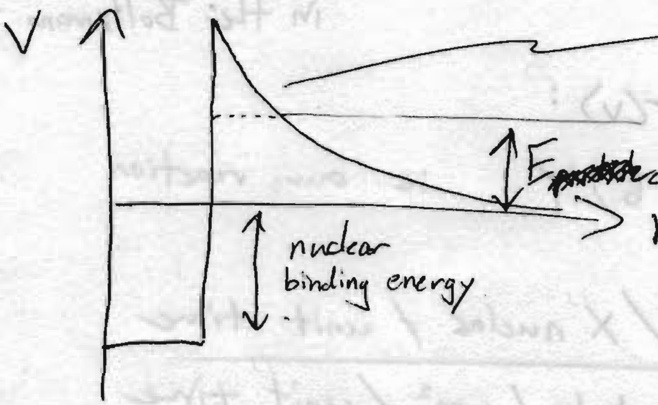
Stars w/ Lars
Lecture 9 - 2/8/12
Nuclear Reactions I

Coulomb barrier to overcome:



$$V(r) = \frac{Z_1 Z_2 e^2}{R} = 1.4\,\text{MeV}\; Z_1 Z_2 \left(\frac{R}{1\,\text{fm}}\right)^{-1}$$

So barrier penetration on the order of MeV's ...

Thermal energy $E_{th} \sim kT = 8.6\ \text{keV}\left(\frac{T}{10^8\,\text{K}}\right)$

If there was no quantum tunneling, we'd need temperatures of $T \sim \frac{Z_1 Z_2 e^2}{kR}$ for fusion to occur

$$\Rightarrow \frac{T}{10^8\,\text{K}} \sim 163\ \frac{Z_1 Z_2}{R/1\text{fm}}$$

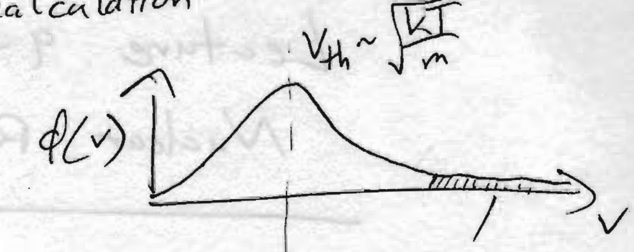
$$\Rightarrow \boxed{T \sim 1.6 \times 10^{10}\,\text{K}\ \frac{Z_1 Z_2}{R/1\text{fm}}}$$

which is much much hotter than the core of the sun

$(T_{c,\odot} \sim 1.5 \times 10^7\,\text{K})$

Two important ingredients in calculation

- Maxwell-Boltzmann distribution
- Tunnelling



still have lots of energetic particles in the Boltzmann tail

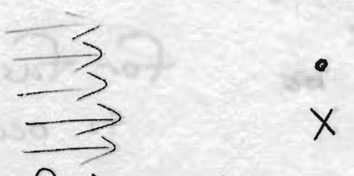
Reaction rates in terms of $\sigma(v)$:

$a + X \rightarrow Y + b$ or $X(a,b)Y$ is our reaction

define σ as $\dfrac{\#\text{ reactions} / X \text{ nucleus} / \text{unit time}}{\#\text{ incident particles} / cm^2 / \text{unit time}}$

$$\Rightarrow [\sigma] = cm^2$$

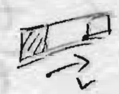
It is the analog for a physical cross section for long-range forces



flux of a's

$[n_a] = cm^{-3}$

$v_{rel} = cm/s$



$n_a \cdot v_{rel} = \frac{\#}{cm^2 \cdot s}$ (flux)

$$\Rightarrow n_a \sigma v_{rel} = \frac{\#\text{reactions}}{X \text{ nucleus} \cdot \text{unit time}}$$

$$\Rightarrow \boxed{r_{ax} = \frac{\#\text{reactions}}{cm^3 \cdot s} = n_a n_X \sigma v_{rel}}$$

