UNIVERSIDADE DE LISBOA
INSTITUTO SUPERIOR TÉCNICO

Processes with jumps in Finance

Rita Duarte Pimentel

Supervisor: Doctor Cláudia Rita Ribeiro Coelho Nunes Philippart

Co-Supervisor: Doctor Raquel Maria Medeiros Gaspar

Thesis specifically prepared to obtain the PhD degree in
Statistics and Stochastic Processes

Draft

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Everything should be made as simple as possible,
but not simpler.

Albert Einstein

Abstract

This thesis addresses three particular investment problems that share, in particular, the following feature: the processes that model the uncertainty exhibit discontinuities in their sample paths. These discontinuities - or jumps, as also called along the thesis - are driven by jump processes, hereby modelled by Poisson processes. In some cases (as it is the case of the first problem presented in the thesis), we extend existing results, and the main objective is to develop and build upon existing analysis frameworks for such investment problems, taking now into account the jumps. In that case, we also assess the impact of the jump process in the resulting decision.

Above all, the problems here addressed are all problems that fall in the category of *optimal stopping problems*: choose a time to take a given action (in particular, the time to decide to invest, as here we consider investment problems) in order to maximize an expected payoff.

The three problems that we consider in this thesis define three different situations that an investor may face.

In the first problem, we assume that a firm is currently receiving a profit stream from an already operational project, and has the option to invest in a new project, with impact in its profitability. Moreover, we assume that there are two sources of uncertainty that influence the firm's decision about when to invest: the random fluctuations of the revenue (depending on the random demand) and the changing investment cost. And, as already mentioned, both processes exhibit discontinuities in their sample paths.

The second and third problems are developed in the scope of *technology adoption*. The technology innovation is, by far, an example of a discountinuous process: the technological level does not increase in a steady pace, but instead from now and then some improvement or breakthrough happens. Thus it is natural to assume that technology innovations are driven by jump processes. As such, in the second problem addressed in the thesis we consider a firm that is producing in a declining market, but with the option to undertake an innovation investment and thereby to replace the old product by a new one, paying a constant sunk cost. As the first product is a well established one, its price is determistic. Upon investment in the second product, the price may fluctuate, according to a geometric Brownian motion. The decision is when to invest in a new product.

In the third and last problem the firm faces a similar decision, in the sense that the firm is producing a well established product and can decide to start producing a new product, with a better technology, and thus more profitable. At the decision moment, the firm should choose between replacing the established product by the new one, or keep on producing the established product so that it produces both products at the same time. Moreover, at this moment the firm should also decide the capacity it

will allocate to the new product.

Overall, the goal of this thesis is two-fold: contribute to the literature of mathematical finance with new models and approaches; and provide to practitioners, in particular firms, new tools to solve real life problems.

Keywords

Investment decision; Real options; Optimal stopping problems; Jump-diffusion process; compound Poisson process; Euler-Cauchy equation.

Resumo

Esta tese aborda três problemas de investimento particulares que partilham a seguinte característica: os caminhos dos processos que modelam a incerteza apresentam descontinuidades. Essas descontinuidades - ou saltos, como também são designados ao longo da tese - são impulsionadas por processos de salto, neste caso modelados por um processos de Poisson.

Em alguns casos (em particular no primeiro problema apresentado na tese), estendemos os resultados existentes. Nessa situação o principal objetivo é desenvolver metodologias já disponíveis, de forma a contemplar a existência de saltos no processo de incerteza.

Acima de tudo, os problemas aqui abordados enquadram-se na categoria de *problemas de paragem ótima*: escolher o tempo para tomar uma determinada ação (em particular, o momento de decidir investir, pois aqui consideramos problemas de investimento) de forma a maximizar o retorno esperado.

Os três problemas que consideramos nesta tese definem três situações diferentes com que um decisor se pode deparar.

No primeiro problema, assumimos que uma empresa está atualmente em produção (e, como retorno, tem um fluxo de lucros) e tem a opção de investir num novo projeto, com impacto na sua rentabilidade. Além disso, assumimos que existem duas fontes de incerteza que influenciam a decisão da empresa sobre quando investir: as flutuações aleatórias da receita (as quais dependem da procura, que também é, por sua vez, aleatória) e o custo pago no momento do investimento. E, como já foi mencionado, ambos os processos têm caminhos descontínuos.

O segundo e o terceiro problemas são desenvolvidos no âmbito da *adoção de tecnologia*. A inovação tecnológica é naturalmente um processo descontínuo: o nível tecnológico não aumenta a um ritmo constante; pelo contrário, de vez em quando ocorre alguma melhoria ou um avanço. Assim, é natural supor que as inovações tecnológica são impulsionadas por processos de salto. Como tal, no segundo problema abordado na tese consideramos uma empresa que está a produzir num mercado em declínio, mas com a opção de fazer um investimento em inovação e assim substituir o produto antigo por um novo, pagando um custo constante. Como o produto original está instalado no mercado, o seu preço decai de forma determinística. Após o investimento no segundo produto, o preço pode flutuar, de acordo com um movimento geométrico Browniano. A decisão neste caso é quando se deve investir no novo produto.

No terceiro e último problema, a empresa enfrenta uma decisão semelhante, no sentido em que a empresa está a produzir um produto instalado no mercado e pode decidir começar a produzir um novo produto, com uma tecnologia melhor e, consequentemente, mais rentável. No momento da decisão, a empresa deve escolher entre substituir o produto original pelo novo ou produzir ambos os produtos.

Além disso, no momento da decisão, a empresa também deve decidir a capacidade que irá afectar ao novo produto.

Esta tese tem dois grandes objetivos: contribuir para a literatura em Matemática Financeira, propondo novos modelos e abordagens; e fornecer aos profissionais, em particular às empresas, novas ferramentas para resolver problemas num contexto real.

Palavras Chave

Decisão de investimento; Opções reais; Problemas ótimos de paragem; Processos de difusão com saltos; Processo de Poisson composto; Equação de Cauchy-Euler.

Acknowledgments

To my grandmother Joana,
who have dreamed with this thesis more than anyone else.

Contents

1
Introduction

Contents

1.1 Motivation

An investment is the act of incurring an immediate cost in the expectation of future rewards. For most firms, a substantial part of their market value is attribute to their options to invest and grow in the future, as opposed to the capital they already have in place (Dixit & Pindyck (1994)).

A specific framework to study investment opportunities is the real options approach, which appeared as an alternative to the Net Present Value (NPV, for short), with the seminal work of Dixit & Pindyck (1994). A real option is the right — but not the obligation — to undertake certain business initiatives, such as investing, deferring, abandoning, expanding, among others, upon payment of a cost in the future. It is distinguished from conventional financial option because it is not traded as a security, and do not usually involve decisions on an underlying asset that is traded as a financial security. Nevertheless, many problems are similar and the methodical tools are identical.

A real option has three fundamental characteristics: the investment is partially or completely irreversible (the initial cost of investment is at least partially sunk); there is uncertainty over the future rewards from the investment; there is some leeway about the timing of the investment. As it can be realized, timing plays a crucial role in the decisions in an uncertain economic environment. On the one hand, the possibility to delay the investment may increase the value of the project once the decision maker will act with more information. On the other hand, waiting for too long may lead to losses. Therefore, the main goal in real options context is to find the best time to make the decision.

Recently, some authors working in real options framework have been arguing that, besides choosing the timing of the investment, the firm should also choose the size of the investment (see Hagspiel *et al.* (2016)). The real options literature has a standard result that with more uncertain environment, firms invest later. However, when also capacity size needs to be determined, this result is not straightforward anymore.

Regarding the uncertainty, it is the prime reason for construction of mathematical models to study the investment decisions. The uncertainty is modelled by a stochastic process. In the early times, finance works usually assumed a geometric Brownian motion (GBM, for short) for the underlying process. This is mainly due to the great analytic properties arising (Eberlein (2010)). However, there is the need to explore more realistic situations, using models that capture, for instance, discontinuous behaviors observed in real life. This may be due to occurrence of rare events, which can be modelled by jump processes (Cont & Tankov (2004)). The inclusion of jumps was first considered by Merton (1976) for stock prices, where each jump represented the arrival of an important piece of information about the stock.

In investment problems, for instance, the demand level may be subject to abrupt changes caused by unexpected conjecture shocks. The jumps may also represent uncertainties about the arrival and impact of new information concerning technological innovation, competition, political risk, regulatory effects and other sources (Martzoukos & Trigeorgis (2002), Wu & Yen (2007)). This is the case, for example, of high-technology companies, where the advances in technology continue to challenge managerial decision-makers in terms of the timing of investment (Wu *et al.* (2012)).

The main objective is to develop and build upon existing analysis frameworks for modelling investment problems.

The investment decisions fall in the context of optimal stopping problems, where a firm seeks to find the time to exercise its investment option in such a way that its expected profit is maximized. This investment decision is in fact a free boundary problem. One way to solve this problem is to find the solution of a variational inequality, the so called Hamilton-Jacobi-Bellman (HJB, for short). In the HJB equation one of the members accounts for the payoff associated with the investment exercise, and the other member accounts for the value of the option (before its exercise). This last part depends strongly on the process that models the uncertainty.

Indeed, under certain conditions, one is able to find a solution to the HJB equation, that is the solution to the initial problem. The type of HJB equation strongly depends on the processes driven the uncertainty.

1.2 Investment options

This thesis addresses three different investment problems, which are described in Chapters 2, 4 and 5. Furthermore, in Chapter 3 we provide a technical result that will be useful, in particular in Chapter 4.

In Chapter 2 we assume that a firm is currently receiving a profit stream from an already operational project and enjoying an iso-elastic revenue function. If the firm decides to invest, then the revenue after the investment changes, but it is still iso-elastic. We also assume that there are two sources of uncertainty: demand (reflected in the revenue of the firm), and investment cost, both driven by a jump diffusion process. Thus the investor needs to optimize his investment decision by taking into account the random fluctuations of the revenue and the changing investment cost. Assuming certain conditions on the parameters, we are able to derive a closed expression for the value of the firm, extending the approach used in Dixit & Pindyck (1994) (Chapter 6.5) to deal with two sources of uncertainty.

The other two papers are related with technology adoption, where we assume that technology arrivals are driven by a pure jump process. It is seen that rapid technological developments are inducting the shift in consumer demand from existing products towards new alternatives. When operating in a declining market, the profitability of incumbent firms is largely dependent on the ability to correctly time the introduction of product innovations. In Chapter 4, we contribute to the existing literature, by considering the optimal innovation investment in the context of the declining market. We study the problem of a firm that has an option to undertake the innovation investment and thereby to replace the established product by the new one. We are able to quantify the value of the option to adopt a new technology, as well as the optimal timing to exercise it.

Finally, in Chapter 5, we consider a firm that is producing a well established product and can decide to start producing a new product, using a better technology and thus, a more profitable product. While doing so, such a firm should decide what to do with its existing production process after the innovation. Essentially it can choose between replacing the established production process by the new one, or keep on producing the established product, so that it produces both products at the same time. Moreover, at the investment moment the firm should decide the capacity it will allocate to the new product. Under some conditions, we are able to present the value of the option and in which conditions it is better to add the innovative product or to replace the established product by the new one.

1.3 Thesis outline

The core items of this thesis are Chapters 2 through 5, which correspond to four articles that have been prepared to be submitted to peer-reviewed journals, as it is shown below.

- **Chapter 2**: Accepted for publication in *European Journal of Operation Research*, available online since January 2017, co-authors: Cláudia Nunes;
- **Chapter 3**: Being prepared to be submitted in *Discrete and Continuous Dynamical Systems*, co-authors: Cláudia Nunes, Ana Prior;
- **Chapter 4**: Being prepared to be submitted in *Journal of Economic Theory*, co-authors: Verena Hagspiel, Kuno J. M. Huisman, Peter M. Kort, Maria N. Lavrutich, Cláudia Nunes;
- **Chapter 5**: Being prepared to be submitted in *Operations Research*, co-authors: Verena Hagspiel, Kuno J. M. Huisman, Peter M. Kort, Maria N. Lavrutich, Cláudia Nunes.

Since some of those papers were produced at the same time, they extend the literature in different directions. In fact, the path that this thesis took was defining itself over time.

The first work corresponds to Chapter 2, where two sources of uncertainty are considered. The firm is exposed to a market where demand may suffer negative shocks and investment cost may suffer positive shocks. This implies that in the optimal stopping problem, the path of the bi-dimensional stochastic process only achieve the stopping region through a continuous movement. In case the jumps have a different sign (in particular, positive in the demand and/or negative in the investment cost), the stopping region could be reached though a jump. Therefore one would need a different approach to solve such problem, that would lead, in particular, to a problem involving a second order Euler-Cauchy equation with Log non-homogenities, for which the analytical particular solution was not yet derived.

