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Dedicated to someone special...

Acknowledgments

A few words about the university, financial support, research advisor, dissertation readers, faculty or other professors, lab mates, other friends and family...

Resumo

Inserir o resumo em Português aqui com o máximo de 250 palavras e acompanhado de 4 a 6 palavras-chave...

Palavras-chave: palavra-chave1, palavra-chave2,...

Abstract

Insert your abstract here with a maximum of 250 words, followed by 4 to 6 keywords...

Keywords: keyword1, keyword2,...

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Chapter 1

Introduction

Insert your chapter material here...

1.1 Motivation

Relevance of the subject... Example Goals in the end of this mini chapter.

1.2 Optimal Stopping Problems

Definição.

Princípio de Programação Dinâmica.

HJB + Gerador infinitesimal de GBM

Investimento: região de continuação e de paragem

Outro tipo de problemas (para além dos de investimento)

Teorema da Verificação

1.3 Thesis Outline

In this thesis we will treat different situations related with the decision of investing in a new product under uncertainty.

On Chapter **??**, the main concepts and state-of-the-art (common to all problems) will be introduced. More specifically:

- On Section **??** we introduce the class of optimal stopping problems, following a similar approach as presented in [**?**].

- On Section **??**, we show how optimal stopping problems relate to investment decisions under uncertainty, following a similar approach as presented in [4].

On Chapter **??**, the first model is derived. Here we consider the situation in which a firm wants to find the optimal time to introduce a new product, having none being produced at the moment. More specifically:

- On Section **??**, we derive the optimal decision regarding the original cash-flow.
- On Section **??**, we derive the optimal decision regarding the maximized cash-flow with respect to production capacity.
- On Section **??**, we study the behaviour of the decision threshold with the different parameters.

On Chapter **??**, we consider the situation in which a firm has a *stable* and recognized product in the market and wants to find the optimal time to invest in a new product and, at the same instant, replace the *stable* product by the new one. More specifically:

- On Section **??**, we derive the optimal decision regarding the original cash-flow.
- On Section **??**, we derive the optimal decision regarding the maximized cash-flow with respect to production capacity of the new product.
- On Section **??**, we study the behaviour of the decision threshold with the different parameters.

On Chapter **??**, we consider a similar situation as in Chapter **??**, but with the extra possibility of producing both products simultaneously. Therefore, now the firm wants to find the optimal time to invest in a new product and the optimal time to stop producing the *stable* product. More specifically:

- On Section **??**, we derive the optimal decisions regarding the original cash-flows.

On Chapter **??**, we derive the optimal R&D investment, by maximizing the expected value function function with respect to the innovation process. More specifically:

- On Section **??**, we derive the optimal R&D investment considering that the innovation process only takes one jump to achieve the breakthrough level.
- On Section **??**, we generalize the previous section, by considering that the innovation process takes $n \in \mathbb{N}$ jumps to achieve the breakthrough level.
- On Section **??**, we study the behaviour of the decision threshold with the different parameters.

On Chapter **??**, we perform some simulations.

Finally, on Chapter **??**, we summarize the relevant findings of the work done and how it can be extended.

1.3.1 Some Notation

Throughout the chapters, many terms will appear and their explanation will come along. However most of them will be always the same, since they do not depend on the chapter that we are working one. Therefore, to promote a better understanding in the context of the problem, the major notation (and its restriction) will be now introduced:

- $\{W(t), t \geq 0\}$: Standard Brownian Motion (or Wiener Process) which is a stochastic process that has the following characteristics:

  1. $W(0) = 0$ with probability 1;
  2. $W(t) - W(s) \sim N(0, t-s)$. Notice that $\mathbb{E}[W(t)] = 0$ and $Var[W(t)] = t$;
  3. Independent increments: $\forall 0 < s_i < t_i < s_j < t_j : \; W(t_i) - W(s_i) \amalg W(t_j) - W(s_j)$;
     Stationary increments: $\forall t, s \geq 0 : \; W(t+s) - W(s) \stackrel{d}{=} W(t)$;
  4. $W(t)$ is continuous in t (however is nowhere differentiable).

  It is also seen as the continuous version of a Random Walk with Normal increments.

- $\{X(t), t \geq 0\}$: Geometric Brownian Motion (GBM) that represents the demand for a certain product. It is the solution of the following stochastic differential equation (SDE)

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \;\; X_0 = x,$$

  where μ represents the drift and σ the volatility, both associated to the demand.

- R: R&D costs such as size of laboratories, wages of the scientists, their computers/machines, *etc.*

- $\{\theta(t), t \geq 0\}$: innovation process assumed to be a homogeneous Compound Poisson Process, that is

$$\theta_t = \theta_0 + uN_t$$

  where θ_0 corresponds to the initial innovation level, $u > 0$ is the jump size and $\{N_t, t \geq 0\}$ follows a Poisson process with rate $\lambda(R) = R^\gamma, \; \gamma \in (0, 1)$. This rate function is such that $\lambda(0) = 0$ (no R&D means zero probability of innovating); $\lambda'(R) > 0$ (bigger investment means the higher probability of success) and $\lambda''(R) > 0$.

- θ: innovation breakthrough level. That is, the level of innovation for which we decide to invest in the new product.

- K_i: capacity of production of product i. Note that when a single product is considered, there is no mention to index i. The firm is considered to produce always up to its capacity. Since profit functions need to be positive, we have the following restrictions regarding capacities of *old* and *new* product, respectively, $K_0 < 1/\alpha$ and $K_1 < \theta/\alpha$.

- α: constant parameter that reflects the sensitivity of the quantity with respect to the price.

- δ: constant parameter that reflects the sensitivity of the quantity with respect to investment sunk costs, that is, costs that cannot be recovered after investing. These sunk costs will be denoted by δK_1 (or δK, on Chapter 3).

- η: cannibalization parameter corresponding to the crossed effect between the old and the new product. It's seen as a penalty representing how the quantity associated to a product will influence the price of the other. On Chapter **??**, we consider that this influence is the same for both products, so we there is a unique cannibalisation parameter. This cannot be greater than the sensibility parameter $\eta < \alpha$.

Chapter 2

Investing and entering the market with a new product (Firm is not active before investing)

Insert your chapter material here...

2.1 Introduction

In this chapter we consider a firm that wants to invest in a product, after a certain innovation level θ is reached, and to produce it in long term. To do so, the firm needs to incur an investment cost proportional to the capacity production K_1. This cost is given by δK_1, with $\delta > 0$ a sensibility parameter related to the investment. We consider here that, at the investing time, the firm needs to pay the investment cost and that the production starts immediately.

The demand function associated to the product to be introduced evolves stochastically with the demand process $\mathbf{X}$ and it is given by

$$p(X_t) = (\theta - \alpha K)X_t \geq 0 \tag{2.1}$$

where $\alpha > 0$ is a sensibility parameter and X_t corresponds to the demand level observed at the instant $t \geq 0$.

Multiplying the demand function by production capacity we obtain the associated instantaneous profit π, that is given by

$$\pi(X_t) = (\theta - \alpha K)KX_t \geq 0. \tag{2.2}$$

The firm is considered to produce always up to its capacity and that the variable costs are constant. Also, time is set to start when the innovation process reaches θ, since before it it's useless to make an investment decision. Therefore X_0 refers to the demand level observed when the desired innovation level is reached. These facts will simplify our notation without losing the applicability of the model.

As mentioned before, two models will be derived. The first one corresponds to the benchmark model. The simplest model to be considered. The second one will take into account the maximized instantaneous profit in function of the production capacity K.

2.2 Stopping Problem

2.2.1 Benchmark Model

We start with the simplest model. In the benchmark model we want to find when is the optimal time invest in the product (in the sense that maximizes the expected long-term profit), taking into account all the information previously referred.

Denoting the time of investment in the new product as τ, our optimization problem can be written as

In the benchmark model we want to find when is the optimal time invest in the product (in the sense that maximizes the expected long-term profit), taking into account all the information previously referred.

Denoting the time of investment in the new product as τ, our optimization problem can be written as

$$\sup_{\tau} \mathbf{E}^{X_0=x} \left[e^{-r\tau} \left(\int_{\tau}^{\infty} e^{-r(\tau-s)} \pi(X_s)\ ds - \delta K \right) \mathbf{1}_{\{\tau<\infty\}} \right] \tag{2.3}$$

for $\theta, x \in R^+$. Here $\mathbf{E}^{X_0=x}[\ .\ ]$ corresponds to the expected value conditional to $X_0 = x$, that is, $\mathbf{E}[\ .\ |\ X_0 = x]$.

We can simplify this expression. Using Tower rule and conditioning on the instant when we exercise τ, we obtain that (2.3) may be written as

$$\sup_{\tau} \mathbf{E}^{X_0=x} \left[e^{-r\tau} \left(\mathbf{E}^{\tau=t} \left[\int_{t}^{\infty} e^{-r(\tau-s)} \pi(X_s)\ ds \right] - \delta K \right) 1_{\{\tau<\infty\}} \right]. \tag{2.4}$$

Let's focus now on the inner expected value $\mathbf{E}^{\tau=t} \left[\int_t^{\infty} e^{-(\tau-s)} \pi(X_s)\ ds \right]$. Changing the integration variable follows

$$\mathbf{E}^{\tau=t} \left[\int_0^{\infty} e^{-rv} \pi(X_{v+\tau})\ dv \right]. \tag{2.5}$$

Considering $r - \mu > 0$, we have that $\int_0^{\infty} \int_{\Omega} e^{-rv} \pi(X_{v+\tau})\ dv < \infty$. Since $e^{-rv}\pi(X_{v+\tau})$ is a continuous funtion it's also a measurable function. By both conditions we obtain that it is $[0,\infty) \times \Omega$-integrable. Therefore by the Dominated Convergence Theorem we can interchange the integrals, from which follows by (2.5), that

$$\int_0^{\infty} \mathbf{E}^{\tau=t} \left[e^{-rv} \pi(X_{v+\tau}) \right]\ dv = (\theta - \alpha K) K \int_0^{\infty} \mathbf{E}^{\tau=t} \left[e^{-rv} X_{v+\tau} \right]\ dv, \tag{2.6}$$

where we took into account the expression of the profit function π.

Let's now focus on the expected value $\mathbf{E}^{\tau=t}\left[e^{-rv}X_v\right]$. It follows that

$$\begin{aligned}
\mathbf{E}^{\tau=t}\left[e^{-rv}X_{v+\tau}\right] &= \mathbf{E}^{\tau=t}\left[X_\tau e^{\left(\mu-\frac{\sigma^2}{2}-r\right)(\tau+v-\tau)+\sigma(W_{\tau+v}-W_\tau)}\right] \\
&= x_\tau e^{\left(\mu-\frac{\sigma^2}{2}-r\right)v}\ \mathbf{E}^{\tau=t}\left[e^{\sigma W_v}\right] \\
&= x_\tau e^{\left(\mu-\frac{\sigma^2}{2}-r\right)v} e^{\frac{\sigma^2}{2}v} \\
&= x_\tau e^{(\mu-r)v}.
\end{aligned} \tag{2.7}$$

In the first step we used the expression associated to the GBM and the fact that knowing the investment time τ, we also know the demand level at that time, here represented as $X_\tau = x_\tau$. In the second step, the fact that the Brownian Motion has stationary increments implies that $W_{\tau+v} - W_\tau \overset{d}{=} W_v - W_0 = W_v$, since we assumed $\mathbf{W}$ to be a standard Brownian Motion which implies $W_0 = 0$. In the third step we used the fact that $W_v \sim \mathcal{N}(0, v)$ and the expression for the moment generating function associated to the Normal distribution, from which follows $\mathbf{E}\left[e^{sW_v}\right] = e^{\frac{1}{2}sv^2}$. Simplifying the expression we obtain (2.7).

Plugging the resultant expression (2.7) in (2.6) and solving it, we obtain the formula of the terminal cost function associated to this problem - corresponding to the expression between parenthesis in (3.1). We will denote it by h and its expression corresponds to

$$h(x) = \frac{(\theta - \alpha K)Kx}{r-\mu} - \delta K. \tag{2.8}$$

The terminal cost function h represents the discounted long-term profit by acquiring a product when the demand level is x. As one can note, it already includes the investment cost of such decision.

