TÉCNICO LISBOA

Thesis Title

Ana Branca Medeiros Pais

Thesis to obtain the Master of Science Degree in

Mathematics and Applications

Supervisor(s): Doctor Cláudia Rita Ribeiro Coelho Nunes Philippart

Examination Committee

Chairperson: Prof. Full Name

Supervisor: Prof. Full Name 1 (or 2)

Member of the Committee: Prof. Full Name 3

Month 2018

Dedicated to someone special...

Acknowledgments

A few words about the university, financial support, research advisor, dissertation readers, faculty or other professors, lab mates, other friends and family...

Resumo

Inserir o resumo em Português aqui com o máximo de 250 palavras e acompanhado de 4 a 6 palavras-chave...

Palavras-chave: palavra-chave1, palavra-chave2,...

Abstract

Insert your abstract here with a maximum of 250 words, followed by 4 to 6 keywords...

Keywords: keyword1, keyword2,...

Contents

List of Tables

List of Figures

Chapter 1

Introduction

Insert your chapter material here...

1.1 Motivation

Relevance of the subject... Example Goals in the end of this mini chapter.

1.2 Optimal Stopping Problems

Definição.

Princípio de Programação Dinâmica.

HJB + Gerador infinitesimal de GBM

Investimento: região de continuação e de paragem

Outro tipo de problemas (para além dos de investimento)

Teorema da Verificação

1.3 Thesis Outline

O que se vai falar em cada capítulo. Referir que trabalhos extendemos ou se assemelham?

Chapter 2

Investing and entering the market with a new product (Firm is not active before investing)

Insert your chapter material here...

2.1 Introduction

Situação do problema. Trabalhos já realizados e de maneira os extendemos.

Some overview of the underlying theory about the topic...

2.2 Stopping Problem

2.2.1 Benchmark Model

2.2.2 Capacity Optimization Model

2.3 Maximization Problem

PASSOU PARA UM NOVO CAPÍTULO

Having calculated the expression of optimized value function F^*, our goal now is to calculate the optimal level of investment R, taking into account that it influences the waiting time for the breakthrough to happen. In order to do it, we need to maximize the expected value of the optimized value function.

Notice that the distribution of the waiting time is given by an Exponential with parameter $\lambda(R)$. Also, since we are interested to find the optimal level of investment made now, one may not forget to discount the optimized value function. Thus we obtain that our optimal level of investment leads to a value function given by

$$
\begin{aligned}
V(x) &= \max_R E\left[e^{-rt}F^*(x) - R\right] \\
&= \max_R \left\{ \int_0^\infty \lambda(R) e^{-\lambda(R)t} e^{-rt} F^*(x) dt - R \right\} \\
&= \max_R \left\{ \int_0^\infty \lambda(R) e^{-\lambda(R)t} e^{-rt} \sup_\tau E^{X_0=x}\left[\max_K e^{-r\tau} h(X_\tau, K) 1_{\{\tau<\infty\}}\right] dt - R \right\} \\
&= \max_R \left\{ \int_0^\infty \lambda(R) e^{-(\lambda(R)+r)t} \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x} - R \right\}
\end{aligned}
$$

Since $F^*(x)$ does not depend on investment R nor time t, and it only depends on the drift μ, the volatility σ of GBM, discount rate r, innovation level after the jump θ and sensibility parameters α and δ and noticing that $R^\gamma + r > 0$, since we have no negative investment, we obtain

$$
V(X) = \max_R \left\{ \frac{R^\gamma}{R^\gamma + r} F^*(x) - R \right\}.
$$

The optimal value of the investment to make, R^*, is found by analyzing the first and the second partial derivatives of the expression to maximize.

$$
\begin{aligned}
\frac{\partial}{\partial R}\left(\frac{R^\gamma}{R^\gamma + r}F^*(x) - R\right) &= \frac{\gamma R^{\gamma-1}F^*(x)r - (R^\gamma + r)^2}{(R^\gamma + r)^2} \\
\frac{\partial^2}{\partial R^2}\left(\frac{R^\gamma}{R^\gamma + r}F^*(x) - R\right) &= -\frac{F^*(x)\gamma r R^{-2+\gamma}(r - \gamma r + (1+\gamma)R^\gamma)}{(R^\gamma + r)^3}
\end{aligned}
$$

2.3.1 Case I: $\gamma = 1 \Leftrightarrow \lambda(R) = R$

Analysing the roots of the first partial derivative in order to parameter R, we get a quadratic polynomial for which we can calculate obtain the expression of the zeros, obtaining

$$
R = -\sqrt{F^*(X)r} - r \ \vee \ R = \sqrt{F^*(X)r} - r
$$

The first solution is not admissible, since it's not possible to have negative investment. Thus, we only have to check if $R = \sqrt{F(x)r} - r$ corresponds to a minimum or a maximum. To do that, we analyse the second partial derivative as stated above.

$$
\frac{\partial^2}{\partial R^2}\left(\frac{R}{R+r}F(x) - R\right) = -\frac{2rF^*(x)}{(R+r)^3} < 0
$$

where we used the fact that $R,\ r,\ F(x) > 0$.

We get then that, in order to have

$$
\arg\max_R V(x) = \sqrt{F^*(x)r} - r
$$

we need to verify

$$
F(x) > r \tag{2.1}
$$

2.3.2 Case II: $\gamma \in (0,1)$

Since the second derivative is negative for considered values of γ and $r, R > 0$, any positive root of the first partial derivative in order to R accomplishes our goal of maximizing the expression above.

When analyzing the roots of the first derivative we obtain the following polynomial,

$$R^{\gamma-1}F^*(x)r - R^{2\gamma} - 2rR^{\gamma} - r^2 = 0,$$

which unfortunately, we are not able to solve analytically for every value $\gamma \in (0,1)$. We considered some numerical illustrations for values $\gamma \in (0,1)$ presented in Section 2.4.2.

2.4 Comparative Statics

2.4.1 Benchmark and Capacity Optimization Model

efeitos de $\mu,\ \sigma,\ \delta$. In this section we study the behaviour of the decision threshold x_B^* and x_C^* and K^* as described in (**??**), with the different parameters.

Comparisons between the benchmark and capacity optimization models will be made.

Proposition: Decision thresholds x_B^* and x_C^* increase with $\sigma,\ \delta$, decrease with θ and do not have a monotonic behaviour with μ.

Proof: First note that x_B^* increases with K and it verifies $\lim_{K\to 0} x_B^*(K) = x_C^*$, $\lim_{K\to\theta/\alpha} x_B^*(K) = \infty$ and that $\forall K \in [0, \theta/\alpha) : x_B^*(K) \geq x_C^*$, where θ/α is considered to be the maximum value that the capacity can take due to restriction MENCIONAR. Thus x_C^* is a particular case of x_B^*. Since the capacity does not depend on the other parameters, we have that results that hold for x_B^*, will also hold for x_C^*.

Regarding σ, we observe that

$$\frac{\partial x_B^*(\sigma)}{\partial \sigma} = \frac{16\delta(r-\mu)\left(2\mu^2 - \mu\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} + 1\right) + 2r\sigma^2\right)}{\sigma(\theta - \alpha K)\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1}\left(\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} - 1\right) - 2\mu\right)^2} > 0$$

The numerator is positive since, using the fact that $r > 0 \Rightarrow 2r > r$, and $r - \mu > 0$, it follows that

$$2\mu^2 - \mu\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} + 1\right) + 2r\sigma^2 \ \geq\ 2\mu^2 - \mu\sigma^2 + 2r\sigma^2 \ =\ 2\mu^2 + \sigma^2(2r - \mu) \ >\ 2\mu^2 + \sigma^2(r - \mu) > 0. \tag{2.2}$$

On the other side, the denominator is positive (and real) since all its expressions are positive. In particular $\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} > 0$. This can be showed using the fact that $d_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1$. Manipulating the expression we obtain that

$$d_1 - \frac{1}{2} + \frac{\mu}{\sigma^2} = \sqrt{\frac{\mu^2}{\sigma^4} - \frac{\mu}{\sigma^2} + \frac{2r}{\sigma^2} + \frac{1}{4}} > 0, \tag{2.3}$$

since $d_1 - \frac{1}{2} > \frac{1}{2} \Rightarrow d_1 - \frac{1}{2} + \frac{\mu}{\sigma^2} > 0$, for values of μ and σ such that $\mu > \left(\frac{1}{2} - d_1\right)\sigma^2$. Thus, by multiplying the square root in (2.3) by 4, we obtain $\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1}$, from which our result holds.

Regarding δ, since d_1 doesn't depend on δ, it follows that

$$\frac{\partial x_B^*(\delta)}{\partial \delta} = \frac{d_1}{d_1 - 1} \frac{\delta(r-\mu)}{\theta - \alpha K} > 0,$$

meaning that x_B^* increases linearly with δ.

Regarding θ, we observe that if its value increases, the denominator of x_B^* increases leading x_B^* to decrease.

Regarding μ, we obtain that

$$\frac{\partial x_B^*(\mu)}{\partial \mu} = \frac{\sqrt{\sigma^4 + 4\mu^2 - 4\mu\sigma^2 + 8r\sigma^2}(\sigma^2 - 2\mu)\delta}{(\sigma^4 + 4\mu^2 - 4\mu\sigma^2 + 8r\sigma^2)(\theta - \alpha K)} = \begin{cases} < 0 & \text{for } \mu < \frac{\sigma^2}{2} \\ > 0 & \text{for } \mu > \frac{\sigma^2}{2} \end{cases}.$$

From (2.3), and in a similar way as previously done, we obtain that

$$2\sigma^2 \sqrt{\frac{\mu^2}{\sigma^4} - \frac{\mu}{\sigma^2} + \frac{2r}{\sigma^2} + \frac{1}{4}} = \sqrt{\sigma^4 + 4\mu^2 - 4\mu\sigma^2 + 8r\sigma^2} > 0$$

and that $\sigma^4 + 4\mu^2 - 4\mu\sigma^2 + 8r\sigma^2 > 0$. Then we have that, holding condition CONDICAO, the denominator is positive. Since $\delta > 0$, the non-monotone behaviour, represented above, comes for the region where $\sigma^2 - 2\mu$ is negative and positive, respectively. Note that we obtain that $\mu = \frac{\sigma^2}{2}$ is a stationary point, where the minimum value of x_B^* is observed.

