

# Notes for PHYS 232: Stellar Structure

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# 1 Introduction

*Monday, January 9, 2012*

## 1.1 The HR Diagram

Most stars shine predominantly in the optical range of the electromagnetic spectrum. As a result, we get most of our information about stars by observing their optical output. It makes sense, then, that we might organize stars by their color, which is indicative of their surface temperature. When plotting a population of stars' luminosities against their surface temperature, we note a strong correlation between the two. As it turns out, the controlling parameter for these quantities is the mass of the star, at least while the star is on the **Main Sequence** (stars burning hydrogen to helium). The correlation between the mass of a main-sequence star and its luminosity is incredibly strong (see HR diagram examples).

## 1.2 Conditions for a Star on the HR Diagram

We are interested in knowing what defines the regime where a star can reside in a particular  $L$ ,  $T_{\text{eff}} = [L/(4\pi\sigma_{\text{SB}}R^2)]^{1/4}$ . Why, for example, is there a dynamical range in the luminosity spanning over six orders of magnitude, while only a range of a factor of about 5 in the effective temperature? To gain some perspective, we might observe the number of stars as a function of brightness. We organize these stars by their **spectral type** (a rough measure of how big the star is) and find their approximate **mass density** (the amount of mass contained in these stars per unit volume):

Spectral Type	$\rho$ ( $M_{\odot}/\text{pc}^3$ )
O-B	0.4
A-F	4
G-M	40
WD's	20

Here we've used the standard labels for different spectral types, O, B, A, F, G, K, M, L, T, which are roughly in decreasing order of size and temperature (the reasoning for this scale is historical rather than logical, and the ordering is often remembered by the mnemonic, "Oh Be A Fine Gal/Guy, Kiss Me! Less Talk!"). We see that the big, bright stars form an exceedingly small portion of the amount of stellar mass in our galaxy. We will find that this is because large stars exhaust their fuel much more quickly than smaller stars, and thus live and die much faster. We'll now observe another type of classification of stars used in our neighborhood, the Milky Way Galaxy.

### 1.2.1 Population I Stars

From Earth, the center of the galaxy is approximately 8.5 kpc away. Meanwhile, the disk is only about 100 pc wide. We've observed that stars in the thin disk (commonly known as **Population I Stars**) are orbiting at the orbital velocity with only a small amount of axial and radial motion. They are essentially dynamically cold and in nearly circular orbits. This is indeed where most of the **interstellar medium** (ISM) resides, causing much of the star formation in the galaxy. This region is also very metal rich. That is, compared to other parts of the universe, there is a much

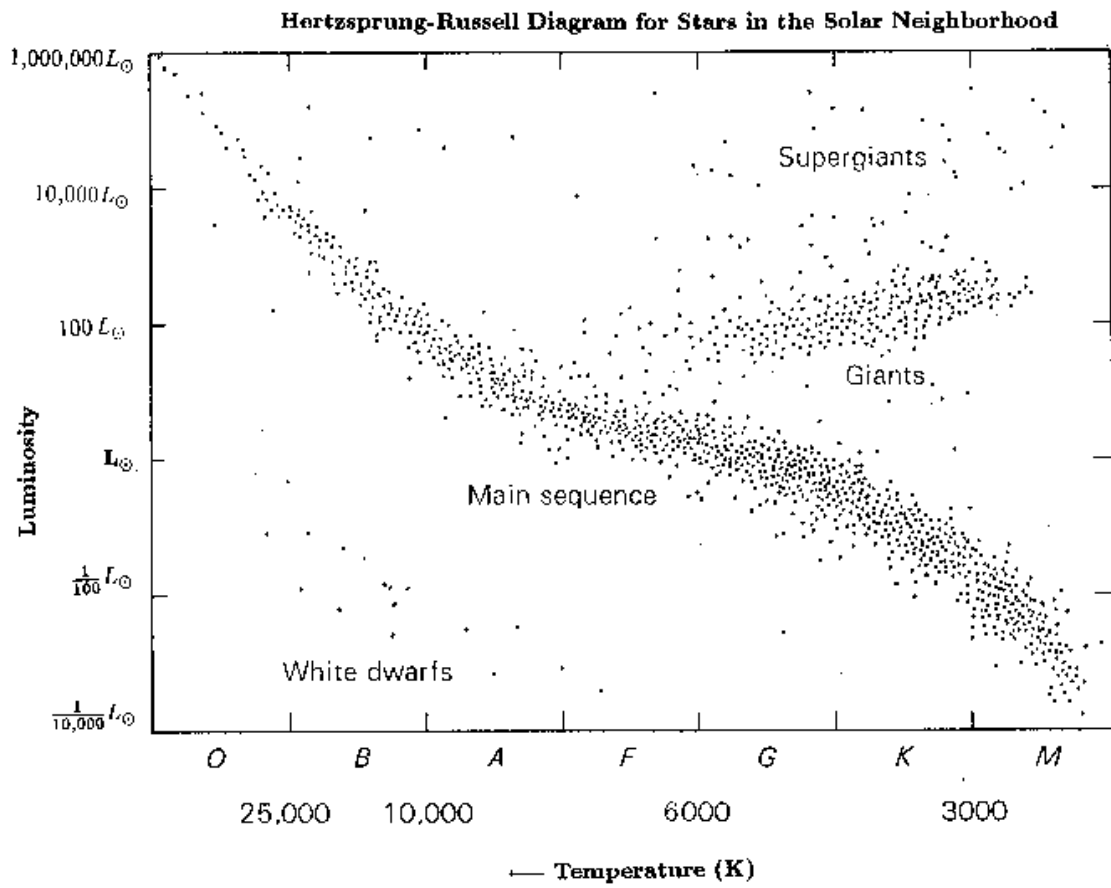


Figure 1: An HR diagram for stars in the local neighborhood (shamelessly stolen from Google Images)

higher concentration of elements heavier than helium present. We will denote the mass fraction of these “metals” with the letter  $Z$ , and in this region, we have  $Z \sim 1 - 2\%$ . These metals come from a previous generation of stars, who died in the past, giving off the metals we now have.

### 1.2.2 Population II Stars

**Population II Stars** reside mostly in the spheroid in the center of the galaxy. These are older stars in regions where star formation is largely shut down. Typically they are metal poor, with metallicities as low as  $Z = 10^{-4}Z_{\odot}$ . Kinematically, they are often on radial orbits (rather than their more azimuthal population I counterparts in the disk). We typically say that the globular clusters are part of this population. Sometimes these stars are seen passing through the disk at velocities comparable to the orbital velocities, and are easily identified due to their high velocities and unique spectra (due to the low metallicities).

## 2 A Simple Hydrostatic Atmosphere

### 2.1 Scale Height and Column Depth

Before tackling the physics of stars, we first consider a simple toy model– the isothermal, plane parallel atmosphere. This model is somewhat applicable to the thin stellar atmosphere near the surface of a star, where curvature can be neglected and the acceleration due to gravity is nearly uniform.

Consider an atmosphere where the local acceleration due to gravity,  $\mathbf{g}$  is constant in value and direction. The atmosphere is composed of an isothermal ideal gas with temperature  $T$ . We wish to find the distribution of particles in this atmosphere.

In a strictly statistical sense, we would expect the energy distribution to be comparable to  $e^{-E/kT}$  (recall that the atmosphere is isothermal, so the average kinetic energy is uniform throughout). In our case, the energy of particles is linear in height, so we expect this probability to be proportional to  $e^{-mgh/kT}$ .

We will let  $m_B \approx m_p$  be the baryon mass,  $\mu$  be the mean molecular mass (measured in AMU), and  $\rho$  is the density in  $\text{g cm}^{-3}$ . We suppose that the gas is in hydrostatic balance, so we have

$$\frac{dP}{dz} = -\rho g \quad (2.1)$$

Combining this with the ideal gas law,

$$P = nkT \quad (2.2)$$

We find that

$$kT \frac{dn}{dz} = -m_p \mu n g \quad (2.3)$$

which in turn gives us

$$\frac{d \ln n}{dz} = -\frac{m_p \mu g}{kT} \quad (2.4)$$

Solving this differential equation gives the expected result

$$n(z) = n(0) \exp\left(-\frac{m_p \mu g z}{kT}\right) = n(0) \exp\left(-\frac{z}{h}\right) \quad (2.5)$$

where we have defined the **scale height**  $h \equiv kT/(\mu m_p g)$ , which is the e-folding distance in number density. As it turns out, the scale height for earth's atmosphere is approximately 10 km. This model is really only valid in cases where  $h \ll R$ , (where  $R$  is the size of the object), so we now investigate the ratio of these two quantities:

$$\frac{h}{R} = \frac{kT}{\mu m_p \frac{GM}{R^2} R} = \frac{1}{\mu} \frac{kT/m_p}{GM/R} \sim \frac{v_{\text{th}}^2}{v_{\text{esc}}^2} \ll 1 \quad (2.6)$$

So if this approximation is valid, the thermal velocities of particles are typically much smaller than the escape velocity of the central body, so a star could retain its own atmosphere (thankfully, Earth does the same to its atmosphere!) For stars, we will find that  $kT_c/m_p \sim GM/R$ . Then, (2.6) tells us that

$$\frac{h}{R} \sim \frac{T_{\text{eff}}}{T_c}. \quad (2.7)$$

This tells us that stars are quite sharp-edged (their scale heights are very small compared to their radii). We can also deduce a physical meaning for the scale height as being how far a particle needs to fall to gain an energy comparable to  $kT$ .

From the ideal gas law, we can easily see that the pressure will also fall off exponentially in this isothermal atmosphere. However, let's explore the pressure a bit more. First, we return to the ideal gas law,

$$P = \frac{\rho}{\mu m_p} kT = nkT \quad (2.8)$$

and the condition for hydrostatic equilibrium,

$$dP = -\rho g dz. \quad (2.9)$$

We now integrate (2.9) from  $z = z$  to  $z \rightarrow \infty$ :

$$P(\infty) - P(z) = - \int_z^\infty \rho(z') g dz' \quad (2.10)$$

$$P(z) = g \int_z^\infty \rho(z') dz \quad (2.11)$$

Note that it is okay to take the integral to infinity so long as we are dealing with a constant  $g$ . This result suggests the definition of the **column density**:

$$y(z) \equiv \int_z^\infty \rho(z) dz \quad (2.12)$$

On the surface of the earth, the column density is approximately  $y = 1000 \text{ g cm}^{-2}$ . Think of it as the amount of mass sitting above you per unit area at a given altitude. The column density is an important number (for us) to determine the details of heat transport. For now though, we can write the pressure in this isothermal atmosphere in a compact form:  $P(z) = gy(z)$ .

## 2.2 Mean Molecular Weights

We'll now make a useful definition for calculating pressures and other useful quantities. For an ideal gas, the total pressure of a mixed gas is simply

$$P = \sum_{i=1}^N n_i kT \quad (2.13)$$

where  $n_i$  are just the number densities for each ion. The number density is computed via

$$n_i = \frac{X_i \rho}{A_i m_p}. \quad (2.14)$$

where  $X_i$  is the mass fraction of the  $i^{\text{th}}$  ion with mass number  $A_i$  and  $\rho$  is the overall mass density. Then the ion pressure is given by (assuming total ionization)

$$P_{\text{ion}} = kT \sum \frac{X_i \rho}{A_i m_p} = \frac{kT \rho}{m_p} \sum \frac{X_i}{A_i} = \frac{kT \rho}{\mu_{\text{ion}} m_p}. \quad (2.15)$$

Where we have defined the **mean molecular weight** of the ions to be

$$\mu_{\text{ion}} = \left[ \sum \frac{X_i}{A_i} \right]^{-1} \quad (2.16)$$

For the electrons, we have (assuming total ionization)

$$P_e = n_e kT = kT \left( \sum Z_i n_i \right) = \frac{kT \rho}{m_p} \sum \frac{Z_i X_i}{A_i}. \quad (2.17)$$

(here  $Z_i$  is the atomic number, not the metallicity). Then the total pressure is just the sum of these two,

$$P = P_{\text{ion}} + P_e = \frac{\rho kT}{m_p} \left( \frac{1}{\mu_e} + \frac{1}{\mu_i} \right) \quad (2.18)$$

So we define the overall mean molecular weight via

$$\frac{1}{\mu} \equiv \frac{1}{\mu_e} + \frac{1}{\mu_i} \quad (2.19)$$

One might think of this as the average weight of a particle that supplies pressure within a gas. Later, we'll see that this quantity, and its evolution, plays a large and critical role in the the nature of stellar evolution. Since fusion tends to decrease the pressure support, the star must continuously readjust its structure so as to hold itself up.

*Wednesday, January 11, 2012*

## 2.3 Electric Fields in Stars

Imagine a pure, ionized hydrogen atmosphere which is, on the large scale, electrically neutral. We wish to find the scale height in a plasma of ionized hydrogen. In this plasma, we have  $n_p = n_e$  due to electric neutrality. Then the overall pressure in this hydrogen plasma is

$$P = 2n_p kT \quad (2.20)$$



Using hydrostatic equilibrium, we get

$$2kT \frac{dn_p}{dz} = -m_p n_p g \quad (2.21)$$

which in turn gives us the differential equation

$$\frac{d \ln n_p}{dz} = -\frac{m_p g}{2kT} \quad (2.22)$$

Which gives us a scale height of

$$h = \frac{2kT}{m_p g} \quad (2.23)$$

We need to look at both plasmas separately while incorporating the electric field created between the protons and electrons. For electrons, we have

$$\frac{1}{n_e} \frac{dP_e}{dz} = -m_e g - eE. \quad (2.24)$$

Likewise for the protons,

$$\frac{1}{n_p} \frac{dP_p}{dz} = -m_p g + eE \quad (2.25)$$

where we've assumed that the electric field points up (the protons are heavier and would thus sink below the electrons). Now adding (2.24) and (2.25), we recover hydrostatic balance. However, subtracting the two equations will get us the electric field:

$$0 = -m_e g + m_p g - 2eE, \quad (2.26)$$

giving the result,

$$eE = \frac{1}{2} (m_p - m_e) g \quad \text{or} \quad e\mathbf{E} \approx -\frac{m_p \mathbf{g}}{2}. \quad (2.27)$$

So this field does not directly affect hydrostatic balance, but it does dramatically impact the relative difficulty of (?UNREADABLE TEXT IN LECTURE NOTES?) in a white dwarf.

### 3 Self-Gravitating Objects

So far we have only considered systems where the acceleration due to gravity is constant. In any self-gravitating object, this is obviously not true. We will, however, continue to assume that such objects do not rotate. Additionally, we will be ignoring mass loss. Essentially all we must write down are equations of mass conservation, momentum conservation, and energy conservation. We'll start with momentum conservation.

#### 3.1 Momentum Conservation and the Free-Fall Timescale

The momentum equation for a fluid is just

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{g} - \nabla P \quad (3.1)$$

This equation essentially states that a self-gravitating object is neither collapsing nor expanding. If we were to “shut off” gravity or the pressure gradient, the star would either explode or collapse, respectively. Such a collapse would occur on the **free-fall timescale**, which we will now derive. Taking the pressure gradient out of (3.1), we retrieve

$$\mathbf{g} = -\frac{Gm(r)}{r^2}\hat{r} \quad (3.2)$$

For this derivation, we will be using a **Lagrangian coordinate system**. This is a system where the coordinates follow a particular fluid element. In essence, we are making the substitution

$$\frac{d}{dt} \rightarrow \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \quad (3.3)$$

Returning back to the derivation, (3.2) gives us

$$\frac{dv_r}{dt} = -\frac{Gm(r)}{r^2} \quad (3.4)$$

Initially, we have  $t = 0$ ,  $v_r = 0$ , and  $r = r_0$  with the radial velocity given by  $v_r = dr/dt$ . Then our differential equation is

$$\frac{d^2r}{dt^2} = -\frac{Gm(r)}{r^2} \quad (3.5)$$

As an order of magnitude estimate, this gives us

$$\frac{r}{t_{\text{ff}}^2} \sim \frac{Gm}{r^2} \Rightarrow t_{\text{ff}}^2 \sim \frac{1}{Gm/r^3} \quad (3.6)$$

So we define the free-fall timescale to be

$$t_{\text{ff}} = \frac{1}{\sqrt{G\langle\rho\rangle}} \quad (3.7)$$

This is also the same as the Keplerian orbital period, modulo some uninteresting constants. The punchline of this whole argument is that a star that is *not* in hydrostatic balance will respond on a timescale of the free-fall timescale. From this alone, we may conclude that the sun (and any other star not currently exploding) is in hydrostatic balance. We will then assume that all stars are always in hydrostatic balance.

### 3.2 The Virial Theorem

Stars are held up by gas pressure, radiation pressure, or both. The pressure gradients are what will be the “restoring forces” against gravity for our cases. In spherical symmetry, hydrostatic balance tells us

$$\frac{dP}{dr} = -\rho \frac{Gm(r)}{r^2} \quad (3.8)$$

We will use this to derive the **Virial Theorem**, which relates the potential energy to the kinetic energy of a system. The equation of mass conservation states that

$$dm = 4\pi r^2 \rho(r) dr \quad (3.9)$$

Now we multiply both sides of (3.8) by  $4\pi r^3 dr$ :

$$\int 4\pi r^3 dP = - \int \rho \frac{Gm(r)}{r^2} 4\pi r^3 dr \quad (3.10)$$

$$= - \int \frac{Gm(r)dm}{r} = E_{\text{GR}} \quad (3.11)$$

where  $E_{\text{GR}}$  is the gravitational binding energy. Performing a similar analysis to the left side of (3.10) gives

$$\int 4\pi r^3 dr \frac{dP}{dr} = 4\pi r^2 P|_0^R - 3 \left[ 4\pi \int P r^2 dr \right] \quad (3.12)$$

$$= -3 \int P 4\pi r^2 dr \quad (3.13)$$

$$= -3 \langle P \rangle V \quad (3.14)$$

where we've defined the average pressure to be the pressure averaged over volume. Also, in (3.13) we've noted that the boundary terms must vanish since  $P(R) = 0$ . Then the virial theorem tells us that

$$\boxed{\langle P \rangle = -\frac{1}{3} \frac{E_{\text{GR}}}{V}} \quad (3.15)$$

Now we examine the total energy:

$$E_{\text{tot}} = E_{\text{GR}} + E_{\text{KE}} = -3 \langle P \rangle V + E_{\text{KE}} \quad (3.16)$$

We need only relate the kinetic energy to the pressure to finish this equation off. For an ideal gas, we know that  $P = NkT/V$ , so the kinetic energy is  $E_{\text{KE}} = \frac{3}{2}NkT = \frac{3}{2}PV$ . This gives a total energy of

$$E_{\text{tot}} = -3 \langle P \rangle V + \frac{3}{2} \langle P \rangle V = -E_{\text{KE}} \quad (3.17)$$

Interestingly, this requires a negative heat capacity. That is, an increase in the temperature of the system causes a net *decrease* in total energy. However for radiation, pressure is given by  $P = \frac{1}{3}aT^4$  and  $E/V = aT^4$ . Taking this to its conclusion gives us

$$E_{\text{tot}} \rightarrow 0 \text{ as the particles become relativistic} \quad (3.18)$$

The origin of this result is in the momentum-energy relation of relativistic particles and non-relativistic particles. That is,  $E = pc$  for ultra-relativistic particles and  $E = p^2/2m$  for non-relativistic particles.

The limiting energy of ultra-relativistic stars puts an upper level on the mass of large stars, since a total energy of a star being zero means unbinding the star. In the “normal case” of an ideal gas star, the more traditional form of the virial theorem applies:

$$\frac{E_{\text{KE}}}{\text{mass}} \sim \frac{GM}{R} \quad (3.19)$$

This is why stars typically behave with a negative heat capacity. That is, as a star radiates,  $E_{\text{tot}}$  is more negative, meaning that  $R$  must decrease and the temperature  $T$  (essentially the kinetic energy per particle) rises. This behavior would have to continue until a new energy source became available.

