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Investment Policies in Competitive Products

Further Results and Extensions

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Resumo

Nesta tese, derivamos a política de investimento óptima relativa ao lançamento de um novo e mais cativante produto. Além disso, assume-se que a empresa em causa está a produzir e a vender já um produto, mas a mesma empresa está já pronta para lançar o novo produto, que vai competir com o original (que estava já lançado no mercado), causando uma diminuição das vendas do mesmo. O objectivo é derivar o nível ideal da procura que justifica o lançamento deste novo produto, tomando em consideração o custo que está envolvido na implementação do mesmo ('sunk cost'), a quota de mercado de cada produto e outros custos e decisões envolvidas, usando uma abordagem de opções reais.

Numa tese passada, a de Nuno Calaim (2011), o autor deriva a política de investimento óptima no contexto das opções reais, assumindo que o processo de incerteza (relacionado com a procura associada aos produtos) segue um Movimento Browniano Geométrico, e que o preço, quantidade e procura estão relacionados por uma função de procura linear.

Aqui, vamos também assumir uma função de procura iso-elástica com factores multiplicativos. Para este caso, derivamos a politica de investimento óptima, e analizamos os valores numéricos obtidos. Comparamos ainda estes novos resultados com os já existentes, procurando uma espécie de robustez neste contexto das políticas de investimento óptimas.

Além disso, propomos outra função objectivo para a política óptima, uma vez que na tese de Nuno Calaim (2011), o autor maximiza a função que consiste nos lucros obtidos sem ter em consideração os custos associados de produção e, assim sendo, a introdução destes mesmos custos é outra inovação que se tentou neste projecto.

Key-Words: Programação Dinâmica, Problema de Paragem Óptima, Principio da Optimalidade de Bellman.

Abstract

In this thesis, we derive the optimal investment policy regarding the launching of a new and more appealing product. Moreover, we assume that a company is producing and selling an established product, but the company is ready to put a new product into the market, which will compete with the original one, causing a breakdown in the sales of it. The goal is to derive the demand level that justifies the launching of this new product, taking into consideration the investment cost (sunk cost), the market share of each product and other costs and decisions involved, using a real options approach.

In a previous thesis, by Nuno Calaim (2011), the author derives the optimal policy in a real options context, assuming that the uncertainty process (related with the demand) follows a Geometric Brownian Motion, and that price, quantity and demand are related by a linear demand function.

Here, we also assume an iso-elastic demand function with multiplicative factors. For this case, we derive the optimal investment policy, and analyze numerical values. We also compare these new results with the existing ones, seeking for a kind of robustness in the optimal investment policy.

Furthermore, we propose another goal in the optimal policy, as in the already mentioned thesis from Nuno Calaim (2011), the author maximizes the expected revenue counting on the profits but without taking into account the production costs associated, and thus their introduction was another innovation we tried on this project.

Key-Words: Dynamic Programming, Optimal Stopping Problem, Bellman Principle of Optimality.

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Chapter 1

Introduction

1.1 Motivation and Examples

On June 27, 2007, *Apple Inc.* launched the *iPhone*, a high-end mobile phone, which was described by *Time Magazine* as the 'Invention of the Year 2007. On July 11, 2008, the same company launched the *iPhone 3G*, an evolution of the previous one. The original *iPhone* was devalued but *Apple*'s profit did not decrease, as more consumers were willing to purchase the new smartphone.

Since then, and at an approximate annual rate, two other improvements to the smartphone were introduced: *iPhone 3GS* and *iPhone 4*.

The subject of this thesis is to answer questions such as: What is the best time for *Apple* to launch *iPhone 5*? When should an older product stop being produced given the existence of the new one?

In our context, we assume that a company has one established product and at some point an innovative one is available to be introduced. This new product is assumed to have two differences when compared to the old one: it is more valuable and it has the potential to attract other consumers.

There are several factors that influence the decision of the company, and the company wants to decide optimally in terms of the obtained revenue. Those factors include: investment costs, the random demand for the products, and also the negative effect that the new product will cause on the old one.

This question has already been tackled in [2], where the author proposed a first attempt to derive the optimal investment policy under some assumptions. Therefore, the present thesis is somehow a continuation of what has been done in [2], and, for this reason, the notation and terminology that are used are the same. Moreover, the analytical results obtained in [2] are explored in more detail and we propose several extensions after that (developed from sketch).

1.2 Real Options Approach to the Problem

Note that the investment problem is analyzed using two approaches, Net Present Value (NPV) and Dynamic Programming. One of the most interesting points of the study is, in fact, the comparison of the results that are obtained using one approach or the other, taking into consideration the meaning/procedure associated to each of them.

We start by introducing the NPV approach. The Net Present Value method states that an investment should be made when the present value of the cash flows generated by that investment exceeds the investment cost. The net present value method implicitly assumes that an investment is either reversible or, in case it is irreversible, that we have a now or never opportunity (without possibility of postponing the investment). In [1], the main reference for all this introduction as it is a pioneer book in terms of this field and, in particular, in terms of all this discussion around these two different approaches, the idea that most investments do not fulfill the assumptions of this approach is extensively discussed and supported. In the technology adoption framework, it is clear that any investment is irreversible. This means that, in order to use the NPV method, the investment should be reversible. However, in most cases, in this particular context, this is also not true.

To incorporate the irreversibility and the possibility to delay an investment, the real option theory was developed. In [1], a good introduction and also a good overview are presented. In the real option theory investment, we see opportunities as options. The company chooses whether to invest or not and, if it invests, the option is over. The fact that the decision can be postponed gives an extra associated value.

In fact, the type of problem we face is associated to an Optimal Stopping Problem, as it is possible to see in [1]. This is justified by the fact that we have a dynamic problem of a risk-neutral and maximizing firm that discounts against a rate (that in our problem will be assumed to be constant), r. Note that 'stopping' represents the decision to undertake the investment, whereas 'continuation' means postponing the investment decision. As a matter of fact, we have the opportunity to undertake a project whose value depends on a state variable that is, in our case, the demand process for our products. This variable evolves stochastically over time according to an Itō process.

Our demand process, hereby denoted by $\{\theta(t), t \geq 0\}$, does, in fact, behave according to an Itō process because it verifies the sufficient and necessary condition that, for $t \geq 0$:

$$\begin{aligned} d\theta(t) &= f(t, \theta(t))dt + g(t, \theta(t))dW(t) \\ \theta(0) &= \theta_0 \end{aligned}$$

where $dW(t)$ is distributed according to a normal distribution with mean 0 and variance dt, and functions $f(.,.)$ and $g(.,.)$ have some specific properties (see [8]).

Denoting the termination payoff when the firm decides to invest at state $\theta(t)$ by $\Omega(\theta(t))$, and denoting the profit flow received by the firm before the decision to invest is taken by $\pi(\theta(t))$, we are now able to define the value of the project before stopping. This value function is denoted by $F(\theta(t))$. The value of this function, given that the investment has not been done yet (the stopping has not occurred), is:

$$F(\theta(t)) = max\{\Omega(\theta(t)), \pi(\theta(t))dt + exp(-rdt)E\left[F(\theta(t+dt))|\theta(t)\right]\} \tag{1.1}$$

The first argument within the maximization operator, $\Omega(\theta(t))$, corresponds to the value of stopping at time t. The second argument corresponds to the value of continuation (not stopping) at time t, and acting optimally from time $t + dt$ onwards. This is called the Bellman principle of optimality. If we are in the continuation region (which means that we choose not to invest), then this implies that the second argument within the maximization operator is greater than the other one (otherwise we would decide to invest). This implies that our value function satisfies the well-known Bellman equation in this region:

$$rF(\theta(t)) = \pi(\theta(t)) + \lim_{dt \to 0} \frac{1}{dt} E\left[dF(\theta(t))\right] \tag{1.2}$$

Note that the expectation in Equation (1.2) can be calculated using Itō's lemma[1]. This lemma states that, given $\theta(t)$ that behaves according to an Itō process and given a function $G(t, \theta(t))$ that is once differentiable with respect to t and twice differentiable with respect to $\theta(t)$, then:

$$dG(t, \theta(t)) = \frac{\partial G(t, \theta(t))}{\partial t} dt + \frac{\partial G(t, \theta(t))}{\partial \theta(t)} \theta(t) + \frac{1}{2} \frac{\partial^2 G(t, \theta(t))}{\partial (\theta(t))^2} (\theta(t))^2 + o(dt) \tag{1.3}$$

It is important to mention that for an Itō process, $(\theta(t))^2$ is of order dt.

Note that, during this thesis, the value function is used in an equivalent form, to the one presented in Equation (1.1). The explanation for this equivalence can be found in [2]. In fact, the expression we use onwards is the following:

$$F(\theta(t)) = max\{\Omega(\theta(t)), \frac{1}{r}\left(\pi(\theta(t)) + \frac{E\left[dF(\theta(t))\right]}{dt}\right)\} \tag{1.4}$$

If we assume that the termination payoff increases with $\theta(t)$, which is something that seems logical in our context, and also that the profit flow is constant, then intuition suggests that there is a value θ^*, that we call trigger value, that defines a threshold such that deciding to invest is optimal if $\theta(t) > \theta^*$, and deciding to wait is optimal, otherwise.

Given the value function $F(\theta(t))$ that is the solution of Equation (1.2), the mentioned threshold θ^* is obtained using the value matching condition that has to be verified by it. This condition is given by:

$$F(\theta^*) = \Omega(\theta^*) \tag{1.5}$$

This condition represents the fact that the maximum between the continuation and termination values is the termination one, for the point θ^*, which is totally understandable,

[1] We do not discuss, for the time being, neither existence or uniqueness of the solution of Equation (1.2).

as this point represents the first value of the demand process for which we are supposed to invest.

Also, if the demand process can pass the threshold continuously, the smooth pasting condition must be verified by our threshold θ^*:

$$\frac{\partial F(\theta^*)}{\partial \theta(t)} = \frac{\partial \Omega(\theta^*)}{\partial \theta(t)} \tag{1.6}$$

This technical condition is well explained in [1] but the argument basically shows that if this condition does not hold, then the threshold has to be another value, and not θ^*.

A third condition boundary condition is needed in order to determine this threshold, as two more variables are involved. The condition that is used is the most logical one:

$$F(0) = 0 \tag{1.7}$$

This last condition represents the fact that if there is no demand, then the value function of the firm is 0. Finally, in order to obtain conditions under which this threshold is unique, we once again refer [1], where two sufficient conditions are given. In fact, these two conditions can be re-defined for our particular context. If a *GBM* is used as the source of uncertainty, then the second sufficient condition in [1], related to the positive persistence of uncertainty, holds. This is an immediate calculation taking into consideration that $\log(\theta(t))$ is normally distributed, with mean $(\alpha - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$; as for the first condition, given the fact that a *GBM* is used, then this condition can be written as: the expression in Equation (1.8) has to be decreasing in θ. The proof can be checked in [7]. As a consequence, the only condition that we need to guarantee in order to obtain the desired uniqueness of the threshold is:

$$\pi(\theta) - r\Omega(\theta) + \alpha\theta\Omega'(\theta) + \frac{1}{2}\sigma^2\theta^2\Omega''(\theta) \tag{1.8}$$

decreases with θ.

1.3 Outline of the Thesis

In the next chapter, Chapter 2, we present an extended interpretation of the results presented in [2]. This analysis is somehow new, as in [2] not much attention was given to such subject. Moreover, we are able to understand better the impact (both mathematically and economically) of the parameters that represent the relevant quantities.

After that, in the following chapters, the optimal policy of investment for some extensions of this model is derived from sketch. For those cases, the derivation of the optimal policy is followed by a numerical analysis.

In the first of these chapters, Chapter 3, we define a model where two demand processes are considered, one for each product. For this case, we are able to develop the expressions and obtain the trigger value for the *NPV* approach. Also, despite the fact that for this case it is not possible to come up with a final optimal policy using the Dynamic Programming approach, as some technical difficulties arise along the way, the expressions are developed as much as possible and are left in some general forms that are also interesting.

In the following chapter, Chapter 4, we assume the same model as before but with a change in the objective function, the function we want to maximize, introducing unitary costs of production for each product. For this case, we obtain the optimal policy under some specific assumptions and approximations.

After this, in Chapter 5, we assume a multiplicative restriction on the price, which is something that sounds more realistic than the linear one. There are several technical problems to be faced along the way (some of them could only be solved imposing some assumptions), and but all the derivation of the optimal policy is possible to do.

The last chapter, Chapter 6, consists of a global conclusion of the work that was produced, complemented by some possible future research that can be developed as a possible continuation of this project.

For all the models that are studied, something that is very interesting is to observe the way the two approaches (*NPV* and Dynamic Programming) that are used in order to make a decision 'agree', or not, for each case. The conclusion of the thesis includes a comparison between the two approaches.

Throughout the chapters, there are plenty of terms being used and, as it appears, we try to explain it in more detail. However, there are also many concepts that are always the same, which means they do not depend on the chapter (model) we are working with, and this 'core notation' can thus be introduced here. It is also convenient to introduce this same notation in this chapter as it allows a better understanding in the context of the description of the problem. This major notation we want to introduce consists of:

- $\{\theta(t), t \geq 0\}$ - Geometric Brownian Motion (*GBM*) that represents the demand for a certain product. It has drift α and volatility σ;

- $\{W(t), t \geq 0\}$ - Brownian Motion also associated to the demand, as the previous *GBM* is the solution of the Stochastic Differential Equation:

  $$d\theta(t) = \alpha\theta(t)dt + \sigma\theta(t)dW(t)$$

  where α represents the drift and σ represents the volatility. Note that a Brownian Motion (or a Wiener Process) is the continuous version of a standard random walk whose increments are modeled by a Normal distribution. More technically, we say that $W = \{W(t), t \geq 0\}$ is a Brownian Motion if it verifies: $W(0) = 0$ with probability 1; it has stationary and independent increments ($\forall t_1 < t_2 < t_3 < t_4$, $W(t_4) - W(t_3)$ is independent of $W(t_2) - W(t_1)$, and their distributions are normal with mean 0 and variances $t_4 - t_3$ and $t_2 - t_1$, respectively); $\forall t \geq 0$, W(t) follows a normal distribution with mean 0 and variance t; it has continuous paths.

- θ_0 - trigger value for the *NPV* approach. Note that θ_0 also denotes the initial demand of the process, but it has the specific meaning of trigger value when it appears in the final rule for the mentioned approach;

- θ^* - trigger value for the Dynamic Programming approach;

- t^* - denotes the optimal time to invest (for the Dynamic Programming approach). It corresponds to the first passage time of the GBM through θ^*, i.e. $t^* = inf\{t : \theta(t) \geq \theta^*\}$. This is also a random variable whose expected value has an explicit expression given by $\frac{ln\frac{\theta^*}{\theta_0}}{\alpha - \frac{\sigma^2}{2}}$. An important fact that is verified is that this random variable has a large variance and so our optimal policy will focus on the optimal demands (trigger values) and not on the optimal times of investment, as the expected value of t^* has little relevance, after all;

- $\Omega(\theta(t))$ - it is associated to the already mentioned Bellman principle of optimality, and it represents the termination payoff when the firm decides to invest at state $\theta(t)$;

- $\pi(\theta(t))$ - it is also associated to the same principle, and it denotes the profit flow received by the firm before the decision to invest;

- $F(\theta(t))$ - it is the last function that is associated to the Bellman principle, and it denotes the value of the project given that the investment has not been done yet. Its analytical expression is given by:

$$F(\theta(t)) = max\{\Omega(\theta(t)), \pi(\theta(t))dt + exp(-rdt)E\left[F(\theta(t+dt))|\theta(t)\right]\}$$

  where the first argument within the maximization operator corresponds to the value of stopping at time t. The second argument corresponds to the value of continuation from time t to time $t + dt$, and acting optimally from time $t + dt$ onwards.

- RevBef - value that is taken by the objective function (function we want to optimize), before investment, given the optimal quantities;

- RevAft - value that is taken by the objective function (function we want to optimize), after investment, given the optimal quantities;

- q_i^{opt} - optimal quantity for product i (i can be either 1 or 2);

- CashBef - expected discounted cash flow for the situation where we do not invest in the new product, in the context of the NPV approach;

- CashAft - expected discounted cash flow for the situation where we decide to invest in the new product, in the context of the NPV approach.

Chapter 2

Interpretation of Previous Results

Following [2], in this chapter we analyze in further detail the results derived in [2], using the same model and the same assumptions. This model is called, for the sake of simplicity, 'Basic Model', until the end of the thesis.

So let $\{p_1(t), t \geq 0\}$ and $\{q_1(t), t \geq 0\}$ denote, respectively, the price and the quantity of product 1, the product that we are initially producing, 'old product', both evolving with time. Assume that the company wants to maximize, at each time t, the following quantity:

$$p_1(t) \times q_1(t)$$

and that the second product is not in the market yet. We note that this quantity represents the profit obtained by the company with the products that it sells.

We note that the quantity, at each time t, $q_1(t)$, can be chosen by the company, whereas the price, $p_1(t)$, cannot be fixed on the value the company desires, as it depends on the demand (and on the quantity) according to the following linear law:

$$p_1(t) = \theta(t) - q_1(t)$$

where, as seen in Chapter 1, $\{\theta(t), t \geq 0\}$ follows a Geometric Brownian Motion.

Furthermore, we denote by $\{p_2(t), t \geq T\}$ and $\{q_2(t), t \geq T\}$ the processes that model the evolution of the price and the quantity, respectively, of product 2. We assume that as soon as product 2 (the innovative product) is launched in the market; the prices of both products follow the following (linear) relations:

$$p_1(t) = a\theta(t) - q_1(t) - \eta q_2(t) \tag{2.1}$$

$$p_2(t) = \beta\theta(t) - \gamma q_2(t) - \eta q_1(t) \tag{2.2}$$

In Equations (2.1) and (2.2), the parameters a, β, γ and η have the following interpretation:

- γ - value that is smaller than 1, as it represents the fact that the price that can be exercised for the new product is not as affected by the quantity of that same product that is produced as the price of the old product is affected by its own produced quantities. This is a technical detail used to impose in the model the idea that the new product is better;

- β - represents the fact that this new product has more demand than the old product used to have. This variable is worth more than 1, as a consequence. It also imposes in the model the idea that the new product is better than the other;
- a - it is introduced to represent in the model the fact that the demand for the old product will suffer a decrease from the moment the new product is introduced. This variable is between 0 and 1, as a consequence. This also imposes in the model that the new product is better.
- η - value that is even smaller than γ, and corresponds to the crossed effect, as the products are supposed to be interacting in the same market, and so the quantity associated to a product will have influence on the price of the other. We consider that the influence is the same for both products, and that is the reason why there is just one variable for this effect;

Moreover, we also use the following parameters:

- I - cost of introduction of the new product ('sunk cost');
- r - (constant) interest rate.

Furthermore, we also use the following parameters for the two extensions (cannibalization effect and implementation delay, respectively) already proposed in [2]:

- μ - variable related to a in the cannibalization effect, as it represents the fact the customers' appetite decreases as time goes by (given the existence of product 2 in the market), which implies a successive decrease on the demand for the old product with time. In this case, variable a is assumed to have an exponential decrease, given by:
  $$a(t) = \exp[-\mu \times (t - T)], t > T$$
  where T denotes the time at which product 2 is launched in the market;
- Δ - represents a delay on the real implementation of the product after the decision to invest is undertaken. In this case, the company can only predict, using the expected value, the value of the demand (which is a Geometric Brownian Motion) and maximize it counting on the fact that the product is introduced Δ time units after. Note that when both effects are taken into consideration (cannibalization and delay) to the initial model, the value of a is again the one to suffer changes according to:
  $$a(t) = \exp[-\mu \times (t - T - \Delta)], t > T$$

All quantities are taken to be non-negative because the company cannot produce negative quantities of a product, i.e. $q_i(t) \geq 0$, for $i \in \{1, 2\}$.

We denote, from here on, the time of investment by T. Note that from time T onwards, the objective of the company is to maximize:

$$p_1(t)q_1(t) + p_2(t)q_2(t)$$

Finding the optimal investment policy is equivalent to deriving a threshold θ^* for the *GBM* $\{\theta(t), t \geq 0\}$, such that the company invests in product 2 as soon as the value of the demand process is at least θ^*. Therefore, the company only needs to examine the evolution of the *GBM*, and to invest at time $t^* = \inf\{t : \theta(t) \geq \theta^*\}$.

Recall that the purpose of this chapter is to analyze the effect of some parameters in the implementation policy. We try to analyze this not only in view of the mathematical implications but also in economical terms. As the number of parameters is considerably large and some of them are not that interesting to be studied in our context, we consider a small subset of them. The most interesting ones, in economical terms, are the parameters of the *GBM*, α and σ, and also parameter a. As for the extensions, in the cannibalization effect, the behavior of the trigger value as a function of μ is the one that is studied; in the implementation delay effect, we study the behavior of the trigger value as a function of parameter Δ.

Taking into consideration the expressions obtained in [2], it is rather straightforward to understand the impact of some of the parameters on the optimal policy (in particular in the determination of θ_0 or θ^*). This can be done for the variables that are least interesting to study, just to check if our intuition is confirmed. For example, an increase on the constant interest rate leads to an increase on the value of θ^*. This means that if the constant interest rate is increased, the investment is postponed. The same happens with the sunk cost. If the cost we have to pay to implement the new product is increased, the investment is postponed.