$\square$

To illustrate results above mentioned we performed some numerical illustrations, using software *Mathematica* and its function `Manipulate`. However here are only able to present static plots - we leave to the interested ones, to see the results achieved with `Manipulate`.

Unless it is written the opposite, following values were considered:

- $\mu = 0.03$
- $\sigma = 0.005$
- $r = 0.05$
- $\delta = 2$
- $\alpha = 0.01$
- $\theta = 10$
- $K = 100$

We start by illustrating how does x_B^* and x_C^* are related by the capacity level K, on which x_B^* is dependent. One can see on Figure 2.1 that conclusions mentioned on the proof (including that $x_B^*(0) = x_C^*$) hold.

On Figure 2.2 we observe that either for negative or positive values of the GBM's drift, both thresholds increase with volatility. This in accordance with [1] and [2] (VER CADERNO), whose works describe that with uncertainty is high, there is a delay time to invest, which is here reflected on an higher demand level.

Regarding the drift parameter μ we obtained that the threshold values do not have a monotonic behaviour, either for smaller or bigger values of volatility. As showed in Figure 2.3, the smallest value of

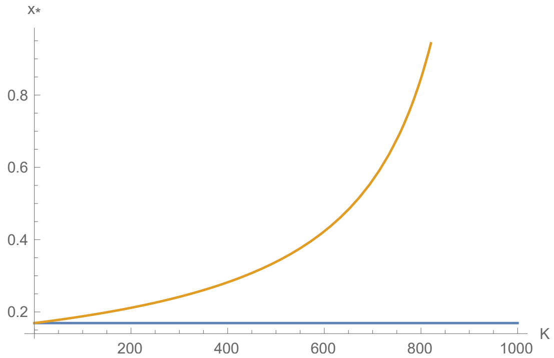
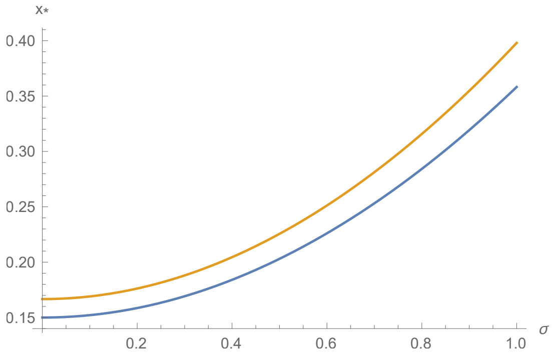
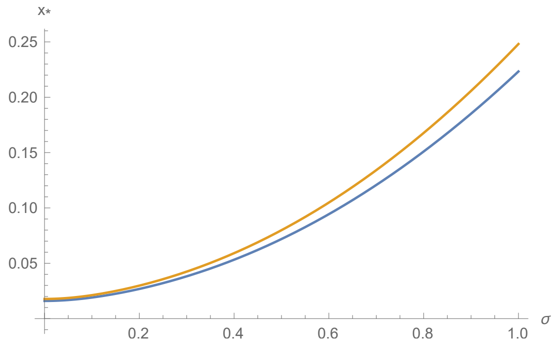


Figure 2.1: Threshold value with respect to the benchmark model (orange) and the capacity optimized model (blue), considering capacity levels $K \in [0, \theta/\alpha)$.



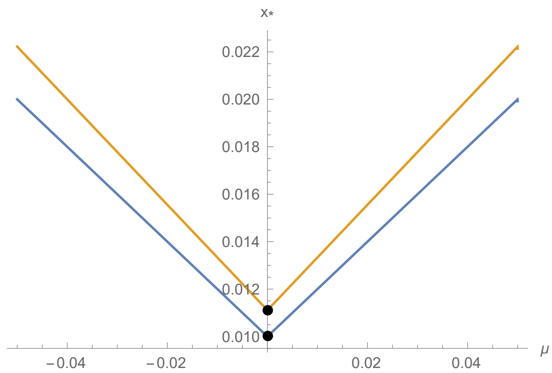
(a) $\mu = -0.7$



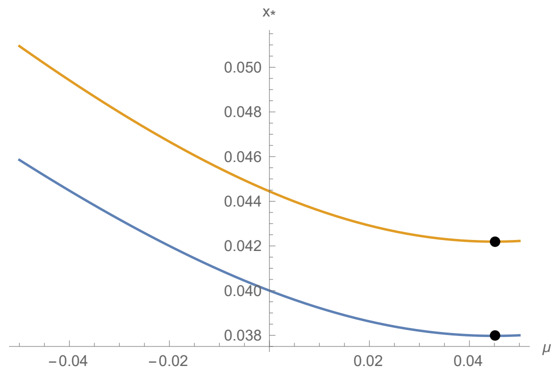
(b) $\mu = 0.03$

Figure 2.2: Threshold value with respect to the benchmark model (orange) and the capacity optimized model (blue), considering volatility $\sigma \in [0.0001, 1]$.

demand level necessary to invest is observed at the stationary point when $\mu = \sigma^2/2$.



(a) $\sigma = 0.0001$

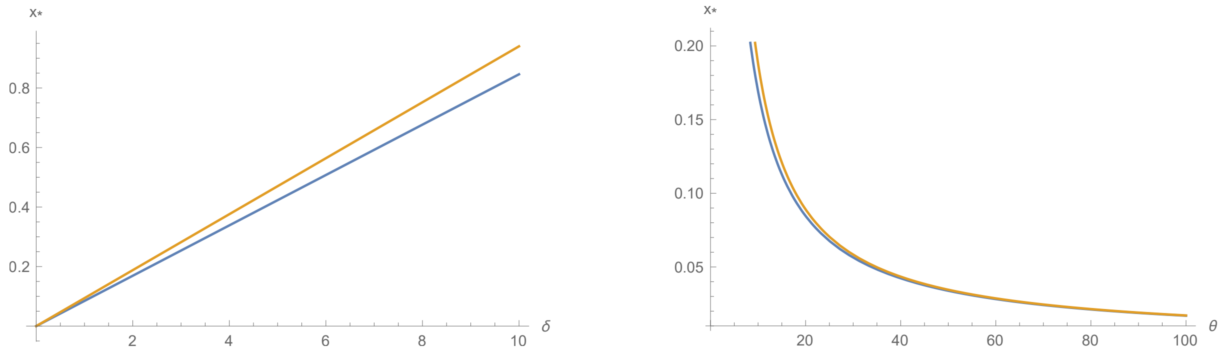


(b) $\sigma = 0.3$

Figure 2.3: Threshold value with respect to the benchmark model (orange) and the capacity optimized model (blue), considering drift $\mu \in [-r, r]$ and corresponding stationary point $\sigma^2/2$ (black).

On Figure 2.4 we observe the behaviour of both threshold levels regarding the two other parameters, sensibility level δ and innovation level θ. We have that the threshold levels increase with δ and decrease with θ, as previously deduced.

Now we analyse optimal capacity level K_C^*, that is given by evaluating K^* as defined in (**??**) on



(a) $\delta \in (0, 10)$

(b) $\theta \in (\alpha K = 1, 50)$

Figure 2.4: Threshold value with respect to the benchmark model (orange) and the capacity optimized model (blue), regarding sensibility parameter δ and innovation level θ.

demand level x_C^*, as done in [3]. Its expression is given by

$$K_C^* = \frac{2\sigma^2\theta}{\alpha\left(\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} + 3\right) - 2\mu\right)}.$$

Proposition: Optimal capacity level K_C^* increases with μ, σ and θ and decreases with r and α.

Proof: The relation between K_C^* and θ, r or α comes immediately by observing K_C^* expression. Now, regarding drift parameter we obtain that

$$\frac{\partial K_C^*(\mu)}{\partial \mu} = \frac{4\theta\left(\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} + 1\right) - 2\mu\right)}{\alpha\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1}\left(\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} + 3\right) - 2\mu\right)^2} > 0.$$

Since from

$$\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} + 1\right) - 2\mu \geq 0 \Leftrightarrow \frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1 \geq \left(\frac{2\mu}{\sigma^4} - 1\right)^2 = \frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + 1 \quad (2.4)$$
$$\Leftrightarrow \frac{8r}{\sigma^2} \geq 0,$$

which is true for $\forall r \geq 0$, and from (2.3) we obtain that both denominator and numerator are positive, from which the result comes.

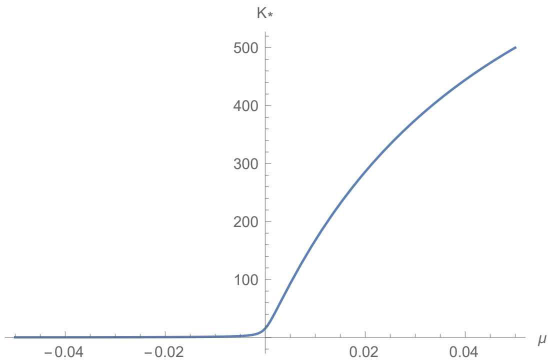
Regarding volatility parameter we obtain that

$$\frac{\partial K_C^*(\sigma)}{\partial \sigma} = \frac{8\theta\left(2\mu^2 - \mu\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} + 1\right) + 2r\sigma^2\right)}{\alpha\sigma\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1}\left(\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} + 3\right) - 2\mu\right)^2} > 0$$

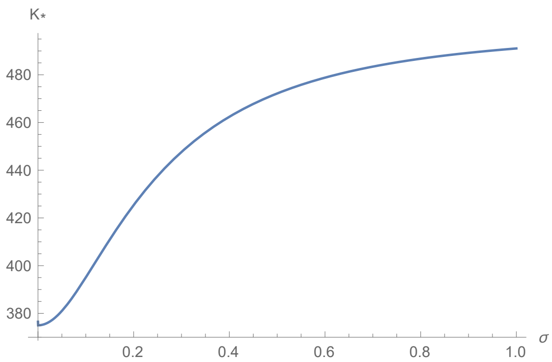
From (2.2) we obtain that the denominator is positive and from (2.3) and (2.4) that the denominator is positive for $\forall r \geq 0$, from which the result holds.

