Investment Decisions Upon Innovative Technological Products

Ana Branca Medeiros Pais

Thesis to obtain the Master of Science Degree in

Mathematics and Applications

Supervisor(s): Prof. Cláudia Rita Ribeiro Coelho Nunes Philippart

Examination Committee

Chairperson: Prof. António Manuel Pacheco Pires
Supervisor: Prof. Cláudia Rita Ribeiro Coelho Nunes Philippart
Member of the Committee: Prof. Maria da Conceição Esperança Amado

November 2018

Acknowledgments

Apesar deste trabalho estar escrito em inglês não resisto em deixar a nota mais pessoal na minha língua materna.

Começo por agradecer à minha orientadora, a Professora Cláudia, que muito em cima da hora e a uma distância de quase 2000 km decidiu guiar-me no culminar de cinco anos de estudo. Os desafios rapidamente surgiram, assim como as resultantes dúvidas, os progressos na direcção errada e os muito celebrados resultados finais. Foi maravilhosa a oportunidade de me ter dado a conhecer as pessoas com quem de perto trabalha e com elas ter partilhado os meus progressos; estupendamente valioso o seu voto de confiança para trabalhar onde me aprouvesse (permitindo-me obter alguns progressos mais relevantes fora de território nacional); e incansável a sua paciência no ping-pong final deste trabalho.

E porque a tese não é fruto apenas dos seis meses em que nela se trabalha, mas antes sim de todo o percurso académico, fora de casa passado, não deixo de agradecer ao pessoal de LMAC que quase família se tornou, pelos muitos cafés, sessões de estudo, churrascos e devaneios; às *soirées do Bob* que sempre serviram para desanuviar das tempestades matemáticas; a todos os amigos que Lausanne me deu e que me ensinaram que as saudades não nos dominam se estivermos em boa companhia e que se consegue realizar (com sucesso!) enormes quantidades de trabalho conjugando muitas horas pelo Rolex e pelo Departamento de Matemática com passeios nos Alpes, idas ao Léman, frutos secos e lutas com colheres de pau no final do dia; e a todos aqueles que surgiram e ficaram (ou se foram), deixando a sua pegada impressa naquilo que hoje sou.

Também, como não poderia deixar de ser, um enorme obrigada a toda a minha família por todo o apoio e conselhos que me dão, por assegurarem que o meu calmo e confortável ninho se encontra sempre de braços abertos para me receber e por todas as infidáveis refeições em que nos reunimos que sempre me deixam de coração cheio.

Quanto ao futuro, esse é um mistério! Contudo, e por enquanto, toda a minha juventude exclama as palavras de Jack London em "The Call of the Wild",

He was mastered by the sheer surging of life,
the tidal wave of being,
the perfect joy of each separate muscle, joint,
and sinew in that it was everything that was not death,
that it was aglow and rampant,
expressing itself in movement,
flying exultantly under the stars.

Resumo

O tema desta tese insere-se na área de Matemática Financeira, em particular em Investimento sob Incerteza. O seu objectivo é definir a política de investimento óptima relativa a um produto tecnológico inovador, através da maximização do seu lucro esperado, relativa aos seguintes cenários:

1. Uma empresa quer investir e entrar no mercado com um produto novo;

2. Uma empresa já activa quer investir num novo produto, substituindo o antigo;

3. Uma empresa já activa quer investir num novo produto, permitindo um período de produção simultânea seguido da substituição total do produto antigo.

Assume-se também que a empresa, quando activa, produz um produto estável no mercado e que a decisão de investimento é irreversível, instantânea e pode ser tomada em qualquer altura após o nível de inovação tecnológica desejado ser atingido e cujos custos não são reembolsáveis.

A metodologia de estudo assume que a procura no mercado evolui de acordo com um Movimento Geométrico Browniano e que o nível de inovação segue um Processo de Poisson Composto. Com base nestas condições, deriva-se o nível de procura que justifica o investimento para cada uma das situações referidas, seguindo-se da sua análise comparativa. Estuda-se também a sensibilidade do tempo até o investimento óptimo e o impacto do investimento em investigação e desenvolvimento (R&D) no processo de investimento, analiticamente e numericamente.

Espera-se que esta tese possa auxiliar as equipas de decisão a avaliar os investimentos relacionados com produtos tecnológicos, indicando a melhor altura para se investir, qual o valor de capacidade de produção óptimo e o valor do projecto.

Palavras-Chave: Problemas de Paragem Óptima; Abordagem de Opções Reais; Investimento Sob Incerteza; Inovação Tecnológica.

Abstract

The scope of this thesis is Financial Mathematics, in particular Investment under Uncertainty. Its goal is to define the optimal investment policy regarding an innovative technological product, by maximizing its expected long run profit, related to the following scenarios:

1. A firm wants to invest and enter the market with a new product;
2. An active firm wants to invest and launch a new product, that totally replaces the old one;
3. An active firm wants to invest and launch a new product, while keeping temporarily the old product.

Moreover, it is assumed that an active firm produces an established product. The investment decision is irreversible, instantaneous, has an associate (sunk) cost and can be made at any time after a desired innovation level is reached.

The methodology of this thesis assumes the demand to evolve as a Geometric Brownian Motion and the innovation level accordingly to a Compound Poisson Process and derives the demand level that justifies the investment decision in each situation along with the respective comparative statics analysis. The sensitivity of optimal investment times and the impact of R&D investment in the innovation process are also analysed both analytically and numerically.

Overall, this thesis expects to support decision teams with technological products' investments, stating when is the best time to invest, the optimal production capacity and the value of the project.

Keywords: Optimal Stopping Problems; Real Options Approach; Investment Under Uncertainty; Technology Innovation.

Contents

List of Tables

List of Figures

Notation

- $\{W(t),\ t \geq 0\}$: Standard Brownian Motion (or Wiener Process) which is a stochastic process that has the following characteristics:

  1. $W(0) = 0$ with probability 1;
  2. $W(t) - W(s) \sim N(0, t-s)$. Notice that $\mathbb{E}[W(t)] = 0$ and $Var[W(t)] = t$;
  3. Independent increments: $\forall\ 0 < s_i < t_i < s_j < t_j :\ W(t_i) - W(s_i) \amalg W(t_j) - W(s_j)$;
     Stationary increments: $\forall t, s \geq 0 :\ W(t+s) - W(s) \stackrel{d}{=} W(t)$;
  4. $W(t)$ is continuous in t (however nowhere differentiable).

- $\{X(t),\ t \geq 0\}$: Geometric Brownian Motion (GBM) represents the demand for a certain product at each instant t. It is the solution of the following stochastic differential equation (SDE)

$$dX_t = \mu X_t dt + \sigma X_t dW_t,\ X_0 = x,$$

  where μ represents the drift and σ the volatility of the demand.

- R: R&D costs such as scientists wages and equipments, which directly related with the innovation process. These are seen as sunk costs, that is, costs that cannot be recovered after being incurred.

- $\{\theta(t),\ t \geq 0\}$: innovation process assumed to be a homogeneous Compound Poisson Process, that is a stochastic process that evolves accordingly to

$$\theta_t = \theta_0 + uN_t$$

  where θ_0 corresponds to the initial innovation level, $u > 0$ is the jump size and $\{N_t,\ t \geq 0\}$ follows a Poisson process with rate $\lambda(R) = R^\gamma,\ \gamma \in (0,1)$.

- θ: innovation breakthrough level. That is, the level of innovation for which we decide to invest in the new product. Considered to be reached in $n \in \mathbb{N}$ jumps, as it will be seen on Chapter 7.

- α: constant parameter that reflects the sensitivity of the quantity with respect to the price, $\alpha > 0$.

- K_i: capacity of production of product i. When a single product is considered, there is no mention to index i. The firm is considered to produce always up to its capacity and, consequently, K_i corresponds as well to the quantity produced. Since profit functions need to be positive, on Chapters

4 and 5, we have the following restrictions regarding capacities of *old* and *new* product, respectively, $K_0 < 1/\alpha$ and $K_1 < \theta/\alpha$. Note that (only) the last restriction will also hold for Chapter 3.

- δ: constant parameter that reflects the sensitivity of the quantity with respect to investment sunk costs. These sunk costs will be denoted by $\delta K_1,\ \delta > 0$ (or δK, on Chapter 3).
- η: cannibalisation parameter corresponding to the crossed effect between the old and the new product and representing how the quantity associated to a product will influence the price of the other. On Chapter 5, we consider that this influence is the same for both products, resulting in a unique cannibalisation parameter. It cannot be greater than the sensitivity parameter α, that is, $\eta < \alpha$.

Chapter 1

Introduction

1.1 Motivation

Nowadays, society is highly attached to technology, from mobile phones to fancy gadgets - and sometimes scratching the absurd [1]. Nevertheless, all this huge demand brought changes on how IT companies should manage their investments. These should, not only pay attention to the product demand on the market, but also to technology evolution. Therefore investors began to require more complex models to support their decisions. Models must not only consider the current value of the firm, such as the ones based on the Net Present Value (NPV), but also must take into account the potential associated to future events.

We may consider as an example ASML, a dutch company considered to be the largest european semiconductor equipment maker which currently is part of Euro Stoxx 50[2]. ASML builds electronic chips producing machines that satisfy the needs of hardware building companies - for instance, Intel and Samsung[3]. Therefore, in order to survive on the competitive market, ASML needs either to develop their own technology, by investing on R&D, or to import a desired one, with an associated cost.

This work is focused on the case where the firm chooses to develop their own technology and, hence, define *a priori* the desired innovation level and investment to be made. Since it is impossible to access that a certain level of technology will be reached in a precise amount of time[4] [5], the innovation process is considered to evolve randomly with time, with a rate that might be influenced by the amount of money invested. More money implies more resources and, hence, an higher evolution rate.

After reaching the desired innovation level, the firm must evaluate under which market conditions it should invest and in which type of products. The final goal is to maximize the expected long run profit.

[1] Parents forget to feed child while playing videogames:
https://www.theguardian.com/world/2010/mar/05/korean-girl-starved-online-game

[2] Stock index representing the 50 most liquid stocks in the Eurozone:
http://en.boerse-frankfurt.de/index/constituents/Euro_Stoxx_50#Constituents

[3] For further details check: https://www.asml.com/company/our-history/en/s277?rid=51985

[4] Flying cars predicted for 2015 in *Back to Future* as an example (!):
http://content.time.com/time/specials/packages/article/0,28804,2024839_2024845_2024855,00.html

[5] Moore's Law - which defends that processor performance doubles (approximately) every two years - is dying:
https://www.forbes.com/sites/forbestechcouncil/2018/03/09/moores-law-is-dying-so-where-are-its-heirs/#53cb24a17a7b

1.2 State-of-the-art

One of the first contributions on Real Options analysis regarding investment decisions was due to McDonald & Siegel (1986) [1]. They model an investment problem where the investor must decide when it is the best time to exercise the option, taking into account that the value of the investment project is stochastically random and evolves accordingly to a Geometric Brownian Motion. Other contributions were due to Dixit (1989) [2], who models the best time to make entry and exit decisions, while considering that the market price evolves accordingly to a Brownian Motion and that each decision has an associated cost.

The study by Dixit & Pindyck (1994) [3], that is nowadays considered the financial by some Bible of Real Options approach, exploits an analogy between real options and financial investment decisions, focusing on many different decision problems (entry, investment and exit, among them) dependent on different stochastically behavioured (diffusion processes and jump diffusion processes, among them) measures, such as demand or market price.

As our understanding about the market of new technological products evolved, the optimal production capacity to be chosen and the impact of technology adoption associated to investment decisions have started to get more relevant.

In the aspect of optimal capacity production Huisman & Kort (2013) [4] proposed a methodology to estimate the best time and capacity to invest in a new product. Both monopoly and a duopoly situations were assumed on their study.

Regarding technology adoption, Farzin *et al.* (1998) [5], presented a comparative study between NPV and Real Option approaches in the context that a firm wants to deduce when is the best time and level to invest in a technology. Also, Hagspiel *et al.* (2016) [6] studied the best time to invest in a new product or exit the market, by considering a firm whose established product has a declining profit and a demand process to evolve accordingly to a GBM.

Recently, Pimentel (2018) [7] explored both optimal capacity level and technology adoption. By considering two sources of uncertainty, related to the demand and the innovation processes, and a firm which is producing an established product, she deduces the optimal times to invest in a new product and to stop the production of the established product.

1.3 Thesis Outline

This thesis is organized in eight chapters.

The first one consists in the present introduction, which includes the state-of-the-art and the main context of the problem considered along this thesis.

The major theoretical concepts are introduced on Chapter 2, following several approaches presented in the literature (references [3], [8] and [9]). We start by exploring the most relevant results of Optimal Stopping Problems (OSP). Then we show how they are related to investment decisions under uncertainty through Real Option analysis.

On the next three chapters we analyse different possible investment decisions, namely:

- Chapter 3: a firm that has no products in the market and wants to find the optimal time to introduce a new product;
- Chapter 4: a firm with an established product in the market wants to find the optimal time to invest in a new product while replacing immediately the old one;
- Chapter 5: a firm with an established product wants to find the optimal time to invest in a new product. Two scenarios are considered:
  1. The old product is immediately replaced;
  2. A simultaneous production period is allowed and followed by the removal of the oldest product.

Two models are developed for situations described on Chapters 3 and 4. The first one corresponds to the benchmark model, which considers the original cash-flow for a chosen production capacity. The second one to the capacity optimization model, which considers the maximized long-run cash-flow with respect to production capacity. Regarding the situation described on Chapter 5, due to its complexity, only the benchmark model is analysed. On the last section of each of the three chapters we study the behaviour of respective thresholds and optimal capacity level with different parameters.

The behaviour of optimal investment times, associated to the situations described on chapters 3 to 5, regarding initial demand values and market's uncertainty is studied on Chapter 6.

On Chapter 7 we derive the optimal R&D investment, by maximizing the expected value function with respect to the innovation process. First, we derive the optimal R&D investment considering that the innovation process only takes one jump to achieve the breakthrough level. Secondly, we generalize the previous situation by considering that the innovation process takes $n \in \mathbb{N}$ jumps to achieve the breakthrough level. In the end of this chapter, the behaviour of the decision threshold with the different parameters is studied.

Finally, on Chapter 8 we summarize the relevant findings and how this work can be extended.

1.4 Problem Context

In the current work we consider a firm that develops its own technology and decides to invest in a new product only after an innovation breakthrough, associated to innovation level θ.

The innovation process associated to the technology level is denoted by $\{\theta(t),\ t \geq 0\}$ and considered to be an homogeneous Compound Poisson process, evolving accordingly to

$$\theta_t = \theta_0 + uN_t$$

where θ_0 corresponds to the initial innovation level, $u > 0$ to the jump size in the technology and $\{N_t,\ t \geq 0\}$ to the jump process, which follows a Poisson process with rate $\lambda(R) = R^\gamma,\ \gamma \in (0,1)$, where R stands for the R&D investment, such as scientist salaries, equipment costs and expenses. The mentioned rate function λ is such that it verifies:

- $\lambda(0) = 0$: no R&D investment means zero probability of innovating;
- $\lambda'(R) > 0$: the larger the investment is, the higher is the probability of observing a jump. Hence, the waiting time for the next technology jump is expect to be shorter;
- $\lambda''(R) < 0$: exists an optimal R&D investment that leads to the maximization of the rate function, that is, $\exists R^* : \lambda(R^*) \geq \lambda(R),\ \forall R$.

When the breakthrough happens, the innovation process stops and the firm is able to decide when it should invest on the new product with the innovation level θ. In order to have the new product available on the market, the firm needs to incur an investment (sunk) cost. This one is related with employees' formation and new equipment, for example, that depends on the production capacity chosen K_1. In this work, it is assumed that the firm produces up to its full capacity. Hence sunk costs will be dependent on the quantity of production chosen.

Since the new product offers a new technology level, not recognized by the market, its profit is strongly dependent on market's demand. The better the reception of the product, the bigger the demand and the corresponding profit.

The goal of this work is to define, the optimal investment time on a new product. The estimation of this time takes into account Real Options analysis, how market conditions influences it and the maximization of the expected long run profit.

In the following sections we will consider the timeline represented in the schematic diagram below in Figure 1.1.

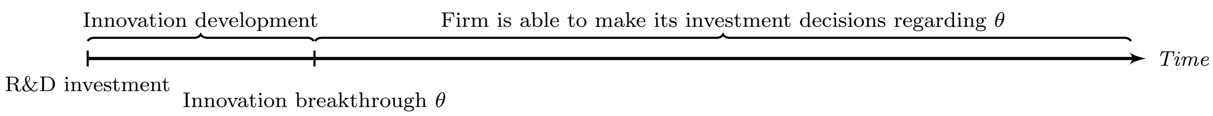


Figure 1.1: Timeline representing the innovation development and the investment period. Time is set to start at the innovation breakthrough.

Henceforth, when determining the optimal investment time on the new product, we consider as the starting time the instant at which occurs the innovation breakthrough. However when evaluating the value function regarding the complete investment (associated to both R&D and new product), the starting time corresponds to the time at which the firm decides to invest in R&D.

Chapter 2

Background concepts

2.1 Introduction

This chapter introduces the field of Optimal Stopping Problems, a subfield of Stochastic Control Problems. It is discussed how it is possible to characterize an optimal stopping time w.r.t. a given scenario defined *a priori*. It is also proposed a general solution of the (standard) optimal problems.

2.2 Optimal Stopping Problems

The main goal of optimal stopping problems consists on finding a stopping time such that a reward or cost function is maximized or minimized, respectively. Taking into account that investment decisions are usually formulated as the maximization of possible gains, we treat along this section the case on which we are dealing with a rewarding function. Nevertheless, we highlight the fact that the minimization problem is easily reductable to a maximization one and *vice-versa* - for further details we recommend the references [8] and [9].

Since investment decisions are usually related to sources of uncertainty, represented on the form of stochastic processes, we start to set our investment environment. We consider a probability space probability space $(\Omega, \mathcal{F}, \mathbb{P})$ associated to the underlying Brownian Motion W, on which $\mathcal{F} = \{\mathcal{F}_t,\ t \geq 0\}$ corresponds to its natural filtration and the unidimensional Itô process $\mathbf{X} = \{X_t,\ t \geq 0\}$ with state space defined on $\mathbb{R}$. $\mathbf{X}$ evolves accordingly to the stochastic differential equation (SDE):

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,\ X_0 = x \in \mathbb{R}, \tag{2.1}$$

where b and σ are functions that satisfy Itô conditions given by:

$$\exists K \in (0,\infty)\ \forall t \in [0,\infty)\ \forall x, y \in \mathbb{R}^n :$$

$$|b(t,x,\alpha) - b(t,y,\alpha)| + ||\sigma(t,x,\alpha) - \sigma(t,y,\alpha)|| \leq K|x-y| \tag{2.2}$$

$$|b(t,x,\alpha)|^2 + ||\sigma(t,x,\alpha)|| \leq K^2(1+|x|^2). \tag{2.3}$$

In the optimal stopping problem context, we define one of the most important concepts regarding optimal stopping problems.

Definition 2.1. *A function $\tau : \Omega \to [0, \infty]$ is called a stopping time with respect to the filtration $\mathcal{F}$ is $\{\omega : \ \tau(\omega) \leq t\} \in \mathcal{F}_t, \ \forall t \geq 0$.*

Intuitively we have that our reward function is strongly influenced by a running cost function g, which accounts for the instantaneous earnings before the decision is taken; a terminal function h, corresponding to the long-run earnings or termination payoff associated to the observed value of the Itô process when the decision is incurred; a stopping time τ, upon which we switch from one stage to another, and a initial given state for the underlying Itô process. Here we denote the reward function by J and it is such that

$$J(x,\tau) = \mathbb{E}^{X_0=x}\left[\left(\int_0^\tau e^{-rs} g(X_s)\ ds + e^{-r\tau} h(X_\tau)\right) \mathbb{1}_{\{\tau<\infty\}}\right].^1 \tag{2.4}$$

Note that $\mathbb{1}_{\{\tau<\infty\}}$ indicates that with probability 1, the decision is taken within a finite time. Also, henceforth on this work, we denote $\mathbb{E}\left[\ .\ |X_0 = x\right]$ by $\mathbb{E}^{X_0=x}\left[\ .\ \right]$.

Denoting V as the value function associated to the reward problem, it is such that

$$V(x) = \sup_\tau J(x,\tau) \tag{2.5}$$

with τ taken to be a stopping time in the set of all $\{\mathcal{F}_t\}$-stopping times.

Therefore we will want to characterize an optimal stopping time τ^* such that

$$J(x,\tau^*) = V(x),\ \forall x \in \mathbb{R} \tag{2.6}$$

In order to accomplish that, we suppose a continuation and a stopping region to be respectively given by $\mathcal{C} = \{x \in \mathbb{R} : x < x^*\}$ and $\mathcal{S} = \{x \in \mathbb{R} : x \geq x^*\}$. These are intuitive guesses: since we want to maximize our reward function, we expect that small values of x lead to a smaller value of $h(x)$, for which we verify $h(x) < V(x)$ meaning that is more attractive to *continue* with an instantaneous earning g; where, on the other hand, large values of x conduce to $h(x) = V(x)$, being preferable to change the strategy. Alternatively, continuation and stopping regions might be also defined as

$$\mathcal{C} = \{x \in \mathbb{R} : h(x) < V(x)\} \tag{2.7}$$

$$\mathcal{S} = \{x \in \mathbb{R} : h(x) = V(x)\}, \tag{2.8}$$

respectively. We note that there is a vast literature about the definition of continuation and stopping problems, that we omit in the work. We simply motivate $\mathcal{C}$ and $\mathcal{S}$, as we present (in a rather informal way) in this section.

As consequence, since the optimal stopping time sets the first instant upon which is advantageous to turnaround, that is when $h(x) = V(x)$ for some $x \in \mathbb{R}$, it might be formally defined as

$$\tau^* = \inf\{t \geq 0 : X_t \notin \mathcal{C}\} = \inf\{t \geq 0 : X_t \in \mathcal{S}\}. \tag{2.9}$$

Although it is not proved here, the uniqueness of an optimal stopping time is a consequence of the Dynamic Principle - quite remarkable on the field of Stochastic Control Problems -, and which briefly states that the optimal solution of a stochastic control problem can be found either by analysing te whole admissible domain or a partition of it. Following [8], in the context of optimal stopping problems, the Dynamic Principle is given by

$$V(x) = \sup_{\tau} \mathbb{E}^{X_0=x}\left[\left(\int_0^{\tau} e^{-rs} g(x_s)\ ds + e^{-r\tau} V(X_\tau)\right) \mathbb{1}_{\{\tau<\infty\}}\right] \tag{2.10}$$

from which we, after comparing with (2.4), conclude that in the stopping region $h(x) = V(x)$, showing that our guess (regarding the stopping region) presented on (2.8) was correct.

In order to obtain a general formulation of our optimal stopping problem regarding continuation and stopping regions, we manipulate the expression associated to the discounted value function (coinciding with the discounted terminal function as seen in (2.10)). Using Itô's lemma and supposing that $e^{-rt}V(x) \in C^2(\mathbb{R}^2)$, we integrate it, obtaining

$$e^{-rt}V(X_\tau) = V(X_0) + \int_0^{\tau} e^{-rs}(\mathcal{L}V(X_s) - rV(X_s))ds$$

where $\mathcal{L}$ denotes the infinitesimal generator of the Itô process X, given by $\mathcal{L}f(x, u(x)) = b(x, u(x))f'(x) + \frac{1}{2}\sigma^2(x, u(x))f''(x)$.

Taking the expectation on both sides and adding the term $\mathbb{E}^{X_0=x}\left[\int_0^{\tau} e^{-rs} g(X_s)ds\right]$ on both sides we get

$$\underbrace{\mathbb{E}^{X_0=x}\left[\int_0^{\tau} e^{-rs} g(X_s)ds + e^{-rt}V(X_\tau)\right]}_{J(x,\tau)\ \text{by}\ (2.10)} = V(x) + \mathbb{E}^{X_0=x}\left[\int_0^{\tau} e^{-rs}(\mathcal{L}V(X_s) - rV(X_s) + g(X_s))ds\right].$$

Since τ^* is considered to be the optimal time associated to the problem, it follows

$$\begin{aligned} J(x,\tau^*) \underset{(2.6)}{=} V(x) &= V(x) + \mathbb{E}^{X_0=x}\left[\int_0^{\tau} e^{-rs}(\mathcal{L}V(X_s) - rV(X_s) + g(X_s))ds\right] \\ &\Rightarrow \mathbb{E}^{X_0=x}\left[\int_0^{\tau} e^{-rs}(\mathcal{L}V(X_s) - rV(X_s) + g(X_s))ds\right] = 0. \end{aligned} \tag{2.11}$$

When $x \notin \mathcal{C}$, we have that $x \in \mathcal{S}$ and, as previously seen, $V(x) = h(x)$ holds.

However when considering an arbitrary $x \in \mathcal{C}$, $\tau > 0$ and therefore taking the limit $\tau \to 0$ in (2.11), it follows that

$$0 = -rV(x) + \mathcal{L}V(x) + g(x).$$

Therefore we obtain that for an arbitrary τ (that might or not be optimal)

$$\begin{cases} -rV(x) + \mathcal{L}V(x) + g(x) \leq 0 \\ V(x) \geq J(x,0) = h(x) \end{cases} \Leftrightarrow \begin{cases} rV(x) - \mathcal{L}V(x) - g(x) \geq 0 \\ V(x) - h(x) \geq 0 \end{cases}, \ \forall x \in \mathbb{R} \tag{2.12}$$

Summarizing the above result , we obtain the Hamilton-Jacobi-Bellman (HJB) variational equation which, in case of a reward problem, is given by

$$\min\{rV(x) - \mathcal{L}V(x) - g(x), V(x) - h(x)\} = 0, \ x \in \mathbb{R}. \tag{2.13}$$

Although continuation and stopping regions are not explicitly defined, we obtained that as stated in (2.7) and (2.8), our guess regarding the continuation region[2] was right and more, that it verifies the leftmost side of (2.13), that is, $\mathcal{C} = \{x \in \mathbb{R} : rV(x) - \mathcal{L}V(x) - g(x) = 0\}$.

Notwithstanding, the HJB variational inequality provides us the main tool to characterize the continuation (and stopping) regiom, we need to make use of our intuition to construct it - in this case it manifests by the assumption that larger values of x lead to the choice of termination payoff.

In order to verify that our guesses concerning $\mathcal{C}$ and $\mathcal{S}$ were the right ones, one could use Itô lemma and manipulate the result achieved. However this last one needs a very strong assumption: $V \in C^1(\mathbb{R})$ - which is not always verified, in particularly, in the boundary between $\mathcal{C}$ and $\mathcal{S}$ due to the quite different behaviour of V deduced by (2.13). Fortunately, this hypothesis can be relaxed by requiring (along with other conditions), V to be *sufficiently smooth.* This idea is present on the following theorem, whose proof can be checked on reference [8]:

Theorem 2.1 (Verification Theorem). *Suppose $\exists\ \phi : \mathbb{R} \to \mathbb{R}$ such that:*

1. $\phi \in C^1(\mathbb{R})$.
2. $\exists \psi : \mathbb{R} \to \mathbb{R}$ *measurable function:* a) $\forall a > 0 :$ ψ *is Lebesgue integrable in* $[-a, a]$*;*
   b) $\forall y \in \mathbb{R} :$ $\phi'(y) - \phi'(0) = \int_0^y \psi(z)dz$.
3. $\forall x \in \mathbb{R} :$ $\min\{rV(x) - \mathcal{L}V(x) - g(x), V(x) - h(x)\} = 0$.
4. $\forall x \in \mathbb{R} :$ $\lim_{t \to \infty} e^{-rt}\mathbb{E}[\phi(X_t)] = 0$.

Then,

1. $\forall x \in \mathbb{R} :$ $\phi(x) \geq J(x,\tau)\ \forall \tau \in \mathcal{S} \ \Rightarrow \ \phi(x) \geq V(x)$.
2. *Let $\mathcal{C}$ be defined as in* (2.7) *and τ^* as in* (2.9).
   Then, $\phi(x) = J(x, \tau^*) = V(x)$ iff τ^* is an optimal stopping time.

Combining the HJB variational inequality with the Verification theorem we obtain a powerful tool to characterize continuation and stopping regions, by the free-boundary problem:

[2] Recall that our guess about the stopping region was already checked to be correct.

$$\begin{cases} V(x) - h(x) = 0 & , \ x \in \mathcal{S} \\ rV(x) - \mathcal{L}V(x) - g(x) = 0 & , \ x \in \mathcal{C} \\ V(x) = h(x) & , \ x \in \partial\mathcal{C} \\ V'(x) = h'(x) & , \ x \in \partial\mathcal{C} \end{cases} \tag{2.14}$$

This way we assure that our guess concerning the continuation region is the right one so as the value that triggers our decision, that is, x^*. Finally, the optimal stopping time is deduced from (2.9) to be such that $\tau^* = \inf\{t \geq 0 : X_t \geq x\}$, which is characterized by (2.7)

2.3 Real Options approach

To incorporate the irreversibility and the possibility to delay an investment the real option approach was extended to support investment decisions. This chapter was based on the pioneer work of Dixit and Pindyck [3].