### 3.3 Applications of the Virial Theorem

The gravitational energy of an object is typically given by

$$E_{\text{GR}} \approx -\frac{GM^2}{R} \quad (3.20)$$

Using the virial theorem, we have

$$-E_{\text{GR}} = 3 \langle P \rangle V = 3Nk \langle T \rangle \quad (3.21)$$

Or,

$$\frac{GM}{R} (Nm_p) \sim 3NkT \quad (3.22)$$

So we have

$$\boxed{kT \sim \frac{GMm_p}{R}} \quad (3.23)$$

This temperature is the temperature of most of the material and is  $T \sim T_c$  (temperature in the core). For the sun, we then have  $T \sim 10^7$  K. Interestingly, assuming hydrostatic equilibrium was all we needed to get a rough estimate of the sun's core temperature! One might note, though, that the surface temperature is significantly lower than the core temperature, so we must assume that there is heat loss taking place in the sun. Today the luminosity of the sun is

$$L_{\odot} = 4 \times 10^{33} \text{ erg/s} \quad (3.24)$$

If we assume there is no energy source for the sun other than gravitational energy, we can come up with a timescale (called the **Kelvin-Helmholtz timescale**)

$$t_{\text{KH}} = \left| \frac{E_{\text{GR}}}{L} \right| \approx 3 \times 10^7 \text{ years} \quad (3.25)$$

for the sun. This has been known for awhile and since the Earth is known to have existed much longer than  $t_{\text{KH}}$ , scientists deduced that another energy source within the sun was needed to explain its longevity. We now know that this energy source is, of course, fusion. Note that at the center of the sun, the temperature of  $10^7$  K corresponds to an energy per particle of about 1 keV. The binding energy of helium is approximately 7MeV, approximately 7000 times bigger than the thermal content. Thus, the sun could last approximately 7000 times longer, bringing the projected lifetime of the sun up to a more reasonable (but still wrong) number of about 200 billion years. We conclude that nuclear energy is a more promising form of energy for the sun than chemical energy.

*Wednesday, January 18, 2012*

### 3.4 Gas Pressure and Radiation Pressure

Recall from the case of the constant density star that the gravitational energy is given by

$$E_{\text{GR}} = -\frac{3}{5} \frac{GM^2}{R} \quad (3.26)$$

And the average pressure is given by the virial theorem to be

$$\langle P \rangle = -\frac{1}{3} \frac{E_{\text{GR}}}{V} = (n_e + n_p)kT = 2n_p kT = \frac{2\rho kT}{m_p} \quad (3.27)$$

Here we've sort of assumed that the star is isothermal. This tells us that the average thermal energy is given by

$$kT = \frac{1}{10} \frac{GMm_p}{R} \quad (3.28)$$

Where the mass is given by

$$M = \rho \frac{4\pi}{3} R^3 \quad (3.29)$$

and the central temperature is given approximately by (scaled by solar units)

$$T_c \approx 2 \times 10^6 \text{ K} \left( \frac{\rho_c}{1 \text{ g cm}^{-3}} \right)^{1/3} \left( \frac{M}{M_\odot} \right)^{2/3} \quad (3.30)$$

These scalings are actually recovered in simulations (see MESA plot from class). Here we've only considered the case of the pressure due to an ideal gas, thus far ignoring the contributions from radiation pressure. We then want to know when radiation pressure becomes comparable to gas pressure. That is,

$$P_{\text{rad}} = \frac{1}{3} a T^4 \geq P_{\text{gas}} \quad (3.31)$$

The temperature in the star is approximately

$$kT \sim \frac{GMm_p}{R} \quad (3.32)$$

and the pressure gradient is, (again, very approximately)

$$\frac{dP}{dr} = -\rho g \approx \frac{P_c}{R} \sim \frac{M}{R^3} \frac{GM}{R^2} \quad (3.33)$$

Then the condition we are seeking is

$$\frac{1}{3} a \left( \frac{GMm_p}{Rk} \right)^4 \gtrsim \frac{GM^2}{R^4} \quad (3.34)$$

Interestingly,  $R$  cancels in (3.34), so this condition is dependent only on the mass of the star. Thus, we can get a hard limit that is independent of any other properties of the star. Dropping tons more constants, this gives

$$M^2 > \frac{k^4}{aG^3m_p^4} \quad (3.35)$$

Recall that the radiation constant is given by  $a = \frac{\pi^2}{15} \frac{k^4}{(\hbar c)^3}$ . Putting this in (3.35), we have

$$\frac{M^2}{m_p^2} > \frac{k^4 (\hbar c)^3}{G^3 m_p^6 k^4} \sim \left( \frac{\hbar c}{G m_p^2} \right)^3 \quad (3.36)$$

Then the limit on the mass is then

$$M > m_p \left( \frac{\hbar c}{G m_p^2} \right)^{3/2} \quad (3.37)$$

stars above this mass (approximately) have significant radiation pressure. Recall the fine structure constant

$$\alpha = \frac{1}{137} = \frac{e^2}{\hbar c} \quad (3.38)$$

Noting that the Coulomb energy is

$$E_{\text{Coulomb}} = \frac{e^2}{r} \quad (3.39)$$

Analogous to (3.39), we define a dimensionless measure of the strength of gravity, which appears in (3.37):

$$\alpha_G = \frac{G m_p^2}{\hbar c} \approx 6 \times 10^{-39} \quad (3.40)$$

Then the fundamental stellar mass given in (3.37) is

$$M > m_p \frac{1}{\alpha_G^{3/2}} \approx 2 M_\odot \quad (3.41)$$

After all that work... it turns out that the mass where radiation pressure *actually* starts to matter is closer to  $60 - 90 M_\odot$ .

Of interest in this case is that as  $P_{\text{rad}} \gg P_{\text{gas}}$ , then  $E_{\text{tot}} \rightarrow 0$  from the virial theorem. In this state, the star has enough energy to unbind itself, so radiation pressure sets an upper limit on the mass of a star.

### 3.5 Summary

Note that here we have used hydrostatic balance to find the central temperature as a function of mass and radius. Additionally we have realized that energy losses from the surface require the radius of a star to decrease and the core temperature to increase (at least until another energy source is present). We will later show that the main sequence is just that place where the power generated by nuclear reactions is equal to that released by the star so that the star need not contract. What we have *not* done yet is to derive the rate of heat loss from the star.

## 4 Heat Transfer in Stars

In studying the ways in which heat moves outward through a star, we will first be ignoring convection, though it is a powerful mechanism when it is available to the star. To move heat through a star, there are electrons, ions, and photons available. Recall (or perhaps you don't) **Fick's Law**, which tells us that the heat flux can be determined via

$$F = \text{ergs cm}^{-2} \text{s}^{-1} = -\frac{1}{3} v \ell \frac{d}{dx}(E) \quad (4.1)$$

where here  $E$  is the energy density. We'll start first with the energy density of an ideal gas of electrons, but before we do that, let's derive (4.1).

### 4.1 Heat Flux Derivation (not done in class)

Imagine a medium with a gradient in temperature across a surface membrane. Let's say "above" the membrane, the temperature is  $T_1$  and "below" the membrane, the temperature is  $T_2$ , with  $T_2 > T_1$ . Particles from region 2 then transport heat when they travel from region 2 to region 1. Let's call  $E$  the internal energy per unit volume as defined before, and then let's see what happens at the surface. Particles, on average, will be coming from a distance  $x + \ell$  above the membrane, where  $\ell = (\sigma n)^{-1}$  is the mean free path of the particles. Then the downward flow of energy is

$$F_{\text{down}} \approx \frac{1}{6} v E(x + \ell) \quad (4.2)$$

whereas particles from beneath move upward and carry heat from below at

$$F_{\text{up}} \approx \frac{1}{6} v E(x - \ell) \quad (4.3)$$

Think of the factor of  $1/6$  as being the portion of the flux moving through a particular face of a cube, in this case, the face pointing up or down. So the net flux in the positive  $\hat{x}$  direction is

$$F_x = -\frac{1}{6} v E(x + \ell) + \frac{1}{6} v E(x - \ell) \quad (4.4)$$

Writing the energy densities as linear functions,

$$E(x + \ell) = E(x) + \ell \frac{dE}{dx} \quad (4.5)$$

$$E(x - \ell) = E(x) - \ell \frac{dE}{dx} \quad (4.6)$$

so

$$F_x = -\frac{1}{3} v \ell \frac{dE}{dx} \quad (4.7)$$

### 4.2 Heat Transport by Electrons

For electrons, the energy density is

$$E = \frac{3}{2} k T n_e \quad (4.8)$$

Then the energy density gradient is

$$\frac{dE}{dx} = \frac{dE}{dT} \frac{dT}{dx} = \frac{3}{2} n k \frac{dT}{dx} \quad (4.9)$$

Then from Fick's Law, we have

$$F = -\frac{1}{3} v \ell \frac{3}{2} n k \frac{dT}{dx} = -\frac{1}{2} v \ell n k \frac{dT}{dx} \quad (4.10)$$

Where the mean free path is

$$\ell = \frac{1}{n \sigma} \quad (4.11)$$

for the scattering cross section  $\sigma$ . Then (4.10) becomes

$$F = - \left[ \frac{1}{2} v \frac{k}{\sigma} \right] \nabla T \quad (4.12)$$

From the theory of Coulomb scattering, the cross section would be

$$\sigma_{\text{Coulomb}} \sim b^2 \sim \frac{e^4}{(kT)^2} \quad (4.13)$$

Which gives a flux of

$$L = 4\pi R^2 F = 4\pi R \frac{(kT)^{7/2}}{m_e^{1/2} e^4} \quad (4.14)$$

For the sun, this would give

$$L \approx 5 \times 10^{31} \text{ erg s}^{-1} \quad (4.15)$$

which is two orders of magnitude too small, so we conclude that the sun does not transmit its heat via conduction through electrons. Now we'll move on to photons

### 4.3 Radiative Diffusion

For photons, the main scatterer will be electrons, so the cross section in question of the mean free path is the Thomson cross section. Additionally, the energy density is now  $E = aT^4$ . Additionally the speed of photons is obviously the speed of light. Then the ratio of the fluxes due to photons and electrons is

$$\frac{F_\gamma}{F_e} = \frac{c}{(kT/m_e)^{1/2}} \frac{e^4/(kT)^2}{\sigma_{\text{Th}}} \frac{E_{\text{rad}}}{E_{\text{gas}}} \quad (4.16)$$

Remember that the Thomson cross section is given by

$$\sigma_{\text{Th}} = \frac{8\pi}{3} \frac{e^4}{(m_e c^2)^2} \quad (4.17)$$

Then (4.16) becomes

$$\frac{F_\gamma}{F_e} = \left( \frac{m_e c^2}{kT} \right)^{1/2} \left( \frac{m_e c^2}{kT} \right)^2 \frac{P_{\text{rad}}}{P_{\text{gas}}} \quad (4.18)$$

Comparing the pressures gives

$$\frac{P_{\text{rad}}}{P_{\text{gas}}} \approx 10^{-4} \left( \frac{M}{M_\odot} \right)^2 \quad (4.19)$$

Then plugging (4.19) into (4.18), we see that heat transport by photons dominates heat transport by electrons whenever

$$M > 0.03 M_\odot \left( \frac{T}{10^7 \text{ K}} \right)^{5/4} \quad (4.20)$$

So if conduction ever dominates, it is in very low mass stars (also white dwarfs in their late lives). For our cases, photons are always going to be the dominant transport mechanism. We still haven't found out what the actual luminosity will be in the case of radiative diffusion. We do so now.

$$L = 4\pi R^2 F = 4\pi R^2 \frac{1}{3} c \ell \frac{d}{dr} (aT^4) \approx R^2 c \frac{1}{n_e \sigma_{\text{Th}}} \frac{1}{R} aT^4 \approx R^2 \frac{cm_p}{\rho \sigma_{\text{Th}}} \frac{1}{R} a \left( \frac{GMm_p}{Rk} \right)^4 \quad (4.21)$$



where we have noted that  $kT = \frac{GMm_p}{R}$ . Continuing on,

$$L \approx \frac{cm_p a (GMm_p)^4}{M \sigma_{\text{Th}} k^4} \propto M^3 \quad (4.22)$$

This relation is surprisingly accurate for stars with masses greater than a solar mass. Note that we have derived the stellar luminosity with *no knowledge* of the source of energy. The luminosity is set by the modes of heat transport available to the star.

Monday, January 23, 2012

#### 4.3.1 The Eddington Limit and the Eddington Standard Model

Continuing with heat transfer via radiation with electron scattering being the primary source of opacity, the flux is given via Fick's law as

$$F = \frac{1}{3} v \ell \frac{d}{dz} (aT^4) \quad (4.23)$$

where now  $v = c$  and  $\ell = (n_e \sigma_{\text{Th}})^{-1}$ . Plugging these in to (4.23), the flux becomes

$$F = \frac{1}{3} c \frac{1}{n_e \sigma_{\text{Th}}} \frac{d}{dz} (aT^4) = \frac{4}{3} \frac{acT^3}{\rho \kappa} \frac{dT}{dz} \quad (4.24)$$

where we've defined the **opacity** via

$$\kappa_{\text{es}} \equiv \frac{\sigma_{\text{Th}}}{m_p} \quad (4.25)$$

The opacity is measured in units of area per unit mass, indicating it is the cross-sectional area per unit mass. For electron scattering, it is simply a constant, but when other processes are relevant, it may depend on temperature and density. With the flux available, we can write the luminosity as

$$L(r) = F(r) 4\pi r^2 = -4\pi r^2 \frac{4}{3} \frac{acT^3}{\rho \kappa} \frac{dT}{dr} = -4\pi r^2 \frac{c}{\kappa \rho} \frac{d}{dr} P_{\text{rad}} \quad (4.26)$$

Noting the product  $\rho dr$  showing up in the denominator of (4.26), we are reminded of hydrostatic equilibrium:

$$\frac{dP}{dr} = -\rho(r)g(r) \quad \Rightarrow \quad dP = -\rho(r)g(r) dr \quad (4.27)$$

Putting in  $g(r)$  explicitly,

$$\frac{dP}{\rho(r)dr} = -\frac{Gm(r)}{r^2} \quad (4.28)$$

Bringing the column depth back in ( $y = \int \rho(r) dr$ ), this becomes

$$\frac{dP}{dy} = -\frac{Gm(r)}{r^2} \quad (4.29)$$

where we've used the fact that  $y = 0$  at the surface and increases *inwards* (hence the sign change). Assuming  $g$  is a constant, this would give our old result

$$\boxed{P = gy} \quad (4.30)$$

Now returning to (4.26):

$$\frac{dP_{\text{rad}}}{dy} = \frac{L(r) \kappa}{4\pi r^2 c} \quad (4.31)$$

and the result we just derived

$$\frac{dP}{dy} = \frac{Gm(r)}{r^2} \quad (4.32)$$

Taking the ratio of (4.31) and (4.32) gives us

$$\frac{dP}{dP_{\text{rad}}} = \frac{4\pi Gm(r)c}{\kappa L(r)} \quad (4.33)$$

Suppose  $M = m(R)$  = total mass of star and  $L = L(R)$  = luminosity of star

$$\frac{dP}{dP_{\text{rad}}} = \left[ \frac{4\pi GcM}{\kappa L} \right] \frac{m(r)}{M} \frac{L}{L(r)} \quad (4.34)$$

We have now introduced the **Eddington Luminosity**,

$$L_{\text{Edd}} = \frac{4\pi GcM}{\kappa} \quad (4.35)$$

In general, the Eddington Luminosity is much larger than the luminosity, since it is the luminosity where the only pressure gradient that matters is the radiation pressure (where the star becomes unstable). Evaluating the Eddington Luminosity in solar units and with electron scattering,

$$L_{\text{Edd}} = 3.1 \times 10^4 L_{\odot} \left( \frac{M}{M_{\odot}} \right) \quad (4.36)$$

Also recall how luminosity scales with mass:

$$L = L_{\odot} \left( \frac{M}{M_{\odot}} \right)^3 \quad (4.37)$$

So we see that for low-mass stars, their luminosities are indeed much smaller than the Eddington Luminosity. Now let's investigate the ratio of the luminosity to the Eddington Luminosity:

$$\frac{L}{L_{\text{Edd}}} \approx 3 \times 10^{-5} \left( \frac{M}{M_{\odot}} \right)^2 \quad (4.38)$$

So until  $M \geq 100 M_{\odot}$ , it will be the case that  $L \ll L_{\text{Edd}}$  and thus  $P_{\text{rad}} \ll P$ . Now returning to (4.34), we define a dimensionless quantity,  $\eta(r)$  by

$$\eta(r) \equiv \frac{L(r)}{L} \frac{M}{m(r)} \quad (4.39)$$

And (4.34) looks like

$$\frac{dP_{\text{rad}}}{dP} = \frac{L}{L_{\text{Edd}}} \eta(r) \quad (4.40)$$

Aside: this model of stars is called the “Eddington Standard Model” and was used to describe stars before their source of energy was known.

### 4.3.2 Polytropic Relations

Now integrating (4.40), we have

$$\int_R^r dP_{\text{rad}} = \frac{L}{L_{\text{Edd}}} \int_R^r \eta(r) dP \quad (4.41)$$

We can integrate the left side, leaving us with

$$P_{\text{rad}}(r) = \frac{L}{L_{\text{Edd}}} \int_R^r \eta(r) dP \quad (4.42)$$

Mathematically speaking, we are now stuck because we do not have any knowledge of  $\eta(r)$ . The formal approach of solving this problem is to define spatial averages of  $\eta(r)$  for different choices of mass and luminosity profiles. We will, however, just let  $\eta(r) \approx 1$ , which gives us

$$P_{\text{rad}} = \frac{L}{L_{\text{Edd}}} P_{\text{gas}}(r) \quad (4.43)$$

And now substituting in our equations of state for the radiation pressure and assuming that the radiation pressure is negligible compared to the gas pressure:

$$\frac{1}{3}aT^4 = \frac{L}{L_{\text{Edd}}} \frac{\rho kT}{\mu m_p} \quad (4.44)$$

Solving this for  $T^3$ , we have

$$T(r)^3 = \frac{L}{L_{\text{Edd}}} \frac{3k}{a\mu m_p} \rho(r) \quad (4.45)$$

So we have gone from a situation where we had only typical or central values to an actual equation that lets us find  $T(r)$ ,  $\rho(r)$ , and  $P(r)$ . It appears then, that in our low radiation pressure limit,  $\rho \propto T^3$ . Additionally, for the ideal gas law, we see that  $P \propto \rho^{4/3}$ . For some given  $L/L_{\text{Edd}}$ , we have a relation between pressure and density via

$$\frac{dP}{dr} = -\rho(r) \frac{Gm(r)}{r^2} \quad (4.46)$$

$$dm(r) = \rho 4\pi r^2 dr \quad (4.47)$$

Let us briefly consider an adiabatic change in an ideal gas. We recall from Freshman physics that  $PV^\gamma = \text{const}$ . For a monatomic, ideal gas,  $\gamma = 5/3$ , so we have  $P \propto \rho^{5/3}$  and  $\rho T \propto P$ , so  $T \propto \rho^{2/3}$ . Then we'll call the "entropy"  $T/\rho^{2/3}$ . For our star, we have  $T^3 \propto \rho \Rightarrow T \propto \rho^{1/3}$ . In our case, the "entropy" is then  $a/\rho^{1/3}$ . Thus the entropy is highest in the outer layers and smallest in the center.