□

Considering some numerical approximations, we observe, on Figure 2.5, that K_C^* increases with both drift and volatility. Note that, regarding the drift parameter, the growth is barely noticeable for negative values of μ, but then it turns to be logarithmic. FINANCIAL INTERPRETATION? This seems to be related with the fact that for small drift values, the future expected demand value is smaller that for positive drift values. Recall that the demand process evolves accordingly to a GBM and its expected value at time t is given by $E^{X_0=x_0}[X_t] = x_0 e^{\mu t}$.



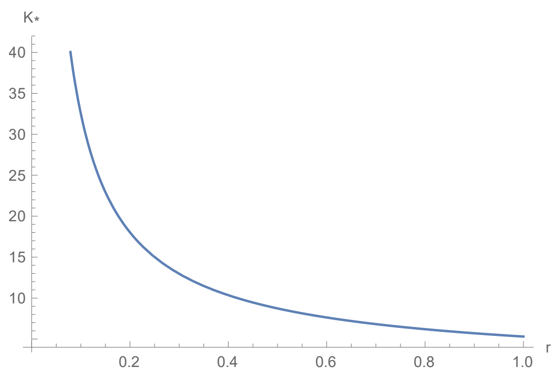
(a) $\mu \in (-r, r)$



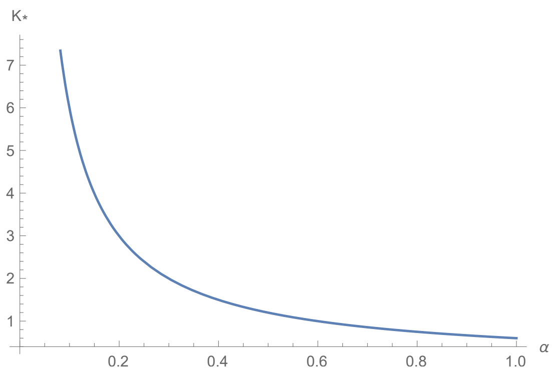
(b) $\sigma \in (0, 1)$

Figure 2.5: Optimal capacity regarding the threshold value x_C^*.

Regarding discount rate r and sensibility parameter α, we have on Figure 2.6 that K_C^* decreases with them, as expected.



(a) $r \in (\mu, 1)$



(b) $\alpha \in (0, 1)$

Figure 2.6: Optimal capacity regarding the threshold value x_C^*.

2.4.2 R&D and Capacity Optimization Model

PASSOU PARA UM NOVO CAPÍTULO

As stated in section **??**, we are not able to solve analytically the polynomial presented in (**??**) for every value $\gamma \in (0, 1)$. However we considered some numerical approximations, using software *Mathematica* and its function `Solve`. For the effect, we considered

- $r = 0.05$;
- $F(X) = 10$;

- $\gamma \in (0, 1]$ incremented by 0.05.

Following results are implemented on script `RVopt.nb`.

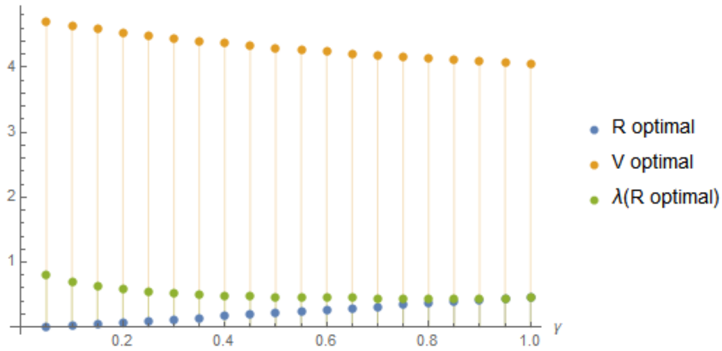


Figure 2.7: Optimal values of R and $V(X)$ for fixed values of F and r

We obtain that, although the optimal investment R grows with exponent γ, the value function for the respective optimal R decreases with exponent γ (and in a different way from the decreasing of $\lambda(R)$). We get that the smaller value of $V(X)$ is approximately 4.05, corresponding to the optimal investment level $R = 1$ and $\lambda(R) = 0.45$ and the biggest value of $V(X)$ is approximately 4.69, corresponding to the optimal investment level $R = 0.014$ and $\lambda(R) = 0.05$

Chapter 3

Adding a new product when already producing one (Firm is already active before investing)

Insert your chapter material here...

3.1 Introduction

Situação do problema. Trabalhos já realizados e de maneira os extendemos.

Some overview of the underlying theory about the topic...

3.2 Stopping Problem

3.2.1 Benchmark Model

3.2.2 Capacity Optimization Model

3.3 Comparative Statics

Since the obtained results cannot reduce to each other, as done in the previous section, we will treat each case separately, starting with the simplest one derived in 4.2.1. Comparisons between the benchmark and capacity optimization models will be made on subsection 3.4.2.

3.3.1 Benchmark Model

Proposition: Decision threshold x_B^* increases with $\delta,\ \sigma$ and α only when $\theta < \frac{K_1}{K_0^2}(K_0+K_1 r\delta)$, decreases with θ and α only when $\theta > \frac{K_1}{K_0^2}(K_0+K_1 r\delta)$ and does not have a monotonic behaviour with $K_0,\ K_1,\ r$.

Proof:

Regarding the formula obtained for x_B^*, we immediately conclude that the decision threshold increases with δ and decreases with θ.

Regarding σ, we observe that

$$\frac{\partial x_B^*(\sigma)}{\partial \sigma} = \frac{(r-\mu)\left(\delta K_1 + \frac{K_0(1-\alpha K_0)}{r}\right)}{(d_1-1)K_1(\theta-\alpha K_1)}\left(\frac{2\mu}{\sigma^3} + \frac{\frac{4\mu\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)}{\sigma^3} - \frac{4r}{\sigma^3}}{2\sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}}\right)\left(1 - \frac{d_1}{(d_1^2-1)}\right).$$

Taking into account our initial assumptions about $r,\ \mu$ and profits associated to the old and the new product, it follows that the leftmost expression is always positive. Since $d_1^2 - d_1 + 1 > 0\ \forall d_1$, the rightmost expression is also positive. Now we need to analyse the expression in between. We chec

Problemas em mostrar quando é que a segunda derivada é positiva.

Regarding K_0, we observe that

$$\frac{\partial x_B^*(K_0)}{\partial K_0} = \frac{d_1(r-\mu)}{r(d_1-1)K_1(\theta-\alpha K_1)}(1-2\alpha K_0) = \begin{cases} > 0 & \text{for } K_0 < \frac{1}{2\alpha} \\ < 0 & \text{for } K_0 > \frac{1}{2\alpha} \end{cases}.$$

since the expression represented in fraction is always positive.

Regarding K_1, we obtain that

$$\frac{\partial x_B^*(K_1)}{\partial K_1} = \frac{d_1(r-\mu)}{(d_1-1)K_1(\theta-\alpha K_1)}\left(\frac{\alpha(\frac{K_0(1-\alpha K_0)}{r} + K_1\delta)}{\theta-\alpha K_1} - \frac{\frac{K_0(1-\alpha K_0)}{r} + K_1\delta)}{K_1} + \delta\right)$$

The leftmost expression is always positive. Thus we only need to evaluate the sign of the expression next to it. Manipulating mentioned expression, taking into account that the capacity level cannot be negative as well as the price function given by $\pi_0 = K_0(1-\alpha K_0)$, it follows that

$$\frac{\partial x_B^*(K_1)}{\partial K_1} = \begin{cases} > 0 & \text{for } K_1 > \frac{-\pi_0 + \sqrt{\alpha\pi_0(\pi_0 + r\delta\theta)}}{r\alpha\delta} \\ < 0 & \text{for } K_1 \in \left[0, \frac{-\pi_0 + \sqrt{\alpha\pi_0(\pi_0 + r\delta\theta)}}{r\alpha\delta}\right] \end{cases},$$

from which we obtain that x_B^* has no monotonic behaviour with K_1.

Regarding parameter α, we obtain that

$$\frac{\partial x_B^*(\alpha)}{\partial \alpha} = \frac{d_1(r-\mu)}{(d_1-1)(\theta-\alpha K_1)}\left(\frac{\frac{K_0(1-\alpha K_0)}{r} + \delta K_1}{\theta - \alpha K_1} - \frac{K_0^2}{rK_1}\right),$$

where the leftmost expression is always positive. Simplifying the expression in the biggest brackets to the same denominator, we obtain that

$$\frac{\partial x_B^*(\alpha)}{\partial \alpha} = \begin{cases} > 0 & \text{for } \theta < \frac{K_0 K_1 + K_1^2 r\delta}{K_0^2} \\ < 0 & \text{for } \theta > \frac{K_0 K_1 + K_1^2 r\delta}{K_0^2}. \end{cases}$$

Note that the sign of the partial derivative does not depend on α.

Regarding parameter r we obtained complex derivates, from which we weren't able to deduce any analytical results. However, as it will be showed right after this proof, using Mathematica we obtained that x_B^* behaves in a non-monotonic way with it.

□

We weren't able to deduce any analytical result regarding the drift value μ. However, after many numerical experiments, we observed that x_B^* decreases with μ, as it's showed on Figure 4.2.