Motivated by problems of this kind, in Chapter 3 we present a more technical result, providing the analytical expression for the solution for this type of Euler-Cauchy equations (second order Euler-Cauchy equation with Log non-homogenities). Clearly this chapter is not dedicated to any particular investment decision problem, but it can be valuable for this class of problems, as we discuss in the introduction of the Chapter. Moreover, it provided also insight to solve some differential equations present in Chapter 4.

Chapters 4 and 5 were started at the same time and initially were thought as an unique work. However, with time we realized that it would be easier to split in two projects: one concentrate in the decline market, only with the possibility of replacing the old product by the new one; and other concentrate in the capacity optimization, allowing adding and replacing strategies. As already mentioned, for Chapter 4 the knowledge gained in Chapter 3 was essential. Indeed, the same type of tools were used to find the particular solution of a first order Euler-Cauchy equation with Log non-homogenities.

Finally, in Chapter 6, the main findings of this thesis are summarized and some new lines of research are also suggested.

Due to the fact that each chapter was composed as self-contained paper, there is some repetition of concepts or methods, and occasionally the notation is not the same in all chapters. Moreover, some of the theoretical and methodological background is not explored in depth in each chapter. This can be due to a limitation of space or to the fact that some contents do not warrant detailing in a contemporary scientific article. Furthermore, in Chapters 4 and 5, given that there are many technical details, the results are presented in the main text but the proofs are left for appendix.

2

Analytical solution for an investment problem under uncertainties with shocks

Contents

2.1 Introduction

In this paper we use the real options framework to study an investment decision when both demand and investment cost are random, evolving according to jump diffusion processes.

A real option means that a firm has the right, but not the obligation, to undertake certain business initiatives such as deferring, abandoning, expanding, staging or contracting a capital investment project. The analysis of real options assumes that there are three driving investment factors underpinning the investment decision. The first is that future rewards are uncertain. The second is that the decision is irreversible in the sense that the investment expenditure cannot be fully recovered. The third and last one is that the timing of the investment is variable and therefore the investor may decide on the best time to invest in order to maximize the value of his firm. We refer to Dixit & Pindyck (1994), Arrow & Fisher (1974), McDonald & Siegel (1986), Dixit (1989) and references therein contained as good examples of seminal works on real options.

One of the early works in this area is McDonald & Siegel (1986) wherein the authors model the investment problem when the value of the investment project evolves following a geometric Brownian motion (GBM, for short), and the investor must decide when he should exercise his investment option. The optimal strategy is to exercise the option as soon as the value of the project exceeds a threshold.

In this paper we assume that there are two sources of uncertainty: demand (reflected in the revenue of the firm), and investment cost. Thus the investor needs to optimise his investment decision by taking into account the random fluctuations of the revenue and the changing investment cost.

Problems with two sources of uncertainty are not new, and some examples can be found in the literature, e.g. in the pioneer book of Dixit & Pindyck (1994) and recently Murto (2007), Zambujal-Oliveira & Duque (2011), Pennings & Sereno (2011) and Alghalith (2016).

One of the most common assumption is that the underlying stochastic processes modelling uncertainty are continuous sample-path processes, such as the GBM. This is specially due to its comprehensive analytical properties, as referred by Eberlein (2010). But as this author mentions, *as a consequence of the distributional deficiencies of the classical models that are driven by Brownian motions, advanced models in finance are now based on processes that include jumps or that consist entirely of jumps*. Empirical and theoretical studies already showed the existence of jumps, from portfolio risk management to option and bond pricing (e.g. Merton (1976), Pan (2002), Liu *et al.* (2003), Johannes (2004), Lee & Mykland (2008), Hagspiel *et al.* (2015), among others).

Nowadays, the world global market encompasses exogenous events that may lead to sudden increase or decrease (shocks) in demand for certain products. Clearly, these shocks will considerably affect investors decisions. Taking into account that the involved processes may exhibit sample-path discontinuities, we consider jump-diffusion processes. Several examples support this assumption.

The chart presented in Figure 2.1 shows the number of worldwide sales of iPhones from 3$^{\text{rd}}$ quarter 2007 to 3$^{\text{rd}}$ quarter 2016, with data aggregated by quarters [1].

It is clear that in this case upward jumps occur with a certain frequency, possibly related with the

[1] Source: Statista. Available at https://www.statista.com/statistics/263401/global-apple-iphone-sales-since-3rd-quarter-2007/.

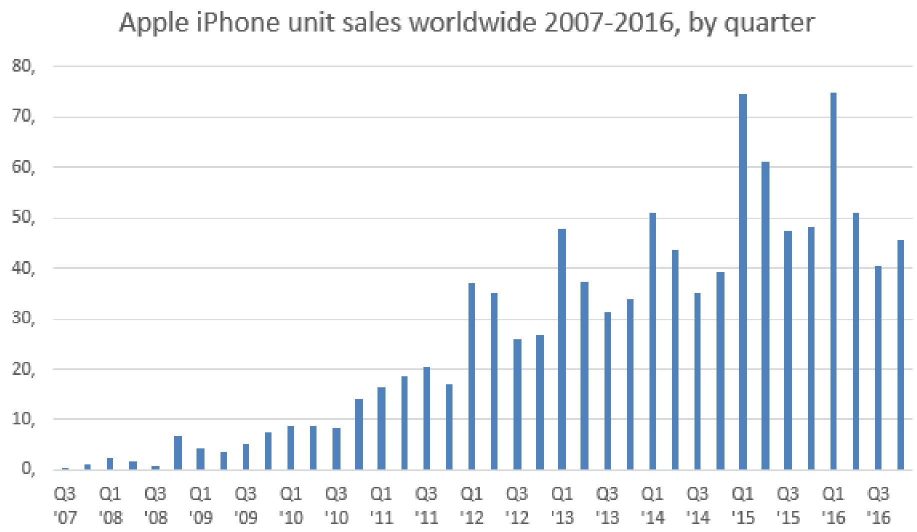


Figure 2.1: Global Apple iPhone unit sales from 3rd quarter 2007 to 3rd quarter 2016 (in million units).

launching of new versions of the iPhone.

Another recent example occurred just after the 9-11 terrorist attack in the USA. One of the headlines of the World Street Journal was *As demand for train service jumps, Amtrak seeks emergency funding.* According to this journal, the Amtrak ridership jumped by 60% [2].

In the previous two examples, events led to an upward jump on demand. But there are also examples pointing in the opposite direction. The following illustrates a case where a downward jump occurred on demand. In 2015 the Volkswagen's sales decreased, due to the Emissions scandal spread in the news. In fact, reports from Volkswagen announced that global sales fell 2 % in 2015.[3] This also had an impact on the price of the company's shares. According to the Wall Street Journal, *shares in Volkswagen AG and Porsche Automobil SE, ..., tumbled by more than 8 % in early trade Wednesday,...*[4].

We notice that after these jumps sale's level tends to return to previous values, meaning that the impact of the jumps has short memory. This observation is in line with Pimentel *et al.* (2012) and Couto *et al.* (2015), where it is assumed that the demand for high speed rail services can be modelled by a jump-diffusion process. In fact, our work is closely related to these two contributions but considers that demand and investment cost are random.

In the current work we are also interested in changing investment cost, like in Oh *et al.* (2016) or Shi (2016). If one assumes the investment cost is constant and known, then the model is being simplified. This problem is even more important when one considers large investments that take several years to be completed (e.g. construction of dams, nuclear power plants or high speed railways).

Examples of changing investment cost can be found in different areas. One of such examples is related to investment in the energy sector. In Hunt & Shuttleworth (1996) we can find a example

[2] Source: The Wall Street Journal, news published on the 21st September 2001. Available at http://www.wsj.com/articles/SB1001021248552406720.

[3] Source: The Guardian, news published on the 8th January. Available at https://www.theguardian.com/business/2016/jan/08/vw-global-sales-fell-year-emissions-scandal-2015.

[4] Source: The Wall Street Journal, news published on the 4th November 2015. Available at http://www.wsj.com/articles/volkswagen-drags-other-auto-shares-lower-on-fears-scandal-could-widen-1446631327.

where the investment cost decreases: *as a result of studies sponsored by space programs, it was possible to build turbines much more efficient and smaller than before, reducing in a drastic way the optimal power plant size, with enormous cost reduction.*

On the other hand, the investment may also suffer a cost escalation. For example, a report from Mark Cooper, entitled "Policy Challenges of Nuclear Reactor Construction, Cost Escalation and Crowding out Alternatives" [5] refers that *... the increasing complexity of nuclear reactors and the site-specific nature of deployment make standardization difficult, so cost reductions have not been achieved and are not likely in the future. More recent, more complex technologies are more costly to construct.*

Another example, also taken from the energy sector, concerns the increasing cost to build new transmission infrastructures. In a report from the USA Energy Information Administration, we find the following statement: *the global economic boom of 2004-07, along with a weakened U.S. dollar, raised the prices of raw materials (such as steel and cement), fuel, and labor faster than the rate of inflation* [6].

Furthermore, besides considering stochastic investment cost, we also allow it to exhibit jumps. Jumps may represent uncertainties arising from the disclosure of new information regarding technological innovation, competition, political risk, regulatory effects and other sources and its impact (Martzoukos & Trigeorgis (2002) and Wu & Yen (2007)).

One paper that is closely related with our work is Murto (2007), where the author characterizes the timing of investment considering two sources of uncertainty. The author assumes that the revenue is a diffusion process and that the technological uncertainty is a pure Poisson process, which influences the investment cost. In our case, we allow both uncertainty factors to be jump-diffusion processes, combining the continuous behaviour (associated with the diffusion) with the discrete behaviour (associated with the jumps). Moreover, Murto (2007) presents analytical solutions only for some particular cases, for instance, assuming that one of the processes is deterministic. In this paper, we are able to derive an analytical solution for a more general class of problems.

We assume that a firm is currently receiving a profit stream from an already operational project and enjoying an iso-elastic revenue function. If the firm decides to invest, then the revenue after the investment is still iso-elastic. While the elasticity parameter is assumed to be the same, the profit stream receives a one-time upward boost once investment is undertaken. Besides that, the firm is exposed to a market where demand may suffer negative shocks and investment cost may suffer positive shocks. Although jumps in demand process are negative and jumps in investment cost are positive, there is no restriction on the sign of drift parameters. We note that with these assumptions we do not restrict ourselves to declining markets neither escalating investment cost, as the drift of these processes may be either positive or negative.

The remainder of the paper is organised as follows. In Section 2 we present the rationale framework and the valuation model. In Section 3 we formalize the problem as an optimal stopping one, present our approach and provide the optimal solution. In Section 4 we derive the results regarding the behaviour

[5] Available at http://www.psr.org/nuclear-bailout/resources/policy-challenges-of-nuclear.pdf

[6] Available at http://www.eia.gov/todayinenergy/detail.php?id=17711

of investment threshold as a function of the most relevant parameters. Finally, Section 5 concludes the work, presenting also some recommendations for future extensions.

2.2 The model

In this section we introduce the mathematical model and the assumptions that used in order to derive the investment policy. We start by presenting the dynamics of the stochastic processes involved, namely demand and investment cost processes and valuation model.

2.2.1 The processes dynamics

We assume that there are two sources of uncertainty, that we will denote by $\boldsymbol{X}$ and $\boldsymbol{I}$, representing demand and investment cost, respectively. Moreover, we consider that both have continuous sample paths almost everywhere but occasional jumps may occur, causing a discontinuity in the path.

Generally speaking, let $\boldsymbol{Y} = \{Y_t : t \geq 0\}$ denote a jump-diffusion process, which has two types of changes: the "normal" vibration, which is represented by a GBM, with continuous sample paths, and the "abnormal", modeled by a jump process, which introduces discontinuities.

Jumps occur at random times and are driven by an homogeneous Poisson process, hereby denoted by $\{N_t, t \geq 0\}$, with rate λ. Moreover, the percentage of jumps are themselves random variables, that we denote by $\{U_i\}_{i\in\mathbb{N}} \overset{i.i.d.}{\sim} U$, i.e., although they may be of different magnitude, they all follow the same probability law and are independent from each other.

