Relative Energetics

- The nuclear force is only "seen" at short distances $\simeq$ fm, so we need to find the overlap of two charged particles at such a small distance. The typical thermal energy is

$$E_{th} = KT = 1\,\text{keV} = \frac{p^2}{2m_n} = $$

- $\hookrightarrow P \simeq \frac{h}{\lambda} \Rightarrow \lambda = \frac{h}{p} \simeq 10^{-10}$ cm

is still >> nuclear size.

So we want to first calculate the overlap probability at zero separation as the proton zip by each other.

-

$$\left[\frac{-\hbar^2}{2m_r}\nabla^2 + V(r)\right]\psi(r) = E\,\psi(r)$$

However, you all have likely seen just how messy the Coulomb Scattering problem is (Merzbacher, Sakurai).

However, much of the essence is captured in a 1D calc, as follows.

~~Now instead what was seen was a continuous e^- spectrum which could only be understood if there was a 3rd particle present that was difficult to detect. ⇒ The ν_0. It interacts very weakly with matter, having~~

$$\sigma \approx 10^{-44} \left(\frac{E_\nu}{m_e c^2}\right)^2 \text{cm}^2$$

~~with n, p, e^- ...~~

Getting Under the Coulomb Barrier

The ions have c.m. energies $\ll E_c$, and so their λ are much $\gg$ than the nuclear size. We start with the reduced mass.

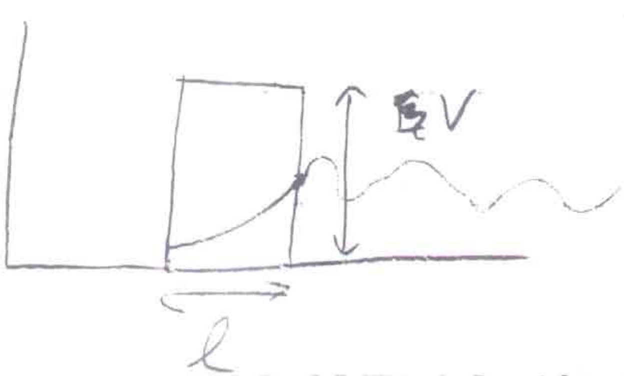
$$m_r = \frac{m_1 m_2}{m_1 + m_2}.$$

For More details, See Clayton

Sch Equation is.

$$\left[-\frac{\hbar^2}{2m_r}\nabla^2 + V(r)\right]\psi(r) = E\psi(r)$$

Lets start with a planar barrier



Outside this $V = 0$

$$\Rightarrow \frac{-\hbar^2}{2m_r}\nabla^2\psi = E\psi$$

which for $\psi = e^{ikx}$ gives

$$\frac{\hbar^2 k^2}{2m_r} = E$$

Inside the barrier, the picture is

$$\frac{-\hbar^2}{2m_r}\frac{d}{dx^2}\psi = (E-V)\psi$$

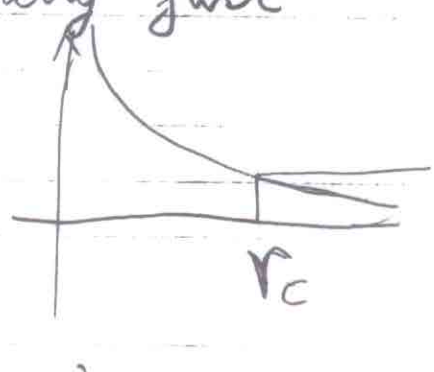
and since $E-V<0$, we get $e^{-k_i x}$
where

$$\frac{\hbar^2 k_i^2}{2m_r} \approx V$$

and so it decays. Indeed for a constant barrier, we get a decay amount of

$$\psi^2 \approx \left[e^{-k_i \Delta x}\right]^2$$

The Coulomb problem is really just



$$\frac{-\hbar^2 k^2}{2m_r} = \frac{-Z_1 Z_2 e^2}{r} + E$$

so

$$\frac{\hbar^2 k^2}{2m_r} = \frac{Z_1 Z_2 e^2}{r} - E$$

and we start at $r_c = \frac{Z_1 Z_2 e^2}{E}$ and go into r_{in}. Lets get the WKB limit, which is that there is a 'local wavenumber'

Should expand ... any
WKB might work + give the full answer.