Denoting F as the value function associated to this problem, we obtain that our optimization problem, as described in (2.3), can be written as a standard optimal stopping problem with null running cost function, given by

$$F(x) = \sup_\tau \mathbf{E}^{X_0=x}\left[e^{-r\tau}h(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] = \sup_\tau \mathbf{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{(\theta-\alpha K)KX_\tau}{r-\mu} - \delta K\right)\mathbf{1}_{\{\tau<\infty\}}\right]. \tag{2.9}$$

Recurring to Bellman principle, we have that the solution F verifies the variational inequality given by Hamilton-Jacobi-Bellman (HJB) equation

$$\min\{-rF(x) + \mathcal{L}F(x), h(x) - F(x)\} = 0. \tag{2.10}$$

Recalling what was discussed in the previous chapter, we have that the value function is given by

$$F(x) = \begin{cases} ax^{d_1}\ ,\ x \in \mathcal{C} \\ h(x)\ ,\ x \in \mathcal{S} \end{cases},$$

where coefficient a and the threshold value x^*, that defines the boundary between continuation and stopping regions, are found by value matching and smooth pasting conditions, expressed by the corresponding system

$$\begin{cases} a(x^*)^{d_1} = \frac{K(\theta-\alpha K)x^*}{r-\mu} - \delta K \\ a d_1 (x^*)^{d_1-1} = \frac{K(\theta-\alpha K)}{r-\mu} \end{cases} \Rightarrow \begin{cases} a = \left(\frac{K(\theta-\alpha K)x^*}{r-\mu} - \delta K\right)(x^*)^{-d_1} = \frac{\delta K (x^*)^{-d_1}}{d_1-1} \\ x^* = \frac{d_1}{d_1-1}\frac{\delta(r-\mu)}{\theta-\alpha K} \end{cases} \tag{2.11}$$

with $d_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1$, being the positive root of the polynomial described in (**??**).

It follows the continuation and stopping regions associated to this problem are given by

$$\mathcal{C} = \left\{ x \in \mathbf{R}^+ : x \leq x^* = \frac{d_1}{d_1 - 1}\frac{\delta(r-\mu)}{\theta - \alpha K} \right\}$$

$$\mathcal{S} = \mathbf{R}^+ \setminus \mathcal{C} = \left\{ x \in \mathbf{R}^+ : x > x^* = \frac{d_1}{d_1 - 1}\frac{\delta(r-\mu)}{\theta - \alpha K} \right\}.$$

Finally, we are now in the position to formally define the stopping time τ, used in (2.9), as $\tau = \inf\{t \geq 0 : X(t) \in \mathcal{S}\} = \inf\{t \geq 0 : X(t) \geq x^*\}$.

2.2.2 Capacity Optimization Model

Now we increase the complexity, by requiring that the production capacity is chosen to be the maximizer of the discounted long-term profit minus the investment cost, while keeping the goal of finding the best time to invest in the product.

Our problem can be written now as

$$\sup_\tau \mathbf{E}^{X_0=x}\left[\max_K \left\{ e^{-r\tau}\left(\int_\tau^\infty e^{-r(\tau-s)}\pi(X_s)\ ds - \delta K\right)\right\} \mathbf{1}_{\{\tau<\infty\}}\right]. \tag{2.12}$$

Manipulating the expression as done in the previous section 2.2.1, we obtain that (2.12) may be written as

$$\sup_\tau \mathbf{E}^{X_0=x}\left[e^{-r\tau}\max_K \{h(X_\tau, K)\}\mathbf{1}_{\{\tau<\infty\}}\right], \tag{2.13}$$

with h corresponding to the terminal function deduced in (2.8), in which we now highlight, not only the dependence on the demand level, but also on the production capacity K chosen at the investing time.

In this section, the capacity optimization model is obtained in two steps. In the first one we calculate the capacity level that optimizes the terminal cost function h, which we will denote by K^*. The second step consists in solving the optimal stopping problem given by $\sup_\tau \mathbf{E}^{X_0=x}\left[e^{-r\tau}h(X_\tau, K^*)\mathbf{1}_{\{\tau<\infty\}}\right]$, in which we are already considering the optimized terminal function.

The optimal capacity level K^* is found by analyzing the behaviour - namely its concavity and stationary points - of the terminal function h, by considering a fix level of demand.

The stationary points are found by calculating the roots of the first partial derivative. We obtain that the first partial derivative is given by

$$\frac{\partial h}{\partial K}(x, K) = \frac{(\theta - 2\alpha K)x}{r-\mu} - \delta,$$

which implies that h has a unique stationary point

$$K = \frac{\theta}{2\alpha} - \frac{\delta(r-\mu)}{2\alpha x}. \tag{2.14}$$

We obtain that its second partial derivative is negative and given by

$$\frac{\partial^2 h}{\partial K^2}(x,K) = -\frac{2\alpha x}{r-\mu} < 0,$$

since we assumed $\alpha > 0$ and $r - \mu > 0$ and the GBM doesn't take negative values. Therefore, h is a concave function and the capacity value found in (2.14) is its global maximizer.

From now on, and to emphasize its maximizer role, we denote (2.14) by K^*. Note that K^* is dependent of the demand level in the sense that the optimal capacity is increasing with the initial observed demand value.

Now we proceed to the second step. Evaluating h at its optimal capacity level K^* we obtain

$$h(x,K^*) = \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x}.$$

Denoting F^* as the value function associated to the optimal stopping problem in (2.15), the optimization problem can be stated as

$$F^*(x) = \sup_\tau \mathbf{E}^{X_0=x}\left[e^{-r\tau}h(X_\tau,K^*)\mathbf{1}_{\{\tau<\infty\}}\right] = \sup_\tau \mathbf{E}^{X_0=x}\left[e^{-r\tau}\frac{(\theta X_\tau - \delta(r-\mu))^2}{4\alpha(r-\mu)X_\tau}\mathbf{1}_{\{\tau<\infty\}}\right], \tag{2.15}$$

which is again a standard optimal stopping problem with null running cost functio. Similarly to the benchmark model, we obtain that the value function associated to (2.15), satisfies the HJB variational inequality as described in (2.10) - although now considering the solution F^* instead of F. Therefore F^* is such that

$$F^*(x) = \begin{cases} bx^{d_1} \ , \ x \in \mathcal{C} \\ h(x,K^*) \ , \ x \in \mathcal{S} \end{cases},$$

where coefficient b and the threshold value x_C^*, that defines the boundary between continuation and stopping regions, are found by value matching and smooth pasting conditions, expressed by the corresponding system

$$\begin{cases} b(x_C^*)^{d_1} = \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x} \\ bd_1(x_C^*)^{d_1-1} = \frac{\theta^2(x_C^*)^2 - \delta^2(r-\mu)^2}{4\alpha(r-\mu)(x_C^*)^2} \end{cases} \tag{2.16}$$

with $d_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1$, being the positive root of the polynomial described in (**??**).

We get two possible positive roots for the threshold level: $x_{C,1}^* = \frac{d_1+1}{d_1-1}\frac{\delta(r-\mu)}{\theta-\alpha K}$ and $x_{C,2}^* = \frac{\delta(r-\mu)}{\theta}$. However, after some manipulation, we exclude the second one $x_{C,2}^*$, since the coefficient b associated to it takes a null value. This is an absurd, since it would lead to a null value function for any demand level smaller than $x_{C,2}^*$, contradicting the fact that the possibility of investing in the future is also valuable.

Therefore we obtain that the threshold level and coefficient b in (2.11) are, respectively, given by

$$x_C^* = \frac{d_1+1}{d_1-1}\frac{\delta(r-\mu)}{\theta} \tag{2.17}$$
$$b = \left(\frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x_C^*}\right)(x_C^*)^{-d_1} = \frac{\delta\theta}{\alpha(d_1^2-1)}\left(\frac{d_1+1}{d_1-1}\frac{\delta(r-\mu)}{\theta}\right)^{-d_1}$$

2.3 Comparative Statics

2.3.1 Benchmark Model

In this section we study the behaviour of the decision threshold x_B^* (2.11) and x_C^* (**??**) and K^* as described in (**??**), with the different parameters.

Comparisons between the benchmark and capacity optimization models will be made.

Proposition: Decision threshold x_B^* increases with $r, \sigma,\ K,\ \alpha,\ \delta$ and decreases with μ, θ.

Proof:

Before showing the main results stated, observe that

$$\phi := \sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} > 0.$$

This is a recurrent expression in most comparative statics sections.

The fact that $\phi > 0$ comes from analyzing the expression inside the square root to infere about ϕ. Since it only has imaginary roots regarding parameter μ and $\frac{\partial^2}{\partial\mu^2}\left(\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1\right) = \frac{8}{\sigma^2} > 0$, it follows that it is always positive, so it is ϕ.

Now we are in position to explain stated results.

Regarding r, we obtain

$$\frac{\partial x_B^*}{\partial r} = \frac{\delta\left(-(d1-1)d1\sigma^2\phi - 2\mu + 2r\right)}{(d1-1)^2\sigma^2(\alpha k - \theta)\sqrt{\frac{4\mu^2}{\sigma^4} - \phi}} > 0,$$

Note that its denominator is positive, giving the constraints of our problem. Analyzing the numerator, we found that it has no possible roots on problem domain. Thus, taking $\mu < 0$, we obtain that $-(d1-1)d1\sigma^2\phi - 2\mu + 2r > 0 \quad \forall r,\ \sigma$ which implies that the numerator is positive for any parameter values taken into the problem domain.

Regarding σ, we obtain

$$\frac{\partial x_B^*}{\partial \sigma} = -\frac{2\delta(\mu - r)\left(-2\mu^2 + \mu\sigma^2(\phi+1) - 2r\sigma^2\right)}{(d_1-1)^2\sigma^5(\alpha k - \theta)\phi} > 0.$$

Note that its denominator is negative, since $\alpha - \theta K < 0$. Analyzing the numerator, we found that it has no possible roots on problem domain. Thus, since $-2\delta(\mu - r)\left(-2\mu^2 + \mu\sigma^2(\phi+1) - 2r\sigma^2\right)|_{\mu=0} > 0$, it follows that the numerator is negative.

Regarding K, we obtain

$$\frac{\partial x_B^*}{\partial K} = \frac{\alpha\delta d_1(r-\mu)}{(d_1-1)(\theta-\alpha k)^2} > 0.$$

Regarding α and δ, we obtain

$$\frac{\partial x_B^*}{\partial \alpha} = \frac{\delta d_1 k(r-\mu)}{(d_1-1)(\theta-\alpha k)^2} > 0$$
$$\frac{\partial x_B^*}{\partial \delta} = \frac{d_1(r-\mu)}{(d_1-1)(\theta-\alpha k)} > 0,$$

from which the result holds.

Regarding μ, we obtain

$$\frac{\partial x_B^*}{\partial \mu} = \frac{\delta\left(\left((d_1-1)d_1\sigma^4\phi - 2\mu^2 + \mu\sigma^2\phi + 1\right) + r\left(2\mu - \sigma^2\left(\phi+1\right)\right)\right)}{(d_1-1)^2\sigma^4(\alpha k-\theta)\phi} < 0.$$

Observe that its denominator is negative. On the other side, taking into account the our restrictions, we obtain that there are no possible roots for the numerator. Then, evaluating for any plausible parameter values we obtain a positive value, from which follows that the numerator is always positive.

Regarding θ, we obtain

$$\frac{\partial x_B^*}{\partial \sigma} = -\frac{\delta d_1(r-\mu)}{(d_1-1)(\theta-\alpha k)^2} < 0,$$

from which the result follows.

□

2.3.2 Capacity Optimization Model

Proposition: Decision threshold x_C^* increases with σ, δ, decreases with θ and has a monotonic behaviour with μ. None of any other parameters have effect on x_C^*.

Proof: Regarding σ, we obtain

$$\frac{\partial x_C^*}{\partial \sigma} = \frac{4\delta(\mu-r)\left(-2\mu^2 + \mu\sigma^2(1+\phi) - 2r\sigma^2\right)}{(d_1-1)^2\theta\sigma^5\phi}.$$

Note that its denominator is positive. Analyzing the numerator, we found that it has no possible roots on problem domain and that it's positive for all parameters.

Regarding δ, we obtain

$$\frac{\partial x_C^*}{\partial \delta} = \frac{(d_1+1)(r-\mu)}{(d_1-1)\theta} > 0.$$

Regarding μ, we obtain

$$\frac{\partial x_C^*}{\partial \mu} = \frac{\delta}{(d_1-1)^2\theta}\left(1 - d_1^2 + (r-\mu)\left(-1 + \frac{2\mu-\sigma^2}{\phi\sigma^2}\right)\right) = \begin{cases} < 0 & \text{for } \mu < \frac{\sigma^2}{2} \\ > 0 & \text{for } \mu > \frac{\sigma^2}{2} \end{cases}.$$

Regarding θ, we obtain

$$\frac{\partial x_C^*}{\partial \theta} = -\frac{\delta(d_1+1)(r-\mu)}{(d_1-1)\theta^2}$$

□

To illustrate results above mentioned we performed some numerical illustrations, using software *Mathematica* and its function Manipulate. However here are only able to present static plots - we leave to the interested ones, to see the results achieved with Manipulate.