In this approach, investment opportunities are seen as real options: *the firm has the right, but not the obligation, to undertake certain initiatives such as deferring, abandoning, expanding, staging or contracting a capital investment project* [10]. There are three factors assumed to hold during the investment decision:

1. Future rewards are random and thus uncertain;
2. The decision is irreversible, in the sense that it is a sunk cost: the investment expenditure cannot be fully recovered;
3. The decision can be made at any time.

Alternatively to the traditional NPV analysis - on which, since the investment is irreversible, it is seen as a now or never opportunity (without the possibility to postpone the investment) -, the decision can be postponed resulting in an extra value associated to its potential.

In accordance to reference [3], the investment decision treated by real options approach can be seen as an optimal stopping problem. This is justified by the fact that we a have a dynamic problem of a risk-neutral firm that discounts against a rate (here assumed to be constant), r, for which we want to find the optimal time to change ruling strategy, by maximizing its value.

Recall that, as stated in Chapter 1, the starting instant ($t = 0$) upon which the firm is able to decide is here considered to be at the innovation breakthrough.

The problems treated on this work are mainly investment and exiting ones, being related to the passage from an established product to an innovative one. Therefore, taking into account what was written in the previous section, they can be written as in (2.5) with reward function J as in (2.4), where the running cost function $g \geq 0$ denotes the current earnings, that is, the cash-flow originated at each instant by the current strategy; the terminal function $h \geq 0$ denotes the long-run cash-flow originated since the decision is taken and by considering the new situation of the firm (and investment costs) and the process X denotes here the demand process, which evolves accordingly to a Geometric Brownian Motion, leading to

$$V(x) = \sup_{\tau \geq 0} J(x,\tau) = \sup_{\tau \geq 0} \mathbb{E}^{X_0=x}\left[\left(\int_0^\tau e^{-rs} g(X_s)\ ds + e^{-r\tau} h(X_\tau)\right) \mathbb{1}_{\{\tau<\infty\}}\right]. \tag{2.15}$$

Note that, by admitting the *old* product to be established in the market, its unitary price is not influenced by the demand. This results on a deterministic running cost function, allowing to easily solve the integral above. On the other hand, the innovative one, since it's not yet recognized in the market, it is strongly dependent on the level of the demand. Hence, we assume that any decision won't be incurred if the demand is very low, because it is more profitable to continue with the established product.

Following this line of thought we are led to admit a continuation region of the form $\mathcal{C} = \{x \in \mathbb{R} : x < x^*\}$, where the value x^* corresponds to the level of demand that triggers the investment decision. $\mathcal{C}$ is characterized by the free-boundary problem stated in (2.14), which, taking into account the definition of $\mathcal{C}$ and the infinitesimal generator of a GBM, it is written as

$$\begin{cases} V(x) - h(x) = 0 & , \ x > x^* \\ \dfrac{\sigma^2}{2} x^2 V''(x) + \mu x V'(x) - rV(x) + g(x) = 0 & , \ x < x^* \quad (2.16a) \\ V(x^*) = h(x^*) & \quad (2.16b) \\ V'(x)|_{x=x^*} = h'(x)|_{x=x^*} & \quad (2.16c) \end{cases}$$

Henceforth, we will refer to (2.16b) as *value matching* condition - since it matches the values of the unknown function V to those of the known termination payoff function h - and to (2.16c) as *smooth pasting* condition. Those are the necessary conditions that allow the first requirement of Theorem 2.1 to be satisfied.

The characterization of the stopping region $\mathcal{S}$ and value function V defined on it are straightforward defined by (2.8) and $V(x) = h(x)$, respectively.

From (2.16a) we obtain that for any demand level in $\mathcal{C}$, the ordinary differential equation of second order, above represented, is satisfied. This is a Cauchy-Euler equation, whose solution (here denoted by $V_{\mathcal{C}}$) might be seen as the sum of the homogeneous solution V_h with a particular solution V_p, that is, $V_{\mathcal{C}}(x) = V_h(x) + V_p(x),\ \forall x \in \mathcal{C}$.

The homogeneous solution is found by solving the (homogenous) Cauchy-Euler equation of second order associated to (2.16a), that is, $\frac{\sigma^2}{2} x^2 V_h''(x) + \mu x V_h'(x) - rV_h(x) = 0$, whose solution has the form of

$$V_h(x) = ax^{d_1} + bx^{d_2}$$

where d_1 and d_2 are the positive and negative solutions of the quadratic equation

$$d^2 + \left(\frac{2\mu}{\sigma^2} - 1\right) d - \frac{2r^2}{\sigma^2} = 0 \qquad \Rightarrow \qquad d_{1,2} = \frac{1}{2} - \frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}. \tag{2.17}$$

Taking into account that $r > \mu$, one can easily check that $d_1 > 1$ and $d_2 < 0$.

On most situations addressed during this work we face a terminal function h that is a non-decreasing function of polynomial type. This is a special case studied in reference [11], from where we conclude that one of the boundary conditions is that the solution for $x = 0$ needs to be zero, ie, $\lim_{x \to 0^+} V_h(x) = 0$. Therefore, we must have $b = 0$ resulting in $V_h(x) = ax^{d_1}$. Also, in this context, by noting that it is an absurd to suppose a project with negative value, it is required that $a > 0$.

Regarding the particular solution V_p, we have that it trivially will depend on the running function g.

Thus, regarding the general context as described in (2.15), we obtain that the value function V is of the form:

$$V(x) = \begin{cases} ax^{d_1} + bx^{d_2} + V_p(x) & , \ x < x^* \\ h(x) & , \ x \geq x^* \end{cases} \tag{2.18}$$

where coefficients a and b and the threshold x^* are found by value matching (2.16b) and smooth pasting (2.16c) conditions along with the ones derived based on the situation treated. Finally, by knowing V we straightforward know how to characterize both continuation and stopping regions.

Fortunately, on most problems here addressed (being the only exception found on Chapter 5), we are able to change the formulation presented in (2.15), in order to get a null running function as follows:

$$\begin{aligned} V(x) &= \sup_{\tau \geq 0} \mathbb{E}^{X_0=x} \left[\int_0^\infty e^{-rs} g(X_s)\, ds + \left(e^{-r\tau} h(X_\tau) - \int_\tau^\infty e^{-rs} g(X_s)\, ds \right) \mathbb{1}_{\{\tau < \infty\}} \right] \\ &= \mathbb{E}^{X_0=x} \left[\int_0^\infty e^{-rs} g(X_s)\, ds \right] + \sup_{\tau \geq 0} \mathbb{E}^{X_0=x} \left[\left(\underbrace{e^{-r\tau} h(X_\tau) - \int_\tau^\infty e^{-rs} g(X_s)\, ds}_{\widetilde{h}(X_\tau)} \right) \mathbb{1}_{\{\tau < \infty\}} \right] \\ &= \mathbb{E}^{X_0=x} \left[\int_0^\infty e^{-rs} g(X_s)\, ds \right] + \sup_{\tau \geq 0} \mathbb{E}^{X_0=x} \left[e^{-rs} \widetilde{h}(X_\tau) \mathbb{1}_{\{\tau < \infty\}} \right]. \end{aligned} \tag{2.19}$$

The above simplification is advantageous when the integral on the left is easily solvable. This way, we only need to solve the rightmost optimal stopping problem, using the methodology presented.

Chapter 3

Investing and entering the market with a new product

3.1 Introduction

In this chapter we consider a firm that is not active in the market and aims to invest in a new product, with an innovation level θ. Therefore, at the investment time, the firm needs to incur an investment cost proportional to the capacity of production K. This cost is given by δK with $\delta > 0$, a sensibility parameter related to the investment[1]. Here we consider that the investment decision is irreversible (meaning that it is not possible to obtain investment costs back) and that the production starts at the same time the investment is undertaken.

The unitary price function p associated to this product is considered to evolve stochastically with the demand process $\mathbf{X}$ and it is given by

$$p(X_t) = (\theta - \alpha K)X_t \geq 0 \tag{3.1}$$

where $\alpha > 0$ is a sensibility parameter and X_t corresponds to the demand level observed at the instant $t \geq 0$. Depending on the authors, p might also be called *demand function*, as it is the case on reference [7], for example. The firm wouldn't fix an unitary negative price to a certain product, since it would imply a negative profit. Therefore p is always positive and hence, it follows that $\theta \geq \alpha K$ must be verified.

The instantaneous profit function π is obtained by multiplying the unitary price function p by the quantity wanted to be produced, which, as previously explained, is assumed to be fixed along the production process. Then, it follows that

$$\pi(X_t) = (\theta - \alpha K)KX_t. \tag{3.2}$$

Recall that investment timeline is set to start at the precise instant the innovation breakthrough

[1] Check Notation section for further details.

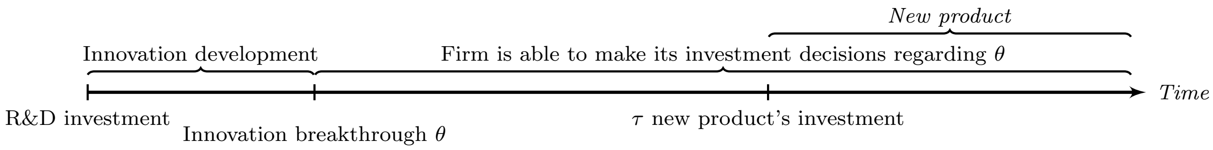


Figure 3.1: Timeline representing the two possible different stages of production and associated decisions. Time is set to start at the innovation breakthrough.

happens, as presented on Figure 3.1. Only after that instant the firm is able to create the product with an in-putted θ innovation level.

Therefore, when determining the optimal investment time on the new product, we consider as the starting time the instant at which occurs the innovation breakthrough. However when evaluating the value function regarding the complete investment (associated to both R&D and new product), the starting time corresponds to the time at which the firm decides to invest in R&D.

We will address two cases: in the first case, we assume that K is given and, in the second case, we optimize K in order to maximize the profit of the firm. The first case is the easiest to derive and therefore we call it the benchmark model.

3.2 Stopping Problem

3.2.1 Benchmark Model

In the benchmark model we want to find when is the optimal investment time such that it leads to the maximization of the expected discounted long-term profit, assuming that this decision is taken in finite time.

Denoting the investment time in the new product by τ, our optimization problem can be written as

$$\sup_{\tau} \mathbb{E}^{X_0=x}\left[\left(\int_{\tau}^{\infty} e^{-rs}\pi(X_s)\ ds - e^{-r\tau}\delta K\right)\ \mathbb{1}_{\{\tau<\infty\}}\right] \tag{3.3}$$

for $\theta, x \in \mathbb{R}^+$. Note that $\mathbb{1}_{\{\tau<\infty\}}$ assures that, with probability 1, the decision is made in a finite amount of time.

Putting the discount term in evidence, from (3.3) follows

$$\sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\int_{\tau}^{\infty} e^{-r(s-\tau)}\pi(X_s)\ ds - \delta K\right)\ \mathbb{1}_{\{\tau<\infty\}}\right]. \tag{3.4}$$

We can simplify this expression. Using Tower rule and conditioning on the instant the firm exercises, we obtain that (3.4) may be written as

$$\sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\mathbb{E}^{\tau}\left[\int_{\tau}^{\infty} e^{-r(s-\tau)}\pi(X_s)\ ds\right] - \delta K\right)\mathbb{1}_{\{\tau<\infty\}}\right]. \tag{3.5}$$

Focusing on the inner expected value $\mathbb{E}^{\tau}\left[\int_t^{\infty} e^{-(\tau-s)}\pi(X_s)\ ds\right]$ and changing its integration variable follows

$$\mathbb{E}^{\tau}\left[\int_0^{\infty} e^{-rv}\pi(X_{v+\tau})\ dv\right]. \tag{3.6}$$

As we assume that $r-\mu>0$ and $e^{-rv}\pi(X_{v+\tau})$ is a measurable function, by Fubini's Theorem we may interchange the order of integration of (3.6), obtaining

$$\int_0^{\infty}\mathbb{E}^{\tau}\left[e^{-rv}\pi(X_{v+\tau})\right]\ dv=(\theta-\alpha K)K\int_0^{\infty}\mathbb{E}^{\tau}\left[e^{-rv}X_{v+\tau}\right]\ dv, \tag{3.7}$$

where we took into account the expression of the profit function π.

Let's now focus on the expected value $\mathbb{E}^{\tau}\left[e^{-rv}X_{v+\tau}\right]$. Using the fact that the process $\mathbf{X}$ is a GBM, it follows that

$$\begin{aligned}\mathbb{E}^{\tau}\left[e^{-rv}X_{v+\tau}\right]&=\mathbb{E}^{\tau}\left[X_{\tau}e^{\left(\mu-\frac{\sigma^2}{2}-r\right)(\tau+v-\tau)+\sigma(W_{\tau+v}-W_{\tau})}\right]\\&=X_{\tau}e^{\left(\mu-\frac{\sigma^2}{2}-r\right)v}\ \mathbb{E}^{\tau}\left[e^{\sigma W_v}\right]\\&=X_{\tau}e^{\left(\mu-\frac{\sigma^2}{2}-r\right)v}e^{\frac{\sigma^2}{2}v}\\&=X_{\tau}e^{(\mu-r)v}.\end{aligned} \tag{3.8}$$

In the first step we use the expression associated to the GBM and the fact that, by knowing the investment time τ, we also know the demand level at that time, here represented as X_{τ}. In the second step, the fact that the Brownian Motion has stationary increments implies that $W_{\tau+v}-W_{\tau}\overset{d}{=}W_v$ In the third step we use the fact that $W_v\sim\mathcal{N}(0,v)$ and the expression for the moment generating function associated to the Normal distribution, from which follows $\mathbb{E}\left[e^{sW_v}\right]=e^{\frac{1}{2}sv^2}$. Simplifying the expression we obtain (3.8).

Plugging the resultant expression (3.8) in (3.7) and solving the integral, we obtain the formula of the terminal cost function associated to this problem - corresponding to the expression between brackets in (3.4) and (3.5). We will denote it by h and its expression corresponds to

$$h(x)=\frac{(\theta-\alpha K)Kx}{r-\mu}-\delta K. \tag{3.9}$$

The terminal cost function h represents the discounted long-term cash-flow by acquiring a product when the demand level is x. It already includes the investment cost of such decision, given by δK.

Denoting F as the value function associated to this problem, we obtain that our optimization problem, as described in (3.3), can be written as a standard optimal stopping problem with null running cost function, given by

$$F(x)=\sup_{\tau}\mathbb{E}^{X_0=x}\left[e^{-r\tau}h(X_{\tau})\mathbb{1}_{\{\tau<\infty\}}\right]=\sup_{\tau}\mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{(\theta-\alpha K)KX_{\tau}}{r-\mu}-\delta K\right)\mathbb{1}_{\{\tau<\infty\}}\right]. \tag{3.10}$$

Then, invoking results presented on Chapter 2, in particular (2.18), it follows that F is defined differently for the continuation and stopping regions, as follows:

$$F(x)=\begin{cases} ax^{d_1} & , \ x\in\mathcal{C} \\ h(x) & , \ x\in\mathcal{S} \end{cases}, \tag{3.11}$$

with d_1 being the positive root of the polynomial described in (2.17).

As motivated before, we propose that $\mathcal{C}=\{x\in\mathbb{R}: x<x_B^*\}$, where x_B^* denotes the demand that triggers the (optimal) decision to invest. Thus we find a and x_B^* using value matching (2.16b) and smooth pasting (2.16c) conditions, expressed by the corresponding system

$$\begin{cases} a(x_B^*)^{d_1}=\frac{K(\theta-\alpha K)x^*}{r-\mu}-\delta K \\ ad_1(x_B^*)^{d_1-1}=\frac{K(\theta-\alpha K)}{r-\mu} \end{cases} \Rightarrow \begin{cases} a=\left(\frac{K(\theta-\alpha K)x_B^*}{r-\mu}-\delta K\right)(x_B^*)^{-d_1}=\frac{\delta K(x_B^*)^{-d_1}}{d_1-1} \\ x_B^*=\frac{d_1}{d_1-1}\frac{\delta(r-\mu)}{\theta-\alpha K} \end{cases} \tag{3.12}$$

Finally, we obtain that the stopping region associated to this problem is defined as $\mathcal{S}=\{x\in\mathbb{R}: x\geq x_B^*\}$ and the optimal stopping time as in (2.9).

3.2.2 Capacity Optimization Model

Now we consider a more realistic case, in which the firm wants to take the best of its investment by choosing an appropriate capacity of production. This can be achieved by requiring that the capacity of production leads to the maximization of the discounted long-term cash-flow. Therefore our goal is now to find when is the optimal (finite) time to invest and which is the optimal capacity associated to it. This can be stated as

$$\sup_{\tau}\mathbb{E}^{X_0=x}\left[\max_K\left\{e^{-r\tau}\left(\int_\tau^\infty e^{-r(\tau-s)}\pi(X_s)\ ds-\delta K\right)\right\}\mathbb{1}_{\{\tau<\infty\}}\right]. \tag{3.13}$$

Manipulating the expression as previously done, we obtain that (3.13) may be written as

$$\sup_{\tau}\mathbb{E}^{X_0=x}\left[e^{-r\tau}\max_K\{h(X_\tau,K)\}\mathbb{1}_{\{\tau<\infty\}}\right], \tag{3.14}$$

with h corresponding to the terminal function deduced in (3.9), in which we now highlight not only the dependence on the demand level, but also on the capacity of production K chosen at the investment time.

In this section, the capacity optimization model is obtained in two steps, as similarly done in [4], for example. In the first step, we calculate the capacity level that optimizes the terminal cost function h, which we will denote by K^*. In the second, we solve the optimal stopping problem given by $\sup_{\tau}\mathbb{E}^{X_0=x}\left[e^{-r\tau}h(X_\tau,K^*)\mathbb{1}_{\{\tau<\infty\}}\right]$, in which we are already considering the optimized terminal function (with respect to K).

The optimal capacity level K^* is found by analysing the behaviour - namely stationary points and concavity - of the terminal function h, while considering a fix level of demand.

Stationary points are found by calculating the roots of the first partial derivative, which is given by

$$\frac{\partial h}{\partial K}(x,K) = \frac{(\theta - 2\alpha K)x}{r-\mu} - \delta, \tag{3.15}$$

following that h has a unique stationary point corresponding to $K = \frac{\theta}{2\alpha} - \frac{\delta(r-\mu)}{2\alpha x}$.

Analysing its second partial derivative of h, we obtain that

$$\frac{\partial^2 h}{\partial K^2}(x,K) = -\frac{2\alpha x}{r-\mu}, \tag{3.16}$$

which is negative, since the every state of the GBM is positive, $\alpha > 0$ and $r - \mu > 0$. Therefore, h is a concave function and thus the zero of the first derivative is its global maximizer. Denoting it by K^*, we have that

$$K^* := \arg\max_K h(x,K) = \frac{\theta}{2\alpha} - \frac{\delta(r-\mu)}{2\alpha x},\ \forall x \tag{3.17}$$

Since the firm can only produce a positive quantity, we need to verify $K^* > 0$ which is equivalent to the condition $x > \frac{\delta(r-\mu)}{\theta}$ for any considered demand level $x \in \mathbb{R}^+$, which holds, in view of our assumptions on $\delta,\ r,\ \mu$ and σ.

Now we proceed to the second step. Evaluating h at its optimal capacity level K^* we obtain

$$h(x,K^*) = \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x}.$$

Denoting F^* as the value function associated to the optimal stopping problem in (3.18), the optimization problem can be stated as

$$F^*(x) = \sup_\tau \mathbb{E}^{X_0=x}\left[e^{-r\tau}h(X_\tau,K^*)\mathbb{1}_{\{\tau<\infty\}}\right] = \sup_\tau \mathbb{E}^{X_0=x}\left[e^{-r\tau}\frac{(\theta X_\tau - \delta(r-\mu))^2}{4\alpha(r-\mu)X_\tau}\mathbb{1}_{\{\tau<\infty\}}\right], \tag{3.18}$$

which is again a standard optimal stopping problem with null running cost function. Similarly to the benchmark model, we obtain that the value function associated to (3.18) satisfies the HJB variational inequality (2.13). Therefore F^* is such that

$$F^*(x) = \begin{cases} bx^{d_1} & ,\ x \in \mathcal{C} \\ h(x,K^*) & ,\ x \in \mathcal{S} \end{cases}, \tag{3.19}$$

with $\mathcal{C} = \{x \in \mathbb{R} : x < x_C^*\}$ the continuation region, where x_C^* stands for the demand level that triggers the optimal investment.

Then, coefficient b and the threshold value x_C^* are found by value matching (2.16b) and smooth pasting conditions (2.16c), expressed by the corresponding system

$$\begin{cases} b(x_C^*)^{d_1} = \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x} \\ bd_1(x_C^*)^{d_1-1} = \frac{\theta^2(x_C^*)^2 - \delta^2(r-\mu)^2}{4\alpha(r-\mu)(x_C^*)^2} \end{cases}. \tag{3.20}$$

We get two possible positive roots for the threshold level: $x^*_{C,1} = \frac{d_1+1}{d_1-1}\frac{\delta(r-\mu)}{\theta-\alpha K}$ and $x^*_{C,2} = \frac{\delta(r-\mu)}{\theta}$. However, after some manipulation, we exclude the second one $x^*_{C,2}$, since the coefficient b associated to it takes a null value. This is an absurd, since it would lead to a null value function for any demand level smaller than $x^*_{C,2}$, neglecting potential investment decisions in the future. Therefore we obtain that the threshold level and coefficient b in (3.12) are given, respectively, by

$$\begin{aligned} x^*_C &= \frac{d_1+1}{d_1-1}\frac{\delta(r-\mu)}{\theta} \\ b &= \left(\frac{(\theta x-\delta(r-\mu))^2}{4\alpha(r-\mu)x^*_C}\right)(x^*_C)^{-d_1} = \frac{\delta\theta}{\alpha(d_1^2-1)}\left(\frac{d_1+1}{d_1-1}\frac{\delta(r-\mu)}{\theta}\right)^{-d_1} \end{aligned} \tag{3.21}$$

Finally the stopping region coincides with $\mathcal{S} = \{x \in \mathbb{R} : x < x^*_C\}$ and the optimal stopping time as in (2.9).

Now we analyse the optimal capacity level K^*_C. If at the innovation breakthrough - that we set to be the initial instant when deducting the optimal investment times - we have $X_0 < x^*_C$, then the investment decision will happen at the same instant the demand reaches level x^*_C. As done in reference[4], the optimal capacity level is obtained by evaluating K^*, as defined in (3.17), at the threshold demand level x^*_C (3.21), leading to

$$K^*_C = \frac{2\sigma^2\theta}{\alpha\left(\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4}-\frac{4\mu}{\sigma^2}+\frac{8r}{\sigma^2}+1}+3\right)-2\mu\right)}. \tag{3.22}$$

However it might be the case that when the breakthrough happens, we observe an equal or higher demand than the threshold x^*_C. This one is not as interesting as the previously mentioned. The firm will invest at the same instant the breakthrough happens choosing an optimal capacity given by evaluating K^*, as defined in (3.17), at the observed demand level at the breakthrough. This case won't be addressed on Comparative Statics (Section 3.3).

3.3 Comparative Statics

In this section we study the behaviour of the decision threshold x^*_B (3.12) and x^*_C (3.21) and K^* as described in (3.22), with the different parameters. Comparisons between the benchmark and capacity optimization models will be also be presented.

3.3.1 Benchmark Model

Proposition 3.1. *The decision threshold x^*_B increases with $r, \sigma,$ $K,$ α and δ, and decreases with θ and μ.*

Proof:

As it will play an important role, we start by defining the following quantity:

$$\phi := \sqrt{\frac{4\mu^2}{\sigma^4}-\frac{4\mu}{\sigma^2}+\frac{8r}{\sigma^2}+1}. \tag{3.23}$$

With simple calculus and in view of the admissible values for μ, σ and r, we obtain that the value inside the square root is always positive and hence ϕ is well defined and non-negative.

Now we are in position to explain the stated results.

Regarding r, we obtain

$$\frac{\partial x_B^*}{\partial r} = \frac{\delta\left(-(d_1-1)d_1\sigma^2\phi - 2\mu + 2r\right)}{(d_1-1)^2\sigma^2(\alpha K-\theta)\sqrt{\frac{4\mu^2}{\sigma^4}-\phi}} > 0,$$

Given the constraints of our problem, its denominator is well-defined and negative so as its numerator[2], from which the result follows.

Regarding σ, we obtain

$$\frac{\partial x_B^*}{\partial \sigma} = \frac{2\delta(r-\mu)\left(-2\mu^2+\mu\sigma^2(\phi+1)-2r\sigma^2\right)}{(d_1-1)^2\sigma^5(\alpha K-\theta)\phi} > 0 \tag{3.24}$$

Again, given the constraints of our problem, the denominator is negative. Since $r-\mu>0$ and $-2\mu^2+\mu\sigma^2(\phi+1)-2r\sigma^2<0$ [3], the denominator is also negative, from which the result follows.

Regarding K, α, δ and μ, we immediately obtain

$$\begin{aligned}
\frac{\partial x_B^*}{\partial K} &= \frac{\alpha\delta d_1(r-\mu)}{(d_1-1)(\theta-\alpha K)^2} > 0\\
\frac{\partial x_B^*}{\partial \alpha} &= \frac{\delta d_1 K(r-\mu)}{(d_1-1)(\theta-\alpha K)^2} > 0\\
\frac{\partial x_B^*}{\partial \delta} &= \frac{d_1(r-\mu)}{(d_1-1)(\theta-\alpha K)} > 0\\
\frac{\partial x_B^*}{\partial \theta} &= -\frac{\delta d_1(r-\mu)}{(d_1-1)(\theta-\alpha K)^2} < 0.
\end{aligned}$$

Regarding μ, we obtain

$$\frac{\partial x_B^*}{\partial \mu} = \frac{\delta\left((d_1-1)d_1\sigma^4\phi - 2\mu^2+\mu\sigma^2\phi+1\right)+r\left(2\mu-\sigma^2(\phi+1)\right)}{(d_1-1)^2\sigma^4(\alpha K-\theta)\phi} < 0.$$

It is possible to conclude that the denominator is negative. On the other side, the numerator is positive[4], from which the result follows.

[2] Numerator's roots are given by $\mu \in \left\{r, \frac{1}{2}\left(\sigma^2 \pm \sqrt{\sigma^4-8r\sigma^2}\right)\right\}$ and $\sigma = \pm i\frac{\mu(\mu-r)}{\sqrt{-\mu^3+2r^3-5\mu r^2+4\mu^2 r}}$.

Regarding the roots in terms of μ we observe that none of them are admissible in our domain. The first one ($r=\mu$) is excluded since we consider $r>\mu$. The second pair $\left(\mu = \frac{1}{2}\left(\sigma^2 \pm \sqrt{\sigma^4-8r\sigma^2}\right)\right)$ is also excluded:

$$\begin{aligned}
\frac{1}{2}\left(\sigma^2-\sqrt{\sigma^4-8r\sigma^2}\right) > r \;\Leftrightarrow\; (\sigma^2-2r)^2 &= \sigma^4-4\sigma^2 r+4r^2 > -8r\sigma^2+\sigma^4\\
&\overset{r\gtrless 0}{\Leftrightarrow} \sigma^2+r>0, \qquad \forall r,\sigma>0\ \forall \mu<r
\end{aligned}$$

is always verified and $\frac{1}{2}\left(\sigma^2+\sqrt{\sigma^4-8r\sigma^2}\right) > \frac{1}{2}\left(\sigma^2-\sqrt{\sigma^4-8r\sigma^2}\right) > r$, resulting that none of them are valid.

Since no imaginary roots are possible regarding our context, none of the two possible roots for σ is admissible in our problem. Hence, it follows that the numerator never changes its sign. Evaluating for any value of μ, σ and r we obtain that $-(d_1-1)d_1\sigma^2\phi-2\mu+2r<0$, showing this way that the numerator is negative.

[3] We obtain that the roots of the polynomial $-2\mu^2+\mu\sigma^2(\phi+1)-2r\sigma^2$ correspond to $r\in\{0,\ \mu\}$ and both are impossible in the domain of our problem. Then evaluating for any value of μ, σ and r we obtain the result stated.

[4] We obtain the same pair of non-admissible roots as stated in the above footnote. Then evaluating for any value of μ, σ and r we obtain the result stated.

$\square$

3.3.2 Capacity Optimization Model

Proposition 3.2. *The decision threshold x_C^* increases with r, σ and δ, decreases with θ and has a non-monotonic behaviour with μ. None of any other parameters have effect on x_C^*.*

Proof:

Recall the expression of x_C^* given in (3.21). One can immediately notice that it is not dependent on parameters α and K. Therefore we proceed with the analysis of x_C^* with respect to the other parameters.

Regarding r, we obtain

$$\frac{\partial x_C^*}{\partial r} = \frac{\delta\left(\left(d_1^2 - 1\right)\sigma^2\phi + 4\mu - 4r\right)}{(d_1 - 1)^2\theta\sigma^2\phi} > 0.$$

Taking into account our problem constraints and after manipulating the expression, we obtain that both denominator and numerator are positive, from which the result holds.

Regarding σ, we obtain

$$\frac{\partial x_C^*}{\partial \sigma} = \frac{4\delta(\mu - r)\left(-2\mu^2 + \mu\sigma^2(1+\phi) - 2r\sigma^2\right)}{(d_1 - 1)^2\theta\sigma^5\phi} > 0.$$

From the negativity of the numerator in (3.24), it follows that the numerator of above expression is also positive. Since the denominator is also positive, the result holds.