3.3.2 Capacity Optimization Model

Proposition: Decision threshold x_C^* increases with δ, decreases asymptotically with θ and do not have a monotonic behaviour with $\mu,\ r, \alpha, K_0$.

Proof:

For the sake of simplicity, we will denote, in this proof, $\phi := \sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} > 0$ and $\psi := 4d_1^2\pi_0(\delta\theta r + \pi_0) + \delta^2\theta^2 r^2 > 0$. Recall also that $\pi_0 > 0$ stands for the profit function associated to the new product.

Regarding δ, we obtain that

$$\frac{\partial x_B^*(\delta)}{\partial \delta} = \frac{(r-\mu)\left(\frac{4d_1^2\theta r\pi_0 + 2\delta\theta^2 r^2}{2\sqrt{\psi}} + d_1\theta r\right)}{(d_1-1)\theta^2 r} > 0$$

from which the result holds.

Regarding θ, we obtain that

$$\frac{\partial x_B^*(\theta)}{\partial \theta} = \frac{\theta(r-\mu)\left(\frac{4\delta d_1^2 r\omega + 2\delta^2\theta r^2}{2\sqrt{\psi}} + \delta d_1 r\right) - 2(r-\mu)\left(d_1(\delta\theta r + 2\omega) + \sqrt{\psi}\right)}{(d_1-1)\theta^3 r}.$$

Manipulating the numerator, it simplifies to

$$\frac{\delta d_1 r\left(2d_1(1-4\theta)\omega + (1-2\theta)\sqrt{\psi}\right) - 4d_1\omega\left(2d_1\omega + \sqrt{\psi}\right) + \delta^2\theta^2\left(-r^2\right)}{\sqrt{\psi}}$$

which is always negative for innovation levels $\theta > 1/2$. COMPLETAR: ARRANJAR SUPOSIÇÃO MAIS FORTE

Since we assume that innovation levels have no upper limit, we evaluated them asymptotically. Denoting $\theta_A := \frac{(r-\mu)}{(d_1-1)r}\left(\sqrt{\delta^2 r^2}+\frac{\delta r\left(\sigma^2(\phi+1)-2\mu\right)}{2\sigma^2}\right) > 0$ we obtain that x_C^* decreases on order of $\frac{\theta_A}{\theta}$, that is,

$$x_C^*(\theta) \sim \frac{\theta_A}{\theta} \Leftrightarrow \lim_{\theta\to\infty}\frac{x_C^*(\theta)}{\frac{\theta_A}{\theta}} = 1.$$

Regarding parameters μ, r α and K_0, we obtained complex derivates, from which we couldn't deduce any analytical result. However, as it will be showed hereunder, x_C^* behaves in a non-monotonic way with all of them.

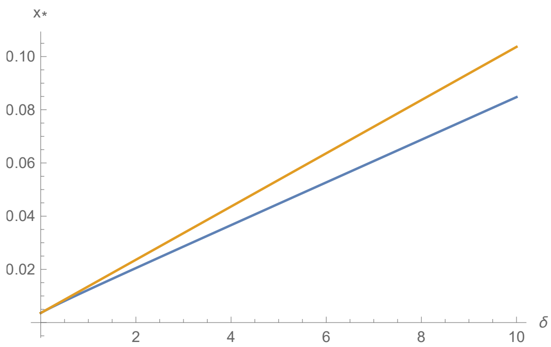
□

Although we couldn't obtain any analytical (strong) evidence, after different experiments done using *Mathematica* and its function `Manipulate`, we obtained that decision thresholds x_B^* and x_C^* increase with volatility σ. An example is showed on Figure 4.1.

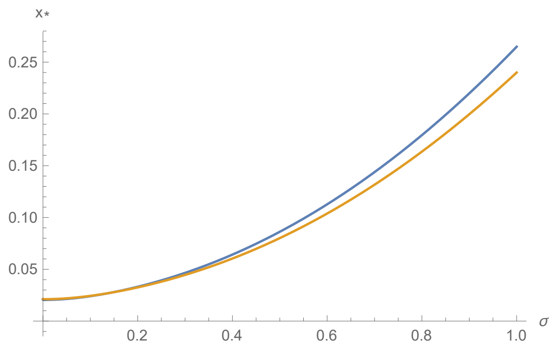
To illustrate results above mentioned we performed some numerical illustrations, using software *Mathematica* and its function `Manipulate`. However here are only able to present static plots - we leave to the interested ones, to see the results achieved with `Manipulate`.

Unless it is written the opposite, following values were considered:

- $\mu = 0.03$
- $\sigma = 0.005$
- $r = 0.05$
- $\delta = 2$
- $\alpha = 0.01$
- $\theta = 10$
- $K_0 = 90$
- $K_1 = 100$



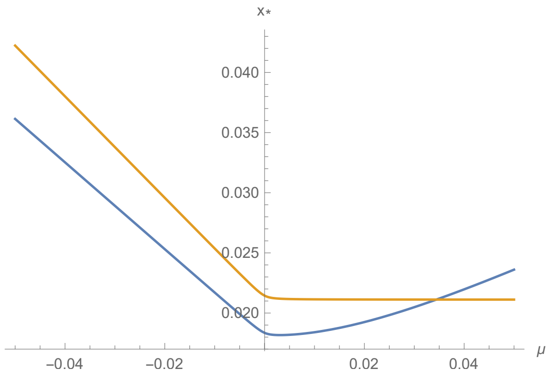
(a) $\delta \in (0, 1)$

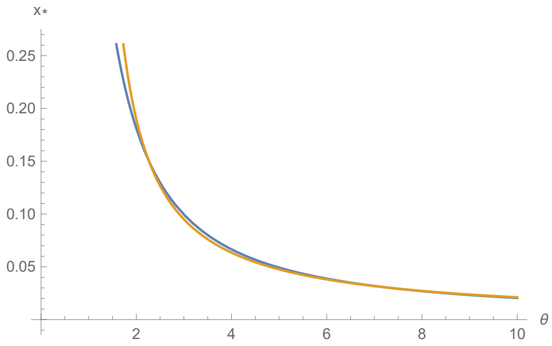


(b) $\sigma \in (0, 1)$

Figure 3.1: Threshold value with respect to the benchmark model (orange) and the capacity optimized model (blue) and parameters with which x_B^* increases.

On Figure 4.1, we obtained similar results to the ones on Section 2. Both threshold values increase with sensibility parameter δ and volatility σ. The first one is justified by the fact that a higher δ means a bigger investment, and thus it will only be made, if there's also a huge demand of the product. The second one is justified by the huge uncertainty of the demand. Since it has a high variance, the demand has a great amplitude of values, which delays the investment decision, only made when the demand reaches a high level. This is in accordance to what is described in [4].

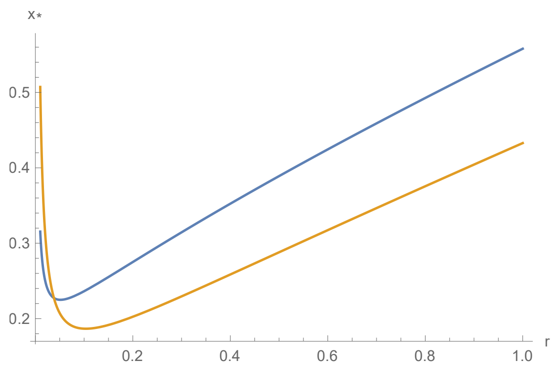




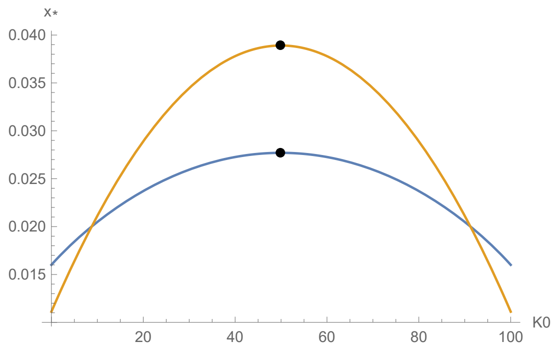
(a) $\mu \in (-r, r)$ (b) $\theta \in (1, 10)$

Figure 3.2: Threshold value with respect to the benchmark model (orange) and the capacity optimized model (blue) and parameter with which x_B^* decreases.

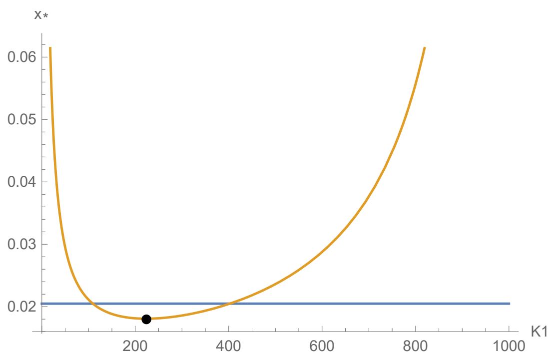
On Figure 4.2 we see that similarly to what happened on Section 2, both threshold levels decrease with innovation level. Although the threshold level associated to the benchmark model decreases with μ in what seems to be a linear way for negative values of μ and almost negligible for positive values of μ, the same doesn't happen to the threshold level associated to the capacity optimization model. This last one, seems to increase for positive values of μ. The same happened considering other values for parameters $K_0,\ \alpha,\ \delta,\ \theta$. Regarding σ, we obtained that when increasing the volatility, the value of μ associated wityh the stationary point happens for values of μ greater than 0.



