Thesis Title

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Dedicated to someone special...

Acknowledgments

A few words about the university, financial support, research advisor, dissertation readers, faculty or other professors, lab mates, other friends and family...

Resumo

Inserir o resumo em Português aqui com o máximo de 250 palavras e acompanhado de 4 a 6 palavras-chave...

Palavras-chave: palavra-chave1, palavra-chave2,...

Abstract

Insert your abstract here with a maximum of 250 words, followed by 4 to 6 keywords...

Keywords: keyword1, keyword2,...

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Chapter 1

Introduction

Insert your chapter material here...

1.1 Motivation

Relevance of the subject... Example Goals in the end of this mini chapter.

1.2 Optimal Stopping Problems

In this thesis we use the Real Options framework to study three different investment decisions in which the demand is considered to be a diffusion process.

1.2.1 Literature

Quite many recent works suggest that Real Option[1] analysis is much more advantageous than the Net Present Value (NPV) analysis when it comes to support investment decisions, since the last one assumes that the company has a now or never approach regarding the investment decision, ignoring the freedom in timing. On the other side a Real Option Analysis takes into account the future potential, as well the respective uncertainty.

The first contributions on the Real Options approach were due to McDonald & Siegel (1986) [2], in whose work they model an investment problem where the investor must decide when it is the best time to exercise, taking into account that the value of the investment project is stochastically random, evolving accordingly to a Geometric Brownian Motion; and due to Dixit (1989) [3], in whose work he models the best time to make entry and exit decisions, while considering that there might be costs associated to each decision and that the market price evolves accordingly to a Brownian Motion.

Years later, Dixit & Pindyck (1994) [4] publish what is considered by some of the experts as the (financial) *Bible of Real Options approach.* It exploits an analogy between real options and financial investment decisions, focusing on many different decision problems (entry, investment and exit, among

[1]Real Option's concept will be defined on Section 2.2.1.

them) dependent on different stochastically behavioured (diffusion processes and jump diffusion processes, among them) measures, such as demand or market price.

Some other topics regarding Real Option increased their relevance. More particularly, the optimal production capacity to be chosen and the impact of technology adoption - being this last one a field of increasing interest nowadays.

Regarding optimal production capacity, Huisman & Kort (2013) [5] consider a competitive market in which a monopoly and a duopoly are inserted and want to deduce the best time and capaciy to invest in a new product.

Regarding technology adoption, Farzin *et al.* (1998) [1], do a comparative study between NPV and Real Option approaches un the context that a company wants to deduce when is the best time and level to invest in a technology. More recently, Hagspiel *et al.* (2016) [6] model the best time for a firm to invest in a new product, having always in mind the option to exit the market, while considering that the firm faces a declining profit for the established product and that the demand level evolves accordingly to a GBM.

In this same year, Pimentel (2018) [7] explored both. By considering two sources of uncertainty (the demand which evolves as a jump diffusion process and the innovation process which evolves as a compound Poisson process), she models the optimal times for a firm, which is producing an established product, to invest in a new product and to stop the production of the established product. Regarding the described situation, two models were developed: the benchmark model and the capacity optimization model.

1.3 Thesis Outline

In this thesis we will treat different situations related with the irreversible decision of investing in a new product under one source of uncertainty.

On Chapter 2, the main concepts and state-of-the-art (common to all problems) will be introduced. More specifically:

- On Section 2.1 we introduce the class of stochastic control problems and then specify in the class optimal stopping problems, following a similar approach as presented in [8].
- On Section 2.2, we show how optimal stopping problems relate to investment decisions under uncertainty, following a similar approach as presented in [4].

On Chapter **??**, the first model is derived. Here we consider the situation in which a firm wants to find the optimal time to introduce a new product, having none being produced at the moment. More specifically:

- On Section **??**, we derive the optimal decision regarding the original cash-flow.
- On Section **??**, we derive the optimal decision regarding the maximized cash-flow with respect to production capacity.

- On Section **??**, we study the behaviour of the decision threshold with the different parameters.

On Chapter **??**, we consider the situation in which a firm has a *stable* and recognized product in the market and wants to find the optimal time to invest in a new product while, at the same instant, replace the *stable* product by the new one. More specifically:

- On Section **??**, we derive the optimal decision regarding the original cash-flow.
- On Section **??**, we derive the optimal decision regarding the maximized cash-flow with respect to production capacity of the new product.
- On Section **??**, we study the behaviour of the decision threshold with the different parameters.

On Chapter **??**, we consider a similar situation as in Chapter **??**, but with the extra possibility of producing both products simultaneously. Therefore, now the firm wants to find the optimal time to invest in a new product and the optimal time to stop producing the *stable* product. More specifically:

- On Section **??**, we derive the optimal decisions regarding the original cash-flows.

On Chapter **??**, we derive the optimal R&D investment, by maximizing the expected value function function with respect to the innovation process. More specifically:

- On Section **??**, we derive the optimal R&D investment considering that the innovation process only takes one jump to achieve the breakthrough level.
- On Section **??**, we generalize the previous section, by considering that the innovation process takes $n \in \mathbb{N}$ jumps to achieve the breakthrough level.
- On Section **??**, we study the behaviour of the decision threshold with the different parameters.

On Chapter **??**, we perform some simulations.

Finally, on Chapter **??**, we summarize the relevant findings of the work done and how it can be extended.

1.3.1 Some Notation

Throughout the chapters, many terms will appear and their explanation will come along. However most of them will be always the same, since they do not depend on the chapter that we are working on. Therefore, to promote a better understanding in the context of the problem, the major notation (and how they are restricted on problem domain) will be now introduced:

- $\{W(t),\ t \geq 0\}$: Standard Brownian Motion (or Wiener Process) which is a stochastic process that has the following characteristics:

  1. $W(0) = 0$ with probability 1;
  2. $W(t) - W(s) \sim N(0, t-s)$. Notice that $\mathbb{E}[W(t)] = 0$ and $Var[W(t)] = t$;

3. Independent increments: $\forall\ 0 < s_i < t_i < s_j < t_j :\ W(t_i) - W(s_i) \amalg W(t_j) - W(s_j)$;

   Stationary increments: $\forall t, s \geq 0 :\ W(t+s) - W(s) \stackrel{d}{=} W(t)$;

4. $W(t)$ is continuous in t (however nowhere differentiable).

It is also seen as the continuous version of a Random Walk with Normal increments.

- $\{X(t),\ t \geq 0\}$: Geometric Brownian Motion (GBM) represents the demand for a certain product at each instant t. It is the solution of the following stochastic differential equation (SDE)

$$dX_t = \mu X_t dt + \sigma X_t dW_t,\ X_0 = x,$$

where μ represents the drift and σ the volatility of the demand.

- R: R&D costs such as size of laboratories, wages of the scientists, their computers/machines, *etecetera*, directly relate with the innovation process. These are seen as sunk costs, that is, costs that cannot be recovered after being incurred.

- $\{\theta(t),\ t \geq 0\}$: innovation process assumed to be a homogeneous Compound Poisson Process, that is a stochastic process that evolves accordingly to

$$\theta_t = \theta_0 + uN_t$$

where θ_0 corresponds to the initial innovation level, $u > 0$ is the jump size and $\{N_t,\ t \geq 0\}$ follows a Poisson process with rate $\lambda(R) = R^\gamma,\ \gamma \in (0,1)$. This rate function is such that $\lambda(0) = 0$: no R&D means zero probability of innovating; $\lambda'(R) > 0$: bigger investment means the higher probability of success and $\lambda''(R) < 0$: exists a amount of R&D costs that maximizes the rate function, that is, $\exists R^* : \lambda(R^*) \geq \lambda(R)\ \forall R$.

- θ: innovation breakthrough level. That is, the level of innovation for which we decide to invest in the new product. Considered to be reached in $n \in \mathbb{N}$ jumps, as it will be seen on Chapter 6.

- K_i: capacity of production of product i. Note that when a single product is considered, there is no mention to index i. The firm is considered to produce always up to its capacity, allowing us to consider K_i as the quantity produced. Since profit functions need to be positive, on Chapters 4 and 5, we have the following restrictions regarding capacities of *old* and *new* product, respectively, $K_0 < 1/\alpha$ and $K_1 < \theta/\alpha$. Note that (only) the last restriction will also hold for Chapter 3.

- α: constant parameter that reflects the sensitivity of the quantity with respect to the price, $\alpha > 0$.

- δ: constant parameter that reflects the sensitivity of the quantity with respect to investment sunk costs. These sunk costs will be denoted by $\delta K_1,\ \delta > 0$ (or δK, on Chapter 3).

- η: cannibalization parameter corresponding to the crossed effect between the old and the new product. It's seen as a penalty representing how the quantity associated to a product will influence the price of the other. On Chapter 5, we consider that this influence is the same for both products, resulting in a unique cannibalisation parameter. This one cannot be greater than the sensibility parameter α, that is, $\eta < \alpha$.

Chapter 2

Background concepts

Before starting to explain the work that is here developed, we introduce the main concepts and tools required.

We will start by introducing the field of Optimal Control Problems: defining its key elements, deducing the dynamic programming principle, respective dynamic programming equation and the verification technique. Secondly, we will explain how they are related with the field of optimal stopping problems as well as how we can find an optimal stopping time. Lastly, we will explain how Optimal Stopping Problems are related with investment decisions under uncertainty, deducing the general solution of the (standard) optimal problems that we will face along this work.

2.1 Stochastic Control and Optimal Stopping Problems

2.1.1 Stochastic Control Problems

Settings

We need to set the key elements related to stochastic control problems. Among these, it is included:

Time Horizon: in the context of investment-consumption problems (as this work is), an infinite time-horizon is considered: $[0, \infty)$. However one may also consider a finite horizon: $[0, T],\ T \in (0, \infty)$, as in the case where a situation is valid until an expiration date T; or and indefinite horizon: $[0, \tau]$ for some τ stopping time, as in the case where a situation is valid until a certain exit time τ.

Controlled State Process: a stochastic process which describes the state of the event that we are interested on and that it is often given as the solution of a stochastic differential equation (SDE) of the form

$$dX_t = b(t, X_t, U_t)dt + \sigma(t, X_t, U_t)dW_t,\ X_0 = x \tag{2.1}$$

where $\{U_t :\ t \in T_u\}$ is a control as defined hereunder; $\{W_t :\ t \geq 0\}$ corresponds to a standard Brownian Motion and b and σ are functions that satisfy the *global Lipschitz* and *linear growth* conditions (also known by *Itô conditions*), respectively given by

$$\begin{aligned} &\exists K \in (0,\infty)\ \forall t \in [0,\infty)\ \forall u \in \mathbb{U}\ \forall x,y \in \mathbb{R}^n : \\ &|b(t,x,\alpha) - b(t,y,\alpha)| + ||\sigma(t,x,\alpha) - \sigma(t,y,\alpha)|| \le K|x-y| \qquad (2.2) \\ &\qquad\qquad |b(t,x,\alpha)|^2 + ||\sigma(t,x,\alpha)|| \le K^2(1+|x|^2). \qquad (2.3) \end{aligned}$$

Control process: a stochastic process chosen to influence the state of the system. Considering the SDE (2.1), $U = \{U_t : \ t \in T_u\}$ is a control that influences the drift and the diffusion of the SDE and that takes values on *control space* $\mathbb{U} \in \mathbb{R}^m$, with T_u representing the *support space.*

Admissible Controls: control processes that verify certain conditions impose by the context of the problem. They will be defined some pages later.

Cost/reward function: denoted as $J(x,U)$, it identifies the costs/rewards of a system with an initial state x and a control process U.

Value Function: denoted as $V(x)$, it corresponds to the minimum/maximum possible cost/reward of the system over all admissible controls. Hence, when dealing with a cost function, V is such that $V(x) = \inf\limits_{U \in \mathbb{U}} J(x,U)$. On the other side, when dealing with a reward function (which will be our case), V is such that $V(x) = \sup\limits_{U \in \mathbb{U}} J(x,U)$.

Therefore we are now in position to state the primary goal of a stochastic control problem: for a given initial state x, we want to find the control process that lead to the optimized value function $V(x)$.

How one is able to derive this optimal control process? That's what will be explained in the next sections, in which we show the main tools used on this sort of problems and proof their correctness.

Dynamic programming principle

We will follow the explanation made in [8]. Hence we assume an infinite horizon time $[0,\infty)$ and a controlled state process as in (2.1) defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ associated to the underlying Brownian Motion, on which Ω corresponds to its state space, $\mathcal{F} = \{\mathcal{F}_t, \ t \ge 0\}$ the natural filtration associated to it and the $\mathbb{P}$ the probability measure. Considering an unidimensional controlled state space and under these assumptions, (2.1) takes the form of

$$dX_t = b(X_t, U_t)dt + \sigma(X_t, U_t)dW_t, \ X_0 = x \in \mathbb{R}, \tag{2.4}$$

with U being the control process defined in the control space $\mathbb{U} = \mathbb{R}$.

Due to the close connection to our problem, we will restrict our attention to Markov controls.

Definition 2.1. *A Markov control process is a stochastic process that takes the form* $U_t = u(X_t), \ \forall t \ge 0$ *for some control function* $u : \mathbb{R} \to \mathbb{U} = \mathbb{R}$.

Also, assume that the state process $X \equiv X_x^u$ (2.27) associated with the Markov control function u is a time-homogeneous Markov process with generator $\mathcal{A} \equiv \mathcal{A}^u$, that is

$$\mathcal{A} = \lim_{t \downarrow 0} \frac{\mathbb{E}^{X_0 = x}(f(X_t)) - f(x)}{t} = b(x, u(x))f'(x) + \frac{1}{2}\sigma^2(x, u(x))f''(x),$$

with $f \in C^2(\mathbb{R})$. Note that X is uniquely determined by the initial state x, the Markov control function u and functions b and σ, which verify conditions (2.2) and (2.3). If functions b and σ are fixed (assumption that will hold from now on), X will be completely determined by x and u.

We now define the cost function J as

$$J(x,u) = \mathbb{E}\left[\int_0^\infty e^{-\gamma s} g(X_s, u(X_s))\ ds \;\middle|\; X_0 = x\right], \tag{2.5}$$

where $\mathbb{E} \equiv \mathbb{E}^u$, and the respective value function V as

$$V(x) = \inf_{u \in \mathbb{U}} J(x,u), \tag{2.6}$$

where the infimum is taken over all the Markov control functions.

For the sake of simplicity, for now we will ignore the admissibility restrictions and consider that always exists a Markov control function u^* such that $V(x) = J(x,u^*)$, $\forall x$ and the conditional expectation as in (2.5) to be $\mathbb{E}[\ .\ |X_0 = x] \equiv \mathbb{E}^{X_0=x}[\ .\ ]$. Also, denote X^* to be the controlled state process associated with u^*. Thus it holds

$$V(x) = J(x,u^*) = \mathbb{E}^{X_0^*=x}\left[\int_0^t e^{-\gamma s} g(X_s^*, u^*(X_s^*))\ ds\right] + \mathbb{E}^{X_0^*=x}\left[\int_t^\infty e^{-\gamma s} g(X_s^*, u^*(X_s^*))\ ds\right] \tag{2.7}$$

Manipulating the expression $\mathbb{E}^{X_0^*=x}\left[\int_t^\infty e^{-\gamma s} g(X_s^*, u^*(X_s^*))\ ds\right]$ we obtain

$$\begin{aligned}
\mathbb{E}^{X_0^*=x}\left[\int_t^\infty e^{-\gamma s} g(X_s^*, u^*(X_s^*))\ ds\right] &= e^{-\gamma t}\mathbb{E}^{X_0^*=x}\left[\int_t^\infty e^{-\gamma(s-t)} g(X_s^*, u^*(X_s^*))\ ds\right] \\
&\underset{v:=s-t}{=} e^{-\gamma t}\mathbb{E}^{X_0^*=x}\left[\int_0^\infty e^{-\gamma v} g(X_{v+t}^*, u^*(X_{v+t}^*))\ dv\right] \\
&= e^{-\gamma t}\mathbb{E}^{X_0^*=x}\left[\int_0^\infty e^{-\gamma v} g(\bar{X}_{v+t}^*, u^*(\bar{X}_{v+t}^*))\ dv\right].
\end{aligned} \tag{2.8}$$

In the last step we introduced the process $\{\bar{X}_v^* = X_{v+t}^* : \ v \geq 0\}$ for some $t \geq 0$. Since X^* is a time-homogeneous Markov process, $\bar{X}^*$ also it is. Therefore its also determined by function b and σ, the same control function u^* and its the initial state. Furthermore, if we consider $\bar{X}_0^* = X_0^* = x$ we have that the conditional expected value on (2.8) is

$$\begin{aligned}
\mathbb{E}^{X_0^*=x}\left[\int_0^\infty e^{-\gamma v} g(\bar{X}_{v+t}^*, u^*(\bar{X}_{v+t}^*))\ ds\right] &= \mathbb{E}^{\bar{X}_0^*=x}\left[\int_0^\infty e^{-\gamma v} g(\bar{X}_{v+t}^*, u^*(\bar{X}_{v+t}^*))\ ds \;\middle|\; \bar{X}_0^* = X_0^* = x\right] \\
&\underset{s=v+t}{=} \mathbb{E}^{\bar{X}_0^*=x}\left[\int_0^\infty e^{-\gamma v} g(X_s^*, u^*(X_s^*))\ ds\right] \\
&= J(x,u^*) \\
&= V(x).
\end{aligned} \tag{2.9}$$

Following a similar approach considering the random starting state $\bar{X}_0^* = X_t^*$, it follows

$$\begin{aligned}\mathbb{E}^{X_0^*=x}\left[\int_0^\infty e^{-\gamma s} g(X_s^*, u^*(X_s^*))\ ds\right] =& e^{-\gamma t}\mathbb{E}^{X_0^*=x}\left[\int_0^\infty g(\bar{X}_v^*, u^*(\bar{X}_v^*))\ dv\right]\\ =& e^{-\gamma t}\mathbb{E}^{X_0^*=x}\left[\mathbb{E}\left[\int_0^\infty g(\bar{X}_v^*, u^*(\bar{X}_v^*))\ dv\ \Big|\bar{X}_0^* = X_t^*\right]\right]\\ =& e^{-\gamma t}\mathbb{E}^{X_0=x}\left[V(X_t^*)\right]. \end{aligned} \tag{2.10}$$

On the first step we used the fact $\{\bar{X}_t^*,\ t \geq 0\} \stackrel{d}{=} \{X_t^*,\ t \geq 0\}$. On the second step we used the Tower Rule, conditioning to $\bar{X}_0^* = X_0^* = x$. On the last step the result on (2.9) was used.

Mencionar a expressão da Tower Rule (Prop ou rodapé)?

Replacing (2.10) on (2.7), we obtain

$$V(x) = \mathbb{E}^{X_0^*=x}\left[\int_0^t e^{-\gamma s} g(X_s^*, u^*(X_s^*))\ ds\right] + e^{-\gamma t}\mathbb{E}^{X_0^*=x}\left[V(X_t^*)\right]. \tag{2.11}$$

Now, consider u to be an arbitrary Markov control function, the respective controlled process X^u, and $\hat{u}$ a Markov control function such that

$$\hat{u} = \begin{cases} u, & 0 \leq s < t \\ u^*, & s \geq t \end{cases}.$$

Following a similar strategy used to derive (2.11) we obtain

$$V(x) \leq \mathbb{E}^{X_0=x}\left[\int_0^t e^{-\gamma s} g(X_s^u, u^*(X_s^u))\ ds\right] + e^{-\gamma t}\mathbb{E}^{X_0=x}\left[V(X_t^u)\right]. \tag{2.12}$$

Combining results (2.11) and (2.12) we obtain the *Dynamic Programming Principle* corresponding to

$$V(x) = \inf_{u\in\mathbb{U}} \mathbb{E}^{X_0=x}\left[\int_0^t e^{-\gamma s} g(X_s^u, u^*(X_s^u))\ ds + e^{-\gamma t} V(X_t^u)\right], \tag{2.13}$$

where $\mathbb{U}$ is considered to be the set of all (admissible) Markov control functions. It states that the optimal strategy either calculated on each of the time intervals $[0, t)$ and $[t, \infty)$ or on the whole time interval $[0, \infty)$, leads to the same result. The same statement holds for any partition of the time interval.

Dynamic Programming Equation

Following a similar approach as in [8], we explain how one can find the optimal control function u^* using the Dynamic Programming Principle.

Consider an arbitrary Markov control u with corresponding state process X with initial state $X_0 = x$ and the respective generator $\mathcal{A}$. Applying Itô's Lemma to the function $f(t, x) = e^{\gamma t}V(x)$ (assumed to be $C(\mathbb{R}^2)$) and integrating we obtain

$$e^{\gamma t}V(X_t) - V(X_0) = \int_0^t e^{\gamma t}\left(-\gamma V(X_s)) + \mathcal{A}V(X_s)\right)\ ds + \sigma\int_0^t \frac{\partial}{\partial x}(e^{-\gamma s}V(x))\Big|_{x=X_s} dW_s. \tag{2.14}$$

Noting that $X_0 = x$ and taking the expect value on both sides it follows that

$$\mathbb{E}^{X_0=x}\left[e^{\gamma t}V(X_t)\right] = V(x) + \mathbb{E}^{X_0=x}\left[\int_0^t e^{\gamma t}\left(-\gamma V(X_s)) + \mathcal{A}V(X_s)\right)\ ds\right], \tag{2.15}$$

where we used the fact that $\int_0^t \frac{\partial}{\partial x}(e^{-\gamma s}V(x))\Big|_{x=X_s} dW_s$ is a martingale - since $\{\frac{\partial}{\partial x}(e^{-\gamma s}V(x)) \in B\} \in \mathcal{F}_s\ \forall B$ Borel-set and $\int_0^t \left(\frac{\partial}{\partial x}(e^{-\gamma s}V(x))\right)^2 ds = \frac{1}{2\gamma}\left(1 - e^{-2\gamma s}\left(\frac{\partial}{\partial x}V(x)\right)^2\right) < \infty \forall V(x), t < \infty$ - and, hence, it's expected value is 0.

Introduzir definiçao de martingale?

Using results deduced in (2.12), (2.15) and by the dynamic programming principle, we obtain

$$V(x) + \mathbb{E}^{X_0=x}\left[\int_0^t e^{\gamma s}\left(-\gamma V(X_s)) + \mathcal{A}V(X_s)\right)\ ds\right] \geq V(x) - \mathbb{E}^{X_0=x}\left[\int_0^t e^{\gamma s}g(X_s^u, u(X_s^u))ds\right]$$

$$\Rightarrow\ \mathbb{E}^{X_0=x}\left[\int_0^t e^{\gamma s}\left(-\gamma V(X_s)) + \mathcal{A}V(X_s) + g(X_s^u, u(X_s^u))\right)\ ds\right] \geq 0. \tag{2.16}$$

Now, we divide the expression in (2.16) by t and we take its limit as $t \downarrow 0$. Taking into account that $X_t \to X_0 = x\ \Rightarrow u(X_t) \to u(x)$ and considering that the different functions are smooth such that we are able to interchange the integral with the expectation (using Fubini's theorem) it follows

$$-\gamma V(x) + \mathcal{A}V(x) + g(x, u(x)) \geq 0. \tag{2.17}$$

Considering now the optimal control function u^*, (2.17) corresponds to

$$-\gamma V(x) + \mathcal{A}^*V(x) + g(x, u^*(x)) = 0. \tag{2.18}$$

Summarizing (2.17) and (2.18), we obtain the *dynamic programming equation* (DPE) that is given by

$$\inf_{u \in \mathbb{U}}\{-\gamma V(x) + \mathcal{A}V(x) + g(x, u(x))\} = 0, \tag{2.19}$$

where the infimum is takenover all Markov control functions. Observe that the DPE establishes a map between $x \in \mathbb{S}$ (an initial observation in the set of states) and $u \in \mathbb{U}$ (corresponding to the optimal control function). Therefore we have that by finding a solution to the DPE, we find the optimal control.

However many assumptions that we made along the explanation - such as the smoothness of the value function V or necessary/sufficient conditions that must hold in order to assure the existence of an optimal control - are hardly observed. Nevertheless there are two main approaches that can be used to obtain a solution to the DPE, while verifying all necessary assumptions: using the theory of *viscosity solutions*, that we will not go further here (for further details check [8], [9] and [11]) or using the *verification* technique, whose main idea we will explain.