---

and that $\exp(-k \Delta x) \Rightarrow \exp\left(-\int k\,dr\right)$
so we want to integrate:

$$k = \left(\frac{2m_r}{\hbar^2}\right)^{1/2}\left(\frac{Z_1 Z_2 e^2}{r} - E\right)^{1/2}$$

define r_c as the classical turning point, then $Z_1 Z_2 e^2/E = r_c$

$$k = \left(\frac{2m_r E}{\hbar^2}\right)^{1/2}\left(\frac{r_c}{r} - 1\right)^{1/2}$$

The integral is just

$$I = \int k\,dr = \left(\frac{2m_r E}{\hbar^2}\right)^{1/2}\int_{r_n}^{r_c}\left(\frac{r_c}{r} - 1\right)^{1/2} dr$$

trick, lets first define $X = r/r_c$, then:

$$I = \left(\frac{2m_r E}{\hbar^2}\right)^{1/2} * r_c \int_{r_n/r_c}^{1}\left(\frac{1}{X} - 1\right)^{1/2} dX$$

So first, ~~the prefactor is~~

$$I = \left(\frac{2m_r E}{\hbar^2}\right)^{1/2} * \frac{Z_1 Z_2 e^2}{E} = \frac{e^2}{\hbar c} Z_1 Z_2 \frac{(2m_r c^2)^{1/2}}{E^{1/2}}$$

$$= \alpha Z_1 Z_2 \left(\frac{2m_r c^2}{E}\right)^{1/2}$$

$$e^2 = \alpha \hbar c$$

So

$$I = \underbrace{\left(\frac{2m_r E}{\hbar^2}\right)^{1/2} \cdot \frac{Z_1 Z_2 e^2}{E}}\int_{r_n/r_c}^{\cancel{\ }1}\left(\frac{1}{X}-1\right)^{1/2} dX$$

$$= \left(\frac{2m_r c^2}{E}\right)^{1/2} \frac{1}{c} \frac{\alpha \hbar \cancel{c}}{\cancel{\hbar}} Z_1 Z_2 = \alpha Z_1 Z_2 \left(\frac{2m_r c^2}{E}\right)^{1/2}$$

And the integral is

$$\int_{r_n/r_c}^{\cancel{\ }1}\left(\frac{1}{X}-1\right)^{1/2} dX \qquad r_n/r_c \to 0 \quad \to 0$$

Ok as $X \to 0$ as rewrite.

$y = 1/x \quad dy = -\frac{1}{x^2}\,dx \Rightarrow dx = -x^2 dy = -\frac{1}{y^2}dy$

so

$$\int_1^\infty (y-1)^{1/2}\left(\frac{-1}{y^2}dy\right) \text{ — fine.}$$

Easier to do the integral as

$X = \cos^2\theta$

$$dX = 2\cos(\theta)\, d(\cos\theta)$$

$$\Rightarrow \quad dX = -2\cos\theta \sin\theta\, d\theta$$

$$\int_{\pi/2}^{0}\left(\frac{1}{\cos^2\theta}-1\right)^{1/2}(-2\cos\theta\sin\theta)\,d\theta$$