Unless it is written the opposite, following values were considered:

- $\mu = 0.03$
- $\sigma = 0.005$
- $r = 0.05$
- $\delta = 2$
- $\alpha = 0.01$
- $\theta = 10$
- $K = 100$

We start by illustrating how does x_B^* and x_C^* are related by the capacity level K, on which x_B^* is dependent. One can see on Figure 2.1 that conclusions mentioned on the proof (including that $x_B^*(0) = x_C^*$) hold.

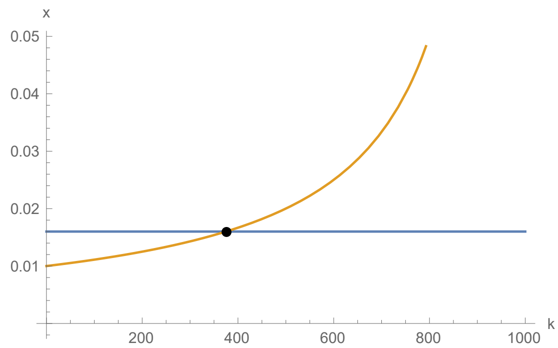


Figure 2.1: Threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange), considering capacity levels $K \in [0, \theta/\alpha)$ and the value that x_B^* takes when considering $K*$ (black).

On Figure 2.2 we observe that both thresholds increase with volatility. This in accordance with [1] and [2], whose works describe that when uncertainty is high, there is a delay time to invest, which is here reflected on an higher demand level.

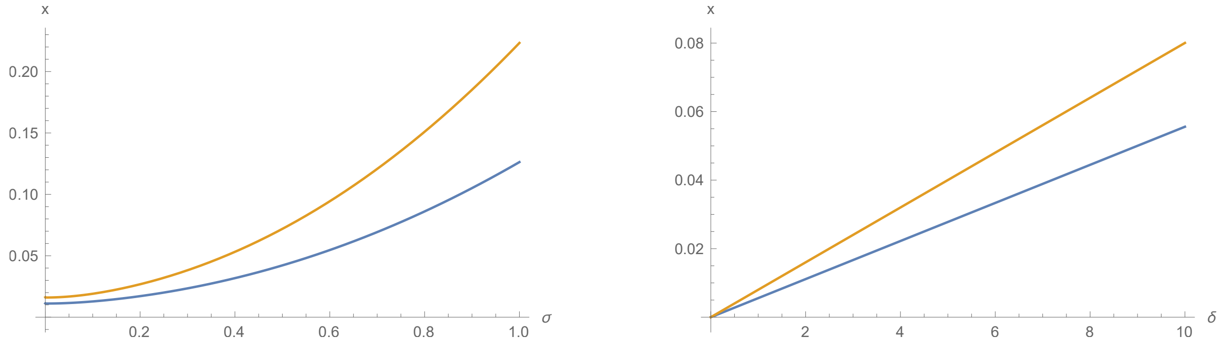
As the sensibility parameter δ increases, the firm needs to pay more sunk costs. Therefore the investment will only be made if higher demand values are observed.

Regarding the drift parameter μ we obtained that the threshold values do not have a monotonic behaviour, either for smaller or bigger values of volatility. As showed in Figure 2.3, the smallest value of demand level necessary to invest is observed at the stationary point when $\mu = \sigma^2/2$.

On Figure 2.4 we observe the behaviour of both threshold levels regarding the two other parameters, sensibility level δ and innovation level θ. We have that the threshold levels increase with δ and decrease with θ, as previously deduced.

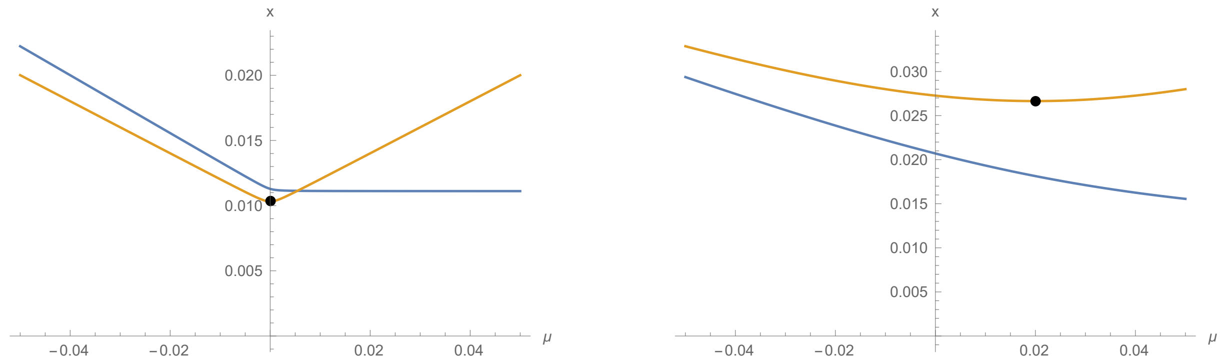
Now we analyse optimal capacity level K_C^*, that is given by evaluating K^* as defined in (2.14) on demand level x_C^*, as done in [3]. Its expression is given by

$$K_C^* = \frac{2\sigma^2\theta}{\alpha\left(\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} + 3\right) - 2\mu\right)}.$$



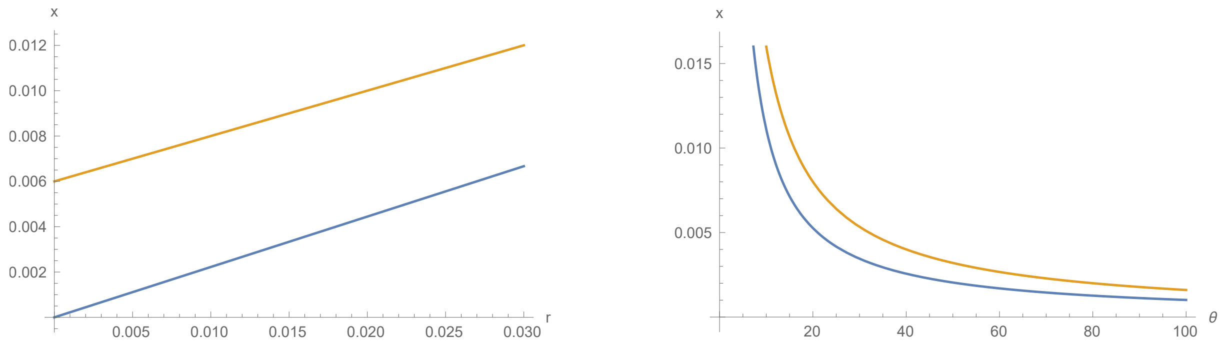
(a) $\sigma \in (0.0001, 1)$ (b) $\delta \in (0, 10)$

Figure 2.2: Threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange) and regarding its increasing parameters σ and δ.



(a) $\sigma = 0.05$ (b) $\sigma = 0.2$

Figure 2.3: Threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange), considering drift $\mu \in [-r, r]$ and corresponding stationary point $\sigma^2/2$ (black).



(a) $r \in (\mu, 1)$ (b) $\theta \in (\alpha K = 1, 50)$

Figure 2.4: Threshold value with respect to the benchmark model (orange) and the capacity optimized model (blue), regarding sensibility parameter δ and innovation level θ.

Proposition: Optimal capacity level K_C^* increases with μ, σ and θ, decreases with r and α and as no relation with δ.

Proof: The relation between K_C^* and θ, r or α comes immediately by observing K_C^* expression.

Now, regarding drift parameter we obtain that

$$\frac{\partial K_C^*(\mu)}{\partial \mu} = \frac{4\theta\left(\sigma^2(\phi+1) - 2\mu\right)}{\alpha\phi\left(\sigma^2(\phi+3) - 2\mu\right)^2} > 0.$$

Since

$$\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4}-\frac{4\mu}{\sigma^2}+\frac{8r}{\sigma^2}+1}+1\right)-2\mu>0 \Leftrightarrow \frac{4\mu^2}{\sigma^4}-\frac{4\mu}{\sigma^2}+\frac{8r}{\sigma^2}+1>\left(\frac{2\mu}{\sigma^4}-1\right)^2=\frac{4\mu^2}{\sigma^4}-\frac{4\mu}{\sigma^2}+1 \tag{2.18}$$

$$\Leftrightarrow \frac{8r}{\sigma^2}>0,$$

which is true for $\forall r>0$ and (2.3.1) we obtain that both denominator and numerator are positive, from which the result comes.

Regarding volatility parameter we obtain that

$$\frac{\partial K_C^*(\sigma)}{\partial\sigma}=\frac{8\theta\left(2\mu^2-\mu\sigma^2\left(\phi+1\right)+2r\sigma^2\right)}{\alpha\sigma\phi\left(\sigma^2\left(\phi+3\right)-2\mu\right)^2}>0$$

One can easily note that the denominator is positive. When it comes to the numerator, we will study the sign of the expression between parenthesis.

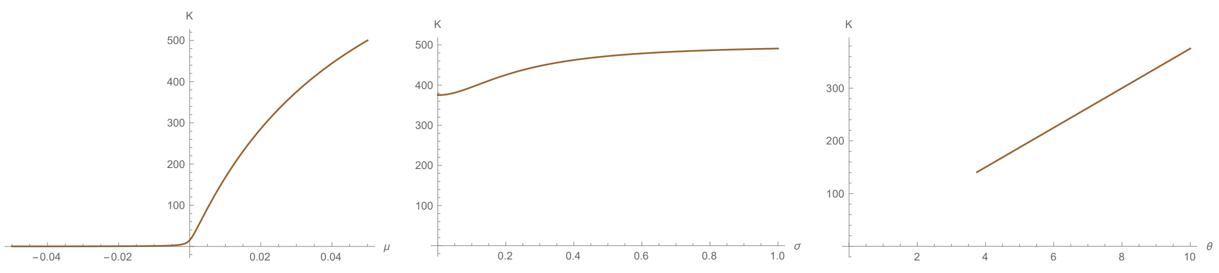
$$2\mu^2-\mu\sigma^2\left(\phi+1\right)+2r\sigma^2>0 \Leftrightarrow \left(\frac{2\mu^2+2r\sigma^2}{\mu\sigma^2}-1\right)^2>\frac{4\mu^2}{\sigma^2}-\frac{4\mu}{\sigma^2}+\frac{8r}{\sigma^2}+1 \tag{2.19}$$

$$\Leftrightarrow r>\mu, \tag{2.20}$$

which always hold, implying that the denominator is always positive.

□

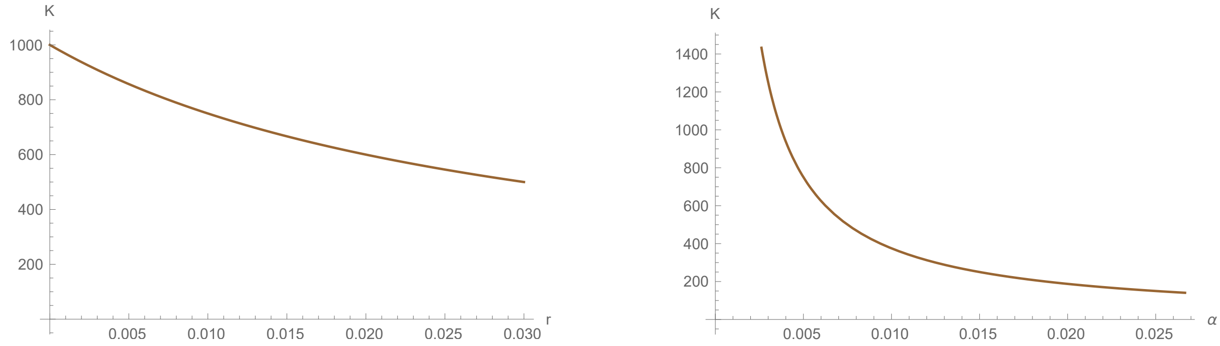
Considering some numerical approximations, we observe, on Figure 2.5, that K_C^* increases with drift, volatility and innovation level, as deduced before. Note that, regarding the drift parameter, the growth is barely noticeable for negative values of μ, but then it turns to be logarithmic. FINANCIAL INTERPRETATION? This seems to be related with the fact that for small drift values, the future expected demand value is smaller that for positive drift values. Recall that the demand process evolves accordingly to a GBM and its expected value at time t is given by $\mathbf{E}^{X_0=x_0}[X_t]=x_0e^{\mu t}$.



(a) $\mu\in(-r,r)$ (b) $\sigma\in(0,1)$ (c) $\theta\in(\alpha K_C^*,10)$

Figure 2.5: Optimal capacity regarding the threshold value x_C^* and increasing parameters μ, σ and θ.

Regarding discount rate r and sensibility parameter α, we have on Figure 2.6 that K_C^* decreases with them, as expected.



(a) $r \in (\mu, 1)$ (b) $\alpha \in (0, 1)$

Figure 2.6: Optimal capacity regarding the threshold value x_C^* and decreasing parameters r and α.