Verification

The verification is seen as a backwards technique. Instead of solving the optimal control problem using the correspondent DPE, we suppose a solution was found and we show that this solution satisfy the admissibility conditions and that it cost/reward corresponds to the value function associated to the problem in hands. Its a very useful method since by finding a solution to the DPE, then that solution gives us what we want, an optimal control.

Once again (but now in order to emphasize that we want to find an ODE's solution) our optimal control problem might be stated as: we want to find $\phi \in C^2(\mathbb{R})$ that satisfies the DPE, that is,

$$\inf_{\alpha \in \mathbb{U}} \{-\gamma\phi(x) + \mathcal{A}^\alpha \phi(x) + g(x, \alpha)\} = 0 \; x \in \mathbb{R}. \tag{2.20}$$

Using the verification technique we are able to find a relationship between the value function V and our solution ϕ, that a priori doesn't exist. Therefore we assume that exists a function $\phi \in C^2(\mathbb{R})$ that satisfies (2.20), we fix $x \in \mathbb{R}$ and the correspondent unique optimal control function $\alpha_x^* \in \mathbb{U}$ that always exists and satisfies

$$\alpha_x^* = \arg\min_{\alpha \in \mathbb{U}} \{-\gamma\phi(x) + \mathcal{A}^\alpha \phi(x) + g(x, \alpha)\} \; x \in \mathbb{R}, \tag{2.21}$$

which corresponds to a map between $x \in \mathbb{R}$ and $\alpha_x^* \in \mathbb{U}$. From now on this map will be denoted by $u^* : \mathbb{R} \to \mathbb{U}$ and we will show that it defines an optimal Markov control function.

However, since any Markov control process is an admissible control process, we need to define the class of admissible control processes. As stated in [8]:

Definition 2.2. *A stochastic process $U = \{U_t, \; t \geq 0\}$ is an admissible control process if:*

1. *U is $\{\mathcal{F}_t\}$-adapted;*
2. *$U_t \in \mathbb{U} \; \forall t \geq 0$;*
3. *The SDE in (2.1) has a unique solution;*
4. *The process $\int_0^t e^{-\gamma s} \phi'(X_s) \sigma(X_s, U_s) dW_s$ is a martingale;*
5. *$e^{-\gamma t} \mathbb{E}^{X_0 = x}[V(X_t)] \to 0$ as $t \to \infty$.*

In order to show that ϕ corresponds to the value function V, consider $\mathbb{A}$ to be the set of admissible controls, that are not necessarily Markov. The cost and value functions are defined in a similar way as before, by taking the infimum over all admissible controls:

$$J(x, U) = \mathbb{E}^{X_0 = x} \left[\int_0^\infty e^{-\gamma t} g(X_t, U_t) dt \right]$$
$$V(x) = \inf_{U \in \mathbb{A}} J(x, U)$$

The verification technique uses the same reasoning as used on the DPE. We define an associated SDE by $e^{-\gamma t}\phi(X_t^*)$ and using Itô's lemma we integrate from 0 to t, obtaining

$$e^{-\gamma t}\phi(X_t^*) = \phi(x) + \int_0^t e^{-\gamma s}[-\gamma\phi(X_s^*) + \mathcal{A}^*\phi(X_s^*)]ds + \int_0^t e^{-\gamma s}\phi'(X_s^*)\sigma(X_s^*, U_s^*)dW_s, \tag{2.22}$$

where $\{X_t^*,\ t \geq 0\}$ denotes the process X associated with the optimal control function α_x^*. This calculation is possible since we assumed $\phi \in C^2(\mathbb{R})$.

Adding $\int_0^\infty e^{-\gamma s}g(X_s^*, U_s^*)ds$ and taking the expected value on both sides of (2.22), it follows that

$$\mathbb{E}^{X_0^*=x}\left[\int_0^\infty e^{-\gamma s}g(X_s^*, U_s^*)ds\right] + \mathbb{E}^{X_0^*=x}\left[e^{-\gamma t}\phi(X_t^*)\right] = \phi(x)+$$
$$+\,\mathbb{E}^{X_0^*=x}\left[\int_0^t e^{-\gamma s}(-\gamma\phi(X_s^*) + \mathcal{A}^*\phi(X_s^*) + g(X_s^*, U_s^*))ds\right], \tag{2.23}$$

where we used the fact that the integral $\int_0^t e^{-\gamma s}\phi'(X_s^*)\sigma(X_s^*, U_s^*)dW_s$ is a martingale and thus its expected value is 0.

Recalling that ϕ is the solution of (2.21), then by the DPE we verify $-\gamma\phi(x) + \mathcal{A}^\alpha\phi(x) + g(x, \alpha) = 0$. Also, since we are considering the set of all admissible control processes, when taking $t \to \infty$, by Definition 2.2, it follows that $e^{-\gamma t}\mathbb{E}^{X_0=x}[V(X_t)] \to 0$.

Hence, in the limit, (2.23) simplifies to

$$\phi(x) = \mathbb{E}^{X_0^*=x}\left[\int_0^\infty e^{-\gamma s}g(X_s^*, U_s^*)ds\right] = J(x, U^*) \tag{2.24}$$

On the other side, when considering an arbitrary admissible control process $U \in \mathbb{A}$, it follows by the DPE that $\phi(X_s) + \mathcal{A}^\alpha\phi(X_s) + g(X_s, U_S \geq 0,\ \forall s \in [0, \infty)$, which taken into the limit as done before ($t \to \infty$) follows that

$$\phi(x) \leq \mathbb{E}^{X_0=x}\left[\int_0^\infty e^{-\gamma s}g(X_s, U_s)ds\right] = J(x, U). \tag{2.25}$$

Joining both results (2.24) and (2.25), we obtain

$$\phi(x) \leq \mathbb{E}^{X_0=x}\left[\int_0^\infty e^{-\gamma s}g(X_s, U_s)ds\right] = J(x, U) \underset{\forall U \in \mathbb{A}}{\Rightarrow} \phi(x) \leq \inf_{U \in \mathbb{A}} J(x, U) = V(x). \tag{2.26}$$

However, since in (2.24) we proved that $\phi(x) = J(x, U^*)$ and by the rightmost relation in (2.26), which states that $\phi(x) = J(x, U^*) = V(x)$, it follows that U^* is an optimal control.

2.1.2 Optimal Stopping Problems

Having treated the general context of optimal control problems, we now restrict to the field of optimal stopping problems. While on an optimal control problem our goal was to find an optimal control, on an optimal stopping problem our goal is to find the optimal time to achieve a certain purpose, optimizing our cost/reward function. Hence, considering an optimal control problem where the set of admissible controls is taken to be formed by controls that, for each state x, they find the optimal time to make an action (if we should continue or stop), this coincides with an optimal stopping problem.

Settings

As it will be seen hereunder, the general formulation is quite identical to the one stated for optimal control problems (now considering an optimal stopping time as the control function). For the sake of simplicity and the close relation with what will be developed in this thesis, we will focus on the unidimensional problem.

We start by considering the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ associated to the underlying Brownian Motion W, on which $\mathcal{F} = \{\mathcal{F}_t,\ t \geq 0\}$ corresponds to its natural filtration and a stochastic process $X = \{X_t,\ t \geq 0\}$ with state space $\mathbb{S} = \mathbb{R}$, which evolves according to the following SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,\ X_0 = x \in \mathbb{R}, \tag{2.27}$$

where b and σ are functions that satisfy Itô conditions (2.2) and (2.3).

One of the main concepts in optimal stopping problems are *optimal stopping times*, which accordingly to [9], we defined as:

Definition 2.3. *A function $\tau : \Omega \to [0, \infty]$ is called a stopping time with respect to the filtration $\mathcal{F}$ is $\{\omega :\ \tau(\omega) \leq t\} \in \mathcal{F}_t\ \forall t \geq 0$.*

We will denote $\mathcal{T}$ to be the set of all $\{\mathcal{F}_t\}-$stopping times.

Observe that now the cost/reward function J depends on a state $x \in \mathbb{S}$ and a stopping time (instead of a control function) and it is such that

$$J(x, \tau) = \mathbb{E}^{X_0 = x} \left[\int_0^\tau \left(e^{-\gamma s} g(X_s)\ ds + e^{-\gamma \tau} h(X_\tau) \right) \mathbb{1}_{\{\tau < \infty\}} \right], \tag{2.28}$$

where g denotes a running function and h a terminal function.

The value function V depends solely on a state $x \in \mathbb{S}$ and it is such that $V(x) = \inf_{\tau \in \mathcal{S}} J(x, \tau)$, if J is a cost function, and $V(x) = \sup_{\tau \in \mathcal{S}} J(x, \tau)$, if J is a reward function.

Accordingly to our goal, we want to find (or characterize) an optimal stopping time $\tau \in \mathcal{T}$ that satisfies $J(x, \tau^*) = V(x),\ \forall x \in \mathbb{R}$. We will see that τ^* can be defined as a complicated functional of the sample paths of X. In order to do so, we split our state space in two regions: a continuation region $\mathcal{C}$ and a stopping region $\mathcal{S}$. Taking their names straightforward, an optimal strategy is the one in which we *continue until its optimal to stop.* Therefore the optimal stopping time is defined as

$$\tau_x^* = \inf\{t \geq 0 :\ X_t^x \notin \mathcal{C}\} = \inf\{t \geq 0 :\ X_t^x \in \mathcal{S}\}, \tag{2.29}$$

where we highlight the dependence on the initial state x chosen. Observe that the continuation region $\mathcal{C}$ as a similar function as a Markov control function on optimal control problems, since it represents the strategy to follow. Considering J to be a cost function, the optimal strategy consists in continuing while the value function V is smaller than the terminal function h. Hence the continuation region is given by $\mathcal{C} = \{x \in \mathbb{R} : V(x) < h(x)\}$.

Hamilton-Jacobi-Bellman Equation

In this section we will give an heuristic motivation to the Hamilton-Jacobi-Bellman equation which, in the context of optimal stopping problems, is the equivalent to the dynamic programming equation.

Define $\tilde{\tau}_x = \inf\{t \geq \tau : X_t \notin \mathcal{C}\}$ to be a stopping time for a fixed initial state $x \in \mathbb{R}$ and an arbitrary $\tau \in \mathcal{T}$. Observe that the stopping rule of $\tilde{\tau}$ - it is optimal to stop after τ, but might not be optimal before - is similar to the idea of dynamic programming principle, by optimizing with respect to different stopping times τ.

Considering the case where J is a cost function, we obtain by the definition of V that

$$\begin{aligned}
V(x) &\leq J(x,\tilde{\tau}) \\
&= \mathbb{E}^{X_0=x}\left[\int_0^{\tilde{\tau}} \left(e^{-\gamma s} g(X_s)\ ds + e^{-\gamma\tilde{\tau}} h(X_{\tilde{\tau}})\right) \mathbb{1}_{\{\tilde{\tau}<\infty\}}\right] \\
&= \mathbb{E}^{X_0=x}\left[\int_0^{\tau} e^{-\gamma s} g(X_s)\ ds + e^{-\gamma\tau}\left(\int_\tau^{\tilde{\tau}} e^{-\gamma(s-\tau)} g(X_s)\ ds + e^{-\gamma(\tilde{\tau}-\tau)} h(X_{\tilde{\tau}})\right) \mathbb{1}_{\{\tilde{\tau}<\infty\}}\right] \\
&= \mathbb{E}^{X_0=x}\left[\int_0^{\tau} e^{-\gamma s} g(X_s)\ ds + e^{-\gamma\tau} J(X_\tau, \tau^*_{X_\tau})\right] \\
&= \mathbb{E}^{X_0=x}\left[\int_0^{\tau} e^{-\gamma s} g(X_s)\ ds + e^{-\gamma\tau} V(X_\tau)\right]. \qquad (2.30)
\end{aligned}$$

For an arbitrary $\tau \in \mathcal{T}:\ \tau \leq \tilde{\tau}_x$, we have $\tilde{\tau} = \tau^*_x$ and hence the following equality holds

$$V(x) = J(x,\tau^*_x) = J(x,\tilde{\tau}) = \mathbb{E}^{X_0=x}\left[\int_0^{\tau} e^{-\gamma s} g(X_s)\ ds + e^{-\gamma\tau} V(X_\tau)\right]. \qquad (2.31)$$

The dynamic principle for optimal stopping is obtained by combining (2.30) and (2.31), being given by

$$V(x) = \inf_{\tau\in\mathcal{T}} \mathbb{E}^{X_0=x}\left[\int_0^{\tau} e^{-\gamma s} g(X_s)\ ds + e^{-\gamma\tau} V(X_\tau)\right]. \qquad (2.32)$$

The dynamic programming equation will be now derived following a similar approach as done for stochastic control problems. Consider a fixed $x \in \mathbb{R}$ and $\tau \in \mathcal{T}$. Assumming that $e^{-\gamma t}V(x) \in C^2(\mathbb{R}^2)$, we use Itô's lemma, to integrate it, (as similarly done in (2.14)); we take the expectation on both sides (as similarly done in (2.15)) and we sum the term $\mathbb{E}^{X_0=x}\left[\int_0^\tau e^{-\gamma t} g(X_t)dt\right]$ on both sides, obtaining

$$\mathbb{E}^{X_0=x}\left[e^{-\gamma\tau} + V(X_\tau)\int_0^{\tau} e^{-\gamma t} g(X_t)dt\right] = V(x) + \mathbb{E}^{X_0=x}\left[\int_0^t e^{-\gamma s}\left(-\gamma V(X_s)) + \mathcal{A}V(X_s) + g(X_s)\right)\ ds\right]. \qquad (2.33)$$

Since $\tilde{\tau}$ is an optimal stopping time and $\tau < \tau^*_x$, we have that

$$V(x) = J(x,\tau^*_x) = J(x,\tilde{\tau}) \underset{(2.30)}{\Rightarrow} V(x) = \mathbb{E}^{X_0=x}\left[e^{-\gamma\tau} + V(X_\tau)\int_0^{\tau} e^{-\gamma t} g(X_t)dt\right]. \qquad (2.34)$$

Combining deduced results (2.33) and (2.34), it follows that

$$\mathbb{E}^{X_0=x}\left[\int_0^t e^{-\gamma s}\left(-\gamma V(X_s)) + \mathcal{A}V(X_s) + g(X_s)\right)\ ds\right] = 0.$$

Let $x \in \mathcal{C}$ and $\tau_x^* > 0$ and assume the continuation region $\mathcal{C}$ to be an open set. By dividing last equation by τ and taking the limit to 0 ($\tau \to 0$) we obtain

$$-\gamma V(x) + \mathcal{A}V(x) + g(x) = 0. \tag{2.35}$$

On the other side, considering $x \notin \mathcal{C}$ (and $\tau_x^* > 0$), we have that the value function V is equal to the terminal cost h, that is,

$$-\gamma V(x) + \mathcal{A}V(x) + g(x) = 0. \tag{2.36}$$

Considering an arbitrary stopping time $\tau \in \mathcal{T}$, by (2.30), using a similar approach as the one use to deduce (2.35) and the definition of the continuation region, we obtain

$$\begin{cases} -\gamma V(x) + \mathcal{A}V(x) + g(x) \geq 0, & x \in \mathcal{C} \\ h(x) - V(x), & x \notin \mathcal{C} \end{cases} \tag{2.37}$$

The dynamic programming equation for optimal stopping problems follows by combining last results, (2.35), (2.36) and (2.37), and corresponds to:

$$\min\{-\gamma V(x) + \mathcal{A}V(x) + g(x), h(x) - V(x)\} = 0, \; x \in \mathbb{R}. \tag{2.38}$$

It also takes the name of *Hamilton-Jacobi-Bellman (HJB) equation*. However instead of an equation, (2.38) is in fact a variational inequality. Also, note that it does not depend explicitly on the continuation, but, since for any state x it recommends us whether to stop or continue, we can deduce the continuation region from which - showing that our previous guess was right.

Observe that all previous deduction were made considering J to be a cost function. However in the context of this thesis, J will be considered to be a reward function. A similar construction as the one already done could be made, but here we will just state the main results that are going to be used.

If J is a reward function, then we obtain that our optimal stopping problem is such that

$$V(x) = \sup_{\tau \in \mathcal{T}} J(x, \tau) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{X_0 = x}\left[(\int_0^\tau e^{-\gamma s} g(X_s)\; ds + e^{-\gamma \tau} h(X_\tau) \mathbb{1}_{\{\tau < \infty\}} \right]^1, \tag{2.39}$$

where g corresponds to the running cost function and h to the terminal function.

The associated HJB equation, as deduced in (2.38), takes the form of

$$\begin{aligned} &\max\{-\gamma V(x) + \mathcal{A}V(x) + g(x), h(x) - V(x)\} = 0 \; x \in \mathbb{R} \\ \Leftrightarrow \;& \min\{\gamma V(x) - \mathcal{A}V(x) - g(x), V(x) - h(x)\} = 0 \; x \in \mathbb{R} \end{aligned} \tag{2.40}$$

from which that we deduce that the associated continuation and stopping regions are respectively given

[1] Recall (again) that we are denoting $\mathbb{E}[\;.\;|X_0 = x]$ by $\mathbb{E}^{X_0=x}[\;.\;]$.

by

$$\mathcal{C} = \{x \in \mathbb{R} : h(x) - V(x) < 0\}, \tag{2.41}$$

$$\mathcal{S} = \{x \in \mathbb{R} : h(x) - V(x) = 0\} = \mathbb{R} \setminus \mathcal{C}. \tag{2.42}$$

The optimal stopping time can be formally defined for any state $x \in \mathbb{R}$ as

$$\tau_x^* = \inf\{\tau \geq 0 : \ x \notin \mathcal{C}\}. \tag{2.43}$$

Verification

As done for optimal control problems, a solution ϕ is found using the verification technique now applied on the context of optimal stopping problems.

We won't go further in detail since the methodology is pretty similar to what was described before. Instead we will give the main idea of it. For more information one should check [8] and [9].

As the verification technique was previously explained, it doesn't give a solution by solving explicitly the HJB variational inequality (2.40), but instead, we assume a solution ϕ and we show that it verifies the HJB variational inequality.

The function ϕ might be defined by parts, in order to verify both sides of the variational inequality (2.40). However the main difficulty appears at on the boundary of $\mathcal{C}$. The strong formulation of the verification technique - as stated on Section 2.1.1 - requires ϕ to be C^2. This assumption is hardly verified since there is an abrupt change on the behaviour from the $\mathcal{C}$ to $\mathcal{S}$. Fortunately, there are generalizations of Itô's rule that only require ϕ to be C^1 (and other weaker assumptions). This condition over ϕ will originate two other conditions referred as *value matching* (2.52a) and *smooth pasting* (2.52b) conditions, as stated in [4], essential to the formulation of ϕ.

Therefore, having ϕ constructed by parts such that it verifies the HJB variational inequality in (2.40) and is C^1 everywhere, we have found the solution to the optimal stopping problem (2.39).

2.2 Optimal Stopping Problems Applied to Investment Decisions under Uncertainty

In this section we will deduce the solution to all standard optimal stopping problems that we will face during this thesis, taking into account the respective financial context.

We will start by presenting some financial concepts related to investment decisions under uncertainty, based on [4]. Then we will return to the mathematical formulation and derive the solution, presenting a detailed explanation.

2.2.1 A Real Options approach

In firm's investment context, a *real option* is seen as a situation on which a firm has the right, but not the obligation, to undertake certain initiatives such as deferring, abandoning, expanding, staging or contracting a capital investment project. There are three factors assumed to hold during the investment decision:

1. Future rewards are random and thus uncertain;
2. The decision is irreversible, in the sense that it is a sunk cost: the investment expenditure cannot be fully recovered;
3. The decision can be made at any time.

Therefore any investment decision might be seen as an optimal stopping problem in which we want to find the best time to make a decision such that the value of the firm is maximized.

By considering $t = 0$ as the starting instant to make the decision, the investment problem can be stated on the form of (2.39), where the running cost function $g \geq 0$ denotes the current earnings, corresponding to the cash-flow originated at each instant by the current situation; the terminal function $h \geq 0$ denotes the long-term earnings after the investment is done, corresponding to the long-term cash-flow originated since the investment is made and considering the new situation of the firm and investment costs; parameter γ denotes the discount rate r and the process X denotes here the demand process, which evolves accordingly to a Geometric Brownian Motion[2]. Since the supremum here is taken over all stopping times after the initial instant, it follows that (2.39) is now written as

$$V(x) = \sup_{\tau \geq 0} J(x,\tau) = \sup_{\tau \geq 0} \mathbb{E}^{X_0=x}\left[\int_0^\tau e^{-rs} g(X_s)\ ds + e^{-r\tau} h(X_\tau) \mathbb{1}_{\{\tau<\infty\}}\right]. \tag{2.44}$$

Recall that V is such that the HJB variational inequality (2.40) is satisfied, where $\mathcal{A}$ is an operator that denotes the infinitesimal generator of the demand level process, represented here as a Geometric Brownian Motion. Accordingly to *Theorem 7.3.3* on [9], its expression is given by

$$\mathcal{A}F(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}^{X_0=x}(F(X_t)) - F(x)}{t} = \frac{\sigma^2}{2} x^2 F''(x) + \mu x F'(x),\ \forall F \in C^2(\mathbb{R}). \tag{2.45}$$

For any state (i.e., demand level) included in the continuation region $\mathcal{C}$, terminal costs are bigger than what we earn by investing, resulting in a negative profit. Thus the firm will prefer to wait until the demand level reaches a value that does not belong to $\mathcal{C}$ and only then, invest. Therefore, the continuation region consists in the set of states attained by the leftmost term above of HJB equation (**??**), that is

$$\mathcal{C} = \{x \in \mathbb{R} : h(x) - V(x) > 0\} = \{x \in \mathbb{R} : rV(x) - \mathcal{A}V(x) - g(x) = 0\}.$$

Our intuition leads us to conjecture that the continuation region consists on the set of demand levels that are under a certain value x^*, which is defined to be the threshold between the continuation region $\mathcal{C}$ and the stopping region $\mathcal{S}$. The threshold value x^* might be greater or equal to the demand value observed immediately after a jump in the innovation process happens, at time t_θ, since at this time we are already in position to invest. If we don't invest for the demand level observed at innovation jump

[2] On Section 1.3.1 the demand process is defined.

x_{t_θ}, then we won't do it for smaller values since $h(x_{t_\theta}) - V(x_{t_\theta}) > 0 \;\Rightarrow\; \forall x < x_{t_\theta} : h(x) - V(x) > 0$, meaning that we will have no profit. Therefore,

$$\mathcal{C} = \{x \in (0,\infty) : x < x^*\} \text{ for some } x^* \in (X_{t_\theta}, \infty). \tag{2.46}$$

On the other hand, the stopping region $\mathcal{S}$, is defined to be the set whose states verify that the terminal function and the value function are equal (equivalent to the set of states attained by the rightmost term above of HJB equation (**??**)), that is,

$$\mathcal{S} = \{x \in \mathbb{R} : h(x) - V(x) = 0\} = \{x \in \mathbb{R} : x \geq x^*\} = \mathbb{R} \setminus \mathcal{C}. \tag{2.47}$$

The value function V, as defined in (2.44), must statements both statements (2.2.1) and (2.47), meaning that V must be defined by parts.

Starting with the condition in (2.2.1), V is such that

$$rV(x) - \mathcal{L}V(x) = rV(x) - \frac{\sigma^2}{2}x^2V''(x) - \mu xV'(x) - g(x) = 0 \tag{2.48}$$

is verified for $\forall x \in \mathcal{C}$.

The solution V above might be seen as the sum of the homogeneous solution V_h with a particular solution V_p, that is, $V(x) = V_h(x) + V_p(x),\ \forall x \in \mathcal{C}$.

A particular solution V_p might be found by considering $V''(x) = 0,\ \forall x$ and then solve the corresponding differential equation $V(x) - \mu xV'(x) - g(x) = 0$. Note that in case the running cost function g is null, $V(x) = V_h(x)$.

The correspondent homogeneous ODE in (2.48) corresponds to an (homogeneous) Cauchy-Euler equation of second order, whose solution has the form

$$V(x) = ax^{d_1} + bx^{d_2}$$

where d_1 and d_2 are the positive and negative solutions of the quadratic equation

$$d^2 + \left(\frac{2\mu}{\sigma^2} - 1\right)d - \frac{2r^2}{\sigma^2} = 0 \qquad \Rightarrow \qquad d_{1,2} = \frac{1}{2} - \frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}. \tag{2.49}$$

Taking into account that $r > \mu$, it follows that $d_1 > 0$ and $d_2 < 0$.

