v \ll c_s \qquad (4.71)$$

where $h$ is agin the scale height. This requirement ensures that the fluid element remains in pressure equilibrium with its surroundings at all times Suppose the element starts in location 1 and travels to location 2. This adiabatic process requires that $PV^\gamma$ be a constant within the bubble, which means that

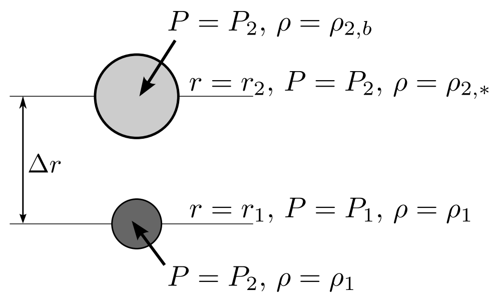


Figure 3: Bubble at initial ($r = r_1$) and perturbed ($r = r_2$) locations. The perturbation happens slow enough so that pressure equilibrium is achieved, but fast enough that there is no heat loss. Thus, the pressure remains equal to the background pressure, but the density need not match the background density.

$$\frac{P_1}{\rho_1^\gamma} = \frac{P_2}{\rho_2^\gamma} \qquad (4.72)$$

where the subscripts refer to where the bubble is located (and the quantities are measured within the bubble, not the surrounding material). Rearranging (4.72) gives us the new density

$$\rho_{2,b}^\gamma = \rho_1^\gamma \frac{P_2}{P_1} \qquad (4.73)$$

Clearly the bubble is now less dense than it was at its starting position, but how dense is it compared to the surrounding material? For *stability*, we require $\rho_{2,b} > \rho_{2,*}$, where the * indicates the density of the star's "background" conditions. For the star, we require

$$\rho_1 \left(\frac{P_2}{P_1}\right)^{1/\gamma} > \rho_{2,*} \qquad (4.74)$$

Note that $\rho_1$, $P_1$, and $P_2$ are the same for both the fluid element and the background. For an infinitesimal change, the pressures are related via

$$P_2 = P_1 + \Delta r \frac{dP}{dr} \qquad (4.75)$$

and likewise the densities,

$$\rho_2 = \rho_1 + \frac{d\rho}{dr}\Delta r \qquad (4.76)$$

(where all quantities are for the star, not the fluid element, as they shall be from now on unless otherwise noted). Then (4.74) requires

$$\frac{1}{\gamma}\frac{d\ln P}{dr} > \frac{d\ln\rho}{dr} \tag{4.77}$$

Now, we recall that

$$P = \frac{\rho kT}{\mu m_p} \quad \Rightarrow \quad d\ln\rho = d\ln P - d\ln T \tag{4.78}$$

(assuming uniform composition) and thus (4.77) can be written as

$$\frac{1}{\gamma}\frac{d\ln P}{dr} > \frac{d\ln P}{dr} - \frac{d\ln T}{dr} \tag{4.79}$$

For an ideal, monatomic gas, this becomes

$$\frac{d\ln T}{d\ln P} < 1 - \frac{1}{\gamma} = \frac{2}{5} \tag{4.80}$$

Recall that earlier we found that for radiative heat transport in the Eddington standard model, $\rho \propto T^3$, and thus $P \propto \rho T \propto T^4$, so $T \propto P^{1/4}$. Then computing the logarithmic derivative in (4.80),

$$\frac{d\ln T}{d\ln P} = \frac{1}{4} < \frac{2}{5} \tag{4.81}$$

so this model is stable to convection. Note that we could also include a gradient in the composition if we wanted. (4.81) would become, more genrally,

$$\boxed{\left(\frac{1}{\gamma} - 1\right)\frac{d\ln P}{dr} - \frac{d\ln\mu}{dr} > -\frac{d\ln T}{dr}} \tag{4.82}$$

This actually weakens the requirement on the temperature gradient. §6.3 of Keppenhahn and Weigert discuss the strange consequences of being in the "twilight zone" between the conditions set up by (4.82) and (4.80). It is possible, in this regime, to have a fluid element oscillate (as in a stable case), but have the oscillation grow in amplitude (not exactly what we would think of as stable).

Barring these strange situations, stars that follow (4.82) and certainly those following (4.80) are stable and do not transport heat by convection. A model that is stable to convection has the entropy increasing with radius. Note that when we say "entropy", we mean the **specific entropy**, or the entropy per unit mass.

To put things in context, we recall the Eddington standard model, where we found that $\rho \propto T^3$, so $P \propto T^4$. Then we have $d\ln T/d\ln P = 1/4 < 2/5$. So then the Eddington standard model star is stable to convection and thus transports heat primarily by radiative diffusion.

### 4.4.1 The Unstable Case

Now we wish to understand what happens when the background model is unstable. The density in the bubble is

$$\rho_{2,b} = \rho_1\left(\frac{P_2}{P_1}\right)^{1/\gamma} \tag{4.83}$$

and for the star,

$$\rho_{2,*} = \rho_1 + \Delta r \frac{d\rho}{dr}\bigg|_* \tag{4.84}$$

The density contrast is then

$$\Delta\rho = \rho_{2,*} - \rho_{2,b} = \Delta r \left[ \frac{d\rho}{dr}\bigg|_* - \frac{1}{\gamma}\frac{\rho}{P}\,\frac{dP}{dr}\bigg|_* \right] \tag{4.85}$$

or

$$\Delta\rho = \rho\Delta r \left[ \frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma} \right] \tag{4.86}$$

where we have invoked hydrostatic equilibrium. Stability requires $\Delta\rho < 0$, or

$$\boxed{\frac{d\ln\rho}{dr} < -\frac{\rho g}{\gamma P} \quad \text{(Stability)}} \tag{4.87}$$

This is the same result we obtained previously, just phrased a bit differently. In the unstable case, though, we require $\Delta\rho > 0$. The acceleration, $a$, or the displaced element will be given by

$$a = \frac{\Delta\rho}{\rho} g = g\Delta r \left[ \frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma} \right] \tag{4.88}$$

This is the equation of motion for the fluid element. Written in terms of the displacement coordinate $x$, (4.88) is

$$\ddot{x} = gx \left( \frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma} \right) \tag{4.89}$$

Strictly speaking this equation holds for unstable or stable cases. Clearly this is a harmonic oscillator, so the physics should be familiar. In any case, we have

$$-\omega^2 = g \left( \frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma} \right) \tag{4.90}$$

If $-\omega^2 < 0$, then the solution is stable, and the element oscillates at the Brunt Väisälä frequency, which is

$$N^2 = -g \left[ \frac{d\ln\rho}{dr} + \frac{\rho g}{\gamma P} \right] \tag{4.91}$$

This is the local frequency of oscillation of a fluid element in a convectively stable atmosphere. In the Earth's atmosphere, this frequency is around five or ten minutes. We should check to see that this formalism doesn't cause supersonic velocities in a stable model, so let's check the velocity of such an oscillating element, which is roughly

$$v_{\text{stable}} \approx N\delta x \approx \delta x \frac{c_s^2}{g} \tag{4.92}$$

Requiring that this be subsonic gives

$$\delta x \frac{c_s^2}{g} < c_s \qquad \Rightarrow \delta x < \frac{g}{c_s} = h \tag{4.93}$$

where $h$ is again the scale height. So as long as perturbations are smaller than the local scale height, our assumptions are valid.

Now if this coefficient is negative, the solution is unstable (i.e., grows exponentially). Note that $N^2 \sim g/h$, so

$$N^2 \sim \frac{g m_p g}{kT} \sim g^2 \frac{1}{v_{\rm th}^2} \quad \Rightarrow N \sim \frac{g}{v_{\rm th}} \sim \frac{g}{c_s} \tag{4.94}$$

So our displacement solution looks like

$$x = x_0 e^{t/\tau} \tag{4.95}$$

where $1/\tau^2 = -N^2$. So the speed is

$$\dot{x} = \frac{x_0}{\tau} e^{t/\tau} = \ell/\tau = \text{speed after moving a distance } \ell \tag{4.96}$$

Generally, the speed is given by

$$v = \ell\sqrt{-N^2} \tag{4.97}$$

Suppose that the stellar interior is strongly unstable, so that $N^2 \sim -g/h$. Then the velocity after traveling some length $\ell$ is

$$v = \ell \left(\frac{g}{h}\right)^{1/2} = \ell \left(\frac{g m_p g}{kT}\right)^{1/2} = \ell \frac{g}{v_{\rm th}} \approx \frac{g}{v_{\rm th}} h = \frac{g}{v_{\rm th}} \frac{v_{\rm th}^2}{g} = v_{\rm th} \tag{4.98}$$

So if $N^2 \sim -g/h$, i.e., the star is strongly unstable, then a runaway fluid element will reach the sound speed after traversing one scale height! Recall, though that the change in density is

$$\left.\frac{\Delta\rho}{\rho}\right|_{\text{at } \ell} \approx \ell \left(\frac{d \ln \rho}{dr} + \frac{\rho g}{P\gamma}\right) \tag{4.99}$$

and thus the velocity is

$$v = \ell \sqrt{\frac{g}{\ell}\left.\left(\frac{\Delta\rho}{\rho}\right)\right|_{\ell}} = (g\ell)^{1/2} \left(\left.\frac{\Delta\rho}{\rho}\right|_{\ell}\right)^{1/2} \tag{4.100}$$

*Wednesday, October 16, 2013*

### 4.4.2 Convective Efficiency

We left off last time with a rough relation between the velocity of an upwardly rising fluid element and the density contrast between the fluid element and the background material at some height $\ell$ above the element's original location. In the unstable case, we found that the element was less dense than the surrounding fluid, by examining the quantity $\Delta\rho/\rho|_\ell$. The velocity at which the element traveled was given by $v = (g\ell)^{1/2}(\Delta\rho/\rho|_\ell)^{1/2}$.

Inserting the scale height in for $\ell$,

$$\ell = h = \frac{kT}{mg} = \frac{c_s^2}{g} \tag{4.101}$$

Then the velocity would be

$$v = c_s \left[ \left. \frac{\Delta\rho}{\rho} \right|_h \right]^{1/2} \tag{4.102}$$

Now we consider the flux due to convection. The flux is just the velocity times the energy density, and so we can write

$$F = v\rho v^2 = \rho v^3 \tag{4.103}$$

Assuming $\Delta\rho/\rho \ll 1$, we may assume that the element is moving subsonically.

The business of heat flux via convection is covered in **Mixing Length Theory**. In this "theory", we imagine a convective element traveling up a **mixing height**, $\ell$ and then allowing the fluid element to equilibrate with the surroundings. Since the preferred lengthscale in a star is the scale height, we typically choose $\ell_{\mathrm{MT}} = \alpha h$ for some dimensionless constant $\alpha$. Applying this to (4.103),

$$F_{\mathrm{conv}} = \rho \frac{kT}{m_p} c_s \left( \left. \frac{\Delta\rho}{\rho} \right|_{\ell=h} \right)^{3/2} = P c_s \left( \left. \frac{\Delta\rho}{\rho} \right|_{\ell=h} \right)^{3/2} \tag{4.104}$$

The measure of how effective convection can be is to compare this to the radiative heat flux. Recall our result for the radiative flux:

$$F_{\mathrm{rad}} = \frac{1}{3} \frac{c}{\kappa\rho} \frac{d}{dr} \left( aT^4 \right) \approx \frac{c}{\kappa\rho} \frac{P_{\mathrm{rad}}}{h} = \frac{cP_{\mathrm{rad}}}{\tau} \tag{4.105}$$

where in the second equality, we're let $d/dr \to 1/h$, and in the third we've used the definition of optical depth, (4.61), to replace the denominator with the optical depth through a scale height. Now we find the ratio in these two fluxes:

$$\frac{F_{\mathrm{conv}}}{F_{\mathrm{rad}}} \sim \frac{\tau P c_s \left( \Delta\rho/\rho|_h \right)^{3/2}}{cP_{\mathrm{rad}}} \sim \frac{c_s}{c} \frac{P}{P_{\mathrm{rad}}} \tau \left( \left. \frac{\Delta\rho}{\rho} \right|_h \right)^{3/2} \tag{4.106}$$

For typical, low-mass stars, we assume that $P/P_{\mathrm{rad}} \sim 10^4 \gg 1$, let's find when both fluxes give equal contributions (also plugging in some solar values):

$$\left( \frac{kT}{m_p c^2} \right)^{1/2} 10^4 \tau \left( \frac{\Delta\rho}{\rho} \right)^{3/2} \sim 1 \tag{4.107}$$

The requirement becomes

$$\frac{\Delta\rho}{\rho} \approx 2 \left( \frac{10^4}{T} \right)^{1/3} \frac{1}{\tau^{2/3}} \tag{4.108}$$

Where $\tau$ and $T$ get smaller, the simple theory implies convection at near the sound speed, $c_s$. In truth, you can't do this problem correctly in one dimension, so this result isn't all that good. What does this imply about the surface, though? Stellar convection at or near the surface becomes sonic, which we say is **inefficient** *and* $\Delta\rho/\rho \to 1$.

> What I'm trying to do here is to write a density model in 1-D that doesn't make me blush.
> *–Lars Bildsten*

To find a situation where this *does* work, we go deep into the interior of the star, where $\tau \ggg 1$ (yes, three greater than signs). The relevant lengthscale here is still the scale height, which begins to approach the radial coordinate:

$$h = \frac{P}{\rho g} \sim \frac{GM^2}{R^4 M} \frac{R^3 R^2}{GM} \sim R \tag{4.109}$$

so we'll just use the radius of the star as our characteristic length scale. The ratio of fluxes is now

$$\frac{F_{\text{conv}}}{F_{\text{rad}}} \sim \frac{c_s}{R} \left[ \frac{P}{P_{\text{rad}}} \tau \frac{R}{c} \right] \left( \left. \frac{\Delta \rho}{\rho} \right|_R \right)^{3/2} \tag{4.110}$$

Note that the time it takes for a photon to diffuse from a star is $t_{\text{rw}} = \tau R/c$. Thus, that is the time it takes to evacuate the radiation field of energy, so multiplying by the ratio in pressures is actually the Kelvin-Helmholtz time. Using this fact, (4.110) can be written as

$$\frac{F_{\text{conv}}}{F_{\text{rad}}} = \frac{t_{\text{KH}}}{t_{\text{dyn}}} \left( \left. \frac{\Delta \rho}{\rho} \right|_R \right)^{3/2} \tag{4.111}$$

Deep in the star, $t_{\text{KH}} \ggg t_{\text{dyn}}$, so convection is efficient when

$$\left. \frac{\Delta \rho}{\rho} \right|_R \sim \left( \frac{t_{\text{dyn}}}{t_{\text{KH}}} \right) \lll 1 \tag{4.112}$$

When convection is efficient, then, the stellar model nearly follows the adiabatic relation! Deep interior convection, in this limit, only implies that we need to know the adiabatic relation very well. The punchline here is that in the deep interior of a star, if the temperature gradient is even slightly super-adiabatic, convection becomes an incredibly powerful form of heat transport.

Note that the ratio of times scales like

$$\frac{t_{\text{dyn}}}{t_{\text{KH}}} \approx 3 \times 10^{-9} \left( \frac{M}{10\ M_\odot} \right)^{3.5} \tag{4.113}$$

Noting that $L \propto M^{3.5}$ and $R = R_\odot (M/M_\odot)$ on the main sequence, this gives us

$$\frac{v}{c_s} \approx 10^{-3} \left( \frac{M}{10\ M_\odot} \right)^{7/6} \tag{4.114}$$

This relatively slow speed reinforces our assumption that convective bubbles move slowly enough to maintain pressure equilibrium with their surroundings. Now remember the adiabatic condition for convection derived earlier:

$$\left. \frac{d \ln T}{d \ln P} \right|_* = \left. \frac{d \ln T}{d \ln P} \right|_{\text{adiabatic}} = \frac{2}{5} \tag{4.115}$$

So in the convective zone, we must have $T \propto P^{2/5} \propto (\rho T)^{2/5}$, or more directly, $T \propto \rho^{2/3}$. So far, we have motivated *two* distinct **polytropic relations** (models where $P \propto \rho^{(n+1)/n}$ for some $n$):

1. **Fully Convective and Efficient**: $P \propto \rho^{5/3}$, where the prefactor is the specific entropy of the star, *which is spatially constant*. Thus if we have the mass of a star and the entropy, we can calculate everything we want to know about this star (in this simple model).
2. **Constant $L/M$** star with constant (electron scattering) opacity, where $P \propto \rho^{4/3}$.

*Monday, October 21, 2013*

## 4.5 Fully Convective Star

Imagine a star of ideal gas *and* from the core to the photosphere is fully convective *and* efficient (this will likely be a bad approximation near the surface). Somewhere near the surface, photons must be made to allow the energy to escape the star. Near the photosphere, the pressure is

$$P_{\rm ph} \approx \frac{g}{\kappa_{\rm ph}} \tag{4.116}$$

If we were to plot $\ln T$ against $\ln P$ (as we did in lecture), for this star, it would have a slope of 2/5. Getting out towards the surface, though, the curve would have to be a bit shallower. We hope this only happens out towards one scale height in from the surface. Incorporating this physics will change $R$, but only on the order of $h|_{\rm location}$. Noting that

$$\frac{h}{R} \sim \frac{T}{T_c} \tag{4.117}$$

we see this is an extremely small error, since $T/T_c$ is very small out towards the surface. We assume $P \propto \rho^{5/3}$ (an $n = 3/2$ polytrope). In this model, the central pressure is

$$P_c = 0.77\frac{GM^2}{R^4} \tag{4.118}$$

The central temperature is

$$T_c = 0.54\frac{GM\mu m_p}{kR} \tag{4.119}$$

Requiring the entropy be the same everywhere requires

$$\frac{T_c}{P_c^{2/5}} = \frac{T_{\rm ph}}{P_{\rm ph}^{2/5}} = \frac{T_{\rm eff}}{P_{\rm ph}^{2/5}} \tag{4.120}$$

Since we've assumed the perfect adiabat (admittedly a lie... also, $\mu$ is changing due to varying ionization conditions in the outer layers of the star). The surface thermal energy is

$$kT_{\rm eff} = 0.6\left(\frac{GM\mu m_p}{R}\right)\left(\frac{R^2}{M\kappa_{\rm ph}}\right)^{2/5} \tag{4.121}$$

Putting in $\kappa_{\rm ph} = \kappa_{\rm es}$ gives us

$$T_{\rm eff} = 200\ {\rm K}\left(\frac{M}{M_\odot}\right)^{3/5}\left(\frac{R_\odot}{R}\right)^{1/5} \tag{4.122}$$

So we have a big problem with assuming electron scattering (or convection) since this is obviously too low for a star like the sun. Also, this temperature is way too low to allow free electrons, which we have been assuming sets the opacity.

## 4.6 The Hayashi Track

It turns out that in this case, the outer body sets the temperature. Electron scattering is *not* the primary source of opacity. Typically $H^-$ opacity is much more important. This is where a very loosely bound ion (ionization energy around 0.75 eV) is "ionized" by a photon (really, it's un-ionized) in the reaction

$$\gamma + \mathrm{H}^- \to e^- + \mathrm{H} \tag{4.123}$$

where the free electrons probably come from an alkali metal due to their very low first ionization energies. Hansen, Kawaler, and Trimble give the value of this opacity as

$$\kappa_{\mathrm{H}^-} = 2.5 \times 10^{-31} \left( \frac{\rho}{1\,\mathrm{g\,cm^{-3}}} \right)^{1/2} \left( \frac{T}{1\ \mathrm{K}} \right)^9 \ \mathrm{cm^2\ g^{-1}} \tag{4.124}$$

Now with the pressure given by

$$P = \frac{\rho k T}{\mu m_p} \tag{4.125}$$

and the density at the photosphere given by

$$\rho = \frac{g}{\kappa} \frac{\mu m_p}{kT} \tag{4.126}$$

Combining (4.125) and (4.126) gives the density as

$$\rho = \frac{g \mu m_p}{kT \kappa_0 \rho^{1/2} T^9} \tag{4.127}$$

Or we could have

$$\rho^{3/2} = \frac{g\mu m_p}{kT^{10}\kappa_0} \quad \Rightarrow \quad \rho = \left( \frac{g\mu m_p}{kT^{10}\kappa_0} \right)^{2/3} \tag{4.128}$$

Comparing these two values for the densities gives an opacity of

$$\kappa_{\mathrm{H}^-} = \kappa_0 T^9 \left( \frac{g\mu m_p}{kT^{10}\kappa_0} \right)^{1/3} \tag{4.129}$$

or more succinctly,

$$\boxed{\kappa_{\mathrm{H}^-} = \kappa_0^{2/3} T^{17/3} \left( \frac{g\mu m_p}{k} \right)^{1/3}} \tag{4.130}$$

Using this opacity in (4.121), we obtain (omitting some algebra), the more reasonable result of

$$T_{\mathrm{eff}} \approx 2500\ \mathrm{K} \left( \frac{M}{M_\odot} \right)^{1/7} \left( \frac{R}{R_\odot} \right)^{1/49} \tag{4.131}$$

So the luminosity goes as

$$\frac{L}{L_\odot} = 0.034 \left( \frac{M}{M_\odot} \right)^{4/7} \left( \frac{R}{R_\odot} \right)^{102/49} \tag{4.132}$$

Writing the effective temperature as a function of the mass and the luminosity gives us

$$T_{\mathrm{eff}} \approx 2600\ \mathrm{K} \left( \frac{L}{L_\odot} \right)^{1/102} \left( \frac{M}{M_\odot} \right)^{7/51} \tag{4.133}$$

Note that this is nearly independent of the luminosity, so it is nearly a straight vertical line on the HR diagram. As a molecular cloud contracts, heat transport is dominated by convection and the opacity in the photosphere is dominated by $H^-$ opacity. This explains how we see stars descending a vertical line in the HR diagram called the **Hayashi Track** that corresponds to (4.133). Note, however, that there is a lower limit on the effective temperature below which no hydrostatic radiating solutions exist because it is impossible to have the same entropy in the photosphere as in the core (recall that that was a requirement for convection to take place).

The contraction of a protostar and its corresponding trip down the Hayashi Track is rather quick at the high-luminosity end but changes as the star contracts. So the initial evolution is down the Hayashi Track until a radiative solution exists, after which point the star evolves along a nearly constant constant luminosity track (the less-famous Henyey Track).

# 5 Star Formation and the Jean's Mass

After discussing a star's trip down the Hayashi track, we take a moment to investigate a bit more about the physics of star formation, which is still a field with many unanswered questions.