$c^2 + s^2 = 1$

$$\frac{1}{c^2} - \frac{c^2}{c^2} = \frac{1-c^2}{c^2} = \frac{s^2}{c^2} = \frac{s}{c}.$$

$$\Rightarrow \int_0^{\pi/2} 2 \sin^2\theta \, d\theta = 2 \cdot \pi/4 = \pi/2 \quad \text{so}$$

$$I = \frac{\pi}{2} \alpha Z_1 Z_2 \left(\frac{2 m_r c^2}{E} \right)^{1/2}$$

Which for the probability of penetration we square to get

$$P = \exp\left(- \pi \alpha Z_1 Z_2 \left(\frac{2 m_r c^2}{E} \right)^{1/2} \right)$$

which allows us to define the Gamow energy

$$\boxed{E_G = (\pi \alpha Z_1 Z_2)^2 (2 m_r c^2)}$$

$$P = \exp\left(- \left(\frac{E_G}{E} \right)^{1/2} \right)$$

$$E_G = 0.988 \; Z_1^2 Z_2^2 \left(\frac{m_r}{m_p}\right) \text{ MeV}$$

This will be the $\boxed{\text{primary}}$ reason why heavier elements are harder to form.

$$m_r = \frac{m_1 m_2}{m_1 + m_2}.$$

P + P	E_G
p+p	0.494 MeV
$p + {}^{12}C$ ~~[illegible]~~	~~[illegible]~~ 33 MeV

So at the center of the sun, the penetration is

$$\left(\frac{494 \text{ keV}}{\text{keV}}\right)^{1/2} \sim 2 \times 10^{-10}$$

and clearly we need to look out most closely for the thermal particles out on the tail.

Can also write this somewhat differently, namely.

$$= \pi \frac{e^2}{\hbar c} Z_1 Z_2 \left(\frac{2 m_r c^2}{\frac{1}{2} m_r v^2}\right)^{1/2}$$

$$= 2\pi \frac{e^2 Z_1 Z_2}{\hbar v}.$$

This is piece in exponent.

In this limit, we can look at this factor as:

$$= \frac{E_{coul}(\lambda)}{\frac{1}{2} m_r v^2}$$

where $$E_{coul}(\lambda) \simeq \frac{e^2 z_1 z_2}{\lambda}$$

where $$\lambda = \frac{h}{p} = \frac{h}{m_r v}$$

so $$\frac{e^2 z_1 z_2}{h} \frac{2 \cancel{m_r} v}{\cancel{m_r} v^2} = \frac{e^2 z_1 z_2}{h v}$$

What is $\frac{\lambda}{r_c} = \frac{h}{\cancel{m_r} v} \frac{\cancel{m_r} v^2}{e^2}$

$$\simeq \frac{hv}{e^2}$$

Dont Say This

So this large # in the exponent is basically the ratio of the classical Coulomb radius to the particle's de Broglie λ outside the barrier.

Another way to say it is that the Coulomb energy cannot be higher than that resolved by Q.M. effects, in which case

$P \propto \exp(-E_c/kT)$ roughly speaking

Lecture 11

Nuclear Cross-Sections

As noted earlier most fusion reaction of interest to us have

$$\lambda = \frac{h}{p} = \frac{h}{(kTm_r)^{1/2}} \approx 10^{-10}\ \text{cm at a keV}$$

$\gg R_{nuc} \simeq 1.3\,\text{fm}\,A^{1/3}$ so Quantum mechanics is the only way to go. The fact that $\lambda \gg R$ is no surprise as the ions are very far from degeneracy.

The reaction physics is completely set by the energy in the COM

$$E_{com} = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\left|\underline{v}_1 - \underline{v}_2\right|^2$$

and since $\lambda \ggg R_{potential}$, we typically carry out a "partial wave" expansion in order to do reactions. In this regime λ is obviously crucial for the nuclear cross-section and $(s=0)$ is all that matters. Thus we expect

$$\sigma(E) \simeq \left(4\pi \lambda\!\!\!^{-2}\right)\left(\begin{matrix}\text{dimen}\\ \text{stuff}\end{matrix}\right)\exp(-\text{Coul})$$