Note that something very important is to ensure that the behaviors of each variable are the same independently of what values we fix for the other variables. This is done using the command *Manipulate* from *Mathematica*[1]. Note that this is not possible to illustrate here, except by fixing specific (different) values for each variable and saving the image to see that the behavior of the image is always the same, which is something we choose not to do. The best idea is just to put an illustrative *3D* plot that exemplifies this situation, as it shows how the behavior of θ^* is always the same as a function of each of the variables we study (drift and volatility of the process, in this case) independently of what value the other variable takes. Our intuition also makes us think that the behavior is not influenced by the values we fix for the other variables, as the arguments we use to justify each behavior are independent of the values taken by any other variable.

Given this, we fix the following values for each variable:

- $I = 1000$
- $\beta = 1.6$;
- $\sigma^2 = 0.0025$, $\alpha = 0.03$;
- $\eta = 0.3$;

[1] Author: Wolfram Research, Inc. Title: Mathematica Edition: Version 7.0 Publisher: Wolfram Research, Inc. Place of publication: Champaign, Illinois Date of publication: 2008

- $\gamma = 0.8$;
- $r = 0.1$;
- $\mu = 0.7$, $\Delta = 30$, $a = 0.8$. Note that this last value is chosen in order that $a\gamma - \beta\eta > 0$, as this condition is the most interesting one. For the other region, product 1 disappears from the market after investment and so the study of the behavior of the trigger value as a function of parameter a makes no sense. Also, the behavior as a function of the other variables of interest does not change.

We proceed with the analysis of such behaviors in the following sections, separating the analysis for the NPV and Dynamic Programming cases.

2.1 *NPV* Rule

This approach consists of taking a now-or-never decision, as we introduced before. For this case, the optimal demand to invest is given by the expression (see [2]):

$$\theta_0 = \sqrt{\frac{4I}{C(K-1)}}$$

where the value of K depends on the parameters of the system, as follows:

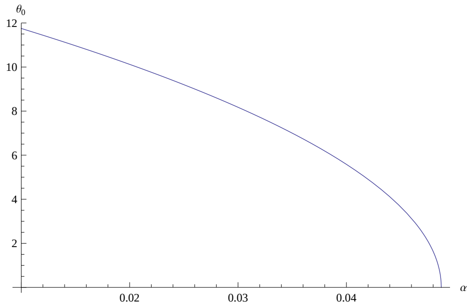
$$K = \begin{cases} \frac{a^2\gamma+\beta^2-2a\beta\eta}{\gamma-\eta^2} & : a\gamma - \beta\eta > 0 \\ \frac{\beta^2}{\gamma} & : a\gamma - \beta\eta \leq 0 \end{cases}$$

Moreover, the value of C is given by:

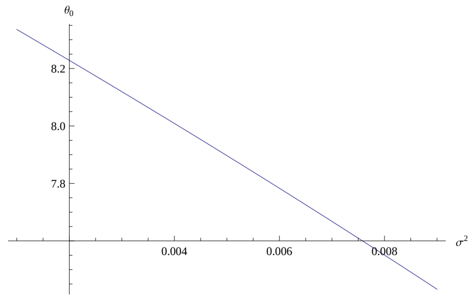
$$C = \frac{1}{r - 2\alpha - \sigma^2}$$

Recall that the most interesting case, and the one we study here, relates to the situation when $a\gamma - \beta\eta > 0$.

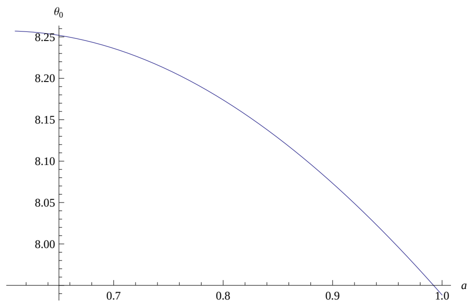
The plots that are obtained by varying each of the variables of interest, along with the corresponding interpretation, are presented below.



(a) θ_0 as a function of α (drift)



(b) θ_0 as a function of σ (volatility)



(c) θ_0 as a function of parameter a

Figure 2.1: θ_0 as a function of the variables of interest

From these plots, we are able to present the following comments:

- **Influence of the drift of the *GBM***: as expected, the bigger the drift α is, the smaller θ_0 becomes (the value of the demand that justifies the now-or-never decision to invest);

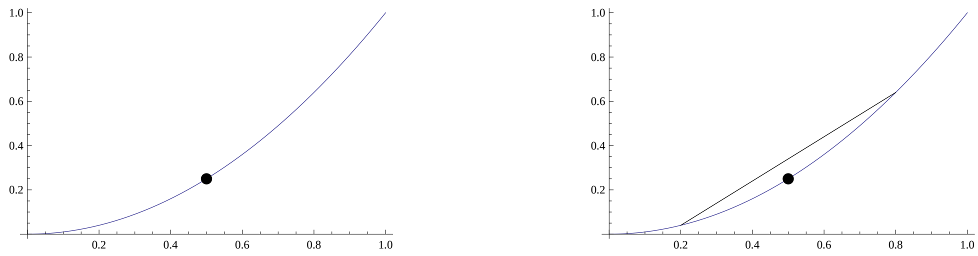
- **Influence of the volatility of the *GBM***: in this case, an increase in the volatility leads to a smaller value of θ_0. We remark that this is not exactly what we would expect intuitively, as an increase in σ^2 is associated to a higher risk. However, that intuition is not correct because in this approach, time is not taken into consideration, and so investing for smaller values of the demand process is not associated to investing earlier. In fact, the decision to invest, or not, is taken at time 0. Moreover, a proper justification for the increasing behavior is given next.

  As the relation $p(t) = \theta(t) - q(t)$ holds, then the objective function can be re-written as follows:
  $$p(t)q(t) = -p^2(t) + p(t)\theta(t)$$
  which reaches its maximum value when $p(t) = \frac{\theta(t)}{2}$, and the associated maximum revenue is worth $\frac{\theta^2}{4}$, a convex function of the demand level.

  At this stage, we recall Jensen's Inequality: *Let g be convex and suppose that X and $g(X)$ are integrable, $E[|X|] < \infty$ and $E[|g(X)|] < \infty$. Then, $g(E[X]) \leq E[g(X)]$.*

Therefore, the expected value of the revenue function increases with σ^2, leading to the conclusion, seen in Figure 2.1, that the trigger value decreases with this quantity.

The particular convex function we are working with is a quadratic one and so, to illustrate the previous conclusions, we provide an example using such functions.



(a) No volatility

(b) With volatility

Figure 2.2: Explanation of the behavior of the trigger value θ_0 as a function of the volatility

In Figure 2.2, we see that for a specific value of the price, if there is no volatility (first graphic), the profit is a specific value, that is represented by a spot on the first image. In the second graphic, we introduce some volatility and we obtain a new image with the straight line illustrating the Jensen's Inequality effect that shows that the profit is now a random variable with an expected value that is larger than the constant value, initially fixed. This is an illustration of something more general than the result we want to prove (for any convex function). This, in fact, illustrates the well-known, and already mentioned, Jensen's Inequality;

- **Influence of parameter $\boldsymbol{a}$**: the influence of this parameter is the expected one, i.e., when it increases, it becomes more appealing to invest for smaller values of the demand, as the presence of product 2 in the market has less effect on the demand for product 1. We remark that, although a is one of the parameters that we use to make product 2 more appealing than product 1 in the model, we also need to take into account parameters β and γ. In fact, if a tends to the value 1 (its maximum possible value), product 2 is still better than product 1.

In terms of conclusions for the *NPV* case, we get the interesting fact that when the volatility gets larger, we are supposed to invest for smaller values of the demand.

On the other hand, the influence of both the drift of the *GBM* and variable a on the trigger value is the expected one.

Before proceeding to the Dynamic Programming approach, we finally exemplify the condition that was obtained using the command *Manipulate*, which allows us to conclude that we can fix all the values of the other variables and vary just one at each time obtaining a general conclusion on the monotonicity. We obtain a *3D* plot of the behavior of the trigger value as a function of two different variables, where we clearly observe the same marginal behavior independently of what value the other variable takes.

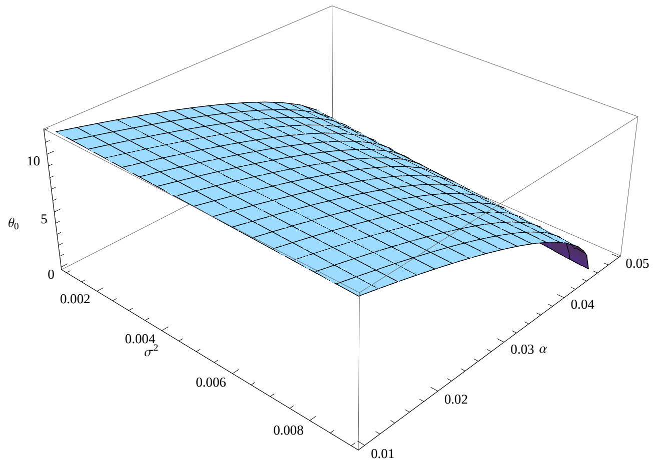


Figure 2.3: θ_0 as a function of both parameters of the *GBM*

Influence of both σ^2 and α on the trigger value θ_0 - we observe, in Figure 2.3, what was already stated before. The behavior of θ_0 is always the same with respect to any of the variables, for any value that we fix for the remaining one. This exemplifies why it is possible to fix every variable as we wish without losing generality in the conclusions we obtain.

2.2 Dynamic Programming

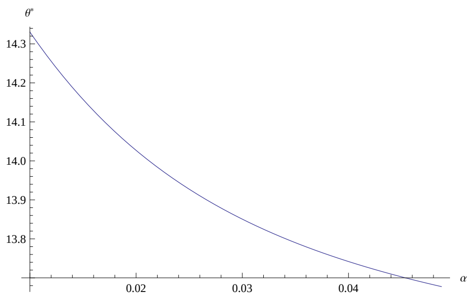
In this case, the expression that was obtained in [2] for the trigger value θ^* is given by:

$$\theta^* = \sqrt{\frac{4I\omega_2}{(\omega_2 - 2)C(K-1)}}$$

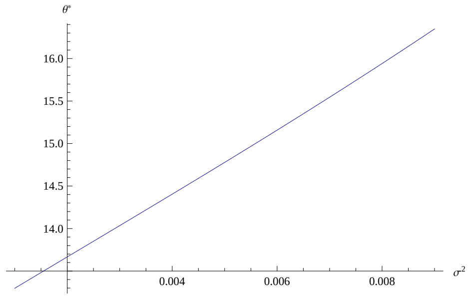
where

$$\omega_2 = \frac{\sigma^2 - 2\alpha + \sqrt{4\alpha^2 - 4(\alpha - 2r)\sigma^2 + \sigma^4}}{2\sigma^2} > 0$$

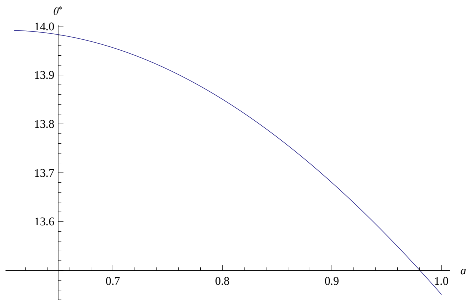
Proceeding the same way as in the previous section, we make the same type of study.



(a) θ^* as a function of α (drift)



(b) θ^* as a function of σ (volatility)



(c) θ^* as a function of parameter a

Figure 2.4: θ^* as a function of the variables of interest

- **Influence of the drift** α: although we have a decreasing behavior of the trigger value θ^*, the type of curve is now different than before. In the NPV case, the function was convex but now it is a concave one;

- **Influence of the volatility** σ: in absolute contrast to what was obtained in the NPV case, θ^* is now increasing with the volatility, in a linear way. Note, however, that this does not necessarily mean that the investment is postponed as the volatility increases. As a matter of fact, the increase on the trigger value that is associated to the increase on the volatility has to be balanced with the fact that the time when that higher value of the process is reached also becomes smaller in order to analyze the condition on time. In [3], it is shown that if we analyze the probability of making the investment in a certain time, that balance can go both ways, depending on the specific situation we are working on. As a result, and returning to the main point of this topic, *the fact that the value of the process for which we must invest becomes higher does not imply that we invest later (in terms of time)*, due to the fact that the variable in study is the volatility, which has influence on the values that are taken by the process at each time t;

- **Influence of parameter a**: the behavior that is obtained is just the same as for the NPV approach, with the same justification.

It is important to highlight the idea that *an increase on the value of the demand on which we decide to invest, when the volatility is the variable in study, does not always* *mean that the ideal time to invest decreases* (despite the fact that the relation holds in most cases). Remember that for the NPV case, time is not taken into discussion, due to being a now-or-never decision (i.e., the decision is taken at time $t = 0$).

We can conclude that: *considering both approaches (*NPV *and Dynamic Programming), the way the thresholds change with the parameters of the demand process (drift and volatility) is different. Regarding the other parameter that is studied, parameter a (influence of product* 2 *in the demand for product* 1*), the threshold for both approaches has the same type of behavior. Also, if variables I and r were studied graphically, the behavior would also be similar for the two approaches.*

2.3 Model with Cannibalization Effect

This is the first of the two extensions developed in [2], and it is based in the idea that the appetite of the customers for the old product decreases with time (exponentially) from the moment the new product is introduced. Note that, in the precious section, this loss of appetite was considered to be constant in time. This extension is something that sounds reasonable as, over the time, people become more and more aware of the fact that the new (better) product exists. In terms of the model, the change occurs in variable a, which is now assumed to have an exponential decrease:

$$a(t) = \exp[-\mu \times (t - T)], t > T$$

Recall that T is the time when the new product is implemented and μ is a variable that was already defined in the beginning of this chapter.

Note that, for this extension, the only approach that was done in [2] was Dynamic Programming and so, as this chapter is devoted to the analysis of the results obtained in [2], that is the only approach we study here. As a matter of fact, it is quite easy to understand why the other approach would not make sense to be used. In fact, in this extension, there is a clear involvement of time, and this is a notion that is not included on such approach, as explained before.

The trigger value that was obtained in [2] is:

$$\theta^* = \sqrt{\frac{4I\omega_2}{(L - C)(\omega_2 - 2)}}$$

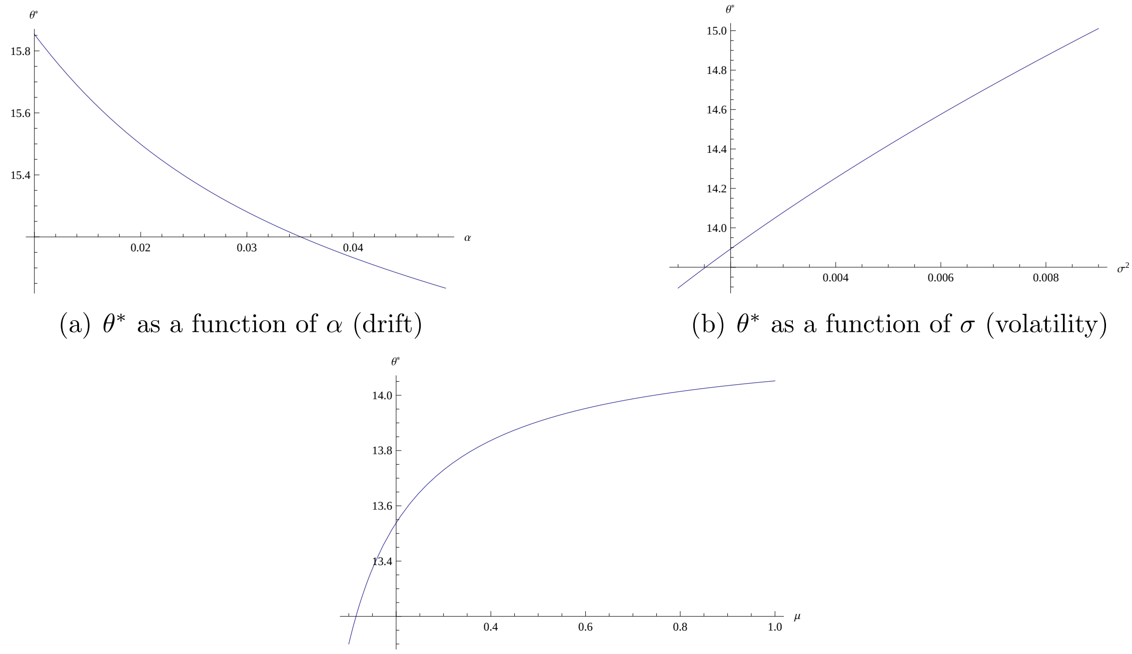
In this case, there is a dependence on a certain value L, for which there are three possible cases. However, there is only one case that is worth studying.

The expression of L for that case is:

$$L = \frac{1}{\gamma - \eta^2}\left(-2E\beta\eta + D\gamma + C\beta^2\left[1 - \frac{2DE(\eta\mu)^2}{\gamma}\left(\frac{\gamma}{\beta\eta}\right)^{-\frac{1}{C\mu}}\right]\right):$$

with $\gamma > \beta\eta$ and $\mu > 0$.

In this case, instead of analyzing the effect of parameter a, we study the effect of parameter μ, on the trigger value. The parameters of the *GBM* are also considered just to show how the behavior of the trigger value as a function of them is unchanged for this new extension:



(a) θ^* as a function of α (drift)

(b) θ^* as a function of σ (volatility)

(c) θ^* as a function of parameter μ

Figure 2.5: θ^* as a function of the variables of interest (cannibalization effect)

- **Influence of the parameters of the *GBM***: the result is just the same as for the model without cannibalization effect, which makes sense, as the intuition behind it is just the same, and there is no reason for a change on this behavior, as both this and the volatility appear exactly the same way in the expressions of this new adaptation of that previous model;

- **Influence of parameter μ**: the faster the loss of appetite for the old product is (the bigger the parameter μ is), the larger the threshold θ^* is. This is the expected result, as an increase in μ decreases the profit of the old product, after investment. If we compare the plot that is obtained here with the one corresponding to the behavior of the trigger value as a function of variable a, in Figure 2.4, we conclude that the relation between the behaviors is possible to understand, as the parameter in study now is exponentially related to parameter a, which justifies that this curve is more pronounced. It is also obvious, taking once again into consideration the relationship between parameters a and μ, why the monotonicity changes. Note that when μ increases too much, product 1 stops being produced, and so we tend to a stationary value, as the expression when we are in the situation where we only sell product 2 does not depend on this variable.

2.4 Model with Implementation Delay

In this section, the second extension presented in [2], introducing a delay on the implementation of the new product, is explored. In fact, in a real situation, when a company decides to invest in a product, it takes a while until it is actually available in the market. As a consequence, this effect sounds realistic. This extension has variable Δ associated with it, as explained before.

Note that the NPV approach is not suitable for this extension for the same reason that was seen for the cannibalization effect extension.

For this extension, the trigger value obtained in [2] is:

$$\theta^* = \sqrt{\frac{4I\omega_2 e^{(r-2\alpha)\Delta}}{(\omega_2 - 2)C(K-1)}}$$

Now, we have a similar situation as for the previous extension, but instead of a adding variable that represents the cannibalization effect to the model, we have a variable that adds a delay in the implementation of the product after the decision is taken. In this case, again like in the previous one, this delay does not change anything in the behavior of the trigger value as a function of the parameters of the GBM, which reduces us to the study of the influence of the variation of the new variable Δ on that same trigger value. Note that there is, like for the model without cannibalization or delay, a bound between producing both products or not, due to the dependence on variable K, but we only study, for the same reason as before, the case where both products are in the market.

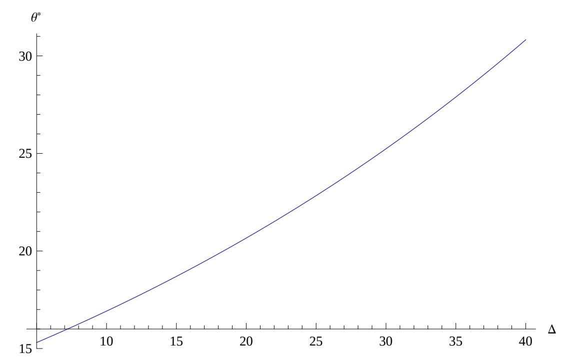


Figure 2.6: θ^* as a function of variable Δ

Influence of Δ: in Figure 2.6, we see that the bigger the delay in the implementation is (represented by variable Δ), the larger the trigger value becomes. This is intuitive as the increase on the delay makes the information we have about the real moment in which the product is implemented more and more restrict, which means more uncertainty and less confidence about the value of the Stochastic Process in the important moment of the release of the product. Note that this argument is similar to the one for the behavior, in the model without any extension, of the trigger value as a function of the volatility, given the same involved factor: uncertainty.

Parameter a was not mentioned during this section because the plot that we would obtain is just the same as for the first model, without extensions, just like for the parameters of the *GBM* (the same way that was observed for the cannibalization effect, and for the same reason).