### 4.3.3 Heat Transfer in the Outer Atmosphere

Near the surface of the star, photons naturally leave (which is why we see them!) We want to know what the place or condition is like. Near the surface, the density must decrease exponentially since  $T \ll T_c$  and thus  $g$  is a constant. Our previous arguments for the scale height and the plane-parallel isothermal atmosphere is largely applicable here, so the length scale of relevance is the scale height:

$$h = \frac{kT}{\mu m_p g} \quad (4.48)$$

It then makes sense that the condition for photons to escape would be for the mean free path to be comparable to the scale height:

$$\ell \sim h \quad \Rightarrow \quad \frac{m_p}{\rho \sigma_{\text{Th}}} \approx \frac{kT}{mg} \quad (4.49)$$

or

$$g \frac{m_p}{\sigma_{\text{Th}}} \sim \frac{\rho kT}{m_p} \sim P_{\text{gas}} \quad (4.50)$$

Again, this can be rewritten as

$$g \kappa^{-1} \sim P_{\text{gas}} \quad \text{where} \quad \kappa = \frac{\sigma_{\text{Th}}}{m_p} \quad (4.51)$$

Then we can say the condition where photons can escape is

$$P_{\text{gas}} \leq \frac{g}{\kappa} \quad (4.52)$$

This argument assumed a constant opacity. In general, we must do a line integral through the depth of the outer atmosphere. Imagine we ask the probability for a photon to reach some depth in the star. We define the **optical depth** to be

$$d\tau = \frac{dr}{\ell} = dr \, \kappa(r) \rho(r) \quad (4.53)$$

Then the probability would be

$$P(\tau) \propto e^{-\tau} \quad (4.54)$$

Then the optical depth at a certain radius would be given by

$$\tau = \int_R^r \kappa(r) \rho(r) dr \quad (4.55)$$

or, invoking hydrostatic equilibrium,

$$\tau = \int \kappa(r) \frac{dP(r)}{g} = \frac{1}{g} \int \kappa(r) dP(r) \quad (4.56)$$

Then the condition for photons to leave becomes when the optical depth is unity, or

$$\tau \sim 1 = \frac{1}{g} \kappa P(r) \quad \Rightarrow \quad P(\tau = 1) = \frac{g}{\kappa} \quad (4.57)$$

Additionally, we can redefine the flux equation in terms of optical depth into a cute form:

$$F = \frac{c}{3} \frac{d}{d\tau} (aT^4) \quad (4.58)$$

#### 4.3.4 The Effective Temperature

At the surface, the flux is given by

$$F = \frac{L}{4\pi R^2} \equiv \sigma_{\text{SB}} T_{\text{eff}}^4 \quad (4.59)$$

Assuming that the flux is constant at or near the surface, we may use the fact that  $a = 4\sigma_{\text{SB}}/c$  and (4.58) to get

$$\frac{d}{d\tau} T^4 = \frac{3}{4} T_{\text{eff}}^4 \quad (4.60)$$

Recall that  $\tau$  is dimensionless, so there is no problem here. For  $\tau \gg 1$ , we have

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \tau \quad (4.61)$$

This implies that as  $\tau$  increases (going deeper and deeper into the star), the radiation field becomes more and more isotropic. Naively, we might assume that the flux would be given by the energy density multiplied by the speed of light, but our result in (4.61) essentially tells us that  $F = acT^4/\tau$ .

*Wednesday, January 25, 2012*

### 4.4 Convection

Another important form of heat transport in stars is that of convection, which is where an instability causes the bulk movement of material (rather than the diffusion of photons or electrons) to transport heat throughout the star. Convection occurs only when the temperature gradient is very steep. Otherwise, radiative diffusion dominates heat transport.

The origin of the instability is that a fluid element that rises adiabatically (faster than the heat transfer time scale) is **lighter** than the surrounding fluid and so it “runs away”. This is a linear instability, in that a displacement exponentially grows in time.

Suppose we have a fluid element in hydrostatic balance. We imagine displacing this fluid element from  $r$  to  $r + dr$ . So we assume that the displaced element responds adiabatically. Thermodynamics tells us that

$$T dS = dE + P dV = 0 \quad (4.62)$$

for an adiabatic process (here these quantities are related to the inside of the fluid element). This is equivalent to the requirement that the timescale of response is much less than that for heat to enter or leave the fluid element. We presume that the timescale is longer than the time for pressure equilibrium as well:

$$t = \frac{h}{c_s} = \text{time for the pressure to equilibrate} \Rightarrow v \ll c_s \quad (4.63)$$

where  $h$  is again the scale height. This requirement ensures that the fluid element remains in pressure equilibrium with its surroundings at all times. Suppose the element starts in location 1 and travels

to location 2. This adiabatic process requires that  $PV^\gamma$  be a constant within the bubble, which means that

$$\frac{P_1}{\rho_1^\gamma} = \frac{P_2}{\rho_2^\gamma} \quad (4.64)$$

where the subscripts refer to where the bubble is located (and the quantities are measured within the bubble, not the surrounding material). Rearranging (4.64) gives us the new density

$$\rho_{2,b}^\gamma = \rho_1^\gamma \frac{P_2}{P_1} \quad (4.65)$$

Clearly the bubble is now less dense than it was at its starting position, but how dense is it compared to the surrounding material? For *stability*, we require  $\rho_{2,b} > \rho_{2,*}$ , where the \* indicates the density of the star's "background" conditions. For the star, we require

$$\rho_1 \left( \frac{P_2}{P_1} \right)^{1/\gamma} > \rho_{2,*} \quad (4.66)$$

Note that  $\rho_1$ ,  $P_1$ , and  $P_2$  are the same for both the fluid element and the background. For an infinitesimal change, the pressures are related via

$$P_2 = P_1 + \Delta r \frac{dP}{dr} \quad (4.67)$$

and likewise the densities,

$$\rho_2 = \rho_1 + \frac{d\rho}{dr} \Delta r \quad (4.68)$$

(where all quantities are for the star, not the fluid element, as they shall be from now on unless otherwise noted). Then (4.66) requires

$$\frac{1}{\gamma} \frac{d \ln P}{d \ln r} > \frac{d \ln \rho}{dr} \quad (4.69)$$

Now, we recall that

$$P = \frac{\rho k T}{\mu m_p} \Rightarrow d \ln \rho = d \ln P - d \ln T \quad (4.70)$$

(assuming uniform composition) and thus (4.69) can be written as

$$\frac{1}{\gamma} \frac{d \ln P}{dr} > \frac{d \ln P}{dr} - \frac{d \ln T}{dr} \quad (4.71)$$

For an ideal, monatomic gas, this becomes

$$\frac{d \ln T}{d \ln P} < 1 - \frac{1}{\gamma} = \frac{2}{5} \quad (4.72)$$

Recall that earlier we found that for radiative heat transport in the Eddington standard model,  $\rho \propto T^3$ , and thus  $P \propto \rho T \propto T^4$ , so  $T \propto P^{1/4}$ . Then computing the logarithmic derivative in (4.72),

$$\frac{d \ln T}{d \ln P} = \frac{1}{4} < \frac{2}{5} \quad (4.73)$$

Note that we could also include a gradient in the composition if we wanted. (4.73) would become, more genrally,

$$\boxed{\left(\frac{1}{\gamma} - 1\right) \frac{d \ln P}{dr} - \frac{d \ln \mu}{dr} > -\frac{d \ln T}{dr}} \quad (4.74)$$

This actually weakens the requirement on the temperature gradient. §6.3 of Keppenahn and Weigert discuss the strange consequences of being in the “twilight zone” between the conditions set up by (4.74) and (4.72). It is possible, in this regime, to have a fluid element oscillate (as in a stable case), but have the oscillation grow in amplitude (not exactly what we would think of as stable).

Barring these strange situations, stars that follow (4.74) and certainly those following (4.72) are stable and do not transport heat by convection. A model that is stable to convection has the entropy increasing with radius. Note that when we say “entropy”, we mean the **specific entropy**, or the entropy per unit mass.

To put things in context, we recall the Eddington standard model, where we found that  $\rho \propto T^3$ , so  $P \propto T^4$ . Then we have  $d \ln T / d \ln P = 1/4 < 2/5$ . So then the Eddington standard model star is stable to convection and thus transports heat primarily by radiative diffusion.

#### 4.4.1 The Unstable Case

Now we wish to understand what happens when the background model is unstable. The density in the bubble is

$$\rho_{2,b} = \rho_1 \left( \frac{P_2}{P_1} \right)^{1/\gamma} \quad (4.75)$$

and for the star,

$$\rho_{2,*} = \rho_1 + \Delta r \left. \frac{d\rho}{dr} \right|_* \quad (4.76)$$

The density contrast is then

$$\Delta\rho = \rho_{2,*} - \rho_{2,b} = \Delta r \left[ \left. \frac{d\rho}{dr} \right|_* - \frac{1}{\gamma} \frac{\rho}{P} \left. \frac{dP}{dr} \right|_* \right] \quad (4.77)$$

or

$$\Delta\rho = \rho \Delta r \left[ \frac{d \ln \rho}{dr} + \frac{\rho g}{P \gamma} \right] \quad (4.78)$$

where we have invoked hydrostatic equilibrium. Stability requires  $\Delta\rho < 0$ , or

$$\boxed{\frac{d \ln \rho}{dr} < -\frac{\rho g}{\gamma P} \quad (\text{Stability})} \quad (4.79)$$

This is the same result we obtained previously, just phrased a bit differently. In the unstable case, though, we require  $\Delta\rho > 0$ . the acceleration,  $a$ , or the displaced element will be given by

$$a = \frac{\Delta\rho}{\rho} g = g \Delta r \left[ \frac{d \ln \rho}{dr} + \frac{\rho g}{P \gamma} \right] \quad (4.80)$$

This is the equation of motion for the fluid element. Written in terms of the displacement coordinate  $x$ , (4.80) is

$$\ddot{x} = gx \left( \frac{d \ln \rho}{dr} + \frac{\rho g}{P \gamma} \right) \quad (4.81)$$

Strictly speaking this equation holds for unstable or stable cases. Clearly this is a harmonic oscillator, so the physics should be familiar. In any case, we have

$$-\omega^2 = g \left( \frac{d \ln \rho}{dr} + \frac{\rho g}{P \gamma} \right) \quad (4.82)$$

If  $-\omega^2 < 0$ , then the solution is stable, and the element oscillates at the Brunt Väisälä frequency is

$$N^2 = -g \left[ \frac{d \ln \rho}{dr} + \frac{\rho g}{\gamma P} \right] \quad (4.83)$$

This is the local frequency of oscillation of a fluid element in a convectively stable atmosphere. In the Earth's atmosphere, this frequency is around five or ten minutes.

Now if this coefficient is negative, the solution is unstable (i.e., grows exponentially). Note that  $N^2 \sim g/h$ , so

$$N^2 \sim \frac{gm_p g}{kT} \sim g^2 \frac{1}{v_{\text{th}}^2} \Rightarrow N \sim \frac{g}{v_{\text{th}}} \sim \frac{g}{c_s} \quad (4.84)$$

So our displacement solution looks like

$$x = x_0 e^{t/\tau} \quad (4.85)$$

where  $1/\tau^2 = -N^2$ . So the speed is

$$\dot{x} = \frac{x_0}{\tau} e^{t/\tau} = \ell/\tau = \text{speed after moving a distance } \ell \quad (4.86)$$

Generally, the speed is given by

$$v = \ell \sqrt{-N^2} \quad (4.87)$$

Suppose that the stellar interior is strongly unstable, so that  $N^2 \sim -g/h$ . Then the velocity after traveling some length  $\ell$  is

$$v = \ell \left( \frac{g}{h} \right)^{1/2} = \ell \left( \frac{gm_p g}{kT} \right)^{1/2} = \ell \frac{g}{v_{\text{th}}} \approx \frac{g}{v_{\text{th}}} h = \frac{g}{v_{\text{th}}} \frac{v_{\text{th}}^2}{g} = v_{\text{th}} \quad (4.88)$$

So if  $N^2 \sim -g/h$ , i.e., the star is strongly unstable, then a runaway fluid element will reach the sound speed after traversing one scale height! Recall, though that the change in density is

$$\left. \frac{\Delta \rho}{\rho} \right|_{\text{at } \ell} \approx \ell \left( \frac{d \ln \rho}{dr} + \frac{\rho g}{P \gamma} \right) \quad (4.89)$$

and thus the velocity is

$$v = \ell \sqrt{\frac{g}{\ell} \left( \frac{\Delta \rho}{\rho} \right) \Big|_{\ell}} = (g\ell)^{1/2} \left( \frac{\Delta \rho}{\rho} \Big|_{\ell} \right)^{1/2} \quad (4.90)$$

*Monday, January 30, 2012*



#### 4.4.2 Convective Efficiency

We left off last time with a rough relation between the velocity of an upwardly rising fluid element and the density contrast between the fluid element and the background material at some height  $\ell$  above the element's original location. In the unstable case, we found that the element was less dense than the surrounding fluid, by examining the quantity  $\Delta\rho/\rho|_\ell$ . The velocity at which the element traveled was given by  $v = (g\ell)^{1/2}(\Delta\rho/\rho|_\ell)^{1/2}$ .

Inserting the scale height in for  $\ell$ ,

$$\ell = h = \frac{kT}{mg} = \frac{c_s^2}{g} \quad (4.91)$$

Then the velocity would be

$$v = c_s \left[ \frac{\Delta\rho}{\rho} \Big|_h \right]^{1/2} \quad (4.92)$$

Now we consider the flux due to convection. The flux is just the velocity times the energy density, and so we can write

$$F = v\rho v^2 = \rho v^3 \quad (4.93)$$

Assuming  $\Delta\rho/\rho \ll 1$ , we may assume that the element is moving subsonically.

The business of heat flux via convection is covered in **Mixing Length Theory**. In this “theory”, we imagine a convective element traveling up a **mixing height**,  $\ell$  and then allowing the fluid element to equilibrate with the surroundings. Since the preferred lengthscale in a star is the scale height, we typically choose  $\ell_{\text{MT}} = \alpha h$  for some dimensionless constant  $\alpha$ . Applying this to (4.93),

$$F_{\text{conv}} = \rho \frac{kT}{m_p} c_s \left( \frac{\Delta\rho}{\rho} \Big|_{\ell=h} \right)^{3/2} = P c_s \left( \frac{\Delta\rho}{\rho} \Big|_{\ell=h} \right)^{3/2} \quad (4.94)$$

Now recall our result for the radiative flux:

$$F_{\text{rad}} = \frac{1}{3} \frac{c}{\kappa\rho} \frac{d}{dr} (aT^4) \approx \frac{cP_{\text{rad}}}{\tau} \quad (4.95)$$

where in the second equality, we're let  $d/dr \rightarrow 1/h$ . Now we find the ratio in these two fluxes:

$$\frac{F_{\text{conv}}}{F_{\text{rad}}} \sim \frac{\tau P c_s (\Delta\rho/\rho|_h)^{3/2}}{c P_{\text{rad}}} \sim \frac{c_s}{c} \frac{P}{P_{\text{rad}}} \tau \left( \frac{\Delta\rho}{\rho} \Big|_h \right)^{3/2} \quad (4.96)$$

For typical, low-mass stars, we assume that  $P/P_{\text{rad}} \sim 10^4 \gg 1$ , let's find when both fluxes give equal contributions (also plugging in some solar values):

$$\left( \frac{kT}{m_p c^2} \right)^{1/2} 10^4 \tau \left( \frac{\Delta\rho}{\rho} \right)^{3/2} \sim 1 \quad (4.97)$$

The requirement becomes

$$\frac{\Delta\rho}{\rho} \approx 2 \left( \frac{10^4}{T} \right)^{1/3} \frac{1}{\tau^{2/3}} \quad (4.98)$$

Where  $\tau$  and  $T$  get smaller, the simple theory implies convection at near the sound speed,  $c_s$ . In truth, you can't do this problem correctly in one dimension, so this result isn't all that good. What does this imply about the surface, though? Stellar convection at or near the surface becomes sonic, which we say is **inefficient** and  $\Delta\rho/\rho \rightarrow 1$ .

What I'm trying to do here is to write a density model in 1-D that doesn't make me blush.  
*–Lars Bildsten*

To find a situation where this *does* work is deep in the interior of the star, where  $\tau \gg 1$  (yes, three greater than signs). The relevant lengthscale here is no longer the scale height, which begins to approach the radial coordinate:

$$h = \frac{P}{\rho g} \sim \frac{GM^2}{R^4 M} \frac{R^3 R^2}{GM} \sim R \quad (4.99)$$

so we'll just use the radius of the star as our characteristic length scale. The ratio of fluxes is now

$$\frac{F_{\text{conv}}}{F_{\text{rad}}} \sim \frac{c_s}{R} \left[ \frac{P}{P_{\text{rad}}} \tau \frac{R}{c} \right] \left( \frac{\Delta\rho}{\rho} \bigg|_R \right)^{3/2} \quad (4.100)$$

Note that the time it takes for a photon to diffuse from a star is  $t_{\text{rw}} = \tau R/c$ . Thus, that is the time it takes to evacuate the radiation field of energy, so multiplying by the ratio in pressures is actually the Kelvin-Helmholtz time. Using this fact, (4.100) can be written as

$$\frac{F_{\text{conv}}}{F_{\text{rad}}} = \frac{t_{\text{KH}}}{t_{\text{dyn}}} \left( \frac{\Delta\rho}{\rho} \bigg|_R \right)^{3/2} \quad (4.101)$$

Deep in the star,  $t_{\text{KH}} \gg t_{\text{dyn}}$ , so convection is efficient when

$$\frac{\Delta\rho}{\rho} \bigg|_R \sim \left( \frac{t_{\text{dyn}}}{t_{\text{KH}}} \right) \ll 1 \quad (4.102)$$

When convection is efficient, then, the stellar model nearly follows the adiabatic relation! Deep interior convection, in this limit, only implies that we need to know the adiabatic relation very well. The punchline here is that in the deep interior of a star, if the temperature gradient is even slightly super-adiabatic, convection becomes an incredibly powerful form of heat transport.

Note that the ratio of times scales like

$$\frac{t_{\text{dyn}}}{t_{\text{KH}}} \approx 3 \times 10^{-9} \left( \frac{M}{10 M_{\odot}} \right)^{3.5} \quad (4.103)$$

Noting that  $L \propto M^{3.5}$  and  $R = R_{\odot}(M/M_{\odot})$  on the main sequence, this gives us

$$\frac{v}{c_s} \approx 10^{-3} \left( \frac{M}{10 M_{\odot}} \right)^{7/6} \quad (4.104)$$

This relatively slow speed reinforces our assumption that convective bubbles move slow enough to maintain pressure equilibrium with their surroundings. Now remember the adiabatic condition for convection derived earlier:

$$\left. \frac{d \ln T}{d \ln P} \right|_* = \left. \frac{d \ln T}{d \ln P} \right|_{\text{adiabatic}} = \frac{2}{5} \quad (4.105)$$

So in the convective zone, we must have  $T \propto P^{2/5} \propto (\rho T)^{2/5}$ , or more directly,  $T \propto \rho^{2/3}$ . So far, we have motivated *two* distinct **polytropic relations** (models where  $P \propto \rho^{(n+1)/n}$  for some  $n$ ):

1. **Fully Convective and Efficient:**  $P \propto \rho^{5/3}$ , where the prefactor is the specific entropy of the star, which is spatially constant. Thus if we have the mass of a star and the entropy, we can calculate everything we want to know about this star (in this simple model).
2. **Constant  $L/M$  star** with constant (electron scattering) opacity, where  $P \propto \rho^{4/3}$ .