Really we need to integrate over possible relative velocities (weighted accordingly via MB distribution)

$$\Rightarrow \quad r_{ax} = \int_0^\infty n_a n_x \, \sigma(v) \, v \, \phi(v) \, dv$$

can depend on v in general

probability of a pair of particles having relative velocity v

$$\left(\int_0^\infty \phi(v) dv = 1 \right)$$

$$\Rightarrow \quad r_{ax} = n_a n_x \langle \sigma v \rangle$$

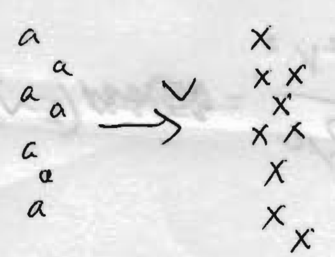
$$\text{where} \quad \langle \sigma v \rangle = \int_0^\infty \sigma(v) \, v \, \phi(v) \, dv$$

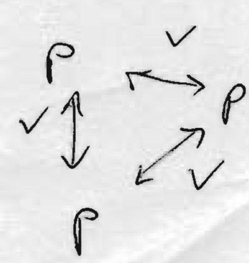
This is what we want to calculate when finding nuclear reaction rates

aside: r_{ax} vaid for $a \neq X$ if we have, say pp reactions then factor changes to $\frac{1}{2} n_p^2$

This is because we assumed we could separate out "target" X's from "incident" a's. We're really counting the number of possible reaction pairs $(X + a)$ so if they're the same, then there are $\frac{1}{2} n_p^2$ unique pairs.

can't make this picture now, it would turn into:





Clayton goes over how to transform
$\phi(v_1)\phi(v_2)$ into $\phi(v_{rel})\phi(v_{cm})$

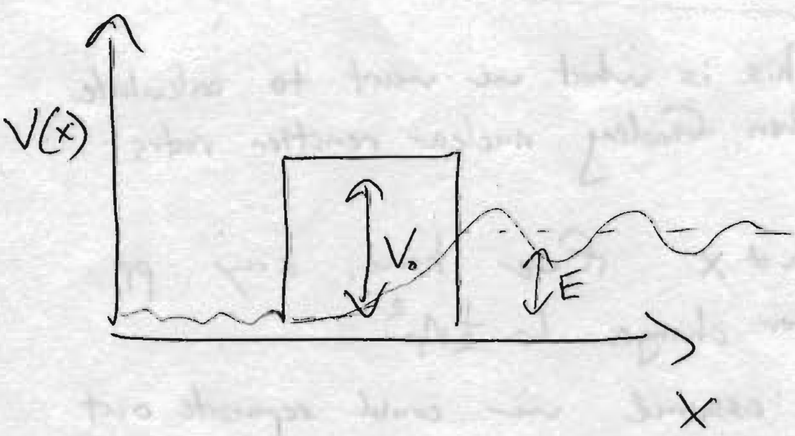
In the end, get

$$\langle \sigma v \rangle = 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(\frac{-\mu v^2}{2kT}\right) dv$$

$$\left(\mu = \frac{m_1 m_2}{m_1 + m_2}, \text{ reduced mass}\right)$$

Barrier penetration probability

Shrö eqn: $\left[\frac{-\hbar^2}{2\mu}\nabla^2 + V\cancel{\;}\right]\psi\cancel{\;} = E\psi$