$\uparrow$

Nuclear Mess

where $4\pi \lambda\!\!\!^{-2} = 4\pi \frac{1}{k^2} = \frac{4\pi}{k^2}$

where $E_c = \frac{\hbar^2 k^2}{2m} \Rightarrow$ $k^{-2} = \frac{\hbar^2}{2mE_c}$

$$= \frac{2\pi\hbar^2}{mE_c} = 2000 \text{ barns} \left(\frac{\text{keV}}{E_c}\right)$$

Hence we introduce an "S" factor' and write:

$$\sigma(E) = \frac{S(E)}{E} \exp\left[-\left(\frac{E_G}{E}\right)^{1/2}\right]$$

so that much of the nuclear physics is captured in one more slowly varying function $S(E)$. If the field piece then

$$S(E) \approx 2000 \text{ barns} \cdot \text{keV}$$

This also helps in extrapolating from the energies obtainable in the lab to those much lower that occur in stars.

Note that this extrap. should be reasonable, as long as the CoM energies are all far below the MeV typical to nuclear excited state structure

End Lecture 10 →

If dimensionless stuff = 1 then:
$S(E) = 2000$ barns · keV,

What are typical values? About this or less, expect for one critical reaction

$$p + p \rightarrow D + e^+ + \nu \qquad (Q = 1.4\,MeV)$$

which has $S \sim 4 \times 10^{-22}$ barn · keV

or 10^{25} times lower than a typical reaction. Where does this small number come from? It is due to the weak interaction and I have one way of looking at it though it is not clearly the correct way or only way.

The interaction lasts a time

$$t_{int} \sim \frac{\lambda}{v} \approx \frac{h}{pv} \approx \frac{h}{E_{com}} \sim 10^{-18} \text{ sec if } E_{com} = 10\,keV$$

But during the interaction, there is some rate for a weak interaction. ~~we stop~~ Since it is highly improbable we can use a Fermi Golden rule approach

I will not go through this in detail, but if E = excess energy (ie. ⇒ what goes into kinetic energy of ion + e^+ + ν_e), then

$$W = \frac{\#}{\text{sec}} = \text{decay-rate}$$

$$\approx \frac{1}{60\pi^3} \frac{G_F^2}{(\hbar c)^7} c\, E^5$$

and $G_F \approx 100\, eV \cdot fm^3$

Check it for β-decay:

$$= \frac{1}{60\pi^3} \frac{(10^{-4}\, MeV \cdot fm^3)^2}{(200\, MeV \cdot fm)^7} c \cdot (1.3\, MeV.)^5$$

$$= 10^{-8} \frac{\cdot \frac{1}{3}}{\underbrace{(200)^7}_{= 10^{-27}}} \left(\frac{c}{fm}\right)$$

$\sim$ ~~4×10^5~~ $s^{-1} = 3 \times 10^{-4}$ ⇒ 3000 sec.

⇒ 55 minutes

Ok, so this works reasonably well.

$$G_F = 10^{-4}\, MeV \cdot fm^3$$

and motivates my guess for

P = probability of weak interaction during the othe..

$$W \cdot t_{int} = \left(\frac{h}{E_{com}}\right) \frac{G_F^2 \, E^5 c}{60\pi^3 (\hbar c)^7}$$

$$= \frac{G_F^2 \, E^5}{30\pi^2 (\hbar c)^6} \frac{1}{E_{com}} = \frac{(10^{-4} \, \mathrm{MeV\,fm^3})^2 \, (E/\mathrm{MeV})^5}{30 \cdot 9 \, (200 \, \mathrm{MeV \cdot fm})^6} \left(\frac{\mathrm{MeV}}{E_c}\right)$$

$$= \cancel{6} \cdot \frac{10^{-8} \, \mathrm{MeV^2 fm^6 \, MeV^5}}{64 \cdot 10^{12} \cdot 270 \, \mathrm{MeV^6 \cdot fm^6}} \left(\frac{E}{\mathrm{MeV}}\right)^5 \left(\frac{\mathrm{MeV}}{E_c}\right)$$

$$= 6 \times 10^{-25} \left(\frac{E}{\mathrm{MeV}}\right)^5 \left(\frac{\mathrm{MeV}}{E_{com}}\right).$$