## 4.5 Fully Convective Star

Imagine a star of ideal gas *and* from the core to the photosphere is fully convective *and* efficient (this will likely be a bad approximation near the surface). Somewhere near the surface, photons must be made to allow the energy to escape the star. Near the photosphere, the pressure is

$$P_{\text{ph}} \approx \frac{g}{\kappa_{\text{ph}}} \quad (4.106)$$

If we were to plot  $\ln T$  against  $\ln P$  (as we did in lecture), for this star, it would have a slope of  $2/5$ . Getting out towards the surface, though, the curve would have to be a bit shallower. We hope this only happens out towards one scale height in from the surface. Incorporating this physics will change  $R$ , but only on the order of  $h|_{\text{location}}$ . Noting that

$$\frac{h}{R} \sim \frac{T}{T_c} \quad (4.107)$$

we see this is an extremely small error, since  $T/T_c$  is very small out towards the surface. We assume  $P \propto \rho^{5/3}$  (an  $n = 3/2$  polytrope). In this model, the central pressure is

$$P_c = 0.77 \frac{GM^2}{R^4} \quad (4.108)$$

The central temperature is

$$T_c = 0.54 \frac{GM\mu m_p}{kR} \quad (4.109)$$

Requiring the entropy be the same everywhere requires

$$\frac{T_c}{P_c^{2/5}} = \frac{T_{\text{ph}}}{P_{\text{ph}}^{2/5}} = \frac{T_{\text{eff}}}{P_{\text{ph}}^{2/5}} \quad (4.110)$$

Since we've assumed the perfect adiabat (admittedly a lie... also,  $\mu$  is changing due to varying ionization conditions in the outer layers of the star). The surface thermal energy is

$$kT_{\text{eff}} = 0.6 \left( \frac{GM\mu m_p}{R} \right) \left( \frac{R^2}{M\kappa_{\text{ph}}} \right)^{2/5} \quad (4.111)$$

Putting in  $\kappa_{\text{ph}} = \kappa_{\text{es}}$  gives us

$$T_{\text{eff}} = 200 \text{ K} \left( \frac{M}{M_{\odot}} \right)^{3/5} \left( \frac{R_{\odot}}{R} \right)^{1/5} \quad (4.112)$$

So we have a big problem with assuming electron scattering (or convection) since this is obviously too low for a star like the sun.

*Wednesday, February 1, 2012*

## 4.6 The Hayashi Track

It turns out that in this case, the outer body sets the temperature. Electron scattering is *not* the primary source of opacity. Typically  $\text{H}^-$  opacity is much more important. This is where a very loosely bound ion (ionization energy around 0.75 eV) is “ionized” by a photon (really, it’s un-ionized) in the reaction



Hansen, Kawaler, and Trimble give the value of this opacity as

$$\kappa_{\text{H}^-} = 2.5 \times 10^{-31} \rho^{1/2} T^9 \text{ cm}^2 \text{ g}^{-1} \quad (4.114)$$

Now with the pressure given by

$$P = \frac{\rho k T}{\mu m_p} \quad (4.115)$$

and the density at the photosphere given by

$$\rho = \frac{g \mu m_p}{\kappa k T} \quad (4.116)$$

Combining (4.115) and (4.116) gives the density as

$$\rho = \frac{g \mu m_p}{k T \kappa_0 \rho^{1/2} T^9} \quad (4.117)$$

Or we could have

$$\rho^{3/2} = \frac{g \mu m_p}{k T^{10} \kappa_0} \Rightarrow \rho = \left( \frac{g \mu m_p}{k T^{10} \kappa_0} \right)^{2/3} \quad (4.118)$$

Comparing these two values for the densities gives an opacity of

$$\kappa_{\text{H}^-} = \kappa_0 T^9 \left( \frac{g \mu m_p}{k T^{10} \kappa_0} \right)^{1/3} \quad (4.119)$$

or more succinctly,

$$\boxed{\kappa_{\text{H}^-} = \kappa_0^{2/3} T^{17/3} \left( \frac{g \mu m_p}{k} \right)^{1/3}} \quad (4.120)$$

Using this opacity in (4.111), we obtain (omitting some algebra), the more reasonable result of

$$T_{\text{eff}} \approx 2500 \text{ K} \left( \frac{M}{M_{\odot}} \right)^{1/7} \left( \frac{R}{R_{\odot}} \right)^{1/49} \quad (4.121)$$

So the luminosity goes as

$$\frac{L}{L_{\odot}} = 0.034 \left( \frac{M}{M_{\odot}} \right)^{4/7} \left( \frac{R}{R_{\odot}} \right)^{102/49} \quad (4.122)$$

Writing the effective temperature as a function of the mass and the luminosity gives us

$$T_{\text{eff}} \approx 2600 \text{ K} \left( \frac{L}{L_{\odot}} \right)^{1/102} \left( \frac{M}{M_{\odot}} \right)^{7/51} \quad (4.123)$$

Note that this is nearly independent of the luminosity, so it is nearly a straight vertical line on the HR diagram. As a molecular cloud contracts, heat transport is dominated by convection and the opacity in the photosphere is dominated by  $\text{H}^-$  opacity. This explains how we see stars descending a vertical line in the HR diagram called the **Hayashi Track** that corresponds to (4.123). Note, however, that there is a lower limit on the effective temperature below which no hydrostatic radiating solutions exist because it is impossible to have the same entropy in the photosphere as in the core (recall that that was a requirement for convection to take place).

The contraction of a protostar and its corresponding trip down the Hayashi Track is rather quick at the high-luminosity end but changes as the star contracts. So the initial evolution is down the Hayashi Track until a radiative solution exists, after which point the star evolves along a nearly constant luminosity track (the less-famous Henyey Track).

## 5 Star Formation and the Jean's Mass

After discussing a star's trip down the Hayashi track, we take a moment to investigate a bit more about the physics of star formation, which is still a field with many unanswered questions.

### 5.1 The Jean's Mass

In the interstellar medium, there are large masses of cold dust and gas that can give rise to isolated regions of stellar formation. Often these manifestations are called **cold cores**. Suppose one of these cold cores is in the ISM with some density  $\rho$  and temperature  $T$  (likely around 10-20 K). The gravitational energy is something like

$$E_{\text{GR}} \approx -\frac{GM^2}{R} \quad (5.1)$$

And the thermal content is something like

$$E_{\text{th}} = \frac{3}{2} \frac{M}{m_p} kT \quad (5.2)$$

In order for this cloud to collapse, we must have the total energy being less than zero. Thus, we must have

$$\frac{GM^2}{R} > \frac{3}{2} \frac{M}{m_p} kT \quad (5.3)$$

If we assume a constant density profile, this tells us that the mass of the object must be

$$M > 500 M_{\odot} \left( \frac{T}{10 \text{ K}} \right)^{3/2} \left( \frac{1 \text{ cm}^{-3}}{n} \right)^{1/2} \quad (5.4)$$

This mass is called the **Jean's Mass**. If a cloud is more massive than this, it is possible to collapse. Now, it may stay as one large blob, or it may fragment... we're not sure yet. The question is then, "How does the Jean's mass change in a collapsing cloud?" We know that the Jeans mass scales as

$$M_J \propto T^{3/2} \rho^{-1/2} \quad (5.5)$$

As the star collapses the density must increase. If the material maintains its entropy, then we must have  $T \propto \rho^{2/3}$ . Then the Jean's mass scales as

$$M_J \propto T^{3/2} \rho^{-1/2} \propto \rho^{1/2} \quad (5.6)$$

So if the collapse is adiabatic, the collapse does not lead to fragmentation. The cloud will reach some critical density, at which point the Jean's mass is higher than the mass of the cloud, halting collapse.

Cooling of the gas (i.e., the temperature remains constant) gives the Jean's mass trivially scaling as

$$M_J \propto \frac{1}{\rho^{1/2}} \quad (5.7)$$

This allows for the possibility of fragmentation since the mass will remain above the Jean's mass. More likely, this will be the case at earlier times, and then later the collapse becomes more adiabatic until a freeze-out. What ends the fragmentation is that at high density, the optical depths are increasing and the gas cannot radiate on the contraction timescale. If we can analyze the microphysics governing these processes, we could determine a minimum Jean's mass, which would explain the fragmentation. At low metal content, the minimum masses are much higher. This is thought to be the reason why the first stars were so large.

When cooling is efficient, then the collapse and fragmentation occurs on the dynamical timescale:

$$t_{\text{dyn}} \approx \frac{1}{\sqrt{G\rho}} = \frac{10^7 \text{ years}}{(n/100 \text{ cm}^{-3})^{1/2}} \quad (5.8)$$

Then it typically takes 1–10 Myrs for the collapse to occur. At the center, an object at hydrostatic balance forms while the outer layers are still collapsing (the collapse is not homologous). This causes an **accretion luminosity** of

$$L \approx \dot{M} \frac{GM_c}{R_c} \quad (5.9)$$

which is powered by a loss of gravitational potential energy. The shocks on the core surface typically have energies of

$$kT_{\text{shock}} \approx \frac{GM_c m_p}{R_c} \approx 0.1 \text{ keV} \quad (5.10)$$

## 5.2 Pre-Main Sequence Stars

Recall the main equations of stellar structure:

$$\frac{dP}{dr} = -\rho(r)g(r) \quad (5.11)$$

$$dm(r) = 4\pi r^2 \rho(r) dr \quad (5.12)$$

Fluxes defined by temperature gradient

We need to be able to have an equation describing energy (the next moment, as it were). Looking at the second law of thermodynamics, we have

$$dQ = TdS = dE + PdV \quad (5.13)$$

The corresponding time-dependent equation in the Lagrangian is

$$T \frac{dS}{dt} = \frac{dE}{dt} + P \frac{dV}{dt} \quad (5.14)$$

In terms of the specific units (per unit mass), where now  $E$  is the specific internal energy,

$$dQ = dE + Pd\left(\frac{1}{\rho}\right) = dE - \frac{P}{\rho^2}d\rho \quad (5.15)$$

The Lagrangian equation becomes

$$T \frac{ds}{dt} = \frac{dQ}{dt} \equiv \text{loss or gain} \quad (5.16)$$

First we have heat gained via nuclear reactions,  $\varepsilon$ . Secondly, there is heat gained or lost due to a gradient in  $\mathbf{F}$ . Then (5.16) becomes our last equation for stellar evolution:

$$\boxed{T \frac{ds}{dt} = \varepsilon_{\text{nuc}} - \frac{\nabla \cdot \mathbf{F}}{\rho}} \quad (5.17)$$

Here we have  $\mathbf{F} = F_r \hat{r}$ . Then

$$\frac{\nabla \cdot \mathbf{F}}{\rho} = \frac{1}{\rho} \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) \quad (5.18)$$

and the luminosity is

$$L(r) = 4\pi r^2 F_r \quad (5.19)$$

Then we can rewrite (5.18) as

$$\frac{\nabla \cdot \mathbf{F}}{\rho} = \frac{1}{\rho 4\pi r^2} \frac{\partial}{\partial r} L(r) = \frac{dL(r)}{dm(r)} \quad (5.20)$$

So in the case of no nuclear burning (a pre-main sequence star), we have

$$T \frac{ds}{dt} = - \frac{dL(r)}{dm(r)} \quad (5.21)$$

We assume that the luminosity gradient is positive due to the temperature gradient, so under the absence of an internal energy source, the entropy will decrease. It will decrease at the rate set by the heat loss to infinity. Adding on the rest to (5.21),

$$T \frac{ds}{dt} = -\frac{dL(r)}{dm(r)} = \frac{dE}{dt} - \frac{P}{\rho^2} \frac{d\rho}{dt} \quad (5.22)$$

Note that the internal energy is

$$E = \frac{3}{2} \frac{kT}{\mu m_p} = \frac{3}{2} \frac{P}{\rho} \quad (5.23)$$

Inserting this into (5.22) gives us

$$T \frac{ds}{dt} = \frac{3}{2} \frac{1}{\rho} \frac{dP}{dt} - \frac{5}{2} \frac{P}{\rho^2} \frac{d\rho}{dt} \quad (5.24)$$

$$= \frac{P}{\rho} \left[ \frac{3}{2} \frac{d \ln P}{dt} - \frac{5}{2} \frac{d \ln \rho}{dt} \right] \quad (5.25)$$

$$= \frac{3}{2} \frac{P}{\rho} \left[ \frac{d \ln P}{dt} - \frac{5}{3} \frac{d \ln \rho}{dt} \right] \quad (5.26)$$

$$= \frac{3}{2} \frac{P}{\rho} \frac{d}{dt} \ln \left( P / \rho^{5/3} \right) = -\varepsilon_{\text{grav}} \quad (5.27)$$

Textbooks will often refer to this as  $-\varepsilon_{\text{grav}}$ . This is done so that

$$\frac{\partial L(r)}{\partial m(r)} = \varepsilon_{\text{nuc}} + \varepsilon_{\text{grav}} \quad (5.28)$$

*Monday, February 6, 2012*

Let's do some dimensional analysis for a contracting star. The pressure is

$$P \sim \frac{GM^2}{R^4} \sim \frac{GM^2}{M^{4/3}} \rho^{4/3} \quad (5.29)$$

Then the quantity in the log in (5.27) would be

$$\frac{P}{\rho^{5/3}} \approx C \frac{1}{\rho^{1/3}} \quad (5.30)$$

If the star is fully convective, then  $P/\rho^{5/3}$  must be spatially constant, and we must have

$$T \frac{ds}{dt} = \frac{3}{2} \frac{P}{\rho} \frac{d}{dt} \left[ \ln \left( C \rho_c^{-1/3} \right) \right] = -\frac{1}{2} \frac{P}{\rho} \frac{d \ln \rho_c}{dt} \quad (5.31)$$

Assuming that  $\varepsilon_{\text{nuc}} = 0$ , (5.28) becomes

$$-\frac{1}{2} \frac{P}{\rho} \frac{d \ln \rho_c}{dt} = -\frac{dL(r)}{dm(r)} \Rightarrow \frac{dL(r)}{dm(r)} = \frac{1}{2} \frac{P}{\rho} \frac{d \ln \rho_c}{dt} \quad (5.32)$$



where  $\rho_c \propto M/R^3$ . Now we want to get (5.32) in terms of more physical variables. Assuming  $M$  is constant in time,

$$\frac{dL(r)}{dm(r)} = \frac{1}{2} \frac{P}{\rho} \frac{d}{dt} \left[ \ln \left( \frac{1}{R^3} \right) \right] \Rightarrow \frac{dL(r)}{dm(r)} = -\frac{3}{2} \frac{P}{\rho} \frac{d}{dt} \ln R \quad (5.33)$$

So, assuming a fully-convective star with no mass loss, we have

$$\boxed{\frac{dL(r)}{dm(r)} = -\frac{3}{2} \frac{P(r)}{\rho(r)} \frac{1}{R} \frac{dR}{dt}} \quad (5.34)$$

Backtracking a bit, earlier we showed that

$$\text{entropy} \propto \frac{P}{\rho^{5/3}} \sim \frac{GM^2}{R^4} \frac{R^5}{M^{5/3}} \propto RM^{1/3} \quad (5.35)$$

for a star in hydrostatic balance. An adiabatic adjustment to this star must require  $R \propto 1/M^{1/3}$  (entropy must remain constant). Thus if the mass decreases, the radius must increase (like in white dwarfs).

Now we want to integrate (5.34). Multiplying both sides by  $dm/dr$  and integrating from the center to the surface gives

$$L = -\frac{3}{2} \frac{1}{R} \frac{dR}{dt} \int_0^R \frac{P(r)}{\rho(r)} \rho(r) 4\pi r^2 dr = -\frac{3}{2} \frac{1}{R} \frac{dR}{dt} \int_0^R P(r) 4\pi r^2 dr \quad (5.36)$$

We know the last integral from the Virial theorem's relation to the gravitational energy. The short answer is that

$$\int d^3r P = \frac{2}{7} \frac{GM^2}{R} \quad (5.37)$$

Then the luminosity is given by

$$L = -\frac{3}{7} \frac{GM^2}{R^2} \frac{dR}{dt} \quad (5.38)$$

This is essentially Kelvin-Helmholtz contraction, but for a star that is fully convective, we can derive the “real equation” (as opposed to our more bogus results earlier).

Note that for stars that are not changing much on the Kelvin-Helmholtz timescale, we may neglect  $T \frac{ds}{dt}$  and simply write

$$\varepsilon_{\text{nuc}} = \frac{\partial L_r}{\partial m_r} \quad (5.39)$$

This is a good approximation for stars on the main sequence, where the burning time is much longer than the Kelvin-Helmholtz time.

### 5.3 Low Mass Stars

Earlier we derived that as stars are on the Hayashi track,

$$\frac{L}{L_\odot} \approx 0.03 \left( \frac{M}{M_\odot} \right)^{4/7} \left( \frac{R}{R_\odot} \right)^2 \quad (5.40)$$

Plugging this result in to our previous result can give us the radius as a function of time:

$$\frac{3}{7} \frac{GM^2}{R^2} \frac{dR}{dt} = -0.03 L_{\odot} \left( \frac{M}{M_{\odot}} \right)^{4/7} \left( \frac{R}{R_{\odot}} \right)^2 \quad (5.41)$$

We can now read off scalings:  $\dot{R} \propto R^4$ ,  $1/t \propto R^3$ , and so  $R \propto t^{-1/3}$ . Explicitly, the radius for a fully-convective star coming down the Hayashi track

$$\frac{R}{R_{\odot}} = \left( \frac{M}{M_{\odot}} \right)^{10/21} \left( \frac{130 \text{ Myr}}{t} \right)^{1/3} \quad (5.42)$$

For masses greater than about a half of a solar mass, convection will dominate heat transport until the luminosity reaches the level that was predicted by earlier opacity arguments that assumed radiative diffusion. The onset of a radiative core is then roughly when  $L \lesssim L_{\text{rad}}$ . The time it takes for this to happen is

$$t > 10^6 \text{ years} \left( \frac{M_{\odot}}{M} \right)^2 \quad (5.43)$$

So massive stars become radiative at a very young age. For low-mass stars, though,

$$t > 2.6 \times 10^6 \text{ years} \left( \frac{M_{\odot}}{M} \right)^{4.4} \quad (5.44)$$

For very low mass stars, this time approaches the Hubble time. However, very low mass stars will remain convective on the main sequence, so nuclear reactions will likely halt the contraction anyway.

## 6 Nuclear Reactions in Stars

The big bang only had (by mass) 75% protons and 25% Helium. In stars, we want to fuse protons into helium and release about 7 MeV per baryon, or  $7 \times 10^{18} \text{ erg g}^{-1}$ , which is *far* higher than the thermal energy content of matter in stars. As it turns out, the thermal energy is too low to bring two protons close enough together to fuse, so tunneling must take place. Additionally, there is no stable nucleus with just two (or more) protons, so weak interactions are required to have any stable reactions.

### 6.1 Liquid Drop Model

The “size” of a nucleus is approximately

$$r_0 = 1.3 \text{ fm } A^{1/3} \quad (6.1)$$

where  $A$  is the mass number ( $A = N + Z$ ). Then the nuclear density is approximately

$$\rho = \frac{Am_p}{\frac{4\pi}{3}R^3} \approx 2 \times 10^{14} \text{ g cm}^{-3} \quad (6.2)$$

There is an energy well with

$$E_{\text{volume}} = -14 \text{ MeV } A \quad (6.3)$$

There must also be a surface term, dictating that surface particles are less bound than their counterparts deeper in

$$N_{\text{surf}} = A^{2/3} \sim \frac{\text{surface}}{\text{volume}} \quad (6.4)$$

So then we have

$$E_{\text{surf}} = 13 \text{ MeV } A^{2/3} \quad (6.5)$$

The Coulomb energy to assemble the nucleus is necessarily

$$E_{\text{Coulomb}} \sim \frac{e^2 Z^2}{r} = \frac{3}{5} \frac{e^2 Z^2}{R} \approx 0.56 \text{ MeV } \frac{Z^2}{A^{1/3}} \quad (6.6)$$

Nuclei typically prefer to have roughly equal numbers of neutrons and protons in order to minimize degeneracy pressure. Thus, often we will have  $N = Z$  since then the kinetic energy is lower. This gives rise to a “symmetry energy”,

$$E_{\text{sym}} = 18 \text{ MeV } \frac{(N - Z)^2}{A} \quad (6.7)$$

*Wednesday, February 8, 2012*

## 6.2 Tunneling through the Coulomb Barrier

In order for two positively charged nuclei to fuse, they must get close enough so that the strong force can overpower the Coulomb potential:

$$V(r) = \frac{Z_1 Z_2 e^2}{r} = 1.4 \text{ MeV } \frac{Z_1 Z_2}{(r/1 \text{ fm})} \quad (6.8)$$

At the center of the sun, where  $T_c \sim 10^7 \text{ K}$ , the thermal energy is only about  $kT \sim 8.6 \text{ keV } (T/10^8 \text{ K})$ . Then clearly the thermal energy in the sun is nowhere near high enough to overcome the Coulomb barrier. If there was no tunneling, we would need temperatures of order

$$T \sim \frac{Z_1 Z_2 e^2}{kR} \sim 1.6 \times 10^{10} \text{ K } \frac{Z_1 Z_2}{(R/1 \text{ fm})} \quad (6.9)$$

Stars pretty much never reach this temperature, so clearly tunneling is how fusion happens in stars.