2.3.3 R&D and Capacity Optimization Model

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As stated in section **??**, we are not able to solve analytically the polynomial presented in (**??**) for every value $\gamma \in (0, 1)$. However we considered some numerical approximations, using software *Mathematica* and its function Solve. For the effect, we considered

- $r = 0.05$;
- $F(X) = 10$;
- $\gamma \in (0, 1]$ incremented by 0.05.

Following results are implemented on script RVopt.nb.

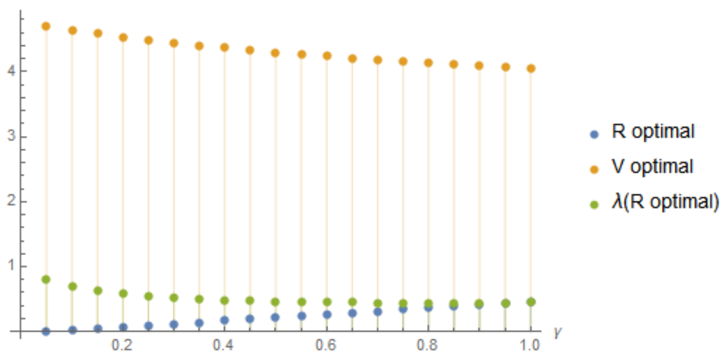


Figure 2.7: Optimal values of R and $V(X)$ for fixed values of F and r

We obtain that, although the optimal investment R grows with exponent γ, the value function for the respective optimal R decreases with exponent γ (and in a different way from the decreasing of $\lambda(R)$). We get that the smaller value of $V(X)$ is approximately 4.05, corresponding to the optimal investment level $R = 1$ and $\lambda(R) = 0.45$ and the biggest value of $V(X)$ is approximately 4.69, corresponding to the optimal investment level $R = 0.014$ and $\lambda(R) = 0.05$

Chapter 3

Adding a new product when already producing one (Firm is already active before investing)

3.1 Introduction

We consider now the case where, even before we decide to invest, the interested firm already produces a certain product. We also consider that the product is *stable* in the market, in the sense that it's a recognized product, and so its demand function is not influenced by the demand level. Instead it's given by

$$p_0 = 1 - \alpha K_0$$

where α stands for a sensibility parameter and K_0 for the capacity of production of the *old* product.

However, the same is not valid for the new product. When the breakthrough takes place, the firm has the option to start to produce the new product. Since this one is a new product, susceptible to the consumers demand, its demand function is considered to be the same as in Section 5.1, by the expression in (2.2), that is,

$$p_1(X_t) = (\theta - \alpha K_1)X_t$$

where θ stands for the innovation level after the breakthrough, α for the same sensitivity parameter as in the *old* product, K_1 for the capacity of production of the *new* product and X_t for the demand level at time t.

Recall that at the moment we decide to invest, we need to pay δK_1 related to sunk costs.

As in the previous section, two models will be derived. The first one is the benchmark model. The second one consists in considering the maximized discount profit, associated with the investment decision, in terms of the capacity associated to the *new* product.

3.2 Stopping Problem

3.2.1 Benchmark Model

We want to find when is the best time to invest in the new product, knowing that the firm produces a *stable*, that it's not influenced by the demand level and whose profit function is given by $\pi_0 = p_0K_0 = (1 - \alpha K_0)K_0$, and when the replacement happens, the firm will be immediately producing a product whose profit function is $\pi_1(X_t) = p_1(X_t)K_1 = (\theta - \alpha K_1)K_1X_t$.

Thus our optimal stopping problem may be written as

$$\sup_{\tau} \mathbf{E}^{X_0=x} \left[\int_0^{\tau} \pi_0 e^{-rs} ds + e^{-r\tau} \left(\int_{\tau}^{\infty} \pi_1(X_s) e^{-rs} ds - \delta K_1 \right) \right]. \tag{3.1}$$

The first integral corresponds to the discounted profit obtain associated to the *old* product from the time when the innovation level θ is reached until the time when the firm decides to invest in the *new* product. The second integral corresponds to the discounted profit associated to the *new* in the long term, after deciding to invest. Sutracting to it δK_1), we obtain the profit associated to the investment decision, that to be evaluated in the precise time when the innovation breakthrough happens, must be discounted (multiplying the resultant expression by $e^{r\tau}$).

We can simplify this problem in order to have a standard optimal stopping problem with null running cost function. Starting by conditioning (3.1) to the time when the investment should happen and using Tower rule it follows that (3.1) is equal to

$$\sup_{\tau} \mathbf{E}^{X_0=x} \left[\mathbf{E}^{\tau=t} \left[\int_0^{\tau} \pi_0 e^{-rs} ds + e^{-r\tau} \left(\int_{\tau}^{\infty} \pi_1(x) e^{-rs} ds - \delta K_1 \right) \right] \right]. \tag{3.2}$$

Since expectation is a linear operator, we can simplify each of the integrals separately.

Note that in the leftmost integral of (3.2), the instantaneous profit associated to the *old* product does not depend on the demand level and all the parameters are deterministic. Then its expression can be simplified as

$$\begin{aligned} \mathbf{E}^{\tau=t} \left[\int_0^t \pi_0 e^{-rs} ds \right] &= \mathbf{E}^{\tau=t} \left[\int_0^t p_0 K_0 e^{-rs} ds \right] \\ &= \mathbf{E}^{\tau=t} \left[\int_0^t (1 - \alpha K_0) K_0 e^{-rs} ds \right] \\ &= \mathbf{E}^{\tau=t} \left[(1 - \alpha K_0) K_0 \frac{1 - e^{-rt}}{r} \right] \end{aligned} \tag{3.3}$$

Following a similar approach as in the previous section, when deducing (2.8), the leftmost expected value of the rightmost integral can also be simplified as

$$
\begin{aligned}
\mathbf{E}^{\tau=t}\left[\int_t^\infty \pi_1(X_s)e^{-rs}ds - \delta K_1\right] &= \mathbf{E}^{\tau=t}\left[\int_t^\infty p_1(X_s)K_1e^{-rs}ds - \delta K_1\right] \\
&= \mathbf{E}^{\tau=t}\left[\int_t^\infty (\theta-\alpha K_1)X_sK_1e^{-rs}ds - \delta K_1\right] \\
&= \mathbf{E}^{\tau=t}\left[\frac{(\theta-\alpha K_1)K_1x_\tau}{r-\mu} - \delta K_1\right]
\end{aligned} \tag{3.4}
$$

where x_τ is taken to be the observed demand level at the time when the *new* product starts being produced. Recall, fromthe previous section, that expression (3.4) only holds if the assumptions of Fubini's Theorem hold, that is $r-\mu>0$.

Denote F as the value function solution to (3.2). Plugging expressions (3.3) and (3.4) in (3.2), getting rid of the expectation conditional to time when the investment decision is made, using (again) Tower rule, we obtain that F may be written as

$$
F(x) = \frac{\pi_0}{r} + \sup_\tau \mathbf{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{(\theta-\alpha K_1)K_1X_\tau}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right)\right)\right], \tag{3.5}
$$

which corresponds to the sum of a constant term with a standard optimal stopping problem with null running cost function, as we wanted. This is the effect of dealing with a very stable product, which doesn't depend on the current demand level, allowing us to simply this much its expression.

Note that the terminal cost function here, given by $h(x) = \frac{(\theta-\alpha K_1)x}{r-\mu} - \left(\delta K_1 + \frac{(1-\alpha K_0)K_0}{r}\right)$, is a non-decreasing function in x of the type $h(x,\underline{\psi}) = k(\underline{\psi})x - j(\underline{\psi})$ with $k(\underline{\psi})$ and $j(\underline{\psi})>0$ functions, $\underline{\psi}$ a vector corresponding to the deterministic arguments and x corresponding to the stochastic argument.

Considering V to be the optimal standard problem present in (3.5), that is

$$
V(x) = \sup_\tau \mathbf{E}^{X_0=x}\left[e^{-r\tau}h(X_\tau)\right] = \sup_\tau \mathbf{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{(\theta-\alpha K_1)K_1X_\tau}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right)\right)\right], \tag{3.6}
$$

V must verify the HJB variational inequality (**??**). As was written in Section , V must then be of the form

$$
V(x) = \begin{cases} a_2(x^*)^{d_1}\ ,\ x\in\mathcal{C} \\ \frac{(\theta-\alpha K_1)K_1X_\tau}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right) = \frac{K(\theta-\alpha K)}{r-\mu}\ ,\ x\in\mathcal{S} \end{cases} \tag{3.7}
$$

and must verify value matching and smooth pasting conditions, leading to the system

$$
\begin{cases} a_2(x^*)^{d_1} = \frac{(\theta-\alpha K_1)K_1X_\tau}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right) = \frac{K(\theta-\alpha K)}{r-\mu} \\ a_2d_1(x^*)^{d_1-1} = \frac{K(\theta-\alpha K)}{r-\mu}, \end{cases} \tag{3.8}
$$

By the system above and according to **??**, the threshold x^* and the coefficient a_2 are respectively given by

$$x^* = \frac{d_1}{d_1-1}\frac{\delta K_1 + \frac{1-\alpha K_0}{r}K_0}{\theta-\alpha K_1}\frac{r-\mu}{K_1} \tag{3.9}$$
$$a_2 = \left(\delta K_1 + \frac{1-\alpha K_0}{r}K_0\right)\frac{(x^*)^{-d_1}}{d_1-1}.$$

Plugging the results stated above on the expression of F (3.5), it leads to

$$F(x) = \frac{\pi_0}{r} + \begin{cases} a_2(x^*)^{d_1} \ , \ x \in \mathcal{C} \\ \frac{(\theta-\alpha K_1)K_1 X_\tau}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right) = \frac{K(\theta-\alpha K)}{r-\mu} \ , \ x \in \mathcal{S}, \end{cases} \tag{3.10}$$

where the continuation and stopping regions associated to this problem are given by

$$\mathcal{C} = \left\{x \in \mathbf{R}^+ : x \leq x^* = \frac{d_1}{d_1-1}\frac{\delta K_1 + \frac{1-\alpha K_0}{r}K_0}{\theta-\alpha K_1}\frac{r-\mu}{K_1}\right\}$$

$$\mathcal{S} = \mathbf{R}^+ \setminus \mathcal{C} = \left\{x \in \mathbf{R}^+ : x > x^* = \frac{d_1}{d_1-1}\frac{\delta K_1 + \frac{1-\alpha K_0}{r}K_0}{\theta-\alpha K_1}\frac{r-\mu}{K_1}\right\}.$$

and the stopping time referred in (3.5), corresponding the optimal time when the investment decision should be made, is formally defined as $\tau = \inf\{t \geq 0 : X(t) \in \mathcal{S}\} = \inf\{t \geq 0 : X(t) \geq x^*\}$.

3.2.2 Capacity Optimization Model

3.3 Comparative Statics

Since the obtained results cannot reduce to each other, as done in the previous section, we will treat each case separately, starting with the simplest one derived in 4.2.1. Comparisons between the benchmark and capacity optimization models will be made on subsection 3.4.2.

3.3.1 Benchmark Model

Proposition: Decision threshold x_B^* increases with δ, σ and α only when $\theta < \frac{K_1}{K_0^2}(K_0 + K_1 r\delta)$, decreases with θ and α only when $\theta > \frac{K_1}{K_0^2}(K_0 + K_1 r\delta)$ and does not have a monotonic behaviour with $K_0,\ K_1,\ r$.

Proof:

Regarding the formula obtained for x_B^*, we immediately conclude that the decision threshold increases with δ and decreases with θ.

Regarding σ, we observe that

$$\frac{\partial x_B^*(\sigma)}{\partial \sigma} = \frac{(r-\mu)\left(\delta K_1 + \frac{K_0(1-\alpha K_0)}{r}\right)}{(d_1-1)K_1(\theta-\alpha K_1)}\left(\frac{2\mu}{\sigma^3} + \frac{\frac{4\mu\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)}{\sigma^3} - \frac{4r}{\sigma^3}}{2\sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}}\right)\left(1 - \frac{d_1}{(d_1^2-1}\right).$$

Taking into account our initial assumptions about r, μ and profits associated to the old and the new product, it follows that the leftmost expression is always positive. Since $d_1^2 - d_1 + 1 > 0 \ \ \forall d_1$, the rightmost expression is also positive. Now we need to analyse the expression in between. We chec

Problemas em mostrar quando é que a segunda derivada é positiva.

Regarding K_0, we observe that

$$\frac{\partial x_B^*(K_0)}{\partial K_0} = \frac{d_1(r-\mu)}{r(d_1-1)K_1(\theta-\alpha K_1)}(1-2\alpha K_0) = \begin{cases} > 0 & \text{for } K_0 < \frac{1}{2\alpha} \\ < 0 & \text{for } K_0 > \frac{1}{2\alpha} \end{cases}.$$

since the expression represented in fraction is always positive.