Before $\sim$ keV

$\hat{=} \, 6 \times 10^{-22}$ when $E \sim \mathrm{MeV}$

Reaction Rates

We are now in a position to calculate the reaction rate. Keep in mind that the nuclear physics depends completely on the com energy. The fusion rate is:

$$\langle \sigma v \rangle = \int_0^\infty v_r \, \sigma(v_r) \, P(v_r) \, dv$$

first give the simple arg. $\lambda = \frac{1}{\sigma n}$ $\tau = ?$

where

$P(v_r)\,dv_r$ = probability that two particles have a relative velocity between v_r & $v_r + dv_r$.

This then gives a local rate

$$R_{fus} = n_1 n_2 \langle \sigma v \rangle_{12}$$

Now, the probability distribution for relative velocities ends up also just being the MB distribution, so

$$P(v_r)\,dv_r = \left[\frac{m_r}{2\pi kT}\right]^{3/2} \exp\left(\frac{-m_r v_r^2}{2kT}\right) d^3v_r$$

Usually we write down a mean free path $\lambda = \frac{1}{\sigma n}$

and a ~~rate, or~~ timescale.

$$t_{coll} = \frac{1}{\sigma n v}.$$

However, ~~[illegible]~~ in a thermal plasma such as we have here and with a very energy sensitive $\sigma(E)$, we need to derive more carefully the rates, so

$$r = n\sigma v \rightarrow = n\langle\sigma v\rangle$$

where $\langle\sigma v\rangle$ is the thermal average.

so

$$\langle \sigma v \rangle = \left[\frac{m_r}{2\pi k_b T}\right]^{3/2} \int_0^\infty \exp\left(\frac{-m_r v_r^2}{2kT}\right) 4\pi v_r^2 \, dv_r$$

$$* \; v_r \, \sigma(v_r).$$

now set $E = \frac{1}{2} m_r v_r^2$, so

$dE = m_r v_r \, dv_r$, then

$$\langle \sigma v \rangle = 4\pi \left[\frac{m_r}{2\pi kT}\right]^{3/2} \int_0^\infty \exp\left(\frac{-E}{kT}\right) \frac{2E}{m_r} \frac{dE}{m_r} \sigma$$

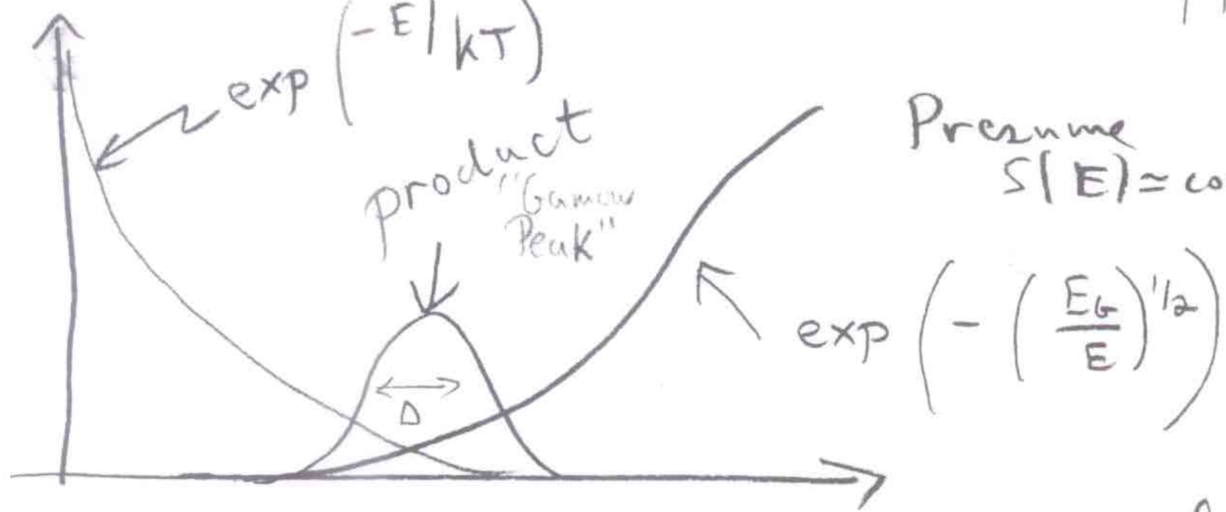
but $\sigma(E) = \frac{S(E)}{E} \exp\left[-\left(\frac{E_G}{E}\right)^{1/2}\right]$