Thus we may express Y_t as follows:

$$Y_t = y_0 \exp\left[\left(\mu - \frac{\sigma^2}{2} - \lambda m\right) t + \sigma W_t\right] \prod_{i=1}^{N_t} (1 + U_i)\,, \tag{2.1}$$

where y_0 is the initial value of the process, i.e., $Y_0 = y_0$, μ and $\sigma > 0$ represent, respectively, the drift and the volatility of the continuous part of the process, m is the expected value of U, and $\{W_t, t \geq 0\}$ is a Brownian motion process. Furthermore, the processes $\{W_t, t \geq 0\}$ and $\{N_t, t \geq 0\}$ are independent and are also both independent of the sequence of random variables $\{U_i\}_{i\in\mathbb{N}}$. We use the convention that if $N_t = 0$ for some $t \geq 0$, then the product in equation (2.1) is equal to 1. Note that, considering the parameters associated to jumps equal to zero (i.e., either $\lambda = 0$, or $U = 0$, with probability one), then we obtain the standard GBM.

As both the Brownian motion and the Poisson processes are Markovian, it follows that $\boldsymbol{Y}$ is also Markovian. It is also stationary and their k^{th} order moments are provided in the next lemma.

Lemma 2.1. *For $\boldsymbol{Y} = \{Y_t : t \geq 0\}$, with Y_t defined as in (2.1), with $t, s \geq 0$ and $k \in \mathbb{N}$:*

$$\mathrm{E}\left[Y_{t+s}^k \middle| Y_t\right] = Y_t^k \ \exp\left\{\left[\left(\mu + (k-1)\frac{\sigma^2}{2} - \lambda m\right) k + \lambda\left(\mathrm{E}\left[(1+U)^k\right] - 1\right)\right] s\right\},$$

where $\mathrm{E}\left[.\middle| Y_t\right]$ denotes the conditional expectation given the value of the process $\boldsymbol{Y}$ at time t.

Proof of Lemma 2.1. *See Appendix (2.A.1) for the proof.* ■

Returning to the investment problem: in the sequel we assume that both processes $\boldsymbol{X}$ and $\boldsymbol{I}$ follow jump-diffusion processes:

$$X_t = x_0 \exp\left[\left(\mu_X - \frac{\sigma_X^2}{2} - \lambda_X m_X\right) t + \sigma_X W_t^X\right] \prod_{i=1}^{N_t^X} \left(1 + U_i^X\right)$$

and

$$I_t = i_0 \exp\left[\left(\mu_I - \frac{\sigma_I^2}{2} - \lambda_I m_I\right) t + \sigma_I W_t^I\right] \prod_{i=1}^{N_t^I} \left(1 + U_i^I\right),$$

where x_0 and i_0 denote the initial values for the processes $\mathbf{X}$ and $\mathbf{I}$, respectively. We use an index X or I for each parameter to distinguish between the two processes and we assume that the processes are independent from each other. We consider that jumps in demand, U_X, take negative values (downward jumps), whereas in investment cost, U_I, take positive values (upward jumps). Therefore $m_X < 0$ and $m_I > 0$.

Next we introduce the valuation model that we consider throughout the paper.

2.2.2 The valuation model

We assume that revenue of the firm depends entirely on demand, and we use V_0(respectively, V_1) to denote the revenue before (respectively, after) the investment, respectively. Moreover, we assume the following specific form for the revenue function:

$$V_j(X_t) = \kappa_j X_t^\theta,$$

for $j \in \{0, 1\}$, with $\kappa_1 > \kappa_0 > 0$ and $\theta > 0$. So investment in the new project only changes the model coefficients, κ_0 and κ_1, and not the elasticity parameter, θ. Furthermore,

$$D(X_t) = (\kappa_1 - \kappa_0) X_t^\theta \tag{2.2}$$

represents the difference in profit difference before and after investment.

Denoting the risk-neutral discount rate by ρ then the value of the firm, if the investment is decided at time τ, is given by:

$$\int_0^{\tau+n} V_0(X_t) e^{-\rho t} dt + \int_{\tau+n}^{+\infty} V_1(X_t) e^{-\rho t} dt - I_\tau e^{-\rho\tau},$$

as long as $\tau < +\infty$. In this equation n denotes the time-to-build, as described in Couto *et al.* (2015).

The goal of the firm is to maximize its expected value, with respect to the time to invest, that is, the firm seeks to maximize the following expected value

$$\mathrm{E}^{(x_0,i_0)}\left[\int_0^{\tau+n} V_0(X_t) e^{-\rho t} dt + \left\{\int_{\tau+n}^{+\infty} V_1(X_t) e^{-\rho t} dt - I_\tau e^{-\rho\tau}\right\} \chi_{\{\tau<+\infty\}}\right],$$

where in the rest of the paper $\mathrm{E}^{(x_0,i_0)}$ denotes the expectation conditional on the event $(X_0, I_0) = (x_0, i_0)$; moreover $\chi_{\{A\}}$ represents the indicator function of set A and τ is a stopping time for the filtration generated by the bidimensional process $(\mathbf{X}, \mathbf{I})$.

Using (2.2) and simple manipulations, we can re-write the previous expectation as

$$\mathrm{E}^{(x_0,i_0)}\left[\int_0^{+\infty} V_0(X_t) e^{-\rho t} dt\right] + \mathrm{E}^{(x_0,i_0)}\left[\left\{\int_{\tau+n}^{+\infty} D(X_t) e^{-\rho t} dt - I_\tau e^{-\rho\tau}\right\} \chi_{\{\tau<+\infty\}}\right].$$

As the first part of the expression does not depend on the time τ to invest, then the goal is to maximize the following functional:

$$J^{\tau}(x_0, i_0) = \mathrm{E}^{(x_0,i_0)}\left[\left\{\int_{\tau+n}^{+\infty} D(X_t)e^{-\rho t}dt - I_{\tau}e^{-\rho\tau}\right\}\chi_{\{\tau<+\infty\}}\right],$$

which represents the value of the option when the firm exercises it at time τ, with $\tau < +\infty$, given that the initial state is (x_0, i_0). Following Øksendal & Sulem (2007), we call J^{τ} the *performance criterion.*

Note that one needs to assume that $\mathrm{E}^{(x_0,i_0)}\left[\int_0^{+\infty} V_0(X_t)e^{-\rho t}dt\right] < +\infty$ in order to have $\tau < +\infty$, which happens if and only if $h > 0$ [7], where

$$h = \rho - \left(\mu_X + (\theta-1)\frac{\sigma_X^2}{2} - \lambda_X m_X\right)\theta - \lambda_X\left(\mathrm{E}\left[(1+U_X)^{\theta}\right] - 1\right) \tag{2.3}$$

since

$$\mathrm{E}^{(x_0,i_0)}\left[\int_0^{+\infty} V_0(X_t)e^{-\rho t}dt\right] = \kappa_0 x_0^{\theta}\int_0^{+\infty} e^{-ht}dt,$$

in view of Lemma 2.1.

From now on we assume that such restriction in the parameters hold, and thus the optimal investment time is finite.

Next we state and prove a result related with the strong Markov property of the involved processes, that we will use subsequently in the optimization problem.

Proposition 2.1. *The performance criterion can be re-written as follows:*

$$J^{\tau}(x, i) = \mathbb{E}^{(x,i)}\left[e^{-\rho\tau}g(X_{\tau}, I_{\tau})\chi_{\{\tau<+\infty\}}\right],$$

with

$$g(x, i) = (\kappa_1 - \kappa_0)\,Ax^{\theta} - i$$

and

$$A = \frac{e^{-hn}}{h}.$$

Proof of Proposition 2.1. *Applying a change of variable in the performance criterion, we obtain*

$$J^{\tau}(x, i) = \mathrm{E}^{(x,i)}\left[e^{-\rho\tau}\left\{\int_0^{+\infty} D(X_{\tau+t+n})e^{-\rho(t+n)}dt - I_{\tau}\right\}\chi_{\{\tau<+\infty\}}\right].$$

Then, using the conditional expectation and the strong Markov property, we get

$$J^{\tau}(x, i) = \mathbb{E}^{(x,i)}\left[e^{-\rho\tau}\left\{\mathrm{E}\left[\int_0^{+\infty} D(X_{\tau+t+n})e^{-\rho(t+n)}dt\,\middle|\,(X_{\tau}, I_{\tau})\right] - I_{\tau}\right\}\chi_{\{\tau<+\infty\}}\right].$$

By Fubini's theorem and the independence between $\boldsymbol{X}$ *and* $\boldsymbol{I}$*, it follows that:*

$$\mathrm{E}\left[\int_0^{+\infty} D(X_{\tau+t+n})e^{-\rho(t+n)}dt\,\middle|\,(X_{\tau}, I_{\tau})\right] = \int_0^{+\infty}\mathrm{E}\left[D(X_{\tau+t+n})|\,X_{\tau}\right]e^{-\rho(t+n)}dt. \tag{2.4}$$

From Lemma 2.1 and simple calculations, we have:

$$\mathrm{E}\left[D(X_{\tau+t+n})|\,X_{\tau}\right] = (\kappa_1 - \kappa_0)\,X_{\tau}^{\theta}\,e^{(\rho-h)(t+n)}. \tag{2.5}$$

Combining (2.4) *and* (2.5) *results in:*

$$\mathrm{E}\left[\int_0^{+\infty} D(X_{\tau+t+n})e^{-\rho(t+n)}dt\,\middle|\,(X_{\tau}, I_{\tau})\right] = (\kappa_1 - \kappa_0)\,X_{\tau}^{\theta}\frac{e^{-hn}}{h}.$$

■

[7] When jumps are not considered in the model, this restriction on the parameters can be written as $\rho > \theta\mu_X + \theta(\theta-1)\frac{\sigma_X^2}{2}$, which is a standard condition to assure that the optimal investment time is finite.

2.3 The optimal stopping time problem

The goal of the firm is to find the optimal time to invest in the new project, i.e., for every $x > 0$ and $i > 0$, we want to find $V(x,i)$ and $\tau^\star \in \mathcal{T}$ such that:

$$V(x,i) = \sup_{\tau \in \mathcal{T}} J^\tau(x,i) = J^{\tau^\star}(x,i), \quad (x,i) \in \mathbb{R}^+ \times \mathbb{R}^+$$

where $\mathcal{T}$ is the set of all stopping times adapted to the filtration generated by the bidimensional stochastic process $(\boldsymbol{X}, \boldsymbol{I})$. The function V is called the *value function*, which represents the optimal value of the firm, and the stopping time $\tau^\star$ is called an *optimal stopping time*.

Using standard calculations from optimal stopping theory (see Øksendal & Sulem (2007)), we derive the following Hamilton-Jacobi-Bellman (HJB, for short) equation for this problem:

$$min\left\{\rho V(x,i) - \mathcal{L}V(x,i), V(x,i) - g(x,i)\right\} = 0, \quad \forall\, (x,i) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{2.6}$$

where $\mathcal{L}$ denotes the infinitesimal generator of the process $(\boldsymbol{X}, \boldsymbol{I})$, which in view of Øksendal & Sulem (2007), is given by:

$$\begin{aligned} \mathcal{L}V(x,i) \quad = \quad & \frac{\sigma_X^2}{2}x^2\frac{\partial^2 V(x,i)}{\partial x^2} + \frac{\sigma_I^2}{2}i^2\frac{\partial^2 V(x,i)}{\partial i^2} + (\mu_X - \lambda_X m_X)\,x\frac{\partial V(x,i)}{\partial x} \\ & + (\mu_I - \lambda_I m_I)\,i\frac{\partial V(x,i)}{\partial i} + \lambda_X\left[\mathrm{E}_{U^X}\left(V(x(1+U^X),i)\right) - V(x,i)\right] \\ & + \lambda_I\left[\mathrm{E}_{U^I}\left(V(x,i(1+U^I))\right) - V(x,i)\right]. \end{aligned} \tag{2.7}$$

Note that in this equation we use the notation E_{U^X} and E_{U^I} to emphasize the meaning of such expected values; for instance E_{U^I} means that we are computing the expected value with respect to the random variable U^I. Whenever the meaning is clear from the context, we simplify the notation and use E instead.

The HJB equation represents the two possible decisions that the firm has. According to Proposition 2.1, if the firm decides to invest when the current demand level is x and the investment cost is i, then its profit is $g(x,i) = (k_1 - k_0)Ax^\theta - i$. Consequently, if the value of the firm, V, is equal to the value that the firms obtains by investing, g, then the firm should invest right away. Otherwise, if the demand x and the investment cost i are such that $V(x,i) > g(x,i)$, then it means that for the firm is more profitable to postpone the investment decision.