Regarding δ, θ and μ, we immediately obtain

$$\begin{aligned}
\frac{\partial x_C^*}{\partial \delta} &= \frac{(d_1+1)(r-\mu)}{(d_1-1)\theta} > 0,\\
\frac{\partial x_C^*}{\partial \theta} &= -\frac{\delta(d_1+1)(r-\mu)}{(d_1-1)\theta^2} < 0\\
\frac{\partial x_C^*}{\partial \mu} &= \frac{\delta}{(d_1-1)^2\theta}\left(1 - d_1^2 + (r-\mu)\left(-1 + \frac{2\mu - \sigma^2}{\phi\sigma^2}\right)\right) = \begin{cases} < 0 & \text{for } \mu < \frac{\sigma^2}{2} \\ > 0 & \text{for } \mu > \frac{\sigma^2}{2} \end{cases}.
\end{aligned}$$

$\square$

Numerical comparisons between the Benchmark and the Capacity Optimization Models

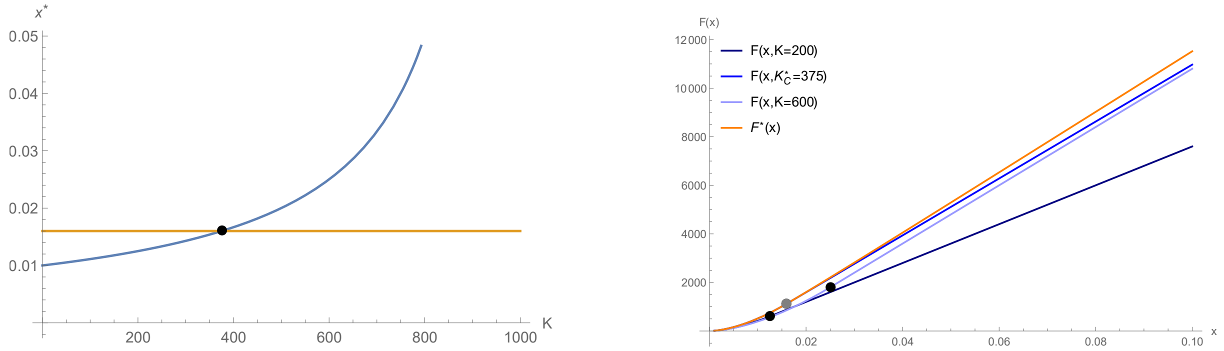
Given the expressions for x_B^* and x_C^*, it follows that

$$\frac{x_B^*}{x_C^*} = \frac{d_1}{d_1 - \alpha K}\frac{\theta}{\theta - \alpha K},$$

from where we conclude that x_B^* may be smaller or larger than x_C^*, depending, in particular, on how large is K. Thus we resort to a numerical illustration, assigning values to the involved parameters.

We use the following values, unless stated otherwise:

- $\mu = 0.03$
- $\sigma = 0.005$
- $r = 0.05$
- $\delta = 2$
- $\alpha = 0.01$
- $\theta = 10$
- $K = 100$



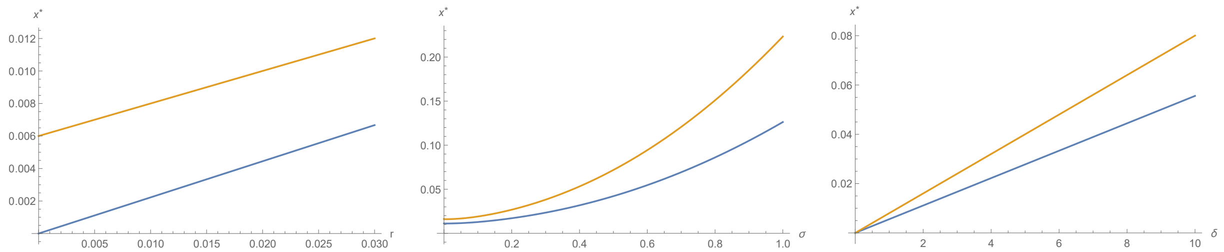
(a) Behaviour of the threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange), considering capacity levels $K \in [0, \theta/\alpha = 1000)$ and the value that x_B^* takes when considering $K_C^* \simeq 375$ (black).

(b) Evaluation of value functions F, considering capacity equal to 200 (darkest blue), $K = K_C^* \simeq 375$ (mid-blue) and 600 (lighest blue), and F^* (orange) with respective demand threshold values presented (black w.r.t. Benchmark Model and grey w.r.t. Capacity Optimization Model)

Figure 3.2: Influence of the chosen capacity K in the threshold values x_B^* and x_C^* and respective value functions F and F^*.

We start by illustrating how x_B^* and x_C^* are related by the capacity level K. On Figure 3.2 (a), the increasing behaviour of x_B^* w.r.t the capacity chosen is presented.

Although the threshold x_B^* may be smaller, greater or equal to x_C^* (depending on the capacity K chosen), we always verify that $F^*(x) \geq F(x, K)$, $\forall x, K$ considered in the domain of our problem. Figure 3.2 (b) confirms this precise fact. Its most intriguing case is $F(x, K_C^*)$, which has the same threshold level as the optimal capacity level and verifies: $F^*(x) = F(x, K_C^*)$, $\forall x \leq x_C^*$, however $F^*(x) > F(x, K_C^*)$, $\forall x > x_C^*$. This is justified by the fact that $F(x, K_C^*)$ considers solely capacity K_C^*[5], which doesn't maximize the terminal function for demand levels greater than x_C^*, while $F^*(x)$ considers the optimal capacity associated to each level of demand, leading to a greater value function.



(a) $r \in (\mu, 1)$ (b) $\sigma \in (0.0001, 1)$ (c) $\delta \in (0, 10)$

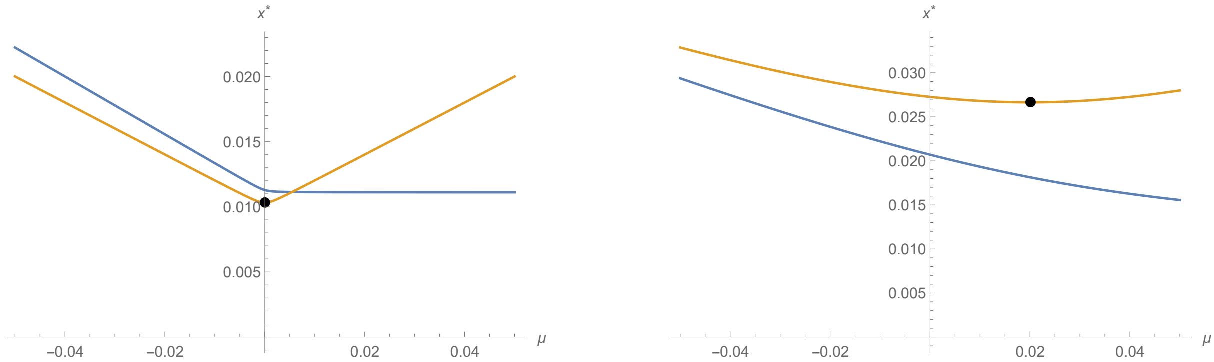
Figure 3.3: Behaviour of the threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange) as a function of increasing parameters r (a), σ (b) and δ (c).

[5]Recall by (3.22) that K_C^* corresponds to the optimal capacity K^*, as described in (3.17), evaluated at x_C^*

On Figure 3.3 (a) we observe that in both situations the decision is postponed with increasing interest rate. This is explainable as: the larger the r, the larger is the expected discounted cash-flow and therefore we need to observe larger values of the demand to balance the money lost due to runway devaluation and future earnings.

Figure 3.3 (b) shows that both thresholds increase with volatility. This in accordance with references [6] and [7], for instance. In fact this is an expected result from real options: by increasing the volatility, the investment is postponed.

Figure 3.3 (c) shows that the greater the sensibility parameter δ is, the larger are the investment sunk cost, leading to a delay on the investment, in the sense that it will only be made if high demand values are observed.



(a) $\sigma = 0.05$

(b) $\sigma = 0.2$

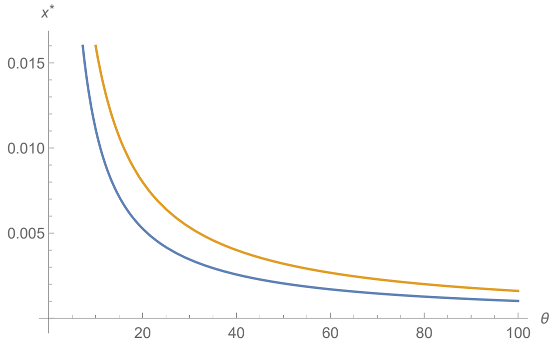
Figure 3.4: Behaviour of the threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange), considering drift $\mu \in [-r, r]$ and corresponding stationary point $\sigma^2/2$ (black).

As proved in Propositions 3.1 and 3.2, the investment threshold does not have a monotonic behaviour w.r.t. μ, as we show in Figure 3.4

For instance, when $\sigma = 0.05$ (low volatility). By inspection of Figure 3.4 (a), we conclude that for $\mu < \frac{\sigma^2}{2}$, $x_B^* > x_C^*$, which leads to postponing the investment decision in the benchmark model. For larger values of μ, the opposite is verified and thus, in the case the market is expected to grow, the investment decision occurs first without capacity optimization. This seems to be related with the fact that an increasing μ also impacts on K_C^* - as can be seen on the leftmost side of Figure 3.6. So, a growing market leads also to an investment in larger capacity and thus higher values of demand are needed. Regarding x_B^*, its value doesn't seem to significantly decrease for a situation when the market is growing. This might whether be related with the low capacity chosen or with the small values of μ considered.

When the volatility is considerably higher, as in Figure 3.4 (b) considering $\sigma = 0.2$, we observe a clear dominance of x_C^* with comparison with x_B^*, at least for the drift range considered.

Regarding the innovation breakthrough level, we obtain that the larger it is, the greater is the unitary price associated to the new product, so as market's expectation, and hence, the firm tends to anticipate its investment decision leading to smaller threshold values for both models, this is illustrated on Figure 3.5.



(a) $\theta \in (1, 100)$

Figure 3.5: Behaviour of the threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange) and decreasing parameter θ.

3.3.3 Optimal Capacity Level

Here we focus on the optimal capacity K_C^* as it is presented in (3.22). We analyse how K_C^* behaves with the different parameters.

Proposition 3.3. *The optimal capacity level K_C^* increases with μ, σ and θ, decreases with r and α, and it is independent on δ.*

Proof: The relation between K_C^* and θ, r or α comes immediately by observing K_C^*'s formula.

Now, regarding drift parameter we obtain that

$$\frac{\partial K_C^*(\mu)}{\partial \mu} = \frac{4\theta\left(\sigma^2(\phi+1)-2\mu\right)}{\alpha\phi\left(\sigma^2(\phi+3)-2\mu\right)^2} > 0.$$

The numerator is positive since

$$\sigma^2(\phi+1) = \sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4}-\frac{4\mu}{\sigma^2}+\frac{8r}{\sigma^2}+1}+1\right)-2\mu > 0 \Leftrightarrow \frac{4\mu^2}{\sigma^4}-\frac{4\mu}{\sigma^2}+\frac{8r}{\sigma^2}+1 > \left(\frac{2\mu}{\sigma^2}-1\right)^2 = \frac{4\mu^2}{\sigma^4}-\frac{4\mu}{\sigma^2}+1$$
$$\Leftrightarrow \frac{8r}{\sigma^2} > 0$$

holds for $\forall r > 0$. As the denominator is clearly positive, the result is verified.

Regarding volatility parameter we obtain that

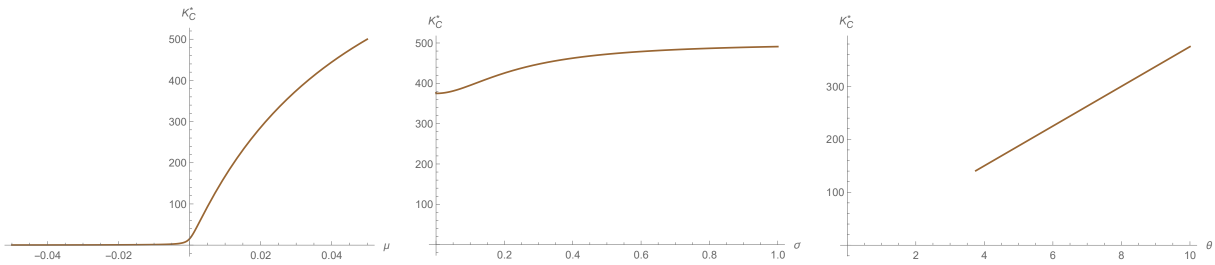
$$\frac{\partial K_C^*(\sigma)}{\partial \sigma} = \frac{8\theta\left(2\mu^2-\mu\sigma^2(\phi+1)+2r\sigma^2\right)}{\alpha\sigma\phi\left(\sigma^2(\phi+3)-2\mu\right)^2} > 0.$$

By similar arguments than the ones used in (3.24), we conclude that the numerator is positive, thus leading to the result.

□

In order to illustrate the previous result and to have an idea about the type of growth of K_C^* w.r.t. the parameters, we add some plots for the numerical values considered in the beginning of this section.

On the leftmost side of Figure 3.6 we study K_C^* as a function of μ. We note that the growth of K_C^* is barely noticed on a decreasing market ($\mu < 0$), but it turns to be logarithmic in a growing market ($\mu > 0$).



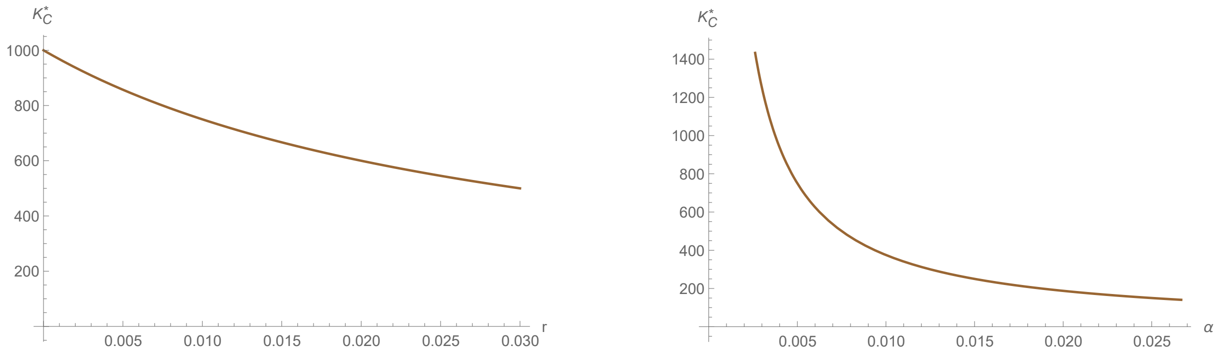
(a) $\mu \in (-r, r)$ (b) $\sigma \in (0, 1)$ (c) $\theta \in (\alpha K_C^*, 10)$

Figure 3.6: Behaviour of the optimal capacity regarding the threshold value x_C^* and increasing parameters μ (a), σ (b) and θ (c).

Mathematically, these behaviours are related with the fact that the denominator of K_C^* decreases with μ, in a weak rate for negative μ and in a significant rate for positive μ. Financially, they are connected with the fact that in a growing market the demand is expected to be larger than in a decreasing market and hence, the capacity should be larger in order to be able to satisfy market needs.

On Figure 3.6 (b), although in a weaker rate, the optimal capacity seems to increase with volatility.

As expected, on Figure 3.6 (c) we observe that K_C^* grows linearly with the chosen innovation level, being this fact related with the expected market response to a newer technology.



(a) $r \in (\mu, 1)$ (b) $\alpha \in (0, 1)$

Figure 3.7: Behaviour of the optimal capacity regarding the threshold value x_C^* and decreasing parameters r (a) and α (b).

Figure 3.7 (a) shows the behaviour of K_C^* w.r.t. the discount rate and, in (b), w.r.t. the sensibility parameter α. Although in different rates, K_C^* decreases with both parameters.

Chapter 4

Investing in a new product when the firm is already active

4.1 Introduction

We consider now the case in which a firm that is already active, with an established product in the market, and has the opportunity to invest in a new and more profitable product, replacing the old one.

By an established product we mean that it is so well recognized in the market that its unitary price is not influenced by the demand level, assuming a fixed price, given by

$$p_0 = 1 - \alpha K_0 \tag{4.1}$$

where α stands for a capacity sensibility parameter and K_0 for the capacity of production of this product, that we will call the old product.

However, the same does not hold for the new product. When the innovation breakthrough takes place, the firm has the option to invest and immediately start to produce the new product. Since this one is a new product, susceptible to the consumers' demand, its unitary price function is considered to be the same as stated in Section 3.1, that is,

$$p_1(X_t) = (\theta - \alpha K_1)X_t \tag{4.2}$$

where θ stands for the innovation level after the breakthrough, α for the same sensitivity parameter as in the old product, K_1 for the capacity of production of the new product and X_t for the demand level at time t.

The instantaneous profit function with respect to each product is given by $\pi_i,\ i \in \{0, 1\}$, and it is obtained by multiplying the unitary price function by the production capacity.

Since the firm is not interested to invest before the innovation process reaches level θ, it is not relevant, for the investment decision, to consider what happened before. Therefore the initial time is set to start

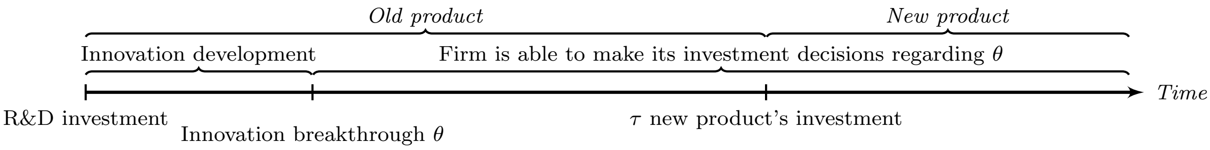


Figure 4.1: Timeline representing the two possible different stages of production and associated decisions. Time is set to start at the innovation breakthrough.

at the instant the breakthrough takes place. Hence the optimal investment time will occur τ^* time units after the breakthrough, as presented on Figure 4.1.

Recall that at the moment we decide to invest, we need to pay δK_1 related to sunk costs. We assume also that upon investment in the new product, the old product is no longer in the market.

As in the previous section, two models will be derived. The first one is the benchmark model, which accounts for the model as explained in the introduction. The second one is the capacity optimization model, which considers the maximized the long-run profit by choosing the optimal capacity K_1 to invest on. Comparative statics w.r.t. both models will be presented afterwards.

4.2 Stopping Problem

4.2.1 Benchmark Model

We want to find when is the best time to invest in the new product, knowing that the firm produces an established product, that it's not influenced by the demand level and which profit function is given by

$$\pi_0 = (1 - \alpha K_0) K_0, \tag{4.3}$$

and when the replacement happens, the firm will be immediately and solely produce a product which profit function corresponds to

$$\pi_1(X_t) = (\theta - \alpha K_1) K_1 X_t. \tag{4.4}$$

Assuming that the investment decision must be made in finite time, our optimal stopping problem may be written as

$$\sup_{\tau} \mathbb{E}^{X_0=x} \left[\int_0^{\tau} \pi_0 e^{-rs} ds + \left(\int_{\tau}^{\infty} \pi_1(X_s) e^{-rs} ds - e^{-r\tau} \delta K_1 \right) \mathbb{1}_{\{\tau<\infty\}} \right]. \tag{4.5}$$

The first integral corresponds to the discounted profit associated to the old product from the time when the innovation level θ is reached (considered here to be its initial instant) until the time when the firm decides to invest in the new product (denoted by τ). The second integral corresponds to the long term discounted profit associated to the new product, obtained after investing. Subtracting to it discounted sunk costs $e^{r\tau}\delta K_1$, we obtain the cash-flow associated to the investment decision.

We can simplify this problem in order to have a standard optimal stopping problem with null running cost function. Starting by conditioning (4.5) to the time when the investment should happen and using

Tower rule it follows that (4.5) is equal to

$$\sup_{\tau} \mathbb{E}^{X_0=x}\left[\mathbb{E}^{\tau}\left[\int_0^{\tau} \pi_0 e^{-rs} ds + \left(\int_{\tau}^{\infty} \pi_1(X_s) e^{-rs} ds - e^{-r\tau}\delta K_1\right) \mathbb{1}_{\{\tau<\infty\}}\right]\right]. \tag{4.6}$$

Since expectation is a linear operator, we can simplify each of the integrals separately.

Note that in the leftmost integral of (4.6), as previously written, the instantaneous profit associated to the old product does not depend on the demand level and all the parameters are deterministic. Then we easily solve the integral, simplifying its expression as

$$\begin{aligned}
\mathbb{E}^{\tau}\left[\int_0^{\tau} \pi_0 e^{-rs} ds\right] &= \mathbb{E}^{\tau}\left[\int_0^{\tau} p_0 K_0 e^{-rs} ds\right] \\
&= \mathbb{E}^{\tau}\left[\int_0^{\tau} (1-\alpha K_0) K_0 e^{-rs} ds\right] \\
&= \mathbb{E}^{\tau}\left[(1-\alpha K_0) K_0 \frac{1-e^{-r\tau}}{r}\right] \\
&= \mathbb{E}^{\tau}\left[\frac{\pi_0}{r}(1-e^{-r\tau})\right] \qquad (4.7)
\end{aligned}$$

Following a similar approach as in Section 3.2.1, when deducing (3.9), and assuming that τ is finite with probability 1, the leftmost expected value of the rightmost integral can also be simplified as

$$\begin{aligned}
\mathbb{E}^{\tau}\left[\int_{\tau}^{\infty} \pi_1(X_s) e^{-rs} ds - e^{-rt}\delta K_1\right] &\underset{v:=s-\tau}{=} \mathbb{E}^{\tau}\left[e^{-rt}\left(\int_0^{\infty} p_1(X_{v+t}) K_1 e^{-rv} dv - \delta K_1\right)\right] \\
&= \mathbb{E}^{\tau}\left[e^{-rt}\left(\int_0^{\infty} (\theta-\alpha K_1) X_{v+t} K_1 e^{-rv} dv - \delta K_1\right)\right] \\
&= \mathbb{E}^{\tau}\left[e^{-rt}\left(\int_0^{\infty} (\theta-\alpha K_1) X_{v+t} K_1 e^{-rv} dv - \delta K_1\right)\right] \\
&= e^{-r\tau}\left(\frac{(\theta-\alpha K_1) K_1 X_{\tau}}{r-\mu} - \delta K_1\right) \qquad (4.8)
\end{aligned}$$

where X_τ is taken to be the demand level at the time when the new product starts being produced. Recall, from the previous section, that expression (4.8) only holds if the assumptions of Fubini's Theorem hold, which particularly implies that $r-\mu>0$.

Let's denote F as the value function solution to (4.6). Plugging expressions (4.7) and (4.8) in (4.6), getting rid of the expectation conditional to time when the investment decision is made and using (again) the Tower rule, we obtain that, assuming the investment decision to be made in finite time, F may be written as

$$F(x) = \frac{\pi_0}{r} + \sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{(\theta-\alpha K_1) K_1 X_{\tau}}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right)\right)\right], \tag{4.9}$$

and therefore this problem can be analysed as in Chapter 3, corresponding to the case of a constant term plus an optimal stopping problem with null running cost function.

Considering V to be the optimal stopping problem present in (4.9), that is

$$V(x) = \sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau}h(X_\tau)\right] = \sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{(\theta-\alpha K_1)K_1 X_\tau}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right)\right)\mathbb{1}_{\{\tau<\infty\}}\right]. \tag{4.10}$$

Following all the steps presented before, we propose as candidate for the continuation region $\mathcal{C} = \{x \in \mathbb{R} : x < x_B^*\}$, with x_B^* being the demand level that triggers the investment. Since V satisfies the HJB variational inequality (2.13) it takes the form of

$$V(x) = \begin{cases} a_2 x^{d_1} & , \ x \in \mathcal{C} \\ \frac{(\theta-\alpha K_1)K_1 x}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right) & , \ x \in \mathcal{S} \end{cases}. \tag{4.11}$$

The coefficient a_2 and the threshold value x_B^* are found by value matching (2.16b) and smooth pasting (2.16c) conditions, expressed by the corresponding system

$$\begin{cases} a_2 (x_B^*)^{d_1} = \frac{(\theta-\alpha K_1)K_1 x_B^*}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right) \\ a_2 d_1 (x_B^*)^{d_1-1} = \frac{(\theta-\alpha K_1)K_1}{r-\mu} \end{cases}, \tag{4.12}$$

leading to

$$x_B^* = \frac{d_1}{d_1-1}\frac{\delta K_1 + \frac{\pi_0}{r}}{\theta-\alpha K_1}\frac{r-\mu}{K_1} \tag{4.13}$$

$$a_2 = \left(\delta K_1 + \frac{\pi_0}{r}\right)\frac{(x_B^*)^{-d_1}}{d_1-1} \tag{4.14}$$

Plugging the results stated above on the expression of F (4.9), it leads to

$$F(x) = \frac{\pi_0}{r} + \begin{cases} a_2 x^{d_1} & , \ x < x_B^* \\ \frac{(\theta-\alpha K_1)K_1 x}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right) & , \ x \geq x_B^* \end{cases} \tag{4.15}$$

with an associated optimal stopping time defined as in (2.9).

4.2.2 Capacity Optimization Model

As considered in Chapter 3, we now extend the previous model, by allowing the firm to optimally choose the capacity of the new product. This optimal stopping problem can be stated as

$$\begin{aligned} F^*(x) &= \sup_{\tau} \mathbb{E}^{X_0=x}\left[\max_{K_1}\left\{\int_0^\tau \pi_0 e^{-rs}ds + e^{-r\tau}\left(\int_\tau^\infty \pi_1(X_s)e^{-rs}ds - \delta K_1\right)\mathbb{1}_{\{\tau<\infty\}}\right\}\right] \\ &= \frac{\pi_0}{r} + \sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau}\max_{K_1}\left\{\frac{(\theta-\alpha K_1)K_1 X_\tau}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right)\right\}\mathbb{1}_{\{\tau<\infty\}}\right], \end{aligned} \tag{4.16}$$

where the expression is simplified in a similar way as done in the previous section, and using the fact that π_0 is deterministic.

Denoting the non-maximized terminal function by h with the dependence on the capacity parameter

K_1 highlighted, we have that

$$h(x, K_1) = \frac{(\theta - \alpha K_1)K_1 x}{r - \mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right). \tag{4.17}$$

Note that h is a second order polynomial with respect to the capacity and its expression corresponds to the terminal function in the previous chapter minus a constant term ($\frac{\pi_0}{r}$). Thus, by studying the first and second derivatives of h, we obtain the same results as achieved in Section 3.2.2: its first and second partial derivatives are, respectively, given by (3.15) and (3.16). Therefore the maximizer of h in (4.17) is the same in (3.17), that is,

$$K_1^* := \arg\max_{K_1} h(x, K_1) = \frac{\theta}{2\alpha} - \frac{\delta(r-\mu)}{2\alpha x}, \ \forall x. \tag{4.18}$$

Evaluating h at its optimal capacity level and denoting by h^*, we obtain that its expression is given by

$$h(x, K^*) = \frac{\pi_0}{r} + \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x}, \tag{4.19}$$

from which follows that our problem, as described in (4.16), can be stated as

$$F^*(x) = \frac{\pi_0}{r} + \sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{\pi_0}{r} + \frac{(\theta X_\tau - \delta(r-\mu))^2}{4\alpha(r-\mu)X_\tau}\right)\mathbb{1}_{\{\tau<\infty\}}\right], \tag{4.20}$$

which consists in a standard stopping optimal problem with null running cost function plus a positive constant term.