Regarding K_1, we obtain that

$$\frac{\partial x_B^*(K_1)}{\partial K_1} = \frac{d_1(r-\mu)}{(d_1-1)K_1(\theta-\alpha K_1)}\left(\frac{\alpha(\frac{K_0(1-\alpha K_0)}{r} + K_1\delta)}{\theta-\alpha K_1} - \frac{\frac{K_0(1-\alpha K_0)}{r} + K_1\delta)}{K_1} + \delta\right)$$

The leftmost expression is always positive. Thus we only need to evaluate the sign of the expression next to it. Manipulating mentioned expression, taking into account that the capacity level cannot be negative as well as the price function given by $\pi_0 = K_0(1-\alpha K_0)$, it follows that

$$\frac{\partial x_B^*(K_1)}{\partial K_1} = \begin{cases} > 0 & \text{for } K_1 > \frac{-\pi_0+\sqrt{\alpha\pi_0(\pi_0+r\delta\theta)}}{r\alpha\delta} \\ < 0 & \text{for } K_1 \in \left[0, \frac{-\pi_0+\sqrt{\alpha\pi_0(\pi_0+r\delta\theta)}}{r\alpha\delta}\right] \end{cases},$$

from which we obtain that x_B^* has no monotonic behaviour with K_1.

Regarding parameter α, we obtain that

$$\frac{\partial x_B^*(\alpha)}{\partial \alpha} = \frac{d_1(r-\mu)}{(d_1-1)(\theta-\alpha K_1)}\left(\frac{\frac{K_0(1-\alpha K_0)}{r} + \delta K_1}{\theta-\alpha K_1} - \frac{K_0^2}{rK_1}\right),$$

where the leftmost expression is always positive. Simplifying the expression in the biggest brackets to the same denominator, we obtain that

$$\frac{\partial x_B^*(\alpha)}{\partial \alpha} = \begin{cases} > 0 & \text{for } \theta < \frac{K_0K_1+K_1^2 r\delta}{K_0^2} \\ < 0 & \text{for } \theta > \frac{K_0K_1+K_1^2 r\delta}{K_0^2}. \end{cases}$$

Note that the sign of the partial derivative does not depend on α.

Regarding parameter r we obtained complex derivates, from which we weren't able to deduce any analytical results. However, as it will be showed right after this proof, using Mathematica we obtained that x_B^* behaves in a non-monotonic way with it.

□

We weren't able to deduce any analytical result regarding the drift value μ. However, after many numerical experiments, we observed that x_B^* decreases with μ, as it's showed on Figure 3.2.

3.3.2 Capacity Optimization Model

Proposition: Decision threshold x_C^* increases with δ, decreases asymptotically with θ and do not have a monotonic behaviour with μ, r, α, K_0.

Proof:

For the sake of simplicity, we will denote, in this proof, $\phi := \sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} > 0$ and $\psi := 4d_1^2\pi_0(\delta\theta r + \pi_0) + \delta^2\theta^2 r^2 > 0$. Recall also that $\pi_0 > 0$ stands for the profit function associated to the new product.

Regarding δ, we obtain that

$$\frac{\partial x_B^*(\delta)}{\partial \delta} = \frac{(r-\mu)\left(\frac{4d_1^2\theta r\pi_0 + 2\delta\theta^2 r^2}{2\sqrt{\psi}} + d_1\theta r\right)}{(d_1-1)\theta^2 r} > 0$$

from which the result holds.

Regarding θ, we obtain that

$$\frac{\partial x_B^*(\theta)}{\partial \theta} = \frac{\theta(r-\mu)\left(\frac{4\delta d_1^2 r\omega + 2\delta^2\theta r^2}{2\sqrt{\psi}} + \delta d_1 r\right) - 2(r-\mu)\left(d_1(\delta\theta r + 2\omega) + \sqrt{\psi}\right)}{(d_1-1)\theta^3 r}.$$

Manipulating the numerator, it simplifies to

$$\frac{\delta d_1 r\left(2d_1(1-4\theta)\omega + (1-2\theta)\sqrt{\psi}\right) - 4d_1\omega\left(2d_1\omega + \sqrt{\psi}\right) + \delta^2\theta^2\left(-r^2\right)}{\sqrt{\psi}}$$

which is always negative for innovation levels $\theta > 1/2$. COMPLETAR: ARRANJAR SUPOSIÇÃO MAIS FORTE

Since we assume that innovation levels have no upper limit, we evaluated them asymptotically. Denoting $\theta_A := \frac{(r-\mu)}{(d_1-1)r}\left(\sqrt{\delta^2 r^2} + \frac{\delta r\left(\sigma^2(\phi+1)-2\mu\right)}{2\sigma^2}\right) > 0$ we obtain that x_C^* decreases on order of $\frac{\theta_A}{\theta}$, that is,

$$x_C^*(\theta) \sim \frac{\theta_A}{\theta} \;\Leftrightarrow\; \lim_{\theta\to\infty} \frac{x_C^*(\theta)}{\frac{\theta_A}{\theta}} = 1.$$

Regarding parameters μ, r α and K_0, we obtained complex derivates, from which we couldn't deduce any analytical result. However, as it will be showed hereunder, x_C^* behaves in a non-monotonic way with all of them.

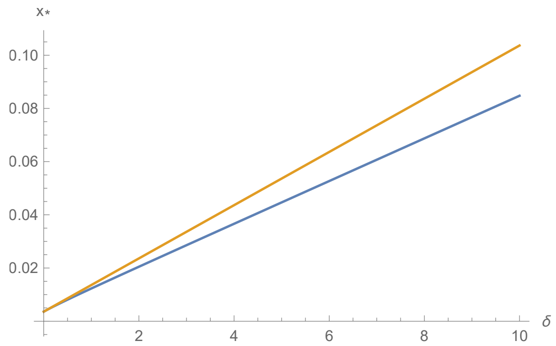
$\square$

Although we couldn't obtain any analytical (strong) evidence, after different experiments done using *Mathematica* and its function Manipulate, we obtained that decision thresholds x_B^* and x_C^* increase with volatility σ. An example is showed on Figure 3.1.

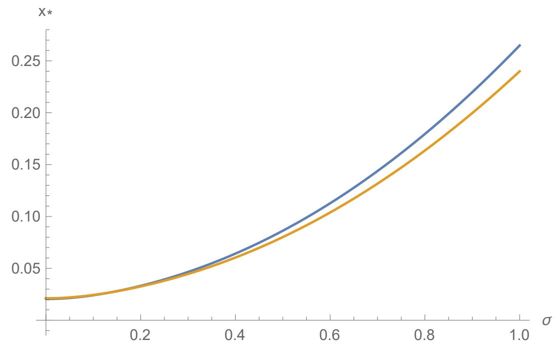
To illustrate results above mentioned we performed some numerical illustrations, using software *Mathematica* and its function Manipulate. However here are only able to present static plots - we leave to the interested ones, to see the results achieved with Manipulate.

Unless it is written the opposite, following values were considered:

- $\mu = 0.03$
- $\sigma = 0.005$
- $r = 0.05$
- $\delta = 2$
- $\alpha = 0.01$
- $\theta = 10$
- $K_0 = 90$
- $K_1 = 100$



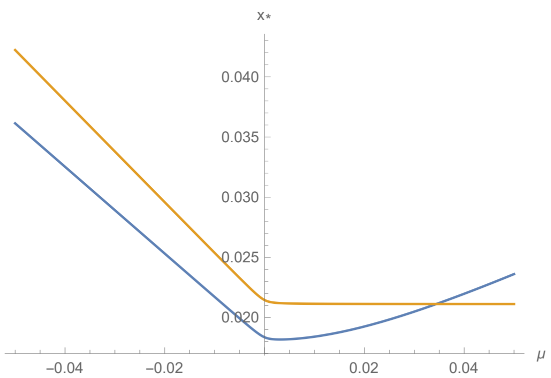
(a) $\delta \in (0, 1)$



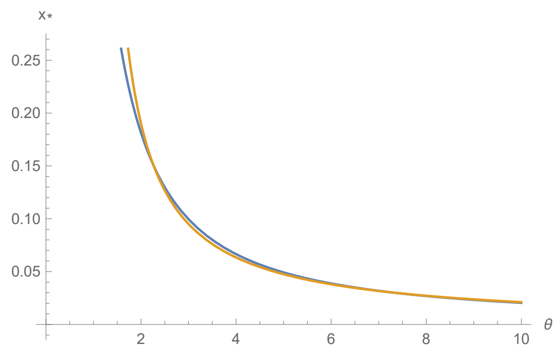
(b) $\sigma \in (0, 1)$

Figure 3.1: Threshold value with respect to the benchmark model (orange) and the capacity optimized model (blue) and parameters with which x_B^* increases.

On Figure 3.1, we obtained similar results to the ones on Section 2. Both threshold values increase with sensibility parameter δ and volatility σ. The first one is justified by the fact that a higher δ means a bigger investment, and thus it will only be made, if there's also a huge demand of the product. The second one is justified by the huge uncertainty of the demand. Since it has a high variance, the demand has a great amplitude of values, which delays the investment decision, only made when the demand reaches a high level. This is in accordance to what is described in [4].



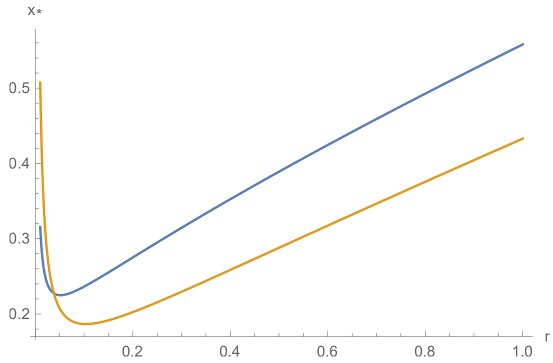
(a) $\mu \in (-r, r)$



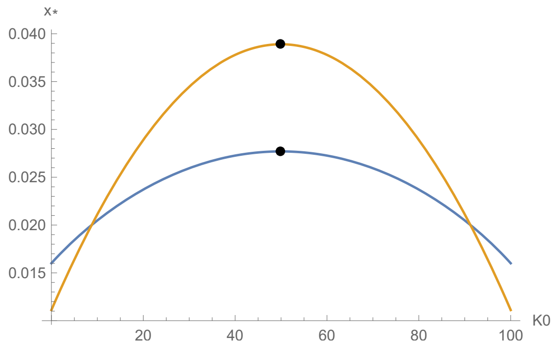
(b) $\theta \in (1, 10)$

Figure 3.2: Threshold value with respect to the benchmark model (orange) and the capacity optimized model (blue) and parameter with which x_B^* decreases.

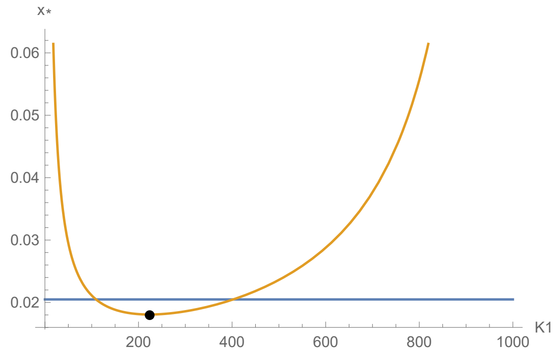
On Figure 3.2 we see that similarly to what happened on Section 2, both threshold levels decrease with innovation level. Although the threshold level associated to the benchmark model decreases with μ in what seems to be a linear way for negative values of μ and almost negligible for positive values of μ, the same doesn't happen to the threshold level associated to the capacity optimization model. This last one, seems to increase for positive values of μ. The same happened considering other values for parameters K_0, α, δ, θ. Regarding σ, we obtained that when increasing the volatility, the value of μ associated wityh the stationary point happens for values of μ greater than 0.



(a) $r \in (\mu, 1)$, $\mu = 0.01$ and $\sigma = 0.9$



(b) $K_0 \in (0, \frac{1}{\alpha} = 100)$



(c) $K_1 \in (0, \frac{\theta}{\alpha} = 1000)$

Figure 3.3: Threshold value with respect to the benchmark model (orange) and the capacity optimized model (blue) and parameters with which x_B^* has a non-monotonic behaviour.