As usual, we call the set $\mathcal{S} = \{(x,i) \in \mathbb{R}^+ \times \mathbb{R}^+ : V(x) = g(x,i)\}$ the *stopping* region, and $\mathcal{C} = \{(x,i) \in \mathbb{R}^+ \times \mathbb{R}^+ : V(x,i) > g(x,i)\}$ the *continuation* region. The sets $\mathcal{C}$ and $\mathcal{S}$ are complementary on $\mathbb{R}^+ \times \mathbb{R}^+$. We define

$$\tau^\star = \inf\left\{t \geq 0 : (X_t, I_t) \notin \mathcal{C}\right\},$$

which is the first time state variables corresponding to demand and investment cost are outside the continuation region. Note that (2.6) means that

$$\rho V(x,i) - \mathcal{L}V(x,i) \geq 0 \ \wedge\ V(x,i) \geq g(x,i),\ \forall\, (x,i) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

Moreover,

$$\rho V(x,i) - \mathcal{L}V(x,i) = 0 \ \wedge\ V(x,i) \geq g(x,i),\ \forall\, (x,i) \in \mathcal{C}$$

whereas

$$\rho g(x,i) - \mathcal{L}g(x,i) \geq 0 \ \wedge \ V(x,i) = g(x,i), \ \forall \ (x,i) \in \mathcal{S}.$$

The solution of the HJB equation, V, must satisfy the following initial condition,

$$V(0^+, i) = 0, \quad \forall i \in \mathbb{R}^+, \tag{2.8}$$

which reflects the fact that the value of the firm will be zero if the demand is also zero.

In the rest of the paper we assume that $\sigma_X > 0$ and $\sigma_I > 0$, and that the jump intensities λ_X and λ_I are both finite. Then the following fit conditions should hold (see Pham (1997), Cont & Tankov (2004) and Larbi & Kyprianou (2005)):

$$V(x,i) = g(x,i) \quad \text{and} \quad \nabla V(x,i) = \nabla g(x,i) \tag{2.9}$$

for $(x,i) \in \partial\mathcal{C}$, where $\partial\mathcal{C}$ denotes the boundary of $\mathcal{C}$, which we call *critical boundary*, and ∇ is the gradient operator. Therefore the solution of the problem is continuous at the critical boundary not only for itself but also for its derivatives. Note that the threshold is curve separating the two regions (the continuation and the stopping regions).

In order to solve the investment problem, we need to find the solution to the following partial differential equation:

$$\rho V(x,i) - \mathcal{L}V(x,i) = 0, \quad (x,i) \in \mathcal{C}. \tag{2.10}$$

and, simultaneously, to identify the continuation and the stopping regions.

Both questions are challenging: on the one hand the solution to (2.10) cannot be found explicitly, and on the other hand we need to guess the form of the continuation region and prove that the solution proposed verifies the HJB equation. Hereafter, we propose an alternative way to solve this problem, that circumvents the difficulties posed by the fact that we have two sources of uncertainty.

2.3.1 Splitting the state space

In this section we start by guessing the shape of the continuation region and later we will prove that our guess is indeed correct.

Our guess about the continuation region comes from intuitive arguments: high demand and low investment cost should encourage firms to invest, whereas in case of low demand and high investment cost it should be optimal to continue, i.e., postpone investment decision. See Figure 2.2 for the general plot of both regions.

We need to define precisely the boundary of $\mathcal{C}$ and for that we use the conditions derived from the HJB equation. We consider the set

$$U = \left\{(x,i) \in \mathbb{R}^+ \times \mathbb{R}^+ : \rho g(x,i) - \mathcal{L}g(x,i) < 0\right\} \tag{2.11}$$

which, taking into account the expression of $\mathcal{L}$ (the infinitesimal generator) and the g function, can be equivalently expressed (after some simple calculations) as

$$U = \left\{(x,i) \in \mathbb{R}^+ \times \mathbb{R}^+ : \frac{x^\theta}{i} < \frac{\rho - \mu_I}{(\kappa_1 - \kappa_0)\, Ah}\right\}. \tag{2.12}$$

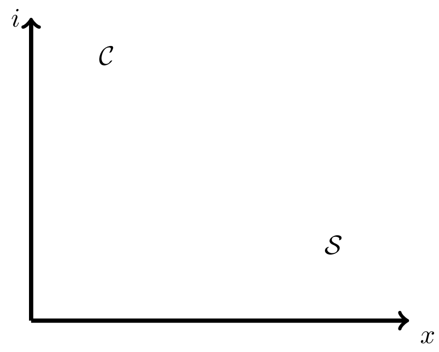


Figure 2.2: Space of process $(\boldsymbol{X}, \boldsymbol{I})$, $\mathcal{C}$ (continuation region) and $\mathcal{S}$ (stopping region).

It follows from Equation (2.12) that this set U depends on the state variables x and i through ratio $\frac{x^\theta}{i}$, meaning that it may be possible to reduce the problem with two state-variables to a one-state variable, using the transformation $Q_t = \frac{X_t^\theta}{I_t}$. We will come back to this remark in section 2.3.2.

Furthermore, by Propositions 3.3 and 3.4 of Øksendal & Sulem (2007), we know that $U \subseteq \mathcal{C}$. Further, if $U = \emptyset$ then $V(x,i) = g(x,i),\ \forall\ (x,i) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $\tau^\star = 0$. This case would mean that it is optimal to invest right away, and thus the problem is trivial. So we need to check under which conditions this trivial situation does not hold.

We have

$$\frac{\rho - \mu_I}{(\kappa_1 - \kappa_0)\, Ah} \leq 0$$

if and only if $\rho - \mu_I \leq 0$, as the denominator is positive; as $\frac{x^\theta}{i} > 0$, this would imply that $U = \emptyset$ and $\tau^\star = 0$. Therefore, if $\mu_I \geq \rho$, the firm should invest right away, which is coherent with a financial interpretation: if the drift of investment cost is higher than the discount rate, the rational decision is to invest as soon as possible.

To avoid such case, we assume the following:

Assumption 2.1. *The following condition on the parameters hold:*

$$\rho > \mu_I.$$

Next we guess that the boundaries of U and $\mathcal{C}$ have the same shape, and therefore $\mathcal{C}$ may be written as follows:

$$\mathcal{C} = \left\{ (x,i) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < \frac{x^\theta}{i} < q^\star \right\},$$

where $q^\star$ satisfies

$$q^\star \geq \frac{\rho - \mu_I}{(\kappa_1 - \kappa_0)\, Ah}. \tag{2.13}$$

In Figure 2.3 we represent the possible form of the boundary of the continuation region $\mathcal{C}$. In this figure, the thick line represents the boundary of the set U. As $U \subseteq \mathcal{C}$, then the dashed lines are possible boundaries of the continuation region $\mathcal{C}$.

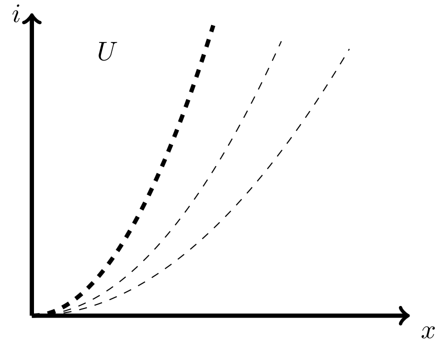


Figure 2.3: Set U (bold line) and some possible boundaries of the continuation region, $\mathcal{C}$.

2.3.2 Reducing to one source of uncertainty

According to the previous discussion about the splitting of the state space and in particular in view of the U set definition (see (2.11)), it seems clear that in fact we are reducing the problem from two-dimensional to one-dimensional. This is precisely the way Dixit & Pindyck (1994) (chapter 6.5) used to solve the optimization problem, when they discussed a problem with two sources of uncertainty both following a GBM. The same approach is followed by Murto (2007), who also proposes a change of variable that results from a combination of both uncertainty processes.

In our case the decision between continuing and stopping depends on demand level and investment cost only through a function of the two of them:

$$q = \frac{x^\theta}{i}.$$

and g function present in the performance criteria can also be written in terms of this new variable, as $g(x,i) = il(q)$, with

$$l(q) = (\kappa_1 - \kappa_0)\, Aq - 1. \tag{2.14}$$

Hence, in view of (2.14), we propose that

$$V(x,i) = if\,(q)\,,$$

where f is an appropriate function, that we still need to derive.

The HJB equation (2.6) can now be re-written with this new variable q and function f as follows:

$$min\,\{\rho f(q) - \mathcal{L}_Q f(q), f(q) - l(q)\} = 0, \quad \forall\, q \in \mathbb{R}^+, \tag{2.15}$$

where the infinitesimal generator of the process $\boldsymbol{Q} = \left\{Q_t = \frac{X_t^\theta}{I_t}, t \geq 0\right\}$ is

$$\begin{aligned}
\mathcal{L}_Q f(q) \quad = \quad & \frac{1}{2}\left(\theta^2\sigma_X^2 + \sigma_I^2\right) q^2 f''(q) \\
& + \left[\left(\mu_X + (\theta-1)\frac{\sigma_X^2}{2} - \lambda_X m_X\right)\theta - (\mu_I - \lambda_I m_I)\right] q f'(q) \\
& + \left[(\mu_I - \lambda_I m_I) - (\lambda_X + \lambda_I)\right] f(q) \\
& + \lambda_X \mathrm{E}\left[f\left(q\left(1+U^X\right)^\theta\right)\right] + \lambda_I \mathrm{E}\left[(1+U^I) f\left(q\left(1+U^I\right)^{-1}\right)\right],
\end{aligned} \tag{2.16}$$

This equation comes from $\mathcal{L}$ defined in (2.7), the relationship between V and f, and simple but extensive computations.

The corresponding continuation and stopping regions (hereby denoted by $\mathcal{C}_Q$ and $\mathcal{S}_Q$, respectively) can now be written as depending only on q:

$$\mathcal{C}_Q = \left\{q \in \mathbb{R}^+ : 0 < q < q^\star\right\} \text{ and } \mathcal{S}_Q = \left\{q \in \mathbb{R}^+ : q \geq q^\star\right\},$$

and thus $q^\star$ is the boundary between the continuation and the stopping regions, as it is usually the case in a problem with one state variable. We note that now we are in a standard case problem of investment with just one state variable (see, for instance, chapter 5 of Dixit & Pindyck (1994)) but with a different dynamics than the usual (the process $\boldsymbol{Q}$ is not a GBM).

Moreover, the following conditions should hold

$$f(0^+) = 0, \quad f(q^\star) = l(q^\star) \quad \text{and} \quad f'(q) = l'(q)|_{q=q^\star}. \tag{2.17}$$

Next, using derivations familiar with the standard case, we are able to derive the analytical solution to equation (2.15).

Proposition 2.2. *The solution of the HJB Equation* (2.15) *verifying the conditions* (2.17)*, hereby denoted by f, is given by*

$$f(q) = \begin{cases} \frac{1}{r_0-1}\left(\frac{q}{q^\star}\right)^{r_0} & 0 < q < q^\star \\ (\kappa_1 - \kappa_0)\, Aq - 1 & q \geq q^\star \end{cases} \tag{2.18}$$

where

$$q^\star = \frac{r_0}{(\kappa_1 - \kappa_0)\, A\, (r_0 - 1)} \tag{2.19}$$

and r_0 is the positive root of the function j:

$$\begin{aligned} j(r) \quad = \quad & \left[\frac{\sigma_X^2}{2}\theta^2 + \frac{\sigma_I^2}{2}\right] r^2 + \left[\left(\mu_X - \frac{\sigma_X^2}{2} - \lambda_X m_X\right)\theta - \left(\mu_I - \frac{\sigma_I^2}{2} - \lambda_I m_I\right)\right] r \\ & + (\mu_I - \lambda_I m_I) - (\lambda_X + \lambda_I) - \rho \\ & + \lambda_X \mathrm{E}\left[(1+U^X)^{r\theta}\right] + \lambda_I \mathrm{E}\left[(1+U^I)^{1-r}\right]. \end{aligned} \tag{2.20}$$

Proof of Proposition 2.2. *When $q \geq q^\star$ (corresponding to the stopping region), the value function is equal to l, the investment cost, defined on* (2.14)*, and thus the result is trivially proved. We then must prove the result for the continuation region, for which $q < q^\star$.*

First, we assume that indeed the solution of the left hand side of the HJB is $\zeta(q) = bq^{r_0}$, where b and r_0 are to be derived. We need to check that

$$\rho\zeta(q) - \mathcal{L}_Q\zeta(q) = 0,$$

where $\mathcal{L}_Q\zeta(q)$ is given by (2.16)*, with ζ instead of f. Computing $\zeta'(q)$ and $\zeta''(q)$ and plugging those expressions in:*

$$\mathcal{L}_Q\zeta(q) = bq^{r_0}(j(r_0) + \rho),$$

after some simple manipulations, we conclude that

$$\rho\zeta(q) - \mathcal{L}_Q\zeta(q) = 0 \Leftrightarrow bq^{r_0} j(r_0) = 0 \Leftrightarrow j(r_0) = 0,$$

where r_0 is a root of the function j defined in (2.20)*. Moreover, as $\zeta(0) = 0$ (from the condition* (2.17)*), it follows that r_0 needs to be positive.*

Next we show that j has one and only one positive root and thus r_0 is unique. The second order derivative of j is equal to

$$j''(r) = \sigma_X^2\theta^2 + \sigma_I^2 + \lambda_X\theta^2 \mathrm{E}\left[\left[\ln(1+U^X)\right]^2 (1+U^X)^{r\theta}\right] + \lambda_I \mathrm{E}\left[\left[\ln(1+U^I)\right]^2 (1+U^I)^{1-r}\right].$$

Given that volatilities and elasticity are positive, $U^X > -1$ a.s. and $U^I > -1$ a.s., it follows that $j''(r) > 0$, $\forall r \in \mathbb{R}$. Thus, j is a strictly convex function. Further, $lim_{r\to+\infty} j(r) = +\infty$ and $j(0) = \mu_I - \rho < 0$ (by assumption (2.21)), which means that $j(0) < 0$. Then, since j is continuous, it has an unique positive root.