There are two important ingredients for calculating reaction rates:

1. Tunneling
2. Maxwell-Boltzmann distribution of particle energies

The likelihood of a particle being able to tunnel through the Coulomb barrier is proportional to its energy, which in turn is determined by the Maxwell-Boltzmann statistics of the ensemble of particles. To get a better handle on what’s going on here, consider a simple reaction:



Sometimes this is written as  $X(a,b)Y$  in astrophysics (no clue why). We define the **cross section** as

$$\sigma \equiv \frac{\text{\#reactions}/X \text{ nucleus/unit time}}{\text{\#incident particles/cm}^2/\text{unit time}} \quad (6.11)$$

which clearly has the dimension of length squared (area). Now suppose the number density of a particles is given by  $n_a$  and the typical relative velocity is given by  $v_{\text{rel}}$ . Then the incoming flux of particles is just  $n_a v_{\text{rel}}$ . From this logic, we may deduce that

$$n_a \sigma v_{\text{rel}} = \frac{\text{\#reactions}}{X \text{ nucleus} \times \text{unit time}} \quad (6.12)$$

Usually we write the reaction rate in the number of reactions per unit time per unit volume, or

$$r_{aX} = n_X n_a \sigma v_{\text{rel}} \quad (6.13)$$

Really we need to integrate over a relative velocity distribution to get an understanding of the actual reaction rate, so

$$r_{aX} = \int_0^\infty dv n_a n_X \sigma(v) \phi(v) \quad (6.14)$$

where  $\phi(v)$  is the probability density of a pair of particles having a relative velocity  $v$ . More often this is written as

$$r_{aX} = n_a n_X \langle \sigma v \rangle \quad (6.15)$$

with

$$\langle \sigma v \rangle = \int_0^\infty dv \sigma(v) v \phi(v) = 4\pi \left( \frac{\mu}{2\pi kT} \right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2kT}\right) dv \quad (6.16)$$

where  $\mu$  is now the reduced mass,  $\mu = m_1 m_2 / (m_1 + m_2)$ . Note that if both reactants are the same species, (6.13) must be divided by a factor of 2 since we cannot imagine having a “target” and a “scatterer” (this screws up relative velocities and other things, see Clayton for the details). Similarly, if you have  $N$  like reactants, we must divide by  $N!$  to adjust the reaction rate appropriately.

### 6.2.1 Barrier Penetration

Recall the Schrödinger equation,

$$\left[ -\frac{\hbar^2}{2\mu} \nabla^2 + V \right] \psi = E \psi \quad (6.17)$$

For simplicity, we will only do a one-dimensional example to model barrier penetration. We suppose a free particle encounters a finite-width and finite-height barrier. Outside the barrier, then, (6.17) reduces to

$$-\frac{\hbar^2}{2\mu} \nabla^2 \psi = E \psi \quad \Rightarrow \quad \psi \propto e^{\pm i k x} \quad (6.18)$$

and the energy is given by  $E = \hbar^2 k^2 / (2\mu)$ . Inside the barrier, though, we have decaying modes:

$$-\frac{\hbar^2}{2\mu} \nabla^2 \psi = (E_V) \psi \quad \Rightarrow \quad \psi \propto e^{\pm \kappa x} \quad (6.19)$$

Where the “energy” is  $E - V = -\hbar^2 \kappa^2 / (2\mu)$ . Now we use this solution in solving for the actual Coulomb potential,

$$V(r) = \frac{Z_1 Z_2 e^2}{r} \quad (6.20)$$

by using the WKB approximation. In this regime, we have

$$\frac{\hbar^2 \kappa^2}{2\mu} = \frac{Z_1 Z_2 e^2}{r} - E \quad (6.21)$$

Using the WKB approximation to calculate the barrier penetration, which assumes that the length-scale of changes in  $V(r)$  is much larger than the De Broglie wavelength of the particle, we get the probability density being

$$\psi^2 \propto \exp\left(-2 \int \kappa \, dr\right) \quad (6.22)$$

where the integral is between the two classical turning points. In this case, there is really only one turning point,

$$r_c = \frac{Z_1 Z_2 e^2}{E} \quad (6.23)$$

Rewriting  $\kappa$  in terms of the turning point,

$$\kappa = \left(\frac{2\mu E}{\hbar^2}\right)^{1/2} \left[\frac{r_c}{r} - 1\right]^{1/2} \quad (6.24)$$

Then the integral in (6.22) becomes

$$\int_{r_c}^{r_{\text{in}}} \kappa \, dr = \int_{r_c}^0 \frac{\pi\alpha}{2} Z_1 Z_2 \left(\frac{2\mu c^2}{E}\right)^{1/2} \quad (6.25)$$

Here we’ve extended the upper limit to zero to make the integral doable, but this is actually a pretty good approximation (see Clayton for the details). This gives the probability of tunneling to be

$$P_{\text{tunnel}} \propto \exp\left[-\pi\alpha Z_1 Z_2 \left(\frac{2\mu c^2}{E}\right)^{1/2}\right] = \exp\left[-\sqrt{E_G/E}\right] \quad (6.26)$$

Where we have defined the **Gamow** energy via

$$E_G \equiv (\pi\alpha Z_1 Z_2)^2 2\mu c^2 \approx 0.98 \text{ MeV } Z_1^2 Z_2^2 \left(\frac{\mu}{m_p}\right) \quad (6.27)$$

For a proton-proton interaction,  $E_G \sim 0.5 \text{ MeV}$ , and for a carbon-proton interaction,  $E_G \sim 33 \text{ MeV}$ . These values are significantly lower than that required for straight-up thermal energy to do the work, but still pretty high compared to  $kT_c$ . The probability for a 1 keV proton to interact with another proton is then

$$P \propto \exp\left[-\left(\frac{500 \text{ keV}}{1 \text{ keV}}\right)^{1/2}\right] \sim 2 \times 10^{-10} \quad (6.28)$$

So we really need to be out on the Boltzmann tail for this to occur.

### 6.2.2 Nuclear Reaction Rates

The De Broglie wavelength of a particle is

$$\lambda = \frac{h}{p} = \frac{h}{(kT\mu)^{1/2}} \approx 10^{-10} \text{ cm in solar core} \quad (6.29)$$

While the “size” of the nucleus is

$$r_{\text{nuc}} \approx 1.3 \times 10^{-13} \text{ cm } A^{1/3} \quad (6.30)$$

Clearly, classical scattering just isn't going to cut it since the De Broglie wavelength is much longer than the target size. Partial wave analysis can get an effective cross-section given by

$$\pi\lambda^2 (\text{dimensionless stuff}) \exp \left[ - \left( \frac{E_G}{E} \right)^{1/2} \right] \quad (6.31)$$

Where the “dimensionless stuff” comes from nasty nuclear physics. Often this is written in terms of the reduced wavelength,  $\lambda/2\pi$ , which I'll denote as  $\bar{\lambda}$  (L<sup>A</sup>T<sub>E</sub>X isn't agreeing with “\lambdabar”).

$$\pi\lambda^2 = 4\pi^3 \bar{\lambda}^2 = \frac{4\pi^3}{k^2} \quad (6.32)$$

So the energy is

$$E \approx \frac{\hbar^2 k^2}{2\mu} \Rightarrow 4\pi^3 \bar{\lambda}^2 = \frac{2\pi^3 \hbar^2}{\mu E} = 2000 \text{ barns } \left( \frac{\text{keV}}{E} \right) \quad (6.33)$$

Then the cross section can be written as

$$\sigma(E) = \frac{S(E)}{E} \exp \left[ - \left( \frac{E_G}{E} \right)^{1/2} \right] \quad (6.34)$$

Where the “stupid factor”,  $S(E)$  is a slowly changing function of the energy that takes into account the details of the nuclear reaction. Typical values of  $S(E)$  are around 2000 keV barns. One important exception is the Deuterium synthesis reaction:

$$p + p \rightarrow D + e^+ + \nu_e \quad (6.35)$$

where  $S \sim 4 \times 10^{-22}$  keV barns. This value is so small because a weak interaction is involved. Now we return back to calculating  $\langle \sigma v \rangle$ , using our newly found cross-section and writing the energy as  $E = \frac{1}{2}\mu v^2$  and  $dE = \mu v dv$ . (Note that the non-relativistic energies are fine for our purposes.) Then we have

$$\langle \sigma v \rangle = \frac{1}{(kT)^{3/2}} \left( \frac{8}{\pi\mu} \right)^{1/2} \int_0^\infty dE S(E) \exp \left[ -\frac{E}{kT} - \left( \frac{E_G}{E} \right)^{1/2} \right] \quad (6.36)$$

Before evaluating this integral, we'll need to know how to use the method of steepest descent. Suppose we have an integral with

$$I = \int_{-1}^\infty g(x) e^{-f(x)} dx \quad (6.37)$$

where  $g(x)$  is slowly varying and  $f(x)$  is sharply peaked. (In our case, the Maxwell-Boltzmann distribution multiplied by the probability of tunneling is sharply peaked. Particles far out on the Maxwell-Boltzmann tail are very likely to tunnel, but incredibly unlikely to exist. Likewise, particles with low energies are in ample supply, but they aren't tunneling anytime soon, so there is some magic window in the middle that is most relevant.) We Taylor expand  $f(x)$  about the peak (where  $f'(x_0) = 0$ ):

$$f(x) \approx f(x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 \quad (6.38)$$

Then (6.37) can be approximated by

$$I = g(x_0)e^{-f(x_0)} \int_{-\infty}^{\infty} \exp \left[ -\frac{f''(x_0)}{2}(x_0)^2 \right] = g(x_0)e^{-f(x_0)} \sqrt{\frac{2\pi}{f''(x_0)}} \quad (6.39)$$

Then for our uses, we have

$$f(E) = \frac{E}{kT} + \left( \frac{E_G}{E} \right)^{1/2} \quad (6.40)$$

Solving for  $E_0$  gives  $E_0^3 = \frac{1}{4}E_G(kT)^2$ . Then the relevant functions and derivatives are

$$f(E_0) = 3 \left( \frac{E_G}{4kT} \right)^{1/3} \quad (6.41)$$

$$f''(E_0) = 3 [2E_G(kT)^5]^{1/6} \quad (6.42)$$

Using this result in (6.36), we get

$$\langle \sigma v \rangle \approx \frac{1}{(kT)^{3/2}} \left( \frac{8}{\pi\mu} \right)^{1/2} S(E_0) \exp \left[ -3 \left( \frac{E_G}{4kT} \right)^{1/3} \right] \int_0^{\infty} \exp \left[ -\frac{(E - E_0)^2}{(\Delta/2)^2} \right] \quad (6.43)$$

$$\Rightarrow \boxed{\Delta = \frac{4}{2^{1/3}\sqrt{3}} E_G^{1/5} (kT)^{5/6}} \quad (6.44)$$

$$= \boxed{2.6 \frac{E_G^{1/6}}{\sqrt{\mu}} \frac{S(E_0)}{(kT)^{2/3}} \exp \left[ -3 \left( \frac{E_G}{4kT} \right)^{1/3} \right] = \langle \sigma v \rangle} \quad (6.45)$$

Now for some numbers. For proton-proton reactions,  $E_G = 494$  keV,  $E_0 = 4.5$  keV  $(T/10^7 \text{ K})^{2/3}$ , and  $\Delta/E_0 = 1 \cdot (T/10^7 \text{ K})^{1/6}$ . These reactions are quite prevalent in the solar core. However, for proton-carbon reactions,  $E_G = 36$  MeV,  $E_0 = 18$  keV  $(T/10^7 \text{ K})^{2/3}$ , and  $\Delta/E_0 = 0.5(T/10^7 \text{ K})^{1/6}$ . These do happen in the sun, but not very much. They are more prevalent in larger stars who derive their energy from the CNO cycle (more on this later).

*Monday, February 13, 2012*

### 6.3 Proton-Proton Interactions

This result makes it evident that there is a hierarchy set by the charges of the particles that are attempting to fuse. That is, elements with lower charges will fuse more easily at low temperatures.

Once that fuel expires, the star contracts, raising the temperature, triggering the next set of nuclear reactions.

The first reaction we might think would occur would be a proton-proton interaction. However, no such nucleus exists, so we instead consider Deuterium formation:



This requires two things:

- Fusion requires getting the two protons within a fermi of each other
- While within one fermi, the reaction needs to have a weak interaction:



(Useful fact:  $(m_n - m_p)c^2 \approx 1.2$  MeV.) As it turns out, this first reaction is the rate limiting step due to the sluggish weak interaction. We find the nuclear power per unit mass via

$$\varepsilon_{\text{nuc}} = \frac{n_1 n_2 \langle \sigma v \rangle E_{\text{nuc}}}{\rho} = \frac{dL(r)}{dm(r)} \quad (6.48)$$

For a proton-proton interaction,

$$\langle \sigma v \rangle_{\text{pp}} = 7 \times 10^{-13} \left( \frac{S}{S_s} \right) \frac{1}{T_7^{2/3}} \exp \left( -\frac{15.7}{T_7^{1/3}} \right) \quad (6.49)$$

Where  $S_s = 2000$  barn keV (1 barn =  $10^{-24}$  cm<sup>2</sup>). Then (6.48) becomes

$$\varepsilon_{\text{nuc}} \approx 3 \times 10^{30} \frac{\text{erg}}{\text{g s}} \left( \frac{S}{S_s} \right) \frac{\rho}{T_7^{2/3}} \exp \left( -\frac{15.7}{T_7^{1/3}} \right) \quad (6.50)$$

Of importance is that this is *most* dependent on the temperature (due to the temperature's presence in the exponential). The luminosity is then set by this value via

$$L = \int \varepsilon dm = L_{\text{rad}} \quad (6.51)$$

This *defines* the main sequence. Scaling to a low-mass star with  $M = 0.1 M_{\odot}$  and  $R = 0.1 R_{\odot}$ , a fully convective star gives the luminosity as

$$L = 3 \times 10^{28} \frac{\text{erg}}{\text{s}} \left( \frac{M}{0.1 M_{\odot}} \right)^{28/51} \left( \frac{R}{0.1 R_{\odot}} \right)^4 \quad (6.52)$$

The “stupid factor” for a proton-proton interaction is

$$S_{\text{pp}} = 4 \times 10^{-22} \text{ keV barns} = 2 \times 10^{-25} S_{\text{Strong}} \quad (6.53)$$

As a first-order estimate, suppose that *all* the mass is available for burning, so

$$L_{\text{nuc}} = M \varepsilon(T_c) \quad (6.54)$$



For a fully-convective star, the central temperature is given by

$$T_c = 0.54 \frac{GM\mu m_p}{k} \quad (6.55)$$

Scaling to the units designated earlier,  $m = M/0.1 M_\odot$ ,  $r = R/0.1 R_\odot$

$$T_7 = \frac{T}{10^7 \text{ K}} = 0.74 \frac{m}{r} \quad (6.56)$$

and

$$\rho = 139m/r^3 \text{ g cm}^{-3} \quad (6.57)$$

Now the luminosity is

$$L_{\text{nuc}} = M\varepsilon(T_c) = 2.4 \times 10^{40} \text{ erg s}^{-1} \frac{m^{4/3}}{r^{7/3}} \exp\left(-\frac{17.35r^{1/3}}{m^{1/3}}\right) = 2.5 \times 10^{28} \text{ erg s}^{-1} m^{1/2} r^4 \quad (6.58)$$

Solving this transcendental equation for  $r$  gives

$$r = m [1.58 + 0.048 \ln m - 0.36 \ln r]^3 \quad (6.59)$$

and for the  $0.1 M_\odot$  star with just proton-proton interactions, the central temperature is

$$T_c = 3.4 \times 10^6 \text{ K} \quad (6.60)$$

So the radius is “chosen” so that the nuclear burning is adequate to match the heat losses due to radiative diffusion and convection.

## 6.4 Sidebar: The Deuterium Main Sequence

Before the onset of main sequence hydrogen burning, protostars will undergo Deuterium burning via



Interestingly, the energy released per gram of the fusion is comparable to the thermal energy. When D “burns”, the energy released per gram is  $kT/m_p$ . This energy release slows the contraction for a finite time until all of the Deuterium is burned up.

## 6.5 The PP Chain and the CN Cycle

So once protons fuse to Deuterium, what happens? There is a well-established chain of events given by



Note that the first reaction is a weak interaction whereas the second two are strong interactions. At the center of the sun, we'll say that the central temperature is around  $T_c \approx 2 \times 10^7$  K. Then the luminosity due to this chain is

$$L_n \propto \frac{S}{S_{\text{Strong}}} \exp \left( -3 \left( \frac{E_G}{4kT} \right)^{1/3} \right) \quad (6.65)$$

where the Gamow energy is

$$E_G = (\pi\alpha Z_1 Z_2)^2 2m_r c^2 \quad (6.66)$$

Now let's compare the PP chain luminosity against a reaction involving fusing a proton to some heavier nucleus with atomic number  $Z$  (a strong reaction):

$$10^{-25} \exp \left( -3 \left( \frac{E_{G,\text{PP}}}{4kT} \right)^{1/3} \right) = (1) \exp \left( -3 \left( \frac{E_{G,?}}{4kT} \right)^{1/3} \right) \quad (6.67)$$

Solving this for the Gamow energy in question, we have

$$E_{G,?} = \left[ E_{G,\text{PP}}^{1/3} + 19.2(4kT)^{1/3} \right]^3 \Rightarrow E_G = \left( 19 + 29T_7^{1/3} \right)^3 \text{ keV} \quad (6.68)$$

This means this happens when  $T \approx 10^7$ , where  $E_G = 50$  MeV. Notably,  $Z = 7$  when  $T_7 = 1$ . In general,

$$\boxed{Z_c = 5 \left( T_7^{1/3} + 0.27 \right)^{3/2}} \quad (6.69)$$

Essentially what's happening here is that we are trading a higher Coulomb barrier in exchange for a purely strong reaction. Of particular interest is the CN cycle:



None of these reactions require a two-particle weak interaction, so when the temperature is high enough, this catalytic chain is competitive with the PP chain. It's not obvious, but the slowest step in this reaction is (6.73). Now we already showed that the nuclear luminosity goes as

$$L_{\text{nuc}} \propto \exp \left( -3 \left( \frac{E_G}{4kT} \right)^{1/3} \right) \quad (6.76)$$

Then the logarithmic derivative with respect to  $\ln T$  is

$$\frac{d \ln L_{\text{nuc}}}{d \ln T} = \left( \frac{E_G}{4kT} \right)^{1/3} \quad (6.77)$$

This effectively tells us the temperature sensitivity of the reaction rates. As it turns out, the PP chain has a lower temperature sensitivity than the CN cycle. As an example, the proton-proton interaction has  $d \ln L / d \ln T \approx 5$  while for the rate-determining step in the CN cycle, we have a value of about 24. For the sun, both of these mechanisms are at work.