- Outside the barrier,

$$\frac{-\hbar^2}{2\mu}\nabla^2\psi = E\psi$$

$$\Rightarrow \psi \propto e^{\pm ikx}$$

with $\frac{\hbar^2 k^2}{2\mu} = E$

- Inside the barrier, $\frac{-\hbar^2}{2\mu}\nabla^2\psi = (E-V)\psi$

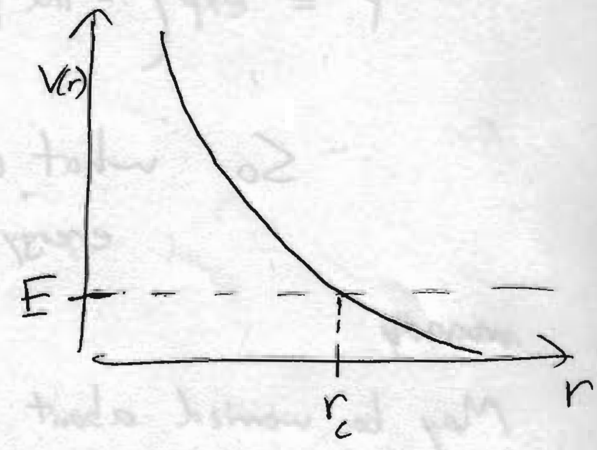
$E - V < 0$

$$\Rightarrow \psi \propto e^{\pm k_i x}$$

with $\frac{-\hbar^2 k_i^2}{2\mu} = \cancel{\;}(E-V)$ for $E < V$

For the Coulomb problem, $V(r) = \frac{Z_1 Z_2 e^2}{r}$

$$\Rightarrow \quad \frac{\hbar^2 K^2}{2\mu} = \frac{Z_1 Z_2 e^2}{r} - E$$



Use WKB to estimate barrier penetration prob:

(assumes lengthscale of $V(r)$ change $\gg \lambda_{particle}$)

$$\Rightarrow \quad \psi^2 \propto \exp\left(-2\int K\,dr\right)$$

$$K = \left(\frac{2\mu}{\hbar^2}\right)^{1/2} \left[\frac{Z_1 Z_2 e^2}{r} - E\right]^{1/2}$$

classical turning pt: r_c where $\frac{Z_1 Z_2 e^2}{r_c} = E$

$$\Rightarrow \quad r_c = \frac{Z_1 Z_2 e^2}{E}$$

$$\Rightarrow \quad K = \left(\frac{2\mu E}{\hbar^2}\right)^{1/2} \left[\frac{r_c}{r} - 1\right]^{1/2}$$

$$\Rightarrow \quad \int_{r_c}^{r_{in}} K\,dr = \left(\frac{2\mu E}{\hbar^2}\right)^{1/2} \int_{r_c}^{r_{in}} \left[\frac{r_c}{r} - 1\right]^{1/2} dr$$

In $r_{in} \to 0$ limit this gives us

$$\int_{r_c}^{0} K\,dr = \frac{\pi\alpha}{2} Z_1 Z_2 \left(\frac{2\mu c^2}{E}\right)^{1/2} \qquad \alpha = \frac{e^2}{\hbar c}$$

And thus a probability of

$$P = \exp\left(-\pi\alpha Z_1 Z_2 \left(\frac{2\mu c^2}{E}\right)^{1/2} \right)$$

So what units are $2\mu c^2 (\pi\alpha Z_1 Z_2)^2$?

energy! define $E_{Gamow} \equiv E_G = 2\mu c^2 (\pi\alpha Z_1 Z_2)^2$

~~[illegible]~~

May be worried about $r_{in} \to 0$ limit, shouldn't there be a strong dependence on how much of the Coulomb barrier we have to tunnel through?

✓ Yes! Calculation is actually done in Clayton Ch. 4

$\hookrightarrow \exp\left(\frac{2\pi Z_1 Z_2 e^2}{\hbar v}\left(1 - \frac{4}{\pi}\left(\frac{E}{E_c}\right)^{1/2} + \frac{2}{3\pi}\left(\frac{E}{E_c}\right)^{3/2} + \ldots\right)\right)$

$$E_G \sim 0.98 \text{ MeV } Z_1^2 Z_2^2 \left(\frac{\mu}{m_p}\right)$$

pp: $E_G \sim 0.5$ MeV

$P \simeq \exp\left(-\left(\frac{E_G}{E}\right)^{1/2}\right)$

$p + {}^{12}C$: $E_G \sim 33$ MeV

(duh, harder to tunnel through larger Coulomb barriers)

In the center of the sun

$kT \sim 1$ keV

$$\Rightarrow \exp\left(- \left(\frac{500 \text{ keV}}{1 \text{ keV}}\right)^{1/2} \right) \sim 2 \times 10^{-10}$$