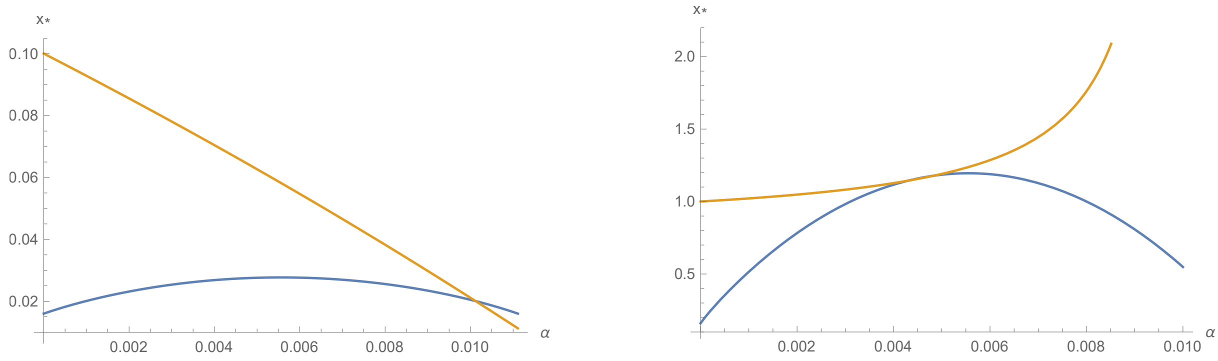
On Figure 3.3, we observe that both threshold level behave in a non-monotonic way with r and K_0.

Interestingly, a maximum value is observed when the capacity level of the *older* product is exactly equal to $\frac{1}{2\alpha}$, whose value comes from the expression $\alpha K_0 \pi_0 = \alpha K_0(1-\alpha K_0)$ included on both expressions of x_B^* (**??**) and x_C^* (**??**).

Regarding parameter K_1, its value doesn't affect threshold x_C^*, since it takes into account the optimal capacity K_C^*. However, when it comes to the threshold x_B^* we have that it achieves a minimum value at $K_1 = \frac{\pi_0 + \sqrt{\alpha \pi_0 (\pi_0 + \delta \theta r)}}{r \alpha \delta}$, as it's represented on the bottom plot.

On Figure 3.4 it's represented the behaviour of x_B^* with α, as written on Proposition (???). Considering fixed values mentioned in REFERIRRRR, we obtain a θ-threshold equal to $\frac{K_1}{K_0^2}(K_0 + K_1 r \delta) = 1.23457$. Testing for innovation levels smaller and greater than the mentioned threshold, we verify what was deduced: that x_C^* behaves differently with α for certain levels of innovation.

Note that on most of the plots you have that the threshold x_C^* has an associated capacity level $(K^*(x_C^*))$ greater than the one considered $(K_1 = 100)$, resulting in values x_B^* smaller than x_C^*, contrarily



(a) $\alpha \in (0,1)\theta = 10 > \frac{K_1}{K_0^2}(K_0 + K_1 r\delta)$ (b) $\alpha \in (0,1)$ and $\theta = 1 < \frac{K_1}{K_0^2}(K_0 + K_1 r\delta)$

Figure 3.4: Threshold value with respect to the benchmark model (orange) and the capacity optimized model (blue) and sensibility parameter α.

to what happened on the previous section.

Now we analyse optimal capacity level K_C^*, that is given by evaluating K^* as defined in (5.1) on demand level x_C^*, as done in [3] and in the previous section. Its expression is given by

$$K_C^* = \frac{\theta}{2\alpha} - \frac{\delta(d_1-1)\theta^2 r}{2\alpha\left(\sqrt{4\alpha d_1^2\pi_0(\alpha\pi_0+\delta\theta r)+\delta^2\theta^2 r^2}+d_1(2\alpha\pi_0+\delta\theta r)\right)}.$$

Proposition: Optimal capacity level K_C^* increases asymptotically with θ and does not have a monotonic behaviour with K_0. Also,

Proof:

Regarding innovation level θ, assuming that it has no upper limit, it's possible to evaluate its behaviour asymptotically. Denoting $\theta_K := \frac{\sigma^2\left(\sqrt{\delta^2 r^2}+\delta r\right)}{\alpha\left(2\sigma^2\sqrt{\delta^2 r^2}+\delta r(\sigma^2(\phi+1)-2\mu)\right)} > 0$, we obtain that K_C^* increases on order of $\theta_K\theta$, that is,

$$K_C^*(\theta) \sim \theta_K\theta \;\Leftrightarrow\; \lim_{\theta\to\infty}\frac{K_C^*}{\theta_K\theta} = 1$$

The non monotonic behaviour of K_C^* with K_0 will be showed hereunder in the obtained plots.

□

Although it wasn't possible to derive any (strong) analytical solution about te behaviour of the other parameters, numerically we obtain robust results. By manipulating each parameter, using command Manipulate, we obtained no different behaviours from the ones showed hereunder.

The results obtained regarding parameters μ, σ, r, α and θ were similar to the ones obtain for the optimal capacity level on the previous section. Since K_C^* depends on value x_C^*, it is expected to observe similiar behaviours regarding the studied parameters.

Starting with the capacity level of the *old* product K_0 and respective considered parameters, on Figure 3.5, we obtained that the highest optimal capacity level K_C^* happens for $K_0 = \frac{1}{2\alpha}$. This is motivated by the results obtained for x_C^*, as seen on Figure **??**,which also reaches its highest value at $\frac{1}{2\alpha}$.

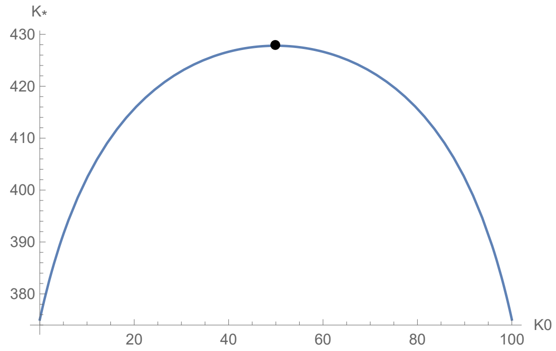
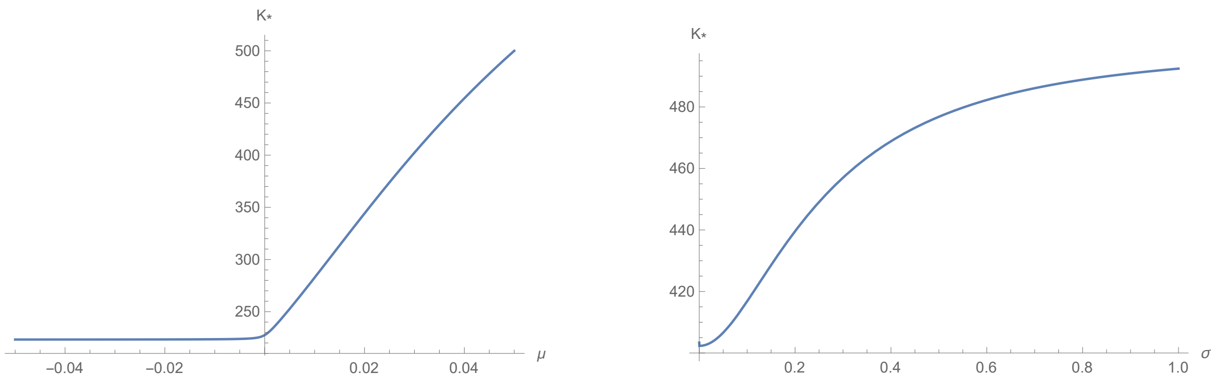


Figure 3.5: Optimal capacity regarding the threshold value x_C^* considering capacity levels $K_0 \in [0, 100)$ and its highest values at $\frac{1}{2\alpha} = 50$.

On Figure 3.6, we obtain that K_C^* increases with both drift and volatility, as it happened in the previous section. Note that again that, contrary to what happens for positive drift values, the growth of K_C^* with μ is barely noticeable for negative values of μ.



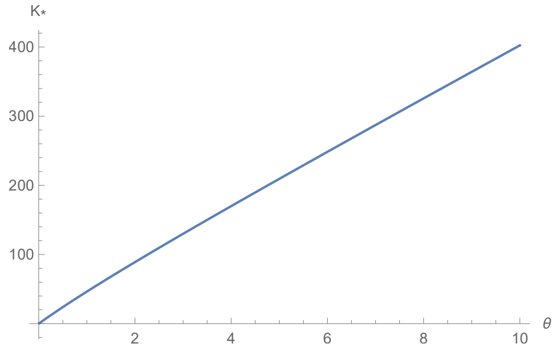
(a) $\mu \in (-r, r)$

(b) $\sigma \in (0, 1)$

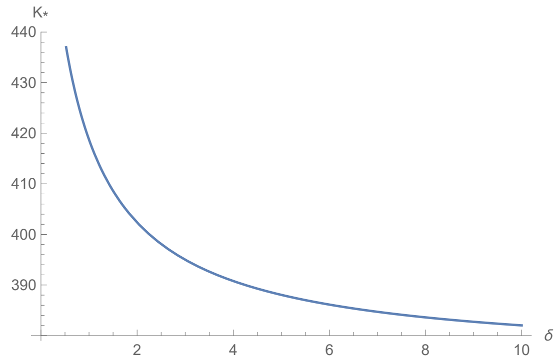
Figure 3.6: Optimal capacity regarding the threshold value x_C^*.

Regarding innovation level θ and sensibility parameter δ, we have on Figure 3.7 that K_C^* increases with them as well. Note that asymptotically, K_C^* seems to increase linearly with θ, as previously deduced.

Regarding discount rate r and sensibility parameter α, we have on Figure 3.8 that K_C^* decreases with them, as happened in the previous section.

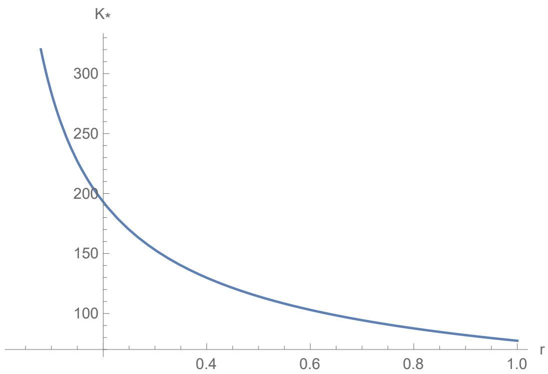


(a) $\theta \in (1, 10)$

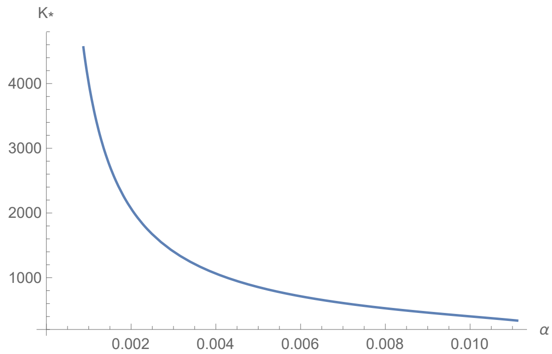


(b) $\delta \in (0, 10)$

Figure 3.7: Optimal capacity regarding the threshold value x_C^*.



(a) $r \in (\mu, 1)$



(b) $\alpha \in (0, 1)$

Figure 3.8: Optimal capacity regarding the threshold value x_C^*.

Chapter 4

Adding a new product when already producing one w/ cannibalisation (Firm is already active before investing)

Insert your chapter material here...

4.1 Introduction

[Introdução de artigos já publicados em semelhante contexto]

We increase the complexity of our problem by considering that the firm has three different states of production.

In the first one, we consider that the firm only produces a (very) stable product, that does not depend on the demand observe. We will call it *old* product. Its instanteneous profit function is given by π_0, which, as defined before, takes the value of

$$\pi_0 = (1 - \alpha K_0)K_0.$$

In the second state, we consider the firm produces simultaneously the *old* product and a new one. We will call it *new* product. This *new* product is inserted in the market since after the innovation process as achieved a certain innovation level, *a priori* defined. Since it's based on a new technology and it's a product that is not know by people, we will consider that its profit depends on the demand level.

The instantaneous profit functions associated to the *old* and the *new* product are given respectively by

$$\pi_0^A(X_t) = (1 - \alpha K_0 - \eta K_1 X_t)K_0,$$

$$\pi_1^A(X_t) = (\theta - \alpha K_1 - \eta K_0 X_t) K_1.$$

We need to consider a cannibalisation (or horizontal differentiation) parameter η that corresponds to the crossed effect between the *old* and the *new* product. As we consider both products to be interacting in the same market, η represents the penalty that the quantity associated to a product will influence the price of the other. We consider here that this influence is the same for both products, so we can have a unique cannibalisation parameter η, however this cannot be greater than the sensibility parameter α ($\eta < \alpha$). Otherwise, the quantity of the other product would have a larger effect on the product price than the quantity of the product itself.