## 5.1 The Jean's Mass

In the interstellar medium, there are large masses of cold dust and gas that can give rise to isolated regions of stellar formation. Often these manifestations are called **cold cores**. Suppose one of these cold cores is in the ISM with some density $\rho$ and temperature $T$ (likely around 10-20 K). The gravitational energy is something like

$$E_{\mathrm{GR}} \approx -\frac{GM^2}{R} \tag{5.1}$$

And the thermal content is something like

$$E_{\mathrm{th}} = \frac{3}{2}\frac{M}{m_p}kT \tag{5.2}$$

In order for this cloud to collapse, we must have the total energy being less than zero. Thus, we must have

$$\frac{GM^2}{R} > \frac{3}{2}\frac{M}{m_p}kT \tag{5.3}$$

If we assume a constant density profile, this tells us that the mass of the object must be

$$M > 500\ M_\odot \left(\frac{T}{10\ \mathrm{K}}\right)^{3/2}\left(\frac{1\ \mathrm{cm}^{-3}}{n}\right)^{1/2} \tag{5.4}$$

This mass is called the **Jean's Mass**. If a cloud is more massive than this, it is possible to collapse. Now, it may stay as one large blob, or it may fragment... we're not sure yet. The question is then, "How does the Jean's mass change in a collapsing cloud?" We know that the Jeans mass scales as

$$M_{\mathrm{J}} \propto T^{3/2}\rho^{-1/2} \tag{5.5}$$

As the star collapses the density must increase. *If* the material maintains its entropy, then we must have $T \propto \rho^{2/3}$. Then the Jean's mass scales as

$$M_\mathrm{J} \propto T^{3/2}\rho^{-1/2} \propto \rho^{1/2} \tag{5.6}$$

So if the collapse is adiabatic, the collapse does not lead to fragmentation. The cloud will reach some critical density, at which point the Jean's mass is higher than the mass of the cloud, halting collapse.

Cooling of the gas (i.e., the temperature remains constant) gives the Jean's mass trivially scaling as

$$M_\mathrm{J} \propto \frac{1}{\rho^{1/2}} \tag{5.7}$$

This allows for the possibility of fragmentation since the mass will remain above the Jean's mass. More likely, this will be the case at earlier times, and then later the collapse becomes more adiabatic until a freeze-out. What ends the fragmentation is that at high density, the optical depths are increasing and the gas cannot radiate on the contraction timescale. If we can analyze the microphysics governing these processes, we could determine a minimum Jean's mass, which would explain the fragmentation. At low metal content, the minimum masses are much higher. This is thought to be the reason why the first stars were so large.

When cooling is efficient, then the collapse and fragmentation occurs on the dynamical timescale:

$$t_\mathrm{dyn} \approx \frac{1}{\sqrt{G\rho}} = \frac{10^7 \text{ years}}{(n/100 \text{ cm}^{-3})^{1/2}} \tag{5.8}$$

Then it typically takes $1-10$ Myrs for the collapse to occur. At the center, an object at hydrostatic balance forms while the outer layers are still collapsing (the collapse is not homologous). This causes an **accretion luminosity** of

$$L \approx \dot{M}\frac{GM_c}{R_c} \tag{5.9}$$

which is powered by a loss of gravitational potential energy. The shocks on the core surface typically have energies of

$$kT_\mathrm{shock} \approx \frac{GM_c m_p}{R_c} \approx 0.1 \text{ keV} \tag{5.10}$$

Note that we've been assuming spherically-symmetric and non-rotating geometry. However, the radius is shrinking several orders of magnitude, so any small amount of rotation is likely balooned by the collapse. So it is hard or impossible to start with such a low specific angular momentum to allow for a spherical collapse. Instead, the accretion onto the protostar is done through an accretion disk where angular momentum can be removed through jets or the disk itself. That's hard to model..., so we'll return to a fantastical spherically-symmetric collapse.

*Wednesday, October 23, 2013*

## 5.2 Pre-Main Sequence Stars

Recall the main equations of stellar structure:

$$\frac{dP}{dr} = -\rho(r)g(r) \tag{5.11}$$

$$dm(r) = 4\pi r^2 \rho(r)\, dr \tag{5.12}$$

Fluxes defined by temperature gradient

We need to be able to have an equation describing energy (the next moment, as it were). Looking at the second law of thermodynamics, we have

$$dQ = TdS = dE + PdV \tag{5.13}$$

The corresponding time-dependent equation in the Lagrangian picture is

$$T\frac{dS}{dt} = \frac{dE}{dt} + P\frac{dV}{dt} \tag{5.14}$$

In terms of the specific units (per unit mass), where now $E$ is the specific internal energy,

$$dQ = dE + Pd\left(\frac{1}{\rho}\right) = dE - \frac{P}{\rho^2}d\rho \tag{5.15}$$

The Lagrangian equation becomes

$$T\frac{ds}{dt} = \frac{dQ}{dt} \equiv \text{loss or gain} \tag{5.16}$$

First we have heat gained via nuclear reactions, $\varepsilon$. Secondly, there is heat gained or lost due to a gradient in $\mathbf{F}$. Then (5.16) becomes our last equation for stellar evolution:

$$\boxed{T\frac{ds}{dt} = \varepsilon_{\text{nuc}} - \frac{\nabla \cdot \mathbf{F}}{\rho}} \tag{5.17}$$

Here we have $\mathbf{F} = F_r\hat{r}$. Then

$$\frac{\nabla \cdot \mathbf{F}}{\rho} = \frac{1}{\rho}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) \tag{5.18}$$

and the luminoisty is

$$L(r) = 4\pi r^2 F_r \tag{5.19}$$

Then we can rewrite (5.18) as

$$\frac{\nabla \cdot \mathbf{F}}{\rho} = \frac{1}{\rho\, 4\pi r^2}\frac{\partial}{\partial r}L(r) = \frac{dL(r)}{dm(r)} \tag{5.20}$$

So in the case of no nuclear burning (a pre-main sequence star), we have

$$T\frac{ds}{dt} = -\frac{dL(r)}{dm(r)} \tag{5.21}$$

We assume that the lumonisty gradient is positive due to the temperature gradient, so under the absence of an internal energy source, the entropy will decrease. It will decrease at the rate set by the heat loss to infinity. Adding on the rest to (5.21),

$$T\frac{ds}{dt} = -\frac{dL(r)}{dm(r)} = \frac{dE}{dt} - \frac{P}{\rho^2}\frac{d\rho}{dt} \tag{5.22}$$

Note that the internal energy is

$$E = \frac{3}{2}\frac{kT}{\mu m_p} = \frac{3}{2}\frac{P}{\rho} \tag{5.23}$$

Inserting this into (5.22) gives us

$$\begin{aligned} T\frac{ds}{dt} &= \frac{3}{2}\frac{1}{\rho}\frac{dP}{dt} - \frac{5}{2}\frac{P}{\rho^2}\frac{d\rho}{dt} && (5.24)\\ &= \frac{P}{\rho}\left[\frac{3}{2}\frac{d\ln P}{dt} - \frac{5}{2}\frac{d\ln\rho}{dt}\right] && (5.25)\\ &= \frac{3}{2}\frac{P}{\rho}\left[\frac{d\ln P}{dt} - \frac{5}{3}\frac{d\ln\rho}{dt}\right] && (5.26)\\ &= \frac{3}{2}\frac{P}{\rho}\frac{d}{dt}\ln\left(P/\rho^{5/3}\right) = -\varepsilon_{\text{grav}} && (5.27) \end{aligned}$$

Textbooks will often refer to this as $-\varepsilon_{\text{grav}}$. This is done so that

$$\frac{\partial L(r)}{\partial m(r)} = \varepsilon_{\text{nuc}} + \varepsilon_{\text{grav}} \tag{5.28}$$

Let's do some dimensional analysis for a contracting star. The pressure is

$$P \sim \frac{GM^2}{R^4} \sim \frac{GM^2}{M^{4/3}}\rho^{4/3} \tag{5.29}$$

Then the quantity in the log in (5.27) would be

$$\frac{P}{\rho^{5/3}} \approx C\frac{1}{\rho^{1/3}} \tag{5.30}$$

If the star is fully convective, then $P/\rho^{5/3}$ must be spatially constant, and we must have

$$T\frac{ds}{dt} = \frac{3}{2}\frac{P}{\rho}\frac{d}{dt}\left[\ln\left(C\rho_c^{-1/3}\right)\right] = -\frac{1}{2}\frac{P}{\rho}\frac{d\ln\rho_c}{dt} \tag{5.31}$$

Assuming that $\varepsilon_{\text{nuc}} = 0$, (5.28) becomes

$$-\frac{1}{2}\frac{P}{\rho}\frac{d\ln\rho_c}{dt} = -\frac{dL(r)}{dm(r)} \quad\Rightarrow\quad \frac{dL(r)}{dm(r)} = \frac{1}{2}\frac{P}{\rho}\frac{d\ln\rho_c}{dt} \tag{5.32}$$

where $\rho_c \propto M/R^3$. Now we want to get (5.32) in terms of more physical variables. Assuming $M$ is constant in time,

$$\frac{dL(r)}{dm(r)} = \frac{1}{2}\frac{P}{\rho}\frac{d}{dt}\left[\ln\left(\frac{1}{R^3}\right)\right] \quad\Rightarrow\quad \frac{dL(r)}{dm(r)} = -\frac{3}{2}\frac{P}{\rho}\frac{d}{dt}\ln R \tag{5.33}$$

So, assuming a fully-convective star with no mass loss, we have

$$\boxed{\frac{dL(r)}{dm(r)} = -\frac{3}{2}\frac{P(r)}{\rho(r)}\frac{1}{R}\frac{dR}{dt}} \tag{5.34}$$

Backtracking a bit, earlier we showed that

$$\text{entropy} \propto \frac{P}{\rho^{5/3}} \sim \frac{GM^2}{R^4}\frac{R^5}{M^{5/3}} \propto RM^{1/3} \tag{5.35}$$

for a star in hydrostatic balance. An adiabatic adjustment to this star must require $R \propto 1/M^{1/3}$ (entropy must remain constant). Thus if the mass decreases, the radius must increase (like in white dwarfs).

Now we want to integrate (5.34). Multiplying both sides by $dm/dr$ and integrating from the center to the surface gives

$$L = -\frac{3}{2}\frac{1}{R}\frac{dR}{dt}\int_0^R \frac{P(r)}{\rho(r)}\rho(r)4\pi r^2\,dr = -\frac{3}{2}\frac{1}{R}\frac{dR}{dt}\int_0^R P(r)4\pi r^2\,dr \tag{5.36}$$

We know the last integral from the Virial theorem's relation to the gravitational energy. The short answer is that

$$\int d^3r\,P = \frac{2}{7}\frac{GM^2}{R} \tag{5.37}$$

Then the luminosity is given by

$$L = -\frac{3}{7}\frac{GM^2}{R^2}\frac{dR}{dt} \tag{5.38}$$

This is essentially Kelvin-Helmholtz contraction, but for a star that is fully convective, we can derive the "real equation" (as opposed to our more bogus results earlier).

Note that for stars that are not changing much on the Kelvin-Helmholtz timescale, we may neglect $T\frac{ds}{dt}$ and simply write

$$\varepsilon_{\text{nuc}} = \frac{\partial L_r}{\partial m_r} \tag{5.39}$$

This is a good approximation for stars on the main sequence, where the burning time is much longer than the Kelvin-Helmholtz time.

*Monday, October 28, 2013*

## 5.3 Transition to Radiation Domination

Earlier we derived that as stars are on the Hayashi track,

$$\frac{L}{L_\odot} \approx 0.03\left(\frac{M}{M_\odot}\right)^{4/7}\left(\frac{R}{R_\odot}\right)^2 \tag{5.40}$$

Plugging this result in to our previous result can give us the radius as a function of time:

$$\frac{3}{7}\frac{GM^2}{R^2}\frac{dR}{dt} = -0.03L_\odot \left(\frac{M}{M_\odot}\right)^{4/7}\left(\frac{R}{R_\odot}\right)^2 \tag{5.41}$$

We can now read off scalings: $\dot{R} \propto R^4$, $1/t \propto R^3$, and so $R \propto t^{-1/3}$. Explicitly, the radius for a fully-convective star coming down the Hayashi track

$$\frac{R}{R_\odot} = \left(\frac{M}{M_\odot}\right)^{10/21}\left(\frac{130\text{ Myr}}{t}\right)^{1/3} \tag{5.42}$$

For masses greater than about a half of a solar mass, convection will dominate heat transport until the luminosity reaches the level that was predicted by earlier opacity arguments that assumed radiative diffusion:

$$L_{\text{rad}}(\kappa = \text{const}) \propto M^3 \tag{5.43}$$

$$L_{\text{rad}}(\kappa \propto \rho T^{-3.5}) \propto M^{5.5}R^{-0.5} \tag{5.44}$$

The onset of a radiative core is then roughly when $L \lesssim L_{\text{rad}}$. The time it takes for this to happen is

$$t > 10^6 \text{ years } \left(\frac{M_\odot}{M}\right)^2 \tag{5.45}$$

So massive stars become radiative at a very young age. For low-mass stars, though,

$$t > 2.6 \times 10^6 \text{ years } \left(\frac{M_\odot}{M}\right)^{4.4} \tag{5.46}$$

For very low mass stars, this time approaches the Hubble time. However, very low mass stars will remain convective on the main sequence, so nuclear reactions will likely halt the contraction anyway.

# 6 Nuclear Reactions in Stars

The big bang only made (by mass) 75% protons and 25% helium nuclei. As stars evolved, they created heavier elements and later relased them into the universe, causing stars like the sun to have a metallicity of around $Z = 0.02$. In hydrogen-burning main sequence stars, we want to fuse protons into helium and release about 7 MeV per baryon, or $7 \times 10^{18}$ erg g$^{-1}$, which is *far* higher than the thermal energy content of matter in stars. As it turns out, the thermal energy is too low to bring two protons close enough together to fuse, so tunneling must take place. Additionally, there is no stable nucleus with just two (or more) protons, so weak interactions are required to have any stable reactions. For example, in the simplest chain of reacitons that fuse hydrogen to helium, the first interaction is a weak interaction between two protons to create a deuteron and a neutrino:

$$\text{p} + \text{p} \rightarrow \text{d}{+}e^+ + \nu_e \tag{6.1}$$

followed rapidly by the coulomb interaction that creates $^3$He:

$$\text{p} + \text{d} \rightarrow {}^3\text{He} + \gamma \tag{6.2}$$

and typically closed via

$$^3\text{He} + {}^3\text{He} \to {}^4\text{He} + 2\text{p} \tag{6.3}$$

which is again, not a weak interaction. The Coulomb potential is obviously crucial in terms of tunneling and we will investigate the exact physics of these processes shortly. First, though, we talk about a general theory of nuclear physics.

## 6.1 Liquid Drop Model

We know from experiment that a large nucleus is basically at constant density, with the radius scaling as $r_n \sim A^{1/3}$ where $A$ is the **mass number** (total number of baryons, $A = N + Z$). More precisely we've found that the "size" of a nucleus is approximately

$$r_n = 1.3 \text{ fm } A^{1/3} \tag{6.4}$$

Then the nuclear density is approximately

$$\rho = \frac{Am_p}{\frac{4\pi}{3} r_n^3} \approx 2 \times 10^{14} \text{ g cm}^{-3} \tag{6.5}$$

Now, let's describe the binding energy of a nucleus with $Z$ protons, $N$ neutrons, and a mass number $A = N + Z$. The main term in the binding energy is a volume term that is

$$E_{\text{volume}} = -14 \text{ MeV } A \tag{6.6}$$

This is essentially just the well depth for each nucleon put in the nucleus. The next term is a "surface" term, due to the phenomenon that a nucleon at the surface of the nucleus is not as "bound" since it interacts with fewer nucleons. Since the volume term is known, the best way to calculate this is to say we are overestimating the volume piece as those at the surface are *less* bound. The nuclear surface area is

$$A_{\text{nuc}} = 4\pi r_n^2 \tag{6.7}$$

Then the number of particles on the surface is given by

$$N_{\text{surf}} = \frac{\text{surface area of nucleus}}{\text{cross-sectional area of nucleon}} = \frac{4\pi r_n^2}{\pi r_0^2} \approx 4A^{2/3} \propto A^{2/3} \tag{6.8}$$

We find the surface energy term to be about

$$E_{\text{surf}} = 13 \text{ MeV } A^{2/3} \tag{6.9}$$

which is positive since we presume this correction acts to unbind the nucleus (a correction to the strongly negative volume term).

We must also consider the energy stored in the electrostatic energy of the assembly of nuclei. Imagine a nucleus of radius $r_n = r_0 A^{1/3}$ that has charge

$$n_Q = \frac{Z}{V} \qquad \text{where} \qquad V = \frac{4\pi}{3} r_n^3 \tag{6.10}$$

spread uniformly throughout. The total Coulomb energy is then

$$E_{\text{Coulomb}} = \int \frac{e^2}{r} q(r)\, dq \tag{6.11}$$

where $q(r) = n_Q\, 4\pi r^3/3$ and $dq = n_Q\, 4\pi r^2\, dr$. Re-expressing the radius in terms of the charge variables, we get

$$r = \left(\frac{3q}{4\pi n_Q}\right)^{1/3} \tag{6.12}$$

Then (6.11) can be re-expressed as

$$E_{\text{Coulomb}} \approx e^2 \int_0^Z \frac{q\, dq}{q^{1/3}} \left(\frac{3}{4\pi n_Q}\right)^{-1/3} = e^2 \left(\frac{3}{4\pi n_Q}\right)^{-1/3} \int_0^Q q^{2/3}\, dq \tag{6.13}$$

so we get

$$E_{\text{Coulomb}} = e^2 \left(\frac{3}{4\pi n_Q}\right)^{-1/3} \frac{3}{5} Q^{5/3} = e^2 \left(\frac{3\frac{4\pi}{3} r_n}{4\pi Z}\right)^{-1/3} \frac{3}{5} Q^{5/3} \approx \frac{3}{5} \frac{e^2 Q^2}{r_n} \tag{6.14}$$

Re-expressing everything in terms of the mass number and the atomic number, we get

$$E_{\text{Coulomb}} = \frac{3}{5} \frac{e^2 Z^2}{R} \approx 0.66 \text{ MeV} \frac{Z^2}{A^{1/3}} \tag{6.15}$$

Now, in reality, there are a few problems with this, namely that

1. Protons do know about each other and thus correlations enter in and reduce the probabilities
2. Protons might prefer to just live at the edge of the nucleus rather than stay in the middle.

Then making a more careful calculation taking the above into account adjusts the result quoted in (6.15) to

$$E_{\text{Coulomb}} = 0.56 \text{ MeV} \frac{Z^2}{A^{1/3}} \tag{6.16}$$

It is observed that the radius of a nucleus increases with $r_n \approx 1.3$ fm $A^{1/3}$ so that the mass density of nucleons is

$$\rho = \frac{3Am_p}{4\pi A\, (1.3 \times 10^{-13})^3} = C = 2 \times 10^{14} \text{ g cm}^{-3} \tag{6.17}$$

Now the neutrons and protons are degenerate particles at these types of densities, and if we just consider that, we find that the number density of neutrons in the nucleus is

$$n_n = \frac{8\pi}{3h^3} p_f^3 = \frac{\rho_N}{2m_p} \tag{6.18}$$

where $\rho_N$ is the number density of all baryons. So the fermi momentum is

$$p_f = \left(\frac{3h^3 \rho_N}{16\pi m_p}\right)^{1/3} = 0.26 m_n C = \hbar k_f = \frac{h}{\lambda} \tag{6.19}$$

so the fermi energy and fermi wavelength are simply

$$E_f = 32 \text{ MeV} \qquad \lambda_f \approx 5 \text{ fm} \tag{6.20}$$

where we've used the fact that $E_f = p_f^2/(2m) = h^2/(\lambda^2 2m)$. The average kinetic energy of these nucleons is then $\frac{3}{5}E_f = 20$ MeV, so then

$$\langle T \rangle \approx 20 \text{ MeV} \quad \Rightarrow \quad U = -40 \text{ MeV} \tag{6.21}$$

The Pauli Exclusion Principle requires nucleons to occupy higher states if the $N \neq Z$, which then drives the nucleus to a less bound state. Let's see how this changes the kinetic energy content in the nucleus relative to $Z = N = A/2$.

Imagine we go to

$$Z' = \frac{A}{2}(1-f) \qquad N' = \frac{A}{2}(1+f) \tag{6.22}$$

where $f$ is the fraction of protons we have changed to neutrons. For a non-relativistic gas, we know that $E_f \propto p_f^2$ and $p_f^3 \propto n$, so the fermi energy and the number density are related via $E_f \propto n^{2/3}$. So let us write

$$\langle T \rangle_n = \frac{3}{5}E_{f,n}N \tag{6.23}$$

and

$$\langle T \rangle_p = \frac{3}{5}E_{f,p}Z \tag{6.24}$$

Initially, $E_{f,n} = E_{f,p} = E_f$ when $N = Z$, so the initial kinetic energy is

$$\langle T \rangle = \frac{3}{5}E_f A \tag{6.25}$$

Afterwards, though, we have

$$\begin{aligned}
\langle T \rangle &= \frac{3}{5}\frac{A}{2}(1+f)E_{f,n} + \frac{3}{5}\frac{A}{2}(1-f)E_{f,p} && (6.26)\\
&= \frac{3}{5}\frac{A}{2}E_f\left[\left(\frac{n_n}{n_0}\right)^{2/3}(1+f) + \left(\frac{n_p}{n_0}\right)^{2/3}(1-f)\right] && (6.27)\\
&= \frac{3}{5}\frac{A}{2}E_f\left[(1+f)^{5/3} + (1-f)^{5/3}\right] && (6.28)\\
&\approx \frac{3}{5}\frac{A}{2}E_f\left[1 + \frac{5}{3}f + \frac{\frac{5}{3}\frac{2}{3}}{2}f^2 + 1 - \frac{5}{3}f + \frac{\frac{5}{3}\frac{2}{3}}{2}f^2\right] && (6.29)\\
&= \frac{3}{5}\frac{A}{2}E_f\left[2 + \frac{10}{9}f^2\right] && (6.30)
\end{aligned}$$

So to second order in the fractional change, $f$, the change in energy due to the change in composition is

$$\Delta E = \frac{1}{3}AE_f f^2 \approx 10 \text{ MeV}\left(\frac{(N'-Z')^2}{A}\right) \tag{6.31}$$

Actually, this is only about half of the symmetry correction. After a full-blown calculation of the ramifications of unequal numbers of protons and neutrons, we find the "symmetry energy" contribution to be

$$E_{\text{sym}} = 18.1 \text{ MeV } \frac{(N-Z)^2}{A} \tag{6.32}$$

And thus the total binding energy of the nucleus is given by

$$\boxed{E = \left[-14A + 18.1\frac{(N-Z)^2}{A} + 0.56\frac{Z^2}{A^{1/3}} + 13.1A^{2/3}\right] \text{ MeV}} \tag{6.33}$$

Now, this tells us much that we need to know as we can calculate the curve of binding energy and all other relevant quantities. What we must first sort out is how fusion can possibly occur at these very low energies.