Since there is no possibility of having a project with negative value and h is a non-negative and non-decreasing function, the solution regarding $x = 0$ must be 0, that is, V must verify

$$\lim_{x \to 0^+} V(x) = 0. \tag{2.50}$$

This fact implies that when the running cost function is null ($V(x) = V_h(x)$), $b = 0$ and hence $V(x) = ax^{d_1}$. Regarding the situation when $V(x) = V_h(x) + V_p(x)$, one should have $V_p(0) = V_h(0)$ and thus coefficients a and b must be determined. However, since we will be able to reduce all problems to the case where the running cost function is null - except on Chapter 5 where V_h is such that $V_h(0) = 0$, implying $\lim_{x\to 0^+} V(x) = \lim_{x\to 0^+} V_h(x) = 0$ - we can focus on the case where

$$V(x) = ax^{d_1} \quad \text{with} \quad d_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1. \tag{2.51}$$

In order to satisfy condition presented in (2.29), V is taken to be equal to terminal cost function h for $\forall x \in \mathcal{S}$.

Despite the two different regions, value function F must be continuous and smooth in all its domain ($F \in C(\mathbb{R})$), particularly at the boundary value x^*. Then, accordingly to [4], *value matching* and *smooth pasting*, respectively given by

$$V(x^*) = V(x^*) \tag{2.52a}$$

$$V'(x^*) = V'(x)|_{x=x^*} \tag{2.52b}$$

must be verified. Solving the respective system, we get values x^* and a, obtainning inally, that our value function takes the form of

$$V(x) = \begin{cases} ax^{d_1} & \text{for } x \in \mathcal{C} \\ h(x) & \text{for } x \in \mathcal{S}, \end{cases}$$

with d_1 as in (2.51) and where $\mathcal{C}$ is defined as in (2.46), $\mathcal{S}$ as in (2.47) and the optimal stopping time as in (2.29).

Chapter 3

Investing and entering the market with a new product (Firm is not active before investing)

3.1 Introduction

In this chapter we consider a firm that wants to invest in a product, after a certain innovation level θ is reached, and to produce it in the long term. To do so, the firm needs to incur an investment cost proportional to the capacity production K_1. This cost is given by δK_1, with $\delta > 0$ a sensibility parameter related to the investment. We consider here that, at the investing time, the firm needs to pay the investment cost and that the production starts immediately.

The demand function associated to the product to be introduced evolves stochastically with the demand process $\mathbf{X}$ and it is given by

$$p(X_t) = (\theta - \alpha K)X_t \geq 0 \tag{3.1}$$

where $\alpha > 0$ is a sensibility parameter and X_t corresponds to the demand level observed at the instant $t \geq 0$.

The instantaneous profit function π is obtained by multiplying the demand function p by the quantity produced. However, as written in Section 1.3.1, we assume that the firm produces up to its capacity. Therefore it follows that π is given by

$$\pi(X_t) = (\theta - \alpha K)KX_t \geq 0. \tag{3.2}$$

In this Chapter (and in the two next ones), time is set to start when the innovation process reaches θ. Since before reaching θ we don't have the desired innovation level, it it's useless to make an investment decision. Consequently, X_0 refers to the demand level observed when the desired innovation level is reached. These assumptions simplify our notation without losing the applicability of the model.

As mentioned before, two models will be derived. The first one corresponds to the benchmark model. The simplest model to be considered. The second one will take into account the maximized instantaneous profit in function of the production capacity K. Comparative statics of both models will be made afterwards.

3.2 Stopping Problem

3.2.1 Benchmark Model

We start with the simplest model. In the benchmark model we want to find when is the optimal time invest in the product, in the sense that it maximizes the expected discounted long-term profit.

Denote the time of investment in the new product as τ. Having in mind that the decision needs to be made in finite time, our optimization problem can be written as

$$\sup_{\tau} \mathbb{E}^{X_0=x}\left[\left(\int_{\tau}^{\infty} e^{-rs}\pi(X_s)\ ds - e^{-r\tau}\delta K\right)\ \mathbb{1}_{\{\tau<\infty\}}\right] \tag{3.3}$$

for $\theta, x \in R^+$.

Putting the discount term in evidence, from (3.3) follows

$$\sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\int_{\tau}^{\infty} e^{-r(s-\tau)}\pi(X_s)\ ds - \delta K\right)\ \mathbb{1}_{\{\tau<\infty\}}\right]. \tag{3.4}$$

We can simplify this expression. Using Tower rule and conditioning on the instant when we exercise τ, we obtain that (3.4) may be written as

$$\sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\mathbb{E}^{\tau=t}\left[\int_{t}^{\infty} e^{-r(s-\tau)}\pi(X_s)\ ds\right] - \delta K\right) 1_{\{\tau<\infty\}}\right]. \tag{3.5}$$

Let's focus now on the inner expected value $\mathbb{E}^{\tau=t}\left[\int_t^\infty e^{-(\tau-s)}\pi(X_s)\ ds\right]$. Changing the integration variable follows

$$\mathbb{E}^{\tau=t}\left[\int_{0}^{\infty} e^{-rv}\pi(X_{v+\tau})\ dv\right]. \tag{3.6}$$

Considering $r-\mu>0$, we have that $\int_0^\infty \int_\Omega |e^{-rv}\pi(X_{v+\tau})|\ \mathbb{P}(d\omega)\ dv < \infty$. Since $e^{-rv}\pi(X_{v+\tau})$ is a continuous function it follows that is also a measurable function. By both conditions we obtain that it is $[0,\infty)\times\Omega$-integrable. Therefore by Fubini's Theorem we can interchange the integrals, from which follows by (3.6), that

$$\int_0^\infty \mathbb{E}^{\tau=t}\left[e^{-rv}\pi(X_{v+\tau})\right]\ dv = (\theta-\alpha K)K\int_0^\infty \mathbb{E}^{\tau=t}\left[e^{-rv}X_{v+\tau}\right]\ dv, \tag{3.7}$$

where we took into account the expression of the profit function π.

Let's now focus on the expected value $\mathbb{E}^{\tau=t}\left[e^{-rv}X_v\right]$. It follows that

$$\begin{aligned}
\mathbb{E}^{\tau=t}\left[e^{-rv}X_{v+\tau}\right] &= \mathbb{E}^{\tau=t}\left[X_\tau e^{\left(\mu-\frac{\sigma^2}{2}-r\right)(\tau+v-\tau)+\sigma(W_{\tau+v}-W_\tau)}\right] \\
&= x_\tau e^{\left(\mu-\frac{\sigma^2}{2}-r\right)v}\ \mathbb{E}^{\tau=t}\left[e^{\sigma W_v}\right] \\
&= x_\tau e^{\left(\mu-\frac{\sigma^2}{2}-r\right)v} e^{\frac{\sigma^2}{2}v} \\
&= x_\tau e^{(\mu-r)v}.
\end{aligned} \tag{3.8}$$

In the first step we used the expression associated to the GBM and the fact that, by knowing the investment time τ, we also know the demand level at that time, here represented as $X_\tau = x_\tau$. In the second step, the fact that the Brownian Motion has stationary increments implies that $W_{\tau+v} - W_\tau \stackrel{d}{=} W_v - W_0 = W_v$, where $W_0 = 0$ holds since we assumed $\mathbf{W}$ to be a standard Brownian Motion. In the third step we used the fact that $W_v \sim \mathcal{N}(0, v)$ and the expression for the moment generating function associated to the Normal distribution, from which follows $\mathbb{E}\left[e^{sW_v}\right] = e^{\frac{1}{2}sv^2}$. Simplifying the expression we obtain (3.8).

Plugging the resultant expression (3.8) in (3.7) and solving the integral, we obtain the formula of the terminal cost function associated to this problem - corresponding to the expression between parenthesis in (3.4) and (3.5). We will denote it by h and its expression corresponds to

$$h(x) = \frac{(\theta - \alpha K)Kx}{r-\mu} - \delta K. \tag{3.9}$$

The terminal cost function h represents the discounted long-term cash-flow by acquiring a product when the demand level is x. It already includes the investment cost of such decision.

Denoting F as the value function associated to this problem, we obtain that our optimization problem, as described in (3.3), can be written as a standard optimal stopping problem with null running cost function, given by

$$F(x) = \sup_\tau \mathbb{E}^{X_0=x}\left[e^{-r\tau}h(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] = \sup_\tau \mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{(\theta-\alpha K)KX_\tau}{r-\mu} - \delta K\right)\mathbf{1}_{\{\tau<\infty\}}\right]. \tag{3.10}$$

Recurring to Bellman principle, we have that the solution F verifies the variational inequality given by Hamilton-Jacobi-Bellman (HJB) variational inequality (2.40) and hence, it is given by

$$F(x) = \begin{cases} ax^{d_1}\ ,\ x \in \mathcal{C} \\ h(x)\ ,\ x \in \mathcal{S} \end{cases}, \tag{3.11}$$

where coefficient a and the threshold value x^*, that defines the boundary between continuation and stopping regions, are found by value matching (2.52a) and smooth pasting (2.52b) conditions, expressed by the corresponding system

$$\begin{cases} a(x^*)^{d_1} = \frac{K(\theta-\alpha K)x^*}{r-\mu} - \delta K \\ ad_1(x^*)^{d_1-1} = \frac{K(\theta-\alpha K)}{r-\mu} \end{cases} \Rightarrow \begin{cases} a = \left(\frac{K(\theta-\alpha K)x^*}{r-\mu} - \delta K\right)(x^*)^{-d_1} = \frac{\delta K(x^*)^{-d_1}}{d_1-1} \\ x^* = \frac{d_1}{d_1-1}\frac{\delta(r-\mu)}{\theta-\alpha K} \end{cases} \tag{3.12}$$

with d_1 being the positive root of the polynomial described in (2.51).

Continuation and stopping regions are respectively described as in (2.46) and (2.47) and the optimal stopping time as in (2.43).

3.2.2 Capacity Optimization Model

Now we consider a more realistic case, in which the firm wants to take the best of its investment, regarding the capcity of production. This can be achieved by requiring that the production capacity is chosen to be the maximizer of the discounted long-term cash-flow. Therefore our goal is now to find when is the best time to invest in the product and which is the optimal capacity associated to it. This can be stated as

$$\sup_{\tau} \mathbb{E}^{X_0=x} \left[\max_{K} \left\{ e^{-r\tau} \left(\int_{\tau}^{\infty} e^{-r(\tau-s)} \pi(X_s)\ ds - \delta K \right) \right\} \mathbb{1}_{\{\tau<\infty\}} \right]. \tag{3.13}$$

Manipulating the expression as previously done, we obtain that (3.13) may be written as

$$\sup_{\tau} \mathbb{E}^{X_0=x} \left[e^{-r\tau} \max_{K} \{h(X_\tau, K)\} \mathbb{1}_{\{\tau<\infty\}} \right], \tag{3.14}$$

with h corresponding to the terminal function deduced in (3.9), in which we now highlight, not only the dependence on the demand level, but also on the production capacity K chosen at the investing time.

In this section, the capacity optimization model is obtained in two steps. In the first one we calculate the capacity level that optimizes the terminal cost function h, which we will denote by K^*. The second step consists in solving the optimal stopping problem given by $\sup_\tau \mathbb{E}^{X_0=x} \left[e^{-r\tau} h(X_\tau, K^*) \mathbf{1}_{\{\tau<\infty\}} \right]$, in which we are already considering the optimized terminal function.

The optimal capacity level K^* is found by analyzing the behaviour - namely stationary points and concavity - of the terminal function h, while considering a fix level of demand.

The stationary points are found by calculating the roots of the first partial derivative. We obtain that the first partial derivative is given by

$$\frac{\partial h}{\partial K}(x, K) = \frac{(\theta - 2\alpha K)x}{r-\mu} - \delta, \tag{3.15}$$

which implies that h has a unique stationary point

$$K = \frac{\theta}{2\alpha} - \frac{\delta(r-\mu)}{2\alpha x}. \tag{3.16}$$

Regarding the concavity's behaviour, we obtain that the second partial derivative of h is negative and it's given by

$$\frac{\partial^2 h}{\partial K^2}(x, K) = -\frac{2\alpha x}{r-\mu} < 0, \tag{3.17}$$

since the GBM doesn't take negative values and we assumed $\alpha > 0$ and $r - \mu > 0$. Therefore, h is a concave function and the capacity value found in (3.16) corresponds to its global maximizer.

From now on, and to emphasize its maximizer role, we denote (3.16) by K^*. Note that K^* is dependent of the demand level in the sense that the optimal capacity is increasing with the initial observed demand value.

Now we proceed to the second step. Evaluating h at its optimal capacity level K^* we obtain

$$h(x, K^*) = \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x}.$$

Denoting F^* as the value function associated to the optimal stopping problem in (3.18), the optimization problem can be stated as

$$F^*(x) = \sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau}h(X_\tau, K^*)\mathbf{1}_{\{\tau<\infty\}}\right] = \sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau}\frac{(\theta X_\tau - \delta(r-\mu))^2}{4\alpha(r-\mu)X_\tau}\mathbf{1}_{\{\tau<\infty\}}\right], \quad (3.18)$$

which is again a standard optimal stopping problem with null running cost function. Similarly to the benchmark model, we obtain that the value function associated to (3.18), satisfies the HJB variational inequality (2.40). Therefore F^* is such that

$$F^*(x) = \begin{cases} bx^{d_1} , & x \in \mathcal{C} \\ h(x, K^*) , & x \in \mathcal{S} \end{cases}, \quad (3.19)$$

where coefficient b and the threshold value x_C^*, that defines the boundary between continuation and stopping regions, are found by value matching (2.52a) and smooth pasting conditions (2.52b), expressed by the corresponding system

$$\begin{cases} b(x_C^*)^{d_1} = \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x} \\ bd_1(x_C^*)^{d_1-1} = \frac{\theta^2(x_C^*)^2 - \delta^2(r-\mu)^2}{4\alpha(r-\mu)(x_C^*)^2} \end{cases} \quad (3.20)$$

with d_1 being the positive root of the polynomial described in (2.51).

We get two possible positive roots for the threshold level: $x_{C,1}^* = \frac{d_1+1}{d_1-1}\frac{\delta(r-\mu)}{\theta-\alpha K}$ and $x_{C,2}^* = \frac{\delta(r-\mu)}{\theta}$. However, after some manipulation, we exclude the second one $x_{C,2}^*$, since the coefficient b associated to it takes a null value. This is an absurd, since it would lead to a null value function for any demand level smaller than $x_{C,2}^*$, contradicting the fact that the possibility of investing in the future is also valuable. Therefore we obtain that the threshold level and coefficient b in (3.12) are, respectively, given by

$$\begin{aligned} x_C^* &= \frac{d_1+1}{d_1-1}\frac{\delta(r-\mu)}{\theta} \\ b &= \left(\frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x_C^*}\right)(x_C^*)^{-d_1} = \frac{\delta\theta}{\alpha(d_1^2-1)}\left(\frac{d_1+1}{d_1-1}\frac{\delta(r-\mu)}{\theta}\right)^{-d_1} \end{aligned} \quad (3.21)$$

and continuation and stopping regions are respectively described as in (2.46) and (2.47) and the optimal stopping time as in (2.43).

Now we analyse the optimal capacity level K_C^*. If at the breakthrough we have $X_0 < x_C^*$, then the investment decision will happen immediately when the demand process reaches level x_C^*, at that time the firm will invest δK_C^*. As done in [5], the optimal capacity level is then given by evaluating K^* as defined in (3.16) at the threshold demand level x_C^* (3.21). This leads to

$$K_C^* = \frac{2\sigma^2\theta}{\alpha\left(\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} + 3\right) - 2\mu\right)}. \tag{3.22}$$

Since the demand process is continuous, there is no possibility on investing $K^*(x_t),\ x_t > x_C^*,\ t > 0$, since we assume the investment to be at the precise instant that x_C^* is observed.

However it might be the case that when the breakthrough happens, we observe an equal or bigger demand than the threshold x_C^*, that is, $X_0 \geq x_C^*$. This case is not as interesting as the previously mentioned, since the investment will be incurred immediately when the breakthrough happens. Nevertheless, in this case the optimal capacity level will be given by evaluating K^* as defined in (3.16) at X_0. This case won't be treated during Comparative Statics made on Section 3.3.

3.3 Comparative Statics

In this section we study the behaviour of the decision threshold x_B^* (3.12) and x_C^* (3.21) and K^* as described in (3.22), with the different parameters. Comparisons between the benchmark and capacity optimization models will be made.

3.3.1 Benchmark Model

Proposition 3.1. *Decision threshold x_B^* increases with $r, \sigma,\ K,\ \alpha$ and δ and decreases with μ and θ.*

Proof:

Before showing the results stated, we focus on ϕ since it will be a recurrent expression in most comparative statics sections. This is given by

$$\phi := \sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} > 0 \tag{3.23}$$

We analyse the expression inside the square root in order to infere about ϕ as well as some extra constraints regarding our problem. However, since that expression only has imaginary roots regarding parameter μ or roots that are not allowed our problem constraints regarding parameters r and σ it follows that it is always positive, so it $\phi > 0,\ \forall r, \mu, \sigma$ in our problem domains.

Now we are in position to explain stated results.

Regarding r, we obtain

$$\frac{\partial x_B^*}{\partial r} = \frac{\delta\left(-(d_1 - 1)d_1\sigma^2\phi - 2\mu + 2r\right)}{(d_1 - 1)^2\sigma^2(\alpha k - \theta)\sqrt{\frac{4\mu^2}{\sigma^4} - \phi}} > 0,$$

Note that its denominator is positive, giving the constraints of our problem. Analyzing the numerator, we found that it has no possible roots on our problem domain. Thus since the expression on the numerator is continuous in each of its parameters and takes a positive value when testing for given values, it follows that it is always positive with respect to our problem domain.

Regarding σ, we obtain

$$\frac{\partial x_B^*}{\partial \sigma} = -\frac{2\delta(\mu - r)\left(-2\mu^2 + \mu\sigma^2(\phi + 1) - 2r\sigma^2\right)}{(d_1 - 1)^2\sigma^5(\alpha k - \theta)\phi} > 0.$$

Note that its denominator is negative, since $\alpha - \theta K < 0$. Analyzing the numerator, we found that it has no possible roots on problem domain. Thus, since $-2\delta(\mu - r)\left(-2\mu^2 + \mu\sigma^2(\phi + 1) - 2r\sigma^2\right)|_{\mu=0} < 0$ and the numerator is continuous with respect to each of its parameters, it follows that it is negative.

Regarding K, α and δ, we obtain immediately

$$\begin{aligned}
\frac{\partial x_B^*}{\partial K} &= \frac{\alpha\delta d_1(r - \mu)}{(d_1 - 1)(\theta - \alpha k)^2} > 0 \\
\frac{\partial x_B^*}{\partial \alpha} &= \frac{\delta d_1 k(r - \mu)}{(d_1 - 1)(\theta - \alpha k)^2} > 0 \\
\frac{\partial x_B^*}{\partial \delta} &= \frac{d_1(r - \mu)}{(d_1 - 1)(\theta - \alpha k)} > 0,
\end{aligned}$$

from which the result stated holds.

Regarding μ, we obtain

$$\frac{\partial x_B^*}{\partial \mu} = \frac{\delta\left(\left((d_1 - 1)d_1\sigma^4\phi - 2\mu^2 + \mu\sigma^2\phi + 1\right) + r\left(2\mu - \sigma^2(\phi + 1)\right)\right)}{(d_1 - 1)^2\sigma^4(\alpha k - \theta)\phi} < 0.$$

Observe that its denominator is negative, since $\alpha K - \theta < 0$. On the other side, taking into account the domain of our problem, we obtain that there are no possible roots for the numerator. Thus since the expression in the numerator is continuous in each of its parameters and takes a positive value when testing for given possible values, it follows that it is always positive with respect to our domain.

Regarding θ, we obtain

$$\frac{\partial x_B^*}{\partial \sigma} = -\frac{\delta d_1(r - \mu)}{(d_1 - 1)(\theta - \alpha k)^2} < 0,$$

from which the result follows.

□

3.3.2 Capacity Optimization Model

Proposition 3.2. *Decision threshold x_C^* increases with r, σ and δ, decreases with θ and has a monotonic behaviour with μ. None of any other parameters have effect on x_C^*.*

Proof: Recall expression of x_C^* given in (3.21). One can immediately notice that it's not dependent on parameters α and K as it is x_B^*. We start the analysis regarding the other parameters.

Regarding r, we obtain

$$\frac{\partial x_C^*}{\partial r} = \frac{\delta\left(\left(d_1^2 - 1\right)\sigma^2\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1} + 4\mu - 4r\right)}{(d_1 - 1)^2\theta\sigma^2\sqrt{\frac{4\mu^2}{\sigma^4} - \frac{4\mu}{\sigma^2} + \frac{8r}{\sigma^2} + 1}} > 0.$$

Its denominator is positive. The numerator is a continuous function with respect to its parameters and has no roots in the domain of our problem. Thus, after testing for given possible values, it follows that it is always positive with respect to our domain.

Regarding σ, we obtain

$$\frac{\partial x_C^*}{\partial \sigma} = \frac{4\delta(\mu - r)\left(-2\mu^2 + \mu\sigma^2(1+\phi) - 2r\sigma^2\right)}{(d_1 - 1)^2\theta\sigma^5\phi} > 0.$$

Note that its denominator is positive. Analyzing the numerator, we found that it has no possible roots on the domain of our problem. Thus since the expression in the numerator is continuous in each of its parameters and takes a positive value when testing for given possible values, it follows that it is always positive with respect to our domain.

Regarding δ, we obtain

$$\frac{\partial x_C^*}{\partial \delta} = \frac{(d_1 + 1)(r - \mu)}{(d_1 - 1)\theta} > 0,$$

from which the result holds.

Regarding θ, we obtain

$$\frac{\partial x_C^*}{\partial \theta} = -\frac{\delta(d_1 + 1)(r - \mu)}{(d_1 - 1)\theta^2} < 0,$$

from which the result holds.

Regarding μ, we obtain

$$\frac{\partial x_C^*}{\partial \mu} = \frac{\delta}{(d_1 - 1)^2\theta}\left(1 - d_1^2 + (r - \mu)\left(-1 + \frac{2\mu - \sigma^2}{\phi\sigma^2}\right)\right) = \begin{cases} < 0 & \text{for } \mu < \frac{\sigma^2}{2} \\ > 0 & \text{for } \mu > \frac{\sigma^2}{2} \end{cases}.$$

□

Numerical comparisons between the Benchmark and the Capacity Optimization Models

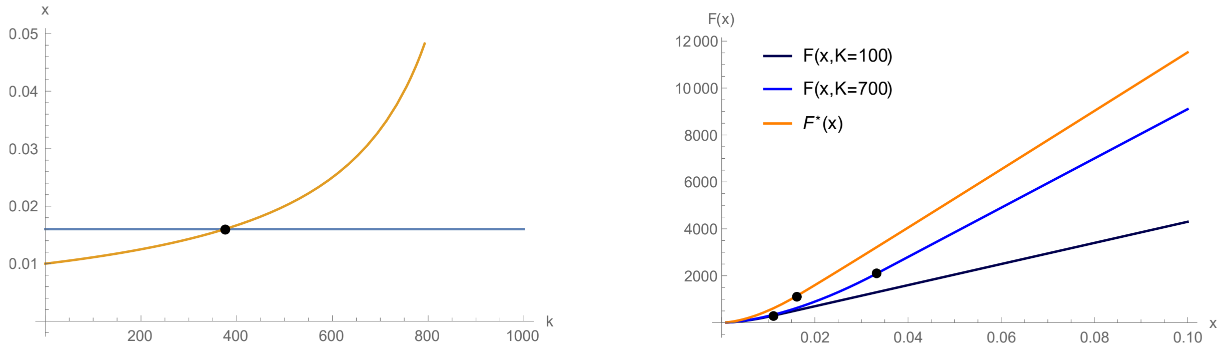
To illustrate results above mentioned we performed some numerical illustrations, using software *Mathematica* and its function Manipulate. However here we are only able to present static plots - we leave to the interested ones, to check the results achieved with Manipulate.