It remains to derive b. For that we use conditions presented in (2.17), and straightforward calculations lead to:

$$b = \frac{1}{r_0 - 1}\left(\frac{1}{q^\star}\right)^{r_0},$$

where $q^\star$ must be given by (2.19). We note that as $q^\star$ is positive, as well as $k_1 - k_0$ and A, one must have $r_0 - 1 > 0$, i.e., $r_0 > 1$. Consequently $b > 0$.

The remainder of the proof is to check that (2.18) is the solution of the HJB Equation (2.15). For that we need

i) To prove that in the continuation region $f(q) \geq l(q)$.

In the continuation region $f(q) = \frac{1}{r_0-1}\left(\frac{q}{q^\star}\right)^{r_0}$. For that purpose, we define the function

$$\varphi(q) = \frac{1}{r_0 - 1}\left(\frac{1}{q^\star}\right)^{r_0} q^{r_0} - (\kappa_1 - \kappa_0) Aq + 1$$

with derivatives

$$\varphi'(q) = \frac{r_0}{r_0 - 1}\left(\frac{1}{q^\star}\right)^{r_0} q^{r_0-1} - (\kappa_1 - \kappa_0) A \quad \text{and} \quad \varphi''(q) = r_0\left(\frac{1}{q^\star}\right)^{r_0} q^{r_0-2}.$$

Since $\varphi''(q) > 0$, φ is a strictly convex function. As $\varphi'(q) = 0 \Leftrightarrow q = q^\star$, then $q^\star$ is the unique minimum of the function φ with $\varphi(q^\star) = 0$.

Summarizing, the function φ is continuous and strictly convex, has an unique minimum at $q^\star$ and $\varphi(q^\star) = 0$, then $\varphi(q) \geq 0, \forall q \in \mathbb{R}^+$. Therefore, $f(q) \geq l(q)$, for $\forall q \in \mathbb{R}$.

ii) To prove that in the stopping region $\rho f(q) - \mathcal{L}_Q f(q) \geq 0$.

In the stopping region $f(q) = l(q)$, then

$$\rho f(q) - \mathcal{L}_Q f(q) = (\kappa_1 - \kappa_0) Ahq + (\mu_I - \rho).$$

Given that $q \geq q^\star$ and using the inequality (2.13), we conclude that

$$\begin{aligned}\rho\zeta(q) - \mathcal{L}_Q\zeta(q) &\geq (\kappa_1 - \kappa_0) Ahq^\star + (\mu_I - \rho) \\ &\geq (\kappa_1 - \kappa_0) Ah\frac{\rho - \mu_I}{(\kappa_1 - \kappa_0) Ah} + \mu_I - \rho \\ &= 0.\end{aligned}$$

Therefore, we conclude that the function f is indeed the solution of the HJB Equation (2.15). ■

Along the previous proof, we had to assume that $r_0 > 1$, which implies that $j(1) < 0$. The following assumption is a consequence of this condition.

Assumption 2.2. *The following condition on the parameters hold:*

$$h > \sigma_I^2 \tag{2.21}$$

where h is defined in (2.3).

We note that this condition on h is more restrictive than $h > 0$. This is a more technical condition for which we cannot find an economical intuition.

2.3.3 Original problem

In this section we prove that the solution of the original problem can be obtained from the solution of the modified problem.

Corollary 2.1. *The solution of the HJB Equation* (2.6) *verifying the initial condition* (2.8) *and the boundary conditions* (2.9)*, hereby denoted by V, is given by*

$$V(x,i) = \begin{cases} \left[\frac{(\kappa_1-\kappa_0)Ax^\theta}{r_0}\right]^{r_0} \left(\frac{r_0-1}{i}\right)^{r_0-1} & 0 < \frac{x^\theta}{i} < q^\star \\ (\kappa_1-\kappa_0)\,Ax^\theta - i & \frac{x^\theta}{i} \geq q^\star \end{cases} \tag{2.22}$$

for $(x,i) \in \mathbb{R}^+ \times \mathbb{R}^+$, where $q^\star$ is defined on (2.19) *and r_0 is the positive root of the function j, defined on* (2.20).

Proof of Corollary 2.1. *Considering the relations of the functions g and f with the function V, it follows that the function V proposed in Equation (2.22) is the obvious candidate to be the solution of the HJB Equation* (2.6). *So next we prove that indeed this is the case. In the following we use some results already presented in proof of Proposition 2.2.*

Let $(x,i) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $q = \frac{x^\theta}{i}$. Using some basic calculations one proves that $\mathcal{L}_Q V(x,i) = i\mathcal{L}_Q f(q)$ and $\rho V(x,i) - \mathcal{L}V(x,i) = i\left[\rho f(q) - \mathcal{L}_Q f(q)\right]$.

Since i is positive, the signal of $\rho V(x,i) - \mathcal{L}V(x,i)$ is exactly the signal of $\rho f(q) - \mathcal{L}_Q f(q)$. By construction, $\rho f(q) - \mathcal{L}_Q f(q) = 0$ in the continuation region and $\rho f(q) - \mathcal{L}f(q) \geq 0$ in the stopping region. Therefore, we just need to prove that in the continuation region $V(x,i) - g(x,i) \geq 0$. This follows immediately because $V(x,i) - g(x,i) = i\left[f(q) - l(q)\right]$, i is positive and we previously proved that $f(q) \geq l(q)$.

Finally, from the initial and boundary conditions of the transformed problem (2.17)*, $V(0^+,i) = if(0^+) = 0$ and for $(x^\star, i^\star) \in \partial C$, i.e., $\frac{x^{\star\theta}}{i^\star} = q^\star$, we obtain the boundary conditions of the original problem* (2.9). ■

2.4 Comparative Statics

In this section we study the behaviour of the investment threshold as a function of the most relevant parameters, in particular the volatilities σ_X and σ_I, the intensity rates of the jumps, λ_X and λ_I, the magnitude of the jumps, m_X and m_I, and the time-to-build, n. In order to be able to derive analytical

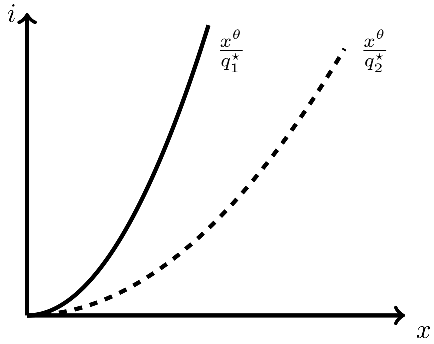


Figure 2.4: Assuming $q_1^\star < q_2^\star$, the full (respectively, dashed) line corresponds to the boundary of the continuation region for $q_1^\star$ (respectively, $q_2^\star$).

results, we assume that the magnitude of the jumps (both in demand and investment) are deterministic and equal to m_X and m_I, respectively [8].

Before we proceed with mathematical derivations, we comment on the meaning of having $q^\star$ monotonic with respect to a specific parameter, as we are dealing with a boundary in a two dimensions space.

For illustration purposes, we consider Figure 2.4. In this figure we are assuming that the threshold $q^\star$ increases with a certain parameter while keeping the others unchanged. In view of the relation $q = \frac{x^\theta}{i}$, where x is the demand and i is the investment cost, if $q^\star$ increases, it means that for any demand value x, the investment cost that triggers the investment decision decreases. Thus the continuation region is larger for higher values of $q^\star$ and, consequently, the decision to invest is postponed (in Figure 2.4 the continuation region corresponding to the smaller threshold function (in full line) is contained in the continuation region corresponding to the larger threshold function (in dashed line)).

In order to ease the presentation, we define the following vectors of parameters:
$\boldsymbol{\phi_X} = (\mu_X, \sigma_X, \lambda_X, m_X)$, $\boldsymbol{\phi_I} = (\mu_I, \sigma_I, \lambda_I, m_I)$, $\boldsymbol{\varsigma} = (n, \boldsymbol{\phi_X})$ and finally $\boldsymbol{\varrho} = (\boldsymbol{\phi_X}, \boldsymbol{\phi_I})$. Furthermore, to emphasize the dependency of j, r_0, h, A and $q^\star$ on the parameters, we introduce the following notation: $j(r, \boldsymbol{\varrho}), r_0(\boldsymbol{\varrho}), h(\boldsymbol{\phi_X}), A(\boldsymbol{\varsigma})$ and $q^\star(\boldsymbol{\varrho})$.

The first result about the monocity of the investment threshold regards the time-to-build n, and it is quite straitgthforward.

Proposition 2.3. *The investment threshold, $q^\star$, increases with the time-to-build, n.*

Proof of Proposition 2.3. *Straightforward calculus leads us to*

$$\frac{dq^\star(\boldsymbol{\varrho})}{dn} = \frac{n}{(\kappa_1 - \kappa_0)\, A(\boldsymbol{\varsigma})} \frac{r_0(\boldsymbol{\varrho})}{r_0(\boldsymbol{\varrho}) - 1},$$

which is always positive. ■

This result was expected: if the time-to-build is higher, than we need better conditions (i.e., lower investment cost or higher demand, which implies higher $q^\star$) to decide to invest.

To prove the behaviour of the investment threshold with respect to the parameters in $\boldsymbol{\varrho}$, we start by proving the following lemma:

[8]Numerical results suggest that the effect of the distribution of jumps is not relevant on investment thresholds.

Lemma 2.2. *The investment threshold changes with the parameters in $\boldsymbol{\varrho}$ according to the following relation:*

$$\frac{dq^\star(\boldsymbol{\varrho})}{dy} = \frac{1}{\Delta(\kappa_1-\kappa_0)A(\boldsymbol{\varsigma})(r_0(\boldsymbol{\varrho})-1)^2}\times$$
$$\begin{cases} \frac{\Delta r_0(\boldsymbol{\varrho})(r_0(\boldsymbol{\varrho})-1)(nh(\boldsymbol{\phi_X})+1)}{h(\boldsymbol{\phi_X})}\frac{\partial h(\boldsymbol{\phi_X})}{\partial y} + \left.\frac{\partial j(r,\boldsymbol{\varrho})}{\partial y}\right|_{r=r_0} & \text{if } y\in\boldsymbol{\phi_X} \\ \left.\frac{\partial j(r,\boldsymbol{\varrho})}{\partial y}\right|_{r=r_0} & \text{if } y\in\boldsymbol{\phi_I} \end{cases}$$

with $\Delta = \left.\frac{\partial j(r,\boldsymbol{\varrho})}{\partial r}\right|_{r=r_0}$. Furthermore, as $\Delta>0$, then the sign of the derivative of $q^\star$ depends only on the derivatives of h and j with respect to the parameter.

Proof of Lemma 2.2. *As*

$$\frac{dq^\star(\boldsymbol{\varrho})}{dy} = \frac{1}{(\kappa_1-\kappa_0)A(\boldsymbol{\varsigma})(r_0(\boldsymbol{\varrho})-1)^2}\times$$
$$\begin{cases} \frac{r_0(\boldsymbol{\varrho})(r_0(\boldsymbol{\varrho})-1)(nh(\boldsymbol{\phi_X})+1)}{h(\boldsymbol{\phi_X})}\frac{\partial h(\boldsymbol{\phi_X})}{\partial y} - \frac{\partial r_0(\boldsymbol{\varrho})}{\partial y} & \text{if } y\in\boldsymbol{\phi_X} \\ -\frac{\partial r_0(\boldsymbol{\varrho})}{\partial y} & \text{if } y\in\boldsymbol{\phi_I} \end{cases}$$

the result follows from the implicit derivative relation $\frac{dr_0(\boldsymbol{\varrho})}{dy} = -\left.\frac{\frac{\partial j(r,\boldsymbol{\varrho})}{\partial y}}{\frac{\partial j(r,\boldsymbol{\varrho})}{\partial r}}\right|_{r=r_0}$.