(a) $r \in (\mu, 1)$, $\mu = 0.01$ and $\sigma = 0.9$



(b) $K_0 \in (0, \frac{1}{\alpha} = 100)$



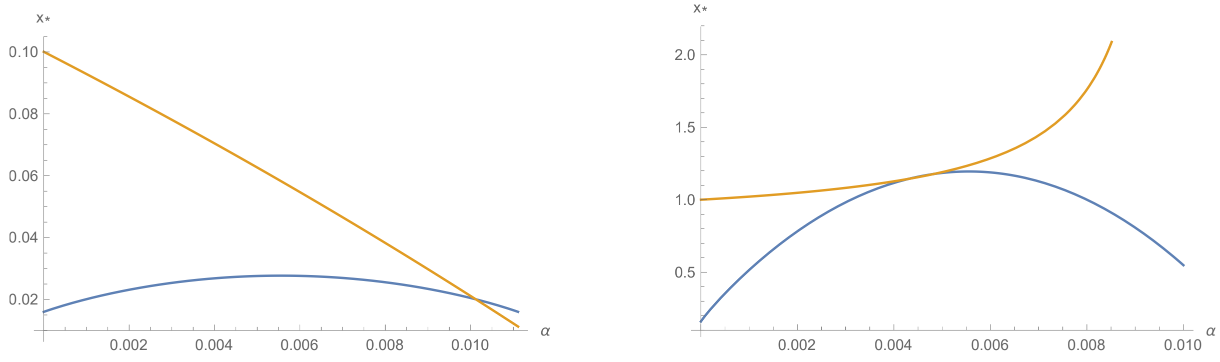
(c) $K_1 \in (0, \frac{\theta}{\alpha} = 1000)$

Figure 3.3: Threshold value with respect to the benchmark model (orange) and the capacity optimized model (blue) and parameters with which x_B^* has a non-monotonic behaviour.

On Figure 4.3, we observe that both threshold level behave in a non-monotonic way with r and K_0.

Interestingly, a maximum value is observed when the capacity level of the *older* product is exactly equal to $\frac{1}{2\alpha}$, whose value comes from the expression $\alpha K_0 \pi_0 = \alpha K_0(1-\alpha K_0)$ included on both expressions of x_B^* (**??**) and x_C^* (**??**).

Regarding parameter K_1, its value doesn't affect threshold x_C^*, since it takes into account the optimal capacity K_C^*. However, when it comes to the threshold x_B^* we have that it achieves a minimum value at $K_1 = \frac{\pi_0+\sqrt{\alpha\pi_0(\pi_0+\delta\theta r)}}{r\alpha\delta}$, as it's represented on the bottom plot.



(a) $\alpha \in (0,1)\theta = 10 > \frac{K_1}{K_0^2}(K_0 + K_1 r\delta)$

(b) $\alpha \in (0,1)$ and $\theta = 1 < \frac{K_1}{K_0^2}(K_0 + K_1 r\delta)$

Figure 3.4: Threshold value with respect to the benchmark model (orange) and the capacity optimized model (blue) and sensibility parameter α.

On Figure 4.4 it's represented the behaviour of x_B^* with α, as written on Proposition (???). Considering fixed values mentioned in REFERIRRRR, we obtain a θ-threshold equal to $\frac{K_1}{K_0^2}(K_0+K_1 r\delta) = 1.23457$. Testing for innovation levels smaller and greater than the mentioned threshold, we verify what was deduced: that x_C^* behaves differently with α for certain levels of innovation.

Note that on most of the plots you have that the threshold x_C^* has an associated capacity level $(K^*(x_C^*))$ greater than the one considered $(K_1 = 100)$, resulting in values x_B^* smaller than x_C^*, contrarily to what happened on the previous section.

Now we analyse optimal capacity level K_C^*, that is given by evaluating K^* as defined in (5.1) on demand level x_C^*, as done in [3] and in the previous section. Its expression is given by

$$K_C^* = \frac{\theta}{2\alpha} - \frac{\delta(d_1-1)\theta^2 r}{2\alpha\left(\sqrt{4\alpha d_1^2\pi_0(\alpha\pi_0+\delta\theta r)+\delta^2\theta^2 r^2} + d_1(2\alpha\pi_0+\delta\theta r)\right)}.$$

Proposition: Optimal capacity level K_C^* increases asymptotically with θ and does not have a monotonic behaviour with K_0. Also,

Proof:

Regarding innovation level θ, assuming that it has no upper limit, it's possible to evaluate its behaviour asymptotically. Denoting $\theta_K := \frac{\sigma^2\left(\sqrt{\delta^2 r^2}+\delta r\right)}{\alpha\left(2\sigma^2\sqrt{\delta^2 r^2}+\delta r(\sigma^2(\phi+1)-2\mu)\right)} > 0$, we obtain that K_C^* increases on order of $\theta_K\theta$, that is,

$$K_C^*(\theta) \sim \theta_K\theta \;\Leftrightarrow\; \lim_{\theta\to\infty}\frac{K_C^*}{\theta_K\theta} = 1$$

The non monotonic behaviour of K_C^* with K_0 will be showed hereunder in the obtained plots.

□

Although it wasn't possible to derive any (strong) analytical solution about te behaviour of the other parameters, numerically we obtain robust results. By manipulating each parameter, using command `Manipulate`, we obtained no different behaviours from the ones showed hereunder.

The results obtained regarding parameters $\mu,\ \sigma,\ r,\ \alpha$ and θ were similar to the ones obtain for the optimal capacity level on the previous section. Since K_C^* depends on value x_C^*, it is expected to observe similiar behaviours regarding the studied parameters.

Starting with the capacity level of the *old* product K_0 and respective considered parameters, on Figure 4.5, we obtained that the highest optimal capacity level K_C^* happens for $K_0 = \frac{1}{2\alpha}$. This is motivated by the results obtained for x_C^*, as seen on Figure **??**,which also reaches its highest value at $\frac{1}{2\alpha}$.

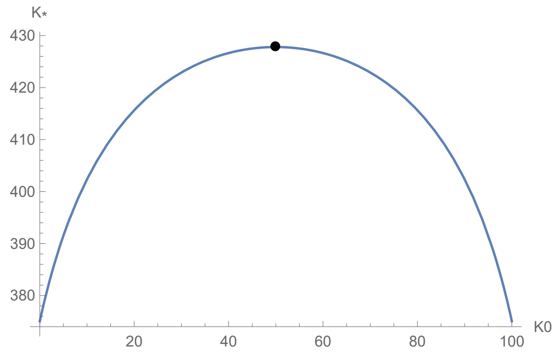
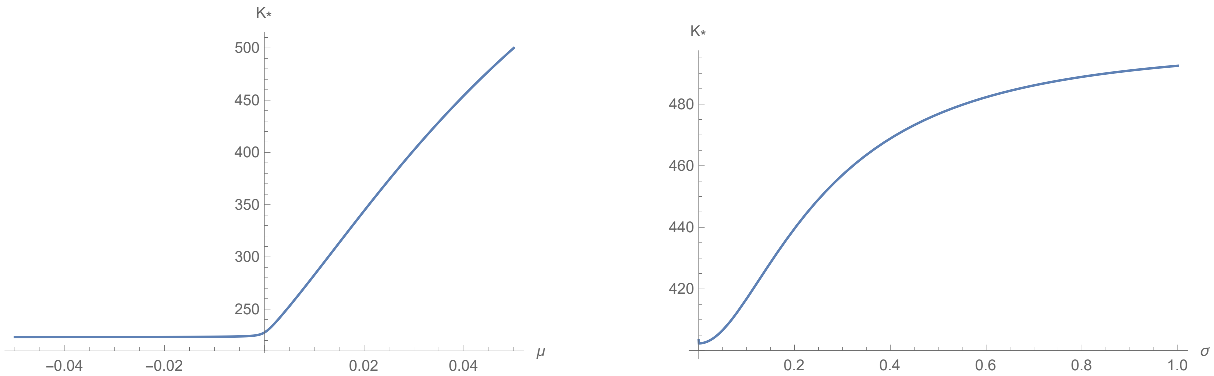


Figure 3.5: Optimal capacity regarding the threshold value x_C^* considering capacity levels $K_0 \in [0, 100)$ and its highest values at $\frac{1}{2\alpha} = 50$.

On Figure 4.6, we obtain that K_C^* increases with both drift and volatility, as it happened in the previous section. Note that again that, contrary to what happens for positive drift values, the growth of K_C^* with μ is barely noticeable for negative values of μ.

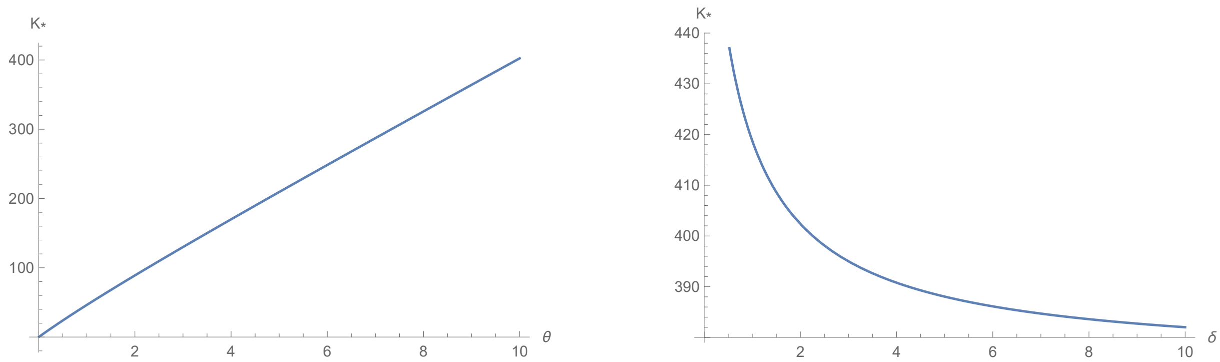
Regarding innovation level θ and sensibility parameter δ, we have on Figure 4.7 that K_C^* increases with them as well. Note that asymptotically, K_C^* seems to increase linearly with θ, as previously deduced.

Regarding discount rate r and sensibility parameter α, we have on Figure 4.8 that K_C^* decreases with them, as happened in the previous section.