Unless it is written the opposite, the following parameter values were considered:

- $\mu = 0.03$
- $\sigma = 0.005$
- $r = 0.05$
- $\delta = 2$
- $\alpha = 0.01$
- $\theta = 10$
- $K = 100$

We start by illustrating how x_B^* and x_C^* are related by the capacity level K, on which x_B^* is dependent. The leftmost side of Figure 3.1 comproves the results stated in Propositions 3.1 and 3.2 are valid: the threshold x_B^* increases with capacity K.

Note that, although the threshold x_B^* may be smaller or bigger (depending on the capacity K chosen), we will always observe $F^*(x) \geq F(x, K)$ $\forall x, K$ parameters considered in the domain of our problem, as

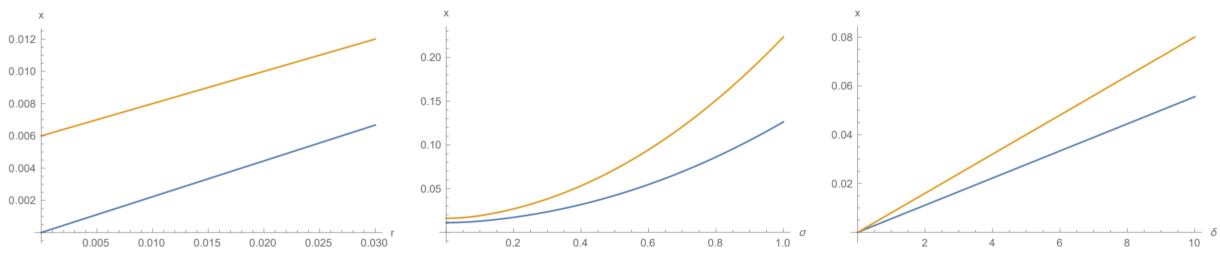


(a) Threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange), considering capacity levels $K \in [0, \theta/\alpha = 1000)$ and the value that x_B^* takes when considering K_C^* (black).

(b) Evaluation of value functions F, considering capacities $K = 100$ (darker blue) and $K = 700$ (lighter blue), and F^* (orange) with respective demand threshold values presented (black)

Figure 3.1: Influence of the chosen capacity K in the threshold values x_B^* and x_C^* and respective value functions F and F^*.

can be seen on the rightmost side of Figure 3.1. Here we have represented value functions $F(x, K)$ defined as in (3.11), with the dependence on the capacity K highlighted, and F^* is as defined in (3.19).



(a) $r \in (\mu, 1)$ (b) $\sigma \in (0.0001, 1)$ (c) $\delta \in (0, 10)$

Figure 3.2: Threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange) and regarding its increasing parameters r, σ and δ.

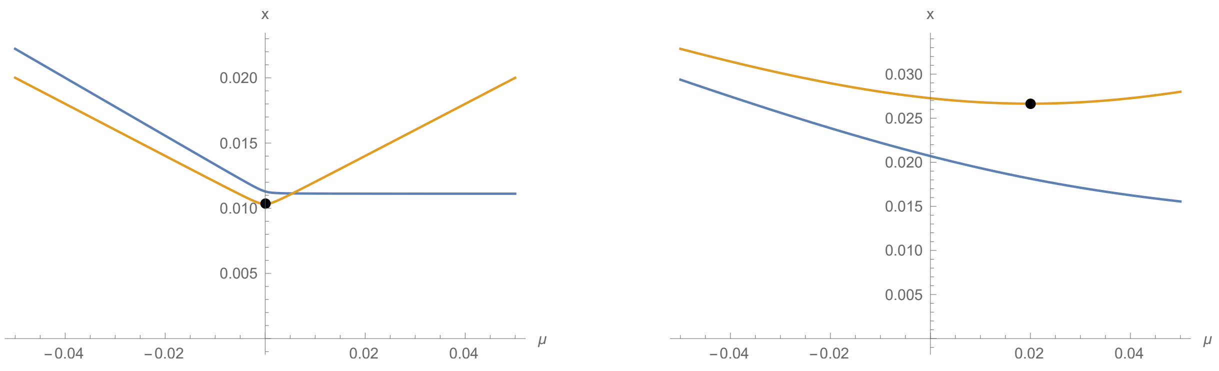
On Figure 3.2 we observe that both thresholds increase with volatility. This in accordance with [7] and [6], whose works describe that when uncertainty is high, there is a delay time to invest, which is here reflected on an higher demand level.

As the sensibility parameter δ increases, the firm needs to pay more sunk costs. Therefore the investment will only be made if higher demand values are observed, which is comproved also by Figure 3.2.

On the other side, as the discount rate r increases, we have a bigger currency devaluation, when evaluating the expected cash-flow to be earned. Thus, in order to justify the irreversible investment that the firm needs to incur on the new product, we need to observe bigger values of demand to balance the money lost due to currency devaluation and the future earnings.

Regarding the drift parameter μ we obtained that the threshold values do not have a monotonic behaviour, either for smaller or bigger values of volatility. As showed in Figure 3.3, the smallest value of demand level necessary to invest is observed at the stationary point when $\mu = \sigma^2/2$.

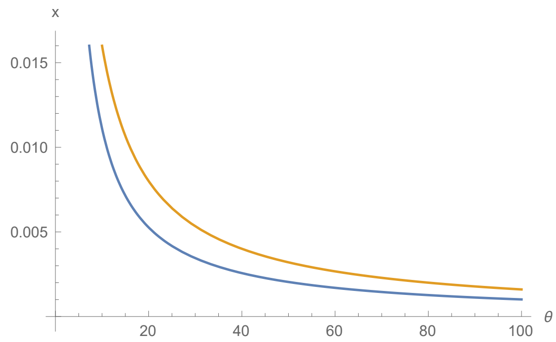
Regarding the innovation breakthrough we observe, on Figure 3.4, that a higher θ implies a smaller demand threshold, regarding both models. This is in accordance to what was done in [7], since as θ



(a) $\sigma = 0.05$

(b) $\sigma = 0.2$

Figure 3.3: Threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange), considering drift $\mu \in [-r, r]$ and corresponding stationary point $\sigma^2/2$ (black).



(a) $\theta \in (1, 10)$

Figure 3.4: Threshold values regarding the benchmark model (blue) and the capacity optimized model (orange) and decreasing parameter θ.

increases, the firm wills to invest in a product with much more advanced technology, that, taking into account our instantaneous profit function π (3.2), will culminate in higher earnings, when fixing the demand value.

3.3.3 Optimal Capacity Level

Now that the threshold values were analysed, we focus on the optimal capacity K_C^* as it is presented in (3.22). We analyse how does K_C^* behaves with the different parameters.

Proposition 3.3. *Optimal capacity level K_C^* increases with μ, σ and θ, decreases with r and α and it is not dependent on δ.*

Proof: The relation between K_C^* and θ, r or α comes immediately by observing K_C^* expression. Now, regarding drift parameter we obtain that

$$\frac{\partial K_C^*(\mu)}{\partial \mu} = \frac{4\theta\left(\sigma^2(\phi+1) - 2\mu\right)}{\alpha\phi\left(\sigma^2(\phi+3) - 2\mu\right)^2} > 0.$$

Since

$$\sigma^2\left(\sqrt{\frac{4\mu^2}{\sigma^4}-\frac{4\mu}{\sigma^2}+\frac{8r}{\sigma^2}+1}+1\right)-2\mu>0 \Leftrightarrow \frac{4\mu^2}{\sigma^4}-\frac{4\mu}{\sigma^2}+\frac{8r}{\sigma^2}+1>\left(\frac{2\mu}{\sigma^4}-1\right)^2=\frac{4\mu^2}{\sigma^4}-\frac{4\mu}{\sigma^2}+1 \tag{3.24}$$

$$\Leftrightarrow \frac{8r}{\sigma^2}>0,$$

which is true for $\forall r>0$. Therefore, it follows that the numerator is positive. Since $\phi>0$ 3.23, it follows that the numerator is positive. From these two conditions, the result follows.

Regarding volatility parameter we obtain that

$$\frac{\partial K_C^*(\sigma)}{\partial\sigma}=\frac{8\theta\left(2\mu^2-\mu\sigma^2\left(\phi+1\right)+2r\sigma^2\right)}{\alpha\sigma\phi\left(\sigma^2\left(\phi+3\right)-2\mu\right)^2}>0$$

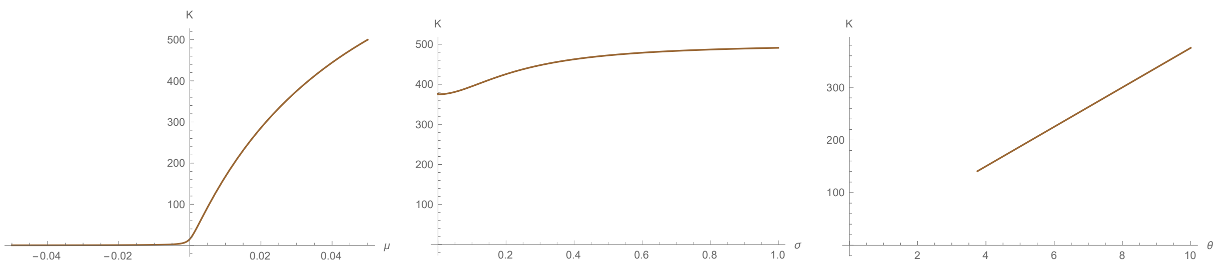
One can easily note that the denominator is positive. When it comes to the numerator, we will study the sign of the expression between parenthesis.

$$2\mu^2-\mu\sigma^2\left(\phi+1\right)+2r\sigma^2>0 \Leftrightarrow \left(\frac{2\mu^2+2r\sigma^2}{\mu\sigma^2}-1\right)^2>\frac{4\mu^2}{\sigma^2}-\frac{4\mu}{\sigma^2}+\frac{8r}{\sigma^2}+1 \tag{3.25}$$

$$\Leftrightarrow r>\mu, \tag{3.26}$$

which always hold, implying that the denominator is always positive.

□



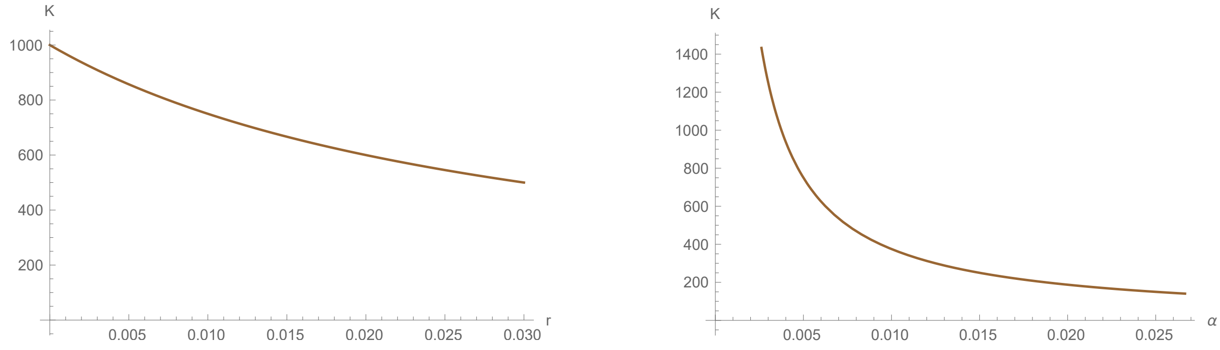
(a) $\mu\in(-r,r)$ (b) $\sigma\in(0,1)$ (c) $\theta\in(\alpha K_C^*,10)$

Figure 3.5: Optimal capacity regarding the threshold value x_C^* and increasing parameters μ, σ and θ.

Considering some numerical approximations, we observe, on Figure 3.5, that K_C^* increases with drift, volatility and innovation level, as deduced before. Note that, regarding the drift parameter, the growth is barely noticeable for negative values of μ, but then it turns to be logarithmic. This is related with the fact that the denominator of K_C^* decreases with μ. For $\mu<0$, the denominator increases in a weak rate, resulting in an almost constant value of K_C^*. However, when $\mu>0$, as it increases, the denominator decreases significantly, resulting in higher values of K_C^*.

Regarding volatility σ, we obtain that the optimal capacity increases with it, however in a weaker rate.

Regarding the innovation breakthrough θ, K_C^* shows to increase linearly with it.



(a) $r \in (\mu, 1)$

(b) $\alpha \in (0, 1)$

Figure 3.6: Optimal capacity regarding the threshold value x_C^* and decreasing parameters r and α.

Regarding discount rate r and sensibility parameter α, we have on Figure 3.6 that K_C^* decreases with them, as presented before in Proposition 3.3.

Chapter 4

Adding a new product when already producing one (Firm is already active before investing)

4.1 Introduction

We consider now the case where, even before we decide to invest, the firm already produces a certain product. We also consider that the product is *stable* in the market, in the sense that it's a recognized product, resulting in a demand function that is not influenced by the demand level. Instead it's given by

$$p_0 = 1 - \alpha K_0 \tag{4.1}$$

where α stands for a capacity sensibility parameter and K_0 for the capacity of production of the *old* product.

However, the same is not valid for the new product. When the breakthrough takes place, the firm has the option to invest and immediately start to produce the new product. Since this one is a new product, susceptible to the consumers' demand, its demand function is considered to be the same as in Section 3.1, by the expression in (3.2), that is,

$$p_1(X_t) = (\theta - \alpha K_1) X_t \tag{4.2}$$

where θ stands for the innovation level after the breakthrough, α for the same sensitivity parameter as in the *old* product, K_1 for the capacity of production of the *new* product and X_t for the demand level at time t.

The instantenous profit function related with each of the products is respectively given by $\pi_i,\ i \in \{0, 1\}$, and it's obtain by multiplying the demand function by the production capacity, that is, $\pi_i(X_t) = p_i(X_t)K_i$.

Recall that at the moment we decide to invest, we need to pay δK_1 related to sunk costs.

As in the previous section, two models will be derived. The first one is the benchmark model. The second one consists takes into account the maximized instantaneous profit in function of the production capacity related with the new product K_1. Comparative statics of both models will be made afterwards.

4.2 Stopping Problem

4.2.1 Benchmark Model

We want to find when is the best time to invest in the new product, knowing that the firm produces a established product, that it's not influenced by the demand level and whose profit function is given by

$$\pi_0 = (1 - \alpha K_0)K_0, \tag{4.3}$$

and when the replacement happens, the firm will be immediately producing a product whose profit function is

$$\pi_1(X_t) = (\theta - \alpha K_1)K_1 X_t. \tag{4.4}$$

Therefore our optimal stopping problem may be written as

$$\sup_{\tau} \mathbb{E}^{X_0=x} \left[\int_0^{\tau} \pi_0 e^{-rs} ds + \left(\int_{\tau}^{\infty} \pi_1(X_s) e^{-rs} ds - e^{-r\tau} \delta K_1 \right) \mathbb{1}_{\{\tau<\infty\}} \right]. \tag{4.5}$$

The first integral corresponds to the discounted profit obtain associated to the *old* product from the time when the innovation level θ is reached (considered to be at $t = 0$) until the time when the firm decides to invest in the *new* product. The second integral corresponds to the long term discounted profit associated to the *new* product, after deciding to invest. Subtracting to it discounted sunk costs $e^{r\tau}\delta K_1$, we obtain the cash-flow associated to the investment decision.

We can simplify this problem in order to have a standard optimal stopping problem with null running cost function. Starting by conditioning (4.5) to the time when the investment should happen and using Tower rule it follows that (4.5) is equal to

$$\sup_{\tau} \mathbb{E}^{X_0=x} \left[\mathbb{E}^{\tau=t} \left[\int_0^{\tau} \pi_0 e^{-rs} ds + \int_{\tau}^{\infty} \pi_1(X_s) e^{-rs} ds - e^{-r\tau} \delta K_1 \right] \right]. \tag{4.6}$$

Since expectation is a linear operator, we can simplify each of the integrals separately.

Note that in the leftmost integral of (4.6), as previously written, the instantaneous profit associated to the *old* product does not depend on the demand level and all the parameters are deterministic. Then we easily solve the integral and its expression can be simplified as

$$
\begin{aligned}
\mathbb{E}^{\tau=t}\left[\int_0^t \pi_0 e^{-rs} ds\right] &= \mathbb{E}^{\tau=t}\left[\int_0^t p_0 K_0 e^{-rs} ds\right] \\
&= \mathbb{E}^{\tau=t}\left[\int_0^t (1-\alpha K_0) K_0 e^{-rs} ds\right] \\
&= \mathbb{E}^{\tau=t}\left[(1-\alpha K_0) K_0 \frac{1-e^{-rt}}{r}\right]
\end{aligned} \tag{4.7}
$$

Following a similar approach as in Section 3.2.1, when deducing (3.9), the leftmost expected value of the rightmost integral can also be simplified as

$$
\begin{aligned}
\mathbb{E}^{\tau=t}\left[\int_t^\infty \pi_1(X_s) e^{-rs} ds - e^{-rt}\delta K_1\right] &\underset{v:=s-\tau}{=} \mathbb{E}^{\tau=t}\left[e^{-rt}\left(\int_0^\infty p_1(X_{v+t}) K_1 e^{-rv} dv - \delta K_1\right)\right] \\
&= \mathbb{E}^{\tau=t}\left[e^{-rt}\left(\int_0^\infty (\theta - \alpha K_1) X_{v+t} K_1 e^{-rv} dv - \delta K_1\right)\right] \\
&= \mathbb{E}^{\tau=t}\left[e^{-rt}\left(\int_0^\infty (\theta - \alpha K_1) X_{v+t} K_1 e^{-rv} dv - \delta K_1\right)\right] \\
&= e^{-r\tau}\left(\frac{(\theta-\alpha K_1) K_1 x_\tau}{r-\mu} - \delta K_1\right)
\end{aligned} \tag{4.8}
$$

where x_τ is taken to be the observed demand level at the time when the *new* product starts being produced. Recall, from the previous section, that expression (4.8) only holds if the assumptions of Fubini's Theorem hold, that is $r - \mu > 0$.

Denote F as the value function solution to (4.6). Plugging expressions (4.7) and (4.8) in (4.6), getting rid of the expectation conditional to time when the investment decision is made, using (again) Tower rule, we obtain that F may be written as

$$
F(x) = \frac{\pi_0}{r} + \sup_\tau \mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{(\theta-\alpha K_1)K_1 X_\tau}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right)\right)\mathbb{1}_{\{\tau<\infty\}}\right], \tag{4.9}
$$

which corresponds to the sum of a constant term with a standard optimal stopping problem with null running cost function, as we wanted. This is the effect of dealing with a very stable product, which doesn't depend on the current demand level, allowing us to simply this much its expression.

Considering V to be the optimal standard problem present in (4.9), that is

$$
V(x) = \sup_\tau \mathbb{E}^{X_0=x}\left[e^{-r\tau} h(X_\tau)\right] = \sup_\tau \mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{(\theta-\alpha K_1)K_1 X_\tau}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right)\right)\right]. \tag{4.10}
$$

V is such that it must verify the HJB variational inequality (2.40) regarding values on the continuation and stopping region. As was written in Section 2.2.1, it follows that V is such that

$$
V(x) = \begin{cases} a_2 (x_B^*)^{d_1}\ , \ x \in \mathcal{C} \\ \frac{(\theta-\alpha K_1)K_1 X_\tau}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right) = \frac{K(\theta-\alpha K)}{r-\mu}\ , \ x \in \mathcal{S} \end{cases} \tag{4.11}
$$

where coefficient a_2 and the threshold value x_B^* are found by value matching (2.52a) and smooth pasting

(2.52b) conditions, expressed by the corresponding system

$$\begin{cases} a_2(x_B^*)^{d_1} = \frac{(\theta-\alpha K_1)K_1 x_B^*}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right) \\ a_2 d_1 (x_B^*)^{d_1-1} = \frac{K(\theta-\alpha K)}{r-\mu}, \end{cases} \tag{4.12}$$

Solving the system above, the threshold x_B^* and the coefficient a_2 are respectively given by

$$x_B^* = \frac{d_1}{d_1-1}\frac{\delta K_1 + \frac{\pi_0}{r}}{\theta - \alpha K_1}\frac{r-\mu}{K_1} \tag{4.13}$$

$$a_2 = \left(\delta K_1 + \frac{1-\alpha K_0}{r}K_0\right)\frac{(x^*)^{-d_1}}{d_1-1}, \tag{4.14}$$

with d_1 being the positive root of the polynomial described in (2.51).

Plugging the results stated above on the expression of F (4.9), it leads to

$$F(x) = \frac{\pi_0}{r} + \begin{cases} a_2(x^*)^{d_1} \ , \ x \in \mathcal{C} \\ \frac{(\theta-\alpha K_1)K_1 X_\tau}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right) \ , \ x \in \mathcal{S}, \end{cases} \tag{4.15}$$

where continuation and stopping regions are respectively described as in (2.46) and (2.47) and the optimal stopping time as in (2.43).

4.2.2 Capacity Optimization Model

As done in Chapter 3, we now extend the previous result, by finding when it is the best time to invest in the *new* product and which is the optimal capacity associated to it. This optimal stopping problem can be stated as

$$\begin{aligned} F^*(x) &= \sup_\tau \mathbb{E}^{X_0=x}\left[\max_{K_1}\left\{\int_0^\tau \pi_0 e^{-rs}ds + e^{-r\tau}\left(\int_\tau^\infty \pi_1(X_s)e^{-rs}ds - \delta K_1\right)\right\}\right] \\ &= \frac{\pi_0}{r} + \sup_\tau \mathbb{E}^{X_0=x}\left[e^{-r\tau}\max_{K_1}\left\{\frac{(\theta-\alpha K_1)K_1 X_\tau}{r-\mu} - \left(\delta K_1 + \frac{\pi_0}{r}\right)\right\}\right], \end{aligned} \tag{4.16}$$

where the expression is simplified in similar way as done in the previous section and using the fact that π_0 is deterministic.

Denoting the non-maximized terminal function by h with the dependence on the capacity parameter K_1 highlighted, we have that

$$h(x, K_1) = \frac{(\theta-\alpha K_1)K_1 x}{r-\mu} - \left(\delta K_1 + \frac{(1-\alpha K_0)K_0}{r}\right). \tag{4.17}$$

Note that, h is a second order polynomial with respect to the capacity and its expression corresponds to the terminal function in the previous chapter minus a constant term ($\frac{(1-\alpha K_0)K_0}{r}$). Thus, by studying the first and second derivatives of h, we obtain the same results as achieved in Section 3.2.2. That is, its first and second partial derivatives are, respectively, given by (3.15) and (3.17). Therefore we have that the maximizer of h in (4.17) is the same in (3.16), that is,

$$K_1^* := \arg\max_{K_1} h(x, K_1) = \frac{\theta}{2\alpha} - \frac{\delta(r-\mu)}{2\alpha x}, \ \forall x. \tag{4.18}$$

Evaluating h at its optimal capacity level and denoting by h^*, we obtain that its expression is given by

$$h(x, K^*) = \frac{K_0}{r}(\alpha K_0 - 1) + \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x}, \tag{4.19}$$

from which follows that our problem, as described in (4.16), can be stated as

$$F^*(x) = \frac{\pi_0}{r} + \sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{K_0}{r}(\alpha K_0 - 1) + \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x}\right)\right], \tag{4.20}$$

which consists in a standard stopping optimal problem with null running cost function plus a constant term.

Denoting V^* as the value function associated to the optimal stopping problem in (4.20), the optimization problem can be stated as

$$V^* = \sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{K_0}{r}(\alpha K_0 - 1) + \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x}\right)\right] \tag{4.21}$$

which is again a standard optimal stopping problem with null running cost function. Similarly to the benchmark model, we obtain that the value function associated to (3.18), satisfies a similar HJB variational inequality as described in (2.40). Therefore V^* is such that

$$V^*(x) = \begin{cases} bx^{d_1} \ , \ x \in \mathcal{C} \\ \frac{K_0}{r}(\alpha K_0 - 1) + \frac{(\theta x - \delta(r-\mu))^2}{4\alpha(r-\mu)x} \ , \ x \in \mathcal{S} \end{cases}, \tag{4.22}$$

where coefficient b and the threshold value x_C^* are found by value matching (2.52a) and smooth pasting (2.52b) conditions, expressed by the corresponding system

$$\begin{cases} b(x_C^*)^{d_1} = \frac{K_0}{r}(\alpha K_0 - 1) + \frac{(\theta x_C^* - \delta(r-\mu))^2}{4\alpha(r-\mu)x^*} \\ bd_1(x_C^*)^{d_1-1} = \frac{\theta^2(x_C^*)^2 - \delta^2(r-\mu)^2}{4\alpha(r-\mu)(x_C^*)^2} \end{cases} \tag{4.23}$$

with d_1 being the positive root of the polynomial described in (2.51).