It remains to prove that $\Delta>0$. As a function only of r, j is a continuous and strictly convex function, with $lim_{r\to+\infty}j(r)=+\infty$, $j(0)<0$ and $j(1)<0$, as we have already stated. Since r_0 is its positive root, in view of the properties of j, it follows that it must be increasing in a neighborhood of r_0, and thus $\Delta>0$. ■

In the following sections, we study the sign of each derivative, whenever possible. Our goal is to inspect if $q^\star$ increases or decreases with each parameter in $\boldsymbol{\phi_X}$ and $\boldsymbol{\phi_I}$.

2.4.1 Comparative statics for the parameters regarding the demand

In this section we state and prove the results concerning the behaviour of $q^\star$ as a function of demand parameters.

We note that when all parameters are maintained, except one, one still needs to verify the assumption on the parameters, namely Assumption 2.2. The following Lemma provides certain bounds on the parameters resulting from this assumption.

Lemma 2.3. *In view of the Assumption 2.2 and considering $m_X<0$, the following restrictions on the demand parameters hold.*

The drift is upper bounded, i.e.,

$$\mu_X < \frac{1}{\theta}\left[\rho - \left((\theta-1)\frac{\sigma_X^2}{2} - \lambda_X m_X\right)\theta - \lambda_X\left((1+m_X)^\theta - 1\right) - \sigma_I^2\right].$$

The domain of the volatility parameter depends on θ. If $0<\theta<1$, σ_X is lower bounded, i.e. $\sigma_X>B$; otherwise it is upper bounded, i.e. $0<\sigma_X<B$, where

$$B = \sqrt{\max\left\{0, \frac{2}{\theta(\theta-1)}\left[\rho - (\mu_X-\lambda_X m_X)\theta - \lambda_X\left((1+m_X)^\theta-1\right) - \sigma_I^2\right]\right\}}.$$

Relatively to the rate of the jumps, the domain also depends on θ. If $0 < \theta < 1$, λ_X is lower bounded, i.e. $\lambda_X > C$; otherwise it is upper bounded, i.e. $0 < \lambda_X < C$, where

$$C = \max\left\{0, \frac{\rho - \left(\mu_X + (\theta-1)\frac{\sigma_X^2}{2}\right)\theta - \sigma_I^2}{(1+m_X)^\theta - (\theta m_X + 1)}\right\}.$$

Regarding jump sizes, let ξ be the negative solution of the equation

$$(1+x)^\theta - (\theta x + 1) = \frac{\rho - \left(\mu_X + (\theta-1)\frac{\sigma_X^2}{2}\right)\theta - \sigma_I^2}{\lambda_X},$$

when there is solution. Also, let

$$D = \frac{\rho - \left(\mu_X + (\theta-1)\frac{\sigma_X^2}{2}\right)\theta - \sigma_I^2}{\lambda_X}.$$

Then the following bounds on m_X should hold:

$$m_X \in \begin{cases} (-1, \xi) & \text{if } 0 < \theta < 1 \text{ and } \theta - 1 < D < 0 \\ (\xi, 0) & \text{if } \theta > 1 \text{ and } 0 < D < \theta - 1 \\ (-1, 0) & \text{otherwise} \end{cases}.$$

Proof of Lemma 2.3. *The sets of admissible values for the parameters μ_X and σ_X come from straightforward manipulation of Assumption 2.2.*

The results regarding jump parameters (λ_X and m_X) rely on the properties of the function

$$\Upsilon_\alpha(x) = \Upsilon_{\alpha,2}(x) - \Upsilon_{\alpha,1}(x), \tag{2.23}$$

with $\Upsilon_{\alpha,1}(x) = \alpha x + 1$ and $\Upsilon_{\alpha,2}(x) = (1+x)^\alpha$, with $x > -1$ and $\alpha > 0$. A representation of function $\Upsilon_{\alpha,1}$ is a tangent line of the representation of function $\Upsilon_{\alpha,2}$ in point $(0,1)$. Also, as $\Upsilon''_{\alpha,2}(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$, function $\Upsilon_{\alpha,2}$ is convex if $\alpha > 1$ and it is concave otherwise. Then, for $x > -1$, $\Upsilon_{\alpha,2}(x) > \Upsilon_{\alpha,1}(x)$ if $\alpha > 1$ and $\Upsilon_{\alpha,1}(x) \geq \Upsilon_{\alpha,2}(x)$ if $0 < \alpha \leq 1$. Then it follows that $\Upsilon_\alpha(x) > 0$ if $\alpha > 1$ and $\Upsilon_\alpha(x) \leq 0$ if $0 < \alpha \leq 1$.

From Assumption 2.2,

$$\lambda_X \Upsilon_\theta(m_X) < \rho - \left(\mu_X + (\theta-1)\frac{\sigma_X^2}{2}\right)\theta - \sigma_I^2$$

and therefore the restriction on λ_X follows.

Proceeding with the analysis with respect to the jump size, m_X, we start by noting that $\Upsilon_\alpha(-1) = \alpha - 1$, $\Upsilon_\alpha(0) = 0$ and $\lim_{x\to\infty} \Upsilon_\alpha(x) = +\infty\chi_{\{\alpha>1\}} - \infty\chi_{\{\alpha<1\}}$. From these properties we know the shape of the function Υ_α for each α. Assumption 2.2 is equivalent to have $\Upsilon_\theta(m_X) < D$.

For $0 < \theta < 1$, given that $m_X \in (-1, 0)$, if $D < \theta - 1$ there are not admissible values for the condition $\Upsilon_\theta(m_X) < D$. At the opposite side, due to the shape of the function $\Upsilon_\theta(m_X)$, if $D > 0$ the domain $(-1, 0)$ is not changed. Likewise, for $\theta > 1$, if $D > \theta - 1$ the domain of m_X does not change and if $D < 0$ the inequality $\Upsilon_\theta(m_X) < D$ is impossible. Therefore, standard study of function $\Upsilon_\theta(m_X)$ leads us to the results presented in the Lemma. ■

Now we are in position to state the main properties of $q^\star$ as a function of the demand parameters. From Lemma 2.2, we can only get straightforward results about the behavior of $q^\star$ as a function of the parameters if $\frac{\partial h(\boldsymbol{\phi_X})}{\partial y}$ and $\left.\frac{\partial j(r,\boldsymbol{\varrho})}{\partial y}\right|_{r=r_0}$ have the same sign. These derivatives only have the same sign for a specific range of values for θ, and thus we are not able to present a comprehensive study for all possible situations.

Proposition 2.4. *Considering the restrictions presented on Lemma 2.3, for* $\frac{1}{r_0(\boldsymbol{\varrho})} \leq \theta \leq 1$*, the investment threshold* $q^\star$*:*

- *increases with demand volatility,* σ_X *and jump intensity,* λ_X*;*
- *decreases with respect to the magnitude of jumps on demand,* m_X*.*

Proof of Proposition 2.4. *The proof for* σ_X *is straightforward knowing the derivatives*

$$\frac{\partial h(\boldsymbol{\phi_X})}{\partial \sigma_X} = -\theta(\theta-1)\sigma_X \quad \text{and} \quad \left.\frac{\partial j(r,\boldsymbol{\varrho})}{\partial \sigma_X}\right|_{r=r_0} = \theta r_0(\boldsymbol{\varrho})(\theta r_0(\boldsymbol{\varrho})-1)\sigma_X.$$

Continuing with λ_X*, we calculate the derivatives*

$$\begin{aligned} \frac{\partial h(\boldsymbol{\phi_X})}{\partial \lambda_X} &= \theta m_X + 1 - (1+m_X)^\theta = -\Upsilon_\theta(m_X) \\ \left.\frac{\partial j(r,\boldsymbol{\varrho})}{\partial \lambda_X}\right|_{r=r_0} &= (1+m_X)^{\theta r_0(\boldsymbol{\varrho})} - (\theta r_0(\boldsymbol{\varrho}) m_X + 1) = \Upsilon_{\theta r_0(\boldsymbol{\varrho})}(m_X), \end{aligned}$$

with Υ_α *given by* (2.23)*. Taking into account the properties of such function, we conclude the result for* λ_X*. Finally, for* m_X *we also calculate the derivatives*

$$\begin{aligned} \frac{\partial h(\boldsymbol{\phi_X})}{\partial m_X} &= \theta\lambda_X\left[1-(1+m_X)^{\theta-1}\right] = \theta\lambda_X\Phi_\theta(m_X) \\ \left.\frac{\partial j(r,\boldsymbol{\varrho})}{\partial m_X}\right|_{r=r_0} &= -\theta r_0(\boldsymbol{\varrho})\lambda_X\left[1-(1+m_X)^{\theta r_0(\boldsymbol{\varrho})-1}\right] = -\theta r_0(\boldsymbol{\varrho})\lambda_X\Phi_{\theta r_0(\boldsymbol{\varrho})}(m_X) \end{aligned}$$

where

$$\Phi_\alpha(x) = 1-(1+x)^\alpha, \tag{2.24}$$

with $x > -1$*. As* $\Phi'_\alpha(x) = -\alpha(1+x)^{\alpha-1}$*,* Φ_α *is decreasing if* $\alpha > 0$*, constant and equal to zero if* $\alpha = 0$*, and increasing if* $\alpha < 0$*. Moreover, as* $\Phi_\alpha(0) = 0$*, then:*

- *if* $\alpha > 0$*,* Φ_α *is positive when* $-1 < x < 0$ *and it is negative for* $x > 0$*;*
- *if* $\alpha < 0$*,* Φ_α *is negative when* $-1 < x < 0$ *and positive for* $x > 0$*.*

Given these results, the proof regarding m_X *is concluded.* ■

It follows from Lemma 2.1 that

$$Var[X_t] = x_0 e^{2\mu_X t}\left(e^{(\lambda_X m_X^2 + \sigma_X^2)t} - 1\right).$$

Looking closer at the variance of the demand reveals that $Var[X_t]$ increases with λ_X and σ_X and decreases with m_X (as $m_X \in [-1, 0]$). Standard results from real options suggest that higher volatility postpones investment decision, which is precisely the result we stated and proved in Proposition 2.4. Therefore, the results obtained in the comparative statics are in fact in line with standard results from real options.

Some papers analyse the effect of time-to-build on the timing of investment, such as Majd & Pindyck (1987), Milne & Whalley (2000) and Bar–Ilan *et al.* (2002). The latter proves that for very short construction times, uncertainty delays investment; on the other hand, if we consider a more conservative number of years for construction, the effect is the opposite. Indeed, Proposition 2.5 shows that our model is in line with these findings, if we also follow a convex revenue function (i.e. $\theta > 1$) as proposed by those authors.

Proposition 2.5. *For $\theta > 1$ and $\sigma_X < B$ (where B is defined at Lemma 2.3), the investment threshold, $q^\star$, for lower (respectively, higher) values of time-to-build, n, increases (respectively, decreases) with the demand volatility, σ_X.*

Proof of Proposition 2.5. *From Lemma 2.2*

$$\begin{aligned}\frac{dq^\star(\boldsymbol{\varrho})}{d\sigma_X} &= \frac{\theta r_0(\boldsymbol{\varrho})\sigma_X}{\Delta(\kappa_1-\kappa_0)A(\boldsymbol{\varsigma})(r_0(\boldsymbol{\varrho})-1)^2 h(\boldsymbol{\phi_X})}\times \\ &\quad \left[(\theta r_0(\boldsymbol{\varrho})-1)h(\boldsymbol{\phi_X})-\Delta(r_0(\boldsymbol{\varrho})-1)(nh(\boldsymbol{\phi_X})+1)(\theta-1)\right],\end{aligned}$$

which it is positive if

$$0 \leq n < \frac{(\theta r_0(\boldsymbol{\varrho})-1)}{\Delta(r_0(\boldsymbol{\varrho})-1)(\theta-1)} - \frac{1}{h(\boldsymbol{\phi_X})}$$

and it is negative for the remaining positive values of n. ■

Although we are not presenting numerical results in the paper, we did perform some experiments to check the behavior of $q^\star$ in respect of the drift. These experiments showed that the investment threshold is not monotonic with the demand's drift, μ_X.