Denoting V^* as the value function associated to the optimal stopping problem in (4.20), the optimization problem can be stated as

$$V^*(x) = \sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{\pi_0}{r} + \frac{(\theta X_\tau - \delta(r-\mu))^2}{4\alpha(r-\mu)X_\tau}\right)\mathbb{1}_{\{\tau<\infty\}}\right] \tag{4.21}$$

which is again a standard optimal stopping problem with null running cost function. Therefore, invoking results presented on Chapter 2, it follows that V^* is such that

$$V^*(x) = \begin{cases} bx^{d_1} \ , \ x \in \mathcal{C} \\ \frac{\pi_0}{r} + \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x} \ , \ x \in \mathcal{S} \end{cases}. \tag{4.22}$$

Considering that the continuation region should be of the form $\mathcal{C} = \{x \in \mathbb{R} : x < x_C^*\}$ - stating that the firm should postpone the investment for small demand values, in particular, smaller than x_C^* -, b and x_C^* are found by value matching (2.16b) and smooth pasting (2.16c) conditions, hereunder expressed

$$\begin{cases} b(x_C^*)^{d_1} = \frac{\pi_0}{r} + \frac{(\theta x_C^* - \delta(r-\mu))^2}{4\alpha(r-\mu)x_C^*} \\ bd_1(x_C^*)^{d_1-1} = \frac{\theta^2(x_C^*)^2 - \delta^2(r-\mu)^2}{4\alpha(r-\mu)(x_C^*)^2} \end{cases}. \tag{4.23}$$

We obtain two possible positive roots for the threshold level:

$$x^*_{C,1} = \frac{r-\mu}{(d_1-1)\theta^2 r}\left(d_1\left(2\alpha pi_0+\delta\theta r\right)+\sqrt{(\delta\theta r)^2+4d_1^2\alpha\pi_0\left(\alpha\pi_0+\delta\theta r\right)}\right)$$
$$x^*_{C,2} = \frac{r-\mu}{(d_1-1)\theta^2 r}\left(d_1\left(2\alpha\pi_0+\delta\theta r\right)-\sqrt{(\delta\theta r)^2+4d_1^2\alpha\pi_0\left(\alpha\pi_0+\delta\theta r\right)}\right).$$

However, since the coefficient b associated to it takes a negative value, we exclude $x^*_{C,2}$. As explained in the previous chapter, considering such b would be an absurd. Consequently, we obtain that the threshold level and coefficient b in (4.23) are, respectively, given by

$$x^*_C = \frac{r-\mu}{(d_1-1)\theta^2 r}\left(d_1\left(2\alpha\pi_0+\delta\theta r\right)+\sqrt{(\delta\theta r)^2+4d_1^2\alpha\pi_0\left(\alpha\pi_0+\delta\theta r\right)}\right) \tag{4.24}$$
$$b = \left(\frac{K_0(\alpha K_0-1)}{r}+\frac{(\theta x^*_C-\delta(r-\mu))^2}{4\alpha x^*_C(r-\mu)}\right)(x^*_C)^{-d_1}.$$

Plugging the results above on (4.20) it follows that F^* corresponds to

$$F^*(x) = \frac{\pi_0}{r}+\begin{cases} bx^{d_1} & , \ x < x^*_C \\ \frac{\pi_0}{r}+\frac{(\theta x-\delta(r-\mu))^2}{4\alpha(r-\mu)x} & , \ x \geq x^*_C \end{cases}, \tag{4.25}$$

leading to an optimal stopping time as defined in (2.9).

Following a similar approach as in Section 3.2.2, we evaluate K_1^* at the threshold demand level x^*_C, obtaining

$$K^*_C = \frac{\theta}{2\alpha}-\frac{\delta(d_1-1)\theta^2 r}{2\alpha\left(\sqrt{4\alpha d_1^2\pi_0(\alpha\pi_0+\delta\theta r)+\delta^2\theta^2 r^2}+d_1(2\alpha\pi_0+\delta\theta r)\right)}. \tag{4.26}$$

4.3 Comparative Statics

In this section we analyse the results obtained regarding x^*_B, x^*_C and K^*_C. As we have considered in Chapter 3, we compare x^*_B and x^*_C.

4.3.1 Benchmark Model

Proposition 4.1. *The decision threshold x^*_B increases with δ, decreases with θ and does not have a monotonic behaviour with K_0, K_1, r. Regarding the sensibility parameter α, x^*_B increases with it when $\theta < \frac{K_1}{K_0^2}(K_0+K_1 r\delta)$, and decreases otherwise. Regarding volatility σ, x^*_B increases with it when $d_1 \in (1, \frac{1}{2}(3+\sqrt{5}))$ and decreases when $d_1 \in \left(\frac{1}{2}(3+\sqrt{5}), \infty\right)$.*

Proof:

Regarding the formula obtained for x^*_B (4.13), we immediately conclude that the decision threshold increases with δ and decreases with θ.

Regarding K_0, we conclude that

$$\frac{\partial x_B^*(K_0)}{\partial K_0} = \frac{d_1(r-\mu)}{r(d_1-1)K_1(\theta-\alpha K_1)}(1-2\alpha K_0) = \begin{cases} >0 & \text{for } K_0 < \frac{1}{2\alpha} \\ <0 & \text{for } K_0 > \frac{1}{2\alpha} \end{cases},$$

since the expression represented in fraction is always positive.

Regarding K_1, we obtain that

$$\frac{\partial x_B^*(K_1)}{\partial K_1} = \frac{d_1(r-\mu)}{(d_1-1)K_1(\theta-\alpha K_1)}\left(\frac{\alpha(\frac{\pi_0}{r}+K_1\delta)}{\theta-\alpha K_1} - \frac{\frac{\pi_0}{r}+K_1\delta}{K_1} + \delta\right).$$

The first term in the equation is always positive. Thus we only need to evaluate the second term (between brackets). Manipulating it and taking into account that the capacity level cannot be negative [1] , it follows that

$$\frac{\partial x_B^*(K_1)}{\partial K_1} = \begin{cases} >0 & \text{for } K_1 > \frac{-\pi_0+\sqrt{\alpha\pi_0(\pi_0+r\delta\theta)}}{r\alpha\delta} \\ <0 & \text{for } K_1 \in \left[0, \frac{-\pi_0+\sqrt{\alpha\pi_0(\pi_0+r\delta\theta)}}{r\alpha\delta}\right] \end{cases},$$

from which we obtain that x_B^* does not have a monotonic behaviour with K_1.

Regarding parameter α, we obtain that

$$\frac{\partial x_B^*(\alpha)}{\partial \alpha} = \frac{d_1(r-\mu)}{(d_1-1)(\theta-\alpha K_1)}\left(\frac{\frac{K_0(1-\alpha K_0)}{r}+\delta K_1}{\theta-\alpha K_1} - \frac{K_0^2}{rK_1}\right).$$

Again, the first term is positive. Simplifying the second term [2], it follows that

$$\frac{\partial x_B^*(\alpha)}{\partial \alpha} = \begin{cases} >0 & \text{for } \theta < \frac{K_0K_1+K_1^2r\delta}{K_0^2} \\ <0 & \text{for } \theta > \frac{K_0K_1+K_1^2r\delta}{K_0^2}. \end{cases}$$

Note that the sign of the partial derivative does not depend on α, but instead on both capacity levels K_0 and K_1, discount rate r and sensibility parameter δ.

[1] We obtain that

$$\frac{\alpha(\frac{\pi_0}{r}+K_1\delta)}{\theta-\alpha K_1} - \frac{\frac{\pi_0}{r}+K_1\delta}{K_1} + \delta = \frac{d_1(r-\mu)(2K_1\pi_0\alpha+K_1^2r\alpha\delta-\pi_0\theta)}{(d_1-1)K_1^2r(\theta-\alpha K_1)^2}$$

which roots are given by

$$2K_1\pi_0\alpha+K_1^2r\alpha\delta-\pi_0\theta=0 \Leftrightarrow K_1 = \frac{-\pi_0 \pm \sqrt{\alpha\pi_0(\pi_0+r\delta\theta)}}{r\alpha\delta}.$$

Since the capacity chosen K_1 must be positive, we obtain a single root, $K_1 = \frac{-\pi_0+\sqrt{\alpha\pi_0(\pi_0+r\delta\theta)}}{r\alpha\delta}$, that leads to the result presented.

[2] Reducing to the same denominator it follows that the rightmost expression corresponds to

$$\frac{\frac{\pi_0}{r}+\delta K_1}{\theta-\alpha K_1} - \frac{K_0^2}{rK_1} = \frac{rK_1\frac{K_0(1-\alpha K_0)}{r}+\delta K_1 - K_0^2(\theta-\alpha K_1)}{rK_1(\theta-\alpha K_1)},$$

which is equal to zero when

$$K_1K_0(1-\alpha K_0)+r\delta K_1^2-K_0^2(\theta-\alpha K_1) = K_0K_1+r\delta K_1^2-K_0^2\theta=0 \Rightarrow \theta = \frac{K_0K_1+K_1^2r\delta}{K_0^2}.$$

Regarding σ, we conclude that

$$\frac{\partial x_B^*(\sigma)}{\partial \sigma} = \underbrace{\frac{(r-\mu)\left(\delta K_1 + \frac{\pi_0}{r}\right)}{(d_1-1)K_1(\theta-\alpha K_1)}}_{(1)} \underbrace{\left(\frac{2\mu}{\sigma^3} + \frac{\frac{4\mu\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)}{\sigma^3} - \frac{4r}{\sigma^3}}{2\sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}}\right)}_{(2)} \underbrace{\left(1 - \frac{d_1}{(d_1-1)^2}\right)}_{(3)} = \begin{cases} > 0 \text{ for } d_1 \in \left(1, \frac{1}{2}(3+\sqrt{5})\right] \\ < 0 \text{ for } d_1 \in \left(\frac{1}{2}(3+\sqrt{5}), \infty\right) \end{cases}.$$

Forthwith, we agree that expression (1) is positive. Manipulating expression (2), we obtain that

$$\begin{aligned} \frac{2\mu}{\sigma^3} + \frac{\frac{4\mu\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)}{\sigma^3} - \frac{4r}{\sigma^3}}{2\sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}} < 0 &\Leftrightarrow \mu d_1 - r < 0 \\ &\Leftrightarrow \frac{\mu}{2} - \frac{\mu^2}{\sigma^2} + \mu\sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} - r < 0 \\ &\Leftrightarrow \frac{r}{\mu}\left(1 - \frac{r}{\mu}\right) < 0, \end{aligned} \tag{4.27}$$

which holds true as $r > \mu$. Analysing expression (3), we have that the polynomial $d_1^2 - d_1 - 1$ has roots on $\left\{\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}(3+\sqrt{5})\right\}$. Since $d_1 > 1$, the first root is not admissible. Consequently, we obtain that expression (3) is negative for $d_1 \in \left(1, \frac{1}{2}(3+\sqrt{5})\right]$, and positive for $d_1 \in \left(\frac{1}{2}(3+\sqrt{5}), \infty\right)$.

Regarding parameter r we obtained complex derivatives, from which we weren't able to derive any analytical result. However, as it will be shown later, numerical results show that x_B^* behaves in a non-monotonic way with r.

Regarding μ, as the sign of the derivatives is not either clearly positive or negative, we could not assess analytically if x_B^* has a monotonic behaviour w.r.t. μ. But as we will present later, numerical illustrations suggest that x_B^* decreases with μ.

□

4.3.2 Capacity Optimization Model

Proposition 4.2. *The decision threshold x_C^* increases with δ, decreases asymptotically with θ and has a non-monotonic behaviour with μ, r, α and K_0.*

Proof:

For the sake of simplicity, in this proof, we consider ϕ as in (3.23), which is always positive and well-defined, and we define $\psi := 4d_1^2\alpha\pi_0(\delta\theta r + \alpha\pi_0) + \delta^2\theta^2 r^2$, which is also positive.

Regarding δ, we immediately conclude that

$$\frac{\partial x_C^*(\delta)}{\partial \delta} = \frac{(r-\mu)\left(\frac{4d_1^2\theta r\pi_0 + 2\delta\theta^2 r^2}{2\sqrt{\psi}} + d_1\theta r\right)}{(d_1-1)\theta^2 r} > 0.$$

Regarding θ, we obtain that

$$\frac{\partial x_C^*(\theta)}{\partial \theta} = \frac{\theta(r-\mu)\left(\frac{4\delta d_1^2 r\alpha\pi_0 + 2\delta^2\theta r^2}{2\sqrt{\psi}} + \delta d_1 r\right) - 2(r-\mu)\left(d_1(\delta\theta r + 2\alpha\pi_0) + \sqrt{\psi}\right)}{(d_1-1)\theta^3 r}.$$

Although we weren't able to evaluate the sign of preceding expression due its the complexity, we analyse its asymptotic behaviour. This is possible since we assume no upper limit for innovation levels θ.

Denoting $\theta_A := \frac{(r-\mu)}{(d_1-1)r}\left(\sqrt{\delta^2 r^2} + \frac{\delta r\left(\sigma^2(\phi+1)-2\mu\right)}{2\sigma^2}\right) > 0$ we obtain that x_C^* asymptotically decreases in order of $\frac{\theta_A}{\theta}$, that is,

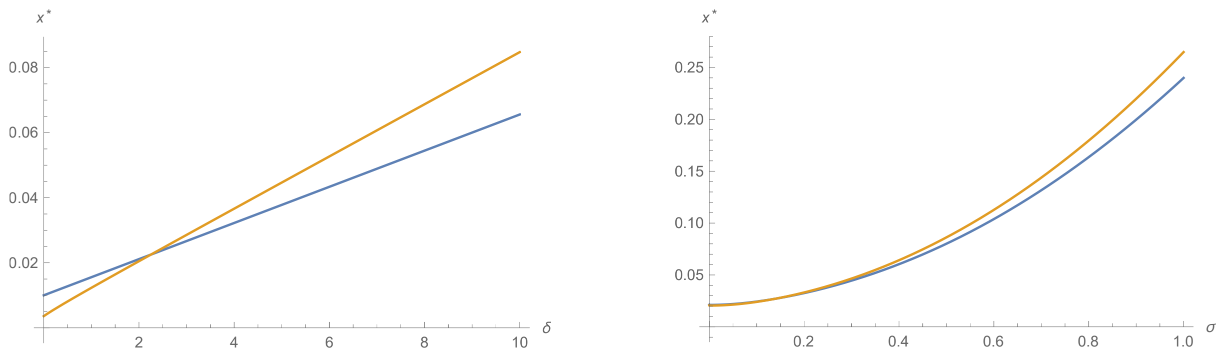
$$x_C^*(\theta) \sim \frac{\theta_A}{\theta} \Leftrightarrow \lim_{\theta\to\infty} \frac{x_C^*(\theta)}{\frac{\theta_A}{\theta}} = 1.$$

Regarding parameters μ, r, α and K_0 we obtained complex derivatives, from which we weren't able to derive any analytical result. However, as it will be shown later, numerical results show that x_C^* behaves in a non-monotonic way with each of them.

□

Numerical comparisons between the Benchmark and the Capacity Optimization Models

In a similar way as done in Chapter 3, using software *Mathematica*, we perform numerical simulations to illustrate the above results. In addition to the values considered in the previous chapter, we also assume $K_0 = 50$ (and $K_1 = 100$).



(a) $\delta \in (0, 10)$ (b) $\sigma \in (0, 1)$

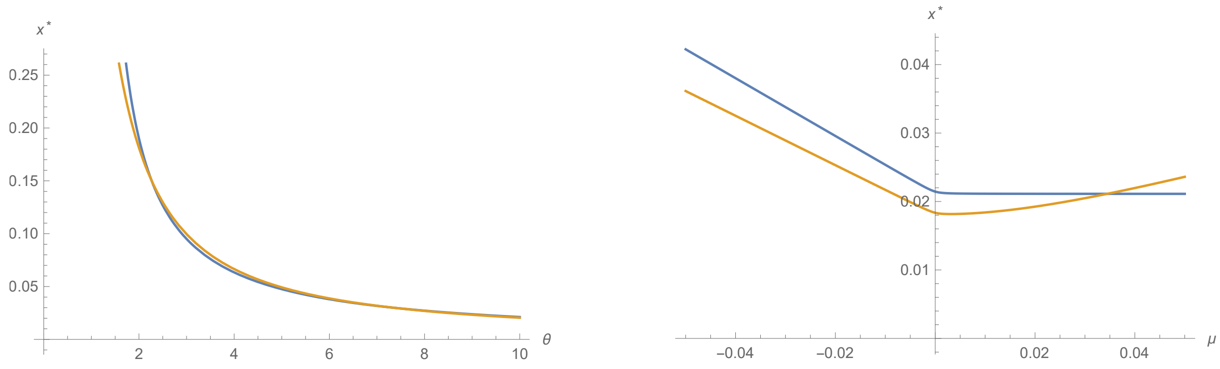
Figure 4.2: Behaviour of the threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange) and parameters with which x_B^* increases δ (a) and σ (b).

Figures 4.2 (a) and (b) show the behaviour of both thresholds by changing δ and σ, respectively. The first result is motivated by the fact that a higher δ reflects a larger investment (sunk) cost, which is only incurred if there is a demand level large enough such that the investment is worthwhile.

The second one is justified by the growing uncertainty (of the demand). Although we were not able to derive analytically this result regarding the threshold x_C^*, it is widely common that a larger volatility postpones the investment decision [3].

On Figure 4.3 (a) we observe that, similarly to what was described on Chapter 3, both threshold levels decrease with innovation level, meaning that the firm is eager to invest when the technology improvement is more significant.

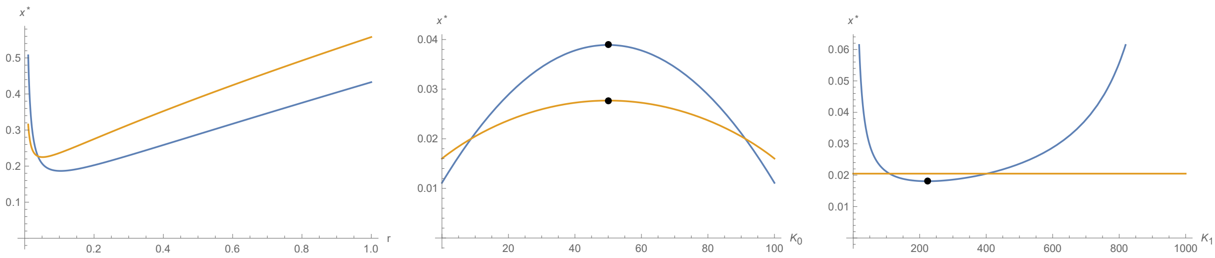
Regarding expectations about the market evolution (illustrated in Figure 4.3 (b)), we see that in a declining market ($\mu < 0$) both thresholds decrease approximately in a linear rate. However the same



(a) $\theta \in (1, 10)$ (b) $\mu \in (-r, r)$

Figure 4.3: Behaviour of the threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange) and parameter with which x_B^* decreases θ (a) and μ (b).

is not verified on a growing market ($\mu > 0$). The threshold associated to the Benchmark Model still decreases, but in a weaker rate than in a declining market, whereas x_C^* increases with μ. Again, we believe this happens in view of the invested capacity associated to each threshold (as, with increasing μ, the optimal capacity K_C^* also increases, and therefore the firm tends to invest later).



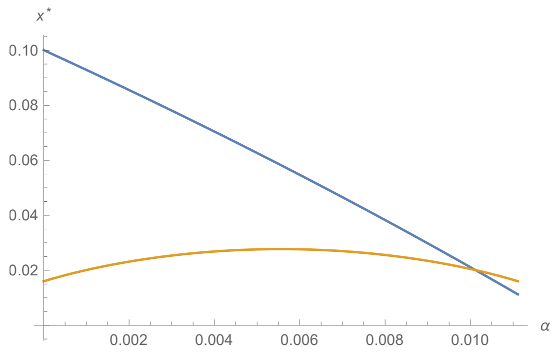
(a) $r \in (\mu, 1)$, $\mu = 0.01$ and $\sigma = 0.9$ (b) $K_0 \in (0, \frac{1}{\alpha} = 100)$ (c) $K_1 \in (0, \frac{\theta}{\alpha} = 1000)$

Figure 4.4: Behaviour of the threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange) and parameters with which x_B^* has a non-monotonic behaviour r (a), K_0 (b) and $K_1(c)$.

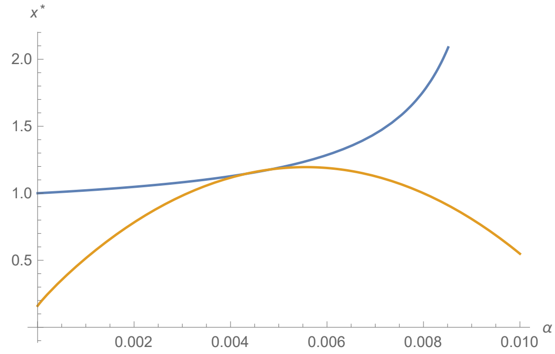
On Figure 4.4 (a) we observe that, concerning each model, exists a discount rate that leads to a minimal demand threshold and that it is associated to small values of r. Unfortunately, we were not able to derive its analytical expression.

Interestingly, as seen on Figure 4.4 (b), a maximum demand threshold is observed on both models when the capacity level K_0 is exactly equal to $\frac{1}{2\alpha}$ (which, on this case, takes the value $K_0 = 50$). This quantity is motivated by the profit function associated to the established product, in the sense that a capacity of $\frac{1}{2\alpha}$ leads to the maximization of π_0. Therefore, for K_0 assuming quite smaller or larger values than $\frac{1}{2\alpha}$, the firm obtains a relative low profit in the old product and, consequently, the idea of investing in a new product gets more attractive to the firm, which is reflected by the small threshold levels.

Regarding the capacity associated to the new product, we notice on Figure 4.4 (c), that (as expected) it only affects x_B^*. We observe that a large x_B^* is associated to either quite small or quite large values of K_1. The first one is explained by the fact that, to invest in a small quantity, the demand needs to be large enough such that the expected long-run π_1 (and investment costs) overcome the profit associated to the established product. The second one supports that the larger the capacity to invest on, the larger the expected π_1 is as well as associated investment costs. Hence, a larger value of demand is needed, in order to justify the investment made. A minimum triggering value is observed at $K_1 = \frac{-\pi_0+\sqrt{\alpha\pi_0(\alpha\pi_0+\delta\theta r)}}{r\alpha\delta}$ (which, on this case, takes the value $K_1 \simeq 223$).



(a) $\alpha \in (0,1)$ and $\theta = 10 > \frac{K_1}{K_0^2}(K_0 + K_1 r\delta)$



(b) $\alpha \in (0,1)$ and $\theta = 1 < \frac{K_1}{K_0^2}(K_0 + K_1 r\delta)$

Figure 4.5: Behaviour of the threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange) and sensibility parameter α.

Figure 4.5 highlights the two possible behaviours of x_B^* regarding parameter α. As stated in Proposition 4.1, we observe that for values of θ greater than $\frac{K_1}{K_0^2}(K_0+K_1 r\delta)$ (which, in this case, is approximately 1.235), x_B^* decreases with α (as represented on (a)). For greater values than that quantity, x_B^* increases (as represented on (b)). We propose that this unexpected behaviour happens due to the influence of both π_0 and π_1. Although we were not able to derive any analytical result concerning x_C^*, we conclude that it behaves non-monotonically with θ, confirming the stated on Proposition 4.2.

4.3.3 Optimal Capacity Level

In this section we analyse the optimal capacity level K_C^*, whose formula is provided in (4.26). Due to its complex expression, we are just able to deduce analytical results concerning the asymptotic behaviour of K_C^* with θ. However, the numerical results obtained lead us to conclude that it has a similar behaviour to the optimal capacity presented on Chapter 3, whose formula corresponds to equation (3.22).

Proposition 4.3. *Asymptotically, the optimal capacity level K_C^* grows in a linear rate with θ. Also, K_C^* has a non-monotonic behaviour with K_0.*

Proof:

Regarding innovation level θ, assuming that it has no upper limit, it's possible to evaluate its behaviour asymptotically. Denoting $\theta_K := \frac{\sigma^2\left(\sqrt{\delta^2 r^2}+\delta r\right)}{\alpha\left(2\sigma^2\sqrt{\delta^2 r^2}+\delta r(\sigma^2(\phi+1)-2\mu)\right)} > 0$, we obtain that K_C^* increases on order of $\theta_K\theta$, that is,

$$K_C^*(\theta) \sim \theta_K\theta \;\Leftrightarrow\; \lim_{\theta\to\infty} \frac{K_C^*}{\theta_K\theta} = 1$$

The non monotonic behaviour of K_C^* with K_0 is illustrated in Figure 4.6.

$\square$

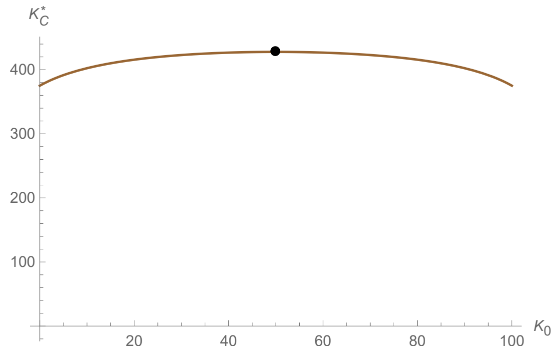
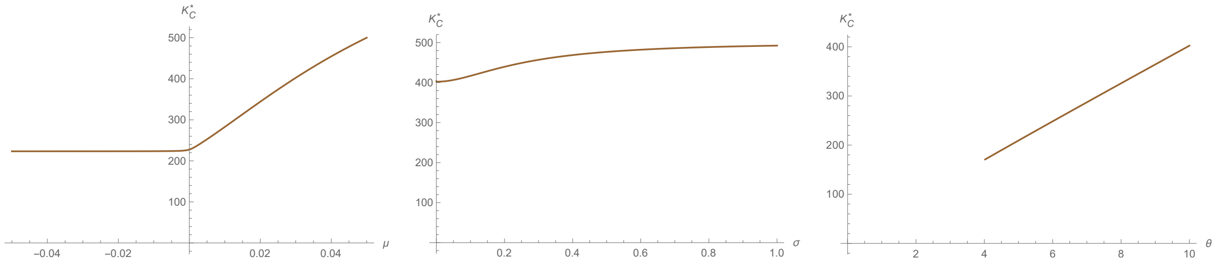


Figure 4.6: Behaviour of the optimal capacity regarding the threshold value x_C^* considering capacity levels $K_0 \in [0, 100)$ and its highest values at $\frac{1}{2\alpha} = 50$ (black).

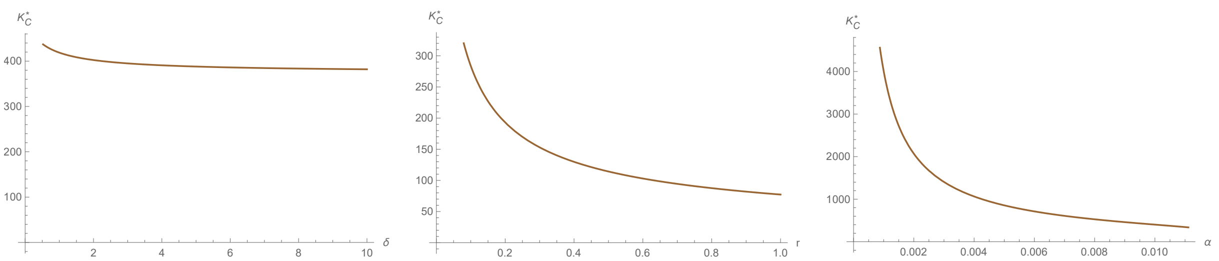
Figure 4.6 shows that the highest optimal capacity level K_C^* happens for $K_0 = \frac{1}{2\alpha}$. We note that this stresses previous observations: if the firm is committed to a larger investment, then it impacts the investing timing, postponing it.



(a) $\mu \in (-r, r)$ (b) $\sigma \in (0, 1)$ (c) $\theta \in (1, 10)$

Figure 4.7: Behaviour of the optimal capacity regarding the threshold value x_C^* and increasing parameters μ (a), σ (b) and θ (c).

Figure 4.7 illustrates that K_C^* increases with both drift, volatility and innovation level, as it happened in the previous section. Note, on Figure 4.7 (a), that, on a declining market ($\mu < 0$), the optimal capacity is significantly smaller than in the growing market ($\mu > 0$). This facts seems to support the increasing behaviour of x_C^* for values $\mu > 0$, as presented on Figure 4.3 (b). Also, note, on Figure 4.7 (c), that K_C^* seems to increase linearly with θ, being in accordance to the asymptotic behaviour of K_C^* with θ previously proved.



(a) $\delta \in (0, 10)$ (b) $r \in (\mu, 1)$ (c) $\alpha \in (0, 1)$

Figure 4.8: Behaviour of the optimal capacity regarding the threshold value x_C^* and decreasing parameters δ (a), r (b) and α (c).

The most relevant result presented above corresponds to the one shown in Figure 4.8 (a). The decreasing behaviour of K_C^* is motivated by the fact that a smaller δ leads to a lower investment cost and therefore the firm is more attracted to invest on a larger capacity.

On Figures 4.8 (b) and (c) we observe that K_C^* behaves in a similar way as K_C^* analysed in the previous Chapter 3, in particular regarding results on Figure 3.7.

Chapter 5

Investing in a new product, allowing a simultaneous production period, when the firm is already active

5.1 Introduction

In this chapter, the case described on the previous chapter is extended by considering a firm, which has an established product in the market, and wants to find the best time to:

1. Invest and introduce in the market an innovative product with technology level θ, allowing the possibility of a simultaneous production of both old and new products;

2. Abandon the production of the old product, maintaining the new one in the market.

Figure 5.1 shows the 3 stages that have to be considered by the firm.

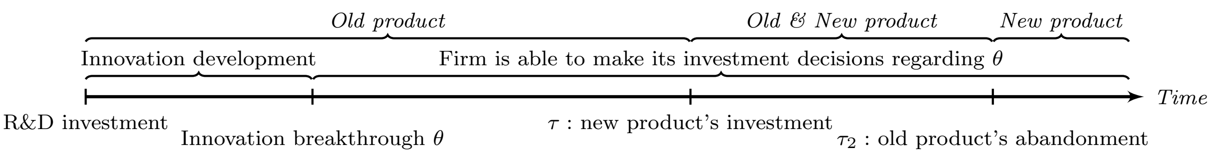


Figure 5.1: Timeline representing the three possible different stages of production and associated decisions.

On the first stage only the established product is produced, whose unitary price p_0 and profit functions π_0 are the same as stated in Chapter 4, and take the values of (4.1) and (4.3), respectively.