We get two possible positive roots for the threshold level:

$$x_{C,1}^* = \frac{r-\mu}{(d_1-1)\theta^2 r}\left(d_1\left(2\alpha K_0 p_0 + \delta\theta r\right) + \sqrt{(\delta\theta r)^2 + 4d_1^2\alpha K_0 p_0\left(\alpha K_0 p_0 + \delta\theta r\right)}\right)$$
$$x_{C,2}^* = \frac{r-\mu}{(d_1-1)\theta^2 r}\left(d_1\left(2\alpha K_0 p_0 + \delta\theta r\right) - \sqrt{(\delta\theta r)^2 + 4d_1^2\alpha K_0 p_0\left(\alpha K_0 p_0 + \delta\theta r\right)}\right).$$

However, after some manipulation, we exclude the second one $x_{C,2}^*$, since the coefficient b associated to it takes a negative value. This is an absurd, since it would lead to a negative value function for any demand level smaller than $x_{C,2}^*$. Therefore we obtain that the threshold level and coefficient b in (4.23) are, respectively, given by

$$x_C^* = \frac{r-\mu}{(d_1-1)\theta^2 r}\left(d_1\left(2\alpha\pi_0+\delta\theta r\right)+\sqrt{(\delta\theta r)^2+4d_1^2\alpha\pi_0\left(\alpha\pi_0+\delta\theta r\right)}\right) \tag{4.24}$$
$$b = \left(\frac{K_0(\alpha K_0-1)}{r}+\frac{(\theta x_C^*-\delta(r-\mu))^2}{4\alpha x_C^*(r-\mu)}\right)(x_C^*)^{-d_1}.$$

Plugging the results above on (4.20) it follows that F^* as the form of

$$F^*(x) = \frac{\pi_0}{r}+\begin{cases} bx^{d_1}\ ,\ x\in\mathcal{C} \\ \frac{K_0}{r}(\alpha K_0-1)+\frac{(\theta x-\delta(r-\mu))^2}{4\alpha(r-\mu)x}\ ,\ x\in\mathcal{S}\end{cases}\ , \tag{4.25}$$

with the continuation and stopping regions are respectively described as in (2.46) and (2.47) and the optimal stopping time as in (2.43).

Now we analyse the optimal capacity level K_C^* as done in Section 3.2.2. By evaluating K_1^* at the threshold demand level x_C^* we obtain

$$K_C^* = \frac{\theta}{2\alpha}-\frac{\delta(d_1-1)\theta^2 r}{2\alpha\left(\sqrt{4\alpha d_1^2\pi_0(\alpha\pi_0+\delta\theta r)+\delta^2\theta^2 r^2}+d_1(2\alpha\pi_0+\delta\theta r)\right)} \tag{4.26}$$

4.3 Comparative Statics

We will start to show some results concerning the Benchmark Model, on Section 4.3.1, and the Capacity Optimization Model, on Section 4.3.2. Comparisons between the benchmark and capacity optimization models will also be made.

4.3.1 Benchmark Model

Proposition 4.1. *Decision threshold x_B^* increases with δ, σ, decreases with θ and does not have a monotonic behaviour with K_0, K_1, r. Regarding sensibility parameter α, x_B^* increases with it when $\theta < \frac{K_1}{K_0^2}(K_0+K_1 r\delta)$ and decreases otherwise.*

Proof:

Regarding the formula obtained for x_B^* (4.13), we immediately conclude that the decision threshold increases with δ and decreases with θ.

Regarding σ, we observe that

$$\frac{\partial x_B^*(\sigma)}{\partial\sigma} = \frac{(r-\mu)\left(\delta K_1+\frac{K_0(1-\alpha K_0)}{r}\right)}{(d_1-1)K_1(\theta-\alpha K_1)}\left(\frac{2\mu}{\sigma^3}+\frac{\frac{4\mu\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)}{\sigma^3}-\frac{4r}{\sigma^3}}{2\sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}}\right)\left(1-\frac{d_1}{d_1^2-1}\right)>0.$$

Taking into account our initial assumptions about r, μ and profits associated to the old and the new product, it follows that the leftmost expression is always positive. Manipulating the expression in between we obtain that

$$
\begin{aligned}
\frac{2\mu}{\sigma^3}+\frac{\frac{4\mu\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)}{\sigma^3}-\frac{4r}{\sigma^3}}{2\sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}}<0 &\Leftrightarrow \mu d_1 - r < 0\\
&\Leftrightarrow \frac{\mu}{2}-\frac{\mu^2}{\sigma^2}+\mu\sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}-r<0\\
&\Leftrightarrow \frac{r}{\mu}\left(1-\frac{r}{\mu}\right)<0,
\end{aligned}
\tag{4.27}
$$

which holds true for the two possible cases, when $\mu < 0$ and $\mu > 0$. Analysing the rightmost expression, we obtain that the polinomial $d_1^2 - d_1 - 1$ has roots for $d_1 \in \left\{\frac{1}{2}(1-\sqrt{5}), \frac{1}{2}(1+\sqrt{5})\right\}$. Since $d_1 > 1$, the first root is impossible to be observed. Therefore it follows that for $d_1 \in \left(1, \frac{1}{2}(1+\sqrt{5})\right]$, the partial derivative above is negative (and positive otherwise).

Regarding K_0, we observe that

$$
\frac{\partial x_B^*(K_0)}{\partial K_0}=\frac{d_1(r-\mu)}{r(d_1-1)K_1(\theta-\alpha K_1)}(1-2\alpha K_0)=\begin{cases}>0 & \text{for } K_0<\frac{1}{2\alpha}\\ <0 & \text{for } K_0>\frac{1}{2\alpha}\end{cases}.
$$

since the expression represented in fraction is always positive.

Regarding K_1, we obtain that

$$
\frac{\partial x_B^*(K_1)}{\partial K_1}=\frac{d_1(r-\mu)}{(d_1-1)K_1(\theta-\alpha K_1)}\left(\frac{\alpha(\frac{K_0(1-\alpha K_0)}{r}+K_1\delta)}{\theta-\alpha K_1}-\frac{\frac{K_0(1-\alpha K_0)}{r}+K_1\delta)}{K_1}+\delta\right)
$$

The leftmost expression is always positive. Thus we only need to evaluate the sign of the expression next to it. Manipulating mentioned expression, taking into account that the capacity level cannot be negative as well as the price function given by $\pi_0 = K_0(1-\alpha K_0)$, it follows that

$$
\frac{\partial x_B^*(K_1)}{\partial K_1}=\begin{cases}>0 & \text{for } K_1>\frac{-\pi_0+\sqrt{\alpha\pi_0(\pi_0+r\delta\theta)}}{r\alpha\delta}\\ <0 & \text{for } K_1\in\left[0,\frac{-\pi_0+\sqrt{\alpha\pi_0(\pi_0+r\delta\theta)}}{r\alpha\delta}\right]\end{cases},
$$

from which we obtain that x_B^* has no monotonic behaviour with K_1.

Regarding parameter α, we obtain that

$$
\frac{\partial x_B^*(\alpha)}{\partial\alpha}=\frac{d_1(r-\mu)}{(d_1-1)(\theta-\alpha K_1)}\left(\frac{\frac{K_0(1-\alpha K_0)}{r}+\delta K_1}{\theta-\alpha K_1}-\frac{K_0^2}{rK_1}\right),
$$

where the leftmost expression is always positive. Simplifying the expression in the biggest brackets to the same denominator, we obtain that

$$\frac{\partial x_B^*(\alpha)}{\partial \alpha} = \begin{cases} > 0 & \text{for } \theta < \frac{K_0 K_1 + K_1^2 r\delta}{K_0^2} \\ < 0 & \text{for } \theta > \frac{K_0 K_1 + K_1^2 r\delta}{K_0^2}. \end{cases}$$

Note that the sign of the partial derivative does not depend on α, but instead on both capacity levels K_0 and K_1, discount rate r and the sensibility parameter δ.

Regarding parameter r we obtained complex derivates, from which we weren't able to deduce any analytical results. However, as it will be showed right after this proof, using Mathematica we obtained that x_B^* behaves in a non-monotonic way with it.

□

We weren't able to deduce any analytical result regarding the drift value μ. However, after many numerical experiments, we observed that x_B^* decreases with μ, as it's showed on Figure 4.2.

4.3.2 Capacity Optimization Model

Proposition 4.2. *Decision threshold x_C^* increases with δ, decreases asymptotically with θ and do not have a monotonic behaviour with μ, r, α and K_0.*

Proof:

For the sake of simplicity, in this proof, we will consider $\phi > 0$ as in (3.23) and we will consider $\psi := 4d_1^2\pi_0(\delta\theta r + \pi_0) + \delta^2\theta^2 r^2 > 0$.

Regarding δ, we obtain that

$$\frac{\partial x_C^*(\delta)}{\partial \delta} = \frac{(r-\mu)\left(\frac{4d_1^2\theta r\pi_0 + 2\delta\theta^2 r^2}{2\sqrt{\psi}} + d_1\theta r\right)}{(d_1-1)\theta^2 r} > 0$$

from which the result holds.

Regarding θ, we obtain that

$$\frac{\partial x_C^*(\theta)}{\partial \theta} = \frac{\theta(r-\mu)\left(\frac{4\delta d_1^2 r\omega + 2\delta^2\theta r^2}{2\sqrt{\psi}} + \delta d_1 r\right) - 2(r-\mu)\left(d_1(\delta\theta r + 2\omega) + \sqrt{\psi}\right)}{(d_1-1)\theta^3 r}.$$

The denominator is positive since $d_1 > 1$. Manipulating the numerator, it simplifies to

$$\frac{\delta d_1\theta r\left(2d_1\alpha\pi_0 - \sqrt{\psi}\right) - 2\left(2d_1\sqrt{\psi}\alpha\pi_0 + \psi\right) + \delta^2\theta^2 r^2}{\sqrt{\psi}},$$

which has two roots associated ($\theta \in \{-\frac{4d_1^2\omega}{\delta r}, 0\}$), which are impossible given the domain of the problem. Thus, since by evaluating for a certain $\theta > 0$, we obtain $\frac{\partial x_B^*(\theta)}{\partial \theta} < 0$, the result holds.

Since we assume that innovation levels have no upper limit, we evaluated them asymptotically. Denoting $\theta_A := \frac{(r-\mu)}{(d_1-1)r}\left(\sqrt{\delta^2 r^2} + \frac{\delta r\left(\sigma^2(\phi+1) - 2\mu\right)}{2\sigma^2}\right) > 0$ we obtain that x_C^* decreases on order of $\frac{\theta_A}{\theta}$, that is,

$$x_C^*(\theta) \sim \frac{\theta_A}{\theta} \Leftrightarrow \lim_{\theta\to\infty} \frac{x_C^*(\theta)}{\frac{\theta_A}{\theta}} = 1.$$

Regarding parameters μ, r α and K_0, we obtained complex derivates, from which we couldn't deduce any analytical result. However, as it will be showed hereunder, x_C^* behaves in a non-monotonic way with all of them.

□

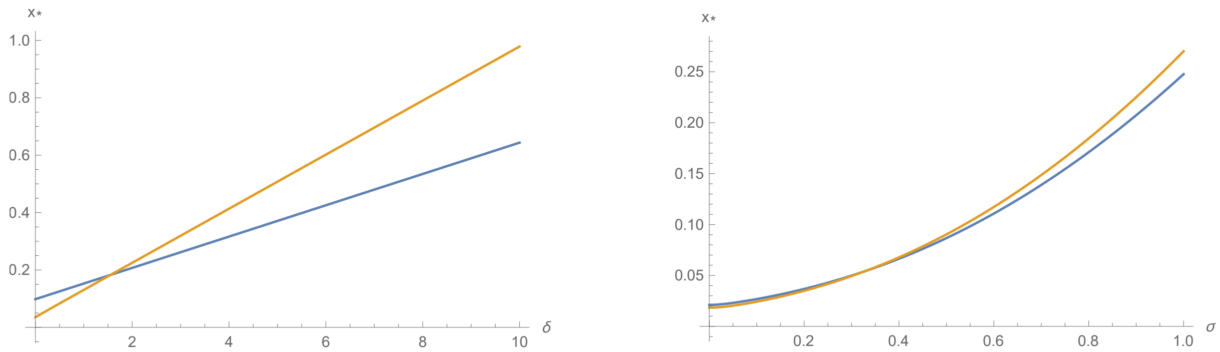
Although we couldn't obtain any analytical (strong) evidence, after different experiments done using *Mathematica* and its function Manipulate, we obtained that decision thresholds x_B^* and x_C^* increase with volatility σ. An example is showed on Figure 4.1.

Numerical comparisons between the Benchmark and the Capacity Optimization Models

To illustrate results above mentioned we performed some numerical illustrations, using software *Mathematica* and its function Manipulate. However here are only able to present static plots - we leave to the interested ones, to see the results achieved with Manipulate.

Unless it is written the opposite, following values were considered:

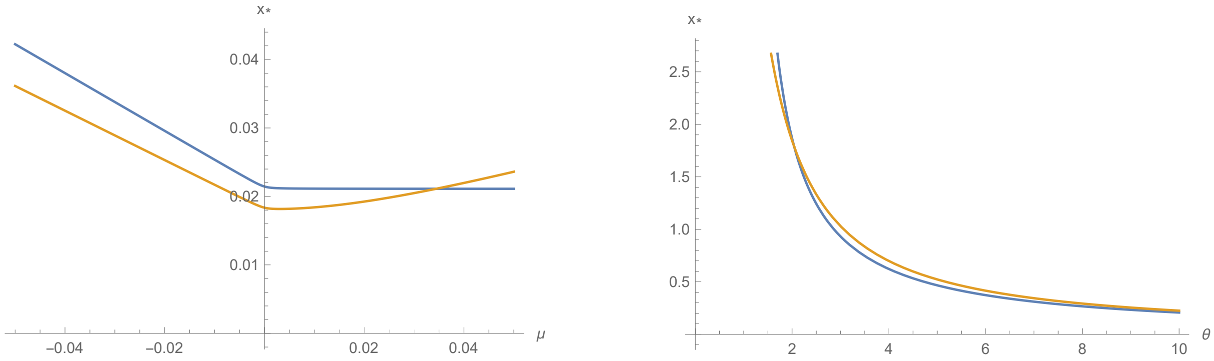
- $\mu = 0.03$
- $\sigma = 0.005$
- $r = 0.05$
- $\delta = 2$
- $\alpha = 0.01$
- $\theta = 10$
- $K_0 = 90$
- $K_1 = 100$



(a) $\delta \in (0, 1)$ (b) $\sigma \in (0, 1)$

Figure 4.1: Threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange) and parameters with which x_B^* increases.

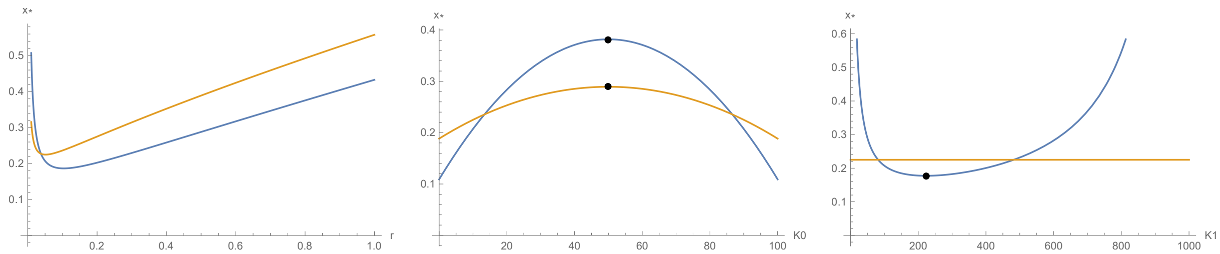
On Figure 4.1, we obtained similar results to the ones on Section 2. Both threshold values increase with sensibility parameter δ and volatility σ. The first one is justified by the fact that a higher δ means a bigger investment, and thus it will only be made, if there's also a huge demand of the product. The second one is justified by the huge uncertainty of the demand. Since it has a high variance, the demand has a great amplitude of values, which delays the investment decision, only made when the demand reaches a high level. This is in accordance to what is described in [4].



(a) $\mu \in (-r, r)$ (b) $\theta \in (1, 10)$

Figure 4.2: Threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange) and parameter with which x_B^* decreases.

On Figure 4.2 we see that, similarly to what happened on Section 2, both threshold levels decrease with innovation level. Although the threshold level associated to the Benchmark Model decreases with μ in what seems to be a linear way for negative values of μ and almost negligible for positive values of μ, the same doesn't happen to the threshold level associated to the capacity optimization model. This last one, seems to increase for positive values of μ.



(a) $r \in (\mu, 1)$, $\mu = 0.01$ and $\sigma = 0.9$ (b) $K_0 \in (0, \frac{1}{\alpha} = 100)$ (c) $K_1 \in (0, \frac{\theta}{\alpha} = 1000)$

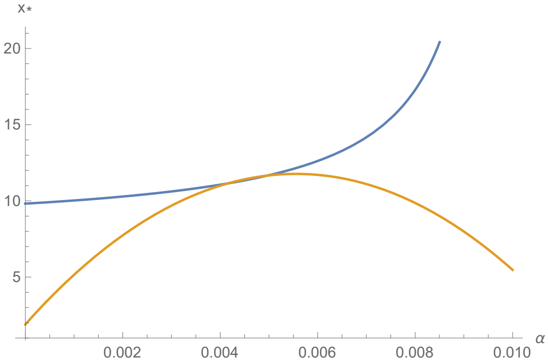
Figure 4.3: Threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange) and parameters with which x_B^* has a non-monotonic behaviour.

On Figure 4.3 we have that a non-monotonic behaviour was also observed on parameters r, K_0 and K_1, being this last one only valid for the Benchmark Model.

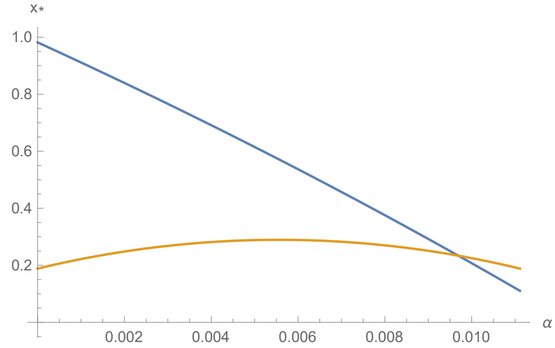
On the leftmost plot we can see that a minimum demand threshold is observed for small values of discount rate r. Unfortunately we couldn't derive its analytical expression.

Interestingly, as seen on the plot located at the center, regarding the capacity of *old* product, a maximum demand threshold is observed for both models when the capacity level K_0 is exactly equal to $\frac{1}{2\alpha}$. This values comes from the expression $\alpha K_0 \pi_0 = \alpha K_0(1 - \alpha K_0)$ included on both expressions of x_B^* (4.13) and x_C^* (4.24).

Regarding parameter K_1, its value doesn't affect threshold x_C^*, since it takes into account the optimal capacity K_C^*. However, when it comes to the threshold x_B^* we have that it achieves a minimum value at $K_1 = \frac{\pi_0 + \sqrt{\alpha \pi_0(\pi_0 + \delta\theta r)}}{r\alpha\delta}$, as it's represented on the bottom plot. Note that the standard value considered was $K_1 = 100$ and that it's associated a threshold x_B^* smaller than the threshold x_C^*. This is one of the reasons why the demand threshold x_B^* appears to be smaller than x_C^* in most cases. However as showed in Figure 3.1 (b), we have that the value function F^* (associated to the threshold x_C^*) will always be greater than the value function F (associated to the threshold x_B^*).



(a) $\alpha \in (0,1)$ and $\theta = 10 > \frac{K_1}{K_0^2}(K_0 + K_1 r\delta)$



(b) $\alpha \in (0,1)$ and $\theta = 1 < \frac{K_1}{K_0^2}(K_0 + K_1 r\delta)$

Figure 4.4: Threshold value with respect to the benchmark model (blue) and the capacity optimized model (orange) and sensibility parameter α.

On Figure 4.4 it's represented the behaviour of x_B^* with α, as written on Proposition . Considering the fixed values considered before, we obtain a θ-threshold equal to $\frac{K_1}{K_0^2}(K_0 + K_1 r\delta) = 1.23457$. One can see on Figure 4.4 when that testing for innovation levels greater (leftmost) and smaller (rightmost) than the mentioned threshold, we verify what was deduced: that x_B^* behaves in either an increasing or decreasing way with α for certain levels of innovation, while x_C^* always behave in a non-monotonic way.

4.3.3 Optimal Capacity Level

Now we analyse optimal capacity level K_C^*, that is given by evaluating K^* on demand level x_C^* and as it is written in (4.26).

Proposition 4.3. *Optimal capacity level K_C^* increases linearly asymptotically with θ and does not have a monotonic behaviour with K_0.*

Proof:

Regarding innovation level θ, assuming that it has no upper limit, it's possible to evaluate its behaviour asymptotically. Denoting $\theta_K := \frac{\sigma^2\left(\sqrt{\delta^2 r^2}+\delta r\right)}{\alpha\left(2\sigma^2\sqrt{\delta^2 r^2}+\delta r(\sigma^2(\phi+1)-2\mu)\right)} > 0$, we obtain that K_C^* increases on order of $\theta_K\theta$, that is,

$$K_C^*(\theta) \sim \theta_K\theta \;\Leftrightarrow\; \lim_{\theta\to\infty} \frac{K_C^*}{\theta_K\theta} = 1$$

The non monotonic behaviour of K_C^* with K_0 will be showed hereunder in the obtained plots.

□

Although it wasn't possible to derive any (strong) analytical solution about te behaviour of the other parameters, numerically we obtain robust results. By manipulating each parameter, using command Manipulate, we obtained no different behaviours from the ones showed hereunder.

The results obtained regarding parameters μ, σ, r, α and θ were similar to the ones obtain for the optimal capacity level on the previous section. Since K_C^* depends on value x_C^*, it is expected to observe similiar behaviours regarding the studied parameters.

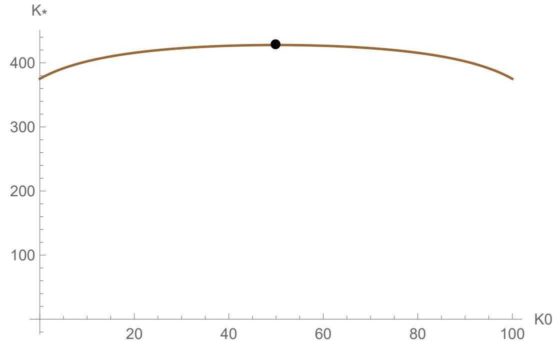
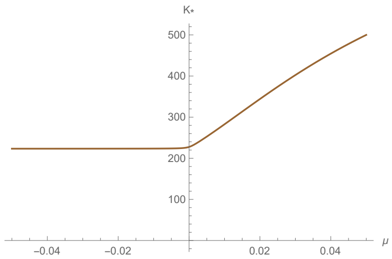
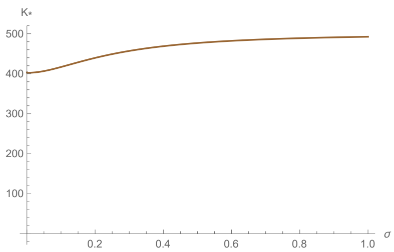


Figure 4.5: Optimal capacity regarding the threshold value x_C^* considering capacity levels $K_0 \in [0, 100)$ and its highest values at $\frac{1}{2\alpha} = 50$ (black).

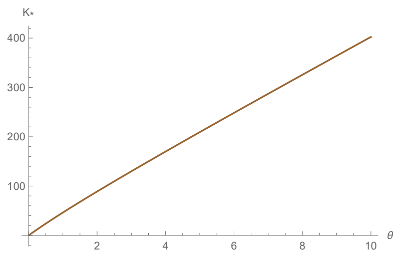
Starting with the capacity level of the *old* product K_0, on Figure 4.5, we obtained that the highest optimal capacity level K_C^* happens for $K_0 = \frac{1}{2\alpha}$. This is motivated by the results obtained for x_C^*, as seen on Figure 4.3, which also reaches its highest value at $\frac{1}{2\alpha}$.