*Wednesday, February 15, 2012*

## 6.6 Power Generation on the Upper Main Sequence

This catalytic chain of carbon and nitrogen reactions allow for the creation of long-lived (in the nuclear sense)  $\beta$ -unstable elements. Thus proton captures on high  $Z$  elements is then competitive with the PP chain. Since the rate-limiting step is (6.73), we would expect an excess of  $^{14}\text{N}$  nuclei in the plasma. Then the nuclear energy generation rate is

$$\varepsilon = \frac{1}{\rho} n_p n_S \langle \sigma v \rangle_{\text{p}+^{14}\text{N}} (28 \text{ MeV}) \quad (6.78)$$

For the Sun (and objects with solar metallicity),

$$10^{-3} n_p \approx n_S = \# \text{ density of } ^{13}\text{C} \text{ and } ^{14}\text{N} \text{ originally} \quad (6.79)$$

The cross-section factor for this rate-determining reaction is  $S = 2.75 \text{ keV barns}$ . Then, we have

$$\langle \sigma v \rangle_{\text{p},\gamma} = 1.5 \times 10^{-15} \frac{1}{T_7^{2/3}} \exp \left( -\frac{72.2}{T_7^{1/3}} \right) \quad (6.80)$$

Plugging all this back in to (6.78), we get

$$\varepsilon_{\text{CN}} = 2.5 \times 10^{25} \text{ erg g}^{-1} \text{ s}^{-1} \rho T_7^{-2/3} \exp \left( -\frac{72.2}{T_7^{1/3}} \right) \quad (6.81)$$

Now using our equation of stellar structure,

$$T \frac{ds}{dt} = \varepsilon - \frac{dL(r)}{dm(r)} \quad (6.82)$$

we can find the main sequence:

$$L = \int \varepsilon(r) dm(r) = M \varepsilon(T_c) \quad (6.83)$$

Thus the nuclear luminosity is given approximately by

$$L_{\text{nuc}} = \int \varepsilon dm = \frac{M^2}{R^3} 5 \times 10^{58} \text{ erg s}^{-1} \frac{1}{T_7^{2/3}} \exp \left( -\frac{72.2}{T_7^{1/3}} \right) = L_{\odot} M^3 \quad (6.84)$$

This implies that for the sun,  $T_7 = 2$ .

### 6.6.1 Scaling Relations on the Upper Main Sequence

What we want to derive is the dependence on stellar mass of  $R$ , since we are asking for  $T_c$ . This process is simplified by expanding the exponential term into a local power law. Then we define the quantity by

$$I = \exp\left(-\frac{72.19}{T_7^{1/3}}\right) \propto T^\nu \quad (6.85)$$

Taking a logarithmic derivative, we get

$$\nu = \frac{72.19}{3T_7^{1/3}} = \frac{d \ln I}{d \ln T_7} = \frac{24}{T_7^{1/3}} \quad (6.86)$$

That is, at  $T_7 = 2$ , we have  $\nu = 19$ .

Now we use  $T \propto M/R$  to eliminate the radius. this gives us

$$\frac{T_7^3}{M^3} \frac{M^2}{T_7^{19}} = 3.2 \times 10^5 M^3 \quad (6.87)$$

This gives a mass dependence on  $T_7$  of

$$\boxed{T_7 = 1.83 M^{1/5}} \quad (6.88)$$

in solar units. Additionally, the radius is given by

$$R \propto \frac{M}{T} \propto M^{0.8} \quad (6.89)$$

as quoted from the literature, this exponent is quoted as 0.75. Then the luminosity is given by

$$L \propto M^3 \propto R^2 T_{\text{eff}}^2 \Rightarrow T_{\text{eff}} \propto M^{1/3} \quad (6.90)$$

Additionally, we can use  $\rho_c \propto M/R^3$  to relate the density to the central density:

$$\rho_c \propto \frac{M}{R^3} \propto \frac{MT^3}{M^3} \propto \frac{T^3}{M^2} \propto \frac{T^3}{T^{10}} \propto \frac{1}{T^7} \quad (6.91)$$

That is,  $T_c \propto \rho_c^{-1/7}$ . For stars larger than the sun, the CN(O) cycle is the primary energy source. The cores are convecting due to the concentrated energy source.  $T_7$  is nearly constant with mass. The above results then define the zero-age main sequence for large stars, assuming that their opacity is set by Thomson scattering.

### 6.6.2 Convection in Large Stars

The cores of large, zero-age main sequence stars are convective. We found earlier that nuclear energy generatin is very temperature sensitive, with  $\varepsilon \propto T^{20}$ , where most of the energy is being generated close to the center. If we were to plot the luminosity coming out of a shell of radius  $r$  against the mass coordinate, we would see the curve rise sharply and then taper off quickly, indicating that the

energy is generated almost entirely by the inner-most mass. Going back to the equation governing radiative diffusion,

$$F = \frac{1}{3} c \frac{1}{\kappa \rho} \frac{d}{dr} (aT^4) \quad (6.92)$$

and the condition for hydrostatic equilibrium,

$$\frac{dP}{dr} = -\rho(r)g(r) \quad (6.93)$$

we may write

$$\frac{d \ln T}{d \ln P} = \frac{P_{\text{tot}}}{aT^4} \frac{F(3\kappa)}{4cg} = \frac{P_{\text{tot}}}{aT^4} \frac{3}{4} \frac{\kappa}{c} \frac{L(r)}{4\pi Gm(r)} \quad (6.94)$$

Note that if all the energy is being generated *exactly* at the center (i.e.,  $L(r) \propto \delta(r)$ ), then we clearly have a problem, since  $m(0) = 0$ . Let's rewrite (6.94) into a more readable form:

$$\frac{d \ln T}{d \ln P} = \frac{P_{\text{tot}}}{aT^4} \frac{3}{4} \frac{L(r)}{L_{\text{Edd}}} \frac{M}{m(r)} \approx 0.1 \frac{M}{m(r)} \quad (6.95)$$

Where we've reused some scaling relations for the the pressure and luminosity ratios. This is essentially the delta function case. We are assuming that  $L(0) = L$ . For convection to occur, we require

$$\frac{d \ln T}{d \ln P} = 0.1 \frac{M}{m(r)} > \frac{2}{5} \quad (6.96)$$

This effectively defines the convective region. If all the burning is inside the convective zone, we don't need to know  $L(r)$  at all. Thus there is a stretch of the star where  $d \ln T / d \ln P \approx 2/5$ . Outside of this convective zone, we would have the Eddington Standard Model:  $T \propto P^{1/4}$ . The convection is in the core, so it is very efficient, with the convective flux being  $F = \rho v^3$ , where  $v_c \lll c_s$ .

Often we denote the fraction of convective material via

$$q_c \equiv \frac{m(r)}{M} \quad \text{where the mass is all the mass in the convective region} \quad (6.97)$$

For relevant data giving  $q_c$  for different models, see the table at the end of Chapter 2 of HKT.

When doing CN burning, we showed that  $\varepsilon \propto Z$  (here  $Z$  is the metallicity). But because the reaction is so incredibly temperature sensitive, this doesn't really matter much. That is, if the metallicity is reduced, the temperature only needs to increase by a very small amount to get back to the same reaction rate.

### 6.6.3 Main Sequence Burning Summary

For star less massive than the sun, the hydrogen fusion is pp dominated. These star typically have radiative cores and convective outer envelopes with the opacity's dominated by Kramers' Rule.

For stars larger than the sun, the hydrogen fusion is dominated by the CN(O) cycle. They typically have convective cores and radiative envelopes with the opacity set by Thomson scattering.

## 6.7 Stellar Evolution

As a star burns hydrogen to helium, its composition naturally changes. This gradual change in composition will affect the properties of the star, ultimately forcing it to turn to new sources of fuel. We can estimate the lifetime of a star while it is burning a particular fuel via

$$t_{\text{MS}} = \frac{E_{\text{fuel}}}{L} \quad (6.98)$$

Where  $E_{\text{fuel}} = \varepsilon_{\text{nuc}} M_{\text{burn}}$  ( $\varepsilon_{\text{nuc}}$  is the nuclear energy available per unit mass of fuel). For massive stars,  $M_{\text{burn}}$  will depend on the evolution (in mass coordinates) of the convective core. Some extreme situations have **convective overshoot**, where convective eddies bring hydrogen rich material from the envelope back into the core to prolong the lifetime of the main-sequence star. Once this fuel runs out, the composition of the core rapidly changes, and Kelvin-Helmholtz contraction sets in to pay for the power loss.

For a lower mass star, there is no convection in the core, so the composition changes much more gradually until the majority of the fuel is expired.

*Monday, February 22, 2012*

## 7 State of the Stellar Interior

When studying stellar interiors, often we assume that the gas is in **Local Thermodynamic Equilibrium**, which is where most thermodynamic variables follow a Maxwell-Boltzmann distribution and thus that all areas of the star are locally in thermodynamic equilibrium.

### 7.1 The Saha Equation

Suppose we have two energy states separated by a energy gap  $\Delta E$ . Then from Boltzmann statistics, the ratio of the probability of particles residing in these two states is given by

$$\frac{n_1}{n_2} = \frac{g_1}{g_0} e^{-\Delta E/(kT)} \quad (7.1)$$

where  $g_1$  and  $g_0$  are the statistical weights of the two configurations. This equation actually describes the ionization states in the stellar interior, though for electrons, it's more accurate to use Fermi-Dirac statistics:

$$n(p) = \frac{g}{h^3} \frac{1}{\exp[(E - p - \mu)/(kT)] + 1} \quad (7.2)$$

where

$$E_p = \frac{p^2}{2m_p} + m_p c^2 \quad (7.3)$$

assuming nonrelativistic particles. To find the number density, we need to integrate this phase space density over all momenta:

$$n = \frac{g}{h^3} \int_0^\infty \frac{4\pi p^2 dp}{\exp[(E - p - \mu)/(kT)] + 1} \quad (7.4)$$

Here we are assuming that the mean free path is much longer than the de Broglie wavelength,

$$\ell \gg \lambda_{\text{DB}} = \frac{h}{p} \quad (7.5)$$

This is equivalent to stating that

$$\exp[(E_p - \mu)/(kT)] \gg 1 \quad (7.6)$$

So the integral in (7.4) simplifies considerably to (missing some explanation here)

$$n = \frac{g_4 \pi}{h^3} \int_0^\infty p^2 \exp[(E_p - \mu)/(kT)] dp \quad (7.7)$$

$$\exp\left(\frac{\mu - m_p c^2}{kT}\right) = \frac{m h^3}{g(2m_p kT \pi)^{3/2}} = \frac{m}{2m_\alpha} \quad (7.8)$$

Here, we are using  $m_\alpha \equiv (2\pi m kT/h^2)^{3/2}$ . Then the chemical potential is necessarily

$$\boxed{\mu_e = kT \ln\left(\frac{m_e}{2m_\alpha}\right) + m_e c^2} \quad (7.9)$$

Now the ionization of Hydrogen is given by



Equating the chemical potentials (the condition for LTE) tells us that

$$\mu_e + \mu_p = \mu_H \quad (7.11)$$

where we've noted that photons have no chemical potential. The chemical potentials are thus

$$\mu_p = m_p c^2 - kT \ln\left(\frac{g_p m_{\alpha,p}}{m_p}\right) \quad (7.12)$$

$$\mu_e = m_e c^2 - kT \ln\left(\frac{g_e m_{\alpha,e}}{m_e}\right) \quad (7.13)$$

$$\mu_H = m_H c^2 - kT \ln\left(\frac{g_H m_{\alpha,H}}{m_H}\right) \quad (7.14)$$

An important fact here is the difference between the rest mass of hydrogen and that of the proton and electron, which is just the ionization energy of the hydrogen atom:

$$m_H c^2 = (m_p + m_e) c^2 - 13.6 \text{ eV} \quad (7.15)$$

Then the condition that the chemical potentials are equal can be reduced to

$$\ln \left[ \frac{g_e m_{\alpha,e}}{m_e} \frac{g_p m_{\alpha,p}}{m_p} \frac{m_H}{g_H m_{\alpha,H}} \right] = \frac{13.6 \text{ eV}}{kT} \quad (7.16)$$

Rearranging this, we get the **Saha Equation**

$$\frac{n^- n^+}{n} = \frac{g_e g_p}{g_H} \left[ \frac{2\pi m_e kT}{h^2} \right]^{3/2} \exp \left( -\frac{13.6 \text{ eV}}{kT} \right) \quad (7.17)$$

Often times we'll see this in terms of the **Quantum Concentration**:

$$n_Q = \left[ \frac{2\pi m kT}{h} \right]^{3/2} \quad (7.18)$$

Now, suppose we are in a state where exactly half of the hydrogen is ionized. In this case,  $n_p = n_e = n_H$ . Solving the Saha equation for temperatures gives us

$$T_{1/2} \approx 12400 \text{ K} \quad (7.19)$$

If we define the ionization fraction via

$$y \equiv \frac{n_p}{n} = \frac{n_e}{n} \quad (7.20)$$

Then the Saha equation is reduced to

$$\frac{y^2}{1-y} = \frac{4 \times 10^{-9}}{\rho} T^{3/2} e^{-1.6 \times 10^5 / T} \quad (7.21)$$

Solving for  $y$  at the center of the sun, we find that  $y \sim 0.75$ , which is not good since we “know” that the solar core is completely ionized. The problem is that the central density is sufficiently high that the  $+1$  in the denominator of the Fermi-Dirac distribution cannot be ignored. In this regime, things like pressure ionization become important, which helps cause the interior to be completely ionized.

## 7.2 Stellar Spectra

Recall the pressure scale height, which in the photosphere, is

$$h = \frac{kT}{m_p g} \ll R \quad (7.22)$$

where the local acceleration due to gravity is (again, in the photosphere),

$$g = \frac{GM}{R^2} = 2.7 \times 10^4 \frac{M}{R_\odot} = 2.7 \times 10^4 \left( \frac{M_\odot}{M} \right)^{1/2} \quad (7.23)$$

Where we have used the fact that  $R \propto M^{0.75}$ . Recalling the optical depth,  $\tau = \int \rho \kappa dr$ , we will say that at the photosphere,  $\tau_{\text{ph}} = 2/3$ . Then we find

$$\tau_{\text{ph}} = \kappa \underbrace{\int \rho dr}_x = \kappa x \quad \Rightarrow \quad x_{\text{ph}} = \frac{2}{3\kappa} \quad (7.24)$$



Then the pressure at the photosphere is

$$P_{\text{ph}} = g x_{\text{ph}} = n_{\text{ph}} k T_{\text{ph}} \quad (7.25)$$

So the density in the photosphere is

$$n_{\text{ph}} = \frac{2gm_{\text{p}}}{3\sigma kT} \approx 10^{16} \text{ cm}^{-3} \left( \frac{\kappa_{\text{es}}}{\kappa} \right) \left( \frac{10^4 \text{ K}}{T} \right) \left( \frac{g}{2.7 \times 10^4} \right) \quad (7.26)$$

For various spectral types, we have a range of masses and effective temperatures

Spectral Type	$T_{\text{eff}}$	$M/M_{\odot}$
O3	52000 K	120
B0	30000 K	17
A0	9500 K	3
G0	6000 K	$\sim 1$

So there is around a factor of ten spread in these effective temperatures. Thus, the different emission and absorption lines that we see will vary greatly among these classes (hence their name of **spectral classes**). Elements with first ionization potential less than 5 eV include Li, Na, Mg, and Al. Those between 10 and 20 eV include H, C, N, and O. Those higher than 20 eV include He and Ne. Depending on the effective temperature and composition of a photosphere, we will observe different relative strengths in these lines.

Going back to the hydrogen atom, recall how transitions to the  $n = 1$  state for an electron are called **Lyman Transitions**. The transition from 1 to 2 corresponds to a wavelength of 1215 Å, and the Lyman limit, from 1 out to infinity is at 911 Å. Probably more well-known are transitions to and from the  $n = 2$ , called the **Balmer Transitions**. The most famous of these is the  $\text{H}\alpha$  line, which is the  $n = 2$  to  $n = 3$  transition, at a wavelength of 6563 Å. Note that this is in the visible and thus easily observable, whereas the Lyman transitions are all in the ultraviolet.

We can predict the relative numbers of electrons in various energy levels using the Boltzmann equation. For instance, the ratio of electrons in the  $n = 2$  state to the  $n = 1$  state is

$$\frac{n_2}{n_1} = \exp \left( -\frac{E_{12} \approx 10.2 \text{ eV}}{kT} \right) = 10^{-5} \quad (7.27)$$

at a temperature of  $10^4$  K. Is this strong enough to produce an observable line? First we look at the cross section of scattering:

$$\pi \lambda^2 \sim 10^{-16} \text{ cm}^{-2} = \sigma_{\text{line}} \gg \sigma_{\text{Th}} = 6.65 \times 10^{-25} \text{ cm}^{-2} \quad (7.28)$$

So comparing this cross section to the Thomson cross section, we have

$$\frac{\sigma_{\text{line}}}{\sigma_{\text{Th}}} \sim 10^8 \quad (7.29)$$

Now we can determine the relative strength of this source of opacity to electron scattering via

$$\exp \left( -\frac{10.2 \text{ eV}}{kT} \right) \frac{\sigma_{\text{line}}}{\sigma_{\text{Th}}} \frac{m_{\text{H}}}{m_{\text{p}}} > 1 \quad (7.30)$$

Or,

$$\exp\left(-\frac{10.2 \text{ eV}}{kT}\right) > \frac{\sigma_{\text{Th}}}{\sigma_{\text{line}}} \Rightarrow T > 6000 \text{ K} \quad (7.31)$$

So for stars with effective temperatures above 6000 K, this transition creates an observable line. Different emission and absorption sources in stars peak at various temperatures (and thus are unique to specific spectral classes), so we can identify stars quite well from their spectra alone.

**H<sup>-</sup> Opacity** With the Boltzmann equation, we can compute what fraction of hydrogen atoms are actually ionized into the H<sup>-</sup> ion:

$$\frac{n(\text{H})}{n(\text{H}^-)} = \frac{n_Q}{n_e} \exp\left(-\frac{0.75 \text{ eV}}{kT}\right) \quad (7.32)$$

or

$$\frac{n(\text{H}^-)}{n(\text{H})} = \frac{n_e}{n_Q} \exp\left[\frac{8700}{T}\right] \quad (7.33)$$

For the sun, this gives  $n_e = 10^{13} \text{ cm}^{-3}$  and thus

$$\frac{n(\text{H}^-)}{n(\text{H})} \approx 10^{-8} e^{8700/T} \quad (7.34)$$

This source also peaks at a particular effective temperature and is actually the dominant source of opacity in the sun's photosphere.

*Wednesday, February 24, 2012*

## 8 Stellar Evolution

Now we wish to know what happens to a star as  $X$  decreases and  $Y$  increases. This change in composition will require a change in the overall behavior of the star. The time a star spends on the main sequence,

$$t_{\text{MS}} = \frac{E_{\text{nuc}}}{L} \quad (8.1)$$

is much larger than the Kelvin Helmholtz time since  $E_{\text{nuc}} \gg E_{\text{th}}$ . This is because the reaction converting hydrogen to helium liberates around 7 MeV per hydrogen atom burned, whereas the thermal energy of each atom is around 1 keV. Thus, it makes sense that the main sequence time is on the order of  $10^3$  times longer than the Kelvin-Helmholtz time.

In fact, even if the star only burns 10% of the available hydrogen, we still have  $t_{\text{MS}} > 1000 t_{\text{KH}}$ . This contrast in timescales allowed for early calculations of stellar evolution because at each time step needed to resolve  $X$ , the star would be in complete thermal balance. One big caveat to what we will do today is that the concentration of the fusion to the center of the star “breaks” our homogeneous assumptions.