2.4.2 Comparative statics for the parameters regarding investment cost

Now we state and prove the results concerning the behavior of $q^\star$ as a function of the investment cost parameters. In view of Lemma 2.2, we just need to assess the sign of the derivative of j in respect of each parameter.

Proposition 2.6. *For values of investment cost parameters such that Assumptions 2.1 and 2.2 hold and $m_I > 0$, investment threshold $q^\star$*

- *decreases with investment cost drift μ_I;*
- *increases with investment cost volatility, σ_I, jumps intensity, λ_I, and magnitude of the jumps in the intensity, m_I.*

Proof of Proposition 2.6. *The proof for μ_I and σ_I is trivial and for that reason we omit it. In order to prove the result for λ_I, we study the behaviour of the following function:*

$$\Theta(r) = \frac{\partial j(r,\boldsymbol{\varrho})}{\partial \lambda_I} = m_I(r-1) + (1+m_I)^{1-r} - 1.$$

We compute the second derivative,

$$\Theta''(r) = [\ln(1+m_I)]^2 (1+m_I)^{1-r} > 0,$$

whence Θ is a strictly convex function. Furthermore, $\Theta(0) = \Theta(1) = 0$, which leads us to conclude that $\Theta(r) > 0$, for $r > 1$. Therefore, $\left.\frac{\partial j(r,\boldsymbol{\varrho})}{\partial \lambda_I}\right|_{r=r_0} = \Theta(r_0(\boldsymbol{\varrho})) > 0$, and the result follows.

Finally, in order to study the behavior of $q^\star$ in respect of m_I, we need to assess the sign of the following derivative:

$$\left.\frac{\partial j(r,\boldsymbol{\varrho})}{\partial m_I}\right|_{r=r_0} = \lambda_I(r_0(\boldsymbol{\varrho})-1)\left[1-(1+m_I)^{-r_0(\boldsymbol{\varrho})}\right].$$

Taking into account Φ_α function defined on (2.24), we notice that $\left.\frac{\partial j(r,\boldsymbol{\varrho})}{\partial m_I}\right|_{r=r_0} = \lambda_I(r_0(\boldsymbol{\varrho})-1)\Phi_{-r_0(\boldsymbol{\varrho})}(m_X)$. As $r_0(\boldsymbol{\varrho}) > 1$ and in view of the properties of function Φ_α, the result follows. ∎

As jumps in investment cost are positive and the expected investment cost increases with the drift μ_I, a firm with a larger drift anticipates its investment decision, in order to avoid larger investment cost.

Regarding the remainder investment parameters the same reasoning presented for Proposition 2.4 applies, taking now into account that $m_I > 0$ and

$$Var[I_t] = i_0 e^{2\mu_I t}\left(e^{(\lambda_I m_I^2+\sigma_I^2)t}-1\right).$$

2.5 Conclusion

In this paper we propose a way to derive analytically the solution to a decision problem with two uncertainties, both following jump-diffusion processes. This method relies on a change of variable that reduces the problem from a two dimensions problem to a one dimension, which we can solve analytically.

Moreover, we also present an extensive comparative statics for the most relevant parameters. It is interesting to note that the behaviour of investment threshold changes with volatility and jump intensity in the same way as in respect demand and investment cost. But when we study the monotonicity of the threshold in respect of the drift and expected value of the jumps, the results are not qualitatively the same. In particular they decrease with the investment drift but they are not monotonic when the drift of the demand changes.

For some parameters the result is as expected, and similar to the one discussed in earlier works (like the behaviour of the investment threshold with volatility of both the demand and the investment cost), whereas in other cases the results are not so straightforward. For instance, the impact of jumps of demand depend on analytical conditions whose economical interpretation is far from being obvious.

Appendix 2.A Proofs

2.A.1 Proof of Lemma 2.1

Applying expectation and re-writting Y_{t+s}, we have

$$\begin{aligned}\mathrm{E}\left[Y_{t+s}^k \middle| Y_t\right] =& Y_t^k \exp\left[\left(\mu-\frac{\sigma^2}{2}-\lambda m\right)ks\right]\times \\ & \mathrm{E}\left[\exp\left[\sigma k\left(W_{t+s}-W_t\right)\right]\prod_{i=N_t+1}^{N_{t+s}}(1+U_i)^k\middle| Y_t\right].\end{aligned}$$

Using the fact that the Brownian motion and the Poisson process have independent and stationary increments and that the processes are independent, we obtain

$$\mathrm{E}\left[Y_{t+s}^k \middle| Y_t\right] = Y_t^k \exp\left[\left(\mu - \frac{\sigma^2}{2} - \lambda m\right) ks\right] \mathrm{E}\left[\exp\left(\sigma k W_s\right)\right] \mathrm{E}\left[\prod_{i=1}^{N_s} (1+U_i)^k\right]. \tag{2.25}$$

As $\{U_i\}_{i\in\mathbb{N}} \overset{i.i.d.}{\sim} U$, then using the probability generating function of a Poisson distribution and the tower property of the conditional expectation, we conclude that

$$\mathrm{E}\left[\prod_{i=1}^{N_s} (1+U_i)^k\right] = \exp\left[\lambda s \left(\mathrm{E}\left[(1+U)^k\right] - 1\right)\right]. \tag{2.26}$$

The result follows using the moment generating function of the Normal distribution, and replacing (2.26) in (2.25). ■

3

Particular solution to the second order Euler-Cauchy equation with Log non-homogeneities

Contents

3.1 Introduction

Consider the following problem

$$y(t) = \sup_{\tau} E\left[e^{-r\tau} g(X_\tau) | X_0 = t\right], \tag{3.1}$$

where g is such that the problem presented in (3.1) is well-defined (in the sense that it is not trivial), and $\boldsymbol{X} = \{X_s : s > 0\}$ being a geometric Brownian motion (GBM) with jumps driven by a homogeneous Poisson process, with intensity $\lambda > 0$, and the continuous part follows a GBM with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$, i.e.

$$X_s = x_0 e^{(\mu - \frac{\sigma^2}{2} - \lambda k)s + \sigma W_t}(1+k)^{N_s}, \quad s > 0,$$

where N_s follows a Poisson distribution with parameter λs, and accounts for the number of jumps that have occurred until time s, $\boldsymbol{W} = \{W_s : s > 0\}$ is a Brownian motion, and x_0 is the initial state of process $\boldsymbol{X}$. Here we assume that the jumps are multiplicative of magnitude k.

The problem (3.1) is an optimal stopping problem, with applications in many fields. For instance, in financial options context, it can be seen as the price of a perpetual American call option (see Merton (1976)). In the Real options literature, it is known as an investment problem (see Dixit & Pindyck (1994)).

In particular, in the context of Real options, the stochastic process $\boldsymbol{X}$ can represent the demand, τ is the time at which the firm decides to invest, and g is the profit of the firm upon investment. In this paper, we consider the case that g is an iso-elastic demand function (Nunes & Pimentel (2017)), so that $g(t) \propto t^\alpha$, with α denoting the elasticity parameter with respect to the process $\boldsymbol{X}$. For simplicity and without loss of generality, we assume that $g(t) = t^\alpha$.

In this framework $y(t)$ represents the value of a firm that optimizes its investment decision (meaning that it maximizes its expected value) when the current value of the process $\boldsymbol{X}$ is t. Standard arguments from real options suggest that in case g is a non-decreasing function, the firm does not undertake the investment as long as the level of uncertainty is below a threshold w^*. Therefore, the optimal stopping time, $\tau^\star$, is such that $\tau^\star = \inf\{s > 0 : X_s \geq w^\star\}$. For obvious reasons, the set $(0, w^\star)$ is usually called the *continuation region*, whereas $[w^\star, +\infty)$ is the *stopping region*. Moreover, in the continuation region the value of the firm is a solution of the following equation

$$\frac{1}{2}\sigma^2 t^2 y''(t) + \mu t y'(t) - (r+\lambda) y(t) = -\lambda G(t(1+k)), \tag{3.2}$$

where G is a function that changes along the domain.

The equation (3.2) is an Euler-Cauchy equation of order two. This equation appears in a wide range of problems, notably in sorting and searching algorithms (Chern *et al.* (2002)), physics and engineering applications (Chen & Wang (2002)).

The solution of the homogeneous Euler-Cauchy equation is well-known. This solution, hereby denoted by y_h, strongly depends on the roots of the characteristic polynomial

$$Q(\beta) = \beta\,(\beta - 1) + a\beta + b. \tag{3.3}$$

The solution is either $y_h(t) = \delta_1 t^{\beta_1} + \delta_2 t^{\beta_2}$, $y_h(t) = t^{\beta}\left(\delta_1 + \delta_2 \ln t\right)$ or $y_h(t) = t^{\lambda}\left[\delta_1 \cos(\mu \ln t) + \delta_2 \sin(\mu \ln t)\right]$, depending if Q has two distinct real roots (β_1 and β_2), one double real root (β) or two complex roots ($\lambda \pm \mu i$), respectively (see Boyce & DiPrima (2008); Kohler & Johnson (2006)).

In this paper, we are concerned with the non-homogeneous part of (3.2) when G has a certain form, as we motivate next.

As the sample path of $\boldsymbol{X}$ exhibits jumps, we need to take into account first the case $t \in \left[\frac{w^*}{1+k}, w^\star\right)$, as with infinitesimal probability λdt, in the next time the state process will be in the stopping region. Therefore, for values in this interval

$$G(t(1+k)) = g(t(1+k)).$$

Then one needs to solve the following differential equation

$$\frac{1}{2}\sigma^2 t^2 y''(t) + \mu t y'(t) - (r+\lambda) y(t) = -\lambda(1+k)^{\alpha} t^{\alpha}.$$

The solution to this equation can be found using Theorem 3.5 of Sabuwala & Leon (2011), leading to the following expression

$$y(t) = \delta_1 t^{\beta_1} + \delta_2 t^{\beta_2} + \delta_3 t^{\alpha} (\log t)^n,$$

where n is the multiplicity of α as a root of the characteristic polynomial Q, and β_1 and β_2 are the (distint and real) roots of (3.3). So for this case, the results provided in the literature are enough, and one can solve the problem to derive the value of the firm at any point $t \in \left[\frac{w^*}{1+k}, w^\star\right)$.

When $t \in \left[\frac{w^*}{(1+k)^2}, \frac{w^*}{1+k}\right)$, with infinitesimal probability λdt, the process will be $\left[\frac{w^*}{1+k}, w^\star\right)$ in the next time, where G is equal to y. Therefore $G(t(1+k)) = a(1+k)^{\beta_1} t^{\beta_1} + b(1+k)^{\beta_2} t^{\beta_2} + c(1+k)^{\alpha} t^{\alpha} (\log(1+k) + \log t)^n$, and thus in this region the value function must be solution of the following differential equation

$$\begin{aligned} \frac{1}{2}\sigma^2 t^2 y''(t) + \mu t y'(t) - (r+\lambda) y(t) &= -\lambda\left[a(1+k)^{\beta_1} t^{\beta_1} + b(1+k)^{\beta_2} t^{\beta_2}\right. \\ &\quad \left. + c(1+k)^{\alpha} t^{\alpha}\left(\log(1+k) + \log t\right)^n\right]. \end{aligned} \tag{3.4}$$

In order to find the solution of (3.4), by the superposition principle, it only remains to solve the following type of equation

$$t^2 y''(x) + a t y'(t) + b y(t) = A t^{\alpha} (\log t)^n. \tag{3.5}$$

This equation is an extension of the one solved in Sabuwala & Leon (2011), because here we have a log non-homogeneity term. We note that when $n = 0$, the previous equation is the same as the one considered by Sabuwala & Leon (2011).

In this paper we address the derivation of the particular solution, hereby denoted by y_p, such that

$$y(t) = y_h(t) + y_p(t).$$

The rest of the paper is organized as follows. In Section 2 we present the solution to (3.5). In Section 3 we provide some examples and Section 4 concludes the paper.

3.2 The solution

We start deriving a recursive expression for the solution. Later, using this result, we will be able to present the non-recursive solution.

Theorem 3.1 (recursive)**.** *Consider the second order Euler-Cauchy equation presented in* (3.5)*, with the corresponding characteristic polynomial Q given by* (3.3)*.*

- *If α is not a root of Q, the particular solution is*

$$y_p(t) = t^\alpha \sum_{i=0}^{n} c_i (\log t)^i ,$$

where $c_n = \frac{A}{Q(\alpha)}$, $c_{n-1} = -nA\frac{Q'(\alpha)}{Q(\alpha)^2}$ and $c_i = -\frac{i+1}{Q(\alpha)}\left[Q'(\alpha)c_{i+1} + (i+2)c_{i+2}\right]$ for $i = 0, 1, 2, ..., n-2$.