(a) $\mu \in (-r, r)$



(b) $\sigma \in (0, 1)$

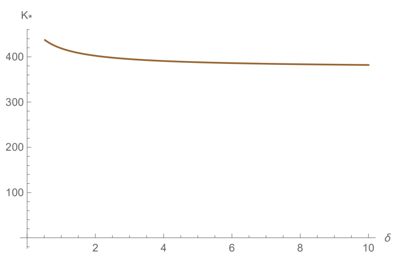


(c) $\theta \in (1, 10)$

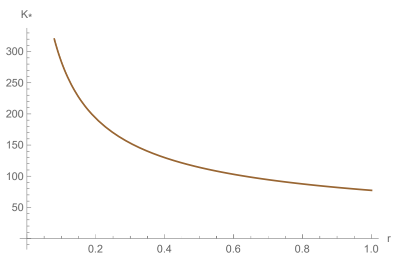
Figure 4.6: Optimal capacity regarding the threshold value x_C^* and increasing parameters μ, σ and θ

On Figure 4.6, we obtain that K_C^* increases with both drift, volatility and innovation level, as it happened in the previous section. Note that again that, contrary to what happens for positive drift values, the growth of K_C^* with μ is barely noticeable for negative values of μ. This is related to the inverse relationship between K_C^* and x_C^*.

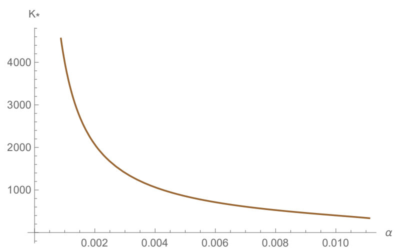
Also note that K_C^* seems to increase linearly with θ, being in accordance to the asymptotical behaviour of K_C^* with θ previously deduced.



(a) $\delta \in (0, 10)$



(b) $r \in (\mu, 1)$



(c) $\alpha \in (0, 1)$

Figure 4.7: Optimal capacity regarding the threshold value x_C^* and decreasing parameters δ, r and α.

Regarding sensibility parameter δ, discount rate r and sensibility parameter α, we have on Figure 4.7 that K_C^* decreases with them, as happened in the previous section.

Chapter 5

Adding a new product when already producing one w/ cannibalisation (Firm is already active before investing)

5.1 Introduction

We increase the complexity of our problem by considering that the firm has three different states of production.

In the first state, we consider that the firm only produces a (very) *stable* product, that does not depend on the demand observe. We will call it *old* product. Its demand function p_0 and instaneous profit function π_0 are the same as stated in Chapter 4 and take the values of (4.1) and (4.3), respectively.

In the second state, we consider the firm produces simultaneously the *old* product and a new one. We will call it *new* product. This *new* product is inserted in the market since after the innovation process as achieved a certain innovation level, *a priori* defined. Since it's based on a new technology and it's a product that is not know by people, we will consider that its profit depends on the demand level.

The instantaneous profit functions associated to the *old* and the *new* product, during the simultaneous production, are given respectively by

$$\pi_0^A(X_t) = (1 - \alpha K_0 - \eta K_1 X_t) K_0 \tag{5.1}$$

$$\pi_1^A(X_t) = (\theta - \alpha K_1 - \eta K_0 X_t) K_1. \tag{5.2}$$

We need to consider a *cannibalisation* (or *horizontal differentiation*) parameter η that corresponds to the crossed effect between the *old* and the *new* product. As we consider both products to be interacting in the same market, η represents the penalty that the quantity associated to a product will influence the price of the other. We consider here that this influence is the same for both products, so we can have a unique cannibalisation parameter η, however this one cannot be greater than the sensibility parameter α ($\eta < \alpha$). Otherwise, the quantity of the other product would have a larger effect on the product price than the quantity of the product itself.

The instantaneous profit function associated to this second state of production is denoted by π_A and it is such that

$$\pi_A(X_t) = \pi_0^A(X_t) + \pi_1^A(X_t) = \pi_0 + \pi_1(X_t) - 2\eta K_0 K_1 X_t = (1-\alpha K_0)K_0 + (\theta - \alpha K_1)K_1 X_t - 2\eta K_0 K_1 X_t. \tag{5.3}$$

In the third (and last) state, we consider that the firm abandons the *old* product and starts producing solely the *new* product, which is not considered to be a stable product. Its demand function p_1 and instaneous profit function π_1 are the same as stated in Chapter 4 and take the values of (4.2) and (4.4), respectively.

Therefore we want to find two optimal times to make different (but maybe simultaneous) decisions. We want to find the best time τ_1 to go from the first to the second state - that is, to invest in the *new* product and start producing, simultaneously, the *old* and the *new* product - and we also want to find the best time τ_2 to go from the second to the third state - that is, to replace the production of the *old* product by the *new* one. Note that $\tau_2 \geq \tau_1$ are both stopping times adapted to the natural filtration of the demand process $\{X_t,\ t \geq 0\}$ and there is no chance on return the production of the *old* product, once the firm had abandoned it in τ_2. Thus these are irreversible choices.

The strategy followed to calculate τ_1 and τ_2 is the one presented in [4]. Dixit and Pindyck suggest to firstly calculate the value of the project, secondly calculate the value of the investment in the second stage and lastly the value of the investment in the first stage. Although the contexts were different, a similar strategy was used in [7] (but here having a decision threshold associated with the innovation process, not only the demand as we do) and in [6] (but considering an entering-or-exit the market situation).

5.2 Stopping Problem

As made in previous sections, we still consider that at the moment we adapt the new product, we need to pay δK_1 related to sunk costs and that at the precise moment we adapt the new product, we are totally able to produce it. Once again, we set the instant $t = 0$ to be the instant immediately after the desired innovation threshold θ happens.

Taking into account the different profits associated to each state of production, as described before, our the optimal stopping problem may be formulated as finding the value function F such that

$$\begin{aligned} F(x) = \sup_{\tau_1} \mathbb{E}^{X_0=x}\Big[\int_0^{\tau_1} \pi_0 e^{-rs} ds + \sup_{\tau_2} \mathbb{E}^{X_{\tau_1}=x_{\tau_1}}\Big[\int_{\tau_1}^{\tau_2} \pi_A(X_s) e^{-rs} ds + \\ + \int_{\tau_2}^{\infty} \pi_1(X_s) e^{-rs} ds \ \mathbb{1}_{\{\tau_2<\infty\}} - e^{-r\tau_1}\delta K_1\Big]\mathbb{1}_{\{\tau_1<\infty\}}\Big] \end{aligned}$$

Manipulating (5.4) by changing the region of integration of the first integral, solving it and omitting indicator functions - to simplify the notation - we obtain

$$\begin{aligned}
F(x) &= \sup_{\tau_1} \mathbb{E}^{X_0=x}\left[\int_0^\infty \pi_0 e^{-rs}ds + \sup_{\tau_2}\mathbb{E}^{X_{\tau_1}=x_{\tau_1}}\left[\int_{\tau_1}^{\tau_2}\left(\pi_A(X_s)-\pi_0\right)e^{-rs}ds + \int_{\tau_1}^{\infty}\left(\pi_1(X_s)-\pi_0\right)e^{-rs}ds - e^{-r\tau_1}\delta K_1\right]\right] \\
&= \frac{\pi_0}{r} + \sup_{\tau_1}\mathbb{E}^{X_0=x}\left[\sup_{\tau_2}\mathbb{E}^{X_{\tau_1}=x_{\tau_1}}\left[\int_{\tau_1}^{\tau_2}\left(\pi_A(X_s)-\pi_0\right)e^{-rs}ds + \int_{\tau_2}^{\infty}\left(\pi_1(X_s)-\pi_0\right)e^{-rs}ds\right] - e^{-r\tau_1}\delta K_1\right]
\end{aligned} \tag{5.4}$$

Changing integration variables of both integrals on optimization problem related with τ_2 we obtain

$$\begin{aligned}
F(x) = \frac{\pi_0}{r} + \sup_{\tau_1}\mathbb{E}^{X_0=x}\Big[e^{-r\tau_1}\Big(\sup_{\tau_2}\mathbb{E}^{X_{\tau_1}=x_{\tau_1}}\Big[\int_0^{\tau_2-\tau_1}\left(\pi_A(X_{\tau_1+s})-\pi_0\right)e^{-rs}ds + \\
+\int_{\tau_2-\tau_1}^{\infty}\left(\pi_1(X_{\tau_1+s})-\pi_0\right)e^{-rs}ds\Big] - \delta K_1\Big)\Big].
\end{aligned} \tag{5.5}$$

where since the term $e^{-r\tau_1}$ does not depend on τ_2, it can be put in evidence as made above.

τ_2 – Optimal Stopping Problem:

Considering F_2 to be the value function associated to the optimal stopping problem related to τ_2 we have that its expression is given by

$$F_2(x) = \sup_{\tau_2}\mathbb{E}^{X_{\tau_1}=x_{\tau_1}}\left[\int_0^{\tau_2-\tau_1}\left(\pi_A(X_{\tau_1+s})-\pi_0\right)e^{-rs}ds + \int_{\tau_2-\tau_1}^{\infty}\left(\pi_1(X_{\tau_1+s})-\pi_0\right)e^{-rs}ds\right], \tag{5.6}$$

from which follows that our optimal stopping problem, initially given by (5.4), is now given by

$$F(x) = \frac{\pi_0}{r} + \sup_{\tau_1}\mathbb{E}^{X_0=x}\left[e^{-r\tau_1}(F_2(X_{\tau_1})-\delta K_1)\right]. \tag{5.7}$$

We have two different optimal stopping problems that we should solve starting on the *latest* stopping time, τ_2, by considering that we know what happened until the instant that the firm invests, τ_1. In order to do that, let $\{Y_t,\ t\geq 0\}$ be the stochastic process that represents the demand level after occuring the investment at τ_1 (being that its initial time) and which evolves stochastically accordingly to a GBM with the same drift μ and volatility σ as $\{X_t,\ t\geq 0\}$, that is $\{Y_t,\ t\geq 0\} = \{X_{\tau_1+t},\ t\geq 0\}$. Note that it's initial value is the same as observed at the instant τ_1, that is $Y_0 = X_{\tau_1}$.

Consider as well τ to be the stopping time, adapted to the natural filtration of the process $\{Y_t,\ t\geq 0\}$, that represents the optimal time for which the firm should make the replacement of the *old* product by the *new* one, after having invested at time τ_1. This means that if $\tau = 0$, then the old product is replaced by the new one at the precise instant when the investment happens τ_1. Note that τ is also adapted to the natural filtration of $\{X_{\tau+t},\ t\geq 0\}$ and that $\tau_2 = \tau_1 + \tau$. Thus, by knowing τ_1 and finding τ, we can calculate τ_2.

Therefore, problem F_2, as written in (5.6), is equivalent to

$$F_2(x_{\tau_1}) = \sup_{\tau}\mathbb{E}^{Y_0=x_{\tau_1}}\left[\int_0^{\tau}\left(\pi_A(Y_s)-\pi_0\right)e^{-rs}ds + \int_{\tau}^{\infty}\left(\pi_1(Y_s)-\pi_0\right)e^{-rs}ds\right], \tag{5.8}$$

meaning that from time 0 to time τ the firm is producing both products and that from time τ on the firm only produces the *new* product, where the instante 0 corresponds to the instant when the firm decides to invest, τ_1.

Fortunately we can simplify the notation of (5.8). Since the Strong Markov property states that after a stopping time, the future path of the GBM depends only on the value at the stopping time (knowing this one), it follows that

$$\{(Y_t|Y_0 = x_{\tau_1}),\ t \geq 0\} = \{(X_{\tau_1+t}|X_{\tau_1} = x_{\tau_1}),\ t \geq \tau_1\} \stackrel{d}{=} \{(X_t, |X_0 = x_{\tau_1}),\ t \geq 0\}.$$

Therefore we can keep the same notation as before and thus from (5.8) follows

$$F_2(x_{\tau_1}) = \sup_{\tau} \mathbb{E}^{X_0=x_{\tau_1}} \left[\int_0^{\tau} (\pi_A(X_s) - \pi_0)\, e^{-rs} ds + \int_{\tau}^{\infty} (\pi_1(X_s) - \pi_0)\, e^{-rs} ds \right]. \tag{5.9}$$

Using the fact that the expectation is a linear operator, we treat the expectation of the rightmost integral of (5.9) separately, that is

$$\mathbb{E}^{X_0=x_{\tau_1}} \left[\int_{\tau}^{\infty} (\pi_1(X_s) - \pi_0)\, e^{-rs} ds \right] = e^{-r\tau} \mathbb{E}^{X_0=x_{\tau_1}} \left[\int_0^{\infty} (\pi_1(X_{\tau+s}) - \pi_0)\, e^{-rs} ds \right]. \tag{5.10}$$

Conditioning to the stopping time τ and using Tower Rule we obtain from (5.10)

$$e^{-r\tau} \mathbb{E}^{X_0=x_{\tau_1}} \left[\mathbb{E}^{\tau=t} \left[\int_0^{\infty} (\pi_1(X_{t+s}) - \pi_0)\, e^{-rs} ds \right] \right]. \tag{5.11}$$

Using the similar arguments as in Section 3.2.1, we interchange the integral with expectation using Fubini's theorem and the fact that $r - \mu > 0$, obtaining

$$e^{-r\tau} \mathbb{E}^{X_0=x_{\tau_1}} \left[\int_0^{\infty} \mathbb{E}^{\tau=t} \left[\pi_1(X_{t+s}) e^{-rs} \right] ds - \frac{\pi_0}{r} \right] = e^{-r\tau} \mathbb{E}^{X_0=x_{\tau_1}} \left[(\theta - \alpha K_1) K_1 \int_0^{\infty} \mathbb{E}^{\tau=t} \left[X_{t+s} e^{-rs} \right] ds - \frac{\pi_0}{r} \right], \tag{5.12}$$

where the term $(\theta - \alpha K_1)K_1$ is constant over time.

We focus now on the expected value conditional to the stopping time τ above. Since the demand level evolves accordingly to a GBM and, by knowing the instant τ, we know its value at time τ - here denoted as x_τ -, it follows

$$\begin{aligned} \mathbb{E}^{X_\tau=x_\tau} \left[X_{\tau+s} e^{-rs} \right] &= \mathbb{E}^{X_\tau=x_\tau} \left[x_\tau e^{\left(\mu - \frac{\sigma^2}{2} - r\right)(\tau+s-\tau) + \sigma(W_{\tau+s} - W_\tau)} \right] \\ &= \mathbb{E}^{X_\tau=x_\tau} \left[x_\tau e^{\left(\mu - \frac{\sigma^2}{2} - r\right)s + \sigma W_s} \right] \\ &= x_\tau e^{(\mu - r)s}, \end{aligned} \tag{5.13}$$

where in the second equality we used the fact that Brownian Motion $\{W_t,\ t \geq 0\}$ has stationary increments, that is

$$W_{\tau+s} - W_\tau \stackrel{d}{=} W_{\tau+s-\tau} - W_0 \stackrel{d}{=} W_s \sim \mathcal{N}(0, s).$$

Plugging (5.13) in (5.12), we obtain

$$e^{-r\tau}\mathbb{E}^{X_0=x_{\tau_1}}\left[(\theta-\alpha K_1)K_1\int_0^{\infty}x_{\tau_1}e^{(\mu-r)s}ds-\frac{\pi_0}{r}\right]=e^{-r\tau}\mathbb{E}^{X_0=x_{\tau_1}}\left[\frac{(\theta-\alpha K_1)K_1}{r-\mu}x_\tau-\frac{\pi_0}{r}\right]. \quad (5.14)$$

Therefore we have found the terminal function associated to the optimal stopping problem F_2. Denoting it by h_2, it's given by

$$h_2(x)=\frac{(\theta-\alpha K_1)K_1}{r-\mu}x-\frac{\pi_0}{r}.$$

Accordingly to (5.9), we may also denote g_2 as the running cost function associated to this problem, that is given by

$$g_2(x)=\pi_A(x)-\pi_0.$$

Thus, plugging expression of running and terminal functions on (5.9), we have that F_2 as initially written in (5.6), is equivalent to

$$F_2(x)=\sup_\tau \mathbb{E}^{X_0=x}\left[\int_0^\tau g(X_s)e^{-rs}ds+e^{-r\tau}h(X_\tau)\right] \quad (5.15)$$

$$=\sup_\tau \mathbb{E}^{X_0=x}\left[\int_0^\tau \left(\pi_0^A(X_s)+\pi_1^A(X_s)-\pi_0\right)e^{-rs}ds+e^{-r\tau}\left(\frac{(\theta-\alpha K_1)K_1}{r-\mu}X_\tau-\frac{\pi_0}{r}\right)\right], \quad (5.16)$$

being this a standard optimal stopping problem.

Since the HJB variational inequality associated to (5.16), implies that the non-homogeneous PDE

$$\frac{\sigma^2}{2}x^2F_2''(x)+\mu xF_2'(x)-rF_2(x)+g(x)=0 \quad (5.17)$$

holds for any demand value in the continuation region, it follows that its solution F_2 takes the form of

$$F_2(x)=F_{2,h}(x)+F_{2,p} \quad \forall x\in\mathcal{C}, \quad (5.18)$$

where $F_{2,h}$ corresponds to the solution to the homogeneous version of the PDE in (5.17) and $F_{2,p}$ corresponds a particular solution of (5.17).

We first calculate the particular solution $F_{2,p}$. By considering $F''_{2,p}(x)=0$ and using expression (5.17), a particular solution is found to be

$$F_{2,p}(x)=\frac{(\theta-\alpha K_1)K_1-2\eta K_0K_1}{r-\mu}x. \quad (5.19)$$

Note that $\lim_{x\to 0}F_{2,p}(x)=0$. This fact is an useful detail when it comes to calculate F_2.

Since there is no possibility of having a project having a negative value it follows $F_2(x)\geq 0\ \forall x\in\mathcal{C}$. Taking also into account the *detail* stated, we obtain that $F_{2,h}=a_2x^{d_1}+b_2x^{d_2}$ simplifies to $F_{2,h}=a_2x^{d_1}$ guaranteeing this way that $\lim_{x\to 0}F_2(x)=\lim_{x\to 0}F_{2,h}(x)+F_{2,p}(x)=0$.

Once again, the constant term a_2 and the threshold value x_2^* are found by value matching (2.52a) and smooth pasting (2.52b) conditions. Thus, by solving the following system

$$\begin{cases} a_2(x_2^*)^{d_1} + \frac{(\theta-\alpha K_1)K_1-2\eta K_0 K_1}{r-\mu}x_2^* = \frac{(\theta-\alpha K_1)K_1}{r-\mu}x_2^* - \frac{\pi_0}{r} \\ ad_1(x_2^*)^{d_1-1} + \frac{(\theta-\alpha K_1)K_1-2\eta K_0 K_1}{r-\mu} = \frac{(\theta-\alpha K_1)K_1}{r-\mu}, \end{cases} \tag{5.20}$$

we obtain

$$\begin{aligned} a_2 &= \left(\frac{2\eta K_0 K_1}{r-\mu}x^* - \frac{\pi_0}{r}\right)(x_2^*)^{-d_1} \\ x_2^* &= \frac{d_1}{d_1-1}\frac{1-\alpha K_0}{2\eta K_1}\frac{r-\mu}{r} \end{aligned} \tag{5.21}$$

Taking into account that F_2 is also defined in the stopping region, we obtain that its expression is given by

$$F_2(x) = \left(ax^{d_1} + \frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu}x\right)\mathbb{1}_{\{x<x_2^*\}} + \left(\frac{(\theta-\alpha K_1)K_1}{r-\mu}x - \frac{\pi_0}{r}\right)\mathbb{1}_{\{x\geq x_2^*\}}, \tag{5.22}$$

where continuation and stopping regions are respectively described as in (2.46) and (2.47) and the optimal stopping time as in (2.43).

τ_1 − **Optimal Stopping Problem:**

We are in position now to return to the main problem, as lastly described in (5.7). It takes the form of

$$F(x) = \frac{\pi_0}{r} + \sup_{\tau_1}\mathbb{E}^{X_0=x}\left[e^{-r\tau_1}\left(\frac{(\theta-\alpha K_1)K_1}{r-\mu}X_{\tau_1} + \left(aX_{\tau_1}^{d_1} - \frac{2\eta K_0 K_1}{r-\mu}X_{\tau_1}\right)\mathbb{1}_{\{X_{\tau_1}<x_2^*\}} - \frac{\pi_0}{r}\mathbb{1}_{\{X_{\tau_1}\geq x_2^*\}} - \delta K_1\right)\right], \tag{5.23}$$

which again an optimal problem with null running cost function, however this time with two terminal cost functions defined on disjoint domains.

We still want to find the optimal time τ_1 to make the investment decision. However as the problem is stated in (5.23), we obtain two different demand threshold levels:

- $x_{1,A}^*$: when we add the *new* product and have a period of simultaneously production. It is associated to the region $X_{\tau_1} < x_2^*$ and to the stopping problem

$$\sup_{\tau_1}\mathbb{E}^{X_0=x}\left[e^{-r\tau_1}\left(\frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu}X_{\tau_1} + a_2X_{\tau_1}^{d_1} - \delta K_1\right)\mathbb{1}_{\{\tau_1<\infty\}}\right] \tag{5.24}$$

- $x_{1,R}^*$: when we add the *new* product and immediately replace the old one. It is associated to the region $X_{\tau_1} \geq x_2^*$ and to the stopping problem

$$\sup_{\tau_1}\mathbb{E}^{X_0=x}\left[e^{-r\tau_1}\left(\frac{(\theta-\alpha K_1)K_1}{r-\mu}X_{\tau_1} - \frac{\pi_0}{r} - \delta K_1\right)\mathbb{1}_{\{\tau_1<\infty\}}\right] \tag{5.25}$$

We will treat both thresholds $x_{1,A}^*$ and $x_{1,R}^*$ by its respective order.

- **Demand threshold $x_{1,A}^*$:**

The optimal stopping problem stated in (5.24) is a standard optimal stopping problem, for which we don't have a solution (yet). Denoting by $F_{1,A}$ its correspondent solution we have that $F_{1,A}$ verifies the HJB variational inequality (2.40), where we take the terminal function to be $h(x) = \frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu} X_{\tau_1} + a_2 X_{\tau_1}^{d_1} - \delta K_1$. Therefore it follows that $F_{1,A}$ takes the form

$$F_{1,A}(x) = \begin{cases} a_{1,A} x^{d_1} \ , \ x \in \mathcal{C}_{1,A} \\ \frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu} x + a_2 x^{d_1} - \delta K_1 \ , \ x \in \mathcal{S}_{1,A} \end{cases} , \tag{5.26}$$

where coefficient $a_{1,A}$ and the threshold value $x_{1,A}^*$ are found by value matching (2.52a) and smooth pasting (2.52b) conditions, expressed by the corresponding system

$$\begin{cases} a_{1,A} (x_{1,A}^*)^{d_1} = \frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu} x_{1,A}^* + a_2 (x_{1,A}^*)^{d_1} - \delta K_1 \\ a_{1,A} d_1 (x_{1,A}^*)^{d_1 - 1} = \frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu} + a_2 d_1 (x_{1,A}^*)^{d_1 - 1} \end{cases} , \tag{5.27}$$

from which we obtain

$$x_{1,A}^* = \frac{d_1}{d_1 - 1} \frac{\delta K_1 (r-\mu)}{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1} \tag{5.28}$$

$$a_{1,A} = a_2 x_2^* + \frac{((\theta-\alpha K_1)K_1 - 2\eta K_1 K_0)}{r-\mu} (x_{1,A}^*)^{1-d_1} - \delta K_1 (x_{1,A}^*)^{-d_1}, \tag{5.29}$$

with d_1 being the positive root of the polynomial described in (2.51).

Observe that since the threshold $x_{1,A}^*$ needs always to be positive, the following restriction must hold

$$(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1 > 0 \ \Leftrightarrow \ K_1 < \frac{\theta}{\alpha + 2\eta K_0} \leq \frac{\theta}{\alpha}, \tag{5.30}$$

where the equality (in the last inequality) holds for $\eta = 0$.