2.5 Model with Both Extensions

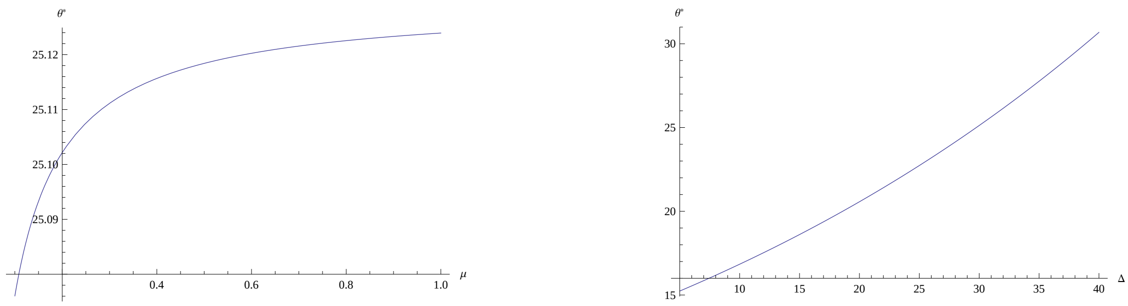
In [2], the combination of both extensions was also explored. The only important alteration in the model is that, as seen in [2], parameter a is now given by the expression:

$$a(t) = \exp[-\mu \times (t - T - \Delta)], t > T$$

The trigger value that is associated is (again for the Dynamic Programming approach), according to [2]:

$$\theta^* = \sqrt{\frac{4I\omega_2 e^{(r-2\alpha)\Delta}}{(L-C)(\omega_2 - 2)}}$$

For this model, we expect the behavior of the trigger value as a function of the parameters of the *GBM* to be the same as for the previous cases and so we do not do the corresponding graphics. We also expect a coherent behavior of the trigger value as a function of Δ and μ. This is something that is worth checking:



(a) θ^* as a function of parameter μ (b) θ^* as a function of parameter Δ

Figure 2.7: θ^* as a function of the variables of interest (both effects)

- **Influence of parameter** μ: the type of behavior of θ^* given μ is exactly the same as for the model without implementation delay;
- **Influence of parameter** Δ: the type of behavior of θ^* given Δ is exactly the same as for the model without cannibalization effect.

The conclusions for the previous graphics are according to what was expected.

2.6 Conclusions of the Previous Results

In most cases, the behavior of the plots is the expected one. Graphics are, in fact, a good way to verify some facts that would not be possible (or, at least, easy) to prove using technical arguments, but that we believe to be true.

The most interesting aspect is the behavior of the trigger value as a function of some variables when we are in the context of the NPV approach. For instance, the behavior of the trigger value as a function of the volatility is not intuitive and a mathematical argument has to be used to justify it. Also, the behavior of the trigger values as a function of the drift is similar in terms of monotonicity for both approaches, but in one case the function is concave and in the other it is convex, which is an interesting phenomenon also.

Recall that the conclusions of the studies for each variable are independent of the values that are fixed for the other variables not being studied, and so the interpretations we provide are robust, in this sense.

2.7 Final Remarks

2.7.1 Time to Stop Producing Product 1

We analyze, during this subsection, the behavior of a particular variable as a function of some others. This variable is the point in time when we stop producing both products and start producing only the new one, in the context of the first extension in [2], associated to the cannibalization effect. As there is a loss of appetite that is stronger and stronger as time goes by, there is the possibility to have a point in time, after the investment, when we stop producing product 1. This is not verified for the model without this effect of cannibalization as, in that case, we either stop producing product 1 from the moment we invest, or not, depending on the values of the parameters. This is a consequence of the fact that parameter a is, in that case, constant in time, and therefore, if at the launching time of product 2, there is no economical advantage of taking product 1 off the market, then the firm will never abandon it completely.

The expression that gives the optimal time we must wait until abandoning product 1 is given by, according to [2]:

$$t^* = \frac{\log(\frac{\gamma}{\beta \times \eta})}{\mu}$$

Thus, the effect of each involved parameter is straightforward:

- γ - if this variable increases, it means that the influence of the quantities that are produced on the price increases, for the new product, and we have less and less margin to play around with the price. This means that product 2 becomes worse and worse in terms of opportunity of investment, which means that the difference between the two products gets smaller and smaller (even knowing that the new one is always better than the other), and so product 1 stays longer in the market;

- β - in the opposite direction, when we increase this quantity, we are increasing the demand for the second (new) product, which makes it more profitable to stick to product 2 exclusively;
- η - this is a measure of the crossed influence of the products. The higher this value is, the more the quantity of one product influences (negatively) the price on the other. This implies that if we increase this variable, the crossed influence becomes more powerful, which is bad for the case where both products are in the market, as the price of the products will be negatively affected;
- μ - this variable has the same interpretation as β, as it also measures the relation between the demands for both products, and it also has the effect of increasing the difference between the two products. The bigger this value is, the faster is the loss of appetite for product 1.

2.7.2 Market Share

In this subsection, we study the market share associated to each product. By market share, we mean the percentage that is produced of each product (taking into consideration the optimal quantities). In order to study this quantity, we have to work in a setting where both products co-exist in the market.

Note that we work with the market share of product 1 but it is just the same as working with the one of product 2, as they have to sum the value 1 (the market share of one product uniquely determines the market share of the other). Also note that the variable that we use to study the behavior of this quantity is parameter a.

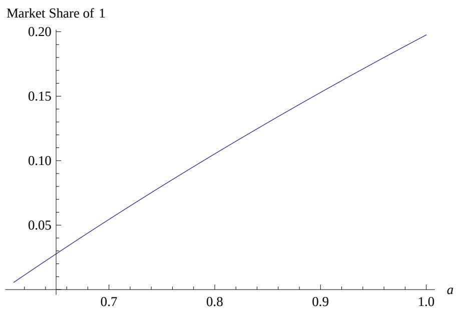


Figure 2.8: Market Share of product 1

In Figure 2.8, we observe that the market share tends to 0, as we approach the value for which we stop producing product 1. On the other hand, when a tends to 1, the market share hits its maximum value. This is due to the fact that the increase on the value of a represents an increase on the demand for the first product. The value is, however, always smaller than 0.5, which is justified by the fact that there are still variables β and γ in the model to impose that the new product is better than the old one. The first variable implies that there is more demand for the new product than for the old one; and the second one allows the new product to be sold for a higher price, if the same quantities of each product are produced. In fact, if the value for variable γ is 1, and a and β also take value 1, then the market share is the same (0.5) for each product, but in that case the investment does not makes sense as the new product is not better than the other one.

2.7.3 Generalization of the Law for the Price

We show, in this subsection, the results that come from a generalization in the model derived in [2] and explored during this chapter. We now assume that the law for the price, when product 1 is the only one in the market (before investment), is:

$$p_1(t) = d\theta(t) - q_1(t)$$

with $d < 1$. Therefore, the price depends on the demand by a fraction of it. When both products are in the market, after investment, the model for the price is defined the same way as before:

$$\begin{aligned} p_1(t) &= a\theta(t) - q_1(t) - \eta q_2(t) \\ p_2(t) &= \beta\theta(t) - \gamma q_2(t) - \eta q_1(t) \end{aligned}$$

For this situation, we obtain:

$$RevBef = \max_{q_1(t)}\{q_1(t)p_1(t)\} = \max_{q_1(t)}\{q_1(t)\left[d\theta(t) - q_1(t)\right]\} = \frac{d^2\theta^2(t)}{4}$$

The quantity that corresponds to this revenue is $q_1^{opt}(t) = \frac{d\theta(t)}{2}$.

Note that this follows trivially assuming a change of variable in which the $\theta(t)$ used throughout this chapter is replaced by $d\theta(t)$.

On the other hand, if the company is operating with both products, the optimal revenue and the optimal quantities are the ones derived in [2], as after investment everything is the same as for the initial model. That is obvious given that the only change in the model was done in the law for the price, *for the situation where the investment has not been done.*

Considering the expected discounted cash flow before and after investment, their comparison leads to the *NPV* Rule. This rule says that we should invest if and only if:

$$K\frac{\theta_0^2}{4}\frac{1}{r-\sigma^2-2\alpha} - I > \frac{d^2\theta_0^2}{4}\frac{1}{r-\sigma^2-2\alpha} \iff \theta_0 > \sqrt{\frac{(r-\sigma^2-2\alpha)4I}{(K-d^2)}}$$

We note that, as d increases, the trigger value increases, which is something that makes sense, as if the market for product 1 gets stronger (declines less), the less appealing will it be to implement product 2.

As for the Dynamic Programming approach, the following values for the functions of interest are obtained:

- $\pi(\theta) = \frac{d^2\theta^2}{4}$

- $\Omega(\theta) = K \times \frac{\theta^2}{4}\left(\frac{1}{r-\sigma^2-2\alpha}\right) - I$

In order to guarantee uniqueness of the threshold, θ^*, we check if Equation (1.8) is decreasing in θ, which, in view of the values taken by $\pi(\theta)$ and $\Omega(\theta)$, means that:

$$\frac{d^2\theta^2}{4} - rK \times \frac{\theta^2}{4}\left(\frac{1}{r-\sigma^2-2\alpha}\right) + rI + \alpha K \times \frac{\theta^2}{2}\left(\frac{1}{r-\sigma^2-2\alpha}\right) + \sigma^2 K \times \frac{\theta^2}{4}\left(\frac{1}{r-\sigma^2-2\alpha}\right)$$

should be decreasing in θ, which holds, because its derivative is negative:

$$\frac{\theta}{2}(d^2 - K) < 0$$

as $\frac{\theta}{2}(1-K) < 0$ is also negative, and $d^2 < 1$ $(0 < d < 1)$. Therefore we can guarantee that the value function has the following form:

$$F(\theta) = \begin{cases} \frac{1}{r}\left(\frac{\theta^2}{4} + \frac{E[dF(\theta)]}{dt}\right) & : \theta < \theta^* \\ K \times \frac{\theta^2}{4}\left(\frac{1}{r-\sigma^2-2\alpha}\right) - I & : \theta \geq \theta^* \end{cases}$$

In the continuation region, the value function $F(.)$ is the solution of a non-homogeneous second order differential equation whose general solution is:

$$F(\theta) = \frac{1}{r-2\alpha-\sigma^2}\frac{d^2\theta^2}{4} + A\theta^{\omega_1} + B\theta^{\omega_2}$$

where

$$\omega_1 = \frac{\sigma^2 - 2\alpha - \sqrt{4\alpha^2 - 4(\alpha - 2r)\sigma^2 + \sigma^4}}{2\sigma^2} < 0$$

$$\omega_2 = \frac{\sigma^2 - 2\alpha + \sqrt{4\alpha^2 - 4(\alpha - 2r)\sigma^2 + \sigma^4}}{2\sigma^2} > 0$$

Using the usual boundary conditions, defined in the introduction, we obtain the system:

$$\begin{cases} C\frac{d^2\theta^{*2}}{4} + B\theta^{*\omega_2} = KC\frac{\theta^{\times 2}}{4} - I \\ C\frac{d^2\theta^*}{2} + B\omega_2\theta^{*\omega_2 - 1} = KC\frac{\theta^*}{2} \end{cases} \tag{2.3}$$

and the explicit solution is:

$$\theta^* = \sqrt{\frac{4I\omega_2}{(\omega_2 - 2)C(K - d^2)}}$$

where $C = \frac{1}{r-2\alpha-\sigma^2}$.

Note that it becomes clear how the effect of this variable appears, not only for this approach but also for the Dynamic Programming one. In fact, the way the effect appears in both cases is very similar, as expected.

As for the cannibalization effect, the final policy is not possible to obtain, as a dependence on time appears in the value function $F(.)$, and the type of methods used to solve these problems assume that there is no such dependency.

As for the case of the implementation delay, where we assume a delay Δ on the implementation after we have decided to launch the second product, the time of decision t is based on the idea of maximizing the expected income at time t', which would be the real time of implementation $(t + \Delta)$.

We start by deriving the profit flow of the company before operating with product 2.

For each time t, knowing the value of $\theta(t)$, the maximization can only be done on the quantities that are produced, assuming the rest is fixed. The expected value has to be used, as the time for which we are doing our calculations is not the time where we are now, and for which we know the value of the demand, due to the delay. The optimal quantity is given by:

$$\begin{aligned}
q_1^{opt}(t') &= \arg\max_{q_1} E[p_1(t') \times q_1 | \theta(t)] \\
&= \arg\max_{q_1} E[(d\theta(t') - q_1)q_1 | \theta(t)] \\
&= \frac{E[d\theta(t')|\theta(t)]}{2} \\
&= \frac{dE[\theta(t')|\theta(t)]}{2}
\end{aligned}$$

Thus, the expected profit at time $t + \Delta = t'$ is equal to:

$$RevBef = e^{-r\Delta} E\left[p_1(t') \times q_1^{opt}(t')|\theta(t)\right] = e^{-r\Delta} \frac{d^2 E\left[\theta(t')|\theta(t)\right]^2}{4}$$

Since $\{\theta(t), t \geq 0\}$ is a Geometric Brownian Motion, we know that:

$$E\left[\theta(t')|\theta(t)\right] = \theta(t)e^{\alpha(t'-t)} = \theta(t)e^{\alpha\Delta}$$

As a consequence, we can now obtain the following expression for the associated profit:

$$\pi(\theta) = \frac{d^2\theta^2}{4} e^{(2\alpha - r)\Delta}$$

Once again, for the case where both products are in the market, all the derivation is totally equal to the one in [2], as the generalization we are trying does not have any influence on this situation.

Thus, we can skip to the computation of the Termination Payoff:

$$
\begin{aligned}
\Omega(\theta) &= E\left[\int_{T+\Delta}^{\infty} e^{-r(s'-T)}\left(p_1(s')\times q_1^{opt}(s') + p_2(s')\times q_2^{opt}(s')\right)ds' - I\middle|\theta(T)=\theta\right]\\
&= \int_{T+\Delta}^{\infty} e^{-r(s'-T)}K\left(\frac{\theta^2}{4}e^{2\alpha\Delta}e^{(2\alpha+\sigma^2)(s'-\Delta-T)}\right)ds' - I\\
&= e^{-r\Delta}\left(K\frac{\theta^2}{4}\frac{e^{2\alpha\Delta}}{r-2\alpha-\sigma^2} - Ie^{r\Delta}\right)
\end{aligned}
$$

Furthermore, $F(\theta)$ must be the solution of an *ODE* of the same type as for the case without implementation delay, subject to the same three boundary conditions.

The result that a unique θ^* separates the θ space in two regions, continuation and stopping, is still valid for this case as the following expression is decreasing in θ (expression that comes from Equation (1.8)):

$$
\frac{d^2\theta^2}{4}e^{(2\alpha-r)\Delta} - re^{-r\Delta}\left(K\frac{\theta^2}{4}\frac{e^{2\alpha\Delta}}{r-2\alpha-\sigma^2} - Ie^{r\Delta}\right) + \alpha e^{-r\Delta}\left(K\frac{\theta^2}{2}\frac{e^{2\alpha\Delta}}{r-2\alpha-\sigma^2}\right)
$$
$$
+\sigma^2 e^{-r\Delta}\left(K\frac{\theta^2}{4}\frac{e^{2\alpha\Delta}}{r-2\alpha-\sigma^2}\right)
$$

In fact, its derivative with respect to θ is negative because:

$$
\frac{\theta}{2}e^{(2\alpha-r)\Delta}(1-K) < 0
$$

and so we immediately get:

$$
\frac{\theta}{2}e^{(2\alpha-r)\Delta}(d^2-K) < 0
$$

as $0 < d^2 < 1$.

The general solution for the value function is given by:

$$
F(\theta) = \frac{d^2\theta^2 e^{(2\alpha-r)\Delta}}{4(r-2\alpha-\sigma^2)} + A\theta^{\omega_1} + B\theta^{\omega_2}
$$

and using the usual boundary conditions, we get that the expression for the trigger value is:

$$
\theta^* = \sqrt{\frac{4I\omega_2 e^{(r-2\alpha)\Delta}}{(\omega_2-2)C(K-d^2)}}
$$

Note that the type of dependence of the trigger value on this new variable is the same that was observed for the model without delay.

For similar reasons that for the case of cannibalization effect, when we consider both implementation delay and cannibalization effect, we obtain explicit dependency on time in the value function, and so this case is not presented also.

Chapter 3

Two Demand Processes

In this chapter, we address the following extension of the Basic Model. We assume that the demand processes for each product is not the same. From now on, we let $\theta_1 = \{\theta_1(t), t \geq 0\}$ and $\theta_2 = \{\theta_2(t), t \geq 0\}$ denote the demand for products 1 and 2, respectively.

We follow the same procedure as the one used for the Basic Model in order to derive the optimal investment policy, but, as we shall see, we get to the conclusion that the complete deduction is not possible to obtain for this new model, due to some technical difficulties that are seen in detail during this chapter.

In this extension, the demand processes are not necessarily the same, although they are driven by the same Brownian Motion. Therefore, we let:

$$d\theta_i(t) = \alpha_i \theta_i(t)dt + \sigma\theta_i(t)dW(t)$$

for $i \in \{1, 2\}$, depending on the product we are referring to.

To be coherent with the context we are dealing with, we consider the drift α_1 to be smaller than α_2. In order to simplify the computations, we assume that the volatility of each process is the same, and we call it σ. In fact, this allows us to go further on the development of the deduction of the optimal policy. This is also an acceptable assumption in terms of economical context.

Also note that, given the fact that the Brownian Motion is the same for both processes, we still only have one source of uncertainty.

3.1 Optimal Quantities and Revenues

We start by deriving the optimal quantities (and corresponding optimal revenues) of each product, for the case where product 1 is the only one in the market, and then also for the case where both products are in the market.

We start with the situation before investment. Given the objective function, written as $\max_{q_1(t)}\{q_1(t)p_1(t)\}$, and the model for the price ($p_1(t) = \theta_1(t) - q_1(t)$), the maximum revenue for a specific time t can thus be written as a function of one only variable, $q_1(t)$

and so in order to maximize the revenue at time t, we just have to determine $q_1(t)$ by solving:

$$RevBef = \max_{q_1(t)}\{q_1(t)p_1(t)\} = \max_{q_1(t)}\{q_1(t)\,[\theta_1(t) - q_1(t)]\} = \frac{\theta_1^2(t)}{4}$$

where the quantity that optimizes it is $q_1^{opt}(t) = \frac{\theta_1(t)}{2}$. Note that $q_1^{opt}(t) \geq 0$ since $\{\theta_1(t), t \geq 0\}$ is a Geometric Brownian Motion, which always takes positive values.

For the case where both products are in the market, the situation is not that simple, as now $\{\theta_1(t), t \geq 0\}$ and $\{\theta_2(t), t \geq 0\}$ are two processes (despite not being independent, as they are driven by the same stochastic process, $\{W(t), t \geq 0\}$). The law for the prices is basically the same as in the Basic Model:

$$\begin{aligned}
p_1(t) &= a\theta_1(t) - q_1(t) - \eta q_2(t) \\
p_2(t) &= \beta\theta_2(t) - \gamma q_2(t) - \eta q_1(t)
\end{aligned}$$

There are now two variables to maximize and we get:

$$\begin{aligned}
RevAft &= \max_{\{q_1(t),q_2(t)\}}\{q_1(t)p_1(t) + q_2(t)p_2(t)\} \\
&= \max_{\{q_1(t),q_2(t)\}}\{q_1(t)[\theta_1(t) - q_1(t) - \eta q_2(t)] + q_2(t)[\theta_2(t) - \gamma q_2(t) - \eta q_1(t)]\} \\
&= \max_{\{q_1(t),q_2(t)\}}\{\theta_1(t)q_1(t) - q_1^2(t) - 2\eta q_1(t)q_2(t) + \theta_2(t)q_2(t) - \gamma q_2^2(t)\}
\end{aligned}$$

We obtain as optimal values for the quantities:

- if $\theta_1(t)\gamma - \theta_2(t)\eta \geq 0$:

$$\begin{aligned}
q_1^{opt}(t) &= \frac{\theta_1(t)\gamma - \theta_2(t)\eta}{2(\gamma - \eta^2)} \\
q_2^{opt}(t) &= \frac{\theta_2(t) - \theta_1(t)\eta}{2(\gamma - \eta^2)}
\end{aligned}$$

- if $\theta_1(t)\gamma - \theta_2(t)\eta < 0$:

$$\begin{aligned}
q_1^{opt}(t) &= 0 \\
q_2^{opt}(t) &= \frac{\theta_2(t)}{2\gamma}
\end{aligned}$$

The corresponding revenue for these optimal quantities is given by:

$$RevAft = \begin{cases} \frac{\theta_1(t)^2\gamma + \theta_2(t)^2 - 2\theta_1(t)\theta_2(t)\eta}{4(\gamma - \eta^2)} & : \theta_1(t)\gamma - \theta_2(t)\eta \geq 0 \\ \frac{\theta_2(t)^2}{4\gamma} & : \theta_1(t)\gamma - \theta_2(t)\eta < 0 \end{cases}$$

Given the fact that we are assuming one only Brownian Motion for both demands, the following relation holds:

$$\theta_2(t) = \frac{\theta_{02}}{\theta_{01}}\theta_1(t)e^{(\alpha_2 - \alpha_1)t} \tag{3.1}$$

Note that all the expressions that include the new product (product 2) are only valid for $t \geq T$, where T denotes, once again, the time of investment. Also note, in the previous expression, that the following simplification of notation is used: $\theta_1(0) = \theta_{01}$ and $\theta_2(T) = \theta_{02}$.