*Wednesday, October 30, 2013*

## 6.2 Tunneling through the Coulomb Barrier

In order for two positively charged nuclei to fuse, they must get close enough so that the strong force can overpower the Coulomb potential:

$$V(r) = \frac{Z_1 Z_2 e^2}{r} = 1.4 \text{ MeV} \frac{Z_1 Z_2}{(r/1 \text{ fm})} \tag{6.34}$$

This is schematically shown in Figure 4

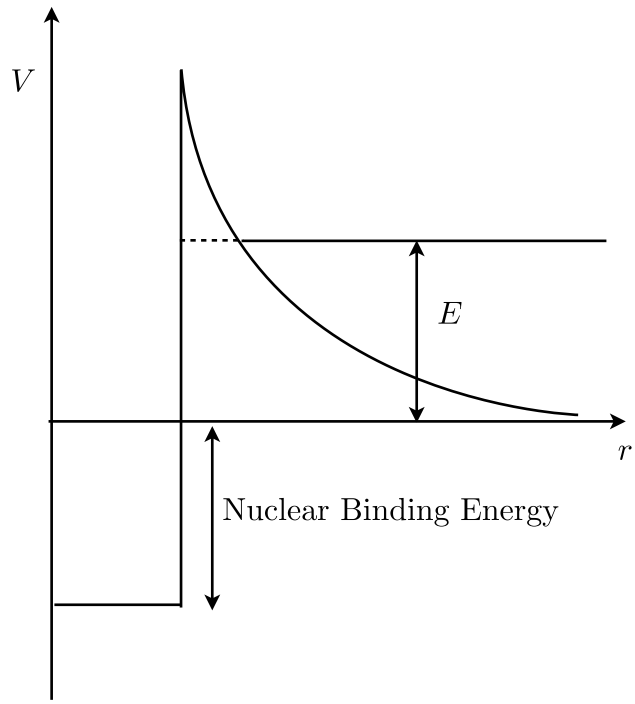


Figure 4: Schematic diagram of the potential of a two-particle interaction. The Coulomb potential rises steadily until the strong force takes over. A typical particle is shown with an energy $E$ less than the peak of the Coulomb potential, so it may only fuse with the target particle if it tunnels through the barrier.

At the center of the sun, where $T_c \sim 10^8$ K, the thermal energy is only about $kT \sim 8.6\,\mathrm{keV}\,\big(T/10^8\ \mathrm{K}\big)$. Then clearly the thermal energy in the sun is nowhere near high enough to overcome the Coulomb barrier. If there was no tunneling, we would need temperatures of order

$$T \sim \frac{Z_1 Z_2 e^2}{kR} \sim 1.6 \times 10^{10}\ \mathrm{K}\ \frac{Z_1 Z_2}{(R/1\ \mathrm{fm})} \tag{6.35}$$

Stars pretty much never reach this temperature, so clearly tunneling is how fusion happens in stars.

There are two important ingredients for calculating reaction rates:

1. Tunneling
2. Maxwell-Boltzmann distribution of particle energies

The liklihood of a particle being able to tunnel through the Coulomb barrier is proportional to its energy, which in turn is determined by the Maxwell-Boltzmann statistics of the ensemble of particles. To get a better handle on what's going on here, consider a simple reaction:

$$\mathrm{a} + \mathrm{X} \to \mathrm{Y} + \mathrm{b} \tag{6.36}$$

Sometimes this is written as X(a,b)Y in astrophysics (no clue why). We define the **cross section** as

$$\sigma \equiv \frac{\#\text{reactions/X nucleus/unit time}}{\#\text{incident particles/cm}^2\text{/unit time}} \tag{6.37}$$

which clearly has the dimension of length squared (area). Now suppose the number density of a particles is given by $n_\mathrm{a}$ and the typical relative velocity is given by $v_\mathrm{rel}$. Then the incoming flux of particles is just $n_\mathrm{a} v_\mathrm{rel}$. From this logic, we may decuce that

$$n_\mathrm{a}\sigma v_\mathrm{rel} = \frac{\#\text{reactions}}{\text{X nucleus} \times \text{unit time}} \tag{6.38}$$

Usually we write the reaction rate in the number of reactions per unit time per unit volume, or

$$r_\mathrm{aX} = n_\mathrm{X} n_\mathrm{a} \sigma v_\mathrm{rel} \tag{6.39}$$

Really we need to integrate over a relative velocity distribution to get an understanding of the actual reaction rate, so

$$r_\mathrm{aX} = \int_0^\infty dv\ n_\mathrm{a} n_\mathrm{X} \sigma(v) v \phi(v) \tag{6.40}$$

where $\phi(v)$ is the probability density of a pair of particles having a relative velocity $v$ (assumed to be Maxwellian). More often this is written as

$$r_\mathrm{aX} = n_\mathrm{a} n_\mathrm{X} \langle \sigma v \rangle \tag{6.41}$$

with

$$\langle \sigma v \rangle = \int_0^\infty dv\ \sigma(v) v \phi(v) = 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2kT}\right) dv \tag{6.42}$$

where $\mu$ is now the reduced mass, $\mu = m_1 m_2/(m_1 + m2)$. Note that if both reactants are the same species, (6.39) must be divided by a factor of 2 since we cannot imagine having a "target" and a "scatterer" (this screws up relative velocities and other things, see Clayton for the details). Similarly, if you have $N$ like reactants, we must divide by $N!$ to adjust the reaction rate appropriately. Often the problem of determining nuclear reactio rates boils down to calculating $\langle \sigma v \rangle$.

### 6.2.1 Barrier Penetration

Recall the Schrödinger equation,

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V\right]\psi = E\psi \tag{6.43}$$

For simplicity, we will only do a one-dimensional example to model barrier penetration. We suppose a free particle encounters a finite-width and finite-height barrier. Outside the barrier, then, (6.43) reduces to

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi = E\psi \quad \Rightarrow \quad \psi \propto e^{-\pm ikx} \tag{6.44}$$

and the energy is given by $E = \hbar^2 k^2/(2\mu)$. Inside the barrier, though, we have decaying modes:

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi = (E - V)\psi \quad \Rightarrow \quad \psi \propto e^{\pm\kappa x} \tag{6.45}$$

Where the "energy" is $E - V = -\hbar^2\kappa^2/(2\mu)$. Now we use this solution in solving for the actual Coulomb potential,

$$V(r) = \frac{Z_1 Z_2 e^2}{r} \tag{6.46}$$

by using the WKB approximation. In this regime, we have

$$\frac{\hbar^2\kappa^2}{2\mu} = \frac{Z_1 Z_2 e^2}{r} - E \tag{6.47}$$

Using the WKB approximation to calculate the barrier penetration, which assumes that the length-scale of changes in $V(r)$ is much larger than the De Broglie wavelength of the particle, we get the probability density being

$$\psi^2 \propto \exp\left(-2\int \kappa \; dr\right) \tag{6.48}$$

where the integral is between the two classical turning points. In this case, there is really only one turning point (where $E = V(r_c)$),

$$r_c = \frac{Z_1 Z_2 e^2}{E} \tag{6.49}$$

Rewriting $\kappa$ in terms of the turning point,

$$\kappa = \left(\frac{2\mu E}{\hbar^2}\right)^{1/2}\left[\frac{r_c}{r} - 1\right]^{1/2} \tag{6.50}$$

Then the integral in (6.48) becomes

$$\int_{r_c}^{r_{\text{in}}} \kappa \, dr = \left(\frac{2\mu E}{\hbar^2}\right)^{1/2} + \int_{r_c}^{r_{\text{in}}}\left[\frac{r_c}{r} - 1\right]^{1/2} dr \approx \frac{\pi\alpha}{2} Z_1 Z_2 \left(\frac{2\mu c^2}{E}\right)^{1/2} \tag{6.51}$$

where $\alpha = e^2/(\hbar c)$. Here we've extended the upper limit to zero to make the integral doable, but this is actually a pretty good approximation (see Clayton for the details). This gives the probability of tunneling to be

$$P_{\text{tunnel}} \propto \exp\left[-\pi\alpha Z_1 Z_2 \left(\frac{2\mu c^2}{E}\right)^{1/2}\right] = \exp\left[-\sqrt{E_{\text{G}}/E}\right] \tag{6.52}$$

Where we have defined the **Gamow** energy via

$$E_{\text{G}} \equiv (\pi\alpha Z_1 Z_2)^2\, 2\mu c^2 \approx 0.98 \text{ MeV } Z_1^2 Z_2^2 \left(\frac{\mu}{m_p}\right) \tag{6.53}$$

For a proton-proton interaction, $E_{\text{G}} \sim 0.5$ MeV, and for a carbon-proton interaction, $E_{\text{G}} \sim 33$ MeV. These values are significantly lower than that required for straight-up thermal energy to do the work, but still pretty high compared to $kT_c$. The probabiliy for a 1 keV (typical of the sun) proton to interact with another proton is then

$$P \propto \exp\left[-\left(\frac{500 \text{ keV}}{1 \text{ keV}}\right)^{1/2}\right] \sim 2 \times 10^{-10} \tag{6.54}$$

So we really need to be out on the Boltzmann tail for this to occur.

### 6.2.2 Nuclear Reaction Rates

To get a better handle on the probability of penetration, we need to integrate over all the possible center of mass energies in a thermal plasma. The reaction physics only really cares about this center of mass energy,

$$E_{\text{com}} = \frac{1}{2}\mu\left|\mathbf{v}_1 - \mathbf{v}_2\right|^2 \tag{6.55}$$

To understand the distribution of the center of mass energies, we need to know the relative velocity (velocity between the two particles in the center of mass frame) distribution, which we'll denote via

$$P(v_r)\,dv_r = \text{Probability of } v_r \text{ between } v_r \text{ and } v_r + dv_r \tag{6.56}$$

Assuming that all particles are at the same temperature (i.e. a Maxwellian velocity distribution), then the thermally averaged cross section for interaction, weighted by the velocity is

$$\begin{aligned} \langle\sigma v\rangle &= \left[\frac{\mu}{2\pi kT}\right]^{3/2}\int_0^\infty \exp\left(\frac{-\mu v_r^2}{2kT}\right) v_r\sigma(v_r)4\pi v_r^2\,dv_r & (6.57)\\ &= 4\pi\left[\frac{\mu}{2\pi kT}\right]^{3/2}\int_0^\infty \exp\left(-\frac{E}{kT}\right)\frac{2E}{\mu}\sigma(E)\frac{dE}{\mu} & (6.58)\end{aligned}$$

Where in the second form we have converted to energy units via the transformations $E = \frac{1}{2}\mu v_r^2$ and $dE = \mu v_r\,dv_r$. Now we only need to quantify the cross section, $\sigma(E)$. A first approach would be to assume that hte cross section is set by the de Broglie wavelength of the incident particles:

$$\lambda = \frac{h}{p} = \frac{h}{(kT\mu)^{1/2}} \approx 10^{-10} \text{ cm in solar core} \tag{6.59}$$

While the "size" of the nucleus is

$$r_{\text{nuc}} \approx 1.3 \times 10^{-13} \text{ cm } A^{1/3} \tag{6.60}$$

Clearly, classical scattering just isn't going to cut it since the De Broglie wavelength is much longer than the target size. Partial wave analysis can get an effective cross-section given by

$$\sigma \approx \pi\lambda^2\,(\text{dimensionless stuff})\exp\left[-\left(\frac{E_{\text{G}}}{E}\right)^{1/2}\right] \tag{6.61}$$

Where the "dimensionless stuff" comes from nasty nuclear physics. Often this is written in terms of the reduced wavelength, $\lambda/2\pi$, which I'll denote as $\bar{\lambda}$ (LaTeX isn't agreeing with "`\lambdabar`").

$$\pi\lambda^2 = 4\pi^3\bar{\lambda}^2 = \frac{4\pi^3}{k^2} \tag{6.62}$$

So the energy is

$$E \approx \frac{\hbar^2 k^2}{2\mu} \quad \Rightarrow \quad 4\pi^3 \bar{\lambda}^2 = \frac{2\pi^3 \hbar^2}{\mu E} = 2000 \text{ barns } \left(\frac{\text{keV}}{E}\right) \tag{6.63}$$

where 1 barn $= 10^{-24}$ cm$^2$. Then the cross section can be written as

$$\sigma(E) = \frac{S(E)}{E} \exp\left[-\left(\frac{E_{\rm G}}{E}\right)^{1/2}\right] \tag{6.64}$$

Where the "stupid factor", $S(E)$ is a slowly changing function of the energy that takes into account the details of the nuclear reaction. Typical values of $S(E)$ are around 2000 keV barns. One important exception is the Deuterium synthesis reaction:

$$\mathrm{p} + \mathrm{p} \to \mathrm{D} + \mathrm{e}^+ + \nu_{\rm e} \tag{6.65}$$

where $S \sim 4 \times 10^{-22}$ kev barns. This value is so small because a weak interaction is involved. For comparison, a strong reaction typically has a stupid factor about 25 orders of magnitude larger. Now we return back to calculating $\langle \sigma v \rangle$, using our newly found cross-section and writing the energy as $E = \frac{1}{2}\mu v^2$ and $dE = \mu v \; dv$. (Note that the non-relativistic energies are fine for our purposes.) Then we have

$$\langle \sigma v \rangle = \frac{1}{(kT)^{3/2}} \left(\frac{8}{\pi\mu}\right)^{1/2} \int_0^\infty dE \; S(E) \exp\left[-\frac{E}{kT} - \left(\frac{E_{\rm G}}{E}\right)^{1/2}\right] \tag{6.66}$$

Before evaluating this integral, we'll need to know how to use the method of steepest descent. Suppose we have an integral with

$$I = \int_{-1}^{\infty} g(x) e^{-f(x)} dx \tag{6.67}$$

where $g(x)$ is slowly varying and $f(x)$ is sharply peaked. (In our case, the Maxwell-Boltzmann distribution multiplied by the probability of tunneling is sharply peaked. Particles far out on the Maxwell-Boltzmann tail are very likely to tunnel, but incredibly unlikely to exist. Likewise, particles with low energies are in ample supply, but they aren't tunneling anytime soon, so there is some magic window in the middle that is most relevant.) We Taylor expand $f(x)$ about the peak (where $f'(x_0) = 0$):

$$f(x) \approx f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 \tag{6.68}$$

Then (6.67) can be approximated by

$$I = g(x_0) e^{-f(x_0)} \int_{-\infty}^{\infty} \exp\left[-\frac{f''(x_0)}{2}(x - x_0)^2\right] dx = g(x_0) e^{-f(x_0)} \sqrt{\frac{2\pi}{f''(x_0)}} \tag{6.69}$$

Then for our uses, we have

$$f(E) = \frac{E}{kT} + \left(\frac{E_{\rm G}}{E}\right)^{1/2} \tag{6.70}$$

Solving for $E_0$ (where $f'(E_0) \equiv 0$) gives $E_0^3 = \frac{1}{4} E_\mathrm{G} (kT)^2$. Then the relevant functions and derivatives are

$$f(E_0) = 3 \left( \frac{E_\mathrm{G}}{4kT} \right)^{1/3} \tag{6.71}$$

$$f''(E_0) = 3 \left[ 2 E_\mathrm{G} (kT)^5 \right]^{-1/3} \tag{6.72}$$

Using this result in (6.66), we get

$$\langle \sigma v \rangle \approx \frac{1}{(kT)^{3/2}} \left( \frac{8}{\pi\mu} \right)^{1/2} S(E_0) \exp\left[ -3 \left( \frac{E_\mathrm{G}}{4kT} \right)^{1/3} \right] \int_0^\infty dE \, \exp\left[ -\frac{(E-E_0)^2}{(\Delta/2)^2} \right] \tag{6.73}$$

(6.74)

where

$$\boxed{\Delta = \frac{4}{2^{1/3}\sqrt{3}} E_\mathrm{G}^{1/6} (kT)^{5/6}} \tag{6.75}$$

Doing the now-simplified integral, we get

$$\boxed{\langle \sigma v \rangle = 2.6 \frac{E_\mathrm{G}^{1/6}}{\sqrt{\mu}} \frac{S(E_0)}{(kT)^{2/3}} \exp\left[ -3 \left( \frac{E_\mathrm{G}}{4kT} \right)^{1/3} \right]} \tag{6.76}$$

Note that $\Delta$ was introduced strictly as a matter of convenience. You can think of it as an effective width of the Gaussian or something like that. Now (6.76) is valid for pretty much any *non-resonant* reaction that we might encounter.

Now for some numbers. For proton-proton reactions, $E_\mathrm{G} = 494$ keV, $E_0 = 4.5 \text{ keV} \left(T/10^7 \text{ K}\right)^{2/3}$, and $\Delta/E_0 = 1 \cdot (T/10^7 \text{ K})^{1/6}$. These reactions are quite prevalent in the solar core. However, for proton-carbon reactions, $E_\mathrm{G} = 36$ MeV, $E_0 = 18 \text{ keV}(T/10^7 \text{ K})^{2/3}$, and $\Delta/E_0 = 0.5(T/10^7 \text{ K})^{1/6}$. These do happen in the sun, but not very much. They are more prevalent in larger stars who derive their energy from the CNO cycle (more on this later).

*Monday, November 4, 2013*

## 6.3 Proton-Proton Interactions

This result makes it evident that there is a hierarchy set by the charges of the particles that are attempting to fuse. That is, elements with lower charges will fuse more easily at low temperatures. Once that fuel expires, the star contracts, raising the temperature, triggering the next set of nuclear reactions.

The first reaction we might think would occur would be a proton-proton interaction. However, no such nucleus exists, so we instead consider Deuterium formation:

$$\mathrm{p} + \mathrm{p} \to \mathrm{D} + e^+ + \nu \tag{6.77}$$

This requires two things:

- The two protons must be within a fermi of each other
- While within one fermi, the system needs to undergo a weak interaction:

$$\mathrm{p} \to \mathrm{n}{+}e^{+} + \nu \tag{6.78}$$

(Useful fact: $(m_n - m_p)c^2 \approx 1.2$ MeV.) As it turns out, this first reaction is the rate limiting step due to the sluggish weak interaction. We find the nuclear power per unit mass via

$$\varepsilon_{\text{nuc}} = \frac{n_1 n_2 \langle \sigma v \rangle E_{\text{nuc}}}{\rho} = \frac{dL(r)}{dm(r)} \tag{6.79}$$

For a proton-proton interaction,

$$\langle \sigma v \rangle_{\text{pp}} = 7 \times 10^{-13} \left( \frac{S}{S_s} \right) \frac{1}{T_7^{2/3}} \exp\left( -\frac{15.7}{T_7^{1/3}} \right) \tag{6.80}$$

Where $S_s = 2000$ barn keV (1 barn$= 10^{-24}$ cm$^2$). Then (6.79) becomes

$$\varepsilon_{\text{nuc}} \approx 3 \times 10^{30} \frac{\text{erg}}{\text{g s}} \left( \frac{S}{S_s} \right) \frac{\rho}{T_7^{2/3}} \exp\left( -\frac{15.7}{T_7^{1/3}} \right) \tag{6.81}$$

Of importance is that this is *most* dependent on the temperature (due to the temperature's presence in the exponential). The luminosity is then set by this value via

$$L = \int \varepsilon \, dm = L_{\text{rad}} \tag{6.82}$$

This *defines* the main sequence. Scaling to a low-mass star with $M = 0.1\ M_\odot$ and $R = 0.1\ R_\odot$, a fully convective star gives the luminosity as

$$L = 3 \times 10^{28} \frac{\text{erg}}{\text{s}} \left( \frac{M}{0.1\ M_\odot} \right)^{28/51} \left( \frac{R}{0.1\ R_\odot} \right)^{2} \tag{6.83}$$

The "stupid factor" for a proton-proton interaction is

$$S_{\text{pp}} = 4 \times 10^{-22} \text{ keV barns} = 2 \times 10^{-25} S_{\text{Strong}} \tag{6.84}$$

This gives a nuclear energy generation rate of

$$\varepsilon_{\text{pp}} = 5.6 \times 10^{5} \text{erg s}^{-1}\ \text{g}^{-1} \frac{\rho_1}{T_7^{2/3}} \exp\left( -\frac{15.7}{T_7^{1/3}} \right) \tag{6.85}$$

As a first-order estimate, suppose that *all* the mass is available for burning, so

$$L_{\text{nuc}} = M \varepsilon(T_c) \tag{6.86}$$

For a fully-convective star, the polytropic relations and $\mu = 0.6$ gives us the central temperature as

$$T_c = 0.54 \frac{G M \mu m_p}{k} \tag{6.87}$$

Scaling to the units designated earlier, $m = M/0.1\ M_\odot$, $r = R/0.1\ R_\odot$

$$T_7 = \frac{T}{10^7\ \mathrm{K}} = 0.74\frac{m}{r} \tag{6.88}$$

and

$$\rho = 139m/r^3\ \mathrm{g\ cm^{-3}} \tag{6.89}$$

Now the luminosity is

$$L_{\mathrm{nuc}} = M\varepsilon(T_c) = 2.4 \times 10^{40}\ \mathrm{erg\ s^{-1}}\ \frac{m^{4/3}}{r^{7/3}} \exp\left(-\frac{17.35r^{1/3}}{m^{1/3}}\right) = 2.5 \times 10^{28}\ \mathrm{erg\ s^{-1}}\ m^{1/2}r^4 \tag{6.90}$$

Solving this transcendental equation for $r$ gives

$$r = m\left[1.58 + 0.048\ln m - 0.36\ln r\right]^3 \tag{6.91}$$

and for the 0.1 $M_\odot$ star with just proton-proton interactions, the central temperature is

$$T_c = 3.4 \times 10^6\ \mathrm{K} \tag{6.92}$$

So the radius and central temperature are "chosen" so that the nuclear burning is adequate to match the heat losses due to radiative diffusion and convection.