Thus the solution associated to (5.24) is given by

$$F_{1,A}(x) = \left(a_{1,A} x^{d_1} \mathbb{1}_{\{x < x_{1,A}^*\}} + \left(\frac{(\theta-\alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu} x + a_2 x^{d_1} - \delta K_1 \right) \mathbb{1}_{\{x \geq x_{1,A}^*\}} \right) \mathbb{1}_{\{x < x_2^*\}}, \tag{5.31}$$

continuation and stopping regions are respectively described as in (2.46) and (2.47) and the optimal stopping time as in (2.43).

- **Demand threshold $x_{1,R}^*$:**

Note that the optimal stopping problem (5.25) is the same as the one explored and analysed in Chapter 4, consisting in the Benchmark Model (Section 4.3.1). Thus it follows that the threshold level is given by (4.13), that is,

$$x_{1,R}^* = \frac{d_1}{d_1 - 1} \frac{\delta K_1 + \frac{\pi_0}{r}}{\theta - \alpha K_1} \frac{r-\mu}{K_1}. \tag{5.32}$$

On the other side the solution associated to (5.25) is given by (4.15), that is, (changing the name of the associated solution)

$$F_{1,R}(x) = \left(a_{1,R}(x^*)^{d_1} \mathbb{1}_{\{x<x^*_{1,R}\}} + \left(\frac{(\theta - \alpha K_1)K_1 x}{r-\mu} - \delta K_1 - \frac{\pi_0}{r} \right) \mathbb{1}_{\{x \geq x^*_{1,R}\}} \right) \mathbb{1}_{\{x \geq x^*_2\}}, \tag{5.33}$$

with $a_{1,R}$ taking the same value as a_2 in (4.14).

Note that from (5.31) it's impossible to observe a demand threshold x^*_2 smaller than $x^*_{1,A}$. This would mean that at the precise instant $x^*_{1,A}$ was reached, we should immediately replace the *old* product by the *new* one. However, this case is not related with thresholds $x_{1,A}$ or x^*_2, but with $x^*_{1,R}$. Therefore we are able to constraint the domains of both functions $F_{1,A}$ and $F_{1,R}$ by analyzing when does $x^*_{1,A} < x^*_2$ happen or not, respectively. That is

$$\begin{aligned} x^*_{1,A} < x^*_2 \;&\Leftrightarrow\; \frac{d_1}{d_1 - 1}\frac{1-\alpha K_0}{2\eta K_1}\frac{r-\mu}{r} = \frac{d_1}{d_1-1}\frac{\delta K_1 (r-\mu)}{(\theta - \alpha K_1)K_1 - 2\eta K_0 K_1} \\ &\Leftrightarrow\; (1-\alpha K_0)((\theta - \alpha K_1)K_1 - 2\eta K_0 K_1) = 2\eta \delta K_1^2 r \\ &\Leftrightarrow\; \eta < \frac{(1-\alpha K_0)(\theta - \alpha K_1)}{2(\delta K_1 r + \pi_0)} =: \eta^*, \end{aligned} \tag{5.34}$$

where we denoted η^* to be the cannibalization threshold that states which value function stands.

Therefore we obtain that our problem, as stated in (5.23), may be written as

$$F(x) = \frac{\pi_0}{r} + \begin{cases} a_{1,A} x^{d_1} & , \; x < x_{1,A} \;\wedge\; \eta < \eta^* \\ a_2 x^{d_1} + \frac{(\theta - \alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu} x - \delta K_1 & , \; x_{1,A} \leq x < x^*_2 \;\wedge\; \eta < \eta^* \\ a_{1,R} x^{d_1} & , \; x < x^*_{1,R} \;\wedge\; \eta \geq \eta^* \\ \frac{(\theta - \alpha K_1)K_1 x}{r-\mu} - \frac{\pi_0}{r} - \delta K_1 & , \; (x > x^*_{1,A} \;\wedge\; \eta < \eta^*) \;\dot{\vee}\; (x > x^*_{1,R} \;\wedge\; \eta \geq \eta^*) \end{cases} \tag{5.35}$$

with $a_{1,A}$ as described in (5.29), $a_{1,R}$ as described in (4.14), $x^*_{1,A}$ as described in (5.28), $x^*_{1,R}$ as described in (4.13), x^*_2 as described in (5.21) and η^* as described in (5.34) (and π_0 as described in (4.3)).

5.3 Comparative Statics

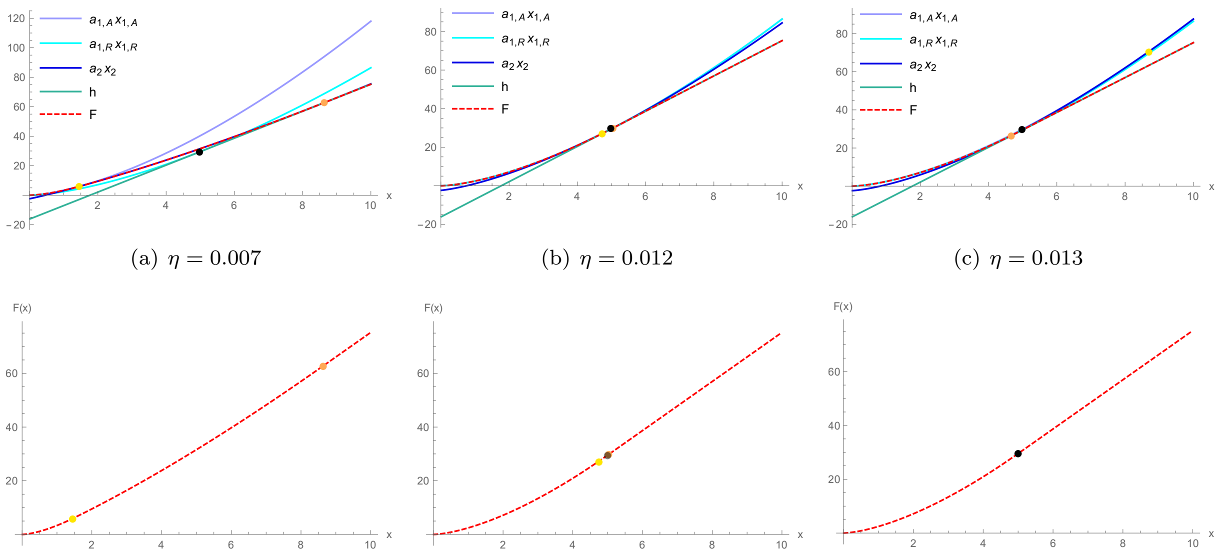
Before we start to analyse how parameters influence the three different threshold levels, we will show how one should interpret the results obtained in (5.35).

Considering parameters and playing with the cannibalization parameter η, we obtain the six plots

- $\mu = 0.45$
- $\sigma = 0.6$
- $r = 0.85$
- $\delta = 0.95$
- $\alpha = 0.015$
- $\theta = 1.5$
- $K_0 = 51.5$
- $K_1 = 2.5$

represented on Figure 5.1.

Each column corresponds a different value of η set. In each plot of the first row it's represented the different possible parts associated to the value function F (5.35), accordingly to the respective legend, being the red dashed line corresponding to the valid value function associated to the parameters considered. It is also represented the three possible thresholds: $x^*_{1,A}$ by the yellow dot, x^*_2 by the orange dot and $x^*_{1,R}$ by the black dot. Finally on the second row it's represented solely the value function F associated (the same as showed in the respective plot above) and respective threshold values.



(a) $\eta = 0.007$ (b) $\eta = 0.012$ (c) $\eta = 0.013$

(d) $\eta = 0.007$ (e) $\eta = 0.012$ (f) $\eta = 0.013$

Figure 5.1: Value function F and respective subfunctions associated to different settings of parameter η.

First, note that for this set of values, we obtain $\eta^* \simeq 0.0121121$. Therefore we have, for the plots in the first and second columns, that it's better to invest and have a period of simultaneous production and only after the second threshold being reached, we change to produce solely the *new* product. On the other side, in the third column, we have the situation where, when we invest, we should immediately produce solely the *new* product.

In the first column, we observe that the thresholds to be considered are $x^*_{1,A}$ and x^*_2. The first one states upon which demand level we should invest in the *new* product and start the simultaneous production. The second one states upon which demand level we should produce only the *new* product. The respective value function associated to this problem has 3 parts and it's given by

$$\begin{aligned} F(x) = \frac{\pi_0}{r} + a_{1,A} x^{d_1} \mathbb{1}_{\{x<x^*_{1,A}\}} + \left(a_2 x^{d_1} + \frac{(\theta - \alpha K_1)K_1 - 2\eta K_0 K_1}{r-\mu} x - \delta K_1 \right) \mathbb{1}_{\{x_{1,A} \leq x < x^*_2\}} \\ + \left(\frac{(\theta - \alpha K_1)K_1 x}{r-\mu} - \frac{\pi_0}{r} - \delta K_1 \right) \mathbb{1}_{\{x > x^*_2\}}. \end{aligned} \tag{5.36}$$

In the second column, we are in a similar situation as in the first column, where we have two thresholds and the value function is also defined in three parts, as in (5.36). Observe also that as $\eta \to \eta^*$, the three thresholds tend to admit the same value. This is accordance with the analytical result that replacing η by η^* in $x^*_{1,A}$ (5.28) and in $x^*_{1,R}$ (4.13) we get

$$x^*_{1,A}(\eta^*) = x^*_2(\eta^*) = x^*_{1,R}.$$

In the third column, we observe that the threshold to be considered is $x^*_{1,R}$. In this case the value function is only defined by two parts, being given by

$$F(x) = \frac{\pi_0}{r} + a_{1,R}x^{d_1}\mathbb{1}_{\{x<x^*_{1,R}\}} + \left(\frac{(\theta-\alpha K_1)K_1 x}{r-\mu} - \frac{\pi_0}{r} - \delta K_1\right)\mathbb{1}_{\{x>x^*_{1,R}\}}$$

5.3.1 Demand threshold $x^*_{1,A}$

Proposition 5.1. *Decision threshold $x^*_{1,A}$ increases with η, δ, σ, K_0 and K_1 and decreases with θ and α.*

Proof:

The results regarding parameters η, δ, θ and α come immediately by the expression of $x^*_{1,A}$ (5.28).

Regarding σ, we obtain that

$$\frac{\partial x^*_{1,A}(\sigma)}{\partial \sigma} = \frac{2\delta(\mu-r)\left(-2\mu^2+\mu\sigma^2(\phi+1)-2r\sigma^2\right)}{(d_1-1)^2\sigma^5\phi(\theta-\alpha K_1-2\eta K_0)} > 0,$$

where the denominator is positive since $d_1 > 1$, $\phi > 0$ and the denominator is positive since $r > \mu$ and by noting that $-2\mu^2+\mu\sigma^2(\phi+1)-2r\sigma^2 < 0 \Leftrightarrow \mu d_1 - r < 0$, which holds always, as previously showed in (4.27).

Regarding K_0 and K_1, we obtain that

$$\begin{aligned}
\frac{\partial x^*_{1,A}(K_0)}{\partial K_0} &= \frac{2\delta d_1\eta(r-\mu)}{(d_1-1)(-\theta+\alpha K_1+2\eta K_0)^2} > 0\\
\frac{\partial x^*_{1,A}(K_1)}{\partial K_1} &= \frac{\alpha\delta d_1\eta(r-\mu)}{(d_1-1)(-\theta+\alpha K_1+2\eta K_0)^2} > 0.
\end{aligned}$$

□

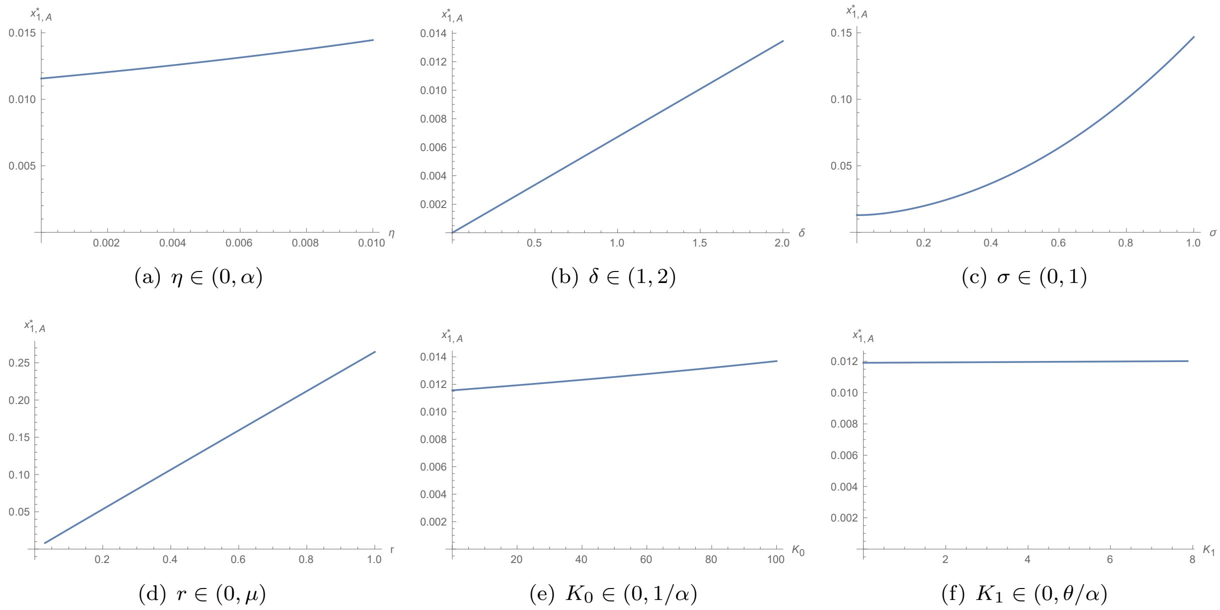
Considering the following parameters,

- $\mu = 0.03$
- $\sigma = 0.05$
- $r = 0.05$
- $K_0 = 90$
- $\eta = 0.007$
- $\alpha = 0.01$
- $\theta = 10$
- $\delta = 2$
- $K_1 = 100$

we perform some numerical approximations concerning $x^*_{1,A}$.

On Figure 5.2 we present some plots that comprove what was stated on Proposition 5.1. We also add another one, regarding the discount rate r, for which we couldn't derive any analytical solution, but studying the behaviour of $x^*_{1,A}$ we conclude that it also increases with r.

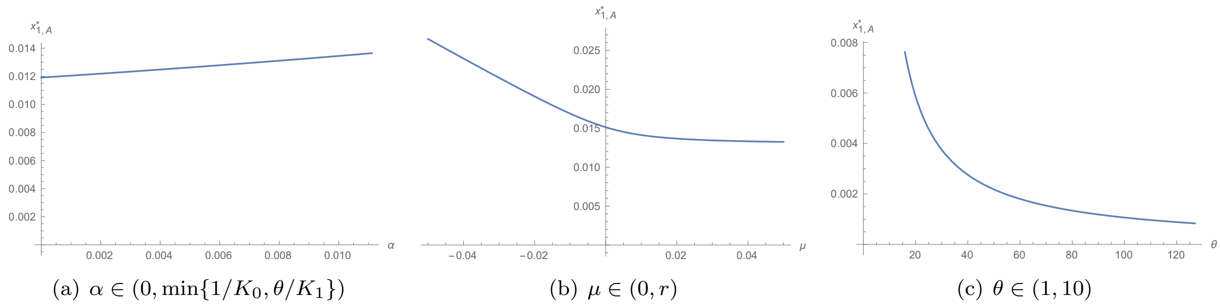
On Figure 5.3 we present some plots that also comprove what was stated on Proposition 5.1, but with respect to the parameters with which $x^*_{1,A}$ increases. Again, we add the plot concerning the drift parameter μ, for which we couldn't derive any analytical solution, but studying the behaviour of $x^*_{1,A}$ we conclude that it also decreases with μ.



(a) $\eta \in (0, \alpha)$ (b) $\delta \in (1, 2)$ (c) $\sigma \in (0, 1)$

(d) $r \in (0, \mu)$ (e) $K_0 \in (0, 1/\alpha)$ (f) $K_1 \in (0, \theta/\alpha)$

Figure 5.2: Threshold value $x^*_{1,A}$ with respect to parameters with which it increases.



(a) $\alpha \in (0, \min\{1/K_0, \theta/K_1\})$ (b) $\mu \in (0, r)$ (c) $\theta \in (1, 10)$

Figure 5.3: Threshold value $x^*_{1,A}$ with respect to parameters with which it decreases.

5.3.2 Demand threshold $x^*_{1,R}$

All results come from Section 4.3.1 and thus they won't repeated here.

5.3.3 Demand threshold x^*_2

Proposition 5.2. *Decision threshold x^*_2 increases with σ and decreases with η, α, K_0 and K_1.*

Proof:

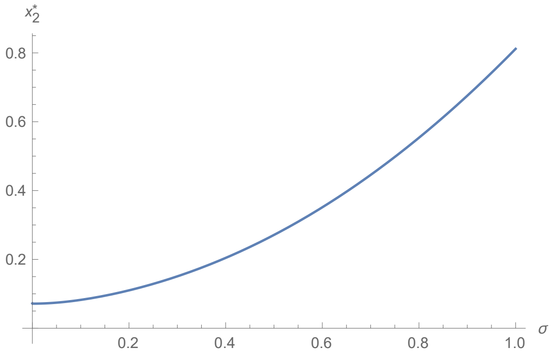
The results regarding parameters η, α, K_0 and K_1 come immediately by the expression of x^*_2 (5.21). Regarding σ, we obtain that

$$\frac{\partial x^*_2(\sigma)}{\partial \sigma} = \frac{(\alpha K_0 - 1)(r - \mu)\left(-2\mu^2 + \mu\sigma^2(\phi + 1) - 2r\sigma^2\right)}{(d_1 - 1)^2 \eta K_1 r \sigma^5 \phi} > 0$$

where the denominator is positive since $d_1 > 1$, $\phi > 0$ and the denominator is positive since $r > \mu$, $1 - \alpha K_0 > 0$ and by noticing (again) that $-2\mu^2 + \mu\sigma^2(\phi + 1) - 2r\sigma^2 < 0 \Leftrightarrow \mu d_1 - r < 0$, which holds always, as previously showed in (4.27).

□

Considering the same parameters as in Section 5.3.1, we perform some numerical approximations concerning x_2^*.

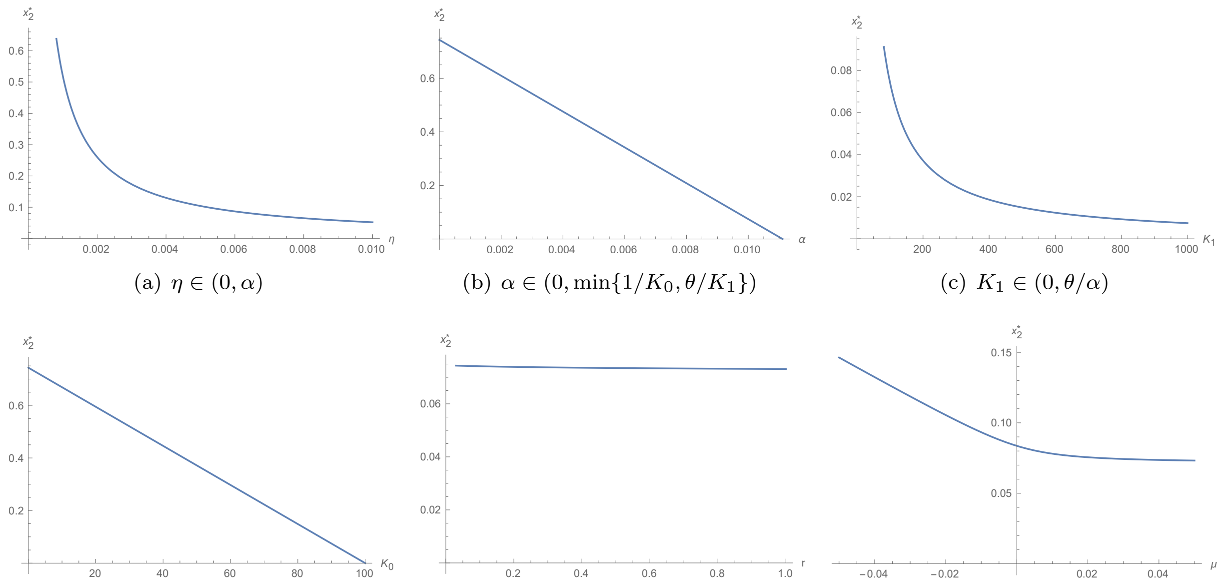


(a) $\eta \in (0, \alpha)$

Figure 5.4: Threshold value x_2^* with respect to parameters with which it increases.

From Figure 5.4 we comprove what was stated on Proposition 5.2 regarding σ.

We also obtained that the volatility σ was the unique parameter we found to increase both thresholds $x_{1,A}^*$ and x_2. Note that by Section 4.3.1, that the same holds for the threshold $x_{1,R}$. Thus we conclude that a situation where the demands shows to have a high volatility, both investment and replacement decisions tend to be postponed.



(a) $\eta \in (0, \alpha)$ (b) $\alpha \in (0, \min\{1/K_0, \theta/K_1\})$ (c) $K_1 \in (0, \theta/\alpha)$

(d) $K_0 \in (0, 1/\alpha)$ (e) $r \in (0, \mu)$ (f) $\mu \in (0, r)$

Figure 5.5: Threshold value $x_{1,A}^*$ with respect to parameters with which it decreases.

On Figure 5.5 we show how the threshold x_2 decreases with parameters stated on Proposition 5.2. We also add plots concerning the drift parameter μ and the discount rate r, for which we couldn't derive any analytical solution, but studying how x_2^* behaves with them, we conclude that it also increases with μ and r, from which the behaviour showed may be generalized.

Chapter 6

Value of the project: the influence of the number of innovation jumps

6.1 Introduction

In the three previous sections we have calculated the value function associated to three different contexts (introduction of a new product; introduction of a new product with the immediate replacement of the *old* one; introduction of a *new* product with the possibility of immediate replacement of the *old* one or simultaneous production followed by the replacement of the *old* product), while considering a fixed level of innovation θ.

Now we are interested to calculate the maximized expected value function, associated to each of the three contexts, with respect to the R&D investment made and taking into account the waiting time until the desired threshold value θ is achieved. This allows us to give a weight depending on the time we need to wait until we are able to introduce the *new* product.

Here we will consider the case when θ is achieved at the next jump and, the generalization of it, when only at the n-th jump ($n \in \mathbb{N}$) we obtain the desired threshold θ.

As stated in Section 1.3.1, recall that the innovation process $\Theta = \{\theta(t),\ t \geq 0\}$ is defined as a Compound Poisson Process with constant rate given by $\lambda(R) = R^{\gamma},\ \gamma \in (0, 1)$. Here R corresponds to the investment in the R&D department and it is related with the size of laboratory, wages of scientists and some other aspects necessary to the research process. The higher the R&D investment, the smaller the expected waiting time until the desired level θ occurs is, so the higher the value function will be.

Note that the investment in R&D R is different from the investment costs δK in the sense that R influences directly the innovation process Θ, while δK is related with the costs that the firm needs to incur (in its factories) to adapt its production to the innovation level θ, like new machines or workshops to teach workers how to use the new technology.

The innovation process can be then expressed as

$$\theta_t = \theta_0 + uN_t,\ t \geq 0.$$

with θ_0 denoting the state of technology at the initial point in time, $u > 0$ is a fixed jump size and $\{N_t,\ t \geq 0\}$ follows a Poisson Process with rate $\lambda(R)$.

Consider now S_n as the random variable that represents the waiting time until the n-th jump is observed, that is,

$$S_n = \min\{t \geq 0 : N(t) = n\}.$$

Accordingly to [8], we have that S_n is distributed according to Erlang distribution with shape parameter n and rate parameter being the same as the Poisson Process, that is, $\lambda(R)$. From this it follows that the expected waiting time for the n-th jump to be reached is given by $\mathbb{E}(S_n) = \frac{n}{\lambda(R)}$. Thus, the higher the investment, the bigger the quantity of jumps observed is. Note that, in the situation in which there is no R&D investment, there is also no evolution in the innovation process, since it implies a null rate $\lambda(0) = 0$ followed by an infinite waiting time for any amount of jumps wanted to be observed.