$$= 8\pi \frac{1}{m_r^{1/2}} \left(\frac{1}{2\pi kT}\right)^{3/2} \int_0^\infty \exp\left(\frac{-E}{kT}\right) S(E) \exp\left(-\left(\frac{E_G}{E}\right)^{1/2}\right) dE$$

$$\frac{8\pi}{2\pi\sqrt{2\pi}\, m_r^{1/2}} = \frac{4}{m_r^{1/2}\sqrt{2\pi}} = \left(\frac{16}{2\pi m_r}\right)^{1/2} =$$

$$\langle \sigma v \rangle = \frac{1}{(kT)^{3/2}} \left(\frac{8}{\pi m_r}\right)^{1/2} \int_0^\infty S(E) \exp\left(\frac{-E}{kT} - \left(\frac{E_G}{E}\right)^{1/2}\right) dE$$

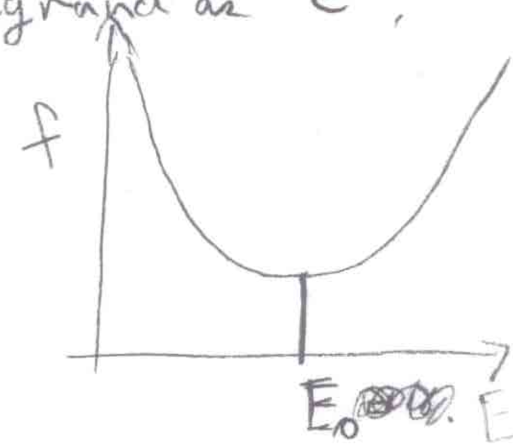
Now, the stuff in the exponent is most impt, and lets look at this carefully under the presumption that $S(E)$ is constant.



Presume $S(E) \simeq$ const

Lets write the integrand as e^f, where

$$f(E) = \frac{E}{KT} + \left(\frac{E_G}{E}\right)^{1/2}$$



$$\frac{\partial f}{\partial E} = \frac{1}{KT} - \frac{1}{2}\frac{E_G^{1/2}}{E^{3/2}}$$

$$\frac{\partial f}{\partial E} = 0 \Rightarrow \quad E^{3/2} = \frac{1}{2} KT\, E_G^{1/2}$$

$$\Rightarrow \quad \boxed{E_0^3 = \frac{1}{4} E_G (kT)^2}$$

Now, we want to expand around the minimum, so

$$f(E) = f(E_0) + \underbrace{\cancel{\frac{df}{dE}\Big|_{E_0}}}_{0 \text{ by df'n.}}(E - E_0) + \frac{1}{2}(E - E_0)^2 \frac{d^2 f}{dE}$$

First

$$f(E_0) = \frac{E_0}{KT} + \left(\frac{E_G}{E_0}\right)^{1/2} = 3\left(\frac{E_G}{4kT}\right)^{1/3}$$

$$\frac{\partial^2 f}{\partial E^2} = + \frac{3}{4}\frac{E_G^{1/2}}{E_0^{5/2}}$$,

~~so we get~~

SPARE Algebra. 143a

So we get

$$\langle \sigma v \rangle = \frac{1}{(kT)^{3/2}} \left(\frac{8}{\pi m_r}\right)^{1/2} S(E_0) \exp\left(-3\left(\frac{E_G}{4 k_B T}\right)^{1/3}\right)$$

$$* \int_0^\infty dE \exp\left(-\frac{3}{8} \frac{E_G^{1/2}}{E_0^{5/2}} (E-E_0)^2\right)$$