An important trick in dealing with nuclear luminosities is to expand the exponential into a local power law for some sensible range of temperatures. For instance, if we want to write

$$\exp\left(-\frac{a}{T_7^{1/3}}\right) \approx T^\nu \tag{6.93}$$

We take the log of both sides:

$$-\frac{a}{T_7^{1/3}} = -a\exp\left(-\frac{\ln T_7}{3}\right) = \nu\ln T \tag{6.94}$$

Now we just differentiate both sides with respect to $\ln T$:

$$\boxed{\frac{a}{3T_7} = \nu} \tag{6.95}$$

## 6.4 Sidebar: Fusion Before Hydrogen Burning

### 6.4.1 Deuterium Burning

Before hydrogen burning takes place, the "easier" Deuterium is usually burned. This process effectively delays gravitational contraction for stars that will eventually move on to the ZAMS. However, for brown dwarfs, this is the only nuclear reaction they will undergo before descending to degeneracy. The reaction in question is

$$\mathrm{p} + \mathrm{D} \rightarrow {}^3\mathrm{He} + \gamma \tag{6.96}$$

The "stupid factor" for this reaction is $S \approx 2.5 \times 10^{-4}$ keV barn, and the other relevant quantities are

$$\mu = m_r = \frac{1 \cdot 2}{3} m_p \tag{6.97}$$

$$E_{\mathrm{G}} = (\pi \alpha Z_1 Z_2)^2 \left(2 m_r c^2\right) = 655 \text{ keV} \tag{6.98}$$

And thus, from our developed theory, the reaction rate is

$$\langle \sigma v \rangle = 8 \times 10^{-20} \frac{1}{T_7^{2/3}} \exp\left(-\frac{17.24}{T_7^{1/3}}\right) \text{ cm}^3 \text{ s}^{-1} \tag{6.99}$$

Then the nuclear energy generation rate due to Deuterium burning is

$$\varepsilon_{\mathrm{D}} = \frac{E}{\rho} n_p n_{\mathrm{D}} \langle \sigma v \rangle = \frac{E_{\mathrm{nuc}}}{m_p} n_{\mathrm{D}} \langle \sigma v \rangle \tag{6.100}$$

However, we know that from big bang nucleosynthesis, $n_{\mathrm{D}} = 10^{-5} n_p$, so we may write

$$\varepsilon_{\mathrm{D}} = \frac{E_{\mathrm{nuc}} \times 10^{-5}}{m_p} n_p \langle \sigma v \rangle \tag{6.101}$$

The lifetime of Deuterium burning is then approximately

$$t_{\mathrm{D}} = \frac{1}{n_p \sigma v} = \frac{2 \times 10^{-5}}{\rho} T_7^{2/3} \exp\left(\frac{17.24}{T_7^{1/3}}\right) \tag{6.102}$$

Suppose we want to know when $t_{\mathrm{D}} = t_{\mathrm{collapse}}$, and since we want to burn all of the deuterium, we will presume for now that it all occurs at the center, where for an $n = 3/2$ polytrope (fully convective protostar), we know that

$$\rho_c = 6 \langle \rho \rangle = 6 \frac{3M}{4\pi R^3} = 8.3 \frac{M}{R^3} \tag{6.103}$$

and the central temperature is

$$T_c = 0.54 \frac{G M \mu m_p}{k R} = 7.43 \times 10^6 \left(\frac{M}{M_\odot}\right) \left(\frac{R}{R_\odot}\right)^{-1} \tag{6.104}$$

Our earlier calcuations gave us

$$t_{\mathrm{collapse}} = \frac{R}{\left|\frac{dR}{dt}\right|} = \frac{\frac{3}{7}\frac{GM^2}{R}}{L} \tag{6.105}$$

with effective temperature

$$T_{\mathrm{eff}} = 4000 \left(\frac{M}{M_\odot}\right)^{7/51} \text{ K} \tag{6.106}$$

and luminosity

$$L = 9 \times 10^{32} \left(\frac{R}{R_\odot}\right)^2 \left(\frac{M}{M_\odot}\right)^{28/51} \text{ erg s}^{-1} \tag{6.107}$$

Putting this all together, we get

$$\boxed{t_{\text{collapse}} = 1.8 \times 10^{15}\ \text{s} \left(\frac{M}{M_\odot}\right)^{3/2} \left(\frac{R}{R_\odot}\right)^{-3}} \tag{6.108}$$

as ealier derived. For the temperature scaling, we'll use

$$T_7 = 0.74 \left(\frac{M}{M_\odot}\right) \left(\frac{R}{R_\odot}\right)^{-1} \tag{6.109}$$

Then our condition that $t_{\text{D}} = t_{\text{collapse}}$ gives us (now letting $r = R/R_\odot$ and $m = M/M_\odot$ for brevity)

$$\frac{2 \times 10^{-5}}{8.3m} r^3 \left(0.74\right)^{2/3} \frac{m^{2/3}}{r^{2/3}} \exp\left[\frac{17.24 r^{1/3}}{(0.74)^{1/3} m^{1/3}}\right] = 1.8 \times 10^{15} \frac{m^{3/2}}{r^3} \tag{6.110}$$

Simplifying, this gives

$$9.1 \times 10^{20} \frac{m^{3/2}}{r^3} \frac{m r^{2/3}}{m^{2/3} r^3} = \exp\left(19.06 r^{1/3}/m^{1/3}\right) \tag{6.111}$$

Taking natural logs on both sides gives

$$18.26 + \frac{11}{6} \ln m - \frac{16}{3} \ln r = 19.06 \frac{r^{1/3}}{m^{1/3}} \tag{6.112}$$

and, "solving" for $r$ gives

$$r = m\left[2.53 + 0.096 \ln m - 0.28 \ln r\right]^3 \tag{6.113}$$

| $M$ | $R$ | $T_c$ | $0.13r/m$ | $t_{\text{D}}$ (years) |
|---|---|---|---|---|
| 0.03 | 0.430 | $5.16 \times 10^5$ | 1.86 | $7.5 \times 10^6$ |
| 0.1 | 1.17 | $6.3 \times 10^5$ | 1.52 | $1.7 \times 10^6$ |
| 0.3 | 2.86 | $7.7 \times 10^5$ | 1.24 | $5 \times 10^5$ |
| 1.0 | 7.58 | $9.76 \times 10^5$ | 0.98 | $1.3 \times 10^5$ |

Table 2: Parameters for various protostars and brown dwarfs undergoing Deuterium burning.

Parameters for vaious contracting, fully-convective models are shown in Table 2.

We might ask whether or not deuterium can even support a star. It can, as we can see by comparing the available nuclear energy,

$$E_{\text{nuc}} = (5.5\ \text{MeV}) N_p (2 \times 10^{-5}) = \frac{M}{m_p} \left(2 \times 10^{-5}\right) (5.5\ \text{MeV}) \tag{6.114}$$

to the gravitational potential energy,

$$E_{\text{GR}} = \frac{3}{7} \frac{GM}{R} \tag{6.115}$$

The stable deuterium burning time (time spent burning Deuterium in hydrostatic balance) is, compared to the overall contraction time, given by

$$t_{\rm D,MS} = t_{\rm cont}\frac{(2\times10^{-5})(5.5\ \mathrm{MeV})}{\frac{3}{7}G\mu m_p/R} = t_{\rm cont}\left[0.13\left(\frac{R}{R_\odot}\right)\left(\frac{M}{M_\odot}\right)^{-1}\right] \tag{6.116}$$

So the star sits on the Deuterium main sequence for a time less than about $10^7$ years. The lowest mass star which can do this is about 0.015 $M_\odot$, while those of lower mass collapse into degeneracy before igniting Deuterium.

It is a bit of a coincidence that the D/H ratio is large enough so that this burning can temporarily hold the star up. Obvioulsy Deuterium-rich environments can lead to longer lifetimes. However, the pre-hydrogen burning doesn't stop there.

### 6.4.2 Lithium Burning

Next up are $^6$Li and $^7$Li. via

$$\mathrm{p} + {}^7\mathrm{Li} \rightarrow 2{}^4\mathrm{He} \tag{6.117}$$

With $S_0 = 120$ keV barn and $m_r = \frac{7}{8}m_p$. Then the Gamow energy for this reaction is

$$E_{\rm G} = (\pi\alpha Z_1 Z_2)^2(2m_r c^2) = 7.736\ \mathrm{MeV} \tag{6.118}$$

and the reaction rate is

$$\langle\sigma v\rangle = \frac{5.1\times10^{-14}}{T_7^{2/3}}\exp\left(-\frac{39.27}{T_7^{1/3}}\right) \tag{6.119}$$

| $M$ | $R$ | $T$ | $t_{\rm cont}$ |
|---|---|---|---|
| 0.08 | 0.26 | $2.27\times10^6$ K | 74 Myr |
| 0.20 | 0.57 | $2.6\times10^6$ K | 26 Myr |
| 1.0 | 2.286 | $3.2\times10^6$ K | - |

Table 3: Parameters for vaious protostars and brown dwarfs undergoing Lithium burning.

Again, everything is still convective, but it is clear that the higher Gamow peak means we must have a higher core temperature. Going through much of the same song and dance as we did with Deuterium, we find

$$t_{\rm cont} = 1.8\times10^{15}\ \mathrm{s}\ \frac{m^{3/2}}{r^3} \tag{6.120}$$

(still) and

$$t_{\rm Li} = \frac{1}{n_p\langle\sigma v\rangle} = \frac{m_p}{\rho\langle\sigma v\rangle} = \left(4\times10^{-12}\frac{r^3}{m}\right)(0.74)^{2/3}\frac{m^{2/3}}{r^{2/3}}\exp\left(\frac{43.4}{m^{1/3}}r^{1/3}\right) \tag{6.121}$$

Doing the same algebra and logarithm tricks, we find that for lithium burning,

$$r = m\left[1.419 + 4.2\times10^{-2}\ln m - 0.123\ln r\right]^3 \tag{6.122}$$

And the corresponding table can be found in Table 3. Clearly we are still not on the main sequence, but much closer, as the low mass main sequence stars have $T_c = 4 \times 10^6$ K and hence all $^7$Li is depleted. The $^7$Li abundance is not large enough to supply the star with any energy at this time to halt collapse, so there is no equivalent Lithium main sequence. What *is* critical about lithium is that it acts as a thermometer, since the burning is so temperature sensitive. If lithium is gone, then we already know that the core temperature has passed a critical point.

## 6.5 Degeneracy in Low-Mass Protostars

Thus far, we have assumed that ideal gas pressure, $P = nkT$ (where $n$ is the number density of particles) sets the pressure at the center of the star. However, in dense enough systems, we must consider electron degeneracy. The De Broglie wavelenths of electrons are given by

$$\lambda_{\rm DB} = \frac{h}{p} = \frac{h}{m_e v_{e,\rm th}} \sim \frac{h}{m_e\left(\frac{kT}{m_e}\right)^{1/2}} \tag{6.123}$$

Quantum mechanical effects become important when $\lambda > n_e^{-1/3}$, below which the wave function must obey the Pauli Exculsion Principle:

$$\frac{1}{n_e^{1/3}} < \frac{h}{m_e^{1/2}(kT)^{1/2}} \tag{6.124}$$

Or, more simply,

$$\frac{1}{\rho} < A\frac{1}{T^{3/2}} \tag{6.125}$$

Beyond this point, electron degeneracy means that the degeneracy pressure exceeds the gas pressure of electrons, and the pressure becomes insensitive to the temperature. The energy density in these degenerate electrons is given roughly by the number density and the Fermi energy,

$$U_e \sim n_e \frac{(hn_e)^{2/3}}{2m_e} \propto n_e^{5/3} \tag{6.126}$$

Again, this is insensitive to the temperature. The onset of electron degeneracy then requires there to be a minimum mass star that can ever burn hydrogen. Stars below this mass go degenerate before hydrogen can ever burn, and the central temperatures never get high enough to generate enough nuclear energy to halt collapse. These are the so-called **Brown Dwarfs**, which have masses $M < 0.08\ M_\odot$.

Note that if we are in the strong degenerate limit, where electron degeneracy is the primary source of pressure, then hydrostatic balance and degeneracy together give us

$$P_c \approx \frac{GM^2}{R^4} \propto \rho^{5/3} \propto \left(\frac{M}{R}\right)^{5/3} \tag{6.127}$$

Simplifying, we get the result

$$\boxed{R \propto \frac{1}{M^{1/3}}} \tag{6.128}$$

as $T \to 0$. So these degnerate objects actually get *smaller* as they become more massive.

## 6.6 The PP Chain and the CN Cycle

So once protons fuse to Deuterium, what happens? There is a well-established chain of events given by

$$\mathrm{p} + \mathrm{p} \to \mathrm{D} + e^+ + \nu_e \tag{6.129}$$

$$\mathrm{p} + \mathrm{D} \to {}^3\mathrm{He} + \gamma \tag{6.130}$$

$${}^3\mathrm{He} + {}^3\mathrm{He} \to {}^4\mathrm{He} + 2\mathrm{p} \tag{6.131}$$

Note that the first reaction is a weak interaction whereas the second two are strong interactions. At the center of the sun, we'll say that the central temperature is around $T_c \approx 2 \times 10^7$ K. Then the luminosity due to this chain is

$$L_n \propto \frac{S}{S_{\text{Strong}}} \exp\left(-3\left(\frac{E_\mathrm{G}}{4kT}\right)^{1/3}\right) \tag{6.132}$$

where the Gamow energy is

$$E_\mathrm{G} = (\pi\alpha Z_1 Z_2)^2\, 2m_r c^2 \tag{6.133}$$

Now let's compare the PP chain luminosity against a reaction involving fusing a proton to some heavier nucleus with atomic number $Z$ (a strong reaction):

$$10^{-25} \exp\left(-3\left(\frac{E_{\mathrm{g,pp}}}{4kT}\right)^{1/3}\right) = (1) \exp\left(-3\left(\frac{E_{\mathrm{G},?}}{4kT}\right)^{1/3}\right) \tag{6.134}$$

Solving this for the Gamow energy in question, we have

$$E_{\mathrm{G},?} = \left[E_{\mathrm{G,pp}}^{1/3} + 19.2(4kT)^{1/3}\right]^3 \quad \Rightarrow \quad E_\mathrm{G} = \left(19 + 29T_7^{1/3}\right)^3 \ \text{keV} \tag{6.135}$$

This means this happens when $T \approx 10^7$, where $E_\mathrm{G} = 50$ MeV. Notably, $Z = 7$ when $T_7 = 1$. In general,

$$\boxed{Z_c = 5\left(T_7^{1/3} + 0.27\right)^{3/2}} \tag{6.136}$$

Essentially what's happening here is that we are trading a higher Coulomb barrier in exchange for a purely strong reaction. Of particular interest is the CN cycle:

$$\mathrm{p} + {}^{12}\mathrm{C} \to {}^{13}\mathrm{N} + \gamma \tag{6.137}$$

$${}^{13}\mathrm{N} \to {}^{13}\mathrm{C} + e^+ + \nu_e \tag{6.138}$$

$$\mathrm{p} + {}^{13}\mathrm{C} \to {}^{14}\mathrm{N} + \gamma \tag{6.139}$$

$$\mathrm{p} + {}^{14}\mathrm{N} \to {}^{15}\mathrm{O} + \gamma \tag{6.140}$$

$${}^{15}\mathrm{O} \to {}^{15}\mathrm{N} + e^+ + \nu_e \tag{6.141}$$

$$\mathrm{p} + {}^{15}\mathrm{N} \to {}^{12}\mathrm{C} + \alpha \tag{6.142}$$

None of these reactions require a two-particle weak interaction, so when the temperature is high enough, this catalytic chain is competitive with the PP chain. It's not obvious, but the slowest step in this reaction is (6.140). Now we already showed that the nuclear luminosity goes as

$$L_{\text{nuc}} \propto \exp\left(-3\left(\frac{E_{\text{G}}}{4kT}\right)^{1/3}\right) \tag{6.143}$$

Then the logarithmic derivative with respect to $lnT$ is

$$\frac{d\ln L_{\text{nuc}}}{d\ln T} = \left(\frac{E_{\text{G}}}{4kT}\right)^{1/3} \tag{6.144}$$

This effectively tells us the temperature sensitivity of the reaction rates. As it turns out, the PP chain has a lower temperature sensitivity than the CN cycle. As an example, the proton-proton interaciton has $d\ln L/d\ln T \approx 5$ while for the rate-determining step in the CN cycle, we have a value of about 24. For the sun, both of these mechanisms are at work.

*Wednesday, November 6, 2013*

## 6.7 Power Generation on the Upper Main Sequence

This catalytic chain of carbon and nitrogen reactions allow for the creation of long-lived (in the nuclear sense) $\beta$-unstable elements. Thus proton captures on high $Z$ elements is then competitive with the PP chain. Since the rate-limiting step is (6.140), we would expect an excess of $^{14}$N nuclei in the plasma. Then the nuclear energy generation rate is

$$\varepsilon = \frac{1}{\rho} n_p n_S \left\langle \sigma v \right\rangle_{\text{p}+^{14}\text{N}} (28 \text{ MeV}) \tag{6.145}$$

For the Sun (and objects with solar metallicity),

$$10^{-3} n_p \approx n_S = \# \text{ density of } ^{13}\text{C and } ^{14}\text{N originally} \tag{6.146}$$

The cross-section factor for this rate-determining reaction is $S = 2.75$ keV barns. Then, we have

$$\left\langle \sigma v \right\rangle_{\text{p},\gamma} = 1.5 \times 10^{-15} \frac{1}{T_7^{2/3}} \exp\left(-\frac{72.2}{T_7^{1/3}}\right) \tag{6.147}$$

Plugging all this back in to (6.145), we get

$$\varepsilon_{\text{CN}} = 2.5 \times 10^{25} \text{ erg g}^{-1} \text{ s}^{-1} \rho T_7^{-2/3} \exp\left(-\frac{72.2}{T_7^{1/3}}\right) \tag{6.148}$$

Now using our equation of stellar structure,

$$T\frac{ds}{dt} = \varepsilon - \frac{dL(r)}{dm(r)} \tag{6.149}$$

we can find the main sequence:

$$L = \int \varepsilon(r)\, dm(r) = M\varepsilon(T_c) \tag{6.150}$$

(that integral is obviously a bit of a lie.) Thus the nuclear luminosity is given approximately by

$$L_{\rm nuc} = \int \varepsilon\, dm = \frac{M^2}{R^3} 5\times 10^{58} \text{erg s}^{-1} \frac{1}{T_7^{2/3}} \exp\left(-\frac{72.2}{T_7^{1/3}}\right) = L_\odot \left(\frac{M}{M_\odot}\right)^3 \tag{6.151}$$

This implies that for the sun, $T_7 = 2$.

### 6.7.1 Scaling Relations on the Upper Main Sequence

What we want to derive is the dependence on stellar mass of $R$, since we are asking for $T_c$. This process is simplified by expanding the exponential term into a local power law. Then we define the quantity $I$ by

$$I = \exp\left(-\frac{72.19}{T_7^{1/3}}\right) \propto T^\nu \tag{6.152}$$

Taking a logarithmic derivative, we get

$$\nu = \frac{72.19}{3T_7^{1/3}} = \frac{d\ln I}{d\ln T_7} = \frac{24}{T_7^{1/3}} \tag{6.153}$$

That is, at $T_7 = 2$, we have $\nu = 19$.

Now we use $T \propto M/R$ to eliminate the radius.this gives us

$$\frac{T_7^3}{M^3}\frac{M^2}{T^2/3}T_7^{19} = 3.2\times 10^5\ M^3 \tag{6.154}$$

This gives a mass dependence on $T_7$ of

$$\boxed{T_7 = 1.83\left(\frac{M}{M_\odot}\right)^{1/5}} \tag{6.155}$$

Additionally, the radius is given by

$$R \propto \frac{M}{T} \propto M^{4/5} \tag{6.156}$$

In the literature, this exponent is often quoted as 0.75. Then the luminosity is given by

$$L \propto M^3 \propto R^2 T_{\rm eff}^4 \quad \Rightarrow \quad T_{\rm eff} \propto M^{1/3} \tag{6.157}$$

Additionally, we can use $\rho_c \propto M/R^3$ to relate the density to the central density:

$$\rho_c \propto \frac{M}{R^3} \propto \frac{MT^3}{M^3} \propto \frac{T^3}{M^2} \propto \frac{T^3}{T^{10}} \propto \frac{1}{T^7} \tag{6.158}$$

That is, $T_c \propto \rho_c^{-1/7}$. For stars larger than the sun, the CN(O) cycle is the primary energy source. The cores are convecting due to the concentrated energy source. $T_7$ is nearly constant with mass. The above results then define the zero-age main sequence for large stars, assuming that their opacity is set by Thomson scattering.

So now we see why in the HR diagram that we have a small dynamical range in effective temperature while having an appreciable spread in luminosities. To increase the effective temperature by a factor of ten, the mass must increase by three orders of magnitude. Meanwhile, the luminosity is highly sensitive to mass, so we get a "slim" HR diagram.

### 6.7.2 Convection in Large Stars

The cores of large, zero-age main sequence stars are convective. We found earlier that nuclear energy generation is very temperature sensitive, with $\varepsilon \propto T^{20}$, where most of the energy is being generated close to the center. If we were to plot the luminosity coming out of a shell of radius r against the mass coordinate, we would see the curve rise sharply and then taper off quickly, indicating that the energy is generated almost entirely by the inner-most mass. Going back to the equation governing radiative diffusion,

$$F = \frac{1}{3}c\frac{1}{\kappa\rho}\frac{d}{dr}\left(aT^4\right) \tag{6.159}$$

and the condition for hydrostatic equilibrium,

$$\frac{dP}{dr} = -\rho(r)g(r) \tag{6.160}$$

we may write

$$\frac{d\ln T}{d\ln P} = \frac{P_{\text{tot}}}{aT^4}\frac{F(3\kappa)}{4cg} = \frac{P_{\text{tot}}}{aT^4}\frac{3}{4}\frac{\kappa}{c}\frac{L(r)}{4\pi Gm(r)} \tag{6.161}$$

Note that if all the energy is being generated *exactly* at the center (i.e., $L(r) \propto \delta(r)$), then we clearly have a problem, since $m(0) = 0$. Let's rewrite (6.161) into a more readable form:

$$\frac{d\ln T}{d\ln P} = \frac{P_{\text{tot}}}{aT^4}\frac{3}{4}\frac{L(r)}{L_{\text{Edd}}}\frac{M}{m(r)} \approx 0.1\frac{M}{m(r)} \tag{6.162}$$

Where we've reused some scaling relations for the the pressure and luminosity ratios. This is essentially the delta function case. We are assuming that $L(0) = L$. For convection to occur, we require

$$\frac{d\ln T}{d\ln P} = 0.1\frac{M}{m(r)} > \frac{2}{5} \tag{6.163}$$

This effectively defines the convective region. If all the burning is inside the convective zone, we don't need to know $L(r)$ at all. Thus there is a stretch of the star where $d\ln T/d\ln P \approx 2/5$. Outside of this convective zone, we would have the Eddington Standard Model: $T \propto P^{1/4}$. The convection is in the core, so it is very efficient, with the convective flux being $F = \rho v^3$, where $v_c \lll c_s$.