During the second stage the firm produces both old and new products, leading to the follow profit functions

$$\pi_0^A(X_t) = (1 - \alpha K_0 - \eta K_1 X_t)K_0 \tag{5.1}$$

$$\pi_1^A(X_t) = (\theta - \alpha K_1 - \eta K_0 X_t)K_1. \tag{5.2}$$

A *cannibalisation* (or *horizontal differentiation*) parameter η is introduced on both expressions to embody the crossed effect between the old and the new product. As we consider both products to be interacting in the same market, η represents the penalty that the quantity associated to a product will influence the price of the other. We consider here that this influence is the same for both products, so we can have a unique cannibalisation parameter η. By observing that $\eta \geq \alpha$ would imply a larger effect on the product price that on the quantity of production itself, we establish η to be such that $\eta < \alpha$. By adding both profits π_0^A and π_1^A we attain the profit associated to this second stage, that is,

$$\pi_A(X_t) = \pi_0^A(X_t) + \pi_1^A(X_t) = (1 - \alpha K_0)K_0 + (\theta - \alpha K_1)K_1 X_t - 2\eta K_0 K_1 X_t. \tag{5.3}$$

Finally, on the third state, we consider that the firm abandons the old product and starts producing solely the new product, which is not considered to be a stable product. Its demand function p_1 and profit function π_1 are as stated in Chapter 4, and take the values of (4.2) and (4.4), respectively.

Therefore we want to find two optimal times to make different (but maybe simultaneous) decisions. We want to find the best time τ_1 to go from the first to the second stage - that is, to invest in the new product and start producing, simultaneously, the old and the new product - and we also want to find the best time τ_2 to go from the second to the third stage - that is, to replace the production of the old product by the new one. Note that $\tau_2 \geq \tau_1$, and both are stopping times adapted to the natural filtration of the demand process $\{X_t,\ t \geq 0\}$. Moreover, we assume that once the old product is abandoned, the decision is irreversible.

The strategy followed to calculate τ_1 and τ_2 is the one presented in [3]. Dixit and Pindyck suggest to calculate the value of the investment in the second stage, and then the value of the investment in the first stage. Although the contexts were different, a similar strategy was used in [7] - but here having a decision threshold associated with the innovation process, not only the demand as we do - and in [6] - but considering an entering-or-exit the market situation.

5.2 Stopping Problem

Similarly to the previous sections, we still consider that at the moment we adapt the new product, we need to pay δK_1 related to sunk costs, and that we are totally able to produce it.

Taking into account the different profits associated to each stage of production, as described on Figure 5.1, our the optimal stopping problem may be formulated as finding the value function F such that

$$\begin{aligned} F(x) = \sup_{\tau_1} \mathbb{E}^{X_0=x}\Big[\int_0^{\tau_1} \pi_0 e^{-rs} ds + \sup_{\tau_2} \mathbb{E}^{X_{\tau_1}=x_{\tau_1}}\Big[\int_{\tau_1}^{\tau_2} \pi_A(X_s) e^{-rs} ds + \\ + \int_{\tau_2}^{\infty} \pi_1(X_s) e^{-rs} ds \ \mathbb{1}_{\{\tau_2<\infty\}} - e^{-r\tau_1}\delta K_1\Big]\mathbb{1}_{\{\tau_1<\infty\}}\Big] \end{aligned} \tag{5.4}$$

Manipulating (5.4) by changing the region of integration of the first integral and admitting that both decisions are made in finite time (with probability 1), we obtain

$$
\begin{aligned}
F(x) &= \sup_{\tau_1} \mathbb{E}^{X_0=x}\Big[\int_0^{\infty} \pi_0 e^{-rs} ds + \sup_{\tau_2} \mathbb{E}^{X_{\tau_1}}\Big[\int_{\tau_1}^{\tau_2} \left(\pi_A(X_s) - \pi_0\right) e^{-rs} ds + \\
&\qquad + \int_{\tau_1}^{\infty} \left(\pi_1(X_s) - \pi_0\right) e^{-rs} ds - e^{-r\tau_1}\delta K_1\Big]\Big] \\
&= \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{X_0=x}\Big[\sup_{\tau_2} \mathbb{E}^{X_{\tau_1}}\Big[\int_{\tau_1}^{\tau_2} \left(\pi_A(X_s) - \pi_0\right) e^{-rs} ds + \\
&\qquad + \int_{\tau_2}^{\infty} \left(\pi_1(X_s) - \pi_0\right) e^{-rs} ds\Big] - e^{-r\tau_1}\delta K_1\Big]
\end{aligned} \tag{5.5}
$$

Changing integration variables of both integrals on the optimization problem related with τ_2 we obtain

$$
\begin{aligned}
F(x) = \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{X_0=x}\Big[e^{-r\tau_1}\Big(\sup_{\tau_2} \mathbb{E}^{X_{\tau_1}}\Big[\int_0^{\tau_2-\tau_1} \left(\pi_A(X_{\tau_1+s}) - \pi_0\right) e^{-rs} ds + \\
+ \int_{\tau_2-\tau_1}^{\infty} \left(\pi_1(X_{\tau_1+s}) - \pi_0\right) e^{-rs} ds\Big] - \delta K_1\Big)\Big].
\end{aligned} \tag{5.6}
$$

since the term $e^{-r\tau_1}$ does not depend on τ_2.

Now we are able to address the stopping problem related with τ_2.

τ_2 – Optimal Stopping Problem:

Considering F_2 to be the value function associated to the optimal stopping problem related to τ_2, we have that

$$
F_2(x) = \sup_{\tau_2} \mathbb{E}^{X_{\tau_1}=x_{\tau_1}}\left[\int_0^{\tau_2-\tau_1} \left(\pi_A(X_{\tau_1+s}) - \pi_0\right) e^{-rs} ds + \int_{\tau_2-\tau_1}^{\infty} \left(\pi_1(X_{\tau_1+s}) - \pi_0\right) e^{-rs} ds\right], \tag{5.7}
$$

from which follows that our optimal stopping problem, initially given by (5.4), is now given by

$$
F(x) = \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{X_0=x}\left[e^{-r\tau_1}\left(F_2(X_{\tau_1}) - \delta K_1\right)\right]. \tag{5.8}
$$

In view of F, as described above, we have in hands two different optimal stopping problems regarding τ_1 and τ_2. We start solving the one concerning the *latest* stopping time, assuming that we know what happened until the instant that the firm invests (τ_1).

In that regard we consider $\{Y_t,\ t \geq 0\}$ as the stochastic process that represents the demand level after occurring the investment at τ_1. Note that it evolves stochastically accordingly to a GBM with the same drift μ and volatility σ as $\{X_t,\ t \geq 0\}$ and whose initial value is the same as observed by X at the instant τ_1, that is, $\{Y_t,\ t \geq 0\} = \{X_{\tau_1+t},\ t \geq 0\}$.

Additionally, we consider as well τ to be the stopping time, adapted to the natural filtration of the process $\{Y_t,\ t \geq 0\}$. Observe that τ represents the optimal time for which the firm should make the replacement of the old product by the new one, after having invested at time τ_1. This means that if $\tau = 0$, then the old product is replaced by the new one at the precise instant when the investment happens τ_1. Note that τ is also adapted to the natural filtration of $\{X_{\tau+t},\ t \geq 0\}$ and that $\tau_2 = \tau_1 + \tau$. Thus, by knowing τ_1 and finding τ, we can calculate τ_2.

Therefore, F_2, as written in (5.7), is equivalent to

$$F_2(x_{\tau_1}) = \sup_{\tau} \mathbb{E}^{Y_0=x_{\tau_1}} \left[\int_0^{\tau} \left(\pi_A(Y_s) - \pi_0\right) e^{-rs} ds + \int_{\tau}^{\infty} \left(\pi_1(Y_s) - \pi_0\right) e^{-rs} ds \right], \tag{5.9}$$

meaning that, in accordance to Figure 5.1, from time 0 to time τ, the firm is producing both products and that from time τ on, the firm only produces the new product, where the instant 0 corresponds to the instant when the firm decides to invest, given in fact by τ_1.

Fortunately it is possible to simplify the notation of (5.9). Since the Strong Markov property states that after a stopping time, the future path of the GBM depends conditionally (only) on the value at the stopping time, it follows that

$$\{(Y_t | Y_0 = x_{\tau_1}),\ t \geq 0\} = \{(X_{\tau_1+t} | X_{\tau_1} = x_{\tau_1}),\ t \geq \tau_1\} \overset{d}{=} \{(X_t, | X_0 = x_{\tau_1}),\ t \geq 0\}.$$

Therefore we can keep the same notation as before and thus from (5.9) it follows

$$F_2(x_{\tau_1}) = \sup_{\tau} \mathbb{E}^{X_0=x_{\tau_1}} \left[\int_0^{\tau} \left(\pi_A(X_s) - \pi_0\right) e^{-rs} ds + \int_{\tau}^{\infty} \left(\pi_1(X_s) - \pi_0\right) e^{-rs} ds \right]. \tag{5.10}$$

Using the fact that the expectation is a linear operator, we treat the expectation of the rightmost integral of (5.10) separately, that is

$$\mathbb{E}^{X_0=x_{\tau_1}} \left[\int_{\tau}^{\infty} \left(\pi_1(X_s) - \pi_0\right) e^{-rs} ds \right] = \mathbb{E}^{X_0=x_{\tau_1}} \left[e^{-r\tau} \int_0^{\infty} \left(\pi_1(X_{\tau+s}) - \pi_0\right) e^{-rs} ds \right]. \tag{5.11}$$

Conditioning to the stopping time τ and using Tower Rule we obtain from (5.11)

$$\mathbb{E}^{X_0=x_{\tau_1}} \left[e^{-r\tau} \mathbb{E}^{\tau} \left[\int_0^{\infty} \left(\pi_1(X_{\tau+s}) - \pi_0\right) e^{-rs} ds \right] \right]. \tag{5.12}$$

Using the similar arguments as in Section 3.2.1, we interchange the integral with expectation using Fubini's theorem and the fact that $r - \mu > 0$, obtaining

$$\begin{aligned} &\mathbb{E}^{X_0=x_{\tau_1}} \left[e^{-r\tau} \left(\int_0^{\infty} \mathbb{E}^{\tau} \left[\pi_1(X_{t+s}) e^{-rs} \right] ds - \frac{\pi_0}{r} \right) \right] = \\ &= \mathbb{E}^{X_0=x_{\tau_1}} \left[e^{-r\tau} \left((\theta - \alpha K_1) K_1 \int_0^{\infty} \mathbb{E}^{\tau} \left[X_{t+s} e^{-rs} \right] ds - \frac{\pi_0}{r} \right) \right], \end{aligned} \tag{5.13}$$

where the term $(\theta - \alpha K_1) K_1$ is constant over time.

We focus now on the expected value conditional to the stopping time τ above. Since the demand level evolves accordingly to a GBM, it follows

$$\begin{aligned}\mathbb{E}^{\tau}\left[X_{\tau+s}e^{-rs}\right] &= \mathbb{E}^{\tau}\left[X_{\tau}e^{\left(\mu-\frac{\sigma^2}{2}-r\right)(\tau+s-\tau)+\sigma(W_{\tau+s}-W_{\tau})}\right] \\ &= \mathbb{E}^{\tau}\left[X_{\tau}e^{\left(\mu-\frac{\sigma^2}{2}-r\right)s+\sigma W_s}\right] \\ &= X_{\tau}e^{(\mu-r)s}, \end{aligned} \tag{5.14}$$

where in the second equality we used the fact that Brownian Motion $\{W_t,\ t \geq 0\}$ has stationary increments, that is

$$W_{\tau+s} - W_{\tau} \overset{d}{=} W_{\tau+s-\tau} - W_0 \overset{d}{=} W_s \sim \mathcal{N}(0, s).$$

Plugging (5.14) in (5.13), we obtain

$$\mathbb{E}^{X_0=x_{\tau_1}}\left[e^{-r\tau}\left((\theta-\alpha K_1)K_1\int_0^{\infty}X_{\tau}e^{(\mu-r)s}ds - \frac{\pi_0}{r}\right)\right] = \mathbb{E}^{X_0=x_{\tau_1}}\left[e^{-r\tau}\left(\frac{(\theta-\alpha K_1)K_1}{r-\mu}X_{\tau} - \frac{\pi_0}{r}\right)\right]. \tag{5.15}$$

Therefore we have found the terminal function associated to the optimal stopping problem F_2, hereby denoted as h_2, which is given by

$$h_2(x) = \frac{(\theta-\alpha K_1)K_1}{r-\mu}x - \frac{\pi_0}{r}.$$

Accordingly to (5.10), we may also denote g_2 as the running cost function associated to this problem, that is given by

$$g_2(x) = \pi_A(x) - \pi_0.$$

Thus, plugging expression of running and terminal functions on (5.10), we have that F_2 as initially written in (5.7), is equivalent to

$$\begin{aligned} F_2(x) &= \sup_{\tau}\mathbb{E}^{X_0=x}\left[\int_0^{\tau}g(X_s)e^{-rs}ds + e^{-r\tau}h(X_{\tau})\right] \qquad (5.16)\\ &= \sup_{\tau}\mathbb{E}^{X_0=x}\left[\int_0^{\tau}\left(\pi_0^A(X_s)+\pi_1^A(X_s)-\pi_0\right)e^{-rs}ds + e^{-r\tau}\left(\frac{(\theta-\alpha K_1)K_1}{r-\mu}X_{\tau}-\frac{\pi_0}{r}\right)\right], \qquad (5.17)\end{aligned}$$

being this a standard optimal stopping problem.

Since the HJB variational inequality associated to (5.17) implies that the non-homogeneous PDE

$$\frac{\sigma^2}{2}x^2F_2''(x) + \mu xF_2'(x) - rF_2(x) + g(x) = 0 \tag{5.18}$$

holds for any demand value in the continuation region, it follows that its solution F_2 takes the form of

$$F_2(x) = F_{2,h}(x) + F_{2,p},\ \forall x \in \mathcal{C}, \tag{5.19}$$

where $F_{2,h}$ corresponds to the solution to the homogeneous version of the PDE in (5.18), $F_{2,p}$ to a particular solution of (5.18) and $\mathcal{C}$ to the continuation region of the form $\{x \in \mathbb{R} : x < x_2^*\}$.

We first calculate the particular solution $F_{2,p}$. By considering $F''_{2,p}(x) = 0$ and using expression (5.18), a particular solution is found to be

$$F_{2,p}(x) = \frac{(\theta - \alpha K_1)K_1 - 2\eta K_0 K_1}{r - \mu} x \quad \xrightarrow[x \to 0]{} \quad 0 \tag{5.20}$$

Since there is no possibility of having a project having a negative value it follows $F_2(x) \geq 0, \ \forall x \in \mathcal{C}$. From (5.20), we obtain that $F_{2,h} = a_2 x^{d_1} + b_2 x^{d_2}$ simplifies to $F_{2,h} = a_2 x^{d_1}$, guaranteeing this way that $\lim_{x \to 0} F_2(x) = \lim_{x \to 0} F_{2,h}(x) + F_{2,p}(x) = 0$.

Once again, the constant term a_2 and the demand level that triggers the abandonment of the old product x_2^* are found by value matching (2.16b) and smooth pasting (2.16c) conditions, expressed by

$$\begin{cases} a_2 (x_2^*)^{d_1} + \frac{(\theta - \alpha K_1)K_1 - 2\eta K_0 K_1}{r - \mu} x_2^* = \frac{(\theta - \alpha K_1)K_1}{r - \mu} x_2^* - \frac{\pi_0}{r} \\ a_2 d_1 (x_2^*)^{d_1 - 1} + \frac{(\theta - \alpha K_1)K_1 - 2\eta K_0 K_1}{r - \mu} = \frac{(\theta - \alpha K_1)K_1}{r - \mu} \end{cases}, \tag{5.21}$$

from which we obtain

$$\begin{aligned} a_2 &= \left(\frac{2\eta K_0 K_1}{r - \mu} x^* - \frac{\pi_0}{r} \right) (x_2^*)^{-d_1} \\ x_2^* &= \frac{d_1}{d_1 - 1} \frac{1 - \alpha K_0}{2\eta K_1} \frac{r - \mu}{r} \end{aligned} \tag{5.22}$$

Taking into account that F_2 is also defined in the stopping region, we obtain that its expression is given by

$$F_2(x) = \left(a_2 x^{d_1} + \frac{(\theta - \alpha K_1)K_1 - 2\eta K_0 K_1}{r - \mu} x \right) \mathbb{1}_{\{x < x_2^*\}} + \left(\frac{(\theta - \alpha K_1)K_1}{r - \mu} x - \frac{\pi_0}{r} \right) \mathbb{1}_{\{x \geq x_2^*\}}, \tag{5.23}$$

τ_1 **– Optimal Stopping Problem:**

We are now in position to return to the main problem, as described in (5.8). Re-evaluating with the expression derived for F_2, we end up with the following:

$$\begin{aligned} F(x) = \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{X_0 = x} \Bigg[e^{-r\tau_1} \Bigg(& \frac{(\theta - \alpha K_1)K_1}{r - \mu} X_{\tau_1} + \left(a X_{\tau_1}^{d_1} - \frac{2\eta K_0 K_1}{r - \mu} X_{\tau_1} \right) \mathbb{1}_{\{X_{\tau_1} < x_2^*\}} - \\ & - \frac{\pi_0}{r} \mathbb{1}_{\{X_{\tau_1} \geq x_2^*\}} - \delta K_1 \Bigg) \Bigg], \end{aligned} \tag{5.24}$$

which is again an optimal problem with null running cost function. However, this time it has two terminal cost functions defined on disjoint domains. Thereafter, we obtain two optimal times associated to the adoption of the innovative product, depending on whether we allow a simultaneous production period or not. These are depending on the following demand thresholds:

- $x_{1,A}^*$: this is the threshold that triggers the investment and the addition of the new product to the market, starting a period of simultaneous production. Therefore it is associated to the region

$X_{\tau_1} < x_2^*$ so as to the following stopping problem

$$\sup_{\tau_1} \mathbb{E}^{X_0=x}\left[e^{-r\tau_1}\left(\frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu}X_{\tau_1} + a_2 X_{\tau_1}^{d_1} - \delta K_1\right)\right], \tag{5.25}$$

where the decision is made on finite time τ with probability 1. Note that when the demand observes larger values than x_2^* its respective value is given by the expression on the rightmost side of (5.24).

- $x_{1,R}^*$: this is the threshold associated to the region $X_{\tau_1} \geq x_2^*$, for which the firm invest in the new product by immediately replacing the one established in the market. Considering that the investment decision is made in finite time with probability 1, this situation leads to the following stopping problem

$$\sup_{\tau_1} \mathbb{E}^{X_0=x}\left[e^{-r\tau_1}\left(\frac{(\theta-\alpha K_1)K_1}{r-\mu}X_{\tau_1} - \frac{\pi_0}{r} - \delta K_1\right)\right] \tag{5.26}$$

We will deduce both thresholds $x_{1,A}^*$ and $x_{1,R}^*$ by its respective order.

- **Demand threshold $x_{1,A}^*$:**

The optimal stopping problem stated in (5.25) is a standard optimal stopping problem, for which we don't have a solution (yet). Denoting by $F_{1,A}$ its correspondent solution, we have that $F_{1,A}$ verifies the HJB variational inequality (2.13), where we take the terminal function to be $h(x) = \frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu}x + a_2 x^{d_1} - \delta K_1$. Therefore it follows that $F_{1,A}$ takes the form

$$F_{1,A}(x) = \begin{cases} a_{1,A}x^{d_1} & , \ x \in \mathcal{C}_{1,A} \\ \frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu}x + a_2 x^{d_1} - \delta K_1 & , \ x \in \mathcal{S}_{1,A} \end{cases}, \tag{5.27}$$

with $\mathcal{C}_{1,A}$ stands for the continuation region of the form $\{x \in \mathbb{R} : x < x_{1,A}^*\}$. The threshold value $x_{1,A}^*$ and coefficient $a_{1,A}$ are found by value matching (2.16b) and smooth pasting (2.16c) conditions, expressed by the corresponding system

$$\begin{cases} a_{1,A}(x_{1,A}^*)^{d_1} = \frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu}x_{1,A}^* + a_2 (x_{1,A}^*)^{d_1} - \delta K_1 \\ a_{1,A}d_1(x_{1,A}^*)^{d_1-1} = \frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu} + a_2 d_1 (x_{1,A}^*)^{d_1-1} \end{cases}, \tag{5.28}$$

from which we obtain

$$x_{1,A}^* = \frac{d_1}{d_1-1}\frac{\delta K_1(r-\mu)}{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1} \tag{5.29}$$

$$a_{1,A} = a_2 x_2^* + \frac{((\theta-\alpha K_1)K_1 - 2\eta K_1 K_0)}{r-\mu}(x_{1,A}^*)^{1-d_1} - \delta K_1 (x_{1,A}^*)^{-d_1}. \tag{5.30}$$

Observe that since the threshold $x_{1,A}^*$ needs always to be positive, the following restriction must hold

$$(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1 > 0 \ \Leftrightarrow \ K_1 < \frac{\theta}{\alpha + 2\eta K_0} \leq \frac{\theta}{\alpha}, \tag{5.31}$$

where the equality (in the last inequality) holds for $\eta = 0$.

Thus the solution associated to (5.25) is given by

$$F_{1,A}(x) = \left(a_{1,A} x^{d_1} \mathbb{1}_{\{x < x^*_{1,A}\}} + \left(\frac{(\theta - \alpha K_1)K_1 - 2\eta K_0 K_1}{r - \mu} x + a_2 x^{d_1} - \delta K_1 \right) \mathbb{1}_{\{x \geq x^*_{1,A}\}} \right) \mathbb{1}_{\{x < x^*_2\}}. \tag{5.32}$$

- **Demand threshold $x^*_{1,R}$:**

Note that the optimal stopping problem (5.26) is the same as the one explored and analysed in Chapter 4, consisting in the benchmark model (Section 4.3.1). Thus it follows that the threshold level is given by (4.13), that is,

$$x^*_{1,R} = \frac{d_1}{d_1 - 1} \frac{\delta K_1 + \frac{\pi_0}{r}}{\theta - \alpha K_1} \frac{r - \mu}{K_1}. \tag{5.33}$$

On the other side the solution associated to (5.26) is given by (4.15), that is, (changing the name of the associated solution)

$$F_{1,R}(x) = \left(a_{1,R} x^{d_1} \mathbb{1}_{\{x < x^*_{1,R}\}} + \left(\frac{(\theta - \alpha K_1)K_1 x}{r - \mu} - \delta K_1 - \frac{\pi_0}{r} \right) \mathbb{1}_{\{x \geq x^*_{1,R}\}} \right) \mathbb{1}_{\{x \geq x^*_2\}}, \tag{5.34}$$

with $a_{1,R}$ taking the same value as a_2 in (4.14).

Note that from (5.32) it is impossible to observe a demand threshold x^*_2 smaller than $x^*_{1,A}$. This would mean that at the precise instant $x^*_{1,A}$ was reached, we should immediately replace the old product by the new one. However, this case is not related with thresholds $x_{1,A}$ or x^*_2, but with $x^*_{1,R}$. Therefore we are able to constraint the domains of both functions $F_{1,A}$ and $F_{1,R}$ by analysing under which conditions $x^*_{1,A} < x^*_2$ holds:

$$\begin{aligned} x^*_{1,A} < x^*_2 \;\Leftrightarrow\; & \frac{d_1}{d_1 - 1} \frac{1 - \alpha K_0}{2\eta K_1} \frac{r - \mu}{r} = \frac{d_1}{d_1 - 1} \frac{\delta K_1 (r - \mu)}{(\theta - \alpha K_1)K_1 - 2\eta K_0 K_1} \\ \Leftrightarrow\; & (1 - \alpha K_0)((\theta - \alpha K_1)K_1 - 2\eta K_0 K_1) = 2\eta\delta K_1^2 r \\ \Leftrightarrow\; & \eta < \frac{(1 - \alpha K_0)(\theta - \alpha K_1)}{2(\delta K_1 r + \pi_0)} =: \eta^*, \end{aligned} \tag{5.35}$$

where we denoted η^* to be the cannibalisation threshold that states which value function stands.

Therefore we obtain that our problem, as stated in (5.24), may be written as

$$F(x)=\frac{\pi_0}{r}+\begin{cases} a_{1,A}x^{d_1} & , \; x < x_{1,A} \;\wedge\; \eta < \eta^* \\ a_2 x^{d_1} + \frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu}x - \delta K_1 & , \; x_{1,A} \leq x < x_2^* \;\wedge\; \eta < \eta^* \\ a_{1,R}x^{d_1} & , \; x < x_{1,R}^* \;\wedge\; \eta \geq \eta^* \\ \frac{(\theta-\alpha K_1)K_1 x}{r-\mu} - \frac{\pi_0}{r} - \delta K_1 & , \; (x > x_2^* \;\wedge\; \eta < \eta^*) \;\dot{\vee}\; (x > x_{1,R}^* \;\wedge\; \eta \geq \eta^*) \end{cases} \tag{5.36}$$

with $a_{1,A}$ as described in (5.30), $a_{1,R}$ as described in (4.14), $x_{1,A}^*$ as described in (5.29), $x_{1,R}^*$ as described in (4.13), x_2^* as described in (5.22) and η^* as described in (5.35) (and π_0 as described in (4.3)).

5.3 Comparative Statics

For a better understanding of numerical experiments' results, in this chapter we consider different values from the ones stated so far. These are given by:

- $\mu = 0.45$
- $\sigma = 0.6$
- $r = 0.85$
- $\delta = 0.95$
- $\alpha = 0.015$
- $\theta = 1.5$
- $K_0 = 51.5$
- $K_1 = 2.5$

On Table 5.1 we assess the cannibalisation threshold η^* and how the value function F relates to it, recurring to expression (5.36). Note that $\frac{\pi_0}{r}$ is a constant positive quantity common to all the different subfunctions, it is not taken into account in the following analysis.

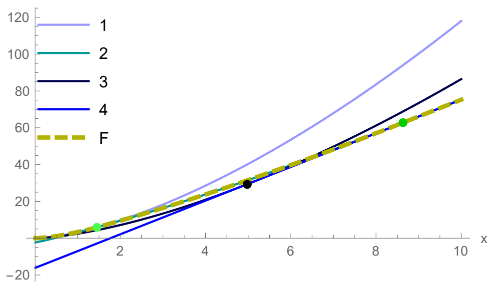
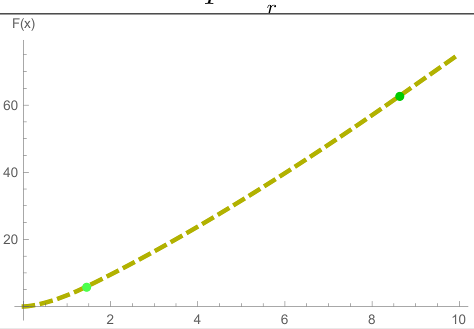
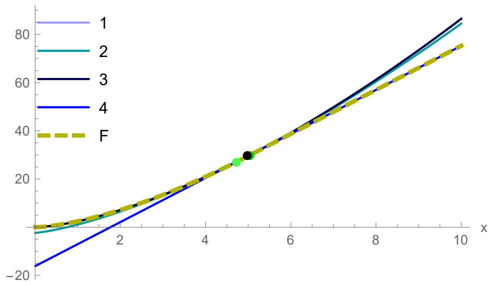
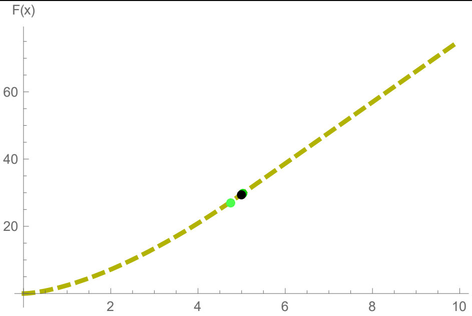
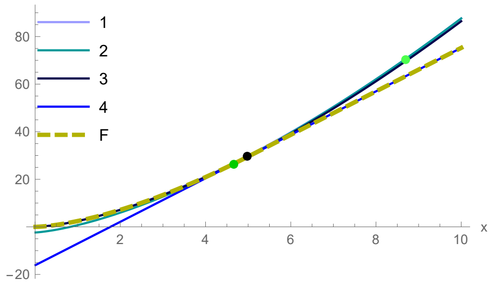
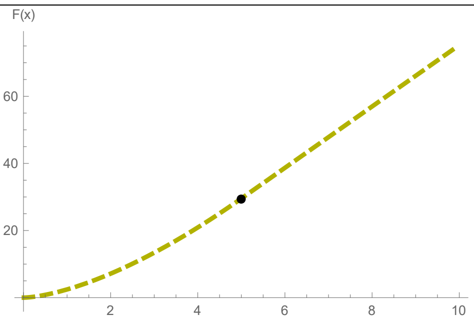
On the leftmost column of plots we present the four different subfunctions - corresponding respectively to the four possible situations on (5.36) - evaluated for the three different levels of cannibalisation chosen. Simultaneously, by the yellow dashed line, we also present $F(x) - \frac{\pi_0}{r}$, which on the rightmost column of plots is solely presented along the relevant demand thresholds that trigger investment decisions.

First, note that regarding this set of values, we obtain a cannibalisation threshold $\eta^* \simeq 0.0121$.

Therefore in the situations represented on the first and second columns the firm is recommended to invest and have a period of simultaneous production and to produce solely the new product when the demand is observed to be x_2^*. On the other side, in the third column, it is represented the situation for which a firm should invest and replace immediately the old established product by the new one as soon as $x_{1,R}^*$ is hit.