The instantaneous profit function associated to this second state of production is denoted by π_A and it is such that

$$\pi_A(X_t) = \pi_0^A(X_t) + \pi_1^A(X_t) = \pi_0 + \pi_1(X_t) - 2\eta K_0 K_1 X_t = (1 - \alpha K_0)K_0 + (\theta - \alpha K_1)K_1 X_t - 2\eta K_0 K_1 X_t.$$

In the third (and last state) we consider that the firm abandons the *old* product and starts producing only the *new* product, which is not considered to be a stable product. The instantaneous profit function associated is given by

$$\pi_1(X_t) = (\theta - \alpha K_1) K_1 X_t.$$

Therefore we want to find two optimal times to make different (but maybe simultaneous) decisions. We want to find the best time τ_1 to go from the first to the second state, that is, to invest in the *new* product and start producing, simultaneously, the *old* and the *new* product. And we also want to find the best time τ_2 to go from the second to the third state, that is, to replace the production of the *old* product by the *new* one. Note that $\tau_2 \geq \tau_1$ are both stopping times adapted to the natural filtration of the demand process $\{X_t,\ t \geq 0\}$ and there is no chance on return the production of the *old* product, once the firm had abandoned it in τ_2. Thus these are irreversible choices.

4.2 Stopping Problem

4.2.1 Benchmark Model

As made in previous sections, we still consider that at the moment we adapt the new product, we need to pay δK_1 related to sunk costs and that at the precise moment we adapt the new product, we are able to produce it. Once again, we set the instant $t = 0$ to be the instant immediately after the desired innovation level happens.

Taking into account the different profits associated to each state of production, as described before, our the optimal stopping problem may be formulated as findng the value function F such that

$$F(x) = \sup_{\tau_1} \mathbb{E}^{X_0 = x} \left[\int_0^{\tau_1} \pi_0 e^{-rs} ds + \sup_{\tau_2} \mathbb{E}^{X_{\tau_1} = x_{\tau_1}} \left[\int_{\tau_1}^{\tau_2} \pi_A(X_s) e^{-rs} ds + \int_{\tau_1}^{\infty} \pi_1(X_s) e^{-rs} ds - e^{-r\tau_1} \delta K_1 \right] \right] \tag{4.1}$$

Manipulating (4.1) by changing the region of integration of the first integral and solving it we obtain

$$\begin{aligned} F(x) &= \sup_{\tau_1} \mathbb{E}^{X_0=x} \left[\int_0^{\infty} \pi_0 e^{-rs} ds + \sup_{\tau_2} \mathbb{E}^{X_{\tau_1}=x_{\tau_1}} \left[\int_{\tau_1}^{\tau_2} (\pi_A(X_s) - \pi_0)\, e^{-rs} ds + \int_{\tau_1}^{\infty} (\pi_1(X_s) - \pi_0)\, e^{-rs} ds - e^{-r\tau_1} \delta K_1 \right] \right] \\ &= \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{X_0=x} \left[\sup_{\tau_2} \mathbb{E}^{X_{\tau_1}=x_{\tau_1}} \left[\int_{\tau_1}^{\tau_2} (\pi_A(X_s) - \pi_0)\, e^{-rs} ds + \int_{\tau_2}^{\infty} (\pi_1(X_s) - \pi_0)\, e^{-rs} ds \right] - e^{-r\tau_1} \delta K_1 \right] \end{aligned} \tag{4.2}$$

Changing integration variables of both integrals on optimization problem related with τ_2 we obtain

$$F(x) = \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{X_0=x} \left[e^{-r\tau_1} \left(\sup_{\tau_2} \mathbb{E}^{X_{\tau_1}=x_{\tau_1}} \left[\int_0^{\tau_2-\tau_1} (\pi_A(X_{\tau_1+s}) - \pi_0)\, e^{-rs} ds + \int_{\tau_2-\tau_1}^{\infty} (\pi_1(X_{\tau_1+s}) - \pi_0)\, e^{-rs} ds \right] - \delta K_1 \right) \right] \tag{4.3}$$

where since the term $e^{-r\tau_1}$ does not depend on τ_2, it can be put in evidence as made above.

Considering F_2 to be the value function associated to the optimal stopping problem related to τ_2 we have that its expression is given by

$$F_2(x) = \sup_{\tau_2} \mathbb{E}^{X_{\tau_1}=x_{\tau_1}} \left[\int_0^{\tau_2-\tau_1} (\pi_A(X_{\tau_1+s}) - \pi_0)\, e^{-rs} ds + \int_{\tau_2-\tau_1}^{\infty} (\pi_1(X_{\tau_1+s}) - \pi_0)\, e^{-rs} ds \right], \tag{4.4}$$

from which follows that our optimal stopping problem, initially given by (4.1), is now given by

$$F(x) = \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{X_0=x} \left[e^{-r\tau_1} (F_2(X_{\tau_1}) - \delta K_1) \right]. \tag{4.5}$$

We have two different optimal stopping problems that we should solve starting on the *latest* stopping time, τ_2, by considering that we know what happened until the instant that the firm invests, τ_1. In order to do that, let $\{Y_t,\ t \geq 0\}$ be the stochastic process that represents the demand level after occuring the investment at τ_1 (being that its initial time) and which evolves stochastically accordingly to a GBM with the same drift μ and volatility σ as $\{X_t,\ t \geq 0\}$, that is $\{Y_t,\ t \geq 0\} = \{X_{\tau_1+t},\ t \geq 0\}$. Note that it's initial value is the same as observed at the instant τ_1, that is $Y_0 = X_{\tau_1}$.

Consider as well τ to be the stopping time, adapted to the natural filtration of the process $\{Y_t,\ t \geq 0\}$, that represents the optimal time for which the firm should make the replacement of the *old* product by the *new* one, after having invested at time τ_1. This means that if $\tau = 0$, then the old product is replaced by the new one at the precise instant when the investment happens τ_1. Note that τ is also adapted to the natural filtration of $\{X_{\tau+t},\ t \geq 0\}$ and that $\tau_2 = \tau_1 + \tau$. Thus, by knowing τ_1 and finding τ, we can calculate τ_2.

Therefore, problem F_2, as written in (4.4), is equivalent to

$$F_2(x_{\tau_1}) = \sup_{\tau} \mathbb{E}^{Y_0=x_{\tau_1}} \left[\int_0^{\tau} (\pi_A(Y_s) - \pi_0)\, e^{-rs} ds + \int_{\tau}^{\infty} (\pi_1(Y_s) - \pi_0)\, e^{-rs} ds \right], \tag{4.6}$$

meaning that from time 0 to time τ the firm is producing both products and that from time τ on the firm only produces the *new* product, where the instante 0 corresponds to the instant when the firm decides to invest, τ_1.

Fortunately we can simplify the notation of (4.6). Since the Strong Markov property states that after a stopping time, the future path of the GBM depends only on the value at the stopping time (knowing this one), it follows that

$$\{(Y_t|Y_0 = x_{\tau_1}),\ t \geq 0\} = \{(X_t|X_{\tau_1} = x_{\tau_1}),\ t \geq \tau_1\} \stackrel{d}{=} \{(X_t, t \geq 0|X_0 = x_{\tau_1}),\ t \geq 0\}.$$

Therefore we can keep the same notation as before and thus from (4.6) follows

$$F_2(x_{\tau_1}) = \sup_{\tau} \mathbb{E}^{X_0=x_{\tau_1}} \left[\int_0^{\tau} (\pi_A(X_s) - \pi_0)\, e^{-rs} ds + \int_{\tau}^{\infty} (\pi_1(X_s) - \pi_0)\, e^{-rs} ds \right]. \tag{4.7}$$

Using the fact that the expectation is a linear operator, we treat the expectation of the rightmost integral separately of (4.7), that is

$$\mathbb{E}^{X_0=x_{\tau_1}} \left[\int_{\tau}^{\infty} (\pi_1(X_s) - \pi_0)\, e^{-rs} ds \right] = e^{-r\tau} \mathbb{E}^{X_0=x_{\tau_1}} \left[\int_0^{\infty} (\pi_1(X_{\tau+s}) - \pi_0)\, e^{-rs} ds \right]. \tag{4.8}$$

Conditioning to the stopping time τ and using Tower Rule we obtain from (4.8)

$$e^{-r\tau} \mathbb{E}^{X_0=x_{\tau_1}} \left[\mathbb{E}^{\tau=t} \left[\int_0^{\infty} (\pi_1(X_{t+s}) - \pi_0)\, e^{-rs} ds \right] \right]. \tag{4.9}$$

We interchange the integral with expectation using Fubini's theorem and the fact that $r - \mu > 0$, obtaining

$$e^{-r\tau} \mathbb{E}^{X_0=x_{\tau_1}} \left[\int_0^{\infty} \mathbb{E}^{\tau=t} \left[\pi_1(X_{t+s}) e^{-rs} \right] ds - \frac{\pi_0}{r} \right] = e^{-r\tau} \mathbb{E}^{X_0=x_{\tau_1}} \left[(\theta - \alpha K_1) K_1 \int_0^{\infty} \mathbb{E}^{\tau=t} \left[X_{t+s} e^{-rs} \right] ds - \frac{\pi_0}{r} \right], \tag{4.10}$$

where the term $(\theta - \alpha K_1)K_1$ is constant over time.

We focus now on the expected value conditional to the stopping time τ above. Since the demand level evolves accordingly to a GBM and, by knowing the instant τ, we know its value at time τ, it follows

$$\begin{aligned} \mathbb{E}^{X_\tau=x_\tau} \left[X_{\tau+s} e^{-rs} \right] &= \mathbb{E}^{X_\tau=x_\tau} \left[x_\tau e^{\left(\mu - \frac{\sigma^2}{2} - r\right)(\tau+s-\tau) + \sigma(W_{\tau+s} - W_\tau)} \right] \\ &= \mathbb{E}^{X_\tau=x_\tau} \left[x_\tau e^{\left(\mu - \frac{\sigma^2}{2} - r\right)s + \sigma W_s} \right] \\ &= x_\tau e^{(\mu - r)s}, \end{aligned} \tag{4.11}$$

where in the second equality we used the fact that Brownian Motion $\{W_t,\ t \geq 0\}$ has stationary increments, that is

$$W_{\tau+s} - W_\tau \stackrel{d}{=} W_{\tau+s-\tau} - W_0 \stackrel{d}{=} W_s \sim \mathcal{N}(0, s).$$

Plugging (4.11) in (4.10), we obtain

$$e^{-r\tau} \mathbb{E}^{X_0=x_{\tau_1}} \left[(\theta - \alpha K_1) K_1 \int_0^{\infty} x_{\tau_1} e^{(\mu-r)s} ds - \frac{\pi_0}{r} \right] = e^{-r\tau} \mathbb{E}^{X_0=x_{\tau_1}} \left[\frac{(\theta - \alpha K_1)K_1}{r - \mu} x_\tau - \frac{\pi_0}{r} \right]. \tag{4.12}$$

Therefore we have found the terminal function associated to the optimal stopping problem F_2. Denoting it by h_2, it's given by

$$h_2(x) = \frac{(\theta - \alpha K_1)K_1}{r-\mu}x - \frac{\pi_0}{r}.$$

Accordingly to (4.7), we may also denote g_2 as the running cost function associated to this problem, that is given by

$$g_2(x) = \pi_A(x) - \pi_0.$$

Thus, plugging expression of running and terminal functions on (4.7), we have that F_2 as initially written in (4.4), is equivalent to

$$\begin{aligned} F_2(x) &= \sup_{\tau} \mathbb{E}^{X_0=x}\left[\int_0^{\tau} g(X_s)e^{-rs}ds + e^{-r\tau}h(X_\tau)\right] && (4.13)\\ &= \sup_{\tau} \mathbb{E}^{X_0=x}\left[\int_0^{\tau} \left(\pi_0^A(X_s) + \pi_1^A(X_s) - \pi_0\right)e^{-rs}ds + e^{-r\tau}\left(\frac{(\theta-\alpha K_1)K_1}{r-\mu}X_\tau - \frac{\pi_0}{r}\right)\right]. && (4.14) \end{aligned}$$

4.2.2 Capacity Optimization Model

4.3 Comparative Statics

Since the obtained results cannot reduce to each other, as done in the previous section, we will treat each case separately, starting with the simplest one derived in 4.2.1. Comparisons between the benchmark and capacity optimization models will be made on subsection 3.4.2.

4.3.1 Benchmark Model

Chapter 5

Maximization Problem: waiting for innovation level desired to be reached)

Insert your chapter material here...

5.1 Introduction

Situação do problema. Trabalhos já realizados e de maneira os extendemos.

Some overview of the underlying theory about the topic...

5.2 One jump

Having calculated the expression of optimized value function F^*, our goal now is to calculate the optimal level of investment R, taking into account that it influences the waiting time for the breakthrough to happen. In order to do it, we need to maximize the expected value of the optimized value function.