Often we denote the fraction of convective material via

$$q_c \equiv \frac{m(r)}{M} \quad \text{where the mass is all the mass in the convective region} \tag{6.164}$$

For relevant data giving $q_c$ for different models, see the table at the end of Chapter 2 of HKT.

When doing CN burning, we showed that $\varepsilon \propto Z$ (here $Z$ is the metallicity). But because the reaction is so incredibly temperature sensitive, this doesn't really matter much. That is, if the metallicity is reduced, the temperature only needs to increase by a very small amount to get back to the same reaction rate.

### 6.7.3 Main Sequence Burning Summary

For star less massive than the sun, the hydrogen fusion is pp dominated. These star typically have radiative cores and convective outer envelopes with the opacity's dominated by Kramers' Rule.

For stars larger than the sun, the hydrogen fusion is dominated by the CN(O) cycle. They typically have convective cores and radiative envelopes with the opacity set by Thomson scattering.

| $M/M_\odot$ | $\log L/L_\odot$ | $T_{c,7}$ | $q_c$ |
|---|---|---|---|
| 1 | 0 | 1.4 | 0 |
| 1.5 | 0.76 | 1.9 | 0.07 |
| 2 | 1.26 | 2.1 | 0.13 |
| 5 | 2.7 | 2.6 | 0.23 |
| 15 | 4.24 | 3.21 | 0.4 |
| 60 | 5.7 | 3.93 | 0.73 |

Table 4: Basic parameters of ZAMS models of various masses (taken from HKT). Provided are the mass in solar masses, the luminosity in solar luminosities, the central temperaures in $10^7$ K, and the mass of the convetive core as a fraction of the total mass.

## 6.8 Introduction to Stellar Evolution

As a star burns hydrogen to helium, its composition naturally changes. This gradual change in composition will affect the properties of the star, ultimately forcing it to turn to new sources of fuel. We can estimate the lifetime of a star while it is burning a particular fuel via

$$t_{\rm MS} = \frac{E_{\rm fuel}}{L} \tag{6.165}$$

Where $E_{\rm fuel} = \varepsilon_{\rm nuc} M_{\rm burn}$ ($\varepsilon_{\rm nuc}$ is the nuclear energy available per unit mass of fuel and $M_{\rm burn}$ is the total amount of mass available for nuclear burning). For massive stars, $M_{\rm burn}$ will depend on the evolution (in mass coordinates) of the convective core. Some extreme situations have **convective overshoot**, where convective eddies bring hydrogen rich material from the envelope back into the core to prolong the lifetime of the main-sequence star. Once this fuel runs out, the composition of the core rapidly changes, and Kelvin-Helmholtz contraction sets in to pay for the power loss.

For a lower mass star, there is no convection in the core, so the composition changes much more gradually until the majority of the fuel is expired.

*Wednesday, November 13, 2013*

# 7 State of the Stellar Interior

Virtually all information about stars come from the observation of their light. Photons reaching the observer transport information about the conditions of the gas they had their last interaction with before leaving the star. This region of "last interaction" is called the **photosphere**. The information about the physical condition of the photosphere (and hence properties of the star like gravity and surface temperature) can be retreived by studying the stellar spectra.

To understand and describe the formation of spectral lines, we must link the degree of ionization and the occupation of orbital energy levels in the gas to the large-scale thermodynamics variables.

When studying stellar interiors, often we assume that the gas is in **Local Thermodynamic Equilibirum**, which is where most thermodynamic variables follow a Maxwell-Boltzmann distribution and thus that all areas of the star are locally in thermodynamic equilibrium.

## 7.1 Conditions at the Photosphere

In a stellar atmosphere with some specified $T_{\rm ef}$, the atmosphere is thin. I.e., the pressure scale height is much smaller than the radius:

$$h = \frac{kT_{\rm eff}}{\mu m_p g} \ll R \tag{7.1}$$

In fact, we earlier showed that, roughly speaking, $H/R \sim T_{\rm eff}/T_c$. At the photosphere, where $\tau = \int \kappa\rho\, dr \approx 1$, we may assume that $g$ is a constant and that the temperature is roughly $T_{\rm eff}$. Understanding the atmosphere requires us to understand the states of the atoms at the stellar surface. From hydrostatic equilibrium, we may approximate

$$\frac{dP}{dz} = -\rho g \qquad \Rightarrow \qquad \int dP = -g\int \rho\, dz = -g\int \frac{d\tau}{\kappa} \qquad \Rightarrow \qquad P_{\rm ph} \approx \frac{g}{\kappa} = \frac{\rho k T_{\rm eff}}{\mu m_p} \tag{7.2}$$

Scaling to solar values, we get

$$g = 2.7\times 10^4\ {\rm cm\,s^{-2}}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R}{R_\odot}\right)^{-2} \tag{7.3}$$

Which gives a number density of

$$n_{\rm ph} \approx 10^{16}\left(\frac{\kappa_{\rm es}}{\kappa}\right)\left(\frac{10^4\ {\rm K}}{T_{\rm eff}}\right)\left(\frac{g}{2.7\times 10^4\ {\rm cm\,s^{-2}}}\right) \tag{7.4}$$

For comparison, the electron density in a solid is closer to $n_e \approx 10^{23}\ {\rm cm^{-3}}$. So, the elctrons and ions are not interacting, but we have not yet determined whether or not quantum effects are important.

## 7.2 The Saha Equation

Suppose we have two energy states separated by a energy gap $\Delta E$. Then from Boltzmann statistics, the ratio of the probility of particles residing in these two states is given by

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{-\Delta E/(kT)} \tag{7.5}$$

where $g_1$ and $g_2$ are the statsitical weights of the two configurations. This equation actually describes the ionization states in the stellar interior, though for electrons, it's more accurate to use Fermi-Dirac statistics:

$$\boxed{n(p) = \frac{g}{h^3} \frac{1}{\exp\left[(E_p - \mu)/(kT)\right] + 1}} \tag{7.6}$$

where

$$\boxed{E_p = \frac{p^2}{2m_p} + m_p c^2} \tag{7.7}$$

assuming nonrelativistic particles. To find the number density, we need to integrate this phase space density over all momenta:

$$n = \frac{g}{h^3} \int_0^\infty \frac{4\pi p^2\, dp}{\exp\left[(E_p - \mu)/(kT)\right] + 1} \tag{7.8}$$

Here we are going to assume that the mean free path is much longer than the de Broglie wavelength,

$$\ell \gg \lambda_{\mathrm{DB}} = \frac{h}{p} \tag{7.9}$$

This is equivalent to stating that

$$\exp\left[(E_p - \mu)/(kT)\right] \gg 1 \tag{7.10}$$

So the integral in (7.8) simplifies considerably to (you remember how to do $\int_0^\infty x^2 e^{-\alpha^2 x^2}$, right?)

$$n = \frac{g4\pi}{h^3} \int_0^\infty p^2 \exp\left[\frac{-(E_p - \mu)}{kT}\right] dp \tag{7.11}$$

$$\exp\left(\frac{\mu - m_p c^2}{kT}\right) = \frac{nh^3}{g(2m_p kT\pi)^{3/2}} = \frac{n}{gn_Q} \tag{7.12}$$

Here, we are using $n_Q \equiv (2\pi mkT/h^2)^{3/2}$. This is the so-called **quantum concentration**, at which particles are degenerate. Then the chemical potential is necessarily

$$\boxed{\mu_e = kT \ln\left(\frac{n_e}{gn_Q}\right) + m_e c^2} \tag{7.13}$$

Now the ionization of Hydrogen is given by

$$e^- + \mathrm{p} \rightleftharpoons \mathrm{H} + \gamma \tag{7.14}$$

We aren't even going to care about the rates of this photoionization/recombinaton reaction. We will simply assume that the rates are equal, mandating the chemical potentials to be equal. Equating the chemical potentials (the condition for chemical balance) tells us that

$$\mu_e + \mu_{\mathrm{p}} = \mu_{\mathrm{H}} \tag{7.15}$$

where we've noted that photons have no chemical potential. The chemical potentials are thus

$$\mu_{\rm p} = m_{\rm p}c^2 - kT \ln\left(\frac{g_{\rm p}n_{Q,p}}{n_{\rm p}}\right) \tag{7.16}$$

$$\mu_e = m_e c^2 - kT \ln\left(\frac{g_e n_{Q,e}}{n_e}\right) \tag{7.17}$$

$$\mu_{\rm H} = m_{\rm H}c^2 - kT \ln\left(\frac{g_{\rm H}n_{Q,{\rm H}}}{n_{\rm H}}\right) \tag{7.18}$$

An important fact here is the difference between the rest mass of hydrogen and that of the proton and electron, which is just the ionization energy of the hydrogen atom:

$$m_{\rm H}c^2 = (m_{\rm p} + m_e)c^2 - 13.6 \text{ eV} \tag{7.19}$$

Then the condition that the chemical potentials are equal can be reduced to

$$\ln\left[\frac{g_e n_{Q,e}}{n_e}\frac{g_{\rm p}n_{Q,{\rm p}}}{n_{\rm p}}\frac{n_{\rm H}}{g_{\rm H}n_{Q,{\rm H}}}\right] = \frac{13.6 \text{ eV}}{kT} \tag{7.20}$$

Rearranging this, we get the **Saha Equation**

$$\boxed{\frac{n_e n_{\rm p}}{n_{\rm H}} = \frac{g_e g_{\rm p}}{g_{\rm H}}\left[\frac{2\pi m_e kT}{h^2}\right]^{3/2} \exp\left(-\frac{13.6 \text{ eV}}{kT}\right)} \tag{7.21}$$

Often times we'll see this in terms of the quantum concentration:

$$n_Q = \left[\frac{2\pi mkT}{h^2}\right]^{3/2} \tag{7.22}$$

Now, suppose we are in a state where exactly half of the hydrogen is ionized. In this case, $n_{\rm p} = n_e = n_{\rm H}$. Solving the Saha equation for temperatures gives us

$$T_{1/2} \approx 12400 \text{ K} \tag{7.23}$$

If we define the ionization fraction via

$$y \equiv \frac{n_{\rm p}}{n} = \frac{n_e}{n} \tag{7.24}$$

Then the Saha equation is reduced to

$$\boxed{\frac{y^2}{1-y} = \frac{4\times 10^{-9}}{\rho}T^{3/2}e^{-1.6\times 10^5/T}} \tag{7.25}$$

Solving for $y$ at the center of the sun, we find that $y \sim 0.75$, which is not good since we "know" that the solar core is completely ionized. The problem is that the central density is sufficiently high that the $+1$ in the denominator of the Fermi-Dirac distribution cannot be ignored. In this regime, things like presure ionization become important, which helps cause the interior to be completely ionized.

## 7.3 Stellar Spectra

Recall the pressure scale height, which in the photosphere, is

$$h = \frac{kT}{m_p g} \ll R \tag{7.26}$$

where the local acceleration due ot gravity is (again, in the photosphere),

$$g = \frac{GM}{R^2} = 2.7 \times 10^4 \frac{M}{R_\odot} = 2.7 \times 10^4 \left(\frac{M_\odot}{M}\right)^{1/2} \tag{7.27}$$

Where we have used the fact that $R \propto M^{0.75}$. Recalling the optical depth, $\tau = \int \rho\kappa \, dr$, we will say that at the photosphere, $\tau_{\rm ph} = 2/3$. Then we find

$$\tau_{\rm ph} = \kappa \underbrace{\int \rho \, dr}_{y} = \kappa y \quad \Rightarrow \quad y_{\rm ph} = \frac{2}{3\kappa} \tag{7.28}$$

Then the pressure at the photosphere is

$$P_{\rm ph} = g y_{\rm ph} = n_{\rm ph} k T_{\rm ph} \tag{7.29}$$

So the density in the photosphere is

$$n_{\rm ph} = \frac{2 g m_{\rm p}}{3\sigma k T} \approx 10^{16} \text{ cm}^{-3} \left(\frac{\kappa_{\rm es}}{\kappa}\right) \left(\frac{10^4 \text{ K}}{T}\right) \left(\frac{g}{2.7 \times 10^4}\right) \tag{7.30}$$

For various spectral types, we have a range of masses and effective temperatures summarized in Table 5.

| Spectral Type | $T_{\rm eff}$ | $M/M\odot$ |
|---|---|---|
| O3 | 52000 K | 120 |
| B0 | 30000 K | 17 |
| A0 | 9500 K | 3 |
| G0 | 6000 K | $\sim 1$ |

Table 5: Parameters of selected spectral types.

So there is around a factor of ten spread in these effective temperatures. Thus, the different emission and absorption lines that we see will vary greatly among these classes (hence their name of **spectral classes**). Elements with first ionization potential less than 5 eV include Li, Na, Mg, and Al. Those between 10 and 20 eV include H, C, N, and O. Those higher than 20 eV include He and Ne. Depending on the effective temperature and composition of a photosphere, we will observe different relative strengths in these lines.

Going back to the hydrogen atom, recall how transitions to the $n = 1$ state for an electron are called **Lyman Transitions**. The transtion from 1 to 2 corresponds to a wavelength of 1215 Å, and the Lyman limit, from 1 out to infinity is at 911 Å. Probaly more well-known are transitions to and from the $n = 2$ state, called the **Balmer Transitions**. The most famous of these is the H$\alpha$ line, which is the $n = 2$ to $n = 3$ transition, at a wavelength of 6563 Å. Note that this is in the visible and thus easily observable, whereas the Lyman transitions are all in the ultraviolet.

We can predict the relative numbers of electrons in various energy levels using the Boltzmann equation. For instance, the ratio of electrons in the $n = 2$ state to the $n = 1$ state is

$$\frac{n_2}{n_1} = \exp\left(-\frac{E_{12} \approx 10.2 \text{ eV}}{kT}\right) = 10^{-5} \tag{7.31}$$

at a temperature of $10^4$ K. Is this strong enough to produce an observable line? First we look at the cross section of scattering:

$$\pi\lambda^2 \sim 10^{-16} \text{ cm}^{-2} = \sigma_{\text{line}} \gg \sigma_{\text{Th}} = 6.65 \times 10^{-25} \text{ cm}^{-2} \tag{7.32}$$

So comparing this cross section to the Thomson cross section, we have

$$\frac{\sigma_{\text{line}}}{\sigma_{\text{Th}}} \sim 10^8 \tag{7.33}$$

Now we can determine the relative strength of this source of opacity to electron scattering via

$$\exp\left(-\frac{10.2 \text{ eV}}{kT}\right) \frac{\sigma_{\text{line}}}{\sigma_{\text{Th}}} \frac{m_{\text{H}}}{m_{\text{p}}} > 1 \tag{7.34}$$

Or,

$$\exp\left(-\frac{10.2 \text{ eV}}{kT}\right) > \frac{\sigma_{\text{Th}}}{\sigma_{\text{line}}} \quad \Rightarrow \quad T > 6000 \text{ K} \tag{7.35}$$

So for stars with effective temperatures above 6000 K, this transition creates an observable line. Different emission and absorption sources in stars peak at various temperatures (and thus are unique to specific spectral classes), so we can identify stars quite well from their spectra alone.

**H$^-$ Opacity** With the Saha equation (or at least something like it), we can compute what fraction of hydrogen atoms are actually ionized into the H$^-$ ion:

$$\frac{n(\text{H})}{n(\text{H}^-)} = \frac{n_Q}{n_e} \exp\left(-\frac{0.75 \text{ eV}}{kT}\right) \tag{7.36}$$

or

$$\frac{n(\text{H}^-)}{n(\text{H})} = \frac{n_e}{n_Q} \exp\left[\frac{8700}{T}\right] \tag{7.37}$$

For the sun, this gives $n_e = 10^{13}$ cm$^{-3}$ and thus

$$\frac{n(\text{H}^-)}{n(\text{H})} \approx 10^{-8} e^{8700/T} \tag{7.38}$$

This source also peaks at a particular effective temperature and is actually the dominant source of opacity in the sun's photosphere.

*Monday, November 18, 2012*

# 8 Stellar Evolution

Now we wish to know what happens to a star as $X$ decreases and $Y$ increases. This change in composition will require a change in the overall behavior of the star. The time a star spends on the main sequence,

$$t_{\mathrm{MS}} = \frac{E_{\mathrm{nuc}}}{L} \approx \frac{E_{\mathrm{nuc}}}{E_{\mathrm{th}}} t_{\mathrm{KH}} \tag{8.1}$$

is much larger than the Kelvin Helmholtz time since $E_{\mathrm{nuc}} \gg E_{\mathrm{th}}$. This is because the reaction converting hydrogen to helium liberates around 7 MeV per hydrogen atom burned, whereas the thermal energy of each atom is around 1 keV. Thus, it makes sense that the main sequence time is on the order of $10^3$ times longer than the Kelvin-Helmholtz time.

In fact, even if the star only burns 10% of the available hydrogen, we still have $t_{\mathrm{MS}} > 1000 t_{\mathrm{KH}}$. This contrast in timescales allowed for early calculations of stellar evolution because at each time step needed to resolve $X$, the star would be in complete thermal balance. One big caveat to what we will do today is that the concentration of the fusion to the center of the star "breaks" our homogeneous assumptions.

## 8.1 Main Sequence Evolution for Massive Stars

We want to construct stellar models as a function of the hydrogen abundance, $X$, which will necessarily change as burning occurs in the star as hydrogen is burned to helium on the main sequence. For now, we will consider the regime of massive stars (say, $M \gtrsim 2M_\odot$), so we have

- Convective core, so there is a "ball" in the center with an evolving composition.
- $\kappa = \kappa_{\mathrm{es}}$
- CNO Burning

Simplifying the helium synthesis to

$$4\mathrm{p} + 2e^- \rightarrow {}^4\mathrm{He} \tag{8.2}$$

We see that, all other things being held constant, this reaction will cause the number density, and thus the pressure to decrease. Due to the high sensitivity of CNO burning to changes in temperature, the star "wants" to keep the temperature constant. To compensate for the loss in pressure, then, the star will have to dynamically adjust its radius. Writing out the pressure explicitly, we have

$$P = \frac{\rho k T}{m_p}\left(2X + \frac{3}{4}Y\right) = \frac{\rho k T}{\mu m_p} \tag{8.3}$$

Then the thermal energy in the core per particle is

$$k_B T_c \sim \frac{G M \mu m_p}{R}, \tag{8.4}$$

where the composition is, from (8.3),

$$\mu = \frac{1}{2X + \frac{3}{4}Y} \tag{8.5}$$

For a cosmic mix, $X = 0.7$ and $Y = 0.25$, giving $\mu_0 = 0.63$. When the core is fully helium, we trivially have $\mu = 4/3$. Let's look at how the luminosity is affected:

$$L \sim R^2 \frac{c}{3\kappa\rho} \frac{d}{dr} \left(aT^4\right) \tag{8.6}$$

where the opacity is given by

$$\kappa = 0.2(1+X) \text{ cm}^2 \text{ g}^{-1} \tag{8.7}$$

Then the luminosity is

$$L \propto R^2 \frac{R^3}{(1+X)M} \frac{1}{R} \frac{\mu^4 M^4}{R^4} \propto \frac{\mu^4 M^3}{1+X} \tag{8.8}$$

This factor increases by a factor of 40 from a cosmic composition to the pure helium composition. Thus, we recover the fact that lumionsity tends to *increase* as a star progresses through the main sequence. Perhaps the factor of 40 is an idealization, but we can be certain that the luminsoity *must increase as hydrogen is used up.*

Now recall that the nuclear luminosity goes as

$$L_{\text{nuc}} \propto T_c^{15\to 20} \tag{8.9}$$

And as the luminosity increases, the temperature will only increase by, at most, 20%. Some scaling relations of importance for the radius can then be obtained:

$$T_c \propto \frac{M\mu}{R} \tag{8.10}$$

So the radius goes as

$$R \propto \frac{\mu}{T_c} \tag{8.11}$$

And so the final radius given a constant central temperature is

$$\frac{R_f}{R_i} \approx \frac{4/3}{0.6} \approx 2 \tag{8.12}$$

Thus the [massive] star doubles in radius as it proceeds through the main sequence.

## 8.2 Main Sequence Evolution for Less Massive Stars

For stars less massive than the sun, we have

- Radiative Core
- $L_{\text{nuc}} \propto T^\nu$ for $\nu \sim 4 - 7$ for p-p chain.
- Kramers' Law Opacity

Again, as the mass increases, the luminosity increases. However, due to the weaker dependence on temperature, the core temperature will rise more significantly than for the high-mass stars to match the luminosity. Again the evolution of the radius is governed via

$$kT_c \sim \frac{Gm_p M\mu}{R} \tag{8.13}$$

Due to the increased dynamicism of $T_c$, it is no longer clear what will happen to the radius. Without proof, we say that the radius does not change nearly as significantly in low-mass stars as it does in high-mass stars.

## 8.3 Post-Main Sequence Evolution

For stars with $M > 2M_\odot$, as $X \to 0$ over the whole convective core, suddenly the core is pure helium. With no nuclear fuel left to support itself, the core undergoes Kelvin-Helmholtz contraction until shell burning of hydrogen picks up and supports the structure above an isothermal helium core.

For stars less than 1.5 $M_\odot$, there is no abrupt transition (the Henyey Hook). Rather, it develops a growing (in mass) helium core. See Figure 5 for a qualitiative description of the evolution of low- and high-mass stars as they leave the main sequence.

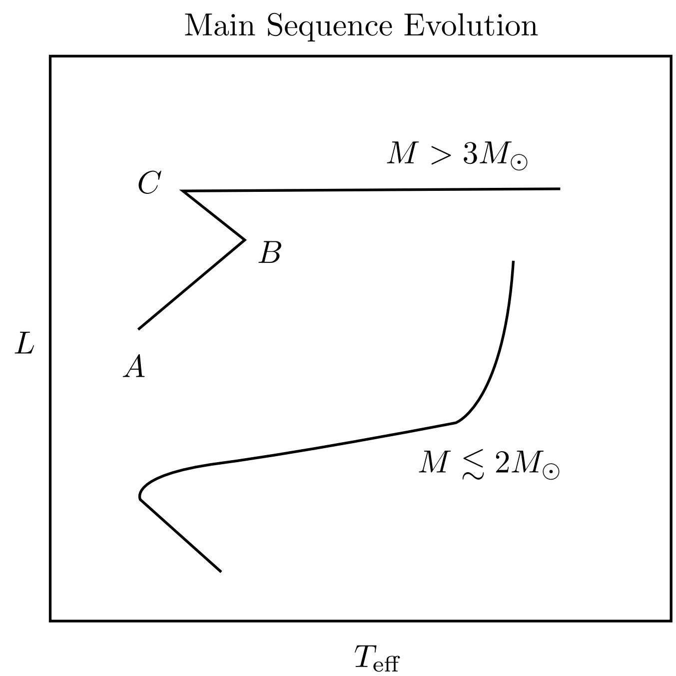


Figure 5: Evolution of a massive star. From $A$ to $B$, the star is undergoing Main Sequence hydrogen burning. Temperature is increasing as $X$ is decreasing until finally all hydrogen is depleted in the core. From $B$ to $C$, the pure helium core undergoes Kelvin-Helmholtz contraction (this is called the "Henyey Hook"). Beyond $C$, hydrogen is burning in a shell.