↑ Lect 13

Lect 14 ↓ Let's rewrite as

$$(E-E_0)^2 * \frac{4}{\Delta^2}$$

so $\frac{4}{\Delta^2} = \frac{3}{8} \frac{E_G^{1/2}}{E_0^{5/2}}$

so $\Delta^2 = \frac{32}{3} \frac{E_0^{5/2}}{E_G^{1/2}}$

$$\Delta = \sqrt{\frac{32}{3}} \frac{E_0^{5/4}}{E_G^{1/4}} = \sqrt{\frac{32}{3}} \left[\frac{E_G (kT)^2}{4}\right]^{\frac{5}{12}} \frac{1}{E_G^{\frac{1}{4}}}$$

$\frac{5}{12} - \frac{3}{12} = \frac{2}{12} = \frac{1}{6}$.

$$= \sqrt{\frac{2}{3}} \frac{4 E_G^{1/6} (kT)^{5/6}}{4^{5/12}} = \sqrt{\frac{2}{3}} \frac{4}{2^{5/6}} \Rightarrow$$

$$E_G = (0.982\,MeV)\, Z_1^2 Z_2^2 \left(\frac{m_r}{m_p}\right)$$

So the Gamow Peak occurs at

$$E_0 = \left[\frac{1}{4} E_G (kT)^2\right]^{1/3}$$

reminder $\to \boxed{E_G = (\pi \alpha Z_1 Z_2)^2 (2 m_r c^2)}$

with a width

$$\Delta = \frac{4}{\sqrt{3}} (E_0 kT)^{1/2} \quad \left[\text{Geometric mean of } E_0 \;\&\; kT\right]$$

so $$\frac{\Delta}{E_0} \simeq \frac{4}{\sqrt{3}} \left(\frac{kT}{E_0}\right)^{1/2} =$$ comes from the Gaussian

How sharp are these and how narrow is the peak?

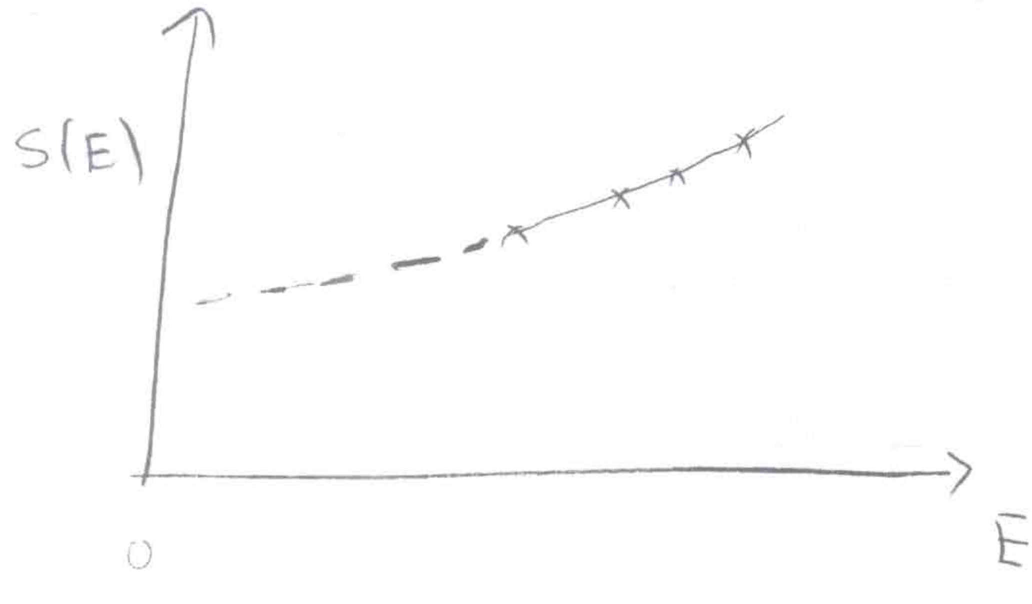
$\boxed{p+p}$	$\boxed{p + {}^{12}C}$
$E_G = 494$ keV	$E_G = 32.63$ MeV
$E_0 = 4.5\, T_7^{2/3}$ keV	$E_0 = 18.2$ keV $T_7^{2/3}$
$\frac{\Delta}{E_0} \approx 1\, T_7^{1/6}$	$\frac{\Delta}{E_0} \approx 0.5\, T_7^{1/6}$