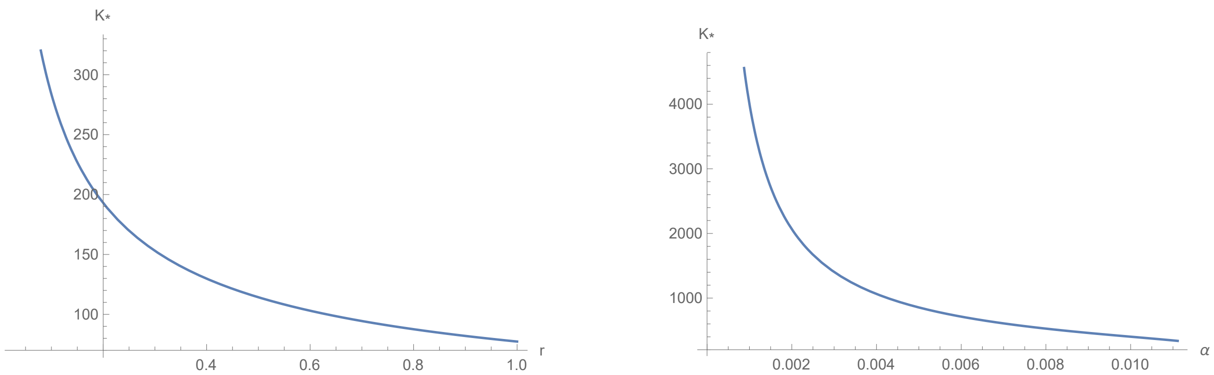
(a) $\mu \in (-r, r)$ (b) $\sigma \in (0, 1)$

Figure 3.6: Optimal capacity regarding the threshold value x_C^*.



(a) $\theta \in (1, 10)$ (b) $\delta \in (0, 10)$

Figure 3.7: Optimal capacity regarding the threshold value x_C^*.



(a) $r \in (\mu, 1)$ (b) $\alpha \in (0, 1)$

Figure 3.8: Optimal capacity regarding the threshold value x_C^*.

Chapter 4

Adding a new product when already producing one w/ cannibalisation (Firm is already active before investing)

Insert your chapter material here...

4.1 Introduction

[Introdução de artigos já publicados em semelhante contexto]

We increase the complexity of our problem by considering that the firm has three different states of production.

In the first one, we consider that the firm only produces a (very) stable product, that does not depend on the demand observe. We will call it *old* product. Its instantaneous profit function is given by π_0, which, as defined before, takes the value of

$$\pi_0 = (1 - \alpha K_0)K_0.$$

In the second state, we consider the firm produces simultaneously the *old* product and a new one. We will call it *new* product. This *new* product is inserted in the market since after the innovation process as achieved a certain innovation level, *a priori* defined. Since it's based on a new technology and it's a product that is not know by people, we will consider that its profit depends on the demand level.

The instantaneous profit functions associated to the *old* and the *new* product are given respectively by

$$\pi_0^A(X_t) = (1 - \alpha K_0 - \eta K_1 X_t)K_0,$$

$$\pi_1^A(X_t) = (\theta - \alpha K_1 - \eta K_0 X_t)K_1.$$

We need to consider a cannibalisation (or horizontal differentiation) parameter η that corresponds to the crossed effect between the *old* and the *new* product. As we consider both products to be interacting in the same market, η represents the penalty that the quantity associated to a product will influence the price of the other. We consider here that this influence is the same for both products, so we can have a unique cannibalisation parameter η, however this cannot be greater than the sensibility parameter α ($\eta < \alpha$). Otherwise, the quantity of the other product would have a larger effect on the product price than the quantity of the product itself.

The instantaneous profit function associated to this second state of production is denoted by π_A and it is such that

$$\pi_A(X_t) = \pi_0^A(X_t) + \pi_1^A(X_t) = \pi_0 + \pi_1(X_t) - 2\eta K_0 K_1 X_t = (1 - \alpha K_0)K_0 + (\theta - \alpha K_1)K_1 X_t - 2\eta K_0 K_1 X_t.$$

In the third (and last state) we consider that the firm abandons the *old* product and starts producing only the *new* product, which is not considered to be a stable product. The instantaneous profit function associated is given by

$$\pi_1(X_t) = (\theta - \alpha K_1)K_1 X_t.$$

Therefore we want to find two optimal times to make different (but maybe simultaneous) decisions. We want to find the best time τ_1 to go from the first to the second state, that is, to invest in the *new* product and start producing, simultaneously, the *old* and the *new* product. And we also want to find the best time τ_2 to go from the second to the third state, that is, to replace the production of the *old* product by the *new* one. Note that $\tau_2 \geq \tau_1$ are both stopping times adapted to the natural filtration of the demand process $\{X_t,\ t \geq 0\}$ and there is no chance on return the production of the *old* product, once the firm had abandoned it in τ_2. Thus these are irreversible choices.

4.2 Stopping Problem

4.2.1 Benchmark Model

As made in previous sections, we still consider that at the moment we adapt the new product, we need to pay δK_1 related to sunk costs and that at the precise moment we adapt the new product, we are able to produce it. Once again, we set the instant $t = 0$ to be the instant immediately after the desired innovation level happens.

Taking into account the different profits associated to each state of production, as described before, our the optimal stopping problem may be formulated as findng the value function F such that

$$F(x) = \sup_{\tau_1} E^{X_0 = x}\left[\int_0^{\tau_1} \pi_0 e^{-rs} ds + \sup_{\tau_2} E^{X_{\tau_1} = x_{\tau_1}}\left[\int_{\tau_1}^{\tau_2} \pi_A(X_s) e^{-rs} ds + \int_{\tau_1}^{\infty} \pi_1(X_s) e^{-rs} ds - e^{-r\tau_1}\delta K_1\right]\right] \tag{4.1}$$

Manipulating (**??**) by changing the region of integration of the first integral and solving it we obtain

$$\begin{aligned} F(x) &= \sup_{\tau_1} E^{X_0=x}\left[\int_0^{\infty} \pi_0 e^{-rs} ds + \sup_{\tau_2} E^{X_{\tau_1}=x_{\tau_1}}\left[\int_{\tau_1}^{\tau_2} (\pi_A(X_s) - \pi_0)\, e^{-rs} ds + \int_{\tau_1}^{\infty} (\pi_1(X_s) - \pi_0)\, e^{-rs} ds - e^{-r\tau_1}\delta K_1\right]\right] \\ &= \frac{\pi_0}{r} + \sup_{\tau_1} E^{X_0=x}\left[\sup_{\tau_2} E^{X_{\tau_1}=x_{\tau_1}}\left[\int_{\tau_1}^{\tau_2} (\pi_A(X_s) - \pi_0)\, e^{-rs} ds + \int_{\tau_2}^{\infty} (\pi_1(X_s) - \pi_0)\, e^{-rs} ds\right] - e^{-r\tau_1}\delta K_1\right] \end{aligned} \tag{4.2}$$

Changing integration variables of both integrals on optimization problem related with τ_2 we obtain

$$F(x) = \frac{\pi_0}{r} + \sup_{\tau_1} E^{X_0=x}\left[e^{-r\tau_1}\left(\sup_{\tau_2} E^{X_{\tau_1}=x_{\tau_1}}\left[\int_0^{\tau_2-\tau_1} (\pi_A(X_{\tau_1+s}) - \pi_0)\, e^{-rs} ds + \int_{\tau_2-\tau_1}^{\infty} (\pi_1(X_{\tau_1+s}) - \pi_0)\, e^{-rs} ds\right] - \delta K_1 \tag{4.3}$$

where since the term $e^{-r\tau_1}$ does not depend on τ_2, it can be put in evidence as made above.

Considering F_2 to be the value function associated to the optimal stopping problem related to τ_2 we have that its expression is given by

$$F_2(x) = \sup_{\tau_2} E^{X_{\tau_1}=x_{\tau_1}}\left[\int_0^{\tau_2-\tau_1} (\pi_A(X_{\tau_1+s}) - \pi_0)\, e^{-rs} ds + \int_{\tau_2-\tau_1}^{\infty} (\pi_1(X_{\tau_1+s}) - \pi_0)\, e^{-rs} ds\right], \tag{4.4}$$

from which follows that our optimal stopping problem, initially given by (**??**), is now given by

$$F(x) = \frac{\pi_0}{r} + \sup_{\tau_1} E^{X_0=x}\left[e^{-r\tau_1}(F_2(X_{\tau_1}) - \delta K_1)\right]. \tag{4.5}$$

We have two different optimal stopping problems that we should solve starting on the *latest* stopping time, τ_2, by considering that we know what happened until the instant that the firm invests, τ_1. In order to do that, let $\{Y_t,\ t \geq 0\}$ be the stochastic process that represents the demand level after occuring the investment at τ_1 (being that its initial time) and which evolves stochastically accordingly to a GBM with the same drift μ and volatility σ as $\{X_t,\ t \geq 0\}$, that is $\{Y_t,\ t \geq 0\} = \{X_{\tau_1+t},\ t \geq 0\}$. Note that it's initial value is the same as observed at the instant τ_1, that is $Y_0 = X_{\tau_1}$.

Consider as well τ to be the stopping time, adapted to the natural filtration of the process $\{Y_t,\ t \geq 0\}$, that represents the optimal time for which the firm should make the replacement of the *old* product by the *new* one, after having invested at time τ_1. This means that if $\tau = 0$, then the old product is replaced by the new one at the precise instant when the investment happens τ_1. Note that τ is also adapted to the natural filtration of $\{X_{\tau+t},\ t \geq 0\}$ and that $\tau_2 = \tau_1 + \tau$. Thus, by knowing τ_1 and finding τ, we can calculate τ_2.