For stars less than about 2 $M_\odot$, as the star leaves the main sequence, it travels horizontally on the HR diagram and eventually starts making its way back up the Hayashi track. This portion of the star's life is called the **Red Giant Branch** (RGB). After hydrogen is depleted in the core, hydrogen continues burning in a shell around the helium core. The radius of this core is much less than the radius of the star. When a one solar mass star "leaves" the main sequence, we typically have $M_{\rm He} = 0.1 M_\odot$. However, the helium core, once degenerate, will *decrease* in radius as the mass increases (where the mass gain is fueld by the hydrogen shell burning).

Then the temperature in the shell is given by

$$kT_s \approx \frac{GM_c}{R_c} m_p \tag{8.14}$$

for a shell where the pressure scale height is $R_c$. Now, assuming electron scattering sets the opacity, the luminoity is

$$L_{\rm rad} = 4\pi R_c^2 \left[\frac{1}{3}\frac{c}{\kappa\rho}\frac{1}{R_c} a T_s^4\right] \tag{8.15}$$

or more succinctly,

$$\boxed{L_{\rm rad} = \frac{4\pi}{3}\frac{R_c}{\kappa\rho_s} ac T_s^4} \tag{8.16}$$

Now requiring that $L_{\rm nuc} = \varepsilon M_{\rm shell} = L_{\rm rad}$ actually tells us that at this stage, CNO burning dominates pp burning. Assuming again that the nitrogen proton scatter is the slow step, we get

$$\varepsilon_{\rm CNO} = 6.6 \times 10^{24} \frac{\rho}{T_7^{2/3}} \exp\left(-70.7 T_7^{-1/3}\right) = \varepsilon_0 \rho T^\nu \tag{8.17}$$

In this case, $\nu = 24/T_7^{1/3} \sim 16$. Using this to find the density in the shell allows us to get the luminosity. The punchline is that

$$\rho = \left[\frac{4}{3}\frac{\sigma T_5^{4-\nu}}{\varepsilon_0 R_c^2 \kappa}\right]^{1/3} \tag{8.18}$$

Then the luminosity scales as

$$L \propto R_c^{5/3} T_5^{(8+\nu)/3} \tag{8.19}$$

But since $T_s \propto M_c/R_c$, we get

$$L \propto R_c^{-(1+\nu/3)} M_c^{(8+\nu)/3} \tag{8.20}$$

Plugging in fiducial values for scaling relations gives us

$$L = 5 \times 10^{31} \text{ erg s}^{-1} \left(\frac{0.1 R_\odot}{R_c}\right)^{25/3} \left(\frac{M_c}{0.2 M_\odot}\right)^{10} \tag{8.21}$$

Since the helium core is degenerate, $R_c \propto M_c^{-1/3}$. This would make the mass dependence of $L$ even stronger. (8.21) is known as the **Luminosity-Core Mass Relation** and is a solution that is available only if the star can evolve to it (i.e., it isn't guaranteed). This state *is* the red giant branch.

*Wednesday, November 20, 2013*

## 8.4 The Schönberg-Chadrasekhar Limit

Returning to stars with masses above around $6M_\odot$, where the hydrogen burning core ends, we have a substantial mass helium core. Figure 6 shows how the hydrogen mass fraction evolves in the profile.

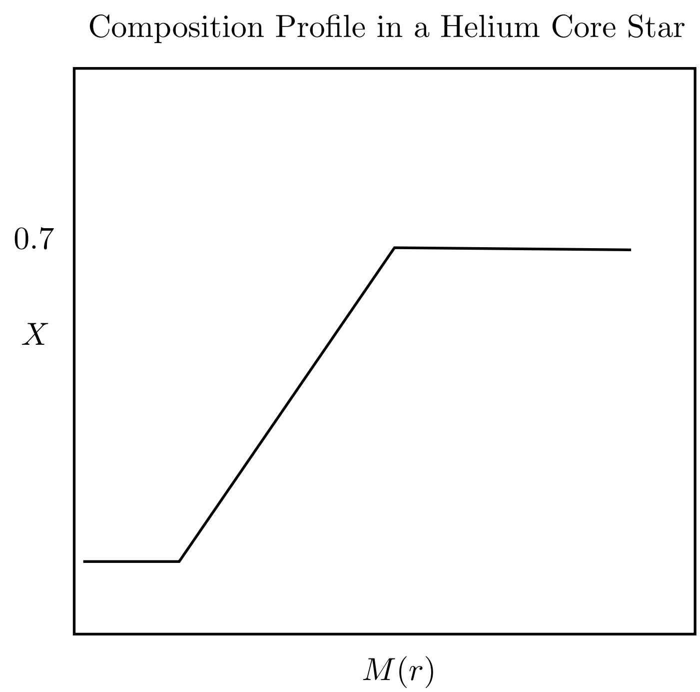


Figure 6: The composition profile of a star with a helium core. In the outer region, we still have $X \sim 0.7$ since that material is largely unburned.

This change in hydrogen mass fraction is the source of the "Henyey Hook" seen in large mass evolutionary tracks. After the hydrogen burning shell is extinguished it undergoes Kelvin-Helmholtz contraction until the Helium ignites. This helium core is nearly isothermal, so using hydrostatic equilibrium and the same old Virial theorem trick,

$$\frac{dP}{dr} = -\rho(r)\frac{Gm(r)}{r^2} \tag{8.22}$$

Now multiplying both sides by $4\pi r^3$ and integrating, we get

$$\int_0^{R_c} 4\pi r^3 \frac{dP}{dr} dr = -G\int_0^{R_c} \frac{4\pi r^2 m(r)\rho(r)}{r} dr \tag{8.23}$$

The term on the left is, integrating by parts,

$$\int_0^{R_c} 4\pi r^3 \frac{dP}{dr} dr = 4\pi R_{\rm core}^3 P_{\rm core}(R_{\rm core}) - 12\pi \int_0^{R_c} r^2 P(r)\, dr \tag{8.24}$$

and the right side is roughly the gravitational potential, so (8.23) becomes

$$4\pi R_{\rm core}^3 P_{\rm core}(R_{\rm core}) - 12\pi \int_0^{R_c} r^2 P(r)\, dr = -\frac{GM_{\rm core}^2}{R_{\rm core}} \tag{8.25}$$

Since we have $P(r) = \rho(r)kT_{\rm core}/(\mu_{\rm core}m_p)$, we can recast (8.25) quite easily to

$$4\pi R_{\rm core}^3 P_{\rm core}(R_{\rm core}) - 3\frac{kT_{\rm core}}{\mu_{\rm core}m_p} M_{\rm core} = -\frac{GM_{\rm core}^2}{R_{\rm core}} \tag{8.26}$$

Solving for the pressure, we have

$$P_{\rm core}(R_{\rm core}) = \frac{3}{4\pi}\frac{kT_{\rm core}}{\mu_{\rm core}m_p}\frac{M_{\rm core}}{R_{\rm core}^3} - \frac{1}{4\pi}\frac{GM_{\rm core}^2}{R_{\rm core}^4} \tag{8.27}$$

Note that this equation has a well-defined maximum at some critical core radius. This radius is

$$R_{\rm core,max} = \frac{GM_{\rm core}\mu_{\rm core}m_p}{kT_{\rm core}}\frac{4}{9} \tag{8.28}$$

and the maximum pressure is

$$P_{\rm core,max} = 0.68\left(\frac{kT_{\rm core}}{\mu_{\rm core}m_p}\right)^4 \frac{1}{G^3 M_{\rm core}^2} \tag{8.29}$$

The overlying envelope must exert a pressure

$$P_{\rm base} \sim \frac{GM^2}{R^4} \tag{8.30}$$

At the interface between the pure helium core and the hydrogen/helium mix envelope, there must be a discontinuity in the density (like a water/air interface, sort of). The $R$ in (8.30) is really just the scale height in the H/He envelope, so we will use

$$kT_{\rm envelope} \approx \frac{GM\mu_{\rm envelope}m_p}{R} = kT_{\rm core} \tag{8.31}$$

where we've mandated a contininous temperature profile at the interface. Then the pressure from the envelope is

$$P_{\rm envelope} = \frac{GM^2}{R^4} = \left(\frac{kT_{\rm core}}{\mu_{\rm envelope}m_p}\right)^4 \frac{1}{G^3 m^2} \tag{8.32}$$

assuming that $M \gg M_{\rm core}$. To have a valid solution, we require

$$0.68\frac{1}{\mu_{\rm core}^4}\frac{1}{M_{\rm core}} > \frac{1}{\mu_{\rm envelope}}\frac{1}{M^2} \tag{8.33}$$

So then the requirement on the core mass is

$$M_{\text{core}} < M \left( \frac{\mu_{\text{envelope}}}{\mu_{\text{core}}} \right)^2 \alpha \tag{8.34}$$

for some numerical constant $\alpha$. Noting that $\mu_{\text{envelope}} = 0.6$ and $\mu_{\text{core}} = 1.33$. The "real" result (not derived here) is

$$q_c = \frac{M_{\text{core}}}{M} = 0.37 \left( \frac{\mu_{\text{envelope}}}{\mu_{\text{core}}} \right)^2 = 0.08 \tag{8.35}$$

If $M_{\text{core}} > 0.08M$ *and* the core is not degenerate, *then* this core is not a solution to hydrostatic balance. The pressure from the envelope would then compress the core until it became hot enough to ignite helium or dense enough for electron degeneracy to support the core.

Now what about electron degeneracy? In smaller stars, the core becomes degnerate before igniting helium. Then the equation of state becomes

$$P(r) = K_{\text{NR}} \rho^{5/3}(r) \tag{8.36}$$

And thus

$$P_{\text{core}}(R_{\text{core}}) = \frac{K_{\text{NR}} M^{5/3}}{R_{\text{core}}^5} - \frac{1}{4\pi} \frac{G M_{\text{core}}^2}{R_{\text{core}}^4} \tag{8.37}$$

Note that this pressure has *no* maximum like the ideal gas situation. That is, the core mass can have any mass whatsoever. Figure 7 shows how degeneracy affects different stars.

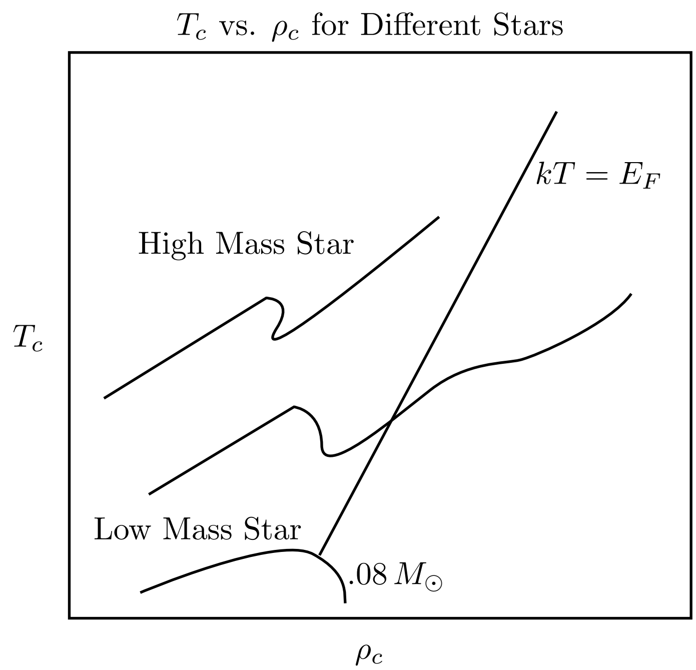


Figure 7: $T_c$ versus $\rho_c$ for various stars. High mass stars reach the helium ignition temperature before becoming degenerate, but lower mass stars do not, so they undergo a helium flash. A $0.08\,M_\odot$ star is shown, and barely (if ever) reaches hydrogen ignition.

Returning to the more massive stars, if $q_{\text{core}} > 0.08$ at the time of hydrogen core depletion, there is no isothermal solution and the star simply contracts on the Kelvin-Helmholtz timescale:

$$t \approx \frac{GM_{\text{core}}^2/R_{\text{core}}}{L} = 10^6 \text{ years} \left(\frac{6\,M_\odot}{M}\right)\left(\frac{5\,R_\odot}{R}\right) \tag{8.38}$$

So these stars contract very rapidly.

Now consider stars in the range of $2\,M_\odot$ and $6\,M_\odot$. Initially, suppose $q < q_c$, but hydrogen shell burning occrs. Now $q$ will increase until $q > q_c$ and the core undergoes Kelvin-Helmholtz contraction. During this time, they shoot over to the red in the HR diagram (the envelope expands while the core contracts), creating the so-called **Hertzsprung Gap** as observed in star clusters. This is a patch in the HR diagram between the TAMS (terminal age main sequence) and the RGB where very few stars are present due to the short evolutionary timescales at this stage of evolution.

For stars with less than $2\,M_\odot$, the core will become degenerate before helium burning can start.

## 8.5 Helium and Higher Burning

Helium burns in advanced stars via the triple-alpha process, according to

$$^4\text{He} + {}^4\text{He} \to {}^8\text{Be} + \gamma \tag{8.39}$$

However, $^8$Be is unstable with a decay time of around $\tau \sim 2 \times 10^{-16}$ sec. The Gamow energy for this interaction is $E_\text{G} = 31.4$ MeV, so the Gamow peak occurs at

$$E_0 = \left(\frac{E_\text{G}(kT)^2}{4}\right) = 83\,\text{keV}\,T_8^{2/3} \tag{8.40}$$

So once $T > 8 \times 10^7$ K or so, the reaction to make $^8$Be proceeds rapidly and can achieve thermal and chemical equilibrium.

*Friday, November 22, 2013*

Equating the chemical potentials,

$$\mu_4 = m_4c^2 - kT\ln\left(\frac{g_4 n_{Q,4}}{n_4}\right) \tag{8.41}$$

$$\mu_8 = m_8c^2 - kT\ln\left(\frac{g_8 n_{Q,8}}{n_8}\right) \tag{8.42}$$

we get

$$\frac{n_8}{n_4^2} = 2^{3/2}\left(\frac{h^2}{2\pi m_\alpha kT}\right)^{3/2} \exp\left(-\frac{(m_8 - 2m_\alpha)c^2}{kT}\right) \tag{8.43}$$

Normalizing to $\rho = 10^4$ g cm$^{-3}$ of pure helium (at the core, obviously), we get

$$\frac{n_8}{n_4} = 3 \times 10^{-6} T_8^{-3/2}\left(\frac{\rho}{10^4\,\text{g}\,\text{cm}^{-3}}\right)\exp\left(-\frac{10.64}{T_8}\right) \tag{8.44}$$

At $T_8 = 2$, about 1 in $10^{-8}$ of the nuclei at any given time are $^8$Be. Then another alpha particle can combine this a $^8$Be nucleus creating a $^{12}$C nucleus. However, this is actually a carbon nucleus in an excited state of carbon that will eventually decay to the ground state through a strange process. First though, we note the Gamow energy and Gamow peak energy for the Carbon synthesis reaction:

$$E_\text{G} = (\pi\alpha(2)(4))^2\, 2m_r c^2 = 168\,\text{MeV} \tag{8.45}$$

$$E_0 = 146\,\text{keV} T_8^{2/3} \tag{8.46}$$

This is for the reaction

$$\alpha + {}^8\text{Be} \to {}^{12}\text{C}^* + \gamma \tag{8.47}$$

Now let us assume that the temperature is high enough so that the excited state of $^{12}$C becomes populated via a chemical equilibrium of (8.47). Now equating *these* chemical potentials and noting that that of the Be isotope is just twice that of the alpha particles:

$$3\mu_4 = \mu_{12^*} \tag{8.48}$$

Messy algebra ensues, with the result that

$$\frac{n_{12^*}}{n_4} = 5 \times 10^{-10} \left(\frac{\rho}{10^5}\right)^2 T_8^3 \exp\left(-\frac{44}{T_8}\right) \tag{8.49}$$

Where we have implicitly used the binding energy of $^{12}$C$^*$, which is 7.644 MeV (notably higher than that of the reactants, Be and the alpha particles, at 7.366 MeV). Note that so far, this reaction has actually *lost* energy. The only way this reaction provides energy is in the two decays it takes for $^{12}$C$^*$ to get down to the ground state of Carbon. For these decays, we have

$$\Gamma_{\text{rad}} = 3.67\,\text{MeV} \tag{8.50}$$

and thus the decay rate is

$$r = \frac{\Gamma}{\hbar} = 5.5 \times 10^{12}\,\text{s}^{-1} \tag{8.51}$$

Putting all of this together, we have that the nuclear energy generation rate is

$$\varepsilon = \frac{n_{12^*} E}{\tau} \frac{1}{\rho} = 5.3 \times 10^{21} \text{erg}\,\text{g}^{-1}\,\text{s}^{-1} \rho_5^2 T_8^{-3} \exp\left(-\frac{44}{T_8}\right) \tag{8.52}$$

where $\tau = r^{-1} \approx 2 \times 10^{-13}$ s. In order to determine this, we had to know the mass difference between $^{12}$C$^*$ and three alpha particles, as well as the photon decay rate of $^{12}$C$^*$ into plain old $^{12}$C. Note that this reaction is even more temperature sensitive than either hydrogen burning source, and the energy per reaction per nucleon is about a tenth of what it is for hydrogen burning. Thus, we expect helium burning to end much faster than hydrogen burning since more reactions have to happen per unit time to match the luminosity, *and* the luminosity is higher due to the change in composition that we found earlier.

### 8.5.1 Higher Burning

Once carbon is present in the core, it can burn to oxygen by the synthesis of an alpha particle and a carbon nucleus. Similarly, the Oxygen can be burned to Neon. The relevant values are shown in Table 6.

| Reaction | $E_{\text{G}}$ (MeV) | $E_0(T_8 = 2)$ | $\exp\left[-3\left(\frac{E_{\text{G}}}{4kT}\right)^{1/2}\right]$ |
|---|---|---|---|
| $\alpha + {}^{12}\text{C}$ | 424 | 315 keV | $1.3 \times 10^{-24}$ |
| $\alpha + {}^{16}\text{O}$ | 804 | 390 keV | $5.5 \times 10^{-38}$ |
| $\alpha + {}^{20}\text{Ne}$ | 1310 | 460 keV | $1.5 \times 10^{-44}$ |

Table 6: Parameters controlling the burning of carbon, oxygen, and neon.

The first reaction in Table 6 does occur fast enough to compete with the triple alpha process, so advanced stars typically have a core consisting of both carbon and oxygen.

### 8.5.2 Helium Burning in a Massive Star

In a convective core, helium burning starts at $T = 10^8\,\mathrm{K}$ due to the high temperature sensitivity of the triple alpha process. Now recall that the luminosity of an evolved star is higher than it was as a ZAMS star, so the burn time is

$$t_{\mathrm{He\,burn}} = \frac{\varepsilon_{\mathrm{nuc}} M_{\mathrm{burn}}}{L} \tag{8.53}$$

which is shorter than the hydrogen-burning main sequence burn time. For stars above around 6 $M_\odot$, carbon burning will eventually occur, and as we move on up to higher and higher masses, more and more heavy elements will get their turn at burning, with the most massive stars stopping at Fe/Si burning (since the binding energy per nucelon peaks there).

For stars greater than around $2M_\odot$, the helium ignites when the electron gas in the core is nondegnenerate. Solar mass stars, however, are not so lucky.

### 8.5.3 Helium Core Flash and Other Instabilities

For stars with masses less than around $2\,M_\odot$, the helium cores have degenerate electrons. Meanwhile, the CN cycle burning increase the mass of the He core. Before we delve into the details of the dynamics of the degenerate core, here's a quick refresher of degeneracy.

The DeBroglie wavelength of particles in a gas of temperature $T$ is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{mkT}} \tag{8.54}$$

The quantum mechanics impacts the equation of state when $\lambda \gtrsim n^{-1/3}$. Then the condition becomes

$$n^{1/3} > \frac{\sqrt{mkT}}{h} \tag{8.55}$$

Note that this is the quantum concentration found in our previous discussion of the Saha equation. And so $n^{2/3} \propto T$. As $T \to 0$, then the DeBroglie wavelenth is going to $\lambda \propto n^{1/3}$ since they are degenerate and that is necessarily the separation distance. Now the pressure for a nonrelativistic gas is given by

$$P = \frac{2}{3}E \tag{8.56}$$

where $E$ is the energy density. The energy is primarily in kinetic energy,

$$E_k = \frac{p^2}{2m_e} = \frac{(\hbar n_e^{1/3})^2}{2m_e} \tag{8.57}$$

So the Fermi energy follows $E_f \propto \rho^{2/3}$, and so the pressure is $P = \rho E_f \propto \rho^{5/3}$. For hydrostatics, we have

$$P = \frac{GM^2}{R^4} \propto \left(\frac{M}{R^3}\right)^{5/3} \quad \Rightarrow \quad R \propto M^{-1/3} \tag{8.58}$$

Thus more massive degenerate objects have *smaller* radii.

Now returning to our intermediate mass star with a helium core, the temperature in the hydrogen burning shell located at about $R_c$ (the radius of the core) is

$$kT_s \approx \frac{GM_s m_p}{\mu_e R_c} \quad \Rightarrow \quad T_s \propto \frac{M_c}{R_c} \propto M_c^{4/3} \tag{8.59}$$

So as the shell deposits more and more helium into the core, the temperature on the outer edge is rising in time. In the degenerate core, the dominant form of heat transport is electron conduction from electron scattering off of the ions. This mode is so efficient that the core is nearly isothermal (though it's actually a bit hotter out towards the hydrogen burning shell since neutrino cooling is lowering the temperature at the center).

When the star is in hydrostatic balance and has an ideal gas equation of state, an injection of entropy would lead to a decrease in temperature (negative heat capacity). For a star, this means that the pressure drops when entropy is added. However, with a degenerate equation of state, an addition of entropy causes the temperature to *increase*. Thus, if helium burning begins in a degenerate core, it acts to increase the temperature, which in turn increases the burning rate. This is a thermonuclear runaway, giving way to the infamous **Helium Core Flash**. The burning continues until degeneracy is lifted (i.e. the specific heat goes back to being negative) and the burning causes a drop in pressure and temperature.