In the first row ($\eta = 0.007$) we observe that the thresholds to be considered are $x_{1,A}^*$ and x_2^*. The first one states upon which demand level we should invest in the new product and start the simultaneous production. The second one states upon which demand level we should produce only the new product.

Table 5.1: Value function F and respective subfunctions (without the quantity $\frac{\pi_0}{r}$), presented following the order in (5.36), associated to different settings of parameter η. There is also represented the threshold values: $x^*_{1,A}$ (light green), x^*_2 (darker green) and $x^*_{1,R}$ (black).

Cannibalisation	Subfunctions of F	$F - \frac{\pi_0}{r}$
$\eta = 0.007$		
$\eta = 0.012$		
$\eta = 0.013$		

The respective value function associated is defined by 3 parts, being given by

$$F(x) = \frac{\pi_0}{r} + a_{1,A}x^{d_1}\mathbb{1}_{\{x<x^*_{1,A}\}} + \left(a_2 x^{d_1} + \frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu}x - \delta K_1\right)\mathbb{1}_{\{x_{1,A}\leq x<x^*_2\}} + \left(\frac{(\theta-\alpha K_1)K_1 x}{r-\mu} - \frac{\pi_0}{r} - \delta K_1\right)\mathbb{1}_{\{x>x^*_2\}}. \tag{5.37}$$

In the second row ($\eta = 0.012$), we are in a similar situation as in the previous row, where we have two thresholds and the value function is also defined in three parts, as in (5.37). Observe also that as $\eta \to \eta^*$, the three thresholds tend to admit the same value. This is accordance with the analytical result that replacing η by η^* in $x^*_{1,A}$ (5.29) and in $x^*_{1,R}$ (4.13) we get

$$x^*_{1,A}(\eta^*) = x^*_2(\eta^*) = x^*_{1,R}.$$

In the third row, we observe that the threshold to be considered is $x^*_{1,R}$. In this case the value function is only defined by two parts, being given by

$$F(x) = \frac{\pi_0}{r} + a_{1,R} x^{d_1} \mathbb{1}_{\{x<x^*_{1,R}\}} + \left(\frac{(\theta - \alpha K_1) K_1 x}{r-\mu} - \frac{\pi_0}{r} - \delta K_1 \right) \mathbb{1}_{\{x>x^*_{1,R}\}}$$

On the following sections we analyse derived thresholds and how they are influenced by the parameters, first analytically and then numerically. Taking into account that the threshold regarding the immediate replace of the established product by the innovative when the investment is incurred ($x^*_{1,R}$) is the same as the one studied in the benchmark model of Chapter 4, we won't go further on it - analysis and conclusion were already stated on Section 4.2.1.

The set of parameters now considered is the same as in previous chapters (more precisely Chapter 4). Additionally we choose $\eta = 0.007$, guaranteeing that we are at the situation where $\eta < \eta^*$, that is, the firm relies on $x^*_{1,A}$ and x^*_2, since it's optimal to have a simultaneous production period.

5.3.1 Demand threshold $x^*_{1,A}$

Proposition 5.1. *The decision threshold $x^*_{1,A}$ increases with η, δ, σ, α, K_0 and K_1 and decreases with θ.*

Proof:

The results regarding parameters η, δ, α and θ come immediately by the expression of $x^*_{1,A}$ (5.29). Regarding σ, we obtain that

$$\frac{\partial x^*_{1,A}(\sigma)}{\partial \sigma} = \frac{2\delta(\mu - r)\left(-2\mu^2 + \mu\sigma^2(\phi+1) - 2r\sigma^2\right)}{(d_1-1)^2\sigma^5\phi(\theta - \alpha K_1 - 2\eta K_0)} > 0,$$

From condition (5.31) and the sign of the numerator in (3.24), it follows, respectively, that the denominator and the numerator are positive.

Regarding K_0 and K_1, we obtain that

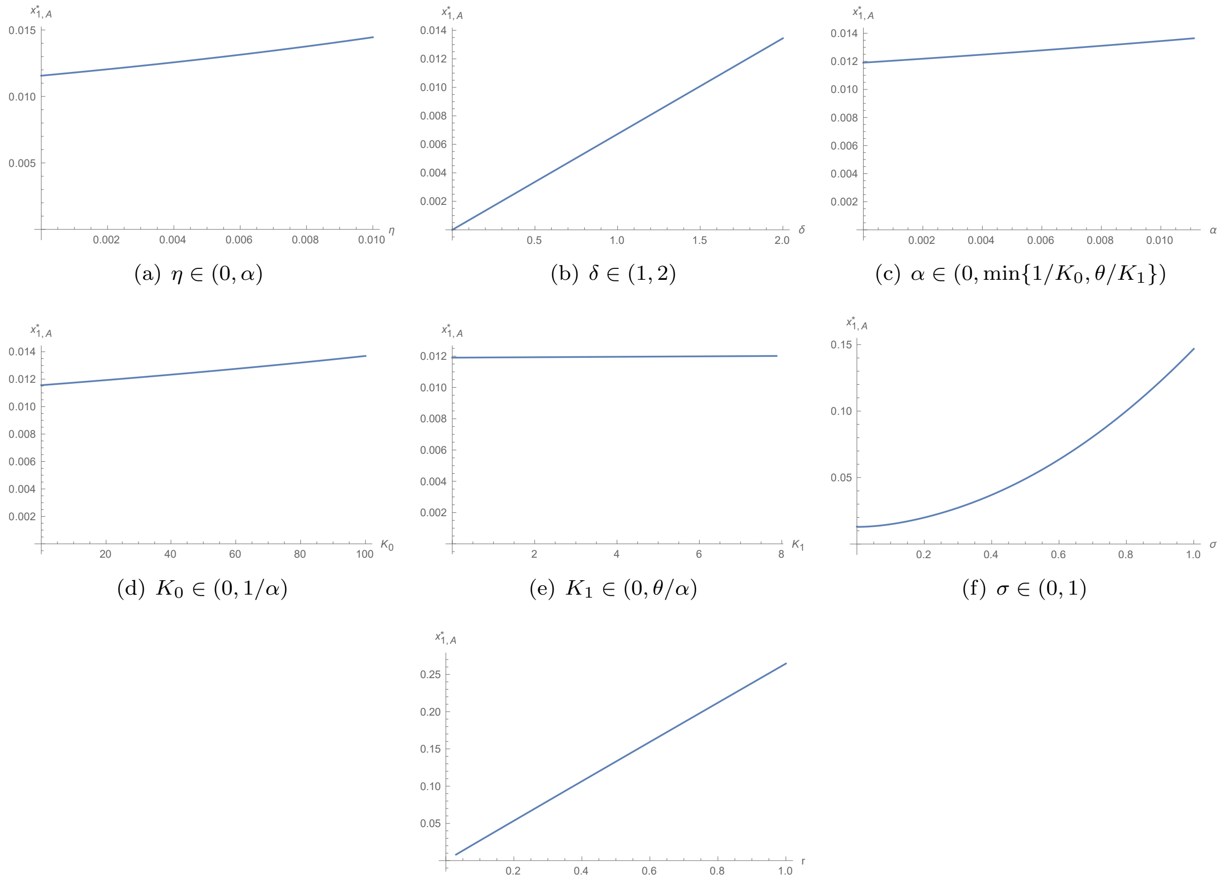
$$\begin{aligned}
\frac{\partial x^*_{1,A}(K_0)}{\partial K_0} &= \frac{2\delta d_1 \eta (r-\mu)}{(d_1-1)(-\theta + \alpha K_1 + 2\eta K_0)^2} > 0 \\
\frac{\partial x^*_{1,A}(K_1)}{\partial K_1} &= \frac{\alpha \delta d_1 \eta (r-\mu)}{(d_1-1)(-\theta + \alpha K_1 + 2\eta K_0)^2} > 0,
\end{aligned}$$

from which the result holds.

□

We performed numerical experiments to assess the changes of $x^*_{1,A}$ with the different parameters. The results obtained are presented on Figures 5.2 and 5.3.

Figure 5.2 (a) shows that the larger the cannibalisation parameter, the later the firm is expected to adopt the new product, since a high demand needs to be observed such that the investment and associated costs of the simultaneous production are overcome. Recall that for quite large values of η, we are in the situation where is preferable to invest on the new product, replacing immediately the old one.



(a) $\eta \in (0, \alpha)$ (b) $\delta \in (1, 2)$ (c) $\alpha \in (0, \min\{1/K_0, \theta/K_1\})$

(d) $K_0 \in (0, 1/\alpha)$ (e) $K_1 \in (0, \theta/\alpha)$ (f) $\sigma \in (0, 1)$

(g) $r \in (0, \mu)$

Figure 5.2: Behaviour of the threshold value $x^*_{1,A}$ with respect to the parameters with which it increases η (a), δ (b), α (c), K_0 (d), K_1 (e), σ (f) and r (g).

The linear relationship between δ and $x^*_{1,A}$, as presented on Figure 5.2 (b), comes immediately by the threshold's expression. This is financially justified by the fact that a larger δ results on larger investmnet (sunk) costs and, therefore, the firm only invests when it could profit.

One can note that $x^*_{1,A}$ shows a similar linear behaviour with α, K_0 and K_1.

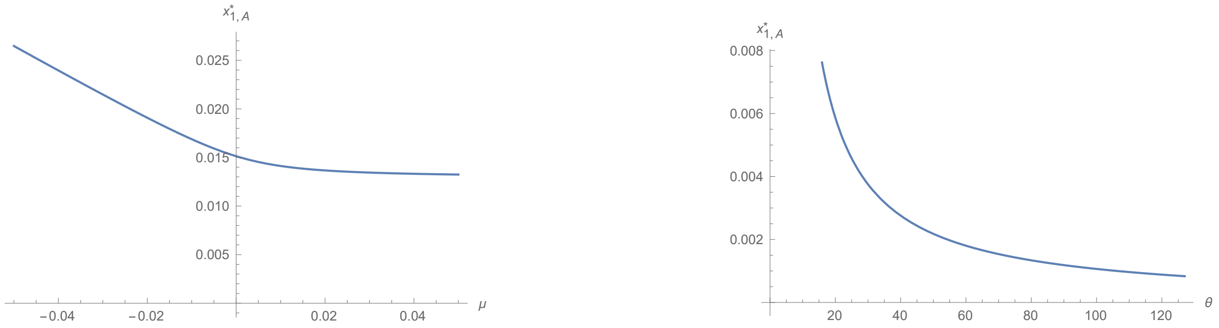
On Figure 5.2 (c), the growth of α suggests that the established product is preferable to the new one, resulting in a larger demand threshold.

Concerning both capacities of production K_0 and K_1, we observe on Figures 5.2 (d) and (e), that the larger they are, the higher the demand level that triggers the invest is. Regarding K_0 this result is justified by the fact that the larger is the value it takes, the higher the profit of the established product is, so the firm gets more relutant about investing for low demand levels. In different circumstances, a larger K_1 induces a longer investment (so as investment costs), wherefore the investment decision will only incur if the observed demand is large enough.

Once again we observe on Figure 5.2 (f) that a growth uncertainty on the demand lead to the postponement of the investment decision.

Although we weren't able to deduce any analytical result concerning the interest rate, we obtain on Figure 5.2 (g) that, as already stated, an increasing interest rate delays the investment decision.

As analysed on previous chapters, we observe on a declining market ($\mu < 0$), the investment threshold



(a) $\mu \in (0, r)$

(b) $\theta \in (1, 130)$

Figure 5.3: Behaviour of the threshold value $x^*_{1,A}$ with respect to parameters with which it decreases μ (a) and θ (b).

decreases linearly in a stronger rate than in a growing market $\mu > 0$, on which the threshold barely changes.

Finally, in view of the results shown in Figure 5.2, we obtain that the larger the innovation level, the higher is the expected profit during the simultaneous production period and, hence, the more attractive is the investment on the new product.

5.3.2 Demand threshold x^*_2

Proposition 5.2. *The decision threshold x^*_2 increases with σ and decreases with η, α, K_0 and K_1.*

Proof:

The results regarding parameters η, α, K_0 and K_1 come immediately by the expression of x^*_2 (5.22). Regarding σ, we obtain that

$$\frac{\partial x^*_2(\sigma)}{\partial \sigma} = \frac{(\alpha K_0 - 1)(r-\mu)\left(-2\mu^2 + \mu\sigma^2(\phi+1) - 2r\sigma^2\right)}{(d_1-1)^2 \eta K_1 r \sigma^5 \phi} > 0$$

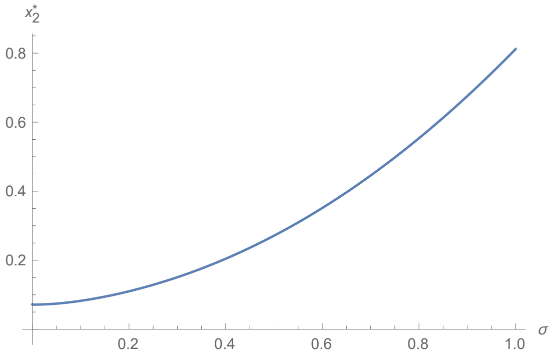
since the denominator is positive and the numerator also, concerning the same reason used in (3.24).

□

Unfortunately we are not able to derive any analytical expression regarding changes influenced by the discount rate and demand's drift. However, supporting on numerical experiments we observe that the behaviour hereunder described holds.

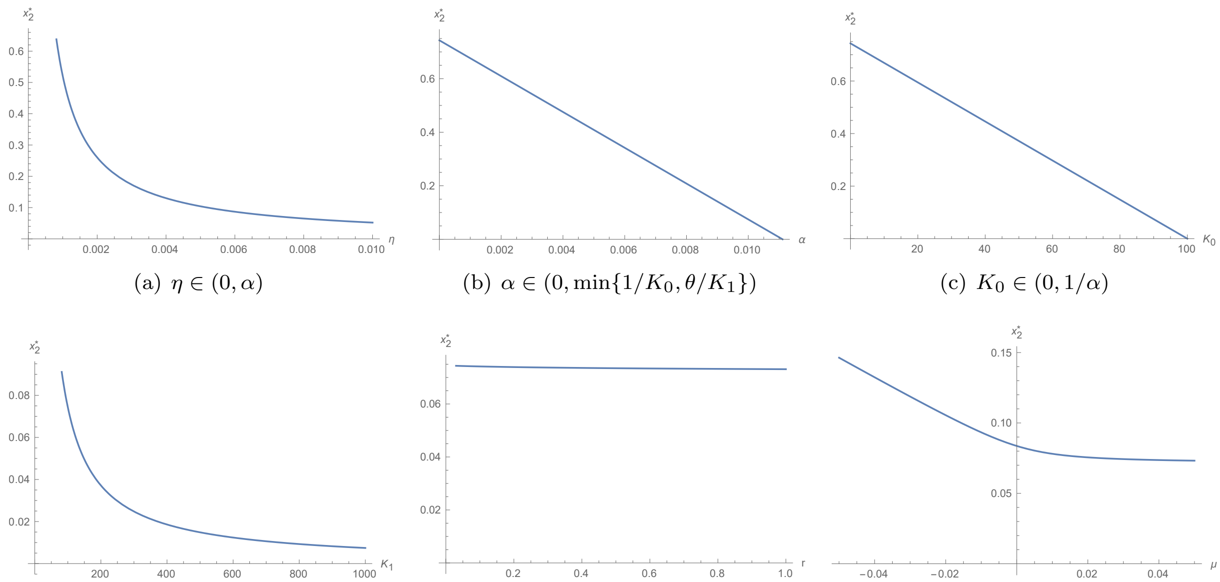
Interestingly, market's uncertainty is the unique aspect that delays the abandonment of the production of the old product, being preferable to keep the firm producing both products. This result is presented on Figure 5.4 and it is justified by the fact that the stable profit associated to the old product, which is not influenced by the uncertainty of the market, benefits the firm from possible low demand levels, leading to a low profit, if the firm was only producing the innovative product.

In view of Figure 5.5 (a) the larger the cannibalisation parameter considered, the smaller is the profit obtained during the simultaneous production period, inducing the firm to antecipate the abandonment decision.



(a) $\sigma \in (0, 1)$

Figure 5.4: Behaviour of the threshold value x_2^* with respect to the parameter with which it increases σ.



(a) $\eta \in (0, \alpha)$ (b) $\alpha \in (0, \min\{1/K_0, \theta/K_1\})$ (c) $K_0 \in (0, 1/\alpha)$

(d) $K_1 \in (0, \theta/\alpha)$ (e) $r \in (0, \mu)$ (f) $\mu \in (-r, r)$

Figure 5.5: Behaviour of the threshold value $x_{1,A}^*$ with respect to parameters with which it decreases η (a), α (b), K_0 (c), K_1 (d), r (e) and μ (f).

Since both cannibalisation η and capacity K_1 appear in the denominator of x_2^*, a similar behaviour is expected from both them. This is verified on Figure 5.5 (d) and comes from the fact that a larger capacity investment (already) incurred - when the demand reached $x_{1,A}^*$ - results on smaller profit by considering a simultaneous production than by solely producing the innovative product, encouraging the firm to invest earlier.

By confronting the results presented on Figures 5.5 (b) and (c) with the expression on (5.22), we observe that both α and K_0 have an akin linear effect: the larger the sensibility α or the capacity K_0, the earlier the firm should cease the production of the old product.

Once again we obtain the same peculiar type of behaviour with respect to changes on market's fortuity, as is showed on Figure 5.5 (f).

Chapter 6

Simulation study of the optimal investment times

6.1 Introduction

In this chapter we analyse the optimal times to invest regarding the three different models developed on previous chapters, using simulation analysis. Following the (usual) studies performed concerning real options, only the influences of the demand level and market's uncertainty are studied.

Regarding the following simulations performed, the optimal stopping times are calculated on different ways. For Chapters 3 and 4 (and accordingly to the diagrams presented on Figures 3.1 and 4.1, respectively) there is a single optimal investment time which is calculated considering as initial instant the time at which the innovation breakthrough takes place. On Chapter 5 (and accordingly to the diagram on Figure 5.1), we have two optimal investment times: the optimal time to enter the market with the new product, τ_1, and the optimal time to abandon the old product, τ_2, considering as initial instant the time when the new product is adopted, that is τ_1. Therefore, if one wants to know the estimated optimal time when a firm should abandon the old product after its entrance in the market, one should sum τ_1 and τ_2, for example.

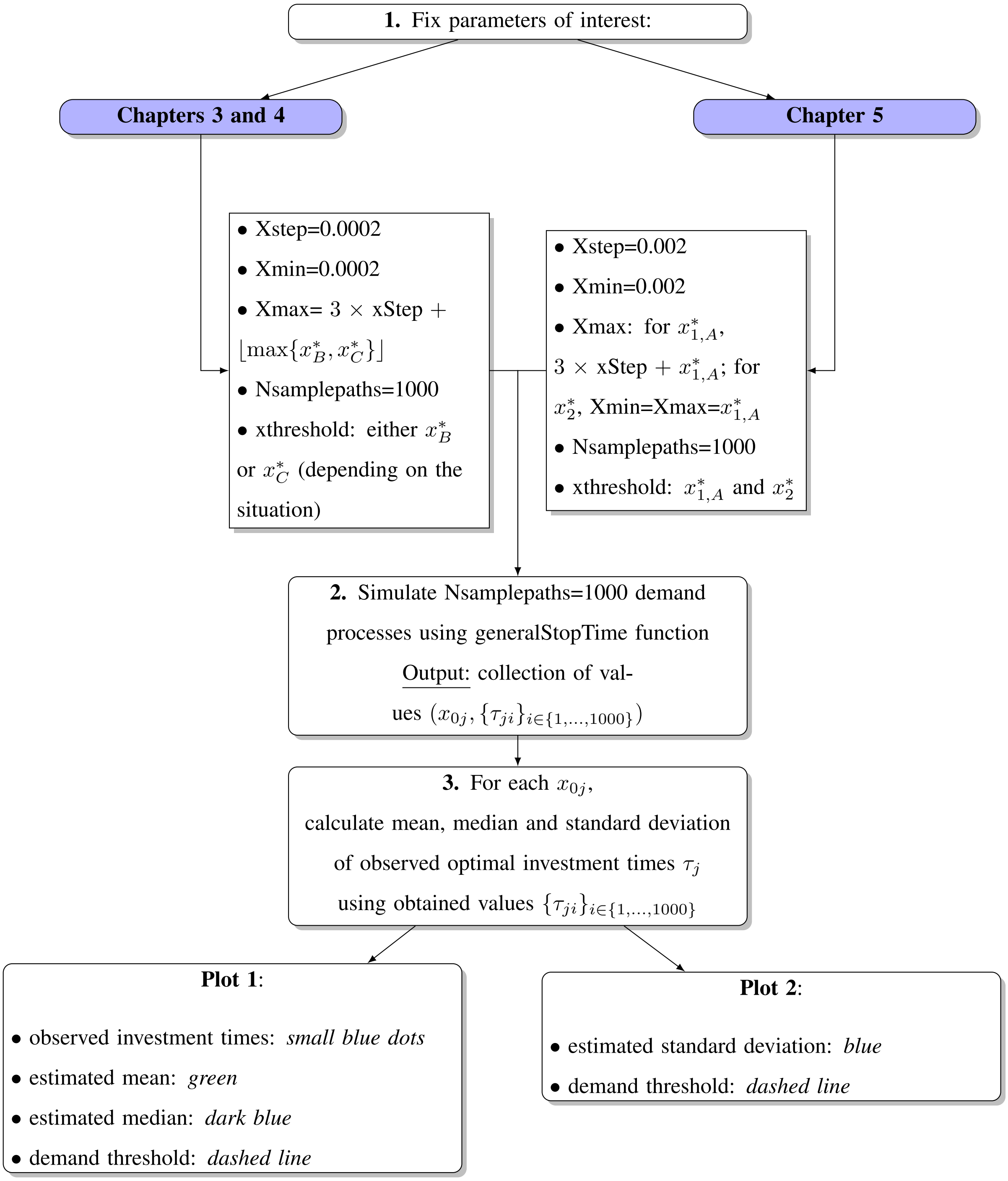
We first consider, by analysing, statistical measures of the simulated series (in particular: mean, median and standard deviation), we study the influence of initial demand values on optimal investing times. Secondly, we analyse how the estimated mean of optimal investment times behaves with volatility and assess its relation with changes either on the associated threshold value or optimal capacity level.

On Appendix A are presented the most important functions implemented in Mathematica.

6.2 Sensitivity of the optimal investment times w.r.t. initial demand value, x_0

Methodology

We analyse model by model, initially fixing relevant parameters to the considered situation, with the exception of the initial point x_0 - which is changed. Considering the values fixed on previous comparative sections, in particular of Chapter 5, with the exception of the volatility, which we increased to $\sigma = 0.1$, the main procedure followed is described on the diagram hereunder:



The range of initial values analysed is between xMin and xMax, considering an increment of xStep. Since the magnitude of thresholds analysed on Chapter 5 is quite larger than the ones on previous two chapters, we consider the demand to be incremented by a larger step: 0.002 instead of 0.0002.

Recall that the most interesting situation on this chapter was obtained by considering $\eta < \eta^*$[1] (5.35), from which we get an optimal investment time regarding the simultaneous production of established and new products and the time for the established product to exit the market, maintaining solely the new one. Regarding the analysis of the optimal stopping time associated to x_2^*, we evaluate for a single initial demand value $x_{1,R}^*$, since this is the first instant, after investing when the demand reaching $x_{1,R}^*$.

The time step associated to the simulation of the demand process[2] can be chosen taking into account how drift and volatility of the demand process are estimated or in view of a certain time window. Since these are mere illustrations concerning deduced models, we are able to choose an arbitrary time step. For the sake of good understanding and simplicity, we consider it to be one time unit.

Results

On Table 6.1 the obtained results are presented, from which we conclude:

1. The smaller the initial demand x_0, the longer the firm needs to wait to invest;
2. The larger the demand value that triggers the investment, the longer the firm needs to wait to be advantageous to invest;
3. The mean and the median of investment times are approximately equal, suggesting that the distribution of investment times is approximately symmetric;
4. Both mean and median are of the same order of magnitude as the standard deviation. This suggests a high variability of the results obtained. Increasing the amount of sample paths simulated we observe no significant decrease of the variability, only a considerable longer computational running time.

We suspect that the noticeable logarithmic behaviour of simulated optimal times' mean comes from the fact that

$$\mathbb{E}^{X_0}[X_\tau] = X_0 e^{\mu\tau} \;\Leftrightarrow\; \tau = \frac{1}{\mu}\log\left|\frac{\mathbb{E}^{X_0}[X_\tau]}{X_0}\right|$$

Considering the estimate of the optimal investment time given by the sample mean of the simulated exercising times (with respect to each initial value tested; here denoted by $\hat{\tau}$) and noticing that X_τ, written above, corresponds to a threshold (with respect to a particular investment situation; here denoted by x^*), it follows that

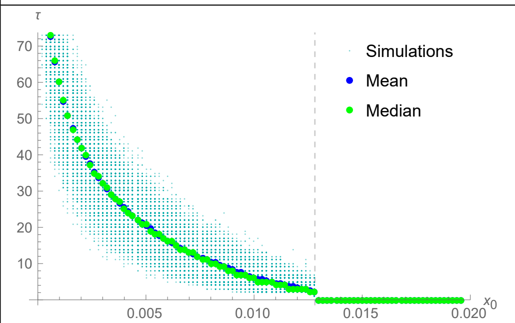
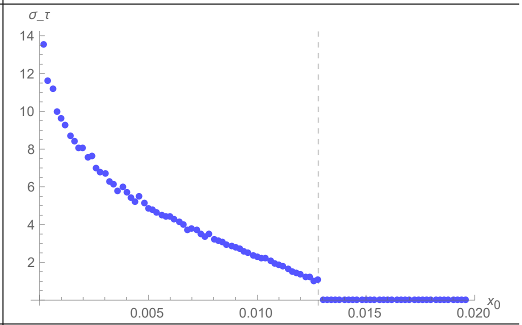
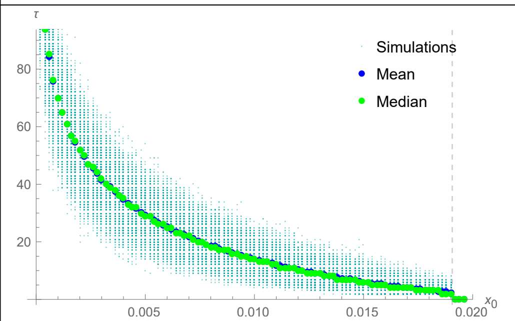
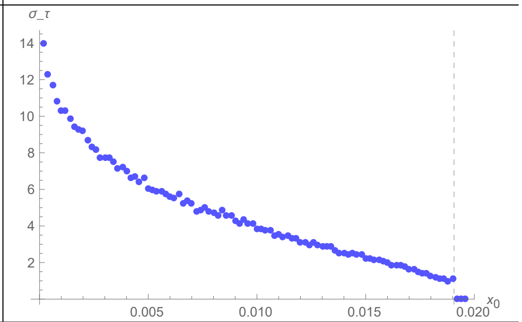
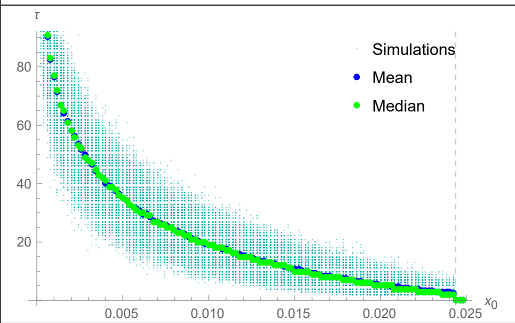
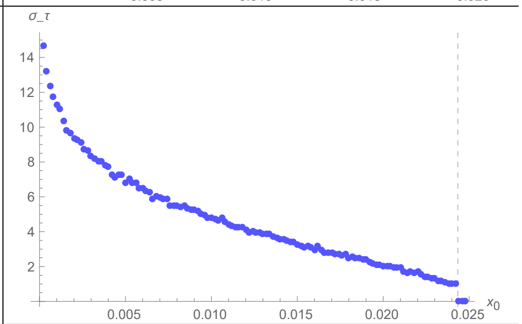
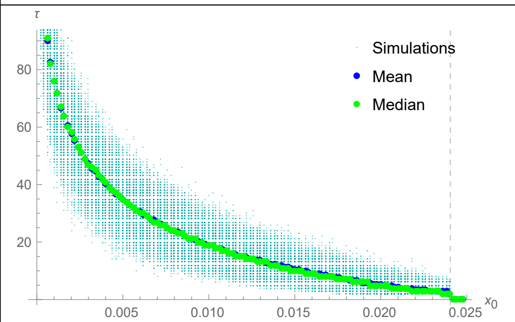
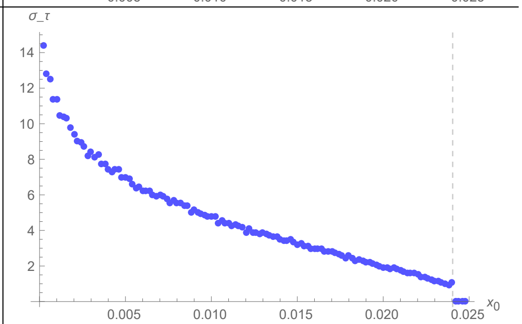
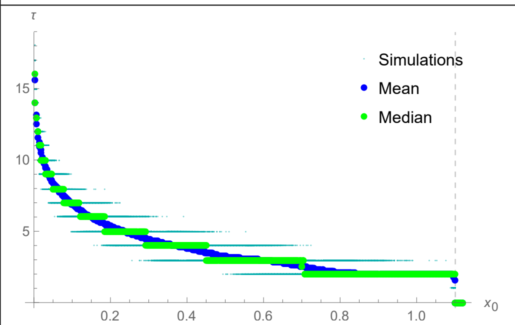
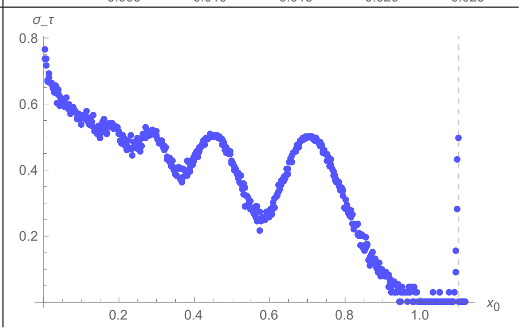
$$\hat{\tau} = \frac{1}{\mu}\log\left|\frac{x^*}{X_0}\right|, \quad \text{considering } x^* > X_0. \tag{6.1}$$

The case $x^* \leq X_0$ is not here addressed since it immediately implies that the firm invests right away, resulting in a null expected waiting time.