Therefore, problem F_2, as written in (**??**), is equivalent to

$$F_2(x_{\tau_1}) = \sup_{\tau} E^{Y_0=x_{\tau_1}}\left[\int_0^{\tau} (\pi_A(Y_s) - \pi_0)\, e^{-rs} ds + \int_{\tau}^{\infty} (\pi_1(Y_s) - \pi_0)\, e^{-rs} ds\right], \tag{4.6}$$

meaning that from time 0 to time τ the firm is producing both products and that from time τ on the firm only produces the *new* product, where the instante 0 corresponds to the instant when the firm decides to invest, τ_1.

Fortunately we can simplify the notation of (**??**). Since the Strong Markov property states that after a stopping time, the future path of the GBM depends only on the value at the stopping time (knowing this one), it follows that

$$\{(Y_t|Y_0 = x_{\tau_1}),\ t \geq 0\} = \{(X_t|X_{\tau_1} = x_{\tau_1}),\ t \geq \tau_1\} \overset{d}{=} \{(X_t, t \geq 0|X_0 = x_{\tau_1}),\ t \geq 0\}.$$

Therefore we can keep the same notation as before and thus from (**??**) follows

$$F_2(x_{\tau_1}) = \sup_{\tau} E^{X_0=x_{\tau_1}} \left[\int_0^{\tau} (\pi_A(X_s) - \pi_0)\, e^{-rs} ds + \int_{\tau}^{\infty} (\pi_1(X_s) - \pi_0)\, e^{-rs} ds \right]. \tag{4.7}$$

Using the fact that the expectation is a linear operator, we treat the expectation of the rightmost integral separately of (**??**), that is

$$E^{X_0=x_{\tau_1}} \left[\int_{\tau}^{\infty} (\pi_1(X_s) - \pi_0)\, e^{-rs} ds \right] = e^{-r\tau} E^{X_0=x_{\tau_1}} \left[\int_0^{\infty} (\pi_1(X_{\tau+s}) - \pi_0)\, e^{-rs} ds \right]. \tag{4.8}$$

Conditioning to the stopping time τ and using Tower Rule we obtain from (**??**)

$$e^{-r\tau} E^{X_0=x_{\tau_1}} \left[E^{\tau=t} \left[\int_0^{\infty} (\pi_1(X_{t+s}) - \pi_0)\, e^{-rs} ds \right] \right]. \tag{4.9}$$

We interchange the integral with expectation using Fubini's theorem and the fact that $r - \mu > 0$, obtaining

$$e^{-r\tau} E^{X_0=x_{\tau_1}} \left[\int_0^{\infty} E^{\tau=t} \left[\pi_1(X_{t+s}) e^{-rs} \right] ds - \frac{\pi_0}{r} \right] = e^{-r\tau} E^{X_0=x_{\tau_1}} \left[(\theta - \alpha K_1) K_1 \int_0^{\infty} E^{\tau=t} \left[X_{t+s} e^{-rs} \right] ds - \frac{\pi_0}{r} \right], \tag{4.10}$$

where the term $(\theta - \alpha K_1)K_1$ is constant over time.

We focus now on the expected value conditional to the stopping time τ above. Since the demand level evolves accordingly to a GBM and, by knowing the instant τ, we know its value at time τ, it follows

$$\begin{aligned} E^{X_\tau=x_\tau} \left[X_{\tau+s} e^{-rs} \right] &= E^{X_\tau=x_\tau} \left[x_\tau e^{\left(\mu - \frac{\sigma^2}{2} - r\right)(\tau+s-\tau) + \sigma(W_{\tau+s} - W_\tau)} \right] \\ &= E^{X_\tau=x_\tau} \left[x_\tau e^{\left(\mu - \frac{\sigma^2}{2} - r\right)s + \sigma W_s} \right] \\ &= x_\tau e^{(\mu - r)s}, \end{aligned} \tag{4.11}$$

where in the second equality we used the fact that Brownian Motion $\{W_t,\ t \geq 0\}$ has stationary increments, that is

$$W_{\tau+s} - W_\tau \overset{d}{=} W_{\tau+s-\tau} - W_0 \overset{d}{=} W_s \sim \mathcal{N}(0, s).$$

Plugging (**??**) in (**??**), we obtain

$$e^{-r\tau} E^{X_0=x_{\tau_1}} \left[(\theta - \alpha K_1) K_1 \int_0^{\infty} x_{\tau_1} e^{(\mu-r)s} ds - \frac{\pi_0}{r} \right] = e^{-r\tau} E^{X_0=x_{\tau_1}} \left[\frac{(\theta - \alpha K_1) K_1}{r - \mu} x_\tau - \frac{\pi_0}{r} \right]. \tag{4.12}$$

Therefore we have found the terminal function associated to the optimal stopping problem F_2. Denoting it by h_2, it's given by

$$h_2(x) = \frac{(\theta - \alpha K_1)K_1}{r - \mu}x - \frac{\pi_0}{r}.$$

Accordingly to (**??**), we may also denote g_2 as the running cost function associated to this problem, that is given by

$$g_2(x) = \pi_A(x) - \pi_0.$$

Thus, plugging expression of running and terminal functions on (**??**), we have that F_2 as initially written in (**??**), is equivalent to

$$F_2(x) = \sup_{\tau} E^{X_0=x}\left[\int_0^{\tau} g(X_s)e^{-rs}ds + e^{-r\tau}h(X_\tau)\right] \tag{4.13}$$

$$= \sup_{\tau} E^{X_0=x}\left[\int_0^{\tau} \left(\pi_0^A(X_s) + \pi_1^A(X_s) - \pi_0\right)e^{-rs}ds + e^{-r\tau}\left(\frac{(\theta - \alpha K_1)K_1}{r - \mu}X_\tau - \frac{\pi_0}{r}\right)\right]. \tag{4.14}$$

4.2.2 Capacity Optimization Model

4.3 Comparative Statics

Since the obtained results cannot reduce to each other, as done in the previous section, we will treat each case separately, starting with the simplest one derived in 4.2.1. Comparisons between the benchmark and capacity optimization models will be made on subsection 3.4.2.

4.3.1 Benchmark Model

Chapter 5

Maximization Problem: waiting for innovation level desired to be reached)

Insert your chapter material here...

5.1 Introduction

Situação do problema. Trabalhos já realizados e de maneira os extendemos.

Some overview of the underlying theory about the topic...

5.2 One jump

Having calculated the expression of optimized value function F^*, our goal now is to calculate the optimal level of investment R, taking into account that it influences the waiting time for the breakthrough to happen. In order to do it, we need to maximize the expected value of the optimized value function.

Notice that the distribution of the waiting time is given by an Exponential with parameter $\lambda(R)$. Also, since we are interested to find the optimal level of investment made now, one may not forget to discount the optimized value function. Thus we obtain that our optimal level of investment leads to a value function given by

$$
\begin{aligned}
V(x) &= \max_R E\left[e^{-rt}F^*(x) - R\right] \\
&= \max_R \left\{\int_0^\infty \lambda(R)e^{-\lambda(R)t}e^{-rt}F^*(x)dt - R\right\} \\
&= \max_R \left\{\int_0^\infty \lambda(R)e^{-\lambda(R)t}e^{-rt}\sup_\tau E^{X_0=x}\left[\max_K e^{-r\tau}h(X_\tau, K)1_{\{\tau<\infty\}}\right]dt - R\right\} \\
&= \max_R \left\{\int_0^\infty \lambda(R)e^{-(\lambda(R)+r)t}\frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x} - R\right\}
\end{aligned}
$$

Since $F^*(x)$ does not depend on investment R nor time t, and it only depends on the drift μ, the volatility σ of GBM, discount rate r, innovation level after the jump θ and sensibility parameters α and δ and noticing that $R^\gamma + r > 0$, since we have no negative investment, we obtain

$$
V(X) = \max_R \left\{\frac{R^\gamma}{R^\gamma + r}F^*(x) - R\right\}.
$$

The optimal value of the investment to make, R^*, is found by analyzing the first and the second partial derivatives of the expression to maximize.

$$
\begin{aligned}
\frac{\partial}{\partial R}\left(\frac{R^\gamma}{R^\gamma + r}F^*(x) - R\right) &= \frac{\gamma R^{\gamma-1}F^*(x)r - (R^\gamma + r)^2}{(R^\gamma + r)^2} \\
\frac{\partial^2}{\partial R^2}\left(\frac{R^\gamma}{R^\gamma + r}F^*(x) - R\right) &= -\frac{F^*(x)\gamma r R^{-2+\gamma}(r - \gamma r + (1+\gamma)R^\gamma)}{(R^\gamma + r)^3}
\end{aligned}
$$

• **Case I:** $\gamma = 1 \Leftrightarrow \lambda(R) = R$

Analysing the roots of the first partial derivative in order to parameter R, we get a quadratic polynomial for which we can calculate obtain the expression of the zeros, obtaining

$$
R = -\sqrt{F^*(X)r} - r \ \vee \ R = \sqrt{F^*(X)r} - r
$$

The first solution is not admissible, since it's not possible to have negative investment. Thus, we only have to check if $R = \sqrt{F(x)r} - r$ corresponds to a minimum or a maximum. To do that, we analyse the second partial derivative as stated above.

$$
\frac{\partial^2}{\partial R^2}\left(\frac{R}{R+r}F(x) - R\right) = -\frac{2rF^*(x)}{(R+r)^3} < 0
$$

where we used the fact that $R,\ r,\ F(x) > 0$.

We get then that, in order to have

$$
\arg\max_R V(x) = \sqrt{F^*(x)r} - r
$$

we need to verify

$$
F(x) > r \tag{5.1}
$$

- **Case II:** $\gamma \in (0,1)$

Since the second derivative is negative for considered values of γ and $r, R > 0$, any positive root of the first partial derivative in order to R accomplishes our goal of maximizing the expression above.