In addition to this degeneracy-induced burning instability, the **thin shell instability** can cause unstable burning in a number of situations. The thin shell instability was discovered in 1965 and is a thermonuclear instability that occurs when temperature sensitive burning is in a geometrically thin shell. By "geometrically thin", we mean that at the location of the shell, $\Delta r \ll R$. To get a better handle on this mechanism, recall the condition for hydrostatic balance (again):

$$\frac{dP}{dr} = -\rho(r)\frac{GM(r)}{r^2} \tag{8.60}$$

Integrating this over a thin shell gives us the plane-parallel atmosphere result:

$$\int_{P_{\rm sh}}^{0} dP = -\int_{R_{\rm sh}}^{\infty} \frac{GM(r)\rho(r)}{r^2} dr \tag{8.61}$$

Or, switching to the mass coordinate,

$$P_{\rm sh} = \int_m^M \frac{Gm(r)}{4\pi r^4} dm = \frac{GM_{\rm sh}}{R_{\rm sh}^2}\frac{\Delta m}{4\pi R_{\rm sh}^2} \tag{8.62}$$

where $dm = 4\pi r^2 \rho(r)\, dr$ and $\Delta m = M - m$, which is the mass in the shell (approximately, since we are ignoring the contribution of the envelope).

When the pressure is constant,

$$T\frac{ds}{dt} = c_P\frac{dT}{dt} = \varepsilon - \frac{1}{\rho}\mathbf{\nabla}\cdot\mathbf{F} \tag{8.63}$$

Then this becomes positive, hence the instability (heat capacity has become positive). The flux term, from radiative diffusion, is

$$\frac{1}{\rho}\mathbf{\nabla}\cdot\mathbf{F} = \frac{1}{dy}\frac{c}{3\kappa}\frac{d}{dy}aT^4 \propto \frac{acT^4}{3\kappa y^2} \tag{8.64}$$

where $y$ is the column depth introduced earlier, $y = \int \rho(r)\,dr$, of the shell. So then (8.64) becomes

$$c_P \frac{dT}{dt} = \varepsilon - \frac{acT^4}{3\kappa y^2} \tag{8.65}$$

For most nuclear energy generation rates, $\varepsilon \propto T^{\nu}$ with $\nu \gg 4$, so this becomes highly unstable. This occurs in stellar evolution as the helium shell flashed in asymptotic giant branch (AGB) stars. A limit cycle can be formed where a thin shell piles up until it burns unstably, starting the process over again. In an accretion context, white dwarfs accrete hydrogen in a thin shell and burn unstably, which is a **Classical Nova**. They can also do this with helium, which causes a larger nova or **.Ia** explosion. Neutron stars accrete hydrogen and burn to make a **Type I X-Ray Burst** (hydrogen burns to helium, helium burns to carbon and oxygen). Additionally, this process can fuel superbursts, where carbon is burned to heavy elements.

Returning to the degenerate electron source of instability, an extremely degenerate electron gas can be the cause of unstable burning. In this regime, the pressure is insensitive to the temperature, so the pressure is constant. The same equation that was relevant to the thin shell instability applies:

$$c_P \frac{dT}{dt} = \varepsilon - \frac{dL(r)}{dm(r)} \tag{8.66}$$

but the reason that $c_P$ is positive is now different (degeneracy versus plane-parallel). Due to the fact that the temperature in the core isn't monotonic, helium ignition starts off-center (remember, the neutrino cooling keeps the center of the core cooler than surrounding areas). In a 1-D model, a convective zone forms at the ignition point. The temperature continues to rise until such a time where the degeneracy is broken and the shell isn't geometrically thin. At this degeneracy breaking-point, we have $kT \sim E_F$. The Fermi energy follows the virial form,

$$E_F \sim \frac{GM_c}{R_c} m_{\mathrm{He}} \sim kT \tag{8.67}$$

At this point, the scale height will now be of the order of the radius of the star, so it is *not* geometrically thin. Then the heat capacity returns to being negative, ending the unstable burning. After this event has passed, degeneracy in the local area has been lifted. A succession of flashes follows, slowly removing the degeneracy and paving the way for stable "main sequence" helium burning.

## 8.6 The Horizontal Branch and The Asymptotic Giant Branch

The core flash lifts degeneracy, leaving behind a nondegenerate helium-burning star. Recalling that $L \propto \mu^4$, the luminosity increases dramatically ($L_\odot \to 10L_\odot$). Once the core flash is over, we would expect stellar evolution to slow down or stop due to the stable nature of helium core burning. The envelope is expanded by a large factor, essentially shutting off the hydrogen shell burning. The luminosity then decreases quickly, creating a tip on an HR diagram (called the tip of the red giant branch, or TRGB). Kelvin-Helmholtz contraction occurs until stable helium core burning is occurring at a rate that balances radiative losses. Essentially the star has become a helium-burning main sequence star, often called a **Red Clump** star. If the star loses part or all of its hydrogen shell, the luminosity is unaffected, but the radius obviously is, so there is a range of colors of red clump stars, depending on how much hydrogen they have retained, forming the **Horizontal Branch** on the HR diagram.

*Monday, November 25, 2013*

The helium core burning lifetime is somewhere around 65-100 million years. This is shorter than the hydrogen burning main sequence since the triple alpha process is less efficient in energy generation *and* the luminosity is greater. After the helium core is spent, there is a carbon/oxygen core and the star with masses between 2 and 6 $M_\odot$ ascend up the **Asymptotic Giant Branch** (AGB), where only hydrogen and helium shell burning can power them. The Hydrogen burning is steady-state, but the rate at which it produces helium was not fast enough to allow steady-state helium burning in the shell below. Instead, as found by Schwarschild and Harm in 1965, the accumulating helium shell undergoes periodic thermonuclear flashes due to the thin shell instability. A series of helium shell flashes occur, but they never get hot enough to ignite carbon, so the star ends its life as a C/O white dwarf.

Stars traveling up the AGB will have a stable hydrogen burning shell dumping helium onto an unstable helium burning shell. That is, the rate of helium supplied by the hydrogen shell cannot be the rate at which helium stably burns in the helium shell. There is a net increase of mass entering the core, $\dot{M}_{\text{in}}$. Additionally, the weakly bound envelope is losing mass due to a wind taking mass to infinity, $\dot{M}_{\text{wind}}$. In the end, mass loss wins the game, and the degenerate core is left behind as a white dwarf, with most of the envelope mass lost into a nebula.

# 9 White Dwarfs

## 9.1 The Physics of Degenerate Objects and the Chandrasekhar Mass

At the late stages of stellar eolution, the electrons become more and more degenerate, eventually dominating the pressure. We now work out the resulting $M(R)$ relation in this limit. The equation of state is just

$$n_e = \frac{2}{h^3}\int_0^{p_F} 4\pi p^2\, dp = \frac{8\pi}{3h^3}p_F^3 \tag{9.1}$$

in the fully degenerate limit (that is, electrons are filling all of momentum space up to the Fermi momentum of the gas). Then the pressure is just (as derived earlier)

$$P_e = \frac{2}{5}n_e E_F \tag{9.2}$$

when things are non-relativistic as we presume for now. Expanding out (9.2), we get

$$P = \frac{2}{5}n_e\frac{1}{2m_e}p_F^2 = \frac{n_e}{5m_e}\left(\frac{3h^3 n_e}{8\pi}\right)^{2/3} = n_e^{5/3}\left(\frac{3h^2}{8\pi}\right)^{2/3}\frac{1}{5m_e} \tag{9.3}$$

If we typically have $\rho = Am_p n_i$ and $n_e = Zn_i$, so that

$$n_e = Z\frac{\rho}{Am_p} \approx \frac{\rho}{2m_p} \qquad \text{(where } A = 2Z\text{)} \tag{9.4}$$

Then the pressure is

$$P = \left(\frac{\rho}{2m_p}\right)^{5/3}\left(\frac{3h^3}{8\pi}\right)^{2/3}\frac{1}{5m_e} \tag{9.5}$$

Now, let's just ask what the rough properties are of an object supported by degenerate electrons. Yet agin, let's just write the pressure as

$$\frac{dP}{dr} = -\rho\frac{Gm(r)}{r^2} \quad \Rightarrow \quad P \sim G\frac{M}{R^2}\frac{M}{R^2} = \frac{GM^2}{R^4} \tag{9.6}$$

But now let's set $\rho R^3 = M$, or equivalently, $R = (M/\rho)^{1/3}$ to eliminate the radius from the pressure. Then linking the pressures from hydrostatic balance and degeneracy pressure, we get

$$P_c \sim \frac{GM^2}{M^{4/3}}\rho^{4/3} = \frac{1}{5m_e}\left(\frac{\rho}{2m_p}\right)^{5/3}\left(\frac{3h^3}{8\pi}\right)^{2/3} \tag{9.7}$$

Solving this for the density gives

$$\rho \sim 4\times 10^5\,\mathrm{g\,cm^{-3}}\left(\frac{M}{0.1\,M_\odot}\right)^2\left(\frac{m_x}{m_e}\right)^3 \tag{9.8}$$

where $m_x$ is presumably the mass of the degenerate particle (in our case, the electron mass).Now this is the crude scaling at low masses and it is important to ask what happens as the mass increases. according to our relation, the density increases. We can get the radius as well from the density, so

$$R \sim 8\times 10^8\,\mathrm{cm}\left(\frac{0.1\,M_\odot}{M}\right)^{1/3}\left(\frac{m_e}{m_x}\right) \tag{9.9}$$

So the star gets smaller as the mass increases. Note that the more proper formula is

$$R = 2\times 10^9\,\mathrm{cm}\left(\frac{0.1\,M_\odot}{M}\right)^{1/3}\left(\frac{m_e}{m_x}\right) \tag{9.10}$$

So for for a neutron star ($m_x \approx m_p$) we find $R \approx 10^6\,\mathrm{cm} = 10\,\mathrm{km}$.

Anyway, returning to white dwarfs, as the mass increases, eventually the electrons start to become relativistic. In particular, we can just ask when the Fermi momentum is just the mass times the speed of light, $p_F \approx m_e c$, which means

$$m_e c = p_F = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3} = \left(\frac{3h^3\rho}{2m_p 8\pi}\right)^{1/3} \tag{9.11}$$

Solving (9.11) for the density gives us

$$\rho = (m_e c)^3\frac{2m_p 8\pi}{3h^3} \approx \frac{16\pi}{3}m_p\left(\frac{m_e c}{h}\right)^3 \tag{9.12}$$

Where we have phrased this in a suggestive way, noting that the De Broglie wavelength in the relativistic limit is $\lambda_c = h/(m_e c)$. This implies that the density reaches a critical level that is independent of any parameter:

$$\boxed{\rho_{\mathrm{crit}} = 2\times 10^6\,\mathrm{g\,cm^{-3}}} \tag{9.13}$$

So at these densities, things get weird. The first important point is that the relativistic gas we know of will be unstable and thus it is impossible to build such an object. In the extreme relativistic limit, the electron gass pressure is

$$P = \frac{1}{4}n_e E_F = \frac{1}{4}n_e p_F c = \frac{1}{4}n_e c\left(\frac{3h^3 n_e}{8\pi}\right)^{1/3} = n_e^{4/3}\frac{c}{4}\left(\frac{3h^3}{8\pi}\right)^{1/3} \tag{9.14}$$

Relating this to our hydrostatic approximation for the central pressure, we find

$$M^{2/3} = \frac{1}{G}\frac{c}{4}\frac{1}{(2m_p)^{4/3}}\left(\frac{3h^3}{8\pi}\right)^{1/3} \tag{9.15}$$

Which predicts a characteristic mass of such objects of

$$M = \frac{1}{G^{3/2}}\left(\frac{c}{4}\right)^{3/2}\frac{1}{(2m_p)^2}\left(\frac{3h^3}{8\pi}\right)^{1/2} \approx 0.3M_\odot \tag{9.16}$$

In reality, the properly detailed calculation using the polytropic relations puts in new factors giving the proper **Chandrasekhar Mass**:

$$\boxed{M_{\text{Ch}} = 1.456\left(\frac{2}{\mu_e}\right)^2 M_\odot} \tag{9.17}$$

Where $\mu_e$ is, as always, the number of baryons per electron. If the star gets heavier than this mass, it undergoes catastrophic collapse, creating a neutron star if it didn't explode as a Type Ia supernova before totally collapsing.

*Wednesday, November 27, 2013*

## 9.2 White Dwarf Cooling

The white dwarf is "burn" hot, with $T \approx 2 \times 10^8$ K or so, with the interior being completely isothermal due to the efficient conductive heat transfer of the degenerate electrons. There is some transition point at $E_F = kT$ which gives a bottom boundary condition of degeneracy. Looking at the Fermi Energy,

$$E_F = \frac{1}{2m_e}\left(\frac{3h^3\rho}{8\pi 2m_p}\right)^{2/3} = 16\,\text{eV}\,\rho^{2/3} = kT \quad \Rightarrow \quad 8625T_8 = 16\rho^{2/3} \tag{9.18}$$

So the transition density is

$$\rho_{\text{tr}} = 1.2 \times 10^4\,\text{g}\,\text{cm}^{-3}T_8^{3/2} \tag{9.19}$$

In the outer, non degenerate layer of the white dwarf, the material is geometrically thin and $M_{\text{outer}} \ll M$. Thus, the plane-parallel atmosphere precription is valid. The surface gravity for a typical white dwarf is

$$g = \frac{GM}{R^2} = 1.6 \times 10^8\,\text{cm}\,\text{s}^{-2} \tag{9.20}$$

where we have assumed $R = 7 \times 10^8$ cm and $M = 0.6\,M_\odot$. Then the pressure is

$$P = g\frac{\Delta m}{4\pi R^2} = \frac{\rho k T}{\mu m_p} \tag{9.21}$$

With this, we can calculate where the electrons become degenerate. We find

$$\Delta M = 10^{-3} M_\odot T_8^{5/2} \tag{9.22}$$

In the hydrogen and helium layer. Now assuming that the flux is constant, and that the opacity follows a Kramers' Law form,

$$F = \text{const} = \frac{1}{3}\frac{c}{\kappa\rho}\frac{d}{dr}\left(aT^4\right) \tag{9.23}$$

or

$$\frac{3F\kappa}{c} = \frac{d}{dy}aT^4 \tag{9.24}$$

Reintroducing the pressure, we get

$$\left[\frac{3F}{4c}\kappa_0\frac{\mu m_p}{kg}\frac{1}{a}\right] P\,dP = T^{7.5}\,dT \tag{9.25}$$

Now integrating this from the point where electrons are degenerate out to the surface, we get

$$\frac{L}{4\pi R^2}\frac{R^2}{GM} = \frac{F}{g} = \frac{2T_8^{8.5}}{8.5P^2}\frac{4cka}{3\kappa_0\mu m_p} \tag{9.26}$$

Solving for the luminosity and scaling, we find

$$\boxed{L = 2.5\,L_\odot\left(\frac{M}{0.6\,M_\odot}\right)\left(\frac{T_c}{10^8\,\mathrm{K}}\right)^{7/2}} \tag{9.27}$$

This is known as **Mestel's Cooling Law**, though it isn't really accurate, since the opacity is changing as we get towards the degeneracy. Now all of the energy source is coming from the thermal content of the ions, which is approximately (fror a pure carbon WD)

$$E_{\text{th}} = \frac{3}{2}N_i kT = 1.25 \times 10^{48}\,\text{ergs}\left(\frac{M}{0.6\,M_\odot}\right)T_8 \tag{9.28}$$

Since the ions form an ideal gas. The luminosity is then obviously

$$L = -\frac{d}{dt}E_{\text{th}} \tag{9.29}$$

Note that the mass dependence cancels here. This gives us a differential equation:

$$\frac{T_8^{7/2}}{\tau} = -\frac{dT_8}{dt} \tag{9.30}$$

where $\tau = 4 \times 10^6$ yrs. We then integrate Equation 9.30

$$-\int_0^t \frac{dt}{\tau} = \int_{T_i}^{T_f}\frac{dT_8}{T_8^{7/2}} \tag{9.31}$$

giving

$$-\frac{t}{\tau} = -\frac{2}{5}\left[\frac{1}{T_f^{5/2}} - \frac{1}{T_i^{5/2}}\right]. \tag{9.32}$$

After a short while $T_f^{-5/2} \gg T_i^{-5/2}$, thus

$$T_8 = \left(\frac{2}{5}\frac{\tau}{t}\right)^{2/5} \tag{9.33}$$

Or more explicitly,

$$T_8 = \left(\frac{1.6 \times 10^6 \,\text{yrs}}{t}\right)^{2/5} \tag{9.34}$$

Note that this solution is not valid at early time since we have ignored the initial data, however after a million years or so, the initial conditions are effectively forgotten anyway. Since we can find the central temperature from the mass and luminosity, we need only the mass and luminosity to determine a white dwarf's age.

We must get the mass through a measure of $\log g$ through the spectrum. At higher gravity, we have

$$P_{\text{ph}} = \frac{g}{\kappa} = \frac{\rho k T}{\mu m_p} \tag{9.35}$$

So as the density goes up, the gravity goes up (and thus the mass goes up). And indeed, as the pressure increases, the line widths in a spectrum increase. This phenomenon is called **Pressure Broadening**.

There are some caveats to this theory, though. Primarily, we have ignored that ions do interact via Coulomb physics. We typically have

$$\frac{e^2}{\langle r \rangle} \gtrsim kT \tag{9.36}$$

Then we define the **Coulomb Parameter**:

$$\Gamma = \frac{Z^2 e^2}{akT} \tag{9.37}$$

where $a$ is the average ion spacing. When $\Gamma > 1$, ions interact strongly via Coulomb interactions.

When in a Coulomb solid, $E_{\text{th}} = 3kT$, since each positional degree of freedom now has an energy. This phase transition introduces a latent heat. The WD goes crystalline at $\Gamma \approx 135$. In a Coulomb liquid, we have short-range order at $\Gamma \sim 1$ and $C_V$ increases relative to $\frac{3}{2}N_i kT$. Lindimann in the 1920's discovered the following rule: when the average displacement at a given site in a lattice is bigeer than $0.1a$, the solid melts.

## 9.3 White Dwarf Taxonomy

Just like main sequence stars have spectral classifications, white dwarfs also have their own classification scheme. **DA** white dwarfs show strong Balmer emission because gravitational sedimentation has caused the surface to be *pure* hydrogen. DA white dwarfs make up $\approx 80$ % of the white dwarf population. Similarly, **DB** white dwarfs are white dwarfs that have no Balmer lines but appreciable helium lines. These are believed to have lost their hydrogen envelopes and developed a pure helium envelope via sedimentation. These make up $\approx 20\%$ of the white dwarf population. There have even been white dwarfs that have pure C/O on their surface, but these are much rarer (SDSS found 2).

*Wednesday, March 7, 2012*

# 10 Evolution of Massive Stars

In stars larger than about 6 $M_\odot$, helium is ignited while the core is non degenerate. There is complete helium burning into a carbon/oxygen core. After that, carbon begins burning to magnesium via

$$
\begin{aligned}
{}^{12}\mathrm{C} + {}^{12}\mathrm{C} &\to {}^{24}\mathrm{Mg} && (10.1)\\
&\to \alpha + {}^{20}\mathrm{Ne} && (10.2)\\
&\to \mathrm{p} + {}^{23}\mathrm{Na} && (10.3)\\
&\to \mathrm{n} + {}^{23}\mathrm{Mg} && (10.4)
\end{aligned}
$$

Note that for heilum burning to take place, we needed a temperature of about $T = 2 \times 10^8\,\mathrm{K}$. The Coulomb barrier for the double carbon ignition will require a higher temperature than helium burning did, but these stars do get hot enough for this to occur.

## 10.1 Onset of Neutrino Emission

These stars, since they are massive enough, gain an additional means of cooling (rather than radiative diffusion): neutrino cooling. Remember our equation of stellar structure

$$T\frac{ds}{dt} = \varepsilon_{\mathrm{nuc}} - \frac{dL(r)}{dm(r)} - \varepsilon_\nu \tag{10.5}$$

where $\varepsilon_\nu$ is the power emitted by the parcel of fluid as neutrinos that leave the star. Since the material in the star is optically thin to neutrinos, their is no transport problem, and so the local rate enters into the entropy equation (i.e., surface conditions are irrelevant, unlinke the situation with radiative diffusion).

Now if we have $\varepsilon_\nu \gg dL(r)/dm(r)$, then the time scale for stellar evolution gets set by $\varepsilon_\nu$ rather than the photon luminosity!

The cross section for neutrinos is approximately

$$\sigma_\nu \sim 10^{-44}\,\mathrm{cm}^2 \left(\frac{E_\nu}{\mathrm{MeV}}\right)^2 \tag{10.6}$$

So we're interested in when

$$\frac{M}{m_p R^2}\sigma_\nu > 1 \tag{10.7}$$

It turns out that this condition is equivalient to

$$R < 30\,\mathrm{km}\left(\frac{E_\nu}{\mathrm{MeV}}\right) \tag{10.8}$$

Interestingly, neutron stars follow this criterion *and* they are optically thick to neutrinos, so you will often hear in the literature about the "neutrinosphere" of a neutron star. Stars are not this dense at this stage, though.

How are neutrinos created, though? The primary source is through positron-electron annihilation:

$$e^+ + e^- \to \nu_e + \overline{\nu}_e \tag{10.9}$$

The temperature and density do get high enough at late times that there is a population of positrons where

$$\frac{e^+ + e^- \to \nu_e + \overline{\nu}_e}{e^+ + e^- \to 2\gamma} \sim 10^{-19} \tag{10.10}$$

where this is essentially a ratio of reaction rates. It is of interest, then, to calculate the equilibrium abundance of positrons from

$$e^+ + e^- \rightleftharpoons 2\gamma \tag{10.11}$$

by equating the chemical potentials:

$$\mu_{e^+} + \mu_{e-} = 0 \tag{10.12}$$

The chemical potentials are given by the usual formula (see Section 7) and we get

$$n_{e^+} = g_e^2 n_{e,Q}^2 \frac{1}{n_e} \exp\left(-\frac{2m_e c^2}{kT}\right) \tag{10.13}$$

Scaling this result, we get

$$n_{e+} = 7.7\times 10^{30}\left(\frac{10^4}{\rho}\right) T_9^3 \exp\left(-\frac{11.84}{T_9}\right) \tag{10.14}$$

| $T_9$ | $n_{e+}/n_{e^-}$ |
|---|---|
| 1 | 0.02 |
| 2 | $> 1$ |

Table 7: Relative densities of positrons and electrons at various temperatures at $\rho = 10^4$ g cm$^{-3}$.