For the optimal quantities, we thus have:

- $\gamma - \frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t}\eta \geq 0$:

$$q_1^{opt}(t) = \frac{\theta_1(t)}{2} \frac{\gamma - \frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t}\eta}{\gamma - \eta^2}$$

$$q_2^{opt}(t) = \frac{\theta_1(t)}{2} \frac{\frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t} - \eta}{\gamma - \eta^2}$$

- $\gamma - \frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t}\eta < 0$:

$$q_1^{opt}(t) = 0$$

$$q_2^{opt}(t) = \frac{\theta_1(t)}{2} \frac{\frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t}}{\gamma}$$

The associated maximal revenue for the two regions is:

$$RevAft = \begin{cases} \frac{\theta_1(t)^2}{4} \frac{\gamma + (\frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t})^2 - 2\frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t}\eta}{\gamma - \eta^2} & : \gamma - \frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t}\eta \geq 0 \\ \frac{\theta_1(t)^2}{4} \frac{(\frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t})^2}{\gamma} & : \gamma - \frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t}\eta < 0 \end{cases}$$

We now start with the derivation of the associated optimal policies.

3.2 ***NPV*** Rule

Given the expressions we have from the end of the previous section, fixing the initial demand for each product, we can calculate the expected discounted cash flow for the situation where we do not invest:

$$\begin{aligned} CashBef &= E\left[\int_0^\infty e^{-rt} p_1(t) q_1^{opt}(t) dt \middle| \theta_1(0) = \theta_{01}\right] \\ &= E\left[\int_0^\infty e^{-rt} \frac{\theta_1^2(t)}{4} dt \middle| \theta_1(0) = \theta_{01}\right] \\ &= \int_0^\infty e^{-rt} \frac{E\left[\theta_1^2(t) | \theta_1(0) = \theta_{01}\right]}{4} dt \\ &= \int_0^\infty e^{(-r+2\alpha_1+\sigma^2)t} \frac{\theta_{01}^2}{4} dt \\ &= \frac{\theta_{01}^2}{4} \frac{1}{r - \sigma^2 - 2\alpha_1} \end{aligned}$$

where we use, in the first equality, the result obtained for the optimal value of the revenue, without investment, at time t.

If, on the other hand, we choose to launch the new product, the expression gets, as expected, more complicated than the one we could observe in the case studied in [2], because of the dependence on time. In fact, we can now have a time at which we stop producing the old product and stick exclusively to the new one.

There are, in fact, two cases to consider. The first one is the simplest one, as from the moment we invest, the quantity for the old product is already 0. In this case, the integral is just the same as for the case with one only *GBM* (derived in [2]). For the other case, there is a time at which the situation changes, which verifies: $\gamma - \frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t}\eta = 0$. This implies that the integral has to be split in two parts. In the first one, both products are still in the market, while in the second one we only produce product 2. The corresponding time of separation is given by $\frac{\log(\frac{\gamma}{\eta}\frac{\theta_{01}}{\theta_{02}})}{\alpha_2-\alpha_1}$. We denote this time as L in the following expressions, for the sake of simplicity.

We now show what happens for the two different cases:

- $\gamma - \frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t}\eta \geq 0$ (case for which for a certain time both products co-exist in the market)

$$\begin{aligned} CashAft =& E[\int_0^L e^{-rt}\left[\frac{\theta_1(t)^2\gamma + \theta_2(t)^2 - 2\theta_1(t)\theta_2(t)\eta}{4(\gamma-\eta^2)}\right] dt \\ &+ \int_L^\infty e^{-rt}\left[\frac{\theta_2(t)^2}{4\gamma}\right] dt - I|\theta_1(0) = \theta_{01}, \theta_2(0) = \theta_{02}] \end{aligned}$$

By linearity of the expected value, we get:

$$\begin{aligned} CashAft =& \frac{1}{4(\gamma-\eta^2)}(\gamma \times E\left[\int_0^L e^{-rt}\left[\theta_1(t)^2\right] dt \middle| \theta_1(0) = \theta_{01}\right] \\ &+ E\left[\int_0^L e^{-rt}\left[\theta_2(t)^2\right] dt \middle| \theta_2(0) = \theta_{02}\right] \\ &- 2\eta \times E\left[\int_0^L e^{-rt}\left[\theta_1(t)\theta_2(t)\right] dt \middle| \theta_1(0) = \theta_{01}, \theta_2(0) = \theta_{02}\right]) \\ &+ E\left[\int_L^\infty e^{-rt}\left[\frac{\theta_2(t)^2}{4\gamma}\right] dt \middle| \theta_2(0) = \theta_{02}\right] - I \end{aligned}$$

Now, in order to be able to develop this expression, the expected value of the product of the *GBM*s has to be obtained (note that the expected value of the product of the *GBMs* is not the same as the product of the expected values, as there is no independence).

Given the initial values $\theta_1(0) = \theta_{01}$ and $\theta_2(T) = \theta_{02}$, the processes are defined:

$$\theta_1(t) = \theta_{01} \times \exp[(\alpha_1 - \frac{\sigma^2}{2})t + \sigma \times W(t)]$$
$$\theta_2(t) = \theta_{02} \times \exp[(\alpha_2 - \frac{\sigma^2}{2})t + \sigma \times W(t)]$$

So, the expression for the expected value of the product is the following:

$$\begin{aligned} E[\theta_1(t)\theta_2(t)|\theta_1(0) = \theta_{01}, \theta_2(0) = \theta_{02}] &= \theta_{01}\theta_{02} \exp[(\alpha_1 + \alpha_2 - \sigma^2)t] \times E[e^{2\sigma W(t)}] \\ &= \theta_{01}\theta_{02} \exp[(\alpha_1 + \alpha_2 - \sigma^2)t] \times \exp[2\sigma^2 t] \\ &= \theta_{01}\theta_{02} e^{(\alpha_1+\alpha_2+\sigma^2)t} \end{aligned}$$

where the step from the first line to the second is due to the fact that $W(t)$ is a random variable with Normal distribution with mean 0 and variance t, and so (Moment Generating Function): $E[e^{kW_t}] = e^{\frac{k^2}{2}t}$.

It can be added that in order to guarantee the convergence of the integrals above, we have to assume the condition $r > 2\alpha_2 + \sigma^2$.

So, we finally obtain the explicit expression for this case:

$$CashAft = \frac{\gamma\theta_{01}^2 \frac{1-e^{-(\frac{\log(\frac{\gamma}{\eta}\frac{\theta_{01}}{\theta_{02}})}{\alpha_2-\alpha_1})(r-2\alpha_1-\sigma^2)}}{r-\sigma^2-2\alpha_1} + \theta_{02}^2 \frac{1-e^{-(\frac{\log(\frac{\gamma}{\eta}\frac{\theta_{01}}{\theta_{02}})}{\alpha_2-\alpha_1})(r-2\alpha_2-\sigma^2)}}{r-2\alpha_2-\sigma^2}}{4(\gamma-\eta^2)}$$
$$- \frac{2\eta\theta_{01}\theta_{02} \frac{1-e^{-(\frac{\log(\frac{\gamma}{\eta}\frac{\theta_{01}}{\theta_{02}})}{\alpha_2-\alpha_1})(r-\alpha_1-\alpha_2-\sigma^2)}}{r-\alpha_1-\alpha_2-\sigma^2}}{4(\gamma-\eta^2)} + \frac{\theta_{02}^2 \frac{e^{-(\frac{\log(\frac{\gamma}{\eta}\frac{\theta_{01}}{\theta_{02}})}{\alpha_2-\alpha_1})(r-2\alpha_2-\sigma^2)}}{r-2\alpha_2-\sigma^2}}{4\gamma} - I$$

- $\gamma - \frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t}\eta < 0$ (after investment, product 1 stops being produced. The two products never co-exist in this situation)

$$\begin{aligned} CashAft &= E\left[\int_0^\infty e^{-rt}\left[\frac{\theta_2(t)^2}{4\gamma}\right] dt - I \middle| \theta_1(0) = \theta_{01}, \theta_2(0) = \theta_{02}\right] \\ &= \frac{\theta_{02}^2}{4\gamma} \frac{1}{r - \sigma^2 - 2\alpha_2} - I \end{aligned}$$

using the result below (that results from the Moment Generating Function, once again):

$$E\left[\theta^2(t) \middle| \theta(s) = \theta_s\right] = \theta_s^2 e^{(2\alpha+\sigma^2)(t-s)}$$

Moreover, the interchange of the expected value with the integral is justified by the Dominated Convergence Theorem, whose assumptions are verified.

The investment must be undertaken if and only if the inequality holds (depending on the region where the values fit):

- $\gamma - \frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t}\eta \geq 0$

$$\frac{\gamma\theta_{01}^2 \frac{1-e^{-(\frac{\log(\frac{\gamma}{\eta}\frac{\theta_{01}}{\theta_{02}})}{\alpha_2-\alpha_1})(r-2\alpha_1-\sigma^2)}}{r-\sigma^2-2\alpha_1} + \theta_{02}^2 \frac{1-e^{-(\frac{\log(\frac{\gamma}{\eta}\frac{\theta_{01}}{\theta_{02}})}{\alpha_2-\alpha_1})(r-2\alpha_2-\sigma^2)}}{r-2\alpha_2-\sigma^2}}{4(\gamma-\eta^2)}$$

$$-\frac{2\eta\theta_{01}\theta_{02}\frac{1-e^{-(\frac{\log(\frac{\gamma}{\eta}\frac{\theta_{01}}{\theta_{02}})}{\alpha_2-\alpha_1})(r-\alpha_1-\alpha_2-\sigma^2)}}{r-\alpha_1-\alpha_2-\sigma^2}}{4(\gamma-\eta^2)} + \frac{\theta_{02}^2 \frac{e^{-(\frac{\log(\frac{\gamma}{\eta}\frac{\theta_{01}}{\theta_{02}})}{\alpha_2-\alpha_1})(r-2\alpha_2-\sigma^2)}}{r-2\alpha_2-\sigma^2}}{4\gamma} - I > \frac{\theta_{01}^2}{4}\frac{1}{r-\sigma^2-2\alpha_1}$$

- $\gamma - \frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t}\eta < 0$

$$\frac{\theta_{02}^2}{4\gamma}\frac{1}{r-\sigma^2-2\alpha_2} - I > \frac{\theta_{01}^2}{4}\frac{1}{r-\sigma^2-2\alpha_1}$$

The inequality represents the comparison between the expected discounted cash flow with and without investment. We basically invest if and only if

$$CashAft > CashBef$$

Note that the relations are written in terms of the initial demands, which is something that characterizes this approach.

3.3 Dynamic Programming

Note that, in this section, function $\Omega(.,.)$ still represents the termination payoff but it is important to mention that it receives two arguments instead of just one, which is the way it is defined in the introduction of this thesis. In fact, in order that it keeps representing the termination payoff, it has to depend on more than one variable now. We should also mention that the same thing happens to $\pi(.,.)$ and, as a consequence, $F(.,.)$.

As a consequence, it is quite obvious which expression does the value function need to have, as far as its definition is concerned, for this new scenario, with two demand processes involved:

$$F(\theta_1,\theta_2) = \max\left\{\Omega(\theta_1,\theta_2), \frac{1}{r}\left(\frac{\theta_1^2}{4} + \frac{E[dF(\theta_1,\theta_2)]}{dt}\right)\right\}$$

Note that $\Omega(\theta_1,\theta_2)$ does not depend directly on time, as we can easily see by computing the value of this function:

- $\gamma - \frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t}\eta \geq 0$

$$\Omega(\theta_1,\theta_2) = \frac{\gamma\theta_1^2 \frac{1-e^{-(\frac{\log(\frac{\gamma}{\eta}\frac{\theta_1}{\theta_2})}{\alpha_2-\alpha_1})(r-2\alpha_1-\sigma^2)}}{r-\sigma^2-2\alpha_1} + \theta_2^2 \frac{1-e^{-(\frac{\log(\frac{\gamma}{\eta}\frac{\theta_1}{\theta_2})}{\alpha_2-\alpha_1})(r-2\alpha_2-\sigma^2)}}{r-2\alpha_2-\sigma^2}}{4(\gamma-\eta^2)}$$

$$-\frac{2\eta\theta_1\theta_2\frac{1-e^{-(\frac{\log(\frac{\gamma}{\eta}\frac{\theta_1}{\theta_2})}{\alpha_2-\alpha_1})(r-\alpha_1-\alpha_2-\sigma^2)}}{r-\alpha_1-\alpha_2-\sigma^2}}{4(\gamma-\eta^2)} + \frac{\theta_2^2 \frac{e^{-(\frac{\log(\frac{\gamma}{\eta}\frac{\theta_1}{\theta_2})}{\alpha_2-\alpha_1})(r-2\alpha_2-\sigma^2)}}{r-2\alpha_2-\sigma^2}}{4\gamma} - I$$

- $\gamma - \frac{\theta_{02}}{\theta_{01}} e^{(\alpha_2-\alpha_1)t}\eta < 0$

$$\Omega(\theta_1, \theta_2) = \frac{\theta_2^2}{4\gamma} \frac{1}{r - \sigma^2 - 2\alpha_2} - I$$

We assume in the previous expressions that we are at time t, and that $\theta_1(t) = \theta_1$ and $\theta_2(t) = \theta_2$.

The next step is to find an explicit solution for the value function, for which we need both $\pi(\theta_1, \theta_2)$ and $\Omega(\theta_1, \theta_2)$. Recall that these two functions only allow us to write the desired function in implicit form, in the continuation region.

In the case of only one demand process for both products, the case developed in [2], the step to convert the implicit value function into an explicit one was immediate, as the expression that separates the θ space in two was easily defined. So, our idea is to reduce our problem to that one, by reducing our problem to one dimension, writing one of the processes as a function of the other, using Equation (3.1). Then, it would be possible to apply the usual methods in order to obtain the value function written in explicit form.

However, from the moment we write one process as a function of the other, a dependence on time appears, which does not allow us to obtain the explicit form of the solution for the value function in the usual way. In fact, the usual simplification in the differential equation that has to be solved, that consists in having the partial derivative of $F(\theta_1, \theta_2)$ with respect to time being equal to 0, no longer holds, and the solution of this new differential equation is not known.

In fact, for the cases where the dependence on time appears, the methods that are known to solve this type of problem do not apply. For instance, using the Bellman principle, we end up with a partial differential equation that depends on two variables and whose solution can only be obtained using numerical methods of some complexity.

3.4 Numerical Analysis

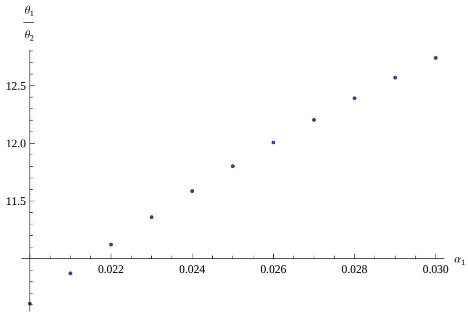
3.4.1 *NPV* Rule

In this section, we analyze the behavior of the ratio of the trigger values as a function of some variables of interest, associated with the demand process. In fact, the response variable could not be, as for the previous cases, just one trigger value, as for this case we have two different processes. Looking back at the expression that defines the condition to be satisfied by the trigger values, we can see that it can be written as a function of $\frac{\theta_{01}}{\theta_{02}}$. This could be, in fact, our new response variable, not only because of the possibility of building the same type of plots as in Chapter 2, given that we would have just one variable again, instead of two, but also because there is a meaning associated with it, and so an interpretation to the resulting plots is possible to provide. For the sake of simplicity, the name of the axes that correspond to the response variable in the plots that follow is just $\frac{\theta_1}{\theta_2}$, instead of $\frac{\theta_{01}}{\theta_{02}}$.

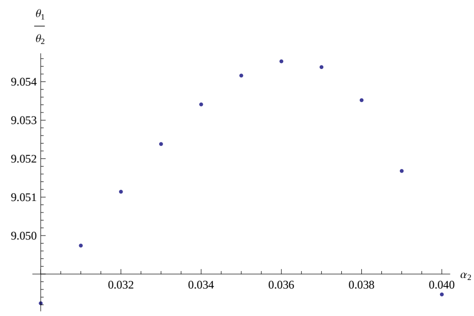
We fix the following values for the drifts:

- $\alpha_1 = 0.02$
- $\alpha_2 = 0.04$

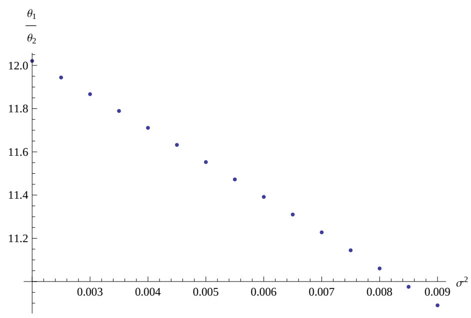
Also note that, the same way that happens for the multiplicative model in Chapter 5, despite the fact that in this case we are working with a linear restriction on the price, the study has to be done for different points and not continuously, as the expression that has to be solved is, in terms of the response variable we want, non-linear.



(a) $\frac{\theta_1}{\theta_2}$ as a function of α_1



(b) $\frac{\theta_1}{\theta_2}$ as a function of α_2



(c) $\frac{\theta_1}{\theta_2}$ as a function of σ

Figure 3.1: $\frac{\theta_1}{\theta_2}$ as a function of the variables of interest

From these plots, we are able to present the following comments:

- **Influence of** α_1: we first note that θ_1 should increase with the drift of the first process. This supports an increasing behavior of the curve. However, there is also an argument supporting the opposite behavior of the curve. When the drift of the first process increases, θ_2 is also supposed to increase, and taking into consideration that this quantity is in the denominator, this should imply a decreasing behavior of the curve. As the two arguments are valid, the result of the plot can go either way. In fact, it seems that the monotonic behavior is just circumstantial and that the function could have a different monotonicity, or could even not to be monotonic at all. What the image says is that the second argument is always stronger than the first one;

- **Influence of** α_2: this is a confirmation of what was stated for the previous image. In this case, both θ_1 and θ_2 should decrease with α_2, and we obtain a region for which one of the arguments is stronger than the other, and then there is a value for which this tendency changes. Therefore, the behavior of the trigger value is not monotonic as a function of α_2;

- **Influence of** σ: it seems to be a coincidence that the behavior is similar to the one obtained when studying the behavior of the trigger value as a function of the volatility in Figure 2.1, as the response variable is different now. What really seems reasonable to be said is that both trigger values are supposed to decrease with the volatility, but, once again, when we take the ratio of those, we are not able to know exactly what happens. We just get the idea of what argument is stronger, just like in the other two plots.

The conclusions that are obtained by observing these previous images are, after all, some of the most interesting results until this point, as normally we just confirm our intuition when looking at the obtained plots, but in this case not only do we obtain unpredictable results, as they also have a specific meaning.

Chapter 4

Introducing Operation Costs

In this chapter, we derive and analyze the optimal investment policy when instead of maximizing the profit, given by $\{p(t)q(t), t \geq 0\}$, we maximize the net profit $\{p(t)q(t) - cq(t), t \geq 0\}$, where c denotes the unitary cost of production.

4.1 Optimal Quantities and Revenues

As product 2 is supposed to be the best one and the innovative one, we can suppose that $c_1 > c_2$, where c_i denotes the cost of production per unit of product i.

The first case to consider is the one where we still do not produce product 2 (situation before investment). The function we want to optimize is now given by $p_1(t)q_1(t) - c_1q_1(t)$. Its maximum value for a specific time t can thus be written as a function of one only variable, $q_1(t)$, as:

$$\begin{aligned} RevBef = \max_{q_1(t)}\{q_1(t)p_1(t) - c_1q_1(t)\} &= \max_{q_1(t)}\{q_1(t)\left[\theta(t) - q_1(t)\right] - c_1q_1(t)\} \\ &= \frac{\theta^2(t)}{4} - \frac{\theta(t)c_1}{2} + \frac{c_1^2}{4} \end{aligned} \tag{4.1}$$

we remark that in order to have this maximum value, the company should produce, at time t, the optimal quantity $q_1^{opt}(t) = \frac{\theta(t)-c_1}{2}$.

As $\{\theta(t), t \geq 0\}$ is a *GBM*, we can only assume that $\theta(t) \geq 0$ with probability one. However, we may have $\theta(t) < c_1$, leading to $q_1^{opt}(t) < 0$, which is a non-admissible value.

Furthermore,

$$
\begin{aligned}
P(q_1^{opt}(t) \geq 0) &= P(\theta(t) > c_1) \\
&= P(\theta_0 \exp{(\alpha - \frac{\sigma^2}{2})t + \sigma W(t)} > c_1) \\
&= P(W(t) > \frac{\log(\frac{c_1}{\theta_0}) - (\alpha - \frac{\sigma^2}{2})t}{\sigma}) \\
&= 1 - \Phi\left(\frac{\log(\frac{c_1}{\theta_0}) - (\alpha - \frac{\sigma^2}{2})t}{\sigma}\right)
\end{aligned}
$$

In order to have this probability close to one, $\forall t$, we have to assume that $\theta_0 >> c_1$, with θ_0 denoting the initial demand, and thus the results that we get are only valid if this approximation holds.