When analyzing the roots of the first derivative we obtain the following polynomial,

$$R^{\gamma-1}F^*(x)r - R^{2\gamma} - 2rR^{\gamma} - r^2 = 0,$$

which unfortunately, we are not able to solve analytically for every value $\gamma \in (0,1)$. We considered some numerical illustrations for values $\gamma \in (0,1)$ presented in Section 2.4.2.

5.3 Multiple jumps

Erlang.

5.4 Comparative Statics

As stated in section **??**, we are not able to solve analytically the polynomial presented in (**??**) for every value $\gamma \in (0,1)$. However we considered some numerical approximations, using software *Mathematica* and its function `Solve`. For the effect, we considered

- $r = 0.05$;
- $F(X) = 10$;
- $\gamma \in (0,1]$ incremented by 0.05.

Following results are implemented on script `RVopt.nb`.

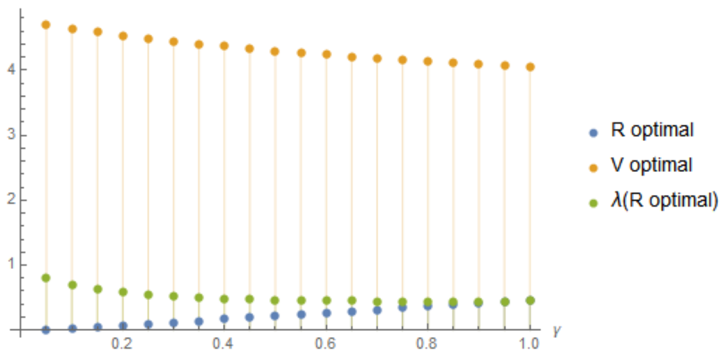


Figure 5.1: Optimal values of R and $V(X)$ for fixed values of F and r

We obtain that, although the optimal investment R grows with exponent γ, the value function for the respective optimal R decreases with exponent γ (and in a different way from the decreasing of $\lambda(R)$). We get that the smaller value of $V(X)$ is approximately 4.05, corresponding to the optimal investment level $R = 1$ and $\lambda(R) = 0.45$ and the biggest value of $V(X)$ is approximately 4.69, corresponding to the optimal investment level $R = 0.014$ and $\lambda(R) = 0.05$

Chapter 6

Add und replace, woop-woop

Mudar o título que não está nada correcto.

Adicionar um introdução bonitinha, citando diversas fontes.

6.1 Numerical Model

Description of the numerical implementation of the models explained in Chapter 5...

6.2 Verification and Validation

Basic test cases to compare the implemented model against other numerical tools (verification) and experimental data (validation)...

Chapter 7

Conclusions

Insert your chapter material here...

7.1 Achievements

The major achievements of the present work...

7.2 Future Work

A few ideas for future work...

Bibliography

[1] R. Pimentel. *Jump Processes in Finance*. PhD thesis, Instituto Superior Técnico, 2018.

[2] V. Hagspiel, K. J. M. Huisman, P. M. Kort, and C. Nunes. How to escape a declining market: Capacity investment or exit? *European Journal of Operational Research*, 254:40–50, 2016.

[3] K. J. M. Huisman and P. M. Kort. Strategic capacity investment under uncertainty. *Tilburg: Operations research*, 2013-003:40–50, 2013.

[4] A. Dixit and R. S. Pindyck. *Investment Under Uncertainty*. Princeton University Press, 1^{st} edition, 1994. ISBN:978-0691034102.

[5] B. Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. Springer, 6^{nd} edition, 2014. ISBN: 978-3540047582.

[6] R. S. Pindyck. Irreversible investment, capacity choice, and the value of the firm. *The American Economic Review*, 78:969–985, 1998.

[7] Y. Farzin, K. J. M. Huisman, and P. M. Kort. Optimal timing of technology adoption. *Journal of Economic Dynamics & Control*, 22:779–799, 1998.

[8] R. McDonald and D. Siegel. The value of waiting to invest. *The Quarterly Journal of Economics*, 101(4):707–728, 1986.

[9] A. Dixit. Entry and exit decisions under uncertainty. *Journal of Political Economy*, 97(3):620–638, 1989.

Appendix A

Vector calculus

In case an appendix if deemed necessary, the document cannot exceed a total of 100 pages...

Some definitions and vector identities are listed in the section below.

A.1 Vector identities

$$\nabla \times (\nabla \phi) = 0 \tag{A.1}$$

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0 \tag{A.2}$$

Appendix B

Technical Datasheets

It is possible to add PDF files to the document, such as technical sheets of some equipment used in the work.

B.1 Some Datasheet

SUNPOWER

C60 SOLAR CELL
MONO CRYSTALLINE SILICON

BENEFITS

Maximum Light Capture
SunPower's all-back contact cell design moves gridlines to the back of the cell, leaving the entire front surface exposed to sunlight, enabling up to 10% more sunlight capture than conventional cells.

Superior Temperature Performance
Due to lower temperature coefficients and lower normal cell operating temperatures, our cells generate more energy at higher temperatures compared to standard c-Si solar cells.

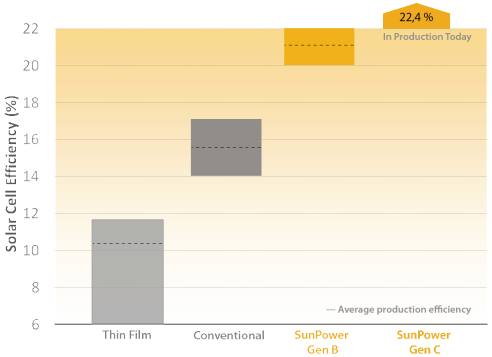
No Light-Induced Degradation
SunPower n-type solar cells don't lose 3% of their initial power once exposed to sunlight as they are not subject to light-induced degradation like conventional p-type c-Si cells.

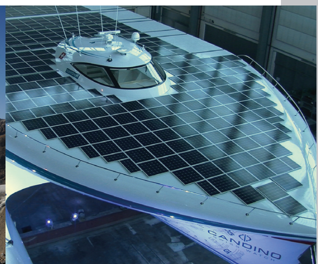
Broad Spectral Response
SunPower cells capture more light from the blue and infrared parts of the spectrum, enabling higher performance in overcast and low-light conditions.

Broad Range Of Application
SunPower cells provide reliable performance in a broad range of applications for years to come.

The SunPower™ C60 solar cell with proprietary Maxeon™ cell technology delivers today's highest efficiency and performance. The anti-reflective coating and the reduced voltage-temperature coefficients provide outstanding energy delivery per peak power watt. Our innovative all-back contact design moves gridlines to the back of the cell, which not only generates more power, but also presents a more attractive cell design compared to conventional cells.

SunPower's High Efficiency Advantage





C60 SOLAR CELL

SUNPOWER

C60 SOLAR CELL
MONO CRYSTALLINE SILICON

Electrical Characteristics of Typical Cell at Standard Test Conditions (STC)

STC: 1000W/m², AM 1.5g and cell temp 25°C

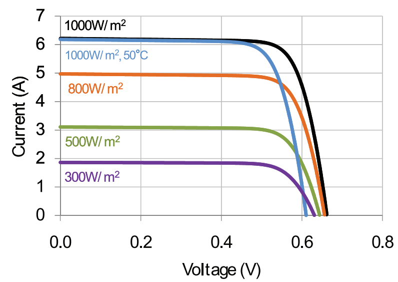
Bin	Pmpp (Wp)	Eff. (%)	Vmpp (V)	Impp (A)	Voc (V)	Isc (A)
G	3.34	21.8	0.574	5.83	0.682	6.24
H	3.38	22.1	0.577	5.87	0.684	6.26
I	3.40	22.3	0.581	5.90	0.686	6.27
J	3.42	22.5	0.582	5.93	0.687	6.28

All Electrical Characteristics parameters are nominal
Unlaminated Cell Temperature Coefficients
Voltage: -1.8 mV / °C Power: -0.32% / °C

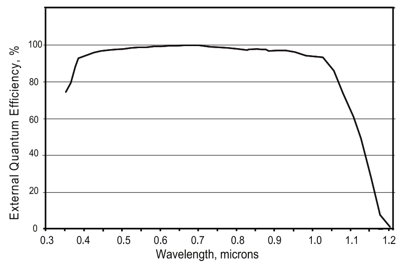
Positive Electrical Ground

Modules and systems produced using these cells must be configured as "positive ground systems".

TYPICAL I-V CURVE



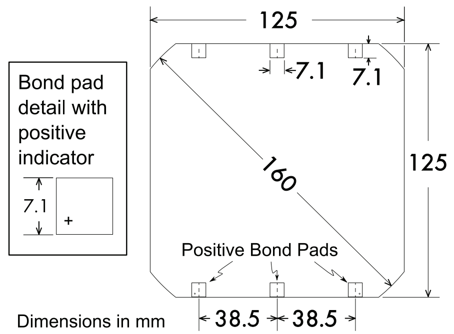
SPECTRAL RESPONSE



Physical Characteristics

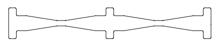
Construction:	All back contact
Dimensions:	125mm x 125mm (nominal)
Thickness:	165µm ± 40µm
Diameter:	160mm (nominal)

Cell and Bond Pad Dimensions



Bond pad area dimensions are 7.1mm x 7.1mm
Positive pole bond pad side has "+" indicator on leftmost and rightmost bond pads.

Interconnect Tab and Process Recommendations



Tin plated copper interconnect. Compatible with lead free process.

Packaging

Cells are packed in boxes of 1,200 each; grouped in shrink-wrapped stacks of 150 with interleaving. Twelve boxes are packed in a water-resistant "Master Carton" containing 14,400 cells suitable for air transport.

Interconnect tabs are packaged in boxes of 1,200 each.

About SunPower

SunPower designs, manufactures, and delivers high-performance solar electric technology worldwide. Our high-efficiency solar cells generate up to 50 percent more power than conventional solar cells. Our high-performance solar panels, roof tiles, and trackers deliver significantly more energy than competing systems.