Notice that the distribution of the waiting time is given by an Exponential with parameter $\lambda(R)$. Also, since we are interested to find the optimal level of investment made now, one may not forget to discount the optimized value function. Thus we obtain that our optimal level of investment leads to a value function given by

$$
\begin{aligned}
V(x) &= \max_R E\left[e^{-rt}F^*(x) - R\right] \\
&= \max_R \left\{ \int_0^\infty \lambda(R) e^{-\lambda(R)t} e^{-rt} F^*(x) dt - R \right\} \\
&= \max_R \left\{ \int_0^\infty \lambda(R) e^{-\lambda(R)t} e^{-rt} \sup_\tau E^{X_0=x}\left[\max_K e^{-r\tau} h(X_\tau, K) 1_{\{\tau<\infty\}}\right] dt - R \right\} \\
&= \max_R \left\{ \int_0^\infty \lambda(R) e^{-(\lambda(R)+r)t} \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x} - R \right\}
\end{aligned}
$$

Since $F^*(x)$ does not depend on investment R nor time t, and it only depends on the drift μ, the volatility σ of GBM, discount rate r, innovation level after the jump θ and sensibility parameters α and δ and noticing that $R^\gamma + r > 0$, since we have no negative investment, we obtain

$$
V(X) = \max_R \left\{ \frac{R^\gamma}{R^\gamma + r} F^*(x) - R \right\}.
$$

The optimal value of the investment to make, R^*, is found by analyzing the first and the second partial derivatives of the expression to maximize.

$$
\begin{aligned}
\frac{\partial}{\partial R}\left(\frac{R^\gamma}{R^\gamma + r}F^*(x) - R\right) &= \frac{\gamma R^{\gamma-1}F^*(x)r - (R^\gamma + r)^2}{(R^\gamma + r)^2} \\
\frac{\partial^2}{\partial R^2}\left(\frac{R^\gamma}{R^\gamma + r}F^*(x) - R\right) &= -\frac{F^*(x)\gamma r R^{-2+\gamma}(r - \gamma r + (1+\gamma)R^\gamma)}{(R^\gamma + r)^3}
\end{aligned}
$$

• **Case I:** $\gamma = 1 \Leftrightarrow \lambda(R) = R$

Analysing the roots of the first partial derivative in order to parameter R, we get a quadratic polynomial for which we can calculate obtain the expression of the zeros, obtaining

$$
R = -\sqrt{F^*(X)r} - r \ \vee \ R = \sqrt{F^*(X)r} - r
$$

The first solution is not admissible, since it's not possible to have negative investment. Thus, we only have to check if $R = \sqrt{F(x)r} - r$ corresponds to a minimum or a maximum. To do that, we analyse the second partial derivative as stated above.

$$
\frac{\partial^2}{\partial R^2}\left(\frac{R}{R + r}F(x) - R\right) = -\frac{2rF^*(x)}{(R+r)^3} < 0
$$

where we used the fact that $R,\ r,\ F(x) > 0$.

We get then that, in order to have

$$
\arg\max_R V(x) = \sqrt{F^*(x)r} - r
$$

we need to verify

$$
F(x) > r \tag{5.1}
$$

- **Case II:** $\gamma \in (0, 1)$

Since the second derivative is negative for considered values of γ and $r, R > 0$, any positive root of the first partial derivative in order to R accomplishes our goal of maximizing the expression above.

When analyzing the roots of the first derivative we obtain the following polynomial,

$$R^{\gamma-1}F^*(x)r - R^{2\gamma} - 2rR^{\gamma} - r^2 = 0,$$

which unfortunately, we are not able to solve analytically for every value $\gamma \in (0, 1)$. We considered some numerical illustrations for values $\gamma \in (0, 1)$ presented in Section 2.3.3.

5.3 Multiple jumps

Erlang.

5.4 Comparative Statics

As stated in section **??**, we are not able to solve analytically the polynomial presented in (**??**) for every value $\gamma \in (0, 1)$. However we considered some numerical approximations, using software *Mathematica* and its function Solve. For the effect, we considered

- $r = 0.05$;
- $F(X) = 10$;
- $\gamma \in (0, 1]$ incremented by 0.05.

Following results are implemented on script RVopt.nb.

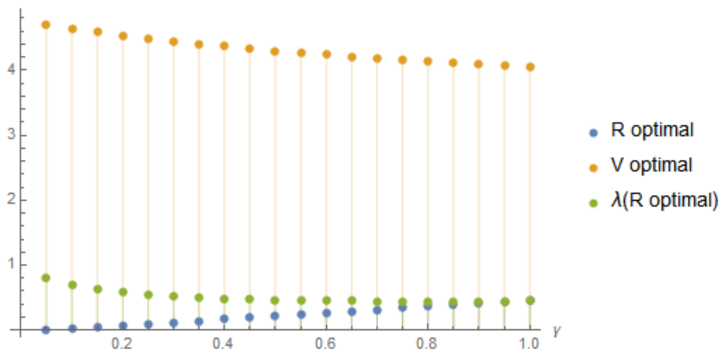


Figure 5.1: Optimal values of R and $V(X)$ for fixed values of F and r

We obtain that, although the optimal investment R grows with exponent γ, the value function for the respective optimal R decreases with exponent γ (and in a different way from the decreasing of $\lambda(R)$). We get that the smaller value of $V(X)$ is approximately 4.05, corresponding to the optimal investment level $R = 1$ and $\lambda(R) = 0.45$ and the biggest value of $V(X)$ is approximately 4.69, corresponding to the optimal investment level $R = 0.014$ and $\lambda(R) = 0.05$

Chapter 6

Add und replace, woop-woop

Mudar o título que não está nada correcto.

Adicionar um introdução bonitinha, citando diversas fontes.

6.1 Numerical Model

Description of the numerical implementation of the models explained in Chapter 5...

6.2 Verification and Validation

Basic test cases to compare the implemented model against other numerical tools (verification) and experimental data (validation)...

Chapter 7

Conclusions

Insert your chapter material here...

7.1 Achievements

The major achievements of the present work...

7.2 Future Work

A few ideas for future work...

Bibliography

[1] R. Pimentel. *Jump Processes in Finance.* PhD thesis, Instituto Superior Técnico, 2018.

[2] V. Hagspiel, K. J. M. Huisman, P. M. Kort, and C. Nunes. How to escape a declining market: Capacity investment or exit? *European Journal of Operational Research*, 254:40–50, 2016.

[3] K. J. M. Huisman and P. M. Kort. Strategic capacity investment under uncertainty. *Tilburg: Operations research*, 2013-003:40–50, 2013.

[4] A. Dixit and R. S. Pindyck. *Investment Under Uncertainty.* Princeton University Press, 1^{st} edition, 1994. ISBN:978-0691034102.

[5] B. Øksendal. *Stochastic Differential Equations: An Introduction with Applications.* Springer, 6^{nd} edition, 2014. ISBN: 978-3540047582.

[6] R. S. Pindyck. Irreversible investment, capacity choice, and the value of the firm. *The American Economic Review*, 78:969–985, 1998.

[7] Y. Farzin, K. J. M. Huisman, and P. M. Kort. Optimal timing of technology adoption. *Journal of Economic Dynamics & Control*, 22:779–799, 1998.

[8] R. McDonald and D. Siegel. The value of waiting to invest. *The Quarterly Journal of Economics*, 101(4):707–728, 1986.

[9] A. Dixit. Entry and exit decisions under uncertainty. *Journal of Political Economy*, 97(3):620–638, 1989.

Appendix A

Vector calculus

In case an appendix if deemed necessary, the document cannot exceed a total of 100 pages...

Some definitions and vector identities are listed in the section below.

A.1 Vector identities

$$\nabla \times (\nabla \phi) = 0 \tag{A.1}$$

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0 \tag{A.2}$$

Appendix B

Technical Datasheets

It is possible to add PDF files to the document, such as technical sheets of some equipment used in the work.

B.1 Some Datasheet

BENEFITS

Maximum Light Capture
SunPower's all-back contact cell design moves gridlines to the back of the cell, leaving the entire front surface exposed to sunlight, enabling up to 10% more sunlight capture than conventional cells.

Superior Temperature Performance
Due to lower temperature coefficients and lower normal cell operating temperatures, our cells generate more energy at higher temperatures compared to standard c-Si solar cells.

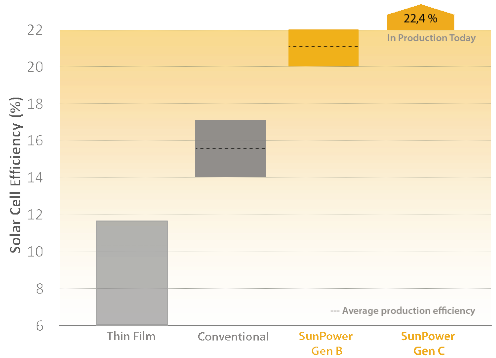
No Light-Induced Degradation
SunPower n-type solar cells don't lose 3% of their initial power once exposed to sunlight as they are not subject to light-induced degradation like conventional p-type c-Si cells.

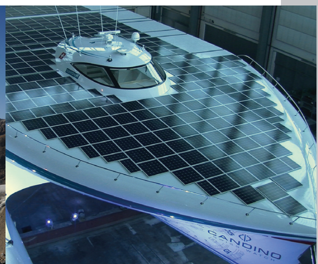
Broad Spectral Response
SunPower cells capture more light from the blue and infrared parts of the spectrum, enabling higher performance in overcast and low-light conditions.

Broad Range Of Application
SunPower cells provide reliable performance in a broad range of applications for years to come.

The SunPower™ C60 solar cell with proprietary Maxeon™ cell technology delivers today's highest efficiency and performance. The anti-reflective coating and the reduced voltage-temperature coefficients provide outstanding energy delivery per peak power watt. Our innovative all-back contact design moves gridlines to the back of the cell, which not only generates more power, but also presents a more attractive cell design compared to conventional cells.

SunPower's High Efficiency Advantage





C60 SOLAR CELL

SUNPOWER™

C60 SOLAR CELL
MONO CRYSTALLINE SILICON

Electrical Characteristics of Typical Cell at Standard Test Conditions (STC)

STC: 1000W/m², AM 1.5g and cell temp 25°C

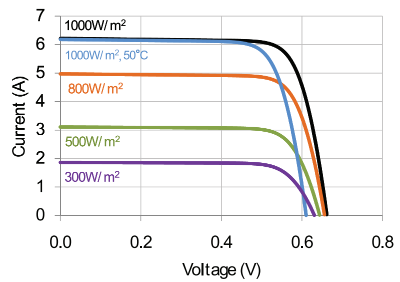
Bin	Pmpp (Wp)	Eff. (%)	Vmpp (V)	Impp (A)	Voc (V)	Isc (A)
G	3.34	21.8	0.574	5.83	0.682	6.24
H	3.38	22.1	0.577	5.87	0.684	6.26
I	3.40	22.3	0.581	5.90	0.686	6.27
J	3.42	22.5	0.582	5.93	0.687	6.28

All Electrical Characteristics parameters are nominal
Unlaminated Cell Temperature Coefficients
Voltage: -1.8 mV / °C   Power: -0.32% / °C

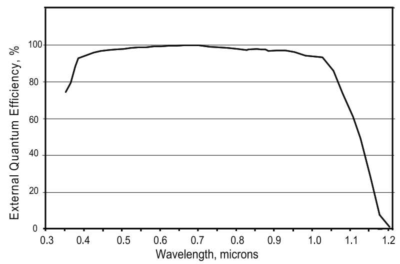
Positive Electrical Ground

Modules and systems produced using these cells must be configured as "positive ground systems".

TYPICAL I-V CURVE



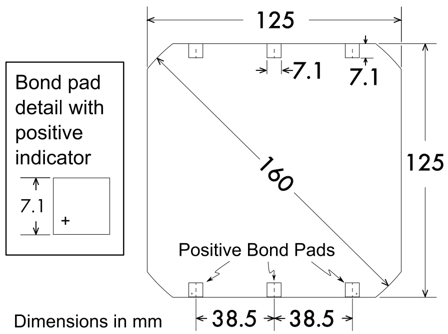
SPECTRAL RESPONSE



Physical Characteristics

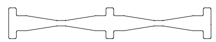
Construction:	All back contact
Dimensions:	125mm x 125mm (nominal)
Thickness:	165μm ± 40μm
Diameter:	160mm (nominal)

Cell and Bond Pad Dimensions



Bond pad area dimensions are 7.1mm x 7.1mm
Positive pole bond pad side has "+" indicator on leftmost and rightmost bond pads.

Interconnect Tab and Process Recommendations



Tin plated copper interconnect. Compatible with lead free process.

Packaging

Cells are packed in boxes of 1,200 each; grouped in shrink-wrapped stacks of 150 with interleaving. Twelve boxes are packed in a water-resistant "Master Carton" containing 14,400 cells suitable for air transport.

Interconnect tabs are packaged in boxes of 1,200 each.

About SunPower

SunPower designs, manufactures, and delivers high-performance solar electric technology worldwide. Our high-efficiency solar cells generate up to 50 percent more power than conventional solar cells. Our high-performance solar panels, roof tiles, and trackers deliver significantly more energy than competing systems.