For the case where both products are already in the market, after investment, we get:

$$
RevAft = \max_{\{q_1(t), q_2(t)\}} \{q_1(t)p_1(t) - c_1 \times q_1(t) + q_2(t)p_2(t) - c_2 \times q_2(t)\}
$$

which is equivalent to writing:

$$
\max_{\{q_1(t), q_2(t)\}} \{a\theta(t)q_1(t) - c_1 \times q_1(t) - q_1^2(t) - 2\eta q_1(t)q_2(t) + \beta\theta(t)q_2(t) - c_2 \times q_2(t) - \gamma q_2^2(t)\}
$$

The corresponding optimal quantities are:

- $\theta(t)(a\gamma - \beta\eta) - c_1\gamma + c_2\eta \geq 0$:

$$
\begin{aligned}
q_1^{opt}(t) &= \frac{\theta(t)}{2}\frac{a\gamma - \beta\eta}{\gamma - \eta^2} + \frac{c_1}{2}\frac{-\gamma}{\gamma - \eta^2} + \frac{c_2}{2}\frac{\eta}{\gamma - \eta^2} \\
q_2^{opt}(t) &= \frac{\theta(t)}{2}\frac{\beta - a\eta}{\gamma - \eta^2} + \frac{c_1}{2}\frac{\eta}{\gamma - \eta^2} + \frac{c_2}{2}\frac{-1}{\gamma - \eta^2}
\end{aligned} \tag{4.2}
$$

- $\theta(t)(a\gamma - \beta\eta) - c_1\gamma + c_2\eta < 0$:

$$
\begin{aligned}
q_1^{opt}(t) &= 0 \\
q_2^{opt}(t) &= \frac{\beta\theta(t) - c_2}{2\gamma}
\end{aligned} \tag{4.3}
$$

We remark that, as generally c_2 is small, one should expect that the assumption that $q_2(t) \geq 0$, $\forall t$, holds with probability close to 1.

When both products are in the market, one also needs to assess the probability that both quantities are non-negative. The quantity related to the old product is never negative and so we just need to see the conditions under which the optimal quantity of the new product is not negative. For the first region (when $\theta(t)(a\gamma - \beta\eta) - c_1\gamma + c_2\eta \geq 0$), it follows that, equivalently, the following condition has to hold:

$$\theta(t) \geq \frac{c_2 - c_1\eta}{\beta - a\eta}$$

note that, as $\beta - a\eta > 0$, by assumption, if $c_2 - c_1\eta < 0$, then the condition immediately holds, and therefore $q_2^{opt}(t) \geq 0$, $\forall t$.

For the other region (when $\theta(t)(a\gamma - \beta\eta) - c_1\gamma + c_2\eta < 0$), we need to assume that:

$$\theta(t) \geq \frac{c_2}{\beta}$$

In terms of the situation where both products coexist in the market, some interesting interpretations are possible to give, related to the optimal quantities that are associated to each product.

Some conclusions are possible to obtain: considering the effect of the cost of a product on the quantity of that same product that is produced, *we note that our intuition is confirmed in the fact that increasing the unitary cost of production of a product leads to a decrease in the quantity of that same product that is produced.*

Another conclusion, now for the effect of the cost of a product on the quantity of the other product that is produced, for the case where both products are in the market ($\theta(t)(a\gamma - \beta\eta) - c_1\gamma + c_2\eta \geq 0$), is that: *an increase on the unitary cost of production for one product leads to an increase on the optimal quantity of the other, and this is the result that one expects, not only given the intuition behind it but also given the fact that, in the model, there is a restriction on the price that is influenced by the quantity that is produced of the other product. If the cost of the other product goes up, then its optimal quantity goes down, which makes the price of the product in study increase, and so it becomes optimal to produce more.*

4.2 *NPV* Rule

In this section, we derive the *NPV* rule, taking into account some of the results obtained in the previous section.

Under the assumption that $q_1^{opt}(t) \geq 0$, $\forall t$, we have the following expected discounted cash flow, for the situation before investment:

$$\begin{aligned} CashBef &= E\left[\int_0^\infty e^{-rt}(p_1(t)q_1^{opt}(t) - c_1 q_1^{opt}(t))dt \middle| \theta(0) = \theta_0\right] \\ &= E\left[\int_0^\infty e^{-rt}(\frac{\theta^2(t)}{4} - \frac{c_1\theta(t)}{2} + \frac{c_1^2}{4})dt \middle| \theta(0) = \theta_0\right] \\ &= \frac{\int_0^\infty e^{-rt}E\left[\theta^2(t)|\theta(0) = \theta_0\right]dt - 2c_1\int_0^\infty e^{-rt}E\left[\theta(t)|\theta(0) = \theta_0\right]dt + \frac{c_1^2}{r}}{4} \end{aligned} \tag{4.4}$$

Note that the step from the first line to the second one is due to the result in Equation (4.1).

Using the relationship:

$$\theta(t) = \theta_0 \times \exp[(\alpha - \frac{\sigma^2}{2})t + \sigma W(t)]$$

and the fact that $\{W(t), t \geq 0\}$ is a Brownian Motion, with $E[e^{\sigma W(t)}] = e^{\frac{\sigma^2 t}{2}}$, it follows that:

$$E[\theta(t)] = \theta_0 e^{\alpha t}$$

and so the final value of the expression in Equation (4.4) is:

$$CashBef = \frac{\theta_0^2}{4} \frac{1}{r - \sigma^2 - 2\alpha} - 2c_1 \frac{\theta_0}{4} \frac{1}{r - \alpha} + \frac{c_1^2}{4} \frac{1}{r} \tag{4.5}$$

Note that this cash flow decreases with c_1, as, for values of c_1 such that $c_1 << \theta_0$, it follows that:

$$2c_1 \frac{\theta_0}{4} \frac{1}{r - \alpha} > \frac{c_1^2}{4} \frac{1}{r}$$

because $c_1\theta_0 > c_1^2$ and $r > (r - \alpha)$.

Moving on to the case where both products are in the market, after investment, we note, before we proceed with the computations, that we have two possible regions that should be defined separately, as the result for the discounted cash flow after investment is for these two cases:

- Region 1: $a\gamma - \beta\eta \geq 0$

  In this situation, the expected discounted cash flow is given by:

$$\begin{aligned} CashAft =& E[\int_0^\infty e^{-rt}(p_1 q_1^{opt} + p_2 q_2^{opt} - c_1 q_1^{opt} - c_2 q_2^{opt})(t) \\ & 1_{\{\theta(t) \geq \frac{c_1\gamma - c_2\eta}{a\gamma - \beta\eta}\}} 1_{\{\theta(t) \geq \frac{c_2 - c_1\eta}{\beta - a\eta}\}} dt] \\ & + E[\int_0^\infty e^{-rt}(p_2 q_2^{opt} - c_2 q_2^{opt})(t) 1_{\{\theta(t) < \frac{c_1\gamma - c_2\eta}{a\gamma - \beta\eta}\}} 1_{\{\theta(t) \geq \frac{c_2}{\beta}\}} dt] - I \\ =& \int_0^\infty e^{-rt}(p_1 q_1^{opt} + p_2 q_2^{opt} - c_1 q_1^{opt} - c_2 q_2^{opt})(t) \\ & P(\theta(t) \geq \frac{c_1\gamma - c_2\eta}{a\gamma - \beta\eta}) P(\theta(t) \geq \frac{c_2 - c_1\eta}{\beta - a\eta}) dt \\ & + \int_0^\infty e^{-rt}(p_2 q_2^{opt} - c_2 q_2^{opt})(t) P(\theta(t) < \frac{c_1\gamma - c_2\eta}{a\gamma - \beta\eta}) P(\theta(t) \geq \frac{c_2}{\beta}) dt - I \end{aligned}$$

  Taking into consideration that the distribution of $\log(\theta(t))$ is $N(\alpha - \frac{\sigma^2}{2}t, \sigma^2 t)$, then it follows that the first probability involved in the final expression that we obtained for the cash flow after investment ($CashAft$) is given by:

$$P(\theta(t) \geq \frac{c_1\gamma - c_2\eta}{a\gamma - \beta\eta}) = 1 - \Phi(\frac{\frac{c_1\gamma - c_2\eta}{a\gamma - \beta\eta} - \alpha}{\sigma t})$$

note that $P(\theta(t) \geq \frac{c_2}{\beta})$ is equally simple to write in this form.

- Region 2: $a\gamma - \beta\eta < 0$

$$\begin{aligned}
CashAft =& E[\int_0^\infty e^{-rt}(p_1 q_1^{opt} + p_2 q_2^{opt} - c_1 q_1^{opt} - c_2 q_2^{opt})(t) \\
& 1_{\{\theta(t) < \frac{c_1\gamma - c_2\eta}{a\gamma - \beta\eta}\}} 1_{\{\theta(t) \geq \frac{c_2 - c_1\eta}{\beta - a\eta}\}} dt] \\
& + E[\int_0^\infty e^{-rt}(p_2 q_2^{opt} - c_2 q_2^{opt})(t) 1_{\{\theta(t) \geq \frac{c_1\gamma - c_2\eta}{a\gamma - \beta\eta}\}} 1_{\{\theta(t) \geq \frac{c_2}{\beta}\}} dt] - I \\
=& \int_0^\infty e^{-rt}(p_1 q_1^{opt} + p_2 q_2^{opt} - c_1 q_1^{opt} - c_2 q_2^{opt})(t) \\
& P(\theta(t) < \frac{c_1\gamma - c_2\eta}{a\gamma - \beta\eta}) P(\theta(t) \geq \frac{c_2 - c_1\eta}{\beta - a\eta}) dt \\
& + \int_0^\infty e^{-rt}(p_2 q_2^{opt} - c_2 q_2^{opt})(t) P(\theta(t) \geq \frac{c_1\gamma - c_2\eta}{a\gamma - \beta\eta}) P(\theta(t) \geq \frac{c_2}{\beta}) dt - I
\end{aligned}$$

Note that some abuse of notation is done in the two expressions for the cash flows after investment, as $q_2^{opt}(t)$ refers to different quantities for the different two main terms that appear in each step of the computations. Each quantity can easily be associated to its actual value by looking at Equations (4.2) and (4.3).

As the expressions are not possible to develop anymore, which means that a final optimal policy for the NPV approach is not possible to obtain for this general scenario, an assumption to allow us to proceed with our computations is considered. We assume that as soon as product 2 is in the market, product 1 is no longer produced. In fact, if the sunk cost is worth paying, it is not that difficult to believe that product 1 becomes insignificant from the moment we invest in the new product. This means that we assume that $\theta(t) \leq \frac{c_1\gamma - c_2\eta}{a\gamma - \beta\eta}$ by assuming that $a\gamma - \beta\eta < 0$ (note that $c_1\gamma - c_2\eta \geq 0$ because $c_1 > c_2$ and $\gamma > \eta$).

Also note that the relation $a\gamma - \beta\eta < 0$ holds when a is small, which means that there is a weak demand for product 1; γ is small, which means that people do not react too negatively (in terms of price they are willing to pay) to an increase on the quantities that are produced for the new product; β is large, which means that the new product is very profitable; and/or η is large, which means a big importance in terms of crossed effect (influence of the quantity of one of the products product in the price of the other). All of these conditions are very likely to be verified.

So, assuming from here on that product 2 is the only one in the market after investment, we start making the usual computations, as now it is possible to derive a final expression for the expected discounted cash flow. We start by calculating the expected discounted cash flow upon investing, as the one without investment was possible to calculate before considering this assumption (see Equation (4.5)):

$$\begin{aligned} CashAft &= E[\int_0^\infty e^{-rt}((p_2 q_2^{opt})(t) - c_2 q_2^{opt}(t))dt|\theta(0)=\theta_0] - I \\ &= E[\int_0^\infty e^{-rt}\frac{\beta^2\theta(t)^2 - 2c_2\beta\theta(t) + c_2^2}{4\gamma}dt|\theta(0)=\theta_0] - I \end{aligned} \tag{4.6}$$

where, in Equation (4.6), we assume that $\theta(t) > \frac{c_2}{\beta}$. In fact, the calculation is not possible to do if the term $P(\theta(t) > \frac{c_2}{\beta})$ is considered this term. Note that this approximation is also used in order to obtain the result of Equation (4.5). The validity of the approximations that we assume in order to be able to obtain a final optimal policy, $\theta(t) > \frac{c_2}{\beta}$ and $\theta(t) > c_1$, that basically depend on the value of the initial demand of the process, is tested in Chapter **??**.

Therefore:

$$\begin{aligned} CashAft =& \frac{1}{4\gamma}(\beta^2 E[\int_0^\infty e^{-rt}\theta(t)^2 dt|\theta(0)=\theta_0] - 2c_2\beta E[\int_0^\infty e^{-rt}\theta(t)dt|\theta(0)=\theta_0] \\ &+ c_2^2\int_0^\infty e^{-rt}dt) - I \\ =& \frac{\beta^2\theta_0^2}{4\gamma}\frac{1}{r-\sigma^2-2\alpha} - 2c_2\frac{\beta\theta_0}{4\gamma}\frac{1}{r-\alpha} + \frac{c_2^2}{4\gamma}\frac{1}{r} - I \end{aligned} \tag{4.7}$$

It can be added that in order to guarantee the convergence of the integrals above, we have to assume the condition that $r > (2\alpha + \sigma^2)$.

As a conclusion, we invest if and only if:

$$\frac{\beta^2\theta_0^2}{4\gamma}\frac{1}{r-\sigma^2-2\alpha} - 2c_2\frac{\beta\theta_0}{4\gamma}\frac{1}{r-\alpha} + \frac{c_2^2}{4\gamma}\frac{1}{r} - I > \frac{\theta_0^2}{4}\frac{1}{r-\sigma^2-2\alpha} - 2c_1\frac{\theta_0}{4}\frac{1}{r-\alpha} + \frac{c_1^2}{4}\frac{1}{r}$$

which is basically a comparison between the values obtained in Equations (4.5) and (4.7), which correspond to the expected discounted cash flow before and after investment, respectively.

4.3 Dynamic Programming

As this is the first time the optimal policy is derived for this approach, we also specify some steps that are not specified for the other cases where this approach is used. The functions whose value matters the most, in order to start solving the problem using this approach, and their corresponding values, are shown below:

- $\pi(\theta) = \frac{\theta^2}{4} - \frac{\theta c_1}{2} + \frac{c_1^2}{4}$
- $\Omega(\theta) = \frac{\beta^2\theta^2}{4\gamma}\frac{1}{r-\sigma^2-2\alpha} - 2c_2\frac{\beta\theta}{4\gamma}\frac{1}{r-\alpha} + \frac{c_2^2}{4\gamma}\frac{1}{r} - I$

Note, in these previous expressions, that we simplify the notation and use just θ instead of $\theta(t)$, which is something that is done for several times during this thesis. Also note that the value of these two functions is based on the results in Equations (4.5) and (4.7). The derivation that leads from the value of *CashAft* (in this case, represented by Equation (4.7)) to the expression of $\Omega(\theta)$ is done in more detail in Section 4.4.

By immediate substitution in the general expression of the value function, in Equation (1.4), we obtain:

$$F(\theta) = \max\{\frac{\beta^2\theta^2}{4\gamma}\frac{1}{r-\sigma^2-2\alpha} - 2c_2\frac{\beta\theta}{4\gamma}\frac{1}{r-\alpha} + \frac{c_2^2}{4\gamma}\frac{1}{r} - I, \frac{1}{r}\left(\frac{\theta^2}{4} - \frac{\theta c_1}{2} + \frac{c_1^2}{4} + \frac{E[dF(\theta)]}{dt}\right)\}$$

Regarding the condition that needs to be verified in order that the uniqueness for the threshold θ^* such that the θ space is divided in two regions (the one regarding the decreasing behavior of Equation (1.8)), this can only be checked numerically, in this case. Moreover, the confirmation that this condition holds was done successfully.

Therefore, the value function has the following form:

$$F(\theta) = \begin{cases} \frac{1}{r}\left(\frac{\theta^2}{4} - \frac{\theta c_1}{2} + \frac{c_1^2}{4} + \frac{E[dF(\theta)]}{dt}\right) & : \theta < \theta^* \\ \frac{\beta^2\theta^2}{4\gamma}\frac{1}{r-\sigma^2-2\alpha} - 2c_2\frac{\beta\theta}{4\gamma}\frac{1}{r-\alpha} + \frac{c_2^2}{4\gamma}\frac{1}{r} - I & : \theta \geq \theta^* \end{cases} \tag{4.8}$$

The expression for $E[dF(\theta)]$ is, using Itō's Lemma:

$$E[dF(\theta)] = \alpha\theta F'(\theta)dt + \frac{1}{2}\sigma^2\theta^2 F''(\theta)dt \tag{4.9}$$

Replacing Equation (4.9) in Equation (4.8), we note that the value function in the continuation region ($\theta < \theta^*$) is a solution of an *ODE* whose general solution is:

$$F(\theta) = \frac{\theta^2}{4}\frac{1}{r-\sigma^2-2\alpha} - 2c_1\frac{\theta}{4}\frac{1}{r-\alpha} + \frac{c_1^2}{4}\frac{1}{r} + A\theta^{\omega_1} + B\theta^{\omega_2}$$

where

$$\omega_1 = \frac{\sigma^2 - 2\alpha - \sqrt{4\alpha^2 - 4(\alpha-2r)\sigma^2 + \sigma^4}}{2\sigma^2} < 0$$

$$\omega_2 = \frac{\sigma^2 - 2\alpha + \sqrt{4\alpha^2 - 4(\alpha-2r)\sigma^2 + \sigma^4}}{2\sigma^2} > 0$$

In addition, the value function must satisfy the following boundary conditions (already defined in the introduction of this thesis):

$$F(0) = 0 \tag{4.10}$$

$$F(\theta^*) = \Omega(\theta^*) \tag{4.11}$$

$$F'(\theta^*) = \Omega'(\theta^*) \tag{4.12}$$

Equation (4.10) leads to $A = 0$. Then, using the other two boundary conditions given by Equations (4.11) and (4.12), we get to the final system of equations:

$$\begin{cases} \frac{\theta^{*2}}{4}\frac{1}{r-\sigma^2-2\alpha} - 2c_1\frac{\theta^*}{4}\frac{1}{r-\alpha} + \frac{c_1^2}{4}\frac{1}{r} + B\theta^{*\omega_2} = \frac{\beta^2\theta^{*2}}{4\gamma}\frac{1}{r-\sigma^2-2\alpha} - 2c_2\frac{\beta\theta^*}{4\gamma}\frac{1}{r-\alpha} + \frac{c_2^2}{4\gamma}\frac{1}{r} - I \\ \frac{\theta^*}{2}\frac{1}{r-\sigma^2-2\alpha} - \frac{c_1}{2}\frac{1}{r-\alpha} + \omega_2 B\theta^{*\omega_2-1} = \frac{\beta^2\theta^*}{2\gamma}\frac{1}{r-\sigma^2-2\alpha} - 2c_2\frac{\beta}{4\gamma}\frac{1}{r-\alpha} \end{cases}$$

Note that an explicit expression for θ^* is possible to find but given its complexity, it is not worth writing it here. Instead, we just give the conditions that it has to verify, and that allow its value to be determined.

We skip to the extension that consists of introducing an implementation delay, as the concept of cannibalization effect does not make sense considering the assumption that product 1 is not in the market from the moment the investment is undertaken.

4.4 Model with Implementation Delay

In this section, the impact of imposing a delay between the decision to invest and the implementation of the new product is studied for this new objective function. The duration of the delay is considered to be Δ.

We start by stating that we are not able to solve the problem without the assumption used for the model without this implementation delay. This means that we assume that product 1 is not in the market from the moment the new product is implemented.

At time t, the company knows $\theta(t)$ and, in order to maximize its expected profit, it needs to decide now the future value of the quantity of product 1 it should produce at time t', $q_1(t')$, in order to maximize its profit. The value that is obtained is:

$$q_1^{opt}(t') = \frac{E[\theta(t')|\theta(t)] - c_1}{2}$$

that is associated to an expected profit at t' $(t + \Delta)$ of:

$$\begin{aligned} RevBef &= e^{-r\Delta}E\left[p_1(t') \times q_1^{opt}(t') - c_1 q_1^{opt}(t')|\theta(t)\right] \\ &= e^{-r\Delta}\left(\frac{E\left[\theta(t')|\theta(t)\right]^2}{4} - \frac{c_1 E\left[\theta(t')|\theta(t)\right]}{2} + \frac{c_1^2}{4}\right) \\ &= \frac{\theta^2(t)}{4}e^{(2\alpha-r)\Delta} - \frac{c_1\theta(t)}{2}e^{(\alpha-r)\Delta} + e^{-r\Delta}\frac{c_1^2}{4} \end{aligned}$$

Note that, as we are calculating a revenue at a point in time that is not the one where we are (it is a future time), the value we obtain is not an exact revenue but an expected revenue. This holds for all the cases where this implementation delay is involved. Both

'RevBef' and 'RevAft' suffer this slight change on their meaning when this effect of delay is introduced.

We assume that the sunk cost I is paid at the time we decide to invest, but even if it was paid at the time of the real implementation, no additional difficulties would arise, and so we consider we are not losing generality with this choice, in this sense.

