How to implement a digital biquadratic notch filter (in software)

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# Background

As presented in the [complimentary white paper](https://github.com/branbick/oscar/blob/main/biquad_notch_filter/doc/damping_ratios.pdf), the continuous-time transfer function of a biquadratic—or “biquad”—notch filter is …

Where Laplace variable, damping ratio (—), center/notch frequency (rad/s), and the subscripts and designate “numerator” and “denominator,” respectively. is a design parameter (i.e., an input), and the aforementioned white paper derives an algorithm for calculating and . That’s great, and can be practically utilized with the “help” of third-party software such as [MathWorks’ Control System Toolbox](https://www.mathworks.com/products/control.html). But what if the goal is to implement a notch filter in our *own* software—software running on an embedded system, for example? If that’s the case, then a *digital* filter is needed, and its coefficients and corresponding difference equation must be determined.

# Derivation

## Bilinear transform

The -transform is the discrete-time counterpart of the [Laplace transform](https://lpsa.swarthmore.edu/LaplaceXform/FwdLaplace/LaplaceXform.html), where ...

And sampling period (s). The first-order, [Maclaurin-series](https://mathworld.wolfram.com/MaclaurinSeries.html) approximation of is . Thus, …

The latter result is known as the bilinear transform: replacing with allows a continuous-time transfer function to be converted into a discrete-time one. Unfortunately, that transformation causes a “warping” of the frequencies—because, after all, it’s based on an approximation. For example, upon applying the bilinear transform to , a Bode plot of the consequent *digital* filter will show that —which was specified for the original, *analog* filter—shifted. Obviously, that’s problematic. Thankfully, it can be remedied via frequency prewarping.

## Frequency prewarping

Let and respectively represent the transfer functions of 1) an analog filter and 2) its equivalent digital version, constructed using the bilinear transform. Then, their respective frequency responses are obtained by evaluating and—employing a previous definition—, where -domain frequency (rad/s) that maps to (rad/s) in the domain. Equating the two frequency responses yields …

(Refer to [Wikipedia](https://en.wikipedia.org/wiki/Bilinear_transform#Frequency_warping) for the “redacted” mathematical manipulation.) Therefore, …

TODO: Add more information about frequency prewarping and necessary equation

Resources:

* <https://www.mathworks.com/help/signal/ref/bilinear.html#mw_8d43fd06-fbf3-4108-be26-62642fa7af82> (accessed 4/22/22)
* <https://en.wikipedia.org/wiki/Bilinear_transform> (accessed 4/22/22)

TODO: Change notation of transfer function (in both white papers) so is not confused between transfer function and sampling period