How to implement a digital biquadratic notch filter (in software)

by [Brandon Bickerstaff](https://www.linkedin.com/in/brandonbickerstaff/)

# Background

As presented in the [complimentary white paper](https://github.com/branbick/oscar/blob/main/biquad_notch_filter/doc/damping_ratios.pdf), the continuous-time transfer function of a biquadratic—or “biquad”—notch filter is …

Where Laplace variable, damping ratio (—), center/notch frequency (rad/s), and the subscripts and designate “numerator” and “denominator,” respectively. is a design parameter (i.e., an input), and the aforementioned white paper derives an algorithm for calculating and . That’s great, and can be practically utilized with the “help” of third-party software such as [MathWorks’ Control System Toolbox](https://www.mathworks.com/products/control.html). But what if the goal is to implement a notch filter in our *own* software—software running on an embedded system, for example? If that’s the case, then a *digital* filter is needed, and its coefficients and corresponding difference equation must be determined.

# Derivation

## Bilinear transform

The -transform is the discrete-time counterpart of the [Laplace transform](https://lpsa.swarthmore.edu/LaplaceXform/FwdLaplace/LaplaceXform.html), where ...

And sampling period (s). The first-order, [Maclaurin-series](https://mathworld.wolfram.com/MaclaurinSeries.html) approximation of is . Thus, …

The latter result is known as the bilinear transform: replacing with allows a transfer function to be converted from continuous time to discrete time. Unfortunately, that transformation causes a “warping” of the frequencies—because, after all, it’s based on an approximation (notably, a nonlinear one). For example, upon bilinearly transforming , a Bode plot of the consequent, *digital* filter would show that —which was specified for the original, *analog* filter—shifted. (Try it for yourself!) Obviously, that’s problematic, as the frequency the filter was designed to “notch out” won’t properly be attenuated. Thankfully, that predicament can be remedied via frequency prewarping.

## Frequency prewarping

Let …

* be an arbitrary (continuous-time) transfer function
* be the bilinearly transformed (discrete-time) equivalent of
* be the defining (analog) frequency of —e.g., the center frequency of a notch filter
* be the warped (digital) frequency corresponding to

The objective is to determine the relationship between and that makes the frequency responses of evaluated at and evaluated at —recalling the previous definition of —equal, i.e., …

Employing the bilinear transform yields …

Where the left-hand side equals …

(Refer to [Wikipedia](https://en.wikipedia.org/wiki/Bilinear_transform#Frequency_warping) for the redacted “mathematical magic.”) Therefore, …

TODO: Add more information about frequency prewarping and necessary equation

Resources:

* <https://en.wikipedia.org/wiki/Bilinear_transform> (accessed 4/22/22)
* <https://www.mathworks.com/help/signal/ref/bilinear.html#mw_8d43fd06-fbf3-4108-be26-62642fa7af82> (accessed 4/22/22)
* <https://youtu.be/NRbGPgcLhU0>

TODO: Change notation of transfer function (in both white papers) so is not confused between transfer function and sampling period