At investment time T, the value for the process $\theta(T) = \theta$ is known, but all future values are unknown. As a consequence, both $\theta(s)$ and $\theta(s')$ (for $s \geq T$ and $s' = s + \Delta$) are random variables. Assuming that we are at time $s > T$ and that $\theta(s)$ is known, the optimal quantity is given by:

$$q_2^{opt}(s') = \frac{\beta E\left[\theta(s')|\theta(s)\right] - c_2}{2\gamma}$$

The expression is similar to the one obtained for the case without delay, with the difference that now we have to work with the expected value of the process at the point in study instead of the actual value (as that value is not known due to the delay).

Now, we can compute the expected profit at time $s + \Delta$ ($s + \Delta > T + \Delta$) using the value for the optimal quantity that was just obtained.

$$\begin{aligned}
RevAft =& E\left[p_2(s').q_2^{opt}(s') - c_2 q_2^{opt}(s')\middle|\theta(T) = \theta\right] \\
=& E\left[\left(\beta\theta(s') - \gamma q_2^{opt}(s')\right) q_2^{opt}(s') - c_2 q_2^{opt}(s')\middle|\theta(T) = \theta\right] \\
=& \beta E\left[\frac{\beta E[\theta(s')|\theta(s)] - c_2}{2\gamma}\theta(s')|\theta(T) = \theta\right] \\
& - \gamma E\left[\frac{\beta^2 E[\theta(s')|\theta(s)]^2 - 2\beta E[\theta(s')|\theta(s)]c_2 + c_2^2}{4\gamma^2}|\theta(T) = \theta\right] \\
& - c_2 E\left[\frac{\beta E[\theta(s')|\theta(s)] - c_2}{2\gamma}|\theta(T) = \theta\right] \\
=& \frac{\beta^2}{2\gamma}E\left[\theta(s)e^{\alpha\Delta}\theta(s')|\theta(T) = \theta\right] - \frac{\beta c_2}{2\gamma}E\left[\theta(s')|\theta(T) = \theta\right] \\
& - \frac{\beta^2}{4\gamma}E\left[e^{2\alpha\Delta}\theta^2(s)|\theta(T) = \theta\right] \\
& + \frac{\beta c_2}{2\gamma}E\left[e^{\alpha\Delta}\theta(s)|\theta(T) = \theta\right] \\
& - \frac{c_2^2}{4\gamma} - \frac{\beta c_2}{2\gamma}E\left[e^{\alpha\Delta}\theta(s)|\theta(T) = \theta\right] + \frac{c_2^2}{2\gamma} \\
=& \frac{\beta^2}{2\gamma}e^{2\alpha\Delta}\theta^2 e^{(2\alpha+\sigma^2)(s-T)} - \frac{\beta c_2}{2\gamma}e^{\alpha(\Delta+s-T)}\theta \\
& - \frac{\beta^2}{4\gamma}e^{2\alpha\Delta}\theta^2 e^{(2\alpha+\sigma^2)(s-T)} + \frac{\beta c_2}{2\gamma}e^{\alpha\Delta}\theta e^{(\alpha+\frac{\sigma^2}{2})(s-T)} \\
& - \frac{\beta c_2}{2\gamma}e^{\alpha\Delta}\theta e^{(\alpha+\frac{\sigma^2}{2})(s-T)} - \frac{c_2^2}{2\gamma}
\end{aligned}$$

Finally, using this last result, it becomes possible to compute the Termination Payoff:

$$
\begin{aligned}
\Omega(\theta) =& E\left[\int_{T+\Delta}^{\infty} e^{-r(s'-T)}\left(p_2(s')\times q_2^{opt}(s') - c_2\times q_2^{opt}(s')\right)ds' - I \middle| \theta(T)=\theta\right]\\
=& \int_{T+\Delta}^{\infty} e^{-r(s'-T)}[\frac{\beta^2}{2\gamma}e^{2\alpha\Delta}\theta^2 e^{(2\alpha+\sigma^2)(s-T)} - \frac{\beta c_2}{2\gamma}e^{\alpha(\Delta+s-T)}\theta - \frac{\beta^2}{4\gamma}e^{2\alpha\Delta}\theta^2 e^{(2\alpha+\sigma^2)(s-T)}\\
&+ \frac{\beta c_2}{2\gamma}e^{\alpha\Delta}\theta e^{(\alpha+\frac{\sigma^2}{2})(s-T)} - \frac{\beta c_2}{2\gamma}e^{\alpha\Delta}\theta e^{(\alpha+\frac{\sigma^2}{2})(s-T)} - \frac{c_2^2}{2\gamma}]ds' - I\\
=& e^{-r\Delta}[\frac{\beta^2\theta^2}{2\gamma}e^{2\alpha\Delta}(\frac{1}{2}-c_2)\frac{1}{r-2\alpha-\sigma^2} - \frac{\beta c_2}{2\gamma}\theta\frac{1}{r-\alpha} + \frac{c_2^2}{4\gamma}\frac{1}{r}\\
&+ \frac{\beta c_2}{2\gamma}\theta e^{\alpha\Delta}(\frac{1}{r-\alpha} - \frac{1}{r-\alpha-\frac{\sigma^2}{2}})] - I
\end{aligned}
$$

Therefore, one can construct the bellman equation, and solve it, in order to derive the value function of the firm.

The usual condition that has to be verified in order that we have a unique solution θ^* is not easy to analyze and its study had to be done, once again, numerically. The result was that the condition is in fact verified for all the admissible values of the different influent variables.

As seen before, in the continuation region, the value function $F(.)$ is the solution of a non-homogeneous second order differential equation whose general solution is (still including constants that have to be determined using the boundary conditions):

$$
F(\theta) = \frac{\theta^2}{4}e^{(2\alpha-r)\Delta}\frac{1}{r-2\alpha-\sigma^2} - \frac{c_1\theta}{2}e^{(\alpha-r)\Delta}\frac{1}{r-\alpha} + \frac{e^{-r\Delta}\frac{c_1^2}{4}}{r} + A\theta^{\omega_1} + B\theta^{\omega_2}
$$

with

$$
\begin{aligned}
\omega_1 &= \frac{\sigma^2 - 2\alpha - \sqrt{4\alpha^2 - 4(\alpha-2r)\sigma^2 + \sigma^4}}{2\sigma^2} < 0\\
\omega_2 &= \frac{\sigma^2 - 2\alpha + \sqrt{4\alpha^2 - 4(\alpha-2r)\sigma^2 + \sigma^4}}{2\sigma^2} > 0
\end{aligned}
$$

To determine the values of the constants A and B, and also the value of θ^*, the three boundary conditions seen in the introduction of this thesis have to be verified. These conditions were also mentioned again in Equations (4.10), (4.12) and (4.11). The system that is obtained has two equations and two variables to determine:

$$
\begin{aligned}
&\frac{\theta^{*2}}{4}e^{(2\alpha-r)\Delta}\frac{1}{r-2\alpha-\sigma^2} - \frac{c_1\theta^*}{2}e^{(\alpha-r)\Delta}\frac{1}{r-\alpha} + \frac{e^{-r\Delta}\frac{c_1^2}{4}}{r} + B\theta^{*\omega_2}\\
&= e^{-r\Delta}[\frac{\beta^2\theta^{*2}}{2\gamma}e^{2\alpha\Delta}(\frac{1}{2}-c_2)\frac{1}{r-2\alpha-\sigma^2} - \frac{\beta c_2}{2\gamma}\theta^*\frac{1}{r-\alpha} + \frac{c_2^2}{4\gamma}\frac{1}{r}\\
&\quad + \frac{\beta c_2}{2\gamma}\theta^* e^{\alpha\Delta}(\frac{1}{r-\alpha} - \frac{1}{r-\alpha-\frac{\sigma^2}{2}})] - I
\end{aligned}
$$

and

$$\frac{\theta^*}{2}e^{(2\alpha-r)\Delta}\frac{1}{r-2\alpha-\sigma^2}-\frac{c_1}{2}e^{(\alpha-r)\Delta}\frac{1}{r-\alpha}+B\omega_2\theta^{*\omega_2-1}$$

$$=e^{-r\Delta}[\frac{\beta^2\theta^*}{\gamma}e^{2\alpha\Delta}(\frac{1}{2}-c_2)\frac{1}{r-2\alpha-\sigma^2}-\frac{\beta c_2}{2\gamma}\frac{1}{r-\alpha}+\frac{\beta c_2}{2\gamma}e^{\alpha\Delta}(\frac{1}{r-\alpha}-\frac{1}{r-\alpha-\frac{\sigma^2}{2}})]$$

Explicit solutions can be obtained for constant B and for our variable of interest, θ^*, but they are not presented, once again, given their complexity.

4.5 Numerical Analysis

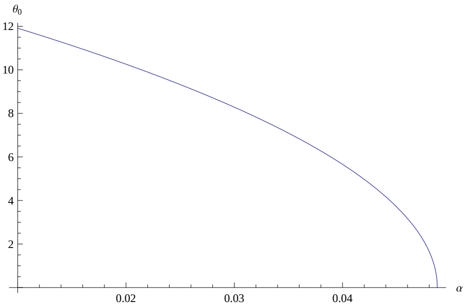
In this section, we analyze the influence of the parameters of the *GBM* (α and σ) and also of the unitary costs in the behavior of the trigger value.

Note that the values that were chosen for c_1 and c_2 (2 and 1, respectively) have to verify the condition that $c_1 > c_2$, as this is a sufficient condition to guarantee that product 1 is out of the market after investment, as seen in Chapter 4.

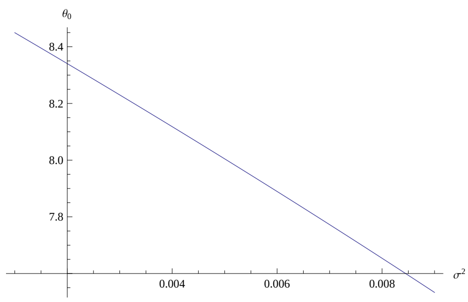
Remember that two approximations were considered; otherwise we would not have able to obtain final results for the optimal policy. These approximations ($\theta(t) > \frac{c_2}{\beta}$ and $\theta(t) > c_1$) depend, after all, on the initial value of the demand of the process.

4.5.1 *NPV* Rule

We start with the study for the *NPV* approach.



(a) θ_0 as a function of α (drift)

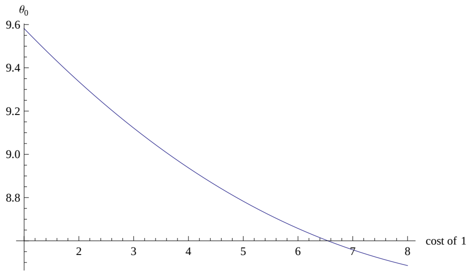


(b) θ_0 as a function of σ (volatility)

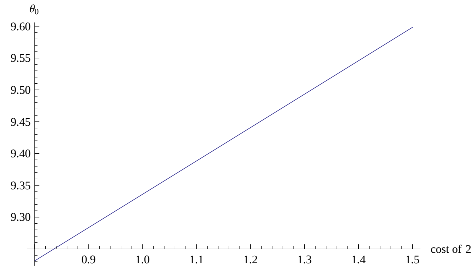
Figure 4.1: θ_0 as a function of the parameters of the *GBM*

In Figure 4.1, we can see that the behavior of the trigger value as a function of both variables is the same as for the objective function used when Figure 2.1 was obtained, which is something that was expected, as the influence of the unitary costs that were introduced is not supposed to affect those behaviors.

Regarding the behavior of θ_0 given the variation of the unitary costs:



(a) θ_0 as a function of c_1



(b) θ_0 as a function of c_2

Figure 4.2: θ_0 as a function of the unitary production costs

From these plots, we are able to present the following comments:

- **Influence of** c_1: When this unitary cost increases, the trigger value decreases, which is something logical as, if the cost for the old product increases, it becomes more appealing to invest in the new one;

- **Influence of** c_2: Given the same argument, the behavior is now the opposite, in terms of monotonicity. If the cost for the second product increases, then it becomes better not to invest in this product. Note that the relation seems to be linear for this case but the range of values that is being studied is smaller than the one for the other cost, which does not allow us to make a fair comparison.

In fact, the restriction that $c_1 > c_2$ lead us to use a smaller range of values for the cost of the new product in the previous image. In order to avoid this situation, we fix c_1 in a new value to allow the range of c_2 to be the same as the one for c_1. The new value of c_1 is 9.

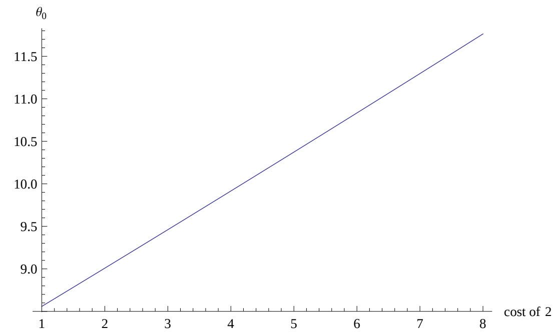


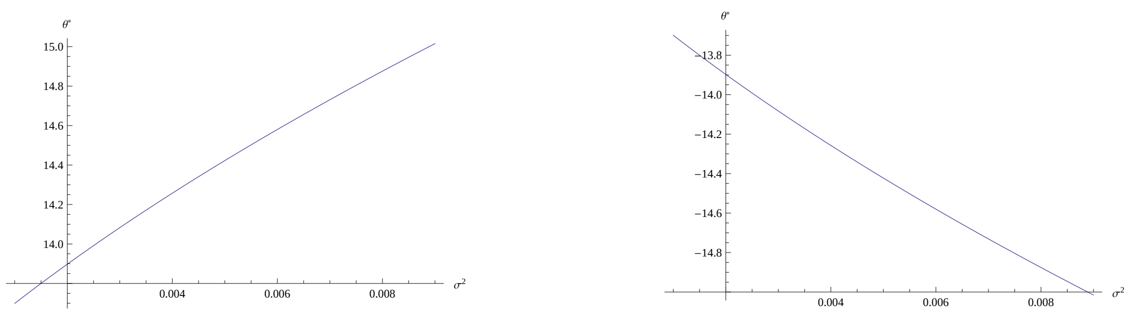
Figure 4.3: θ_0 as a function of c_2

Influence of c_2 **on the trigger value** θ_0 - We verify that the linear behavior seems to be, in fact, verified. The comparison with the image referring to c_1 is now possible and the behaviors are, in fact, different. If this difference is logical in terms of monotonicity, it is not logical in terms of the different form of the curves, as both costs appear in the exact same way in the expressions associated to the derivation of the optimal policy, given the assumption that product 1 is not produced after the investment. This is the first indication that the approximations under which all these results were obtained do not seem to be valid.

4.5.2 Dynamic Programming

In fact, considering now the Dynamic Programming approach, it is easy to confirm that the approximations are clearly not valid.

As an experience to test this possible situation, we fix two different sets of values for the costs. In the figure below, the first image is the result of fixing both costs with the value 0, and the second one is associated to the costs 2×10^{-17} and 1×10^{-17}, for product 1 and 2 respectively.



(a) θ_0 as a function of σ (no costs)

(b) θ_0 as a function of σ (infinitesimal costs)

Figure 4.4: θ_0 as a function of σ (two cases)

We can observe in Figure 4.4 that if we consider the costs to be 0, then the same plot as in Figure 2.4 is obtained. However, if we introduce some infinitesimal costs, the image changes completely and behaves in a non-comprehensible way. This clearly shows how the approximations we are considering are not valid. There is no reason for the occurrence of a change like this in the obtained plot given an infinitesimal change in the unitary costs.

After observing this, the analysis for this model is ended here, with the final conclusion that the approximations that are assumed to be valid in order to solve this case ($\theta(t) > \frac{c_2}{\beta}$ and $\theta(t) > c_1$) are not valid, after all.

Chapter 5

Iso-elastic Inverse Demand Function

In this chapter, we analyze the optimal investment policy assuming a different price model. Whereas until this point in this thesis, and also in [2], a linear relationship is assumed between the price, the quantity and the demand process, in this chapter we assume that the relation is multiplicative.

For this new situation, we derive the optimal investment policy following the same steps as for previous models.

Assuming the same notation as in the previous chapters, we consider now that the price of the product, the demand process and the quantities are related through the following relation:

$$p(t) = \theta(t) q(t)^{-\zeta} \tag{5.1}$$

with $\zeta < 1$. This relation holds in the case where there is only one product in the market. When both products (1 and 2) are being produced, the model for the price is the following:

$$\begin{aligned} p_1(t) &= a\theta(t) q_1(t)^{-\zeta} q_2(t)^{-\eta} \\ p_2(t) &= \beta\theta(t) q_2(t)^{-\gamma} q_1(t)^{-\eta} \end{aligned}$$

with $0 < \eta < \gamma < \zeta < 1$. We remark that these assumptions follow from the same reasoning that was considered for the linear case, seen in the previous chapters and also in [2].

Moreover, as soon as product 1 is no longer in the market ($q_1(t) = 0$), the price of product 2 is simply given by:

$$p_2(t) = \beta\theta(t) q_2(t)^{-\gamma} \tag{5.2}$$

One concept that is worth introducing here is the concept of elasticity, as it has a very important role in the law we are assuming for the price. It is basically, in economy, a measure of how a relative variation on the price affects the relative variation on the quantity that is sold, for a certain product.

We now prove that variable ζ represents, in fact, given Equation (5.1), the 'elasticity parameter'. This means that we have to prove the following relation:

$$\zeta = \frac{p}{q} \frac{dq}{dp}$$

The proof is done now.

$$q(t) = p(t)^{\zeta}\theta(t)^{-1+\zeta} \Rightarrow \frac{dq}{dp} = \zeta p^{\zeta-1}\theta^{-1+\zeta} \Leftrightarrow$$

$$\Leftrightarrow \frac{p}{q}\frac{dq}{dp} = \zeta\frac{p^{\zeta}}{q}\theta^{-1+\zeta} \Leftrightarrow \frac{p}{q}\frac{dq}{dp} = \zeta$$

Note that ζ does not depend on time.

In fact, this multiplicative model for the price is something that is commonly used in the field of economy, given the fact that it is realistic and that we have a nice interpretation for the main parameter that is involved, as explained and proved before.

Note that, for this model, we also assume that there is a unitary production cost c_i, just like we did in the previous chapter, where i corresponds to the number of the product to which the cost is associated. The objective function is thus the same as in Chapter 4.

5.1 Optimal Quantities and Revenues

We can easily derive the optimal quantities to be produced at each time t, before investment.

Assuming that product 1 is the only one in the market, then the optimal revenue is given by:

$$\begin{aligned} RevBef &= \max_{q_1(t)}\{q_1(t)p_1(t) - c_1 q_1(t)\} \\ &= \theta(t)^{\frac{1}{\zeta}} c_1^{1-\frac{1}{\zeta}} (1-\zeta)^{\frac{1}{\zeta}-1}\zeta \end{aligned} \tag{5.3}$$

This is associated to the optimal quantity $q_1^{opt}(t) = (\frac{c_1}{(1-\zeta)\theta(t)})^{-\frac{1}{\zeta}} \geq 0$, with probability 1.

For the situation where both products are in the market, after investment, a problem occurs. When costs were introduced in the previous chapter, the problems started when we tried to determine the optimal policy using NPV approach. However, for this multiplicative model, we cannot even determine the optimal quantities, if the case where both products co-exist in the market. In fact, the maximization of:

$$\max_{\{q_1(t), q_2(t)\}}\{q_1(t)p_1(t) - c_1 q_1(t) + q_2(t)p_2(t) - c_2 q_2(t)\}$$

involves terms in such a way that we are not able to solve it.

This means that, from here on, such as for the previous chapter, we will assume that we abandon product 1 as soon as product 2 appears in the market. Although this is only a partial answer to the problem, it is, in fact, a realistic description, as most of the time companies invest in a new product that is better than a previous one and they simply stop producing that old version after that investment is done. Even if they do not stop the production completely, the major percentage, in terms of profit, is associated to the new product, and so this seems to be a good approximation. This also applies to the previous assumption in the previous chapter.

Given this assumption, it is now possible to obtain the optimal revenue and the associated optimal quantity of product 2 after the investment, as:

$$\begin{aligned} RevAft &= \max_{q_2(t)}\{q_2(t)p_2(t) - c_2q_2(t)\} \\ &= \theta(t)^{\frac{1}{\gamma}} c_2^{1-\frac{1}{\gamma}} \beta^{\frac{1}{\gamma}} (1-\gamma)^{\frac{1}{\gamma}-1}\gamma \end{aligned} \tag{5.4}$$

Recall that we assume that if product 1 is not produced, the model for the price of product 2 is the one given by Equation (5.2).

The associated optimal quantity is $q_2^{opt}(t) = (\frac{c_2}{(1-\gamma)\beta\theta(t)})^{-\frac{1}{\gamma}}$.

Note that the optimal quantity, as well as the optimal revenue, both increase with β and $\theta(t)$, and they decrease with c_2, as expected.

5.2 *NPV* Rule

Given the optimal quantities and the consequent optimal revenues for both situations, before and after investment, it is now possible to calculate the corresponding expected discounted cash flows for each situation.