## 8.1 Main Sequence Evolution for Massive Stars

We want to construct stellar models as a function of the hydrogen abundance,  $X$ , which will necessarily change as burning occurs in the star as hydrogen is burned to helium on the main sequence. For now, we will be in the regime of massive stars (say,  $M = 2M_\odot$ ), so we have

- Convective core, so there is a “ball” in the center with an evolving composition.
- $\kappa = \kappa_{\text{es}}$
- CNO Burning

Simplifying the helium synthesis to



We see that, all other things being held constant, this reaction will cause the number density, and thus the pressure to decrease. Due to the high sensitivity of CNO burning to changes in temperature, the star “wants” to keep the temperature constant. To compensate for the loss in pressure, then, the star will have to dynamically adjust its radius. Writing out the pressure explicitly, we have

$$P = \frac{\rho kT}{m_p} \left( 2X + \frac{3}{4}Y \right) = \frac{\rho kT}{\mu m_p} \quad (8.3)$$

Then the thermal energy in the core,

$$k_B T_c \sim \frac{GM\mu m_p}{R} \quad (8.4)$$

Where the composition is, from (8.3),

$$\mu = \frac{1}{2X + \frac{3}{4}Y} \quad (8.5)$$

For a cosmic mix,  $X = 0.7$  and  $Y = 0.25$ , giving  $\mu_0 = 0.63$ . When the core is fully helium, we trivially have  $\mu = 4/3$ . Let’s look at how the luminosity is affected:

$$L \sim R^2 \frac{c}{3\kappa\rho} \frac{d}{dr} (aT^4) \quad (8.6)$$

where the opacity is given by

$$\kappa = 0.2(1 + X) \text{ cm}^2 \text{ g}^{-1} \quad (8.7)$$

Then the luminosity is

$$L \propto R^2 \frac{R^3}{(1 + X)M} \frac{1}{R} \frac{\mu^4 M^4}{R^4} \propto \frac{\mu^4 M^3}{1 + X} \quad (8.8)$$

This factor increases by a factor of 40 from a cosmic composition to the pure helium composition. Thus, we recover the fact that luminosity tends to *increase* as a star progresses through the main sequence. Perhaps the factor of 40 is an idealization, but we can be certain that the luminosity *must increase as hydrogen is used up*.

Now recall that the nuclear luminosity goes as

$$L_{\text{nuc}} \propto T_c^{15 \rightarrow 20} \quad (8.9)$$

And as the luminosity increases, the temperature will only increase by, at most, 20%. Some scaline relations of importance for the radius can then be obtained:

$$T_c \propto \frac{\mu}{R} \quad (8.10)$$

So the radius goes as

$$R \propto \frac{\mu}{T_c} \quad (8.11)$$

And so the final radius is given by

$$\frac{R_f}{R_i} \approx \frac{4/3}{0.6} \approx 2 \quad (8.12)$$

Thus the star doubles in radius as it proceeds through the main sequence.

## 8.2 Main Sequence Evolution for Less Massive Stars

For stars less massive than the sun, we have

- Radiative Core
- $L_{\text{nuc}} \propto T^\nu$  for  $\nu \sim 4 - 7$  for p-p chain.
- Kramers' Law Opacity

Again, as the mass increases, the luminosity increases. However, due to the weaker dependence on temperature, the core temperature will rise more significantly than for the high-mass stars to match the luminosity. Again the evolution of the radius is governed via

$$kT_c \sim \frac{Gm_p M \mu}{R} \quad (8.13)$$

Due to the increased dynamicism of  $T_c$ , it is no longer clear what will happen to the radius. Without proof, we say that the radius does not change nearly as significantly in low-mass stars as it does in high-mass stars.

## 8.3 Post-Main Sequence Evolution

For high mass stars, see Figure 2 for a qualitative description of the evolution of high-mass stars as they leave the main sequence.

For stars less than  $1.5 M_\odot$ , there is no abrupt transition (the Henyey Hook). Rather, it develops a growing (in mass) helium core.

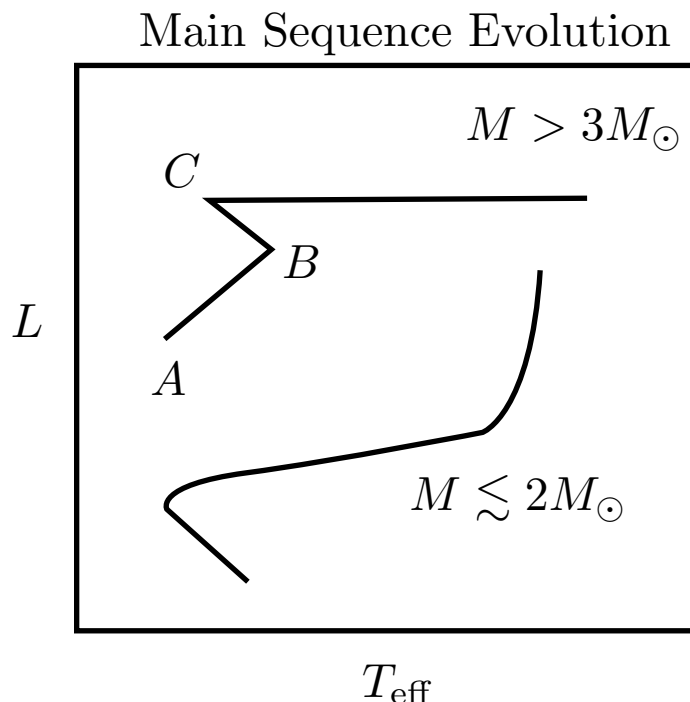


Figure 2: Evolution of a massive star. From  $A$  to  $B$ , the star is undergoing Main Sequence hydrogen burning. Temperature is increasing as  $X$  is decreasing until finally all hydrogen is depleted in the core. From  $B$  to  $C$ , the pure helium core undergoes Kelvin-Helmholtz contraction (this is called the “Heney Hook”). Beyond  $C$ , hydrogen is burning in a shell.

### 8.3.1 The Red Giant Branch (RGB)

For stars less than about  $2 M_{\odot}$ , as the star leaves the main sequence, it travels horizontally on the HR diagram and eventually starts making its way back up the Hayashi track. This portion of the star’s life is called the **Red Giant Branch** (RGB). After hydrogen is depleted in the core, hydrogen continues burning in a shell around the helium core. The radius of this core is much less than the radius of the star. When a one solar mass star “leaves” the main sequence, we typically have  $M_{\text{He}} = 0.1 M_{\odot}$ . However, the helium core, once degenerate, will *decrease* in radius as the mass increases (where the mass gain is fueled by the hydrogen shell burning).

Then the temperature in the shell is given by

$$kT_s \approx \frac{GM_c}{R_c} m_p \quad (8.14)$$

for a shell where the pressure scale height is  $R_c$ . Now, assuming electron scattering sets the opacity,

the luminosity is

$$L_{\text{rad}} = 4\pi R_c^2 \left[ \frac{1}{3} \frac{c}{\kappa \rho} \frac{1}{R_c} a T_s^4 \right] \quad (8.15)$$

or more succinctly,

$$L_{\text{rad}} = \frac{4\pi}{3} \frac{R_c}{\kappa \rho_s} a c T_s^4 \quad (8.16)$$

Now requiring that  $L_{\text{nuc}} = \varepsilon M_{\text{shell}} = L_{\text{rad}}$  actually tells us that at this stage, CNO burning dominates pp burning. Assuming again that the nitrogen proton scatter is the slow step, we get

$$\varepsilon_{\text{CNO}} = 6.6 \times 10^{24} \frac{\rho}{T_7^{2/3}} \exp\left(-70.7 T_7^{-1/3}\right) = \varepsilon_0 \rho T^\nu \quad (8.17)$$

In this case,  $\nu = 24/T_7^{1/3} \sim 16$ . Using this to find the density in the shell allows us to get the luminosity. The punchline is that

$$\rho = \left[ \frac{4}{3} \frac{\sigma T_5^{4-\nu}}{\varepsilon_0 R_c^2 \kappa} \right]^{1/3} \quad (8.18)$$

Then the luminosity scales as

$$L \propto R_c^{5/3} T_5^{(8+\nu)/3} \quad (8.19)$$

But since  $T_s \propto M_c/R_c$ , we get

$$L \propto R_c^{-(1+\nu/3)} M_c^{(8+\nu)/3} \quad (8.20)$$

Plugging in fiducial values for scaling relations gives us

$$L = 5 \times 10^{31} \text{ erg s}^{-1} \left( \frac{0.1 R_\odot}{R} \right)^{25/3} \left( \frac{M_c}{0.2 M_\odot} \right)^{10} \quad (8.21)$$

Since the helium core is degenerate,  $R_c \propto M_c^{-1/3}$ . This would make the mass dependence of  $L$  even stronger.

*Monday, February 27, 2012*

## 8.4 The Schönberg-Chandrasekhar Limit

Returning to stars with masses above around  $6M_\odot$ , where the hydrogen burning core ends, we have a substantial mass helium core. Figure 3 shows how the hydrogen mass fraction evolves in the profile.

This change in hydrogen mass fraction is the source of the “Heney Hook” seen in large mass evolutionary tracks. After the hydrogen burning shell is extinguished undergoes Kelvin-Helmholtz contraction until the Helium ignites. This helium core is nearly isothermal, so using hydrostatic equilibrium and the same old Virial theorem trick,

$$\frac{dP}{dr} = -\rho(r) \frac{Gm(r)}{r^2} \quad (8.22)$$

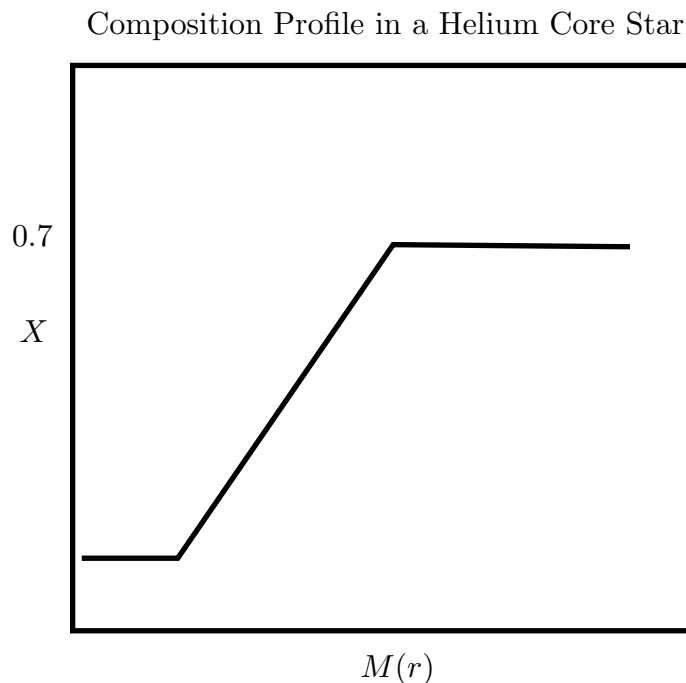


Figure 3: The composition profile of a star with a helium core. In the outer region, we still have  $X \sim 0.7$  since that material is largely unburned.

Now multiplying both sides by  $4\pi r^3$  and integrating, we get

$$\int_0^{R_c} 4\pi r^3 \frac{dP}{dr} dr = -G \int_0^{R_c} \frac{4\pi r^2 m(r) \rho(r)}{r} dr \quad (8.23)$$

The term on the left is, integrating by parts,

$$\int_0^{R_c} 4\pi r^3 \frac{dP}{dr} dr = 4\pi R_{\text{core}}^3 P_{\text{core}}(R_{\text{core}}) - 12\pi \int_0^{R_c} r^2 P(r) dr \quad (8.24)$$

and the right side is roughly the gravitational potential, so (8.23) becomes

$$4\pi R_{\text{core}}^3 P_{\text{core}}(R_{\text{core}}) - 12\pi \int_0^{R_c} r^2 P(r) dr = -\frac{GM_{\text{core}}}{R_{\text{core}}} \quad (8.25)$$

Since we have  $P(r) = \rho(r)kT_{\text{core}}/(\mu_{\text{core}}m_p)$ . Then we can recast (8.25) quite easily to

$$4\pi R_{\text{core}}^3 P_{\text{core}}(R_{\text{core}}) - 3\frac{kT_{\text{core}}}{mu_{\text{core}}m_p}M_{\text{core}} = -\frac{GM_{\text{core}}^2}{R_{\text{core}}} \quad (8.26)$$

Solving for the pressure, we have

$$P_{\text{core}}(R_{\text{core}}) = \frac{3}{4\pi} \frac{kT_{\text{core}}}{\mu_{\text{core}} m_p} \frac{M_{\text{core}}}{R_{\text{core}}^3} - \frac{1}{4\pi} \frac{GM_{\text{core}}^2}{R_{\text{core}}} \quad (8.27)$$

Note that this equation has a well-defined maximum at some critical core radius. The radius is

$$R_{\text{core,max}} = \frac{GM_{\text{core}} \mu_{\text{core}} m_p}{kT_{\text{core}}} \frac{4}{9} \quad (8.28)$$

and the maximum pressure is

$$P_{\text{core,max}} = 0.7 \left( \frac{kT_{\text{core}}}{\mu_{\text{core}} m_p} \right)^4 \frac{1}{G^3 M_{\text{core}}^2} \quad (8.29)$$

The overlying envelope must exert a pressure

$$P_{\text{base}} \sim \frac{GM^2}{R^4} \quad (8.30)$$

At the interface between the pure helium core and the hydrogen/helium mix envelope, there must be a discontinuity in the density (like a water/air interface, sort of). The  $R$  in (8.30) is really just the scale height in the H/He envelope, so we will use

$$kT_{\text{envelope}} \approx \frac{GM \mu_{\text{envelope}} m_p}{R} = kT_{\text{core}} \quad (8.31)$$

where we've mandated a continuous temperature profile at the interface. Then the pressure from the envelope is

$$P_{\text{envelope}} = \frac{GM^2}{R^4} = \left( \frac{kT_{\text{core}}}{\mu_{\text{envelope}} m_p} \right)^4 \frac{1}{G^3 m^2} \quad (8.32)$$

assuming that  $M \gg M_{\text{core}}$ . To have a valid solution, we require

$$0.68 \frac{1}{\mu_{\text{core}}^4} \frac{1}{M_{\text{core}}} > \frac{1}{\mu_{\text{envelope}}} \frac{1}{M^2} \quad (8.33)$$

So then the requirement on the core mass is

$$M_{\text{core}} < M \left( \frac{\mu_{\text{envelope}}}{\mu_{\text{core}}} \right)^2 \alpha \quad (8.34)$$

for some numerical constant  $\alpha$ . Noting that  $\mu_{\text{envelope}} = 0.6$  and  $\mu_{\text{core}} = 1.33$ . The “real” result (not derived here) is

$$q_c = \frac{M_{\text{core}}}{M} = 0.37 \left( \frac{\mu_{\text{envelope}}}{\mu_{\text{core}}} \right)^2 = 0.08 \quad (8.35)$$

If  $M_{\text{core}} > 0.08M$  and the core is not degenerate, then this core is not a solution to hydrostatic balance.



Now what about electron degeneracy? In smaller stars, the core becomes degenerate before igniting helium. Then the equation of state becomes

$$P(r) = K_{\text{NR}} \rho^{5/3}(r) \quad (8.36)$$

And thus

$$P_{\text{core}}(R_{\text{core}}) = \frac{K_{\text{NR}} M^{5/3}}{R_{\text{core}}^5} - \frac{1}{4\pi} \frac{GM_{\text{core}}^2}{R_{\text{core}}^4} \quad (8.37)$$

Note that this pressure has *no* maximum like the ideal gas situation. That is, the core mass can have any mass whatsoever. Figure 4 shows how degeneracy affects different stars.

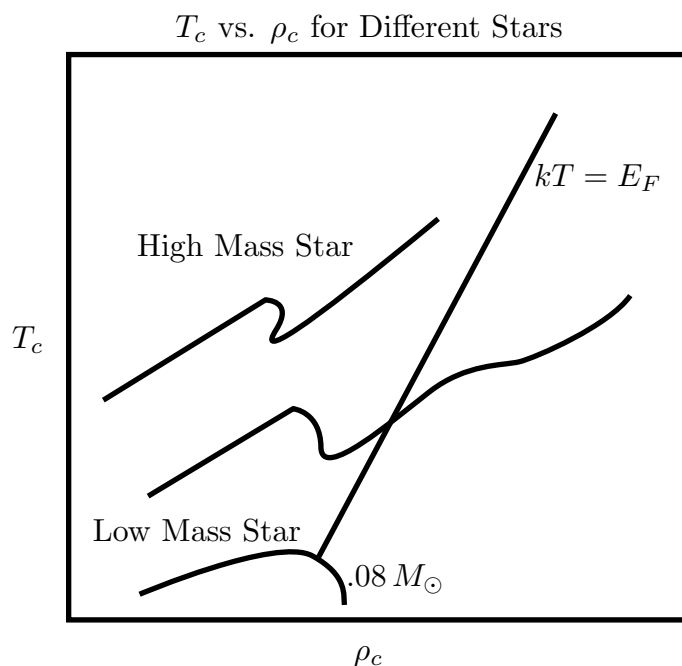


Figure 4:  $T_c$  versus  $\rho_c$  for various stars. High mass stars reach the helium ignition temperature before becoming degenerate, but lower mass stars do not, so they undergo a helium flash. A  $0.08 M_{\odot}$  star is shown, and barely (if ever) reaches hydrogen ignition.

Returning to the more massive stars, if  $q_{\text{core}} > 0.08$  at the time of hydrogen core depletion, there is no isothermal solution and the star simply contracts on the Kelvin-Helmholtz timescale:

$$t \approx \frac{GM_{\text{core}}^2/R_{\text{core}}}{L} = 10^6 \text{ years} \left( \frac{6 M_{\odot}}{M} \right) \left( \frac{5 R_{\odot}}{R} \right) \quad (8.38)$$

So these stars contract very rapidly.

Now consider stars in the range of  $2 M_\odot$  and  $6 M_\odot$ . Initially, suppose  $q < q_c$ , but hydrogen shell burning occurs. Now  $q$  will increase until  $q > q_c$  and the core undergoes Kelvin-Helmholtz contraction. For stars with less than  $2 M_\odot$ , the core will become degenerate before helium burning can start.

*Monday, February 29, 2012*

## 8.5 Helium and Higher Burning

Helium burns in advanced stars via the triple-alpha process, according to



(8.40)

However,  ${}^8\text{Be}$  is unstable with a decay time of around  $\tau \sim 2 \times 10^{-16}$  sec. The Gamow energy for this interaction is  $E_G = 31.4$  MeV, so the Gamow peak occurs at

$$E_0 = \left( \frac{E_G (kT)^2}{4} \right) = 83 \text{ keV} T_8^{2/3} \quad (8.41)$$

So once  $T > 8 \times 10^7$  K or so, the reaction to make  ${}^8\text{Be}$  proceeds rapidly and can achieve thermal and chemical equilibrium.