So all the energies are typically <u>much lower</u> than the characteristic nuclear energy, so that the COM energy is nearly <u>zero</u> at typical T. $\Rightarrow$ (OVER)

This is an important point, as the lab measurements are difficult to make at low energies.

Quite ~~of the~~ often then what is done is to measure the reaction at higher energies and extrapolate down to lower energies.

Here is where the S(E) factor plays an important role, as if there are no resonances etc ~~in it~~ as the COM energy changes, then one can expect it to be a smooth function. In the past, things were that



$$\Rightarrow \boxed{\Delta = \frac{4}{\sqrt{3}\,2^{1/3}} E_G^{1/6} (kT)^{5/6}}$$

The integral is then just.

$$\int_0^\infty dE \exp\left(-\frac{(E_0-E)^2}{(\Delta/2)^2}\right) = 2\int_{-\infty}^{\infty}$$

$$= \frac{\Delta}{2}\sqrt{\pi}$$

so we have:

$9/6 - 5/6 = 4/6$

$$\langle \sigma v \rangle = \frac{1}{(kT)^{3/2}} \left(\frac{\cancel{4}\cdot 2}{\cancel{4} m_r}\right)^{1/2} S(E_0) \exp\left(-3\left(\frac{E_G}{4kT}\right)^{1/3}\right)$$

$$* \Delta$$

So

$$\boxed{\langle \sigma v \rangle = \frac{\sqrt{2}\cdot 4}{\sqrt{3}\,2^{1/3}} \frac{E_G^{1/6} (kT)^{5/6}}{\sqrt{m_r}\,(kT)^{3/2}} S(E_0) \exp.}$$

$9/6 - 5/6 = 4/6 = 2/3$

$$\langle \sigma v \rangle = \frac{1}{(kT)^{3/2}} \left(\frac{8}{\pi m_r}\right)^{1/2} \int_0^\infty S(E)\, dE \exp\left(-3\left(\frac{E_G}{4kT}\right)^{1/3} - \frac{3}{8} \frac{E_G^{1/2}}{E_0^{5/2}} (E - E_0)^2\right]$$

$$= \frac{1}{(kT)^{3/2}} \left(\frac{8}{\pi m_r}\right)^{1/2} \exp\left[-3\left(\frac{E_G}{4kT}\right)^{1/3}\right]$$

the correction to these approx Clayton p307→309

$$* \int_0^\infty S(E) \exp\left[-\frac{3}{8} \frac{E_G^{1/2}}{E_0^{5/2}} (E - E_0)^2\right] dE$$

Now, if we can presume that $S(E)$ = constant over E_0 then we will pull it out [a NON-RESONANT Reaction] and do the integral over the Gaussian.

After much manipulation this yields:

$$\boxed{\langle \sigma v \rangle = 2.6 \frac{E_G^{1/6}}{\sqrt{m_r}} \frac{S(E_0)}{(kT)^{2/3}} \exp\left[-3\left(\frac{E_G}{4kT}\right)^{1/3}\right]}$$

units.
$$\frac{\cancel{erg^{1/6}}\ erg \cdot cm^2}{gr^{1/2}\ \cancel{erg^{4/6}}} = \frac{cm}{s}\ cm^2 = \frac{cm^3}{s}\ \checkmark.$$