We start with the case before investment:

$$\begin{aligned} CashBef &= E\left[\int_0^\infty e^{-rt}(p_1(t)q_1^{opt}(t) - c_1q_1^{opt}(t))dt \middle| \theta(0) = \theta_0\right] \\ &= E\left[\int_0^\infty e^{-rt}(\theta(t)^{\frac{1}{\zeta}} c_1^{1-\frac{1}{\zeta}}(1-\zeta)^{\frac{1}{\zeta}-1}\zeta)dt \middle| \theta(0) = \theta_0\right] \\ &= c_1^{1-\frac{1}{\zeta}}(1-\zeta)^{\frac{1}{\zeta}-1}\zeta \int_0^\infty e^{-rt} E\left[\theta^{\frac{1}{\zeta}}(t) \middle| \theta(0) = \theta_0\right] dt \\ &= \theta_0^{\frac{1}{\zeta}} c_1^{1-\frac{1}{\zeta}}(1-\zeta)^{\frac{1}{\zeta}-1}\zeta \frac{1}{r - \frac{1}{\zeta}\alpha - (\frac{1}{\zeta^2} - \frac{1}{\zeta})\frac{\sigma^2}{2}} \end{aligned} \tag{5.5}$$

where the step that leads to the final result is obtained using the Moment Generating Function of the Normal distribution (and the fact that the demand process is a *GBM*), as this implies the relation:

$$E[\theta(t)^{\frac{1}{\zeta}}|\theta(0) = \theta_0] = \theta_0^{\frac{1}{\zeta}} \exp[(\frac{1}{\zeta}\alpha + (\frac{1}{\zeta^2} - \frac{1}{\zeta})\frac{\sigma^2}{2})t]$$

As for the expected discounted cash flow for the case where we decide to invest, we obtain:

$$
\begin{aligned}
CashAft &= E\left[\int_0^\infty e^{-rt}\left[p_2(t)q_2(t) - c_2 q_2(t)\right]dt - I \middle| \theta(0) = \theta_0\right] \\
&= E\left[\int_0^\infty e^{-rt}\left[\theta(t)^{\frac{1}{\gamma}} c_2^{1-\frac{1}{\gamma}} \beta^{\frac{1}{\gamma}} (1-\gamma)^{\frac{1}{\gamma}-1}\gamma\right] dt - I \middle| \theta(0) = \theta_0\right] \\
&= c_2^{1-\frac{1}{\gamma}} \beta^{\frac{1}{\gamma}} (1-\gamma)^{\frac{1}{\gamma}-1}\gamma E\left[\int_0^\infty e^{-rt}\left[\theta(t)^{\frac{1}{\gamma}}\right] dt \middle| \theta(0) = \theta_0\right] - I \\
&= \theta_0^{\frac{1}{\gamma}} c_2^{1-\frac{1}{\gamma}} \beta^{\frac{1}{\gamma}} (1-\gamma)^{\frac{1}{\gamma}-1}\gamma \frac{1}{r - \frac{1}{\gamma}\alpha - (\frac{1}{\gamma^2} - \frac{1}{\gamma})\frac{\sigma^2}{2}} - I
\end{aligned}
\tag{5.6}
$$

We can guarantee the convergence of the integral if:

$$
r > \frac{1}{\gamma}\alpha + (\frac{1}{\gamma^2} - \frac{1}{\gamma})\frac{\sigma^2}{2}
$$

So, we conclude that for the NPV approach, based on both Equations (5.5) and (5.6), the company should invest if and only if:

$$
\begin{aligned}
&\theta_0^{\frac{1}{\gamma}} c_2^{1-\frac{1}{\gamma}} \beta^{\frac{1}{\gamma}} (1-\gamma)^{\frac{1}{\gamma}-1}\gamma \frac{1}{r - \frac{1}{\gamma}\alpha - (\frac{1}{\gamma^2} - \frac{1}{\gamma})\frac{\sigma^2}{2}} - I \\
&> \theta_0^{\frac{1}{\zeta}} c_1^{1-\frac{1}{\zeta}} (1-\zeta)^{\frac{1}{\zeta}-1}\zeta \frac{1}{r - \frac{1}{\zeta}\alpha - (\frac{1}{\zeta^2} - \frac{1}{\zeta})\frac{\sigma^2}{2}}
\end{aligned}
$$

5.3 Dynamic Programming

Regarding this approach, we skip directly to the expression of both $\pi(\theta)$ and $\Omega(\theta)$:

- $\Omega(\theta) = \beta^{\frac{1}{\gamma}} \theta^{\frac{1}{\gamma}} c_2^{1-\frac{1}{\zeta}} (1-\zeta)^{\frac{1}{\zeta}-1}\zeta \frac{1}{r - \frac{1}{\zeta}\alpha - (\frac{1}{\zeta^2} - \frac{1}{\zeta})\frac{\sigma^2}{2}}$
- $\pi(\theta) = \theta^{\frac{1}{\zeta}} c_1^{1-\frac{1}{\zeta}} (1-\zeta)^{\frac{1}{\zeta}-1}$

We have also seen before that the condition for the uniqueness of the threshold is guaranteed as long as the expression in Equation (1.8) is decreasing in θ. This depends, once again, on the two functions whose results we have just provided. The analytical study of the resulting expression is not easy to do and so, like in all the cases where Dynamic Programming approach was used in the previous chapter, we were forced to check the condition numerically.

The general explicit form of the value function is the same as always, as well as the boundary conditions. In order to obtain θ^*, we have to solve the system 2×2 below (where $A = 0$, and B and θ^* are still to determine) given by:

$$\theta^{*\frac{1}{\zeta}} c_1^{1-\frac{1}{\zeta}} (1-\zeta)^{\frac{1}{\zeta}-1} \zeta \frac{1}{r - \frac{1}{\zeta}\alpha - (\frac{1}{\zeta^2} - \frac{1}{\zeta})\frac{\sigma^2}{2}} + B\theta^{*\omega_2}$$
$$= \beta^{\frac{1}{\gamma}} \theta^{*\frac{1}{\gamma}} c_2^{1-\frac{1}{\gamma}} (1-\gamma)^{\frac{1}{\gamma}-1} \gamma \frac{1}{r - \frac{1}{\gamma}\alpha - (\frac{1}{\gamma^2} - \frac{1}{\gamma})\frac{\sigma^2}{2}} - I$$

and

$$\theta^{*\frac{1}{\zeta}-1} c_1^{1-\frac{1}{\zeta}} (1-\zeta)^{\frac{1}{\zeta}-1} \frac{1}{r - \frac{1}{\zeta}\alpha - (\frac{1}{\zeta^2} - \frac{1}{\zeta})\frac{\sigma^2}{2}} + B\omega_2\theta^{*\omega_2 - 1}$$
$$= \beta^{\frac{1}{\gamma}} \theta^{*\frac{1}{\gamma}-1} c_2^{1-\frac{1}{\gamma}} (1-\gamma)^{\frac{1}{\gamma}-1} \frac{1}{r - \frac{1}{\gamma}\alpha - (\frac{1}{\gamma^2} - \frac{1}{\gamma})\frac{\sigma^2}{2}}$$

An explicit solution is not possible to determine, given the fact that the restriction on the price is now multiplicative, generating a non-linear system, but in Chapter **??** a numerical study is done.

Before proceeding to the extension of the model with delay, it is important to mention that the effect of cannibalization does not affect the solution of the problem, as product 1 is not being produced from the moment product 2 is implemented, and therefore the loss of appetite that is verified for this old product does not have any influence.

5.4 Model with Implementation Delay

In this section, we determine the optimal policy for this new model with the multiplicative restriction, but now assuming that the delay effect that has already been considered in previous chapters is also added. This basically means that the company has to wait Δ units of time from the moment it decides to invest until the product is actually in the market.

Once again, the approach that is considered for this particular extension is the Dynamic Programming one.

Most steps during this section coincide with steps that have been done before for other models and so we frequently omit details.

We start by calculating the optimal quantities for products 1 and 2, and associated expected profits.

For the case before investment,

$$q_1^{opt}(t') = \arg\max_{q_1} E[p_1(t') \times q_1 - c_1 q_1 | \theta(t)]$$
$$= \arg\max_{q_1} E[(\theta(t') q_1^{-\zeta}) q_1 - c_1 q_1 | \theta(t)]$$
$$= (\frac{c_1}{(1-\zeta) E[\theta(t')|\theta(t)]})^{-\frac{1}{\zeta}}$$

This leads to an expected profit at time t' of:

$$RevBef = e^{\Delta(\frac{\alpha}{\zeta}-r)}\theta(t)^{\frac{1}{\zeta}}c_1^{1-\frac{1}{\zeta}}(1-\zeta)^{\frac{1}{\zeta}-1}\zeta$$

For the 'with investment' situation, the corresponding optimal revenue is the value taken by the objective function given the optimal quantity of product 2, as we fix the quantity of product 1 in the value 0 in this situation (where s', s and T represent exactly the same as in Section 4.4):

$$RevAft = E\left[p_2(s').q_2^{opt}(s') - c_2 q_2^{opt}(s')\middle|\theta(T)=\theta\right] \tag{5.7}$$

After replacing $p_2(s')$ in the previous expression using the law given by Equation (5.2), the expression can be maximized and the optimal quantity that corresponds to that maximization is given by:

$$q_2^{opt}(s') = (\frac{c_2}{(1-\gamma)\beta E\left[\theta(s')|\theta(s)\right]})^{-\frac{1}{\gamma}}$$

Replacing this quantity in Equation (5.7), we finally obtain the result:

$$\begin{aligned}
RevAft =& \beta^{\frac{1}{\gamma}}c_2^{1-\frac{1}{\gamma}}[(1-\gamma)^{\frac{1}{\gamma}-1}e^{\alpha\Delta(\frac{1}{\gamma}-1)}E\left[\theta(s')\theta(s)^{\frac{1}{\gamma}-1}|\theta(T)=\theta\right] \\
&-(1-\gamma)^{\frac{1}{\gamma}}e^{\alpha\Delta\frac{1}{\gamma}}E\left[\theta(s)^{\frac{1}{\gamma}}|\theta(T)=\theta\right]] \\
=& \beta^{\frac{1}{\gamma}}c_2^{1-\frac{1}{\gamma}}e^{\alpha\Delta\frac{1}{\gamma}}\theta^{\frac{1}{\gamma}}e^{\frac{\alpha}{\gamma}(s-T)}e^{(\frac{1}{\gamma^2}-\frac{1}{\gamma})\frac{\sigma^2}{2}(s-T)}(1-\gamma)^{\frac{1}{\gamma}-1}\gamma
\end{aligned}$$

There is a result that had to be derived in order to make the last step in the previous computation. The derivation is not shown here but the result is the following:

$$E\left[\theta(s')\theta(s)^{\frac{1}{\gamma}-1}|\theta(T)=\theta\right] = e^{\alpha\Delta}\theta^{\frac{1}{\gamma}}e^{\frac{\alpha}{\gamma}(s-T)}e^{(\frac{1}{\gamma^2}-\frac{1}{\gamma})\frac{\sigma^2}{2}(s-T)}$$

Finally, it becomes possible to compute the Termination Payoff:

$$\begin{aligned}
\Omega(\theta) &= E\left[\int_{T+\Delta}^{\infty} e^{-r(s'-T)}\left(p_2(s')\times q_2^{opt}(s') - c_2\times q_2^{opt}(s')\right)ds' - I\middle|\theta(T)=\theta\right] \\
&= \beta^{\frac{1}{\gamma}}c_2^{1-\frac{1}{\gamma}}\theta^{\frac{1}{\gamma}}(1-\gamma)^{\frac{1}{\gamma}-1}\gamma\frac{e^{\Delta(\frac{\alpha}{\gamma}-r)}}{r-\frac{\alpha}{\gamma}-(\frac{1}{\gamma^2}-\frac{1}{\gamma})\frac{\sigma^2}{2}} - I
\end{aligned}$$

Note that this function behaves as expected, increasing in β, α, θ and Δ, and decreasing in c_2 and σ.

The value function $F(\theta)$ becomes totally defined, given Equation (1.4), after stating that:

$$\pi(\theta) = e^{\Delta(\frac{\alpha}{\zeta}-r)}\theta^{\frac{1}{\zeta}}c_1^{1-\frac{1}{\zeta}}(1-\zeta)^{\frac{1}{\zeta}-1}\zeta$$

The θ space can be divided by the threshold θ^* if the condition in Equation (1.8) is decreasing in θ. As for the model without delay, the verification had to be done numerically. Moreover, we were successful on doing it.

We directly skip to the system that has to be solved in order to determine variables B and θ^*:

$$c_1^{1-\frac{1}{\zeta}}\theta^{*\frac{1}{\zeta}}(1-\zeta)^{\frac{1}{\zeta}-1}\zeta\frac{e^{\Delta(\frac{\alpha}{\zeta}-r)}}{r-\frac{\alpha}{\zeta}-(\frac{1}{\zeta^2}-\frac{1}{\zeta})\frac{\sigma^2}{2}}+B\theta^{*\omega_2}$$

$$=\beta^{\frac{1}{\gamma}}c_2^{1-\frac{1}{\gamma}}\theta^{*\frac{1}{\gamma}}(1-\gamma)^{\frac{1}{\gamma}-1}\gamma\frac{e^{\Delta(\frac{\alpha}{\gamma}-r)}}{r-\frac{\alpha}{\gamma}-(\frac{1}{\gamma^2}-\frac{1}{\gamma})\frac{\sigma^2}{2}}-I$$

and

$$c_1^{1-\frac{1}{\zeta}}\theta^{*\frac{1}{\zeta}-1}(1-\zeta)^{\frac{1}{\zeta}-1}\frac{e^{\Delta(\frac{\alpha}{\zeta}-r)}}{r-\frac{\alpha}{\zeta}-(\frac{1}{\zeta^2}-\frac{1}{\zeta})\frac{\sigma^2}{2}}+B\omega_2\theta^{*\omega_2-1}$$

$$=\beta^{\frac{1}{\gamma}}c_2^{1-\frac{1}{\gamma}}\theta^{*\frac{1}{\gamma}-1}(1-\gamma)^{\frac{1}{\gamma}-1}\frac{e^{\Delta(\frac{\alpha}{\gamma}-r)}}{r-\frac{\alpha}{\gamma}-(\frac{1}{\gamma^2}-\frac{1}{\gamma})\frac{\sigma^2}{2}}$$

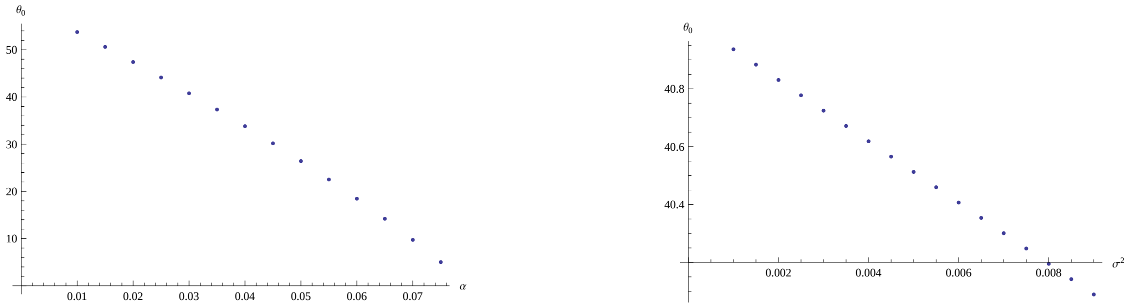
Solutions can then be obtained for specific numerical values.

5.5 Numerical Analysis

Note that, for this model, the condition associated with the trigger value generates a non-linear equation, due to the multiplicative law for the price, and so it is not possible to obtain continuous plots like the ones that are obtained in Chapter 2. It is only possible to obtain the solution for specific points. Nevertheless we just have to consider a sufficiently big number of points, so that the behavior of the curve is still clear, just like for the case where we have two demand processes.

5.5.1 *NPV* Rule

We start with the study of the behavior of the trigger value as a function of the parameters of the *GBM*, for the *NPV* approach.



(a) θ_0 as a function of α (drift)

(b) θ_0 as a function of σ (volatility)

Figure 5.1: θ_0 as a function of the parameters of the *GBM*

From these plots, we are able to present the following comments:

- **Influence of the drift of the *GBM***: despite the big difference between the model associated to this image and the one associated to the corresponding image in Figure 2.1, given the difference on the law for the price, we can see that the behavior of the two curves is very similar. A possible explanation for this fact, as this conclusion is not the expected one at all, is that the behavior would be different if the range of values that was considered was larger. On the other hand, the fact that the type of monotonicity is the same is completely intuitive;

- **Influence of the volatility of the *GBM***: in terms of the type of curve that is obtained, the same comment that was made for the drift is valid. The behaviors are unexpectedly similar. In terms of the type of monotonicity, for the linear restriction on the price, a proper justification was found, using Jensen's Inequality, and the same argument can, in fact, be used once again, as we check right next.

We now prove why the trigger value increases with the volatility. Proceeding the same way that we did for the corresponding situation in Chapter 2, we prove that the objective function can be written as a convex function of the demand process. The objective function can be firstly written as a function of the price and of the demand process, given the fact that the optimal quantity is given by $q(t) = (\frac{\theta(t)}{p(t)})^{\frac{1}{\gamma}}$, as:

$$p(t)(\frac{\theta(t)}{p(t)})^{\frac{1}{\gamma}} - c(\frac{\theta(t)}{p(t)})^{\frac{1}{\gamma}}$$

Now, in order to obtain the optimal value of the price as a function of $\theta(t)$, we derive with respect to the price and then calculate the corresponding zero. The value of the derivative is:

$$\theta(t)(\frac{-p(t)^{-\frac{1}{\gamma}-1}(p(t)-c)}{\gamma} + p(t)^{\frac{-1}{\gamma}})$$

It becomes obvious that the zero of the derivative does not depend on θ, which means that the objective function can be written as:

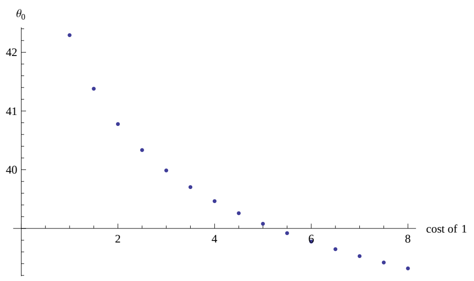
$$C\theta(t)^{\frac{1}{\gamma}}$$

where C denotes a constant and $0 < \gamma < 1$, which implies the convexity of the desired function, as we have θ to the power of a number that is greater than 1.

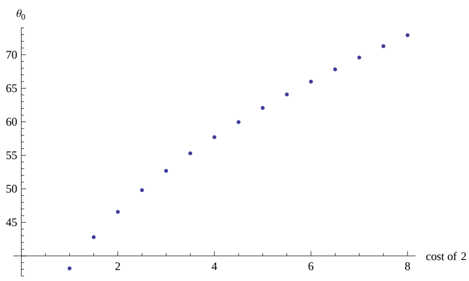
As a consequence, as the profit is a convex function of the demand process, the same argument can be used.

As the plots that were obtained in the previous subsection seem to indicate, as we already mentioned, that the approximations that were considered in the corresponding chapter are not valid, we do not have any term of comparison for the behavior of the trigger value as a function of the costs.

Note, in the figure below, in particular in the second image, that, we fix $c_1 = 9$ once again, in order to work with two ranges of the same size in both images and, therefore, to be able to make fair comparisons.



(a) θ_0 as a function of c_1



(b) θ_0 as a function of parameter a

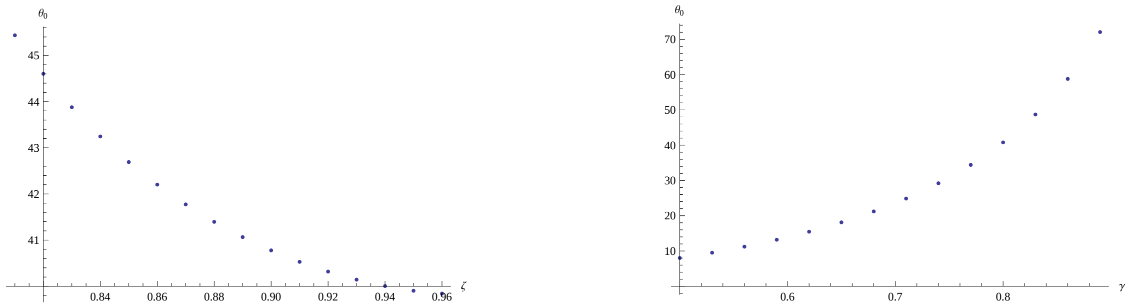
Figure 5.2: θ_0 as a function of the unitary production costs

From these plots, we are able to present the following comments:

- **Influence of c_1**: the type of curve is a direct consequence of the restriction on the price (multiplicative) that we are assuming. As for the type of monotonicity that is obtained, it is justified by the fact that a bigger unitary cost on the old product leads the company to invest more easily;

- **Influence of c_2**: just like for the other image of the previous figure, related to c_1, the type of curve is justified by the law that determines the price. The type of monotonicity is also the expected one, and the justification is also similar to the one for the other cost. One last important thing to be mentioned is the fact that, despite this curve is clearly non-linear like the one for the cost of the old product, this one is less pronounced, as the scale of the θ_0 axes is different.

For this multiplicative model, it is also worth studying the behavior of the trigger value as a function of the elasticities.