[1] For values $\eta \geq \eta^*$, we are in the same case as described on Chapter 4: we have an unique threshold which sets the transition from the production of the established product to the new one.

[2] Necessary to functions demandProcess and stopTime, auxiliary functions of generalStopTime: check Appendix A.

Table 6.1: Sensitivity analysis of estimated mean and median and standard deviation of the optimal investment time for each of the three situations studied, regarding different initial values x_0 and the benchmark (B) and capacity optimization (CO) models.

Chapter	Model	Mean & Median	Standard Deviation
3	B		
	CO		
4	B		
	CO		
5	-		

On the last row we find the results concerning the optimal time to invest in a new product and start to produce, simultaneously, the old and new products, as studied on Chapter 5. The noticeable *step* behaviour of both simulated values and respective mean is a consequence of the unitary time step considered during the simulation and the scale of the results obtained. Note that this implies a more irregular standard deviation as presented on the last row of Table 6.1. Although this behaviour is not easily detectable on the other situations, since the results obtained are in a larger scale, one expects that they behave the same way.

Taking now into account the optimal time to stop the production of the established product, studied on Chapter 5, we obtain the results presented on Table 6.2.

Table 6.2: Sensitivity analysis of estimated mean, median and standard deviation of optimal time to decide stop the production of the established product when considering an observed demand $x^*_{1,A}$.

Mean	Median	Standard Deviation
5.49	5.00	0.77

Recall that we consider the demand process to start at $x^*_{1,A}$, since this is the demand level observed at the instant the firm invests on the innovative product. Therefore, mean and median are taken considering as *initial instant* the time at which the firm decides to invest in the new product and start the simultaneous production, that is, $\tau^*_{1,A} = \inf\{t \geq 0 : X_t \geq x^*_{1,A}\}$. Thus, the mean/median presented should be read as the mean/median time after the investment on the new product was done. To obtain the estimated mean/median time for which the firm produces solely the new product, one should add the value presented on Table 6.2 to the one obtained as the mean/median time to invest in the new product, for a given initial demand value x_0.

We note that, in this case, it seems that the results do not have large volatility, and thus we may suggest that the sample mean and median are good descriptors of the optimal time it takes to decide to stop the old product's production.

6.3 Sensitivity of the optimal investment times with respect to volatility, σ

In this section we are interested to observe how optimal investment times behave with market's uncertainty.

Methodology

We analyse all models concerning the three previous chapters, by studying the respective investment times concerning changes on demand's volatility, which is consider to vary between 0.01 and 0.5. By considering two different initial demand values ($x_0 = 0.001$ and $x_0 = 0.01$), we study its behaviour by analysing its absolute and relative variation, following the procedure described on the undermentioned diagram.

1. Fix parameters of interest:

- σMin=0.01
- σMax=0.5
- timeStep=1
- horizon=1500
- Nsamplepaths=500
- xthreshold: the ones associated to each chapter

2. Simulate Nsamplepaths=500 demand processes using estTau function

Output: collection of values $(\sigma_j, \bar{\tau}_j)$ with $\bar{\tau}_j$ representing the mean of the investment times concerning the 500 sample paths simulated w.r.t. σ_j

Plot 1:

estimated expected optimal investment times w.r.t. σ

- x_0=0.001: *lighter colours*
- x_0=0.001: *darker colours*

Plot 2:

variation rate of estimated expected optimal investment times w.r.t. σ calculated accordingly to (6.2)

- x_0=0.001: *lighter colours*
- x_0=0.001: *darker colours*

The absolute variation is (straightforward) given by the mean of simulated optimal investment times with respect to each value of σ.

The relative variation is analysed accordingly to the variation rate (here denoted by $\triangle_n\%$), which is calculated as

$$\triangle_n\% = \frac{f(\sigma_n) - f(\sigma_{n-1})}{\sigma_n - \sigma_{n-1}} \times 100\% \quad \text{with} \quad \sigma_n = 0.01 + n \times 0.01, \quad n = \left\{0, ..., \frac{\sigma Max}{\sigma Step} = 50\right\}, \qquad (6.2)$$

where f corresponds to the sample mean of optimal times simulated w.r.t. a certain volatility.

Since it's not deterministic that all sample paths reach the demand threshold, we define a time horizon to assure that the algorithm runs in *human-time* and that it ends. On the results showed hereunder a horizon of 1500 time units[3] is considered. As before we consider a time step of one time unit.

Once again the analysis regarding the optimal time to abandon the simultaneous production, referred on Chapter 5 and denoted by τ_2, consider as initial demand value x_1, A (that is the demand level observed when the firm invests in the innovative product). Also, the values presented regarding τ_2 sould be read

[3]This value is used as input in stopTimemod function, which is an auxiliary function of estTau. For further details check Appendix A.

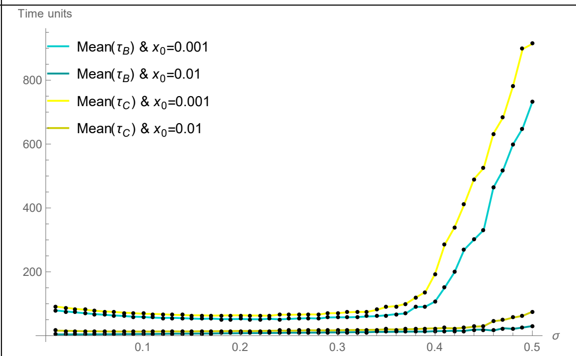
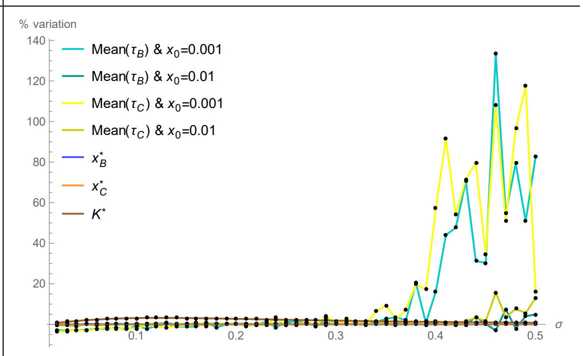
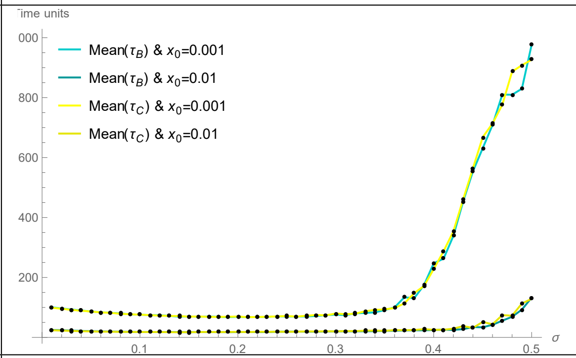
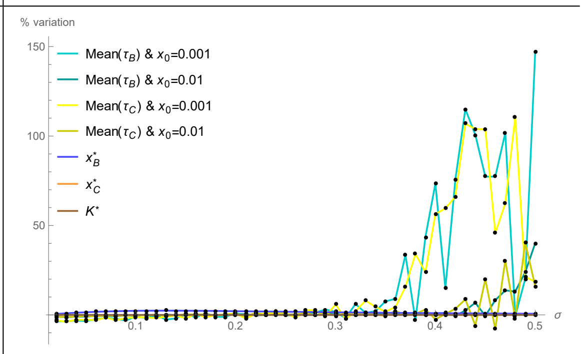
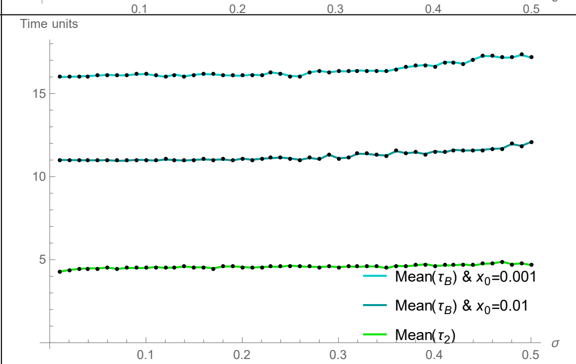
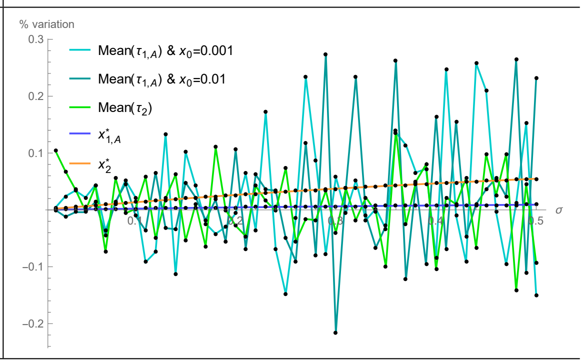
as: *after the investment on the innovative product, the firm is expected to wait τ_2 time units to abandon the production of the old product.*

Results

The numerical results for the investment times related to both models (the benchmark and the optimal capacity models) as a function of σ (the volatility of the demand) obtained are presented on Table 6.3.

For the sake of completeness, we include also the corresponding investment thresholds and the optimal capacity level. These are not obtained by simulation. They are rather the implementation of equations presented on the previous respective chapters. However, due to the contrasting order of magnitude of the optimal investment times with the optimal capacity K^* and demand thresholds, the last ones are only considered for the analysis of the relative variation, on the rightmost column.

Table 6.3: Sensitivity analysis of the (estimated) mean of optimal investment time w.r.t. the demand's volatility, regarding different initial values $x_0 \in \{0.001,\ 0.01\}$.

<table>
<tr><th>Chapter</th><th>Absolute variation</th><th>Relative variation</th></tr>
<tr><td>3</td><td>
</td><td>
</td></tr>
<tr><td>4</td><td>
</td><td>
</td></tr>
<tr><td>5</td><td>
</td><td>
</td></tr>
</table>

Confronting the results of performed simulations, regarding both models on chapters 3 and 4, above presented with the numerical results present on Appendix A, the following conclusions are pointed:

1. The mean time to undertake the decision is stable for a significant range of values σ (i.e., the mean seems not to change significantly for values $\sigma \leq 0.35$). But then it shows a considerable increasing trend, more pronounced on the optimal capacity model;

2. The observed mean of the investment times has a non-monotonic behaviour:

   - For small values of σ (approximately smaller than 0.2), as slight decreasing tendency, which leads to anticipate the investment decision, is observed as the volatility increases. This is not an expected result from real option approach - for which an increasing volatility leads to a late decision [3] -, however it might be justifiable by the crossed-effect of a small rate growing of the thresholds for small values of σ (as seen on Figure 3.3 (b) and Figure 4.2 (b)) and the impact of a small volatility level on the demand process;
   - For large values of σ (approximately greater than 0.2), we observe that a larger uncertainty on the market seems to likely postpone the investment decision;

3. There seems to exist no relation between the behaviour of the mean time to undertake the decision and the demand threshold level or the optimal capacity associated.

For the case when the firm is already in the market and may either replace or add a new product (Chapter 5), the investment times (presented on the last row), have a different behaviour from the previous ones. We note that for this case the expected times (both for the investment and for the replacement time) are more stable that in the other two cases, in the sense that changes on σ do not affect them as significantly as in the previous cases. We do not have any explanation for this fact. However we do suspect that it is related with the fact that quite different values are considered for the numerical simulation of the chapter 5 than for chapters 3 and 4.

6.4 Sensitivity of the optimal investment times with respect to the crossed-effect between initial demand value, x_0, and volatility, σ

Additionally we study the crossed-effect of both initial demand value and volatility. This is achieved by analysing the frequency of the optimal investment times obtained through their histogram representation.

Methodology

Per each set of values chosen, we simulate 500 demand sample paths, using function hist[4], until they reach either the respective demand threshold or the time horizon, here set as 1500 time units. The set of parameters considered was the same as in the respective comparative statics sections.

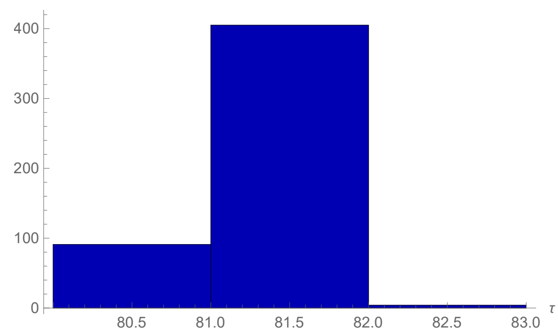
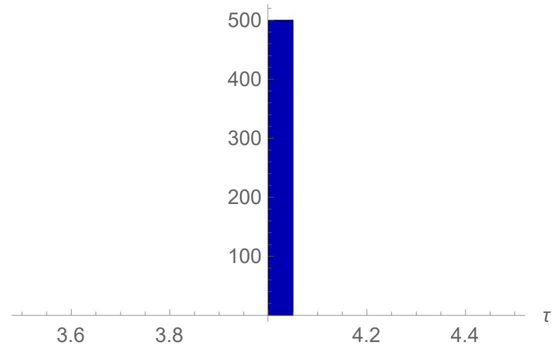
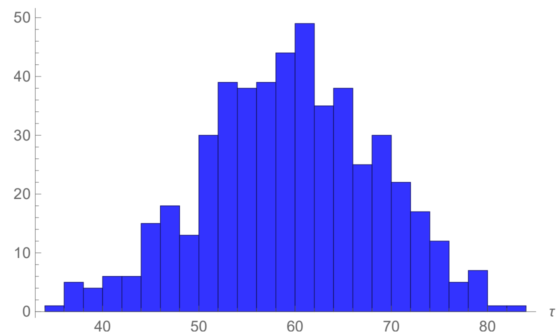
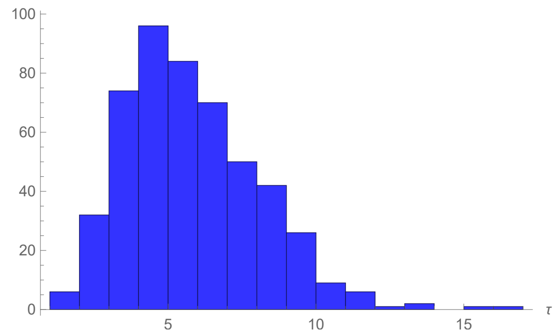
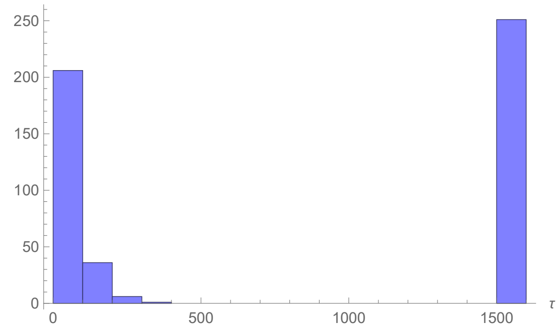
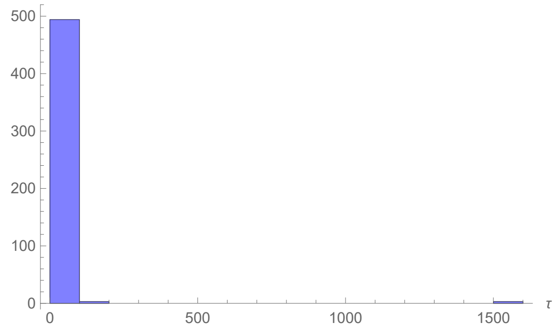
As previously noticed, since the results obtained for chapters 3 and 4 and both benchmark and capacity optimization models are quite similar, we only represent in this thesis here the results regarding chapters 3 and 5.

[4]For further details, check Appendix A.

Results

Table 6.4 shows the results regarding the benchmark model associated to the situation on which a firm is not active in the market and wants to enter it with an innovative product, deduced on Chapter 3. The results obtained can be extended to the optimal investment time deduced on Chapter 4.

Table 6.4: Histograms of simulated optimal investment times of the benchmark model w.r.t. the situation for which a firm is not active in the market and wants to enter it with a new product (Chapter 3), considering initial demand values $x_0 = \{0.001,\ 0.01\}$ and demand's volatilities $\sigma = \{0.001,\ 0.1,\ 0.5\}$.

Volatility	$x = 0.001$	$x = 0.01$
$\sigma = 0.001$		
$\sigma = 0.1$		
$\sigma = 0.5$		

We conclude the following aspects:

1. The smaller the uncertainty in the market, the lower is the variability of the waiting time to invest. However, as previously concluded from Table 6.3, this is not associated to the lowest waiting times. When considering $\sigma = 0.001$ and $x_0 = 0.1$ we observe that for all the demand processes simulated, the firm enters the market after 4 time units. This is not a surprising, as we are choosing a really low value of volatility, leading to an almost deterministic process;

2. The larger the volatility, the larger is the number of demand sample paths that do not reach the demand level that triggers the investment within the horizon defined (of 1500 time units). This result is in accordance to the significant growth of the waiting time when on markets with a large uncertainty, as observed on Table 6.3;

3. A significant reduction on the waiting time until investment is observed when the initial demand value increases. This result is also noticeable in Table 6.1, on which the waiting time seems to have a logarithmic dependence on the initial value chosen;

4. Assesing the normality of the optimal investment times,, recurring to a Shapiro-Wilk test [12][5] , we obtain that solely for $x = 0.001$ and $\sigma = 0.1$, the normality is verified for the usual significance levels (1%, 5%, 10%, 15%) - since we obtain a p-value $\simeq 0.264$.

Now we analyse the situation on which a firm is already active, but wants to enter the market with an innovative product, allowing a simultaneous production period, as described on Chapter 5. Here we only analyse the threshold that triggers the introduction of the innovative product in the market, $x^*_{1,A}$, since it is dependent on the initial demand value (while the threshold that triggers the removal of the old product, x_2, it is not).

From the results presented on Table 6.5, the following conclusions are taken:

1. The larger the uncertainty in the market, the larger the variance observed on the simulated optimal investment times;

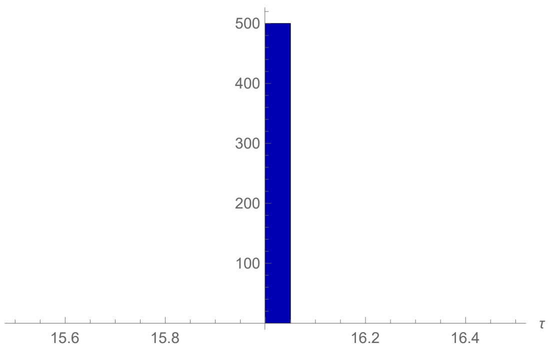
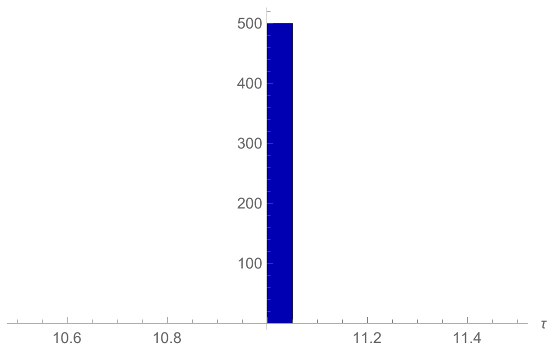
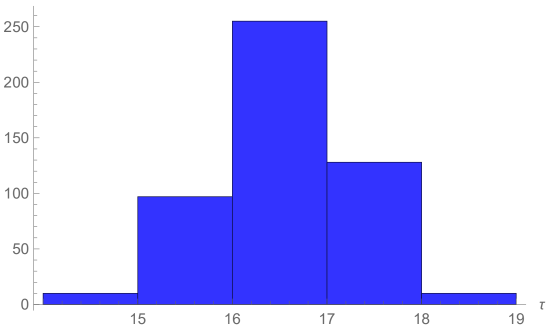
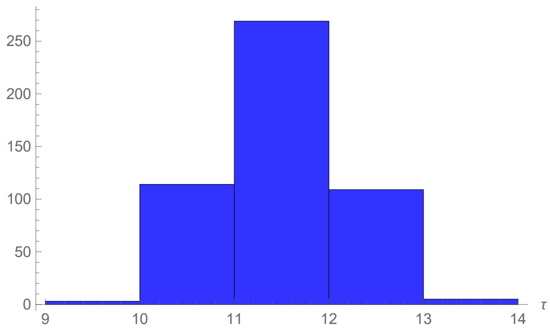
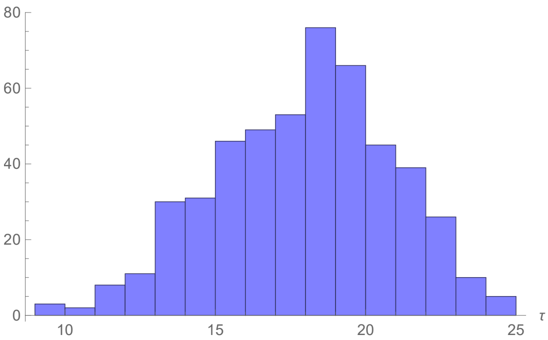
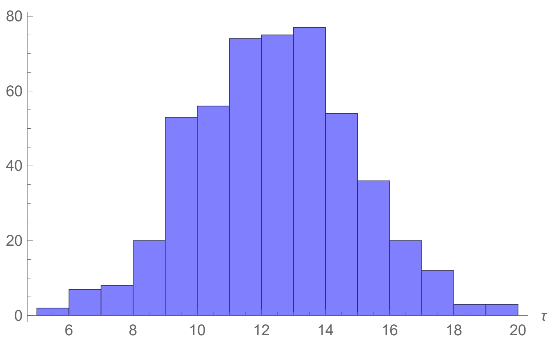
2. There are no significant changes on the behaviour of the simulated optimal investment times with the initial value of the demand observed. Nevertheless, we observe that larger initial demand values reflect on smaller waiting times until it is advantageous for the firm to invest, as concluded from Table 6.1;

3. In none of the situations the time horizon defined (of 1500 time units) was reached. This is in accordance to the uniformity observed on Table 6.3;

4. In none of the situations exists statistical evidence that the waiting times are normally distributed - the largest p-value obtained when performing Shapiro-Wilk test [12]: 4×10^{-5}.

[5] On Mathematica it is given by ShapiroWilkTest function: https://reference.wolfram.com/language/ref/ShapiroWilkTest.html

Table 6.5: Histograms of simulated optimal investment times w.r.t. the situation when an already active firm wants to introduce an innovative product in the market, allowing a simultaneous production period, considering initial demand values $x_0 = \{0.001,\ 0.01\}$ and demand's volatilities $\sigma = \{0.001,\ 0.1,\ 0.5\}$.

Volatility	$x = 0.001$	$x = 0.01$
$\sigma = 0.001$		
$\sigma = 0.1$		
$\sigma = 0.5$		

Chapter 7

Value of the project: the influence of the number of innovation jumps

7.1 Introduction

The previous chapters were focused on a timeline starting at the instant a desired innovation level θ was achieved and hence, ignoring all what happened before it. Following this reasoning we studied the value function and the demand value that triggers the investment associated to three different contexts: entering the market with an innovative product; introducing a new product with the immediate replacement of the *old* one; introducing of a *new* product with the possibility of immediate replacement of the *old* one or simultaneous production followed by the replacement of the *old* product).

On this chapter we are interested in the calculation of the value function with respect to the R&D investment, taking into account the waiting time until the innovation breakthrough takes place. This allows us to state a relation between the R&D investment and the time a firm needs to wait until it is in a favourable situation to invest.

Innovation levels are assumed to increase by jumps. Therefore the innovation process $\Theta = \{\theta_t,\ t \geq 0\}$ is defined as an homogenous Compound Poisson Process, with constant rate given by $\lambda(R) = R^{\gamma}$, with R corresponding to the investment in the R&D department, and γ a sensitivity parameter taking values in $(0, 1]$.

Note that R&D investment (R) is different from investment costs (δK): the first influences directly the innovation Θ, while the second is related with costs the firm needs to incur to adapt its production to the new technology level. For example, R&D costs can be related with scientist wages and laboratory equipment, while investment costs can be related with new equipment the firm needs to purchase or formation necessary to its employees, in order to adapt to the new technology.

The innovation process can be then expressed as

$$\theta_t = \theta_0 + uN_t,\ t \geq 0.$$

with θ_0 denoting the state of technology at the initial point in time, $u > 0$ a fixed jump size and $\{N_t,\ t \geq 0\}$ the jump process which follows a Poisson Process with rate $\lambda(R)$.

Considering now S_n to be the random variable that represents the waiting time until the n-th jump in the process is observed, accordingly to [8], S_n follows an Erlang distribution with shape parameter n and rate parameter $\lambda(R)$, the same as in the jump process, that is,

$$S_n = \min\{t \geq 0 : N(t) = n\} \sim Erlang(n, \lambda(R)).$$

It immediately follows that the expected waiting time for the n-th jump to be reached is given by $\mathbb{E}(S_n) = \frac{n}{\lambda(R)}$, meaning that the larger the R&D investment, the sooner the n-th jump is expected to happen. Also, note that, if there is no investment on R&D, the jumps are supposed to occur at a null rate ($\lambda(0) = 0$), resulting in a infinite expected waiting time. In fact, when $\lambda(0) = 0$, we assume that the technology level does not change and therefore our problem is ill-posed. For this reason, we assume that $\lambda(R) > 0$.

In this chapter we are interested to study how the complete value of the project - reflecting R&D investment so as the innovative product investment - is affected by the R&D investment and the number of jumps needed until it is favourable to invest on the innovative product. First we focus on the situation when the innovation breakthrough happens within one innovation jump and then we generalise this scenario, by considering the breakthrough to occur within $n \in \mathbb{N}$ jumps.

We close this introduction highlighting the fact that every investment value function previously deduced is independent of the R&D investment, depending solely on the technology level θ.

7.2 One jump

We start with the simplest case: the desired innovation level θ is reached within one jump. Therefore S_1 denotes the random variable associated to the waiting time until the jump occurs and it is such that

$$S_1 \sim \text{Erlang}(1, \lambda(R)) \stackrel{d}{=} \text{Exponential}(\lambda(R)).$$

We also denote by F the value function associated to an investment situation and V to be the maximized value of the investment project given by

$$V(x) = \max_R \mathbb{E}[e^{-rS_1} F(x) - R] \geq 0 \tag{7.1}$$

It consists on the maximized expected value of the investment on a new technology minus R&D costs associated. Note that, since we assumed the timeline of F to start when the innovation breakthrough occurs (S_1), we discount its value to the time the R&D investment is made and consider x to be the demand value expected to observe at S_1.

Here we assume, which notoriously leads to a simpler analysis, that no matter when the innovation breakthrough happens, the level of the process X is deterministic and given by x.

We note that although F results from an optimal stopping problem, V is a (standard) optimization problem w.r.t. a deterministic value R, denoting the R%D costs.

Taking into account that the expectation is with respect to S_1, we obtain

$$\begin{aligned} V(x) &= \max_R \left\{ \int_0^\infty f_{S_1}(t) e^{-rt} F(x) dt - R \right\} \\ &= \max_R \left\{ \int_0^\infty \lambda(R) e^{-\lambda(R)t} e^{-rt} F(x) dt - R \right\} \end{aligned} \tag{7.2}$$

where f_{S_1} corresponds to the probability density function of an Exponential distribution with parameter $\lambda(R)$.

As previously written (and deduced), none of the value functions studied depends on the R&D investment or on the time at they are evaluated, only on the demand level observed at the breakthrough. F never depends on R or t. Therefore the integral on (7.2) can be solved, leading to

$$V(x) = \max_R \left\{ \frac{\lambda(R)}{\lambda(R) + r} F(x) - R \right\} = \max_R \left\{ \frac{R^\gamma}{R^\gamma + r} F(x) - R \right\}, \tag{7.3}$$

which is a maximization problem with respect to the R&D investment R. Since by (7.1), V cannot be negative it follows that the restriction

$$R^\gamma F(x) - (R^\gamma + r)R \geq 0 \ \Leftrightarrow F(x) \geq \frac{R^\gamma + r}{R^{\gamma - 1}} \tag{7.4}$$

must always be verified.