In the second entry in Table 7, the density of the electrons are set by positrons rather than the ions, so our previous considerations are invalid. When does this happen? I.e., when does $n_{e^+} = n_e = Zn_i$? Well, it's when this happens:

$$\rho_4^2 = 2566 T_9^3 \exp\left(-\frac{11.84}{T_9}\right) \tag{10.15}$$

Which is satisfied at $\rho_4 = 1$ and $T_9 = 1.35$. Going back to our "Saha" equation,

$$n_{e^+}n_{e^-} = 4\left(\frac{2\pi m_e kT}{h^3}\right)^3 \exp\left(-\frac{2m_e c^2}{kT}\right) \tag{10.16}$$

And so the rate of energy from neutrinos is

$$\frac{dU_\nu}{dt} = n_{e^+}n_{e^-}\left\langle\sigma v\right\rangle W \tag{10.17}$$

where $\langle\sigma v\rangle$ is the "reaction rate" and $W$ is the energy loss from a pair of neutrinos leaving. Writing out the cross section of the neutrinos again,

$$\sigma = 1.4\times 10^{-45}\,\frac{c}{v}\left[\left(\frac{W}{m_e c^2}\right)^2 - 1\right]\,\mathrm{cm}^2 \tag{10.18}$$

Thus $\langle\sigma v\rangle$ is just a constant:

$$\langle\sigma v\rangle = 4\times 10^{-45}c \tag{10.19}$$

Then the cooling rate is, more explicitly

$$\frac{dU_\nu}{dt} = 4.8\times 10^{18} T_9^3 \exp\left(-\frac{11.84}{T_9}\right)\mathrm{erg\,cm^{-3}\,s^{-1}} \tag{10.20}$$

Since the thermal energy is just

$$E_{\mathrm{th}} = nkT = 7\times 10^{19}\,\mathrm{erg\,cm^{-3}}\,T_9\rho_4 \tag{10.21}$$

and thus the cooling timescale is

$$t_{\mathrm{cool}} \sim \frac{E_{\mathrm{th}}}{dU_\nu/dt} = 14\,\mathrm{s}\,\frac{\rho_4}{T_9^2}\exp\left(\frac{11.84}{T_9}\right) \tag{10.22}$$

The neutrino cooling, once "available" quickly becomes the driver for energy losses and "resets"

| $T_9$ | $t_{\mathrm{cool}}$ |
|---|---|
| 0.6 | 160 years |
| 1 | 0.74 months |
| 2 | 0.4 hours |

Table 8: Neutrino cooling time scale for various temperature assuiming $\rho_4 = 1$.

the evolution timescales to *much* shorter values.

## 10.2 Plasmon Decay

One other important neutrino emission process is **plasmon decay**. In a plasma, the dispersion relation for photons becomes

$$\omega^2 = (ck)^2 + \omega_0^2 \tag{10.23}$$

where

$$\omega_0^2 = \frac{4\pi n_e e^2}{m_e} = \text{Plasma Frequency} \tag{10.24}$$

Interestingly, (10.23) looks a lot like the relativistic finite mass energy relation:

$$E^2 = p^2c^2 + m^2c^4 \tag{10.25}$$

We can solve for energy and momentum conservation for this particle to decay to two particles. It turns out that the decay $\gamma \to \nu_e + \overline{\nu}_e$ is an available decay. This then gives another source of neutrino losses. We haven't derived a specific rate or anything, so if you'd like to know more, see Clayton.

## 10.3 Back to Stellar Evolution

When neutrino cooling is dominant, we have

$$L_\nu = \int \varepsilon_\nu \, dM \gg L_\gamma \tag{10.26}$$

Then we have

$$T\frac{ds}{dt} = \varepsilon_{\text{nuc}} - \varepsilon_\nu \tag{10.27}$$

This can become a local condition, so we need to rethink of what we think of as our "main sequence" for massive, carbon-burning stars. We'd like to find where, in the $T_c - \rho_c$ plane, that $\varepsilon_{\text{nuc}} = \varepsilon_\nu$. See Figure 8 for an idea of what's going on here.

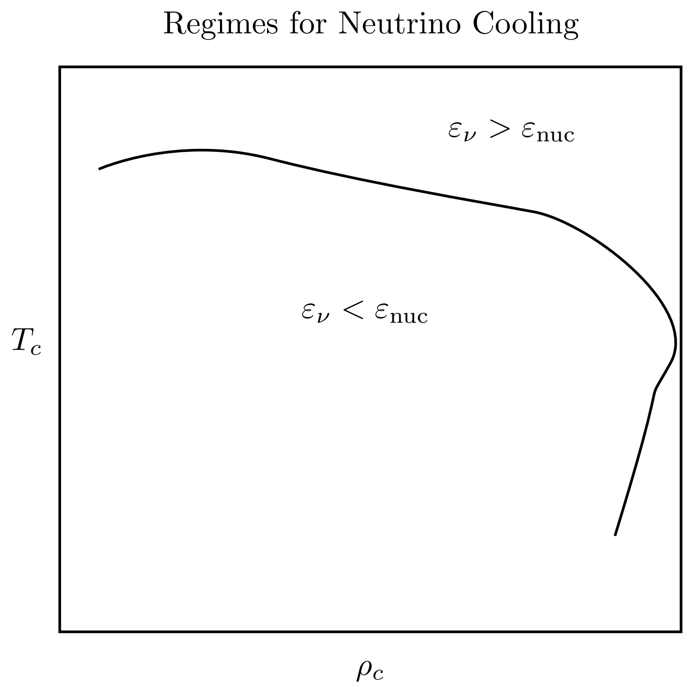


Figure 8: Rough sketch of the limits where neutrino cooling becomes dominant.

If there is just carbon burning taking place, all of the carbon will eventually be burned, leaving oxygen unburned. Then oxygen, neon, and magnesium are the dominant elements. For stars between $6\,M_\odot$ and $8\,M_\odot$ where carbon is burned, electron degeneracy sets in, and they will die as white dwarfs.

After carbon burning, we have $^{16}$O, $^{20}$Ne, and $^{23}$Na available. The $^{16}\text{O}+{}^{16}\text{O}$ reaction requires such a high temperature that prior to this, we have photodisintegration of neon:

$$\gamma + {}^{20}\text{Ne} \rightarrow \alpha + {}^{16}\text{O} \tag{10.28}$$

These alpha particle laying around get scooped up by neon (before Oxygen!):

$$\alpha + {}^{20}\text{Ne} \rightarrow \gamma + {}^{24}\text{Mg} \tag{10.29}$$

This leads to magnesium burning:

$$\alpha + {}^{24}\text{Mg} \rightarrow \gamma + {}^{28}\text{Si} \tag{10.30}$$

This process leaves us with oxygen, magnesium, and silicon.

*Monday, March 12, 2012*

# 11 Core Collapse and Stellar Death

## 11.1 The March to Collapse

Neutrino cooling "resets" the timescale for evolution, thereby accelerating the burning phases. Additionally, for nearly all these phases (carbon, neon, oxygen, etc.), the evolution times still remains longer than

$$t_{\mathrm{dyn}} \sim \frac{1}{\sqrt{G\rho}} = 4\,\mathrm{s}\left(\frac{10^6\,\mathrm{g\,cm^{-3}}}{\rho_c}\right)^{1/2} \tag{11.1}$$

Meanwhile, the star is building up an onion skin shell structure of overlying layers of different elements. The burning up to $^{28}$Si is still fusion driven. However, we find that $^{28}$Si actually burns via **photodisintegration**:

$$^{28}\mathrm{Si} + {}^{28}\mathrm{Si} \rightarrow {}^{56}\mathrm{Ni} \tag{11.2}$$

This reaction has a prohibitive Coulomb barrier, so the following reaction is favored

$$\gamma + {}^{28}\mathrm{Si} \rightarrow \alpha + {}^{24}\mathrm{Mg} \tag{11.3}$$

This is the photodisintegration of $^{28}$Si mentioned above. Remember that at $T_9 \sim 1$, we have $E_\gamma \sim 100\,\mathrm{keV}$.

The most stable element at fixed mass number $A$ are neutron rich ($N > Z$). During silicon "burning", there is time for weak interactions, yielding more neutron-rich isotopes. Using beta captures, the pressure support is decreased.

## 11.2 Stellar Core Collapse

As the electrons in the core become relativistic, the mass of the core approaches the Chandrasekhar mass (appropriately corrected for temperature and the electrons per baryon, so lower than that for white dwarf). As the mass increases, the core compresses, making the electrons become more relativistic and also increasing the temperature. The electrons are captured by nuclei, and the photons photodisintegrate the nuclei, starting the undoing of all of the nucleosynthesis that has been going on in the star.

Now we have to worry about energy *sinks* in the star, since these reactions are endothermic. One such reaction is

$$\gamma + {}^{56}\mathrm{Fe} \rightarrow 13\alpha + 4n \tag{11.4}$$

Which requires an energy of 124 MeV. This energy comes from the ambient thermal energy, so the core is cooled, and thus contracts. The energy lost per unit mass is

$$\frac{124\,\mathrm{MeV}}{56 m_p} = 2\,\mathrm{MeV/baryon} \tag{11.5}$$

The Contraction must get to

$$\frac{GM}{R} \approx 2\,\mathrm{MeV/baryon} \tag{11.6}$$

| $\rho_9$ | $T_{10}$ |
|---|---|
| 0.1 | 0.78 |
| 1 | 0.95 |
| 10 | 1.2 |

Table 9: Density and temperature of a degenerate core at half iron disintegration

Which gives for a star the mass of the sun being $R = 1000\,\text{km}$! Just considering the case where half of the $^{56}$Fe is destroyed, we get the data shown in Table 9.

Photodisintegration gets the radius down to less than 1000 km and the density up to $\rho_{10} \sim 1$. Since the electrons are relativistic, the Fermi energy is $E_F \sim m_e c^2$ (for the electrons). Now recall the Chandrasekhar mass,

$$M_{\text{Ch}} = 1.46\,M_\odot \left[\frac{Y_e}{1/2}\right]^2 \tag{11.7}$$

Where $Y_e$ is the number of electrons per baryon. For a pure C/O object, $Y_e = 1/2$. For $^{56}$Fe, though, $Y_e = 0.46$, which introduces a correction for a *cold* iron core, bringing the mass down to $1.26\,M_\odot$. As the Fermi energy increases, more electron captures occur, increasing the neutron-richness of the plasma.

During the rest of the collapse, the nuclei are being broken up and more and more protons are converted to neutrons via electron capture, which also causes a flurry of neutrino emission. The collapse suddenly “halts” when a density is reached that implies the mean spacing is of order of the neutron radius, or 1 fm (fermi). Here we have the onset of an incompressible equation of state. That is, the pressure rises dramatically as the density approaches the nuclear density of about $\rho \approx \rho_{\text{nuc}} \approx 10^{14}\,\text{g}\,\text{cm}^{-3}$.

At this point, the radius is about 15 km and the collapsed core “halts” at about 10 km. The gravitational potential energy is

$$E_{\text{GR}} = \frac{GM^2}{R} = 10^{53}\,\text{ergs} \tag{11.8}$$

So if a fraction of this energy is tapped, it would be enough to unbind the envelope (around $10^{51}$ ergs). As it turns out, the outgoing shock *does* contain about $10^{52}$ ergs.

## 11.3 Neutron Star and Black Hole Formation

The collapsed core is essentially a proto-neutron star with $E_{\text{th}} = E_{\text{GR}} = 10^{53}$ ergs, which is trying to cool via thermal neutrino emission. The open question is how the outgoing shock succeeds in “punching through” the infalling matter.

Cores in massive stars collapse when $M_{\text{core}} > M_{\text{crit}}$, where $M_{\text{crit}}$ depends on the evolution of the star, details of the star’s rotation, and the mass of the star. Collapses lead to a dense proto-neutron star with a radius of around 20 km and an outgoing shock.

*Monday, December 2, 2013*

# 12 Missing Elements

Thus far, we've avoided discussing several key concepts in stellar astrophysics. Namely, we've missed

- Rotation
- Magnetism
- Dynamics
- Binarity
- Mass Transfer
- Stellar Winds

First, we'll briefly talk about binarity

## 12.1 Binarity

Binarity is, briefly, two or more stars existing in mutual orbits. While the sun is a lone star, about half of all stars are actually in binaries. Their relative prevalence is due to the fact that it is easier to store angular momentum in orbital angular momentum than in rotational angular momentum, so cloud collapse "likes" to form binaries.

When in a binary, a star experiences tidal coupling between the orbit and rotation, so the very act of being in a binary will affect its rotation. Additionally, being in a binary (especially tight binaries) causes mass transfer between the binary members and the surrounding medium. This can explain some weird discrepancies between masses/ages of stars (**Algol Paradox**), massive white dwars, and ISM enrichment.

## 12.2 Dynamics and Rotation

In this course, we have relied heavily on hydrostatic equilibrium:

$$0 = -\nabla P + \rho \mathbf{g} \tag{12.1}$$

In reality, this is a special case of the momentume quation from fluid mechanics:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] = -\nabla P + \rho \mathbf{g} \tag{12.2}$$

If we add in rotation, we recover the centrifugal and coriolis forces:

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\,\mathbf{\Omega}_0 \times \mathbf{u} \right] = -\nabla P + \rho \mathbf{g} + \rho \Omega_0^2 \varpi \hat{\varpi} \tag{12.3}$$

Where, in spherical coordinates, $\varpi = r \sin\theta$, since centrifugal forces act in a cylindrical fashion. $\mathbf{\Omega}_0$ is the bulk rotation rate.

We might also check to compare the strength of the centrifugal force to the surface gravity. If the effect is weak, we may not care about rotation.

$$\frac{\rho\Omega^2 R}{\rho g} = \frac{\Omega^2 R^2}{GM/R} = \frac{v_{\text{eq}}^2}{v_{\text{esc}}^2} = \frac{\text{Kinetic Energy at Surface}}{\text{Gravitational Energy at Surface}} \tag{12.4}$$

So if this quantity is greater than unity, the star is ripped apart (it's spinning at "break-up"). If the quantity less than unity, but still large, we should still care about rotation. If it is very small, though, we are safe to neglect rotation effects. For the sun, $v_{\text{eq}}^2/v_{\text{esc}}^2 \sim 10^{-5}$, so the centrifugal forces do not greatly affect the shape of the sun.

#### 12.2.1 Hydrostatic Rotating Stars

Just because rotation is present doesn't mean there isn't a hydrostatic solution anymore. Such a solution simply won't be spherical anymore. The new equation of hydrostatic equilibrium would be

$$\nabla P = \rho \mathbf{g} + \rho\Omega_0^2 \varpi\,\hat{\varpi} \tag{12.5}$$

We can write the gravitational acceleration as the gradient of a scalar potential:

$$\mathbf{g} = -\nabla\phi_g \tag{12.6}$$

So we can define an effective potential as

$$\phi_{\text{eff}} = \phi_g - \frac{\Omega^2\varpi^2}{2} \tag{12.7}$$

Which can give us an effective gravity:

$$\mathbf{g}_{\text{eff}} = -\nabla\phi_{\text{eff}} = -\nabla\left[\phi_g - \frac{\Omega^2\varpi^2}{2}\right] \tag{12.8}$$

And now the equation of hydrostatic equilibrium is [artifically] cleaner:

$$\nabla P = \rho\mathbf{g}_{\text{eff}} \tag{12.9}$$

The only difference between this and our standard calculation is that $\mathbf{g}_{\text{eff}}$ is not spherical, so constant pressure contours are oblate. This is actually observed in stars that are rapidly rotating. They are puffed out along the equator and thinner at the poles. There is a problem here, though. We've assumed rigid-body rotation, which is fine, but if we also mandate thermal equilibrium (as we have been doing), then the **von Zeipel** paradox kicks in. In 1924, von Zeipel found that a uniformly star cannot be in steady-state thermal equilibrium.

The solution to this proble is to allow the rotation to vary as a function of position *or* to allow velocities in the fluids:

$$\mathbf{\Omega} = \mathbf{\Omega}(r,\theta) \qquad \text{or} \qquad \mathbf{u} \neq 0 \tag{12.10}$$

Many stars take the latter option, inducing **meridional flows** that add extra mixing to a star that we previously would not have expected.

### 12.2.2 The Coriolis Term

In our hydrostatic calculations, we assumed no flows (and thus no coriolis force). However, we've just shown that there are often non-zero velocities in rotating stars, so we need to get a handle on what the coriolis force does. To characterize this effect, we introduce the **Rossby Number**. To do so, we'll use some dirty fluid dynamics math, where derivatives are simply inverse lengthscales: $\nabla \to 1/R$ and velocites are converted to scalars: $\mathbf{u} \to u$.

$$R_o = \frac{\mathbf{u}\cdot\nabla\mathbf{u}}{2\mathbf{\Omega}\times\mathbf{u}} = \frac{u^2/R}{2\Omega u} = \frac{u}{2\Omega R} \tag{12.11}$$

If $R_o \gg 1$, rotation does *not* matter, but if $R_o \ll 1$, the velocities are heavily constrained by rotation, so rotation *does* matter. We can measure $\Omega$ and $R$, but we usually don't have access to the internal velocities ($u$), so this is hard to constrain from observations. Studies on the effects of the coriolis terms are thus very dependent on 3D simulations or helioseismic studies of the sun.

### 12.2.3 Rotation in the Early Sun

We already noted that the sun's rotation doesn't greatly affect it's current shape. However, the same may not be true of the early sun. Ths sun likely collapsed out of an ISM cloud with $n \sim 1$ cm$^{-3}$. The sphere of such ISM that encompasses 1 $M_\odot$ is roughly 2 pc $\sim 6\times10^{18}$ cm. The angular velocity of that cloud is set by the differential rotation in the galactic disk:

$$\Omega_{\rm cloud} \sim \Delta R \frac{d\Omega_g}{dr} \sim \frac{\Delta R}{R}\Omega_g \tag{12.12}$$

And so the period of the cloud is

$$P_c \sim \frac{R}{\Delta R} P_g \approx 8\times10^{11} \text{ years} \tag{12.13}$$

period of the galaxy's rotation is $P_g \approx 2\times10^8$ years. Now during the sun's collapse, the angular momentum must be conserved, which means that after shrinking down to its current radius, the period was

$$P_f = \left(\frac{R_f}{R_i}\right)^2 P_i = \left(\frac{R_\odot}{2\text{ pc}}\right)^2 (8\times10^{11}\text{ years}) \sim 1\text{ hr} \tag{12.14}$$

Such a short period means the equatorial velocity is $v_{\rm eq} \sim 1000$ km s$^{-1}$, and so $v_{\rm eq}^2/v_{\rm esc}^2 > 1$. So evidently angular momentum transport and shedding of angular momentum is an essential part of stellar formation, and the youngest stars must be rotating very rapidly.

*Wednesday, December 4, 2013*

## 12.3 Magnetism in Stars

### 12.3.1 Stellar Spindown

Late in their lives (age $\gtrsim 1$ Gyr for the sun), the period grows approximately with the square root of the age, so the spindown rate is actually decreasing with time. This is likely due to the presence of strong winds in early evolution that help to efficiently remove angular momentum from the star.

Observations (and simulations) show a correlation between X-ray luminosity and rotation period. For small periods, the X-ray luminosity is largely a flat function, but at around 10 days, the X-ray luminosity dies off as a power law. X-Ray emission from stars are expected to arise from magnetic activity in the coronae of such stars, so evidently magnetic fields and rotation have a very strong connection.

In fact, the observed increase of the period of a star with age and accurate measurements of the period together can help us constrain the age of a star. Making age estimates in this way is in the purview of the field of **gyrochronology**.

### 12.3.2 More Magnetism

In stars, we have magnetic fields running through a hot plasma, so we require the machinery of magnetohydrodynamics (MHD) to accurately describe the interplay between the magnetic fields and the plasma. In MHD, we model the plasma as a single fluid where electrons and ions move together as very tightly-coupled particles. The overall charge of said plasma both locally and globally neutral (so no electric fields), though currents are allowed (hence magnetic fields). Finally, we'll assume that the flows in the plasma are slow, in that $v \ll c$. A quick review of Maxwell's equations gives us

$$\nabla \cdot \mathbf{B} = 0 \tag{12.15}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [\mathbf{u} \times \mathbf{B}] = -(\mathbf{u} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{u} - \mathbf{B}(\nabla \cdot \mathbf{u}) \tag{12.16}$$

If we re-express (12.16) to move the advection term to the left side, we get

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{u} - \mathbf{B}(\nabla \cdot \mathbf{u}) \tag{12.17}$$

The first source term tells us that flows that are collinear with magnetic fields tend to strengthen the magnetic fields, and the second term tells us that converging flows also act to strengthen a magnetic field.

There's still one more equation left: the momentum equation, now with a magnetic term

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = -\nabla P + \rho \mathbf{g} + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} \tag{12.18}$$

That magnetic term is a bit difficult to expand, so let's expand it with a vector identity:

$$\frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} = -\frac{1}{4\pi}\nabla\left(\frac{1}{2}B^2\right) + \frac{1}{4\pi}(\mathbf{B} \cdot \nabla)\mathbf{B} \tag{12.19}$$

Ah! Now we recognize the first part as a magnetic pressure, and the second term is [SOMETHING ELSE]. Now let's return to our old problem of the sun's collapse from the galactic ISM. The ISM has magnetic fields running through it at approximately $B \approx 5 \times 10^{-6}$ G. Due to magnetic flux freezing in a plasma, we get $BR^2 = \text{const}$, so as the sun collapses from $R \sim \text{pc}$ to $R \sim R_\odot$, the magnetic field is amplified by a factor of $10^{16}$ (just like we saw with the angular momentum conservation argument). The resulting field would then be

$$B_{\text{collapsed}} \sim 1.8 \times 10^{10} \text{ G} \tag{12.20}$$

This is *crazy* high. For comparison, magnetic white dwarfs have field strengths of about $10^6$ G, neutron stars are sitting at around $10^{12}$ G, and the most magnetic entities in the known universe, magnetars, have fields at a whopping $10^{15}$ G. So, this proposed field of the collapsed sun seems improbable. Note also that magnetic pressure at the center of such a star would be

$$P_B \sim \frac{B^2}{8\pi} \sim 10^{19} \text{ dyne cm}^{-2} \tag{12.21}$$

Compare that to the simple pressure we get from HSE: $P_c \sim GM^2/R^4 \sim 10^{16}$ dyne cm$^{-2}$. Evidently, stellar formation requires not only massive angular momentum loss, but also a tremendous weakening of magnetic fields.