So we need to really be out on the Boltzmann tail ~~for~~ rxn's to occur

Nuclear Reaction Rates in thermal plasma

$$\lambda = \frac{h}{p} = \frac{h}{(kT\mu)^{1/2}} \simeq 10^{-10}\ \text{cm}$$

$$r_{nuc} \simeq 1.3 \times 10^{-13}\ \text{cm}\ \ A^{1/3}$$

typically write

$$\sigma(E) = \cancel{\quad} \ \underline{4\pi\lambda^2} \left(\begin{array}{c}\text{dimensionless}\\ \text{stuff}\end{array}\right) \exp\left(-\left(\frac{E_G}{E}\right)^{1/2}\right)$$

nuclear ~~physics~~ physics is here

motivated by
partial wave scattering in QM

$$\cancel{\quad}\ 4\pi\bar{\lambda}^2 = \frac{4\pi}{k^2} \qquad\qquad E = \frac{\hbar^2 k^2}{2\mu}$$

$$\Rightarrow 4\pi\bar{\lambda}^2 = \frac{2\pi\hbar^2}{\mu E} = 2000 \text{ barns} \left(\frac{\text{keV}}{E}\right)$$

$$1 \text{ barn} = 10^{-24}\ \text{cm}^2$$

$$\Rightarrow \sigma(E) = \frac{S(E)}{E} \exp\left(-\left(\frac{E_G}{E}\right)^{1/2}\right)$$

write this way so most of the quickly varying energy dependence is out of $S(E)$

typical values for $S(E)$ are thus $\simeq 2000$ barns·keV
except for

$$pp \rightarrow D + e^+ + \nu_e \qquad (Q = 1.4\ \text{MeV})$$

$$S = 4 \times 10^{-22}\ \text{barns} \cdot \text{keV}$$

why? Weak interaction!
(β^+-decay)

recall there's about a $10^{25}\times$ difference between weak & strong timescales in nucleus

Now back to:

$$\langle \sigma v \rangle = 4\pi \left(\frac{\mu}{2kT}\right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(\frac{-\mu v^2}{2kT}\right) dv$$

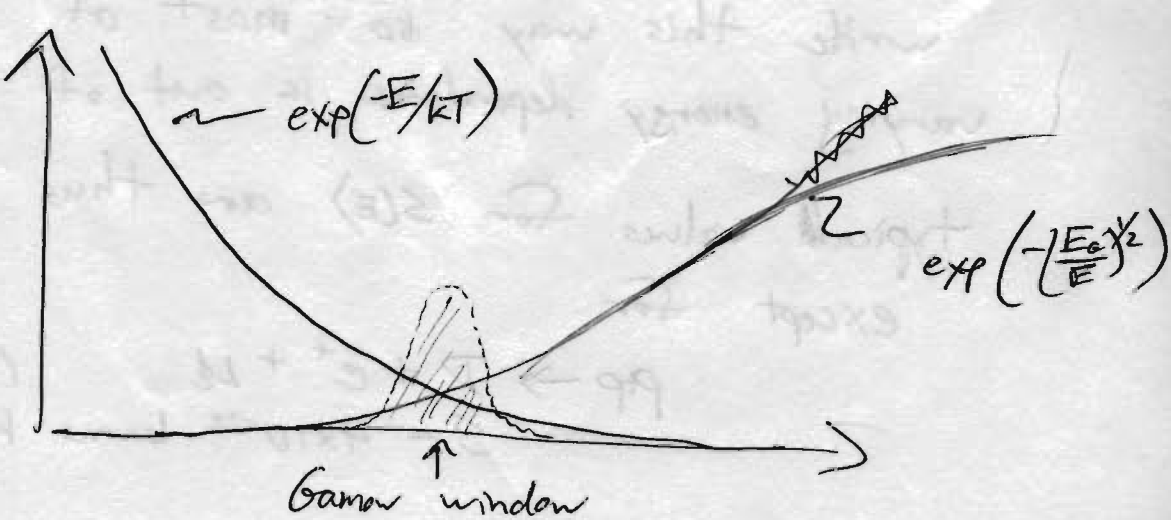
$$E = \frac{1}{2}\mu v^2 \qquad dE = \mu v\, dv$$

at temps we care about, we're <u>highly</u> non-relativistic
still $\left(\frac{1\,\mathrm{keV}}{m_p}\right)^{1/2} \sim 3\times 10^5$ m/s $\Rightarrow \frac{v}{c} \sim 10^{-3}$

$$\Rightarrow \langle \sigma v \rangle = \frac{1}{(kT)^{3/2}} \left(\frac{8}{\pi\mu}\right)^{1/2} \int_0^\infty dE\ S(E) \exp\left[\frac{E}{kT} - \left(\frac{E_G}{E}\right)^{1/2}\right]$$

Boltzmann tail — Tunneling

Let's assume for the moment that $S(E) \sim$ constant over the part of integral that contributes most (as it often is)



How to approximate such an integral?