Thus the following standard optimization technique is used to find V:

$$\frac{\partial}{\partial R} \left(\frac{R^\gamma}{R^\gamma + r} F(x) - R \right) = \frac{\gamma R^{\gamma-1} F(x) r - (R^\gamma + r)^2}{(R^\gamma + r)^2} \tag{7.5}$$

$$\frac{\partial^2}{\partial R^2} \left(\frac{R^\gamma}{R^\gamma + r} F(x) - R \right) = -\frac{F(x) \gamma r R^{-2+\gamma} (r - \gamma r + (1+\gamma) R^\gamma)}{(R^\gamma + r)^3} \leq 0. \tag{7.6}$$

Note that, since $\gamma \in (0, 1]$, $F(x) \geq 0$, $\forall x$ and r, $R > 0$, the second partial derivative with respect to R (7.6) is always negative. Hence the expression to be maximized in (7.3) corresponds to a concave function, meaning that we are always able to find a value $R^* = \arg\max_R \left\{ \frac{R^\gamma}{R^\gamma + r} F(x) - R \right\}$.

Therefore R^* is such that 7.5, computed at R^* is zero. Then it follows that R^* depends intrinsically on γ, and so, next we analyse its effects.

- **Case I:** $\gamma = 1 \Leftrightarrow \lambda(R) = R$

First of all, note that this case means that jumps occur at a rate given precisely by the investment costs.

Analysing the roots of the first partial derivative in order to investment parameter R, we get a quadratic polynomial, whose zeros are given by

$$R = -\sqrt{F(x) r} - r \ \vee \ R = \sqrt{F(x) r} - r$$

Since it is not possible to have negative investment, we discard the leftmost solution.

The second solution is admissible, since it is always true that

$$\begin{aligned}\sqrt{F(x)r} - r \underset{(7.4)}{\geq} \sqrt{(R+r)r} - r \geq 0 \\ \Leftrightarrow rR + r^2 - r^2 = rR \geq 0.\end{aligned} \tag{7.7}$$

Thus, if condition (7.4) holds, the optimal investment is given by

$$R^* = \max_R V(x) = \sqrt{F(x)r} - r. \tag{7.8}$$

- **Case II:** $\gamma \in (0,1)$

Considering now $\gamma \in (0,1)$ and taking into account expression (7.5), potential maximizers of V, will be found by calculating the roots of the following polynomial

$$R^{\gamma-1}F(x)r - R^{2\gamma} - 2rR^{\gamma} - r^2 = 0. \tag{7.9}$$

From condition (7.6) it follows that the maximizer R^* is such that it verifies (7.4) and (7.9).

Unfortunately, we are not able to solve (7.9) analytically for any value $\gamma \in (0,1)$. However, using software Mathematica, we performed some numerical illustrations for values $\gamma \in (0,1)$, presented in Section 7.4.

7.3 Multiple jumps

Now we take another step and generalize the previous idea, by considering that the desired innovation level θ is reached at the n-th innovation jump. Therefore we consider S_n to be the random variable associated to the waiting time until the n-th jump occurs and it is such that

$$S_n \sim \text{Erlang}(n, \lambda(R)).$$

Note that by considering $n = 1$, we are at the situation analysed in the previous section.

For the sake of simplicity we keep the same notation as before, where F denotes the value function associated to a certain investment situation and V the maximized expected discounted value function minus the R&D investment needed to be made, is now given by

$$\begin{aligned}V_n(x) &= \max_R \mathbb{E}[e^{-rS_n}F(x) - R] \\ &= \max_R \left\{ \int_0^\infty f_{S_n}(t)e^{-rt}F(x)dt - R \right\} \\ &= \max_R \left\{ \int_0^\infty \frac{\lambda(R)^n t^{n-1}}{(n-1)!} e^{-\lambda(R)t} e^{-rt} F(x)dt - R \right\}\end{aligned} \tag{7.10}$$

where f_{S_n} corresponds to the probability density function of an Erlang with shape parameter n and rate parameter $\lambda(R)$.

Considering W to be a random variable such that $W \sim \text{Erlang}(n, \lambda(R)+r)$ and f_W the correspondent probability density function, the integral above can be simplified as follows

$$\begin{aligned}\int_0^{\infty} \frac{\lambda(R)^n t^{n-1}}{(n-1)!} e^{-\lambda(R)t} e^{-rt} F(x) dt &= \frac{\lambda(R)^n}{(\lambda(R)+r)} F(x) \int_0^{\infty} \frac{(\lambda(R)+r)^n t^{n-1}}{(n-1)!} e^{-t(\lambda(R)+r)} dt \\ &= \left(\frac{\lambda(R)}{\lambda(R)+r}\right)^n F(x) \int_0^{\infty} f_W(t)\ dt \\ &= \left(\frac{\lambda(R)}{\lambda(R)+r}\right)^n F(x). \end{aligned} \tag{7.11}$$

Plugging the resulting expression (7.11) in (7.10), we obtain that V_n corresponds to a standard maximization problem given by

$$V_n(x) = \max_R \left\{ \left(\frac{\lambda(R)}{\lambda(R)+r}\right)^n F(x) - R \right\} = \max_R \left\{ \left(\frac{R^\gamma}{R^\gamma + r}\right)^n F(x) - R \right\}. \tag{7.12}$$

Since V_n is expected to be greater or equal to 0 $\forall n \in \mathbb{N}$, the following restriction must hold

$$R^{\gamma n} F(x) - R(R^\gamma + r)^n \geq 0 \ \Leftrightarrow \ F(x) \geq \frac{(R^\gamma + r)^n}{R^{\gamma n - 1}}. \tag{7.13}$$

Then the optimal investment in R&D centres, R^*, should be such that the following holds:

$$\frac{\partial}{\partial R}\left(\left(\frac{R^\gamma}{R^\gamma + r}\right)^n F(x) - R\right) = \frac{\gamma F(x) n r \left(\frac{R^\gamma}{r+R^\gamma}\right)^n - rR + R^{\gamma+1}}{rR + R^{\gamma+1}} = 0 \tag{7.14}$$

$$\frac{\partial^2}{\partial R^2}\left(\left(\frac{R^\gamma}{R^\gamma + r}\right)^n F(x) - R\right) = \frac{\gamma F(x) n r \left(\frac{R^\gamma}{r+R^\gamma}\right)^n (r(\gamma n - 1) - (\gamma+1)R^\gamma)}{R^2 (r + R^\gamma)^2} < 0. \tag{7.15}$$

Due to the complexity of both expressions above represented, we are not able to deduce an expression for stationary points neither to assess if the second partial derivative is negative. And, consequently, we need to resort to numerical calculations.

Although, here we do not present a complete study for it, these calculations are quite easy to implement, in order to find an approximation of R^* with respect to several values of the parameter γ. In this view, we developed the function V presented on Appendix B.

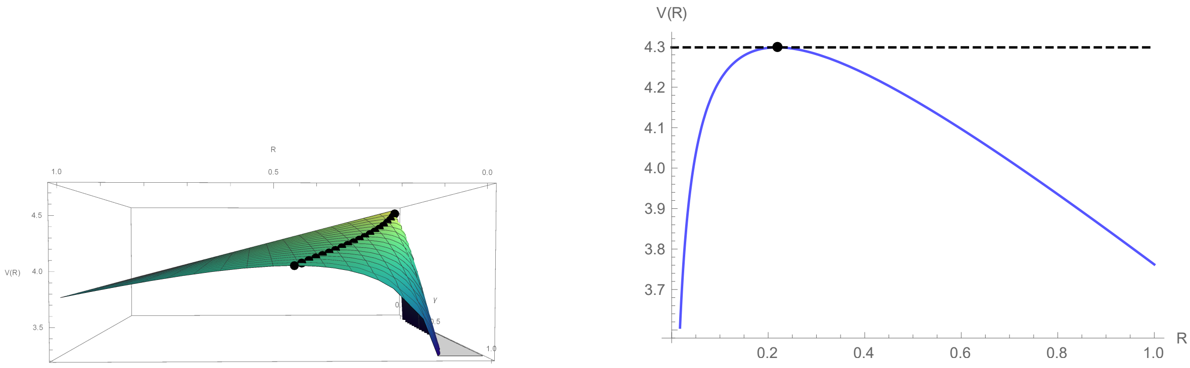
7.4 Comparative Statics

In this Section we assess how optimal investment (R^*), its associated rate jump ($\lambda(R^*)$) and optimal value function (V_n) behave with the number of jumps considered (n) and sensitivity chosen (γ).

Since we are not abe to deduce any analytical expression regarding the optimal R&D investment, we develop a function that returns an approximation to the optimal R&D investment, for a fixed situation. This function is presented in Appendix B and it is named calcR.

In view of the numerical experiments, by the independence of F on parameters R and γ, we consider it to be fixed and to take the value of 10. The discount rate is fixed at $r = 0.05$ and γ is considered to start at 0.05 and incremented 0.05 until it reaches 1.

We start analysing the case when a firm is in a favourable position to invest after an innovation jump (this is the case when $n = 1$).



(a) $R \in (0, 1]$ and $\gamma \in (0, 1]$.

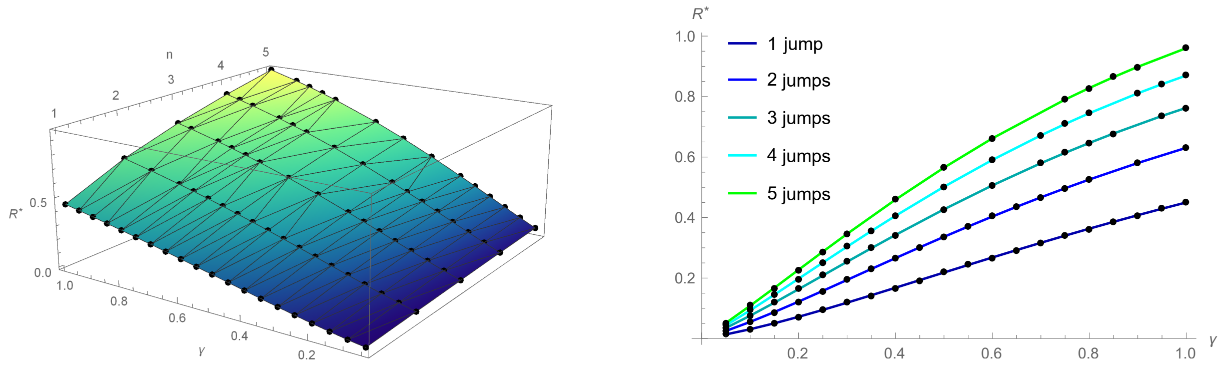
(b) A particular case: $\gamma = 0.5$.

Figure 7.1: Function to be maximized in (7.1) with respect to parameters γ and R and corresponding maximum values V_1 regarding each γ (black)

On Figure 7.1 (a) we observe that the function to be maximized is concave with respect to the R&D investment R. This seems to be related with the balance achieved between the linear decay of V with the R&D investment made and the greater jump rate and, consequently, sooner expected level of innovation, and smaller discount weight on F, both influenced by the investment R. Furthermore, we observe the smaller the γ, the larger optimal R&D investment R^* is.

On Figure 7.1 (b) it is illustrated the particular case when $\gamma = 0.5$ and the associated optimal R&D investment (corresponding to 4.3).

Now we increase the complexity and analyse the behaviour of R^*, $\lambda(R^*)$ e V_n, regarding the occurrence of 1 to 5 innovation jumps.

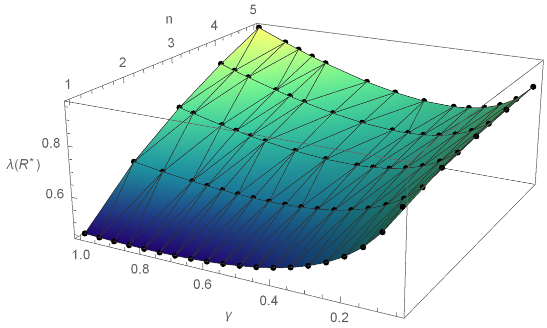


(a) 3D view.

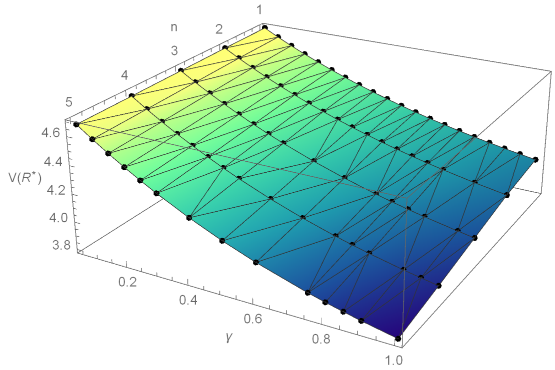
(b) 2D view.

Figure 7.2: Optimal R&D values R^* regarding parameter γ and the occurrence of $n \in \{1, ..., 5\}$ innovation jumps, with corresponding numerical approximations (black).

On the grounds of the results shown on Figure 7.2, we conclude that the optimal R&D investment increases with both sensitivity γ and number of jumps n. Note it is not possible to obtain the optimal value R^* (for instant for $\gamma = 0.45$ and $n \geq 3$). However using the values obtained in their neighbourhood, we are able to estimate its tendency, as represented on both figures 7.1 and 7.2.



(a) Jump rate $\lambda(R^{*})$.



(b) V_n

Figure 7.3: Effect of parameters γ and the occurrence of $n \in \{1, ..., 5\}$ innovation jumps on jump rate $\lambda(R^{*})$ and project's value V_n, with corresponding numerical approximations (black).

We observe on Figure 7.3 (a) that the jump rate $\lambda(R^{*})$ increases with the number of jumps considered n. However, the same does not hold for parameter γ: we observe that the jump rate has a non-monotonic behaviour with λ, a consequence of the exponentiation R^{γ}.

Figure 7.3 (b) shows that the non-monotonic behaviour of $\lambda(R^{*})$ seems not to influence (strongly) the tendency of V_n, since V_n decreases with both n and γ.

We propose that the higher the number of jumps until the firm is able to invest, the greater is the amount of time the firm is expected to wait to be in a favourable position to invest, leading to a larger discount on the value function F and, therefore, to a smaller value of the project V. On the other side, the larger the sensitivity γ, the larger the optimal investment R^{*} and the smaller the jump rate. This leads to a project with comparative smaller value.

Chapter 8

Conclusions

This chapter summarizes the relevant achievements of the work carried out in this thesis.

On Chapter 3, in the view of the case when a firm wants to enter the market with a new product, we conclude that:

- In the benchmark model, the demand value that triggers the investment increases with the interest rate r, the market's uncertainty σ, the production capacity K and the parameter α; it decreases with the innovation level θ and the drift value μ;

- In the capacity optimization model, the demand value that triggers the investment increases with the interest rate r, the market's uncertainty σ and the parameter δ; it decreases with the innovation level θ; it has a non-monotonic behaviour with the drift value μ. Moreover, the optimal capacity level increases with the drift value μ; decreases with the interest rate r and the parameter α.

On Chapter 4, considering the case when an active firm wants to enter the market with a new product, replacing the old one, we conclude that:

- In the benchmark model, the demand value that triggers the investment increases with the parameter δ, the market's uncertainty σ for $d_1 \in \left(1, \frac{1}{2}(3+\sqrt{5})\right)$ and the parameter α for $\theta < \frac{K_1}{K_0^2}(K_0 + K_1 r\delta)$; it decreases with the innovation level θ, the market's uncertainty σ for $d_1 > \frac{1}{2}(3+\sqrt{5})$ and the parameter α for $\theta > \frac{K_1}{K_0^2}(K_0 + K_1 r\delta)$; it has a non-monotonic behaviour with the capacity of the old and new products K_0 and K_1, respectively, and the interest rate r.

- In the capacity optimization model, the demand value that triggers the investment increases with the parameter δ; it decreases asymptotically with the innovation level θ; it has a non-monotonic behaviour with the capacity of the old product K_0, the interest rate r, the drift value μ and the parameter α. Moreover, the optimal capacity level increases in a linear rate with the innovation level θ; it has a non-monotonic behaviour with the capacity of the old product K_0.

On Chapter 5 we conclude that the cannibalisation has a crucial role on the investment decision. For larger values of it, the firm is recommended to invest and immediately replace the *old* by the *new* product, being in the case treated on Chapter 4. On the other side, for smaller values of cannibalisation, a simultaneous production period followed by the total replacement of the *old* product is more favourable that the immediate replacement. We conclude:

- The demand value that triggers the introduction of the new product increases with the cannibalisation parameter η, the parameters α and δ, the market's uncertainty σ and the capacity of the old and new products K_0 and K_1; it decreases with the innovation level θ;

- The demand value that triggers the removal of the old product increases with the market's uncertainty σ; it decreases with the cannibalisation parameter η, the parameter α and the capacity of the old and new products K_0 and K_1.

We extend the usual sensitivity analysis, that is narrowed to the comparative statics of the investment thresholds, by analysing the expected waiting times until the investment through numerical simulations. We observe that the investment waiting time follows an approximate symmetric distribution and decreases in a logarithmic rate with the initial demand value. Furthermore, the expected waiting time is observed to have a non-monotonic behaviour with the market's uncertainty, which indicates that volatility has a different impact on the waiting time and on the investment threshold.

Lastly, when analysing the impact of R&D investment on the overall investment decision (by considering that we know the value of the demand at the instant the innovation breakthrough occurs), we conclude that it has no direct influence on the investment decision. Nevertheless, we are able to maximize the value function associate to the project by finding its optimal R&D investment.

8.1 Suggestions for Future Work

Although on this thesis many situations were explored from different perspectives, further studies can be carried out, by exploring the following aspects:

- The demand evolves accordingly to a mean-revarting process (such as a Cox-Ingersoll-Ross or an Ornstein-Uhlenbeck process);

- The innovation process is modelled by a non-homogenous compound Poisson process or even a more general renewal process;

- Only the demand level at the time the R&D investment is made is known and we (still) want to deduce the optimal amount to invest in R&D as well as the best time to initiate the production of the innovative product, by exploring the situation referred on chapters 3, 4 and 5;

- We are able to estimate the parameters of proposed models with data provided from a real firm and, therefore, to help the firm in its investment decisions.

The above suggestions are feasible and natural continuation of this work and we believe that can contribute to improve the prediction accuracy of the optimal investment time, similarly to what was developed on this thesis.

Bibliography

[1] R. McDonald and D. Siegel. The value of waiting to invest. *The Quarterly Journal of Economics*, 101(4):707–728, 1986.

[2] A. Dixit. Entry and exit decisions under uncertainty. *Journal of Political Economy*, 97(3):620–638, 1989.

[3] A. Dixit and R. S. Pindyck. *Investment Under Uncertainty*. Princeton University Press, 1^{st} edition, 1994. ISBN: 978-0691034102.

[4] K. J. M. Huisman and P. M. Kort. Strategic capacity investment under uncertainty. *Tilburg: Operations research*, 2013-003, 2013.

[5] Y. Farzin, K. J. M. Huisman, and P. M. Kort. Optimal timing of technology adoption. *Journal of Economic Dynamics & Control*, 22:779–799, 1998.

[6] V. Hagspiel, K. J. M. Huisman, P. M. Kort, and C. Nunes. How to escape a declining market: Capacity investment or exit? *European Journal of Operational Research*, 254:40–50, 2016.

[7] R. Pimentel. *Jump Processes in Finance*. PhD thesis, Instituto Superior Técnico, 2018.

[8] K. Ross. *Stochastic Control in Continuous Time*. (unpublished notes), 2018.

[9] B. Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. Springer, 6^{nd} edition, 2014. ISBN: 978-3540047582.

[10] S. A. Ross, R. W. Westerfield, J. F. Jaffe, and B. D. Jordan. *Corporate Finance: Core Principles and Applications*. McGraw-Hill Education, 1^{st} edition, 1988. ISBN: 978-0077861650.

[11] M. Guerra, C. Nunes, P. M. Kort, and C. Oliveira. Exit option for a class of profit functions. *Journal of Computer Mathematics*, 94:2178–2193, 2017.

[12] S. Shapiro and M. Wilk. An analysis of variance test for normality. *Biometrika*, 74(5):3–833, 1965.

[13] S. Ross. *Stochastic Processes*. John Wiley & Sons, Inc, 2^{nd} edition, 1996. ISBN:978-0471120626.

[14] B. Øksendal and A. Sulem. *Applied Stochastic Control of Jump Diffusions*. Springer, 1^{st} edition, 2005. ISBN: 978-3540140239.

[15] R. S. Pindyck. Irreversible investment, capacity choice, and the value of the firm. *The American Economic Review*, 78:969–985, 1998.

Appendix A

Mathematica code to assess the influence of changes on initial demand values and volatility on the optimal decision time and numerical results obtained

In this appendix we present the code used to make the assessments presented on Chapter 6 as well as the numerical results obtained concerning the study of market's uncertainty impact.

A.1 Sensitivity analysis w.r.t. to initial demand values

• Simulation of a value of the demand process at time t knowing value at time $t = 0$:

```
demandProcess[x0_,t_,mu_,sigma_]:=x0 Exp[(mu-sigma^2/2)t+
          sigma RandomVariate[NormalDistribution[0,Sqrt[t]]]];
```

• Simulation of a sample path of the demand process (starting at x_0) until it reaches the threshold stipulated:

```
stopTime[timeStep_, xthreshold_, x0_, mu_, sigma_] := Module[{stop, t, X},
          stop = 0; (* hasn't stopped *)
          t = 0.001;
          X = {x0};
          While[Last[X] < xthreshold,
                    X = Append[X, demandProcess[x0, t, mu, sigma]];
```

```
        t = t + timeStep];
    t
];
```

• Simulation of a number of NsamplePaths demand processes, w.r.t. each initial value tested between xMin and xMax:

```
generalStopTime[Xmin_,Xmax_, xStep_,NsamplePaths_,timeStep_,xthreshold_,mu_,sigma_]:=
    Module[{x0=Xmin,
    stop={},
    X={}
    },
    While[x0<=Xmax,
        stop=Append[stop,Table[stopTime[timeStep,xthreshold,x0,mu,sigma],
            {i,1,NsamplePaths}]];
        X=Append[X,Table[x0,{i,1,NsamplePaths}]];
        x0=x0+xStep;
    ];
{X,stop}
];
```

A.2 Sensitivity analysis w.r.t. to volatility

A.2.1 Mathematica code

Besides function demandProces stated hereunder, we also defined the following functions:

• Simulation of a sample path of the demand process (starting at x_0) until it reaches the threshold stipulated, returning the stopping time and the whole sample path:

```
stopTimemod[timeStep_,xthreshold_,x0_,mu_,sigma_,horizon_]:=Module[{stop,t,X},
    t=0;
    X={x0};
    While[Last[X]<xthreshold &&t <=horizon,
        t=t+timeStep;
        X=Append[X,demandProcess[x0,t,mu,sigma]];
    ];
    X=Append[X,Table[demandProcess[x0,t+timeStep*i,mu,sigma],{i,horizon-t}]];
{t,Flatten@X}];
```

• Estimation of the investment waiting time w.r.t. changes in market's uncertainty:

```
estTau[x0_, threshold_] := Module[{b},
        Table[
                b = Table[stopTimemod[timeStep, threshold@sigma, x0, mu, sigma,
                horizon], {i, NsamplePaths}];
                (*Print["Tempos de paragem=",First/@b];*)
                If[Count[First /@ b, horizon + timeStep] > 0,
                  Print[ "(sigma, # unended sample paths) = ", {sigma,
                    Count[First /@ b, horizon + timeStep]}]
                ];
        Mean[First /@ b],
{sigma, 0.01, sigmaMax, 0.01}]
];
```

A.2.2 Numerical results

Table A.1: Sensitivity analysis of the (estimated) mean of optimal investment time, the threshold level and the optimal capacity level (referred on Chapter 3) with the volatility, regarding different initial values $x_0 \in \{0.001,\ 0.01\}$.

σ	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	
K^*	381.04	395.08	410.92	425.39	437.62	447.66	455.80	462.41	
x_B^*	0.012	0.013	0.015	0.017	0.020	0.023	0.027	0.032	
x_C^*	0.017	0.019	0.022	0.027	0.032	0.038	0.045	0.053	
$\overline{\tau_B}$	67.64	58.71	53.64	52.22	52.46	57.49	63.97	106.98	$x_0 = 0.001$
$\overline{\tau_C}$	78.21	68.43	63.82	63.94	65.55	70.66	90.09	193.65	
$\overline{\tau_B}$	4.29	5.26	6.48	7.81	9.53	10.25	11.90	14.154	$x_0 = 0.01$
$\overline{\tau_C}$	13.88	12.89	13.40	14.92	15.53	16.93	19.77	23.11	

Table A.2: Sensitivity analysis of the (estimated) mean of optimal investment time, the threshold level and the optimal capacity level (referred on Chapter 4) with the volatility, regarding different initial values $x_0 \in \{0.001,\ 0.01\}$.

σ	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	
K^*	406.62	416.81	428.59	439.60	449.10	457.01	463.51	468.84	
x_B^*	0.022	0.024	0.028	0.033	0.038	0.045	0.052	0.060	
x_C^*	0.021	0.024	0.028	0.033	0.039	0.047	0.055	0.064	
$\overline{\tau_B}$	86.86	76.4	71.57	65.01	68.18	73.59	84.79	203.85	$x_0 = 0.001$
$\overline{\tau_C}$	86.45	75.33	69.36	67.75	71.57	76.08	99.44	194.35	
$\overline{\tau_B}$	20.94	18.04	18.04	17.19	17.37	19.55	22.19	25.88	$x_0 = 0.01$
$\overline{\tau_C}$	20.34	18.69	18.18	17.94	19.88	21.21	22.04	23.02	

Table A.3: Sensitivity analysis of the (estimated) mean of optimal investment time, the threshold level and the optimal capacity level (referred on Chapter 5) with the volatility, regarding different initial values $x_0 \in \{0.001,\ 0.01\}$.

σ	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	
$x^*_{1,A}$	1.092	1.101	1.116	1.136	1.161	1.190	1.224	1.261	
x^*_2	6.518	6.572	6.659	6.779	6.928	7.104	7.306	7.530	
$\overline{\tau_{1,A}}$	16.08	16.18	16.13	16.07	16.06	16.34	16.34	16.64	$\}x_0 = 0.001$
$\overline{\tau_{1,A}}$	10.92	11.00	11.01	11.08	11.10	1.10	11.27	11.53	$\}x_0 = 0.001$
$\overline{\tau_2}$	4.48	4.50	4.54	4.56	4.65	4.55	4.49	4.63	

A.3 Sensitivity analysis w.r.t. the crossed-effect between initial demand and volatility

• Histogram of the optimal investment times:

```
hist[timeStep_, xfunc_, x0_, mu_, sigma_, horizon_, NsamplePaths_, color_] :=
 Module[{b, xthreshold = xfunc@sigma},
        b = Table[stopTimemod[timeStep, xthreshold, x0, mu, sigma, horizon],
                {i, NsamplePaths}];
        {Histogram[First /@ b, 20, ChartStyle -> color, AxesLabel -> {"tau", " "}],
        First /@ b,
        ShapiroWilkTest[First /@ b, "TestDataTable"]}
];
```

Appendix B

Mathematica code to assess the influence of the number of innovation jumps

In this appendix we present the code used to make the assessments presented on Chapter 7.

- Calculation of the optimal R&D investment, R^*, for given parameter γ and number of jumps n:

```
V[R_] := (R^gamma/(r + R^gamma))^n *F - R;
ptstat = Flatten[Values[NSolve[r R + R^(1 + gamma) -
          F n r (R^gamma/(r + R^gamma))^n gamma == 0 && R > 0,R]]]; (*raizes
                 da 1a derivada *)
If[ptstat == {},
          ptstat =
          Flatten[Values[FindRoot[r R + R^(1 + gamma) -
                   F n r (R^gamma/(r + R^gamma))^n gamma == 0, {R,
                   0.5}]]]  (* outro metodo para calcular raizes da 1a derivada *)
];
d2[R_] := (F n r (R^gamma/(r + R^gamma))^n gamma (-R^gamma (1 + gamma)
          + r (-1 + n gamma)))/(R^2 (r + R^gamma)^2);
maxrelat = Negative@Map[d2, ptstat] //. False -> 0 //. True -> 1;  (*
          pontos estacionarios com 2a derivada <0 *)
neg = maxrelat*ptstat; (* vector com entradas dadas por pontos est.
          com 2a derivada <0, cc entrada a zero *)
If[Norm[maxrelat] == 1, (* existe apenas 1 ponto est. com 2a derv. <
0 -> maximo global! *)
          max = Norm[ptstat*neg/Norm[neg]],
```

```
            max = 0;
            tam = Length[ptstat];
            While[i <= tam, (* teste para escolher qual dos pontos est.
com 2a derv. <0 -> maximo global *)
                        If[V[maxrelat[[i]]] > max,
                        max = maxrelat[[i]]];
                        i++];
            ];
max];
```

- Calculation of the optimal R&D investment, R^{*}, up to 5 innovation jumps and by varying γ from 0 to 1:

```
n = 1;
RoptSaltos = {};
While[n <= 5,
            gammas = Rest[Range[0, 1, 0.05]];
            Ropt = MapThread[calcR, {gammas, Table[n, {i, 1, Length[gammas]}]}];
                        (* lista R optimais *)
            RoptSaltos = Append[RoptSaltos, Ropt];
            n++
];
```