Note from previous deduced expressions of value function that none of them depend on the R&D investment R. This one only influences the innovation process. Therefore, as it will be showed in the next sections, this is a standard maximization problem, in which we will want to maximize the expected value function with respect to the R&D investment R.

6.2 One jump

We start with the simpler case: the desired innovation level θ is reached in the next innovation jump. Therefore S_1 denotes the random variable associated to the waiting time until the jump occurs and it is such that

$$S_1 \sim \text{Erlang}(1, \lambda(R)) \overset{d}{=} \text{Exponential}(\lambda(R)).$$

We also will denote here F to be the value function for a fixed situation and V to be given by

$$V(x) = \max_R \mathbb{E}[e^{-rS_1} F(x) - R] \geq 0 \tag{6.1}$$

and corresponding to the maximized expected discounted value function minus the R&D investment needed to be made. Recall that, in the previous sections, we deduced all the value functions by setting the initial time to be the time at the innovation threshold θ is reached. Therefore in order to have the real value, we need to discount it to the time when the jump happens, that is, S_1. Also, since the jumps are governed by a stochastic process (and thus they are random), we need to calculate the expected value of the discounted value function. Since the R&D investment R is deterministic, it could be inside or outside the expected value (but for the sake of simplicity we decided to include it when calculating the expectation).

In order to estimate the function V, one could have stayed just with stochastic simulations, using Monte Carlo methods for instance. However we preferred to deduce an analytical result, since its stronger than any estimation made.

Coming back to the expression of V in (6.1), and by considering the expectation with respect to S_1

we obtain that it follows

$$
\begin{aligned}
V(x) &= \max_R \left\{ \int_0^\infty f_{S_1}(t) e^{-rt} F(x) dt - R \right\} \\
&= \max_R \left\{ \int_0^\infty \lambda(R) e^{-\lambda(R)t} e^{-rt} F(x) dt - R \right\}
\end{aligned} \tag{6.2}
$$

where f_{S_1} corresponds to the probability density function of an Exponential with parameter $\lambda(R)$, associated to the random variable S_1.

Since, as previously written (and deduced), none of the value functions studied depend on the R&D investment or the time at they are evaluated, only on the demand level observed at the breakthrough, F never depends on R or t. Therefore from (6.2), we obtain

$$
V(x) = \max_R \left\{ \frac{\lambda(R)}{\lambda(R) + r} F(x) - R \right\} = \max_R \left\{ \frac{R^\gamma}{R^\gamma + r} F(x) - R \right\}, \tag{6.3}
$$

which is a maximization problem with respect to the R&D investment R. Since by (6.1), V is expected to be greater or equal to 0 it follows that the restriction

$$
R^\gamma F(x) - (R^\gamma + r) R \geq 0 \;\Leftrightarrow F(x) \geq \frac{R^\gamma + r}{R^{\gamma-1}} \tag{6.4}
$$

must be always verified.

The optimal value of the investment to be made, from now on denoted by R^*, is found by analyzing the first and the second partial derivatives of the expression to maximize, that is,

$$
\frac{\partial}{\partial R} \left(\frac{R^\gamma}{R^\gamma + r} F(x) - R \right) = \frac{\gamma R^{\gamma-1} F(x) r - (R^\gamma + r)^2}{(R^\gamma + r)^2} \tag{6.5}
$$

$$
\frac{\partial^2}{\partial R^2} \left(\frac{R^\gamma}{R^\gamma + r} F(x) - R \right) = -\frac{F(x) \gamma r R^{-2+\gamma} (r - \gamma r + (1+\gamma) R^\gamma)}{(R^\gamma + r)^3} \leq 0. \tag{6.6}
$$

Note that, since $\gamma \in (0, 1]$, $F(x) \geq 0 \; \forall x$ and $r, \; R > 0$, the second partial derivative with respect to R (6.6) is always negative. Hence the expression to be maximized in (6.3) is a concave function and we always are able to find a $R^* = \arg\max_R \left\{ \frac{R^\gamma}{R^\gamma + r} F(x) - R \right\}$.

Regarding the set of possible values for γ and their interest, we consider two different cases.

Firstly we analyse the solution for $\gamma = 1$, although this implies that the jump's rate is equal to R, it is, nevertheless, an important case since it was the only one for which we were able to obtain an analytical solution. Secondly we analyze the other possible values $\gamma \in (0, 1)$, but only through a numerical solver and using software Mathematica.

The case $\gamma = 0$ will not be considered due to its lack of importance, regarding the problem in hands. Note that

$$
\gamma = 0 \;\Rightarrow\; \lambda(R) = 1,
$$

that is, considering $\gamma = 0$, the innovation rate is not influenced by the size of the R&D investment made (which is not compatible with the real life).

- **Case I:** $\gamma = 1 \Leftrightarrow \lambda(R) = R$

Analysing the roots of the first partial derivative in order to parameter R, we get a quadratic polynomial for which we can calculate obtain the expression of the zeros, obtaining

$$R = -\sqrt{F(x)r} - r \ \vee \ R = \sqrt{F(x)r} - r$$

The first solution is not admissible, since it's not possible to have negative investment ($R > 0$).

The second solution is admissible, if considering the restriction in (6.4), since we obtain

$$\sqrt{F(x)r} - r \underset{(6.4)}{\geq} \sqrt{(R+r)r} - r \geq 0 \tag{6.7}$$

$$\Leftrightarrow rR + r^2 - r^2 = rR \geq 0, \tag{6.8}$$

which always holds since both discount rate r and R&D investment R are positive.

On account of the negativity concerning the second partial derivative (6.6), we obtain that, if (6.4) holds, then the optimal R&D investment to be made corresponds to

$$R^* = \max_R V(x) = \sqrt{F(x)r} - r. \tag{6.9}$$

- **Case II:** $\gamma \in (0, 1)$

Now, considering $\gamma \in (0, 1)$ and taking into account expression (6.5), stationary points will be found by calculating the roots of the following polynomial

$$R^{\gamma-1} F(x)r - R^{2\gamma} - 2rR^{\gamma} - r^2 = 0. \tag{6.10}$$

From condition (6.6) it follows that the maximizer R^* is such that it verifies (6.4) and (6.10).

Unfortunately, we are not able to solve (6.10) analytically for any value $\gamma \in (0, 1)$. However, using software Mathematica, we performed some numerical illustrations for values $\gamma \in (0, 1)$ presented in Section **??**.

6.3 Multiple jumps

Now we generalize the context of the previous section, by considering that the desired innovation level θ is reached at the n-th innovation jump. Therefore we consider S_n to be the random variable associated to the waiting time until the n-th jump occurs and it is such that

$$S_n \sim \text{Erlang}(n, \lambda(R)).$$

By considering $n = 1$, we are in situation analysed in the previous section.

For the sake of simplicity we keep the same notation as before, where F denotes the value function associated to a certain investment situation and V the maximized expected discounted value function minus the R&D investment needed to be made, is now given by

$$\begin{aligned}
V_n(x) &= \max_R \mathbb{E}[e^{-rS_n}F(x) - R] \\
&= \max_R \left\{ \int_0^\infty f_{S_n}(t) e^{-rt} F(x) dt - R \right\} \\
&= \max_R \left\{ \int_0^\infty \frac{\lambda(R)^n t^{n-1}}{(n-1)!} e^{-\lambda(R)t} e^{-rt} F(x) dt - R \right\} \qquad (6.11)
\end{aligned}$$

where f_{S_n} corresponds to the probability density function of an Erlang with shape parameter n and rate parameter $\lambda(R)$, associated to the random variable S_n.

Considering W to be a random variable such that $W \sim \text{Erlang}(n, \lambda(R)+r)$ and f_W the correspondent probability density function, the integral above can be simplified as it follows

$$\begin{aligned}
\int_0^\infty \frac{\lambda(R)^n t^{n-1}}{(n-1)!} e^{-\lambda(R)t} e^{-rt} F(x) dt &= \frac{\lambda(R)^n}{(\lambda(R)+r)^n} F(x) \int_0^\infty \frac{(\lambda(R)+r)^n t^{n-1}}{(n-1)!} e^{-t(\lambda(R)+r)^n)} dt \\
&= \left(\frac{\lambda(R)}{\lambda(R)+r} \right)^n F(x) \int_0^\infty f_W(t)\ dt \\
&= \left(\frac{\lambda(R)}{\lambda(R)+r} \right)^n F(x). \qquad (6.12)
\end{aligned}$$

Plugging the resultant expression (6.12) in (6.11), we obtain that V_n corresponds to a maximization problem given by

$$V_n(x) = \max_R \left\{ \left(\frac{\lambda(R)}{\lambda(R)+r} \right)^n F(x) - R \right\} = \max_R \left\{ \left(\frac{R^\gamma}{R^\gamma + r} \right)^n F(x) - R \right\}. \qquad (6.13)$$

Since V_n is expected to be greater or equal to 0 $\forall n \in \mathbb{N}$, the following restriction must hold

$$R^{\gamma n} F(x) - (R^\gamma + r)^n \geq 0 \ \Leftrightarrow\ F(x) \geq \frac{(R^\gamma + r)^n}{R^{\gamma n - 1}}. \qquad (6.14)$$

The optimal investment to be made considering that at the n-th jump we achieve the desired breakthrough level θ, denoted by R_n^*, is found by analysing the first and the second partial derivatives of the expression to maximize, that is,

$$\begin{aligned}
\frac{\partial}{\partial R} \left(\left(\frac{R^\gamma}{R^\gamma + r} \right)^n F(x) - R \right) &= \frac{\gamma F(x) n r \left(\frac{R^\gamma}{r + R^\gamma} \right)^n - rR + R^{\gamma+1}}{rR + R^{\gamma+1}} \qquad (6.15) \\
\frac{\partial^2}{\partial R^2} \left(\left(\frac{R^\gamma}{R^\gamma + r} \right)^n F(x) - R \right) &= \frac{\gamma F(x) n r \left(\frac{R^\gamma}{r + R^\gamma} \right)^n (r(\gamma n - 1) - (\gamma + 1) R^\gamma)}{R^2 (r + R^\gamma)^2}. \qquad (6.16)
\end{aligned}$$

This time we are not able to deduce if the second partial derivative is negative. Thus, in order to check if a stationary point is a minimum or a maximum, we will replace its value on expression (6.16). If it leads to a negative value, it will be considered as a relative maximum. R^* will be selected among all these values to be one that maximizes the most V_n.

We are not even able to find any stationary point analytically, due to the complexity of (6.15).

6.4 Comparative Statics

In this Section we will check how R^* and $\lambda(R^*)$ behaves with γ and how V_n behaves with γ and $n \in \mathbb{N}$.

As stated in Section 6.3, we are not able to find an analytical expression regarding the roots of expression (6.15) and hence we are not able to derive ana analytical expression for the optimal R&D investment R^*. However, we developed a function that, given a certain parameter γ and an amount of jumps n and considering a fixed discount rate r and the value function evaluated for an initial demand level x $F(x)$, calculates numerically the R&D investment that optimizes expression (6.5), that leads to function V_n. This function is presented in Appendix by the name calcR. All of the next results were obtained using it.

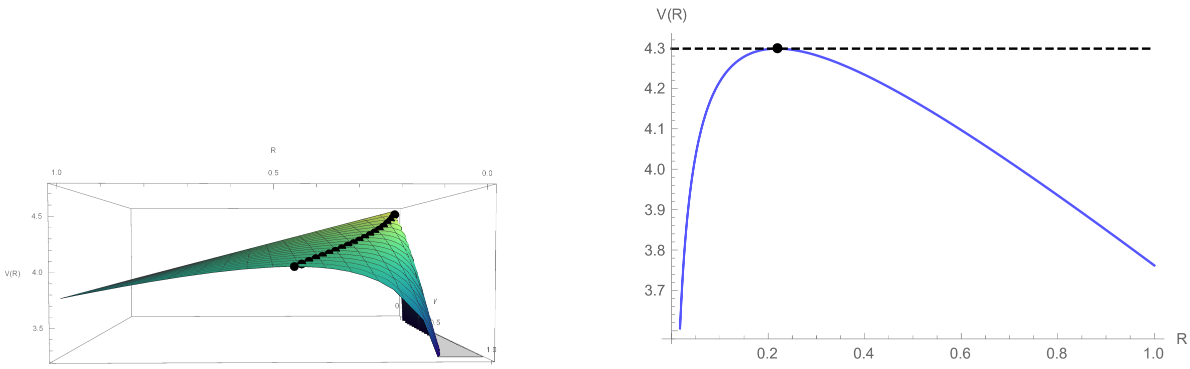
The following parameters were considered regarding all the next results:

- $r = 0.05$;
- $F(X) = 10$;

We also considered different values of $\gamma \in (0, 1]$, starting in 0.05 and, incremented by 0.05, until 1.

Note on expression (6.13), since $F(x)$ is not influenced by γ or R, it can be seen as a constant term representing the value of the project assuming an innovation level θ, which is yet to come. Hence we are able to proceed with our numerical approximations.

We start by analysing the case when we are able to decide when to invest after a jump. That is the case when $n = 1$.



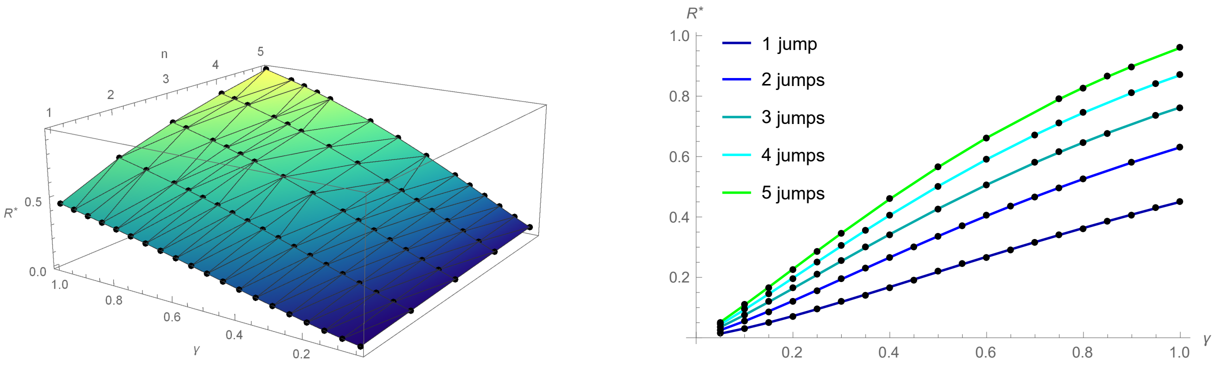
(a) $R \in (0, 1]$ and $\gamma \in (0, 1]$. (b) A particular case: $\gamma = 0.5$.

Figure 6.1: Function to be maximized in (6.1) with respect to parameters γ and R and corresponding maximum values V_1 regarding each γ (black)

On the leftmost side of Figure 6.1 we observe that the function to be maximized is convex regarding the R&D investment R, as stated in (6.6). Note that a smaller value of γ implies a higher value of R^*. This seems to be related with the fact that a smaller γ leads to higher jump rate, which implies a higher expected time. Thus, to balance this increasing on the expected waiting time, one increases the R&D investment.

On the rightmost side of Figure 6.1 we have the particular case when $\gamma = 0.5$ and the optimal R&D investment, calculated numerically. For $n = 1$ we were able to obtain all R^*, but same didn't hold for bigger values of n (as it will be seen in the following figures).

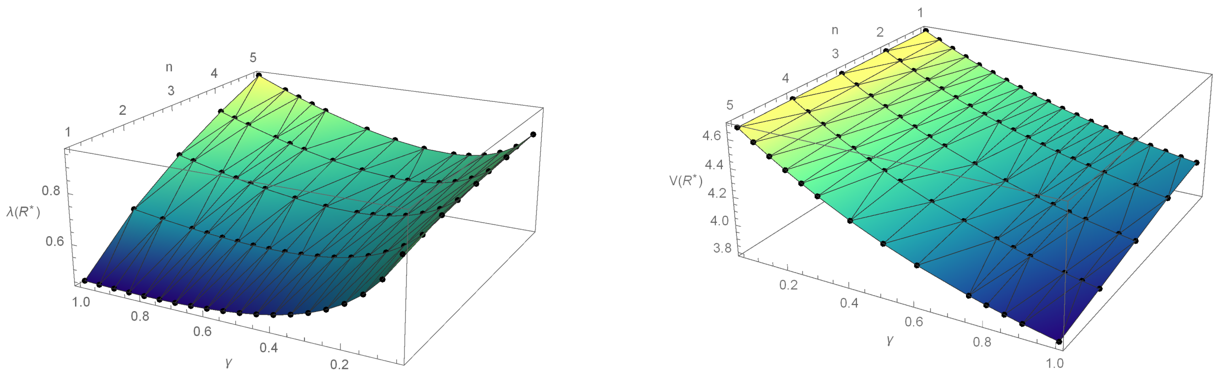
Now we increase the complexity and analyse the behaviour of R^*, $\lambda(R^*)$ e V_n, regarding the occurrence of 1 to 5 innovation jumps.



(a) 3D view. (b) 2D view.

Figure 6.2: Optimal R&D values R^* regarding parameter γ and the occurrence of $n \in \{1, ..., 5\}$ innovation jumps with corresponding numerical approximations (black).

On Figure 6.2 one can notice that the optimal R&D investment increases with both parameter γ and number of jumps n. Note that we weren't able to obtain the optimal value R^* (for instant for $\gamma = 0.45$ and $\forall n \geq 3$). However by connecting the values that we were able to calculate, we are able to check the (mentioned) increasing tendency.



(a) Jump rate $\lambda(R^*)$. (b) V_n

Figure 6.3: Effect of parameters γ and the occurrence of $n \in \{1, ..., 5\}$ innovation jumps on jump rate $\lambda(R^*)$ and project's value V_n with corresponding numerical approximations (black).

On the leftmost side of Figure 6.3 one can notice that although the jump rate $\lambda(R^*)$ increases with the number of jumps considered n, the same doesn't hold for parameter γ (for which it has a non-monotonic behaviour). This is related with the value R^* and its exponentiation with γ, necessary to calculate $\lambda(R^*)$.

On the rightmost side of Figure 6.3 one can notice that the non-monotonic behaviour of $\lambda(R^*)$ seems not to influence (strongly) the tendency of V_n. We have that V_n decreases with both γ and n. The first is explained due to the fact that if the higher the number of jumps we need to wait, before being able to think about the investment, the lower is the value of the project we have in hands (when compared with one of same value F and same parameter γ).

Chapter 7

Conclusions

Insert your chapter material here...

7.1 Achievements

The major achievements of the present work...

7.2 Future Work

A few ideas for future work...

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Appendix A

Vector calculus

In case an appendix if deemed necessary, the document cannot exceed a total of 100 pages...

Some definitions and vector identities are listed in the section below.

A.1 Vector identities

$$\nabla \times (\nabla \phi) = 0 \tag{A.1}$$

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0 \tag{A.2}$$

Appendix B

Technical Datasheets

It is possible to add PDF files to the document, such as technical sheets of some equipment used in the work.

B.1 Some Datasheet

SUNPOWER

C60 SOLAR CELL

MONO CRYSTALLINE SILICON

BENEFITS

Maximum Light Capture
SunPower's all-back contact cell design moves gridlines to the back of the cell, leaving the entire front surface exposed to sunlight, enabling up to 10% more sunlight capture than conventional cells.

Superior Temperature Performance
Due to lower temperature coefficients and lower normal cell operating temperatures, our cells generate more energy at higher temperatures compared to standard c-Si solar cells.

No Light-Induced Degradation
SunPower n-type solar cells don't lose 3% of their initial power once exposed to sunlight as they are not subject to light-induced degradation like conventional p-type c-Si cells.

Broad Spectral Response
SunPower cells capture more light from the blue and infrared parts of the spectrum, enabling higher performance in overcast and low-light conditions.

Broad Range Of Application
SunPower cells provide reliable performance in a broad range of applications for years to come.

The SunPower™ C60 solar cell with proprietary Maxeon™ cell technology delivers today's highest efficiency and performance. The anti-reflective coating and the reduced voltage-temperature coefficients provide outstanding energy delivery per peak power watt. Our innovative all-back contact design moves gridlines to the back of the cell, which not only generates more power, but also presents a more attractive cell design compared to conventional cells.

SunPower's High Efficiency Advantage

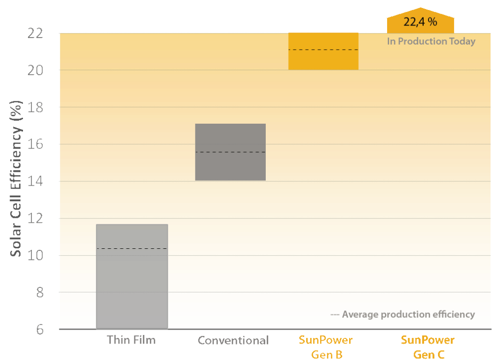
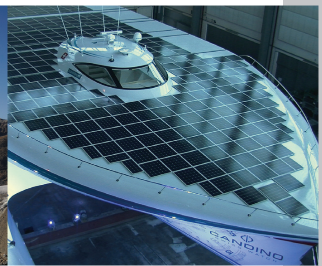



Photo courtesy of 3S Photovoltaics



C60 SOLAR CELL

SUNPOWER

C60 SOLAR CELL

MONO CRYSTALLINE SILICON

Electrical Characteristics of Typical Cell at Standard Test Conditions (STC)

STC: 1000W/m², AM 1.5g and cell temp 25°C

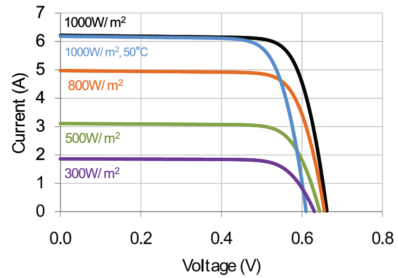
Bin	Pmpp (Wp)	Eff. (%)	Vmpp (V)	Impp (A)	Voc (V)	Isc (A)
G	3.34	21.8	0.574	5.83	0.682	6.24
H	3.38	22.1	0.577	5.87	0.684	6.26
I	3.40	22.3	0.581	5.90	0.686	6.27
J	3.42	22.5	0.582	5.93	0.687	6.28

All Electrical Characteristics parameters are nominal
Unlaminated Cell Temperature Coefficients
Voltage: -1.8 mV / °C Power: -0.32% / °C

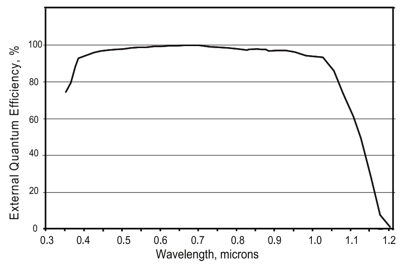
Positive Electrical Ground

Modules and systems produced using these cells must be configured as "positive ground systems".

TYPICAL I-V CURVE



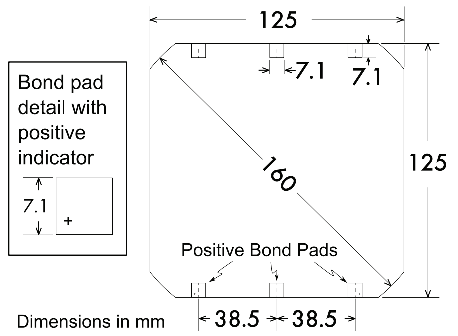
SPECTRAL RESPONSE



Physical Characteristics

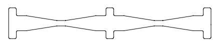
Construction:	All back contact
Dimensions:	125mm x 125mm (nominal)
Thickness:	165μm ± 40μm
Diameter:	160mm (nominal)

Cell and Bond Pad Dimensions



Bond pad area dimensions are 7.1mm x 7.1mm
Positive pole bond pad side has "+" indicator on leftmost and rightmost bond pads.

Interconnect Tab and Process Recommendations



Tin plated copper interconnect. Compatible with lead free process.

Packaging

Cells are packed in boxes of 1,200 each; grouped in shrink-wrapped stacks of 150 with interleaving. Twelve boxes are packed in a water-resistant "Master Carton" containing 14,400 cells suitable for air transport.

Interconnect tabs are packaged in boxes of 1,200 each.

About SunPower

SunPower designs, manufactures, and delivers high-performance solar electric technology worldwide. Our high-efficiency solar cells generate up to 50 percent more power than conventional solar cells. Our high-performance solar panels, roof tiles, and trackers deliver significantly more energy than competing systems.

sunpowercorp.com

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