Equating the chemical potentials, we have

$$\frac{n_8}{n_4^2} = 2^{3/2} \left( \frac{h^2}{2\pi m_\alpha kT} \right)^{3/2} \exp \left( -\frac{m_8 - 2m_\alpha c^2}{kT} \right) \quad (8.42)$$

Normalizing to  $\rho = 10^4 \text{ g cm}^{-3}$  of pure helium (at the core, obviously), we get

$$\frac{n_8}{n_4} = 3 \times 10^{-6} T_8^{-3/2} \left( \frac{\rho}{10^4 \text{ g cm}^{-3}} \right) \exp \left( -\frac{10.64}{T_8} \right) \quad (8.43)$$

At  $T_8 = 2$ , about 1 in  $10^{-8}$  of the nuclei at any given time are  ${}^8\text{Be}$ . Then another alpha particle can combine with this  ${}^8\text{Be}$  nucleus creating a  ${}^{12}\text{C}$  nucleus. However, this is actually a carbon nucleus in an excited state of carbon that will eventually decay to the ground state through a strange process. First though, we note the Gamow energy and Gamow peak energy for the Carbon synthesis reaction:

$$E_G = (\pi\alpha(2)(4))^2 2m_\alpha c^2 = 168 \text{ MeV} \quad (8.44)$$

$$E_0 = 146 \text{ keV} T_8^{2/3} \quad (8.45)$$

This is for the reaction



Now let us assume that the temperature is high enough so that the excited state of  ${}^{12}\text{C}$  becomes populated via a chemical equilibrium of (8.46). Now equating *these* chemical potentials and noting that that of the Be isotope is just twice that of the alpha particles:

$$3\mu_4 = \mu_{12^*} \quad (8.47)$$

Messy algebra ensures, with the result that

$$\frac{n_{12^*}}{n_4} = 5 \times 10^{-10} \left( \frac{\rho}{10^5} \right)^2 T_8^3 \exp \left( -\frac{44}{T_8} \right) \quad (8.48)$$

Where we have implicitly used the binding energy of  $^{12}\text{C}^*$ , which is 7.644 MeV (notably higher than that of the reactants, Be and the alpha particles, at 7.366 MeV). Note that so far, this reaction has actually *lost* energy. The only way this reaction provides energy is in the two decays it takes for  $^{12}\text{C}^*$  to get down to the ground state of Carbon. For these decays, we have

$$\Gamma_{\text{rad}} = 3.67 \text{ MeV} \quad (8.49)$$

and thus the decay rate is

$$r = \frac{\Gamma}{\hbar} = 5.5 \times 10^{12} \text{ s}^{-1} \quad (8.50)$$

Putting all of this together, we have that the nuclear energy generation rate is

$$\varepsilon = \frac{n_{12^*} E}{\tau \rho} = 5.3 \times 10^{21} \text{ erg g}^{-1} \text{ s}^{-1} \rho_5^2 T_8^{-3} \exp \left( -\frac{44}{T_8} \right) \quad (8.51)$$

In order to determine this, we had to know the mass difference between  $^{12}\text{C}^*$  and three alpha particle, as well as the photon decay rate of  $^{12}\text{C}^*$  into plain old  $^{12}\text{C}$ . Note that this reaction is even more temperature sensitive than either hydrogen burning source, and the energy per reaction per nucleon is about a tenth of what it is for hydrogen burning. Thus, we expect helium burning to end much faster than hydrogen burning since more reactions have to happen per unit time to match the luminosity, *and* the luminosity is higher due to the change in composition that we found earlier.

### 8.5.1 Higher Burning

Once carbon is present in the core, it can burn to oxygen by the synthesis of an alpha particle and a carbon nucleus. Similarly, the Oxygen can be burned to Neon. The relevant values are shown in Table 1 The first reaction in Table 1 does occur fast enough to compete with the triple alpha

Reaction	$E_0(T_8 = 2)$	$\exp \left[ -3 \left( \frac{E_G}{4kT} \right)^{1/2} \right]$
$\alpha + ^{12}\text{C}$	315 keV	$1.3 \times 10^{-24}$
$\alpha + ^{16}\text{O}$	390 keV	$5.5 \times 10^{-38}$
$\alpha + ^{20}\text{Ne}$	460 keV	$1.5 \times 10^{-44}$

Table 1: Parameters controlling the burning of carbon, oxygen, and neon.

process, so advanced stars typically have a core consisting of both carbon and oxygen.

### 8.5.2 Helium Burning in a Massive Star

In a convective core, helium burning starts at  $T = 10^8$  K due to the high temperature sensitivity of the triple alpha process. Now recall that the luminosity of an evolved star is higher than it was as a ZAMS star, so the burn time is

$$t_{\text{He burn}} = \frac{E_{\text{nuc}} M_{\text{burn}}}{L} \quad (8.52)$$

which is shorter than the hydrogen-burning main sequence burn time. For stars above around  $6 M_{\odot}$ , carbon burning will eventually occur, and as we move on up to higher and higher masses, more and more heavy elements will get their turn at burning, with the most massive stars stopping at Fe/Si burning (since the binding energy per nucleon peaks there).

For stars greater than around  $2M_{\odot}$ , the helium ignites when the electron gas in the core is nondegenerate. Solar mass stars, however, are not so lucky.

### 8.5.3 Helium Core Flash

For stars with masses less than around  $2 M_{\odot}$ , the helium cores have degenerate electrons. Meanwhile, the CN cycle burning increases the mass of the He core. Before we delve into the details of the dynamics of the degenerate core, here's a quick refresher of degeneracy.

The DeBroglie wavelength of particles in a gas of temperature  $T$  is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{mkT}} \quad (8.53)$$

The quantum mechanics impacts the equation of state when  $\lambda \gtrsim n^{-1/3}$ . Then the condition becomes

$$n^{1/3} > \frac{\sqrt{mkT}}{h} \quad (8.54)$$

Note that this is the quantum concentration found in our previous discussion of the Saha equation. And so  $n^{2/3} \propto T$ . As  $T \rightarrow 0$ , then the DeBroglie wavelength is going to  $\lambda \propto n^{1/3}$  since they are degenerate and that is necessarily the separation distance. Now the pressure for a nonrelativistic gas is given by

$$P = \frac{2}{3} E \quad (8.55)$$

where  $E$  is the energy density. The energy is primarily in kinetic energy,

$$E_k = \frac{p^2}{2m_e} = \frac{(\hbar n_e^{1/3})^2}{2m_e} \quad (8.56)$$

So the Fermi energy follows  $E_F \propto \rho^{2/3}$ , and so the pressure is  $P = \rho E_f \propto \rho^{5/3}$ . For hydrostatics, we have

$$P = \frac{GM^2}{R^4} \propto \left(\frac{M}{R^3}\right)^{5/3} \Rightarrow R \propto M^{-1/3} \quad (8.57)$$

Thus more massive degenerate objects have *smaller* radii.

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Now returning to our intermediate mass star with a helium core, the temperature in the hydrogen burning shell located at about  $R_c$  (the radius of the core) is

$$kT_s \approx \frac{GM_s m_p}{\mu_e R_c} \Rightarrow T_s \propto \frac{M_c}{R_c} \propto M_c^{4/3} \quad (8.58)$$

So as the shell deposits more and more helium into the core, the temperature on the outer edge is rising in time. In the degenerate core, the dominant form of heat transport is electron conduction from electron scattering off of the ions. This mode is so efficient that the core is nearly isothermal (though it's actually a bit hotter out towards the hydrogen burning shell since neutrino cooling is lowering the temperature at the center).

When the star is in hydrostatic balance and has an ideal gas equation of state, an injection of entropy would lead to a temperature (negative heat capacity). For a star, this means that the pressure drops when entropy is added. Here are some ways to cause trouble in such a star:

- Make the electrons degenerate so that the pressure is independent of temperature.
- Thin Shell Instability

The **Thin Shell Instability** was discovered in 1965 and is a thermonuclear instability that occurs when temperature sensitive burning is in a geometrically thin shell. By “geometrically thin”, we mean that at the location of the shell,  $\Delta r \ll R$ . To get a better handle on this mechanism, recall the condition for hydrostatic balance (again):

$$\frac{dP}{dr} = -\rho(r) \frac{GM(r)}{r^2} \quad (8.59)$$

Integrating this over a thin shell gives us the plane-parallel atmosphere result:

$$\int_{P_{\text{sh}}}^0 dP = - \int_{R_{\text{sh}}}^{\infty} \frac{GM(r)\rho(r)}{r^2} dr \quad (8.60)$$

Or, switching to the mass coordinate,

$$P_{\text{sh}} = \int_m^M \frac{Gm(r)}{4\pi r^4} dm = \frac{GM_{\text{sh}}}{R_{\text{sh}}^2} \frac{\Delta m}{4\pi R_{\text{sh}}^2} \quad (8.61)$$

where  $dm = 4\pi r^2 \rho(r) dr$  and  $\Delta m = M - m$ , which is the mass in the shell (approximately, since we are ignoring the contribution of the envelope).

When the pressure is constant,

$$T \frac{ds}{dt} = c_P \frac{dT}{dt} = \varepsilon - \frac{1}{\rho} \nabla \cdot \mathbf{F} \quad (8.62)$$

Then this becomes positive, hence the instability (heat capacity has become positive). The flux term, from radiative diffusion, is

$$\frac{1}{\rho} \nabla \cdot \mathbf{F} = \frac{1}{dy} \frac{c}{3\kappa} \frac{d}{dy} aT^4 \propto \frac{acT^4}{3\kappa y^2} \quad (8.63)$$

where  $y$  is the column depth introduced earlier,  $y = \int \rho(r) dr$ , of the shell. So then (8.63) becomes

$$c_P \frac{dT}{dt} = \varepsilon - \frac{acT^4}{3\kappa y^2} \quad (8.64)$$

For most nuclear energy generation rates,  $\varepsilon \propto T^\nu$  with  $\nu \gg 4$ , so this becomes highly unstable. This occurs in stellar evolution as the helium shell flashed in asymptotic giant branch (AGB) stars. A limit cycle can be formed where a thin shell piles up until it burns unstably, starting the process over again. In an accretion context, white dwarfs accrete hydrogen in a thin shell and burn unstably, which is a **Classical Nova**. They can also do this with helium, which causes a larger nova or **.Ia** explosion. Neutron stars accrete hydrogen and burn to make a **Type I X-Ray Burst** (hydrogen burns to helium, helium burns to carbon and oxygen). Additionally, this process can fuel superbursts, where carbon is burned to heavy elements.

Returning to the degenerate electron source of instability, an extremely degenerate electron gas can be the cause of unstable burning. In this regime, the pressure is insensitive to the temperature, so the pressure is constant. The same equation that was relevant to the thin shell instability applies:

$$c_P \frac{dT}{dt} = \varepsilon - \frac{dL(r)}{dm(r)} \quad (8.65)$$

but the reason that  $c_P$  is positive is now different (degeneracy versus plane-parallel). Due to the fact that the temperature in the core isn't monotonic, helium ignition starts off-center (remember, the neutrino cooling keeps the center of the core cooler than surrounding areas). In a 1-D model, a convective zone forms at the ignition point. The temperature continues to rise until such a time where the degeneracy is broken and the shell isn't geometrically thin. At this degeneracy breaking-point, we have  $kT \sim E_F$ . The Fermi energy follows the virial form,

$$E_F \sim \frac{GM_c}{R_c} m_{\text{He}} \sim kT \quad (8.66)$$

At this point, the scale height will now be of the order of the radius of the star, so it is *not* geometrically thin. Then the heat capacity returns to being negative, ending the unstable burning. After this event has passed, degeneracy in the local area has been lifted. A succession of flashes follows, slowly removing the degeneracy and paving the way for stable “main sequence” helium burning.

## 8.6 The Horizontal Branch and The Asymptotic Giant Branch

The core flash lifts degeneracy, leaving behind a nondegenerate helium-burning star. Recalling that  $L \propto \mu^4$ , the luminosity increases dramatically ( $L_\odot \rightarrow 10L_\odot$ ). Once the core flash is over, we would expect stellar evolution to slow down or stop due to the stable nature of helium core burning. The envelope is expanded by a large factor, essentially shutting off the hydrogen shell burning. The luminosity then decreases quickly, creating a tip on an HR diagram (called the tip of the red giant branch, or TRGB). Kelvin-Helmholtz contraction occurs until stable helium core burning is occurring at a rate that balances radiative losses. Essentially the star has become a helium-burning main sequence star, often called a **Red Clump** star. If the star loses part or all of its hydrogen shell, the luminosity is unaffected, but the radius obviously does, so there is a range of colors of

red clump stars, depending on how much hydrogen they have retained, forming the **Horizontal Branch** on the HR diagram.

The helium core burning lifetime is somewhere around 65-100 million years. This is shorter than the hydrogen burning main sequence since the triple alpha process is less efficient in energy generation *and* the luminosity is greater. After the helium core is spent, there is a carbon/oxygen core and the star with masses between 2 and 6  $M_{\odot}$  ascend up the **Asymptotic Giant Branch** (AGB), where only hydrogen and helium shell burning can power them. A series of helium shell flashes occur, but they never get hot enough to ignite carbon, so will end their lives as C/O white dwarfs.

Stars traveling up the AGB will have a stable hydrogen burning shell dumping helium onto an unstable helium burning shell. That is, the rate of helium supplied by the hydrogen shell cannot be the rate at which helium stably burns in the helium shell. There is a net increase of mass entering the core,  $\dot{M}_{\text{in}}$ . Additionally, the weakly bound envelope is losing mass due to a wind taking mass to infinity,  $\dot{M}_{\text{wind}}$ . In the end, mass loss wins the game, and the degenerate core is left behind as a white dwarf, with most of the envelope mass lost into a nebula.

*Monday, March 5, 2012*

## 9 Endpoints of Stellar Evolution

### 9.1 White Dwarf Cooling

[MISSING DUE TO INABILITY TO BE UP BEFORE NOON.]

In the outer, non degenerate layer of the white dwarf, the material is geometrically thin and  $M_{\text{outer}} \ll M$ . Thus, the plane-parallel atmosphere prescription is valid. The surface gravity for a typical white dwarf is

$$g = \frac{GM}{R^2} = 1.6 \times 10^8 \text{ cm s}^{-2} \quad (9.1)$$

where we have assumed  $R = 7 \times 10^8 \text{ cm}$  and  $M = 0.6 M_{\odot}$ . Then the pressure is

$$P = g \frac{\Delta m}{4\pi R^2} = \frac{\rho k T}{\mu m_p} \quad (9.2)$$

With this, we can calculate where the electrons become degenerate. We find

$$\Delta M = 10^{-3} M_{\odot} T - 8^{5/2} \quad (9.3)$$

In the hydrogen and helium layer. Now assuming that the flux is constant, and that the opacity follows a Kramers' Law form,

$$F = \text{const} = \frac{1}{3} \frac{c}{\kappa \rho} \frac{d}{dr} (a T^4) \quad (9.4)$$

or

$$\frac{3F\kappa}{c} = \frac{d}{dy} a T^4 \quad (9.5)$$

Reintroducing the pressure, we get

$$\left[ \frac{3F}{4c} \kappa_0 \frac{\mu m_p}{kg} \frac{1}{a} \right] P dP = T^{7.5} dT \quad (9.6)$$

Now integrating this from the point where electrons are degenerate out to the surface, we get

$$\frac{L}{4\pi R^2} \frac{R^2}{GM} = \frac{F}{g} = \frac{2T_8^{8.5}}{8.5P^2} \frac{4cka}{3\kappa_0\mu m_p} \quad (9.7)$$

Solving for the luminosity and scaling, we find

$$L = 2.5 L_\odot \left( \frac{M}{0.6 M_\odot} \right) \left( \frac{T_c}{10^8 \text{ K}} \right)^{7/2} \quad (9.8)$$

This is known as **Mestel's Cooling Law**, though it isn't really accurate, since the opacity is changing as we get towards the degeneracy. Now all of the energy source is coming from the thermal content of the ions, which is approximately

$$E_{\text{th}} = \frac{3}{2} N_i kT = 1.25 \times 10^{48} \text{ ergs} \left( \frac{M}{0.6 M_\odot} \right) T_8 \quad (9.9)$$

Since the ions form an ideal gas. The luminosity is then obviously

$$L = -\frac{d}{dt} E_{\text{th}} \quad (9.10)$$

Note that the mass dependence cancels here. This gives us a differential equation:

$$\frac{T_8^{7/2}}{\tau} = -\frac{dT_8}{dt} \quad (9.11)$$

where  $\tau = 4 \times 10^6$  yrs. If we ignore  $t \gg \tau$ , this gives

$$T_8 = \left( \frac{2}{5} \frac{\tau}{t} \right)^{2/5} \quad (9.12)$$

Or more explicitly,

$$T_8 = \left( \frac{1.6 \times 10^6 \text{ yrs}}{t} \right)^{2/5} \quad (9.13)$$

Note that this solution is not valid at early time since we have ignored the initial data, however after a million years or so, the initial conditions are effectively forgotten anyway. Since we can find the central temperature from the mass and luminosity, we need only the mass and luminosity to determine a white dwarf's age.

We must get the mass through a measure of  $\log g$  through the spectrum. At higher gravity, we have

$$P_{\text{ph}} = \frac{g}{\kappa} = \frac{\rho kT}{\mu m_p} \quad (9.14)$$



So as the density goes up, the gravity goes up (and thus the mass goes up). And indeed, as the pressure increases, the line widths in a spectrum increase. This phenomenon is called **Pressure Broadening**. (This is so cool!).

There are some caveats to this theory, though.

1. Ions do interact via Coulomb physics. We typically have

$$\frac{e^2}{\langle r \rangle} \gtrsim kT \quad (9.15)$$

Then we have

$$\Gamma = \frac{Z^2 e^2}{a k T} \quad (9.16)$$

where  $a$  is the average ion spacing. When  $\Gamma > 1$ , ions interact strongly via Coulomb interactions.

When in a Coulomb solid,  $E_{\text{th}} = 3kT$ , since each positional degree of freedom now has an energy. This phase transition introduces a latent heat. The WD goes crystalline at  $\Gamma \approx 173$ . In a coulomb liquid, we have short-range order at  $\Gamma \sim 1$  and  $C_V$  increases relative to  $\frac{3}{2}N - ikT$ . Lindemann in the 1920's discovered the following rule: when the average displacement at a given site in a lattice is bigger than  $0.1a$ , the solid melts.

## 9.2 The Chandrasekhar Mass

There is a maximum mass for an object that is held up in hydrostatic balance with degenerate electrons. Remember that the number of quantum states is given by

$$N = \frac{2}{h^3} \int d^3x d^3p \quad (9.17)$$

So the number density is

$$n_e = \frac{2}{h^3} \int_0^{p_F} 4\pi p^2 dp = \frac{8\pi}{3h^3} p_F^3 \quad (9.18)$$

So the number density of the electrons *set* the Fermi momentum. Now in the non-relativistic limit, we suppose that  $p_F < m_e c$ , the pressure is

$$P = n_e^{5/3} \left( \frac{3h^2}{8\pi} \right)^{2/3} \frac{1}{5m_e} \quad (9.19)$$

For electrons, we typically have  $n_e = \rho/2m_p$  in a C/O white dwarf. Then expressing this in terms of the density (rather than number density), we get

$$P = \left( \frac{\rho}{2m_p} \right)^{5/3} \left( \frac{3h^2}{8\pi} \right)^{2/3} \frac{1}{5m_e} \quad (9.20)$$

From hydrostatic balance, we have

$$P_c \sim \frac{GM^2}{R^4} \propto \rho^{5/3} \quad (9.21)$$

This gives us a mass-radius relation,

$$R \propto M^{-1/3} \quad (9.22)$$

Which can be re-expressed in terms of the density,  $\rho \sim MR^{-3} \propto M^2$  when the electrons are non-relativistic. But what happens if the electrons *are* relativistic? The pressure is

$$P = \frac{1}{4}n_e E_F = \frac{1}{4}n_e p_{Fc} \propto \rho^{4/3} \quad (9.23)$$

since

$$p_F = \left( \frac{3h^3 n_e}{8\pi} \right)^{1/3} \quad (9.24)$$

(Recall in the relativistic limit,  $E = pc$ .) Then in this case, the pressure follows

$$P \propto \rho^{4/3} \propto \left( \frac{M}{R^3} \right)^{4/3} \propto \frac{M^2}{R^4} \quad (9.25)$$

Interestingly, the radius cancels, so this relation of pressure defines a mass, which is the **Chandrasekhar Mass**:

$$M_{\text{Ch}} = 1.45 M_{\odot} \left( \frac{2}{\mu_e} \right)^2 \quad (9.26)$$

where  $\mu_e$  is the number of baryons per electron. Then the maximum possible mass for a white dwarf is this Chandrasekhar mass. As the mass approaches this limit, effectively  $\rho$  is going to infinity much faster than  $\rho \propto M^2$ .

If the outer conditions increase above  $M_{\text{Ch}}$ , then the object will collapse.