(a) θ_0 as a function of the elasticity of product 1

(b) θ_0 as a function of the elasticity of product 2

Figure 5.3: θ_0 as a function of some variables of the elasticities

From these plots, we are able to present the following comments:

- **Influence of the elasticity of product** 1: the type of curve that is obtained is the expected one, given the law for the price. Also, the fact that the function is decreasing is also very intuitive, given the definition of elasticity. In fact, when the elasticity increases, we tend to abandon the old product (invest in the other), as we are increasing the effect of a relative variation on the price on the relative variation of the quantities (contrary effect), leaving a smaller margin for playing with the quantities that are produced;

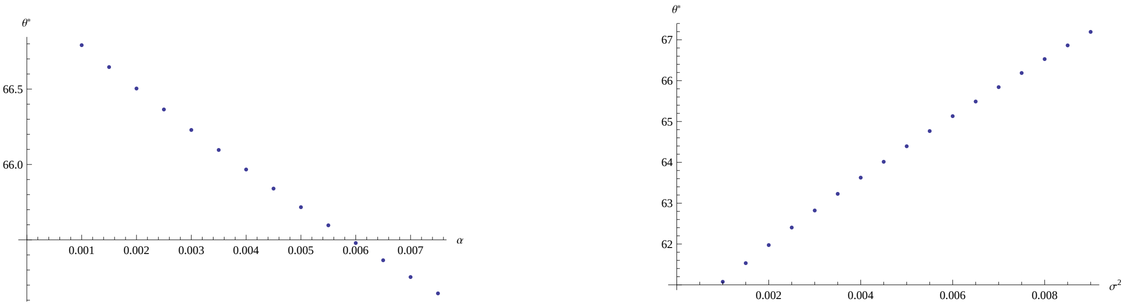
- **Influence of the elasticity of product** 2: The type of curve is the expected one, due to the law for the price, once again. Also, the argument for the monotonicity is just the same as for the other elasticity, as we tend to abandon our product as the elasticity increases, which, in this case, supports the decision not to invest and, thus, increases the trigger value.

In general, the behaviors that were obtained are according to expectation, not only in terms of the non-linearity that is verified but also in terms of the type of monotonicity that is observed.

5.5.2 Dynamic Programming

Now, the same type of relation is studied, for the same variables, but now taking into consideration the results that we have for the Dynamic Programming approach.

We start, once again, by analyzing the behavior associated to the parameters of the *GBM*.

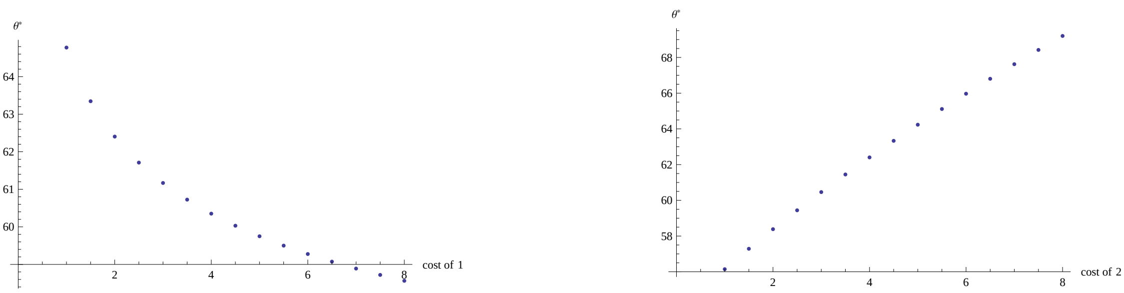


(a) θ^* as a function of α (drift) (b) θ^* as a function of σ (volatility)

Figure 5.4: θ^* as a function of the parameters of the *GBM*

Influence of the parameters of the *GBM*: in the case of this last figure, Figure 5.4, both images are commented at the same time, as the comment is exactly the same for both. The same way as for the correspondent image in the *NPV* approach, the monotonicity is just the one we expect and the type of curve is surprisingly similar to the one that was obtained in Figure 2.4. For both of these situations, the monotonicity and type of curve, the comment for the *NPV* approach is still valid, for both parameters, in terms of the existing relation with the corresponding image for the Basic Model.

We move on to the study of the behavior of the trigger value as a function of the unitary costs of production.

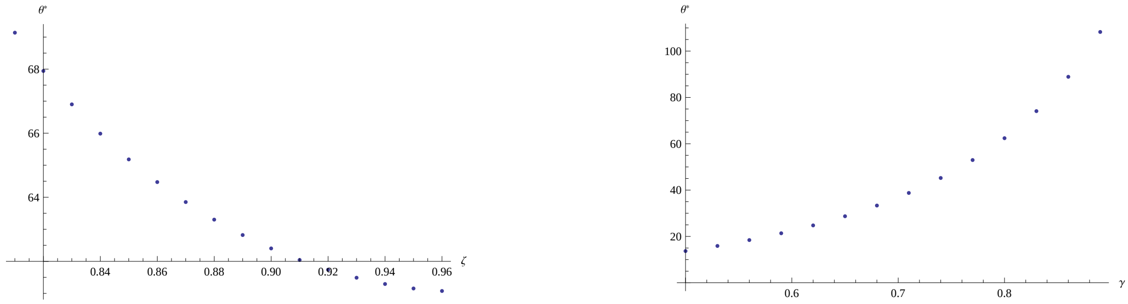


(a) θ^* as a function of c_1 (b) θ^* as a function of c_2

Figure 5.5: θ^* as a function of the unitary production costs

Influence of the costs - just like for Figure 5.4, the comment associated to Figure 5.5 is done for both images at the same time, as the idea is the same for both. The behavior that is observed does not depend on the type of approach we choose to use (the image that is obtained behaves exactly the same way as for the *NPV* case). This means that the comments that are done for the other approach, after Figure 5.5, still apply, for both costs.

We proceed by analyzing the behavior of the trigger value as a function of the elasticities.



(a) θ^* as a function of parameter ζ (b) θ^* as a function of parameter γ

Figure 5.6: θ^* as a function of the elasticities

Influence of the elasticities - There is not much to be commented on this set of images, as the comment is precisely the same that is done for the figure that refers to the costs (Figure 5.5). In fact, the behavior is just the same as for the NPV approach.

The last variable to be studied is Δ, related to the Implementation Delay extension. It is important to mention, in order to avoid any type of doubt, that this variable is studied in the context of an extension of the multiplicative model but the model is not the same as the one that generated the previous images of this subsection. This next image is based on the model in Section 5.4, while the previous ones refer to the same model without implementation delay, associated to the optimal policy in Section 5.3. In fact, it supports the general conclusions that come right after it.

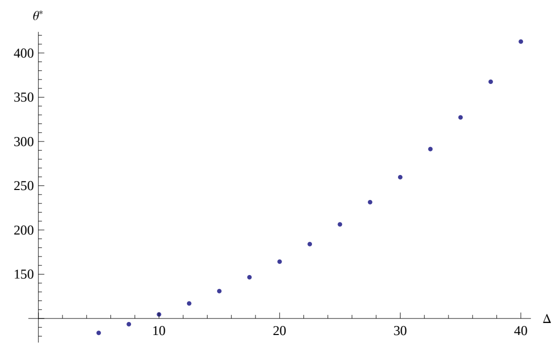


Figure 5.7: θ^* as a function of parameter Δ

Influence of parameter Δ - in Figure 5.7, not only do we observe that the monotonicity is, as expected, the same as for Figure 2.6, but we also observe that the type of curve is very similar to the one in that same mentioned figure. This is something that is less expected, but that can be justified as we did for the cases of the parameters of the *GBM*, for which the similarity with the linear case was also verified.

The main conclusion is that, the same way that was observed in Chapter 2 for the model with a linear law for the price, the behavior of the trigger value as a function of the variables that are parameters of the demand process is different for the two different approaches.

As for the behavior of the trigger value as a function of the other variables of interest, not associated to the demand process, it is the same for both approaches. That is the result that was expected. In fact, having observed this for two completely different models, with different laws for the price, shows how this condition looks completely general. In fact, in [1], a specific relation is obtained between the trigger value for the *NPV* approach and the one for the Dynamic Programming approach, and this relation can be generalized to other models, such as the ones we are dealing with in this project. That relation depends only on the parameters of the *GBM* (α and σ), which justifies the fact that those are the only variables that lead to a change on the behavior of the corresponding trigger value when going from the *NPV* approach to the Dynamic Programming one.

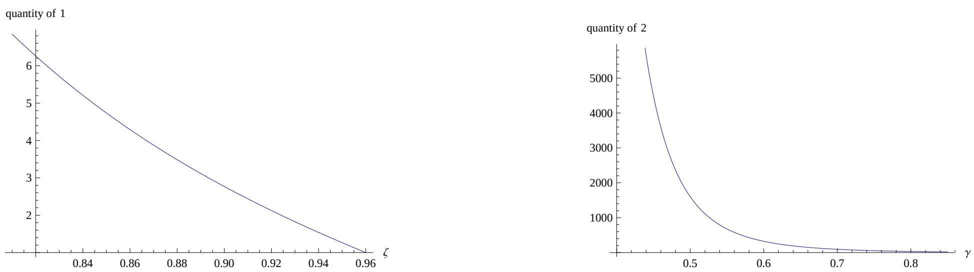
Thus, an important conclusion is derived, taking into consideration the meaning associated to each approach. *The only variables that are influenced by the 'value of waiting', that is taken into consideration for the Dynamic Programming approach but not for the* NPV *one, are the parameters of the* GBM.

Something less expected is the fact that the plots associated to the parameters of the *GBM*, and also to parameter Δ, are very similar to the corresponding plots for the linear model, studied in Chapter 2. In fact, as in this happens not only to the parameters of the *GBM* but also for parameter Δ, we conclude that this is not a characteristic of these parameters (α and σ). If the parameters of the *GBM* were the only ones associated to this particular phenomenon, we would suspect that there would be a logical reason for the phenomenon. Given the actual situation, the idea that the fact that these curves are very similar is justified by the small range of values that is used is supported, because there is no reason for this common type of curve to be verified for these different variables.

5.5.3 Optimal Quantities vs. Elasticities

We now study the relation between the optimal quantities and the values of the elasticities of each product. The idea to do this study comes from the fact that the concept of elasticity is very important and has a close relation to the concept of 'produced quantities'.

Note that a value for the demand process has to be fixed in order to obtain the following plots. The value that is chosen is 50.



(a) Product 1 (b) Product 2

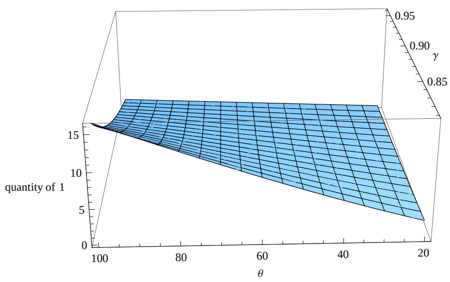
Figure 5.8: Optimal Quantities vs. Elasticities

- **Relation for product** 1: we note, looking at Figure 5.8, that, as intuition suggests, the optimal quantity for product 1 suffers a decrease when its elasticity is increased. This can be justified by the fact that, as the elasticity increases, an increase in the quantity that is produced leads to a larger and larger decrease in the price that we can exercise. In fact, if ζ is low, an additional unit of production hardly reduces the output price, so it is optimal to produce a lot. For large ζ, the opposite holds, as so it is better for the company not to produce too much. The fact that the behavior is non-linear is because of the marginal revenue which is a decreasing function of the quantity. Note that the marginal revenue is constant for the linear model;

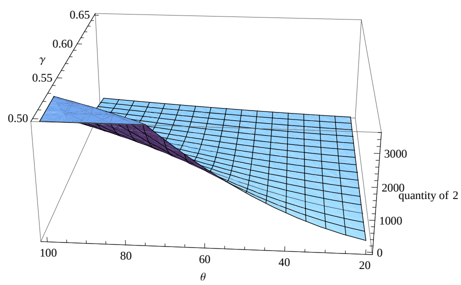
- **Relation for product** 2: we note, looking once again at Figure 5.8, that an increase on the value of the elasticity of the new product leads its optimal quantity to decrease. The explanation is totally similar to the one for the elasticity of the old product.

Note that 'marginal revenue' refers, in analytical terms, to the first derivative of the revenue, which basically corresponds to a rate of variation.

Just like we did in every other study during this project, we have to confirm that these plots in 2 dimensions are valid for every possible value that we fix for the other variables involved. That is the only way to guarantee that our interpretations are valid, independently of what values we fix for the other variables. We did not present more *3D* plots during the project, as they were just illustrative and the official way we used to confirm this condition was not via these plots (we used function *Manipulate* from *Mathematica*, as mentioned before). Also, these plots are usually not easy to interpret. However, in this particular situation, we present two of these plots, one for each product, considering also variable θ, which is a particularly sensitive variable and so it seems worth showing that, in fact, the curves that are obtained in Figure 5.8 do not depend on its value. This means that we can, without loss of generality, assume the value 50 for the demand process.



(a) Product 1



(b) Product 2

Figure 5.9: Optimal Quantities vs. Elasticities + Demand Process

In Figure 5.9, we can clearly see that, no matter what value the θ process takes, it is confirmed that the type of curve that relates any of the optimal quantities with the corresponding elasticity is always the same.

Chapter 6

Conclusions and Further Research

6.1 Conclusions

In this thesis, we derive and interpret the optimal investment policy, based on two different approaches, for a company that has an established product and has the possibility to invest in a new (innovative) product.

The problem is basically solved by obtaining the solution of a second-order differential equation with a free boundary, for the Dynamic Programming approach. This free boundary corresponds to the value θ^*, the trigger value that we want to determine. The computations are possible to do mostly because of the fact that a GBM is used as the source of uncertainty.

Note that, for every model that we study, the variable that represents the optimal time to invest has a large variance, and so its expected value is a measure of little relevance. In fact, the variable that we consider to be the most interesting one to be studied is the trigger value, value that separates the θ space in two. We were able to prove that, for every model that was studied, this value is unique.

We start by analyzing the model for which the optimal policy was already derived in [2], giving an extensive interpretation to the results that were obtained back then. Moreover, we are able to come up with a proper explanation for all the phenomena that were studied for this particular model. The study included the two extensions that are studied in [2]: the cannibalization effect and the implementation delay. For this model, the most interesting relation was the one between the trigger value and the volatility as, for the NPV Rule, we are supposed to invest for smaller values of the demand process if the volatility increases, which is the opposite of what is obtained for the Dynamic Programming approach.

The first extension that was derived from sketch consisted in assuming the existence of two demand processes, one for each product. For this case, the optimal policy was only possible to derive for the NPV approach. For the other approach, Dynamic Programming, the dependence on time that appears in the value function does not allow us to use the typical methods to solve this type of problem. The numerical results that were obtained for this model, for the NPV approach, were related to the relation between the ratio of the trigger values and the parameters of the *GBM*, and the interpretation for that case is not easy to do, but one thing that can be said is that the behaviors did not contradict the ones that had been obtained for the Basic Model.

In the chapters that follow, we try to solve other problems that also came as extensions of the one that we called Basic Model, the one studied in Chapter 2, not only by changing the objective function, but also by changing the law for the price. For these two cases, under some special assumptions, the optimal policies are still possible to derive.

For the first one, the one where we introduce unitary costs in the objective function, we were able to conclude, using the numerical analysis that was done, that the approximations that had to be considered in order to derive the optimal policy were not valid, after all.

For the second one, where not only do we consider the unitary costs of production but we also assume a multiplicative relation for the price, the numerical analysis revealed that the behavior of the trigger values as a function of the parameters of the *GBM* is very similar to the one for the Basic Model. This is something we found quite strange but that can be justified by the fact that the range of values that is considered to build the graphics is small not allowing the differences that are expected to be verified to be possible to observe. Furthermore, the relation between the trigger values and both the costs and the elasticities showed the expected behavior, reflecting the fact that the relation for the price is not linear for that model.

Finally, a conclusion that is not associated to a specific model in particular is the relation between the different approaches (NPV and Dynamic Programming). This was mentioned (in the introduction of this thesis) to be one of the most interesting topics of our study, despite the fact that the Dynamic Programming approach is the most realistic one, and so we finish this conclusion with a small study for a possible relation between the graphics of the trigger values as functions of specific variables of interest for the two different approaches.

The idea to do this study comes from the fact that, given the conclusion that was obtained in the end of the previous section, it should be possible to find a concrete measure to associate them. This does not apply to the relations between trigger values and the parameters of the *GBM*, as the mentioned conclusion for those was different than for the others.

Note that we frequently use the notation 'linear model' for the model studied in Chapter 2 and 'multiplicative model' for the model studied in Chapter 5.

We start by taking into consideration the elasticities, in the context of the multiplicative model. Our idea is that, despite the fact that the type of curve for these variables is the same for both approaches, the curves are less pronounced for the NPV case. As the absolute values for this approach are also smaller, if there is a specific relation to be found, then this relation must be something measured in relative terms, meaning that we should try to relate relative variations, rather than trying to relate absolute values, for the response variable.

In fact, what we verify is that *the relative percentage of variation of the values of the trigger values as a function of the elasticities is very close, for similar intervals, between* NPV *and Dynamic Programming approaches.*

The tables that support this statement are presented below:

Table 6.1: Elasticity of product 1

	NPV	Dynamic Programming
Interval 1	-0.0183488	-0.0172814
Interval 2	-0.00798151	-0.00759214
Interval 3	-0.00154374	-0.00128038

Table 6.2: Elasticity of product 2

	NPV	Dynamic Programming
Interval 1	0.186193	0.161003
Interval 2	0.171909	0.163144
Interval 3	0.225884	0.217737

The notation *Interval 1*, *Interval 2* and *Interval 3* refer to the relative variation between the first and the last point of specific intervals in the plots that were obtained for the two different approaches, where Interval 1 refers to an interval at the beginning of those plots; Interval 2 refers to a middle interval; and Interval 3 to an interval in the end of the plots. Note that this notation is used for all the tables associated with this particular study. Also note that each interval is defined to have length 0.5.

For example, in Table 6.2, the first interval corresponds to the relative variation in $[1, 1.5]$; the second one to the relative variation in the interval $[4.5, 5]$; and the final one to the relative variation in the interval $[7.5, 8]$. In this case, we are talking about specific intervals in Figures 5.3 and 5.6, as these plots refer to the behavior of the trigger values as a function of the elasticities, for both NPV and Dynamic Programming approaches, respectively.

It is logical to try the same thing with the relation between trigger values and production costs, as this could be the desired property of all the variables that are not related to the demand process. We just present the table for one of the costs, as the idea gets clear by observing this mentioned table.

Table 6.3: Cost of product 1

	NPV	Dynamic Programming
Interval 1	-0.0215763	-0.0220583
Interval 2	-0.00458491	-0.00468095
Interval 3	-0.00261064	-0.00266688

In fact, looking at Table 6.3, the same conclusion that was obtained for the elasticities, is now obtained for the costs.

This means that the comment that was done before for the elasticities can now be extended as follows: *The relative percentage of variation for the variables of interest that are not related to the* GBM *is very close for similar intervals in the plots obtained for the two different approaches,* NPV *and Dynamic Programming, for the multiplicative model.*

This relation between the two approaches is justified by the fact that the mentioned variables for which the relation holds appear in the same way in the expressions that determine the trigger value in both approaches.

We started to test this relation for the parameters of the multiplicative model but if this relation is verified for the reason we are assuming, then parameter a from the linear model should also verify the relation. In fact, this is the only parameter that is not related to the *GBM* for which the relation has not been tested, and it is also our secret weapon to allow us to generalize this possible relation even more (extend it to the linear model).

Table 6.4: Parameter a of the linear model

	NPV	Dynamic Programming
Interval 1	-0.00191349	-0.0019095
Interval 2	-0.00559829	-0.00559547
Interval 3	-0.00891535	-0.00891515

It becomes obvious that the idea can be generalized as follows: *The relative percentage of variation of the trigger values as functions of all the variables of interest that are not related to the* GBM *is very close for similar intervals, comparing both* NPV *and Dynamic Programming approaches.*

The only thing that is not so good about the relation is that no analytical proof can be provided for this phenomenon, which, in fact, would be something difficult to be done, given that the relation is not exact.

6.2 Further Research

Recall that, for a certain case, for instant the one studied in Chapter 3, we are not able to derive the optimal policy for the Dynamic Programming approach, given the fact that we have to solve, in a certain step, a differential equation with an explicit time dependency, for which we are not able to find a closed form, analytically. Taking into consideration this last statement, one logical extension is to try to solve this case numerically.

Another possible idea to extend what has been done during this project is to introduce a Jump Process in the demand, and try to obtain the corresponding optimal policies for all these models. Note that a jump process is a type of stochastic process that has discrete movements, called jumps, rather than small continuous movements.

In fact, in real life, not only is the demand for a process always changing in a way that can be considered as continuous, as there are little variations in every instant, but there are also some exceptional events that occur and that lead to sudden strong variations.

Given what has been stated in the previous paragraph, mixing this demand process that we use during this thesis (represented by a GBM), that represents the mentioned little/natural variations that occur over time in the demand, with a jump process, that represents the events in time that sometimes occur and lead to sudden significant variations in the value that represents the demand, seems to be a very realistic setting in the context we are working on.

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