Steepest decent!

$$I = \int_{-\infty}^{\infty} g(x)\, e^{-f(x)}\, dx$$

if $g(x)$ slowly varying and $f(x)$ sharply has min. @ $x = x_0$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \tfrac{1}{2} f''(x_0)(x-x_0)^2 + \dots$$
$$\simeq f(x_0) + \tfrac{1}{2} f''(x_0)(x-x_0)^2$$

$$\Rightarrow \quad I = g(x_0)\, e^{-f(x_0)} \int_{-\infty}^{\infty} \exp\left(-\frac{f''(x_0)}{2}(x-x_0)^2\right) dx$$

$$I = g(x_0)\, e^{-f(x_0)} \sqrt{\frac{2\pi}{f''(x_0)}}$$

For us, have $f(E) = \dfrac{E}{kT} + \left(\dfrac{E_G}{E}\right)^{1/2}$

$$\left.\frac{df}{dE}\right|_{E_0} = 0 \Rightarrow E_0^{\ 3} = \frac{1}{4} E_G (kT)^2$$

and $f(E_0) = \dfrac{E_0}{kT} + \left(\dfrac{E_G}{E_0}\right)^{1/2} = 3\left(\dfrac{E_G}{4kT}\right)^{1/3}$

$$\left.\frac{d^2 f}{dE^2}\right|_{E_0} = \frac{3(kT)^{1/3}}{2^{1/3} E_G^{1/3} (kT)^2} = 3\left(2 E_G (kT)^5\right)^{-1/3}$$

$$\hookrightarrow \exp\left(\frac{-3}{2}\left(2E_G (kT)^5\right)^{-1/3} (E-E_0)^2\right)$$

So our integral becomes

$$\langle \sigma v \rangle \simeq \frac{1}{(kT)^{3/2}}\left(\frac{8}{\pi\mu}\right)^{1/2} S(E_0)\, \exp\left(-3\left(\frac{E_G}{4kT}\right)^{1/3}\right)\int_0^\infty dE\, \exp\left(\frac{-(E-E_0)^2}{(\Delta/2)^2}\right)$$

where $\left(\frac{\Delta}{2}\right)^2 = \frac{2\,(2E_G^{*}(kT)^5)^{1/3}}{3}$

$$\Rightarrow \quad \Delta = \frac{(2E_0^{*}(kT)^5)^{1/6}}{\sqrt{3}}\sqrt{2}\cdot 2$$

$$= E_G^{*1/6}(kT)^{5/6}\,\frac{1}{\sqrt{3}}\,2\cdot 2^{2/3}$$

$$\boxed{\Delta = \frac{4}{\sqrt{3}\,2^{1/3}}\,E_G^{1/6}(kT)^{5/6}}$$

So, for <u>non-resonant reactions</u> we have:

$$\langle \sigma v \rangle \simeq 2.6\,\frac{E_G^{1/6}}{\sqrt{\mu}}\,\frac{S(E_0)}{(kT)^{2/3}}\,\exp\left[-3\left(\frac{E_G}{4kT}\right)^{1/3}\right]$$

$$\text{and} \quad \frac{\Delta}{E_0} = \frac{4}{\sqrt{3}}\left(\frac{kT}{E_0}\right)^{1/2}$$

$p+p$	$p+{}^{12}C$
$E_G = 494$ keV	$E_G = 36$ MeV
$E_0 = 4.5\text{ keV}\left(\frac{T}{10^7 K}\right)^{2/3}$	$E_0 = 18\text{ keV}\left(\frac{T}{10^7}\right)^{2/3}$
$\frac{\Delta}{E_0} = 1\cdot\left(\frac{T}{10^7}\right)^{1/6}$	$\frac{\Delta}{E_0} = 0.5\left(\frac{T}{10^7}\right)^{1/6}$