- *If α is a root of Q with multiplicity one, the particular solution is*

$$y_p(t) = t^\alpha \sum_{i=0}^{n} c_i (\log t)^{i+1} ,$$

where $c_n = \frac{A}{(n+1)Q'(\alpha)}$ and $c_i = -\frac{i+2}{Q'(\alpha)}c_{i+1}$, for $i = 0, 1, 2, ..., n-1$.

- *If α is a root of Q with multiplicity two, the particular solution is*

$$y_p(t) = t^\alpha c_n (\log t)^{n+2} ,$$

where $c_n = \frac{A}{(n+1)(n+2)}$.

Proof of Theorem 3.1. *We start by proposing that the particular solution of equation* (3.5) *is of the form $y_p(t) = t^\alpha P(t)$. Calculating first and second derivatives, we obtain $y_p'(t) = t^{\alpha-1}\left[tP'(t) + \alpha P(t)\right]$ and $y_p''(t) = t^{\alpha-2}\left[t^2P''(t) + 2\alpha tP'(t) + \alpha(\alpha-1)P(t)\right]$. Then*

$$t^2y_p''(t) + aty_p'(t) + by_p(t) \quad = \quad t^\alpha\left[t^2P''(t) + \left(Q'(\alpha)+1\right)tP'(t) + Q(\alpha)P(t)\right].$$

Thus $P(t)$ is such that

$$t^2P''(t) + \left(Q'(\alpha)+1\right)tP'(t) + Q(\alpha)P(t) = A(\log t)^n.$$

Taking into account whether $Q(\alpha)$ is null or not, we end up with different cases, described hereafter.

1. *If α is not a root of Q, we propose that $P(t) = \sum_{i=0}^{n} c_i (\log t)^i$.*
   Calculating first and second derivatives we obtain $P'(t) = \frac{1}{t}\sum_{i=1}^{n} ic_i (\log t)^{i-1}$ and $P''(t) = \frac{1}{t^2}\left[\sum_{i=2}^{n} i(i-1)c_i(\log t)^{i-2} - \sum_{i=1}^{n} ic_i(\log t)^{i-1}\right]$, from where $t^2P''(t)+\left(Q'(\alpha)+1\right)tP'(t)+Q(\alpha)P(t)$ is given by

$$\sum_{i=0}^{n-2}\left[(i+2)(i+1)c_{i+2} + Q'(\alpha)(i+1)c_{i+1} + Q(\alpha)c_i\right](\log t)^i$$
$$+\left[Q'(\alpha)nc_n + Q(\alpha)c_{n-1}\right](\log t)^{n-1} + Q(\alpha)c_n(\log t)^n.$$

In order to have (5.23) *we need to enforce that* $Q(\alpha)c_n = A$, $Q'(\alpha)nc_n + Q(\alpha)c_{n-1} = 0$ *and* $(i+2)(i+1)c_{i+2} + Q'(\alpha)(i+1)c_{i+1} + Q(\alpha)c_i = 0$, *for* $i = 0, 1, ..., n-2$, *which leads us to the conclusion that* $c_n = \frac{A}{Q(\alpha)}$, $c_{n-1} = -\frac{n\, Q'(\alpha)\, A}{Q(\alpha)^2}$ *and* $c_i = -\frac{i+1}{Q(\alpha)}\left[Q'(\alpha)c_{i+1} + (i+2)c_{i+2}\right]$, *for* $i = 0, 1, ..., n-2$.

2. *If* α *is a root of* Q *with multiplicity one, we propose that* $P(t) = \sum_{i=0}^{n} c_i\,(\log t)^{i+1}$.
   Calculating first and second derivatives we obtain $P'(t) = \frac{1}{t}\sum_{i=0}^{n}(i+1)c_i\,(\log t)^i$ *and* $P''(t) = \frac{1}{t^2}\left[\sum_{i=1}^{n}(i+1)ic_i(\log t)^{i-1} - \sum_{i=0}^{n}(i+1)c_i(\log t)^i\right]$. *Given that* $Q(\alpha) = 0$, *then* $t^2P''(t) + (Q'(\alpha)+1)\,tP'(t) + Q(\alpha)P(t)$ *is given by*
   $$\sum_{i=0}^{n-1}\left[(i+2)c_{i+1} + Q'(\alpha)c_i\right](\log t)^i + Q'(\alpha)(n+1)c_n(\log t)^n.$$
   Assuming that α *has multiplicity one we have* $Q'(\alpha) \neq 0$. *Thus in order to have* (5.23), *we need to enforce that* $Q'(\alpha)(n+1)c_n = A$ *and* $(i+2)c_{i+1} + Q'(\alpha)c_i = 0$, *for* $i = 0, 1, ..., n-1$, *which leads us to the conclusion that* $c_n = \frac{A}{(n+1)Q'(\alpha)}$ *and* $c_i = -\frac{i+2}{Q'(\alpha)}c_{i+1}$, *for* $i = 0, 1, ..., n-1$.

3. *If* α *is a root of* Q *with multiplicity two, we propose that* $P(t) = c_n\,(\log t)^{n+2}$.
   Calculating first and second derivatives we obtain $P'(t) = \frac{1}{t}c_n(n+2)\,(\log t)^{n+1}$ *and* $P''(t) = \frac{1}{t^2}c_n(n+2)\left[(n+1)(\log t)^n - (\log t)^{n+1}\right]$. *Since* $Q(\alpha) = 0$ *and* $Q'(\alpha) = 0$, *then* $t^2P''(t) + (Q'(\alpha)+1)\,tP'(t) + Q(\alpha)P(t)$ *is given by*
   $$c_n(n+2)(n+1)(\log t)^n.$$
   Finally, in order to have (5.23) *we conclude that* $c_n = \frac{A}{(n+1)(n+2)}$.

■

The previous theorem provides a particular solution of Equation (3.5), where the coefficients are computed in backwards recursion. Next, we present the particular solution of the problem, this time providing explicit formulas for the involved coefficients.

Theorem 3.2 (non-recursive)**.** *Consider the second order Euler-Cauchy equation presented in* (3.5), *with the corresponding characteristic polynomial* Q *given by (3.3).*

- *If* α *is not a root of* Q, *the particular solution is* $y_p(t) = t^\alpha \sum_{i=0}^{n} c_i\,(\log t)^i$, *with*
  $$c_i = (-1)^{n-i}\,\frac{n!}{i!}\,\frac{A}{Q(\alpha)^{n-i+1}}\,\sum_{\substack{j=0\\ j\in\mathbb{N}_0}}^{\frac{n-i}{2}}(-1)^j\binom{n-i-j}{j}\,Q'(\alpha)^{n-i-2j}\,Q(\alpha)^j, \tag{3.6}$$
  for $i = 0, 1, 2, .., n$, *where* $\binom{k}{r} = \frac{k!}{r!(k-r)!}$, *with* $k \geq r \geq 0$.

- *If* α *is a root of* Q *with multiplicity one, the particular solution is*$y_p(t) = t^\alpha\sum_{i=0}^{n} c_i\,(\log t)^{i+1}$, *with*
  $$c_i = (-1)^{n-i}\,\frac{n!}{(i+1)!}\,\frac{A}{Q'(\alpha)^{n-i+1}},\ \textit{for } i = 0, 1, 2, .., n. \tag{3.7}$$

- *If* α *is a root of* Q *with multiplicity two, the particular solution is* $y_p(t) = t^\alpha c_n\,(\log t)^{n+2}$, *with* $c_n = \frac{A}{(n+1)(n+2)}$.

Proof of Theorem 3.2. *The last case coincides with the one presented in Theorem 3.1. For the other two cases, we use backwards mathematical induction to prove it, taking advantage of the recursive solutions presented in Theorem 3.1.*

1. *If α is not a root of Q the particular solution is of the form*

$$y_p(t) = t^\alpha \sum_{i=0}^{n} c_i \left(\log t\right)^i ,$$

where $c_n = \frac{A}{Q(\alpha)}$, $c_{n-1} = -nA\frac{Q'(\alpha)}{Q(\alpha)^2}$ and $c_i = -\frac{i+1}{Q(\alpha)}\left[Q'(\alpha)c_{i+1} + (i+2)c_{i+2}\right]$ for $i = 0,1,2,...,n-2$. We want to prove that, for $i = 0,1,2,..,n$, the coefficients c_i can be written in the general form presented in (3.6)*.*

Using backwards mathematical induction we have two base cases to be verified, c_n and c_{n-1}, which we know from Theorem 3.1 that are $\frac{A}{Q(\alpha)}$ and $-nA\frac{Q'(\alpha)}{Q(\alpha)^2}$, respectively. Taking into account (3.6)*, we have*

$$\begin{aligned} c_n &= (-1)^0 \, \frac{n!}{n!} \, \frac{A}{Q(\alpha)} \, (-1)^0 \binom{0}{0} \, Q'(\alpha)^0 \, Q(\alpha)^0 = \frac{A}{Q(\alpha)}, \\ c_{n-1} &= (-1) \, \frac{n!}{(n-1)!} \, \frac{A}{Q(\alpha)^2} \, (-1)^0 \binom{1}{0} \, Q'(\alpha)^1 \, Q(\alpha)^0 = -nA\frac{Q'(\alpha)}{Q(\alpha)^2}, \end{aligned}$$

which means that the base cases are verified. For the inductive step, we assume that, for $i = 0,1,2,...,n-2$, c_{i+1} and c_{i+2} are given by (3.6)*, and we want to prove that c_i is also given by* (3.6)*.*

From Theorem 3.1, we know that $c_i = -\frac{i+1}{Q(\alpha)}\left[Q'(\alpha)c_{i+1} + (i+2)c_{i+2}\right]$ for $i = 0,1,2,...,n-2$. Plugging in c_i the expressions of c_{i+1} and c_{i+2} given by (3.6)*, we obtain*

$$\begin{aligned} -\frac{i+1}{Q(\alpha)} \Bigg[& Q'(\alpha)(-1)^{n-i-1}\frac{n!}{(i+1)!}\frac{A}{Q(\alpha)^{n-i}} \sum_{\substack{j=0 \\ j\in\mathbb{N}_0}}^{\frac{n-i}{2}-\frac{1}{2}} (-1)^j \binom{n-i-j-1}{j} Q'(\alpha)^{n-i-2j-1} Q(\alpha)^j \\ & +(i+2)(-1)^{n-i-2}\frac{n!}{(i+2)!}\frac{A}{Q(\alpha)^{n-i-1}} \sum_{\substack{j=0 \\ j\in\mathbb{N}_0}}^{\frac{n-i}{2}-1} (-1)^j \binom{n-i-j-2}{j} Q'(\alpha)^{n-i-2j-2} Q(\alpha)^j \Bigg] . \end{aligned}$$

Rearranging the terms and changing the variable in the second sum, we get

$$\begin{aligned} (-1)^{n-i}\frac{n!}{i!}\frac{A}{Q(\alpha)^{n-i+1}} \Bigg[& \sum_{\substack{j=0 \\ j\in\mathbb{N}_0}}^{\frac{n-i}{2}-\frac{1}{2}} (-1)^j \binom{n-i-j-1}{j} Q'(\alpha)^{n-i-2j} \, Q(\alpha)^j \\ & + \sum_{\substack{j=1 \\ j\in\mathbb{N}_0}}^{\frac{n-i}{2}} (-1)^j \binom{n-i-j-1}{j-1} Q'(\alpha)^{n-i-2j} \, Q(\alpha)^j \Bigg] . \end{aligned}$$

Joining the two sums and taking into account some permutation's properties, we end up with the following expression

$$\begin{aligned} (-1)^{n-i}\frac{n!}{i!}\frac{A}{Q(\alpha)^{n-i+1}} \Bigg[& \sum_{\substack{j=1 \\ j\in\mathbb{N}_0}}^{\frac{n-i}{2}-\frac{1}{2}} (-1)^j \binom{n-i-j}{j} Q'(\alpha)^{n-i-2j} \, Q(\alpha)^j \\ & + Q'(\alpha)^{n-i} + (-1)^{\frac{n-i}{2}} \, Q(\alpha)^{\frac{n-i}{2}} \chi_{\{n-i \text{ is even}\}} \Bigg] . \end{aligned}$$

Finally, we conclude that

$$c_i = (-1)^{n-i}\frac{n!}{i!}\frac{A}{Q(\alpha)^{n-i+1}}\sum_{\substack{j=0\\ j\in\mathbb{N}_0}}^{\frac{n-i}{2}}(-1)^j\binom{n-i-j}{j}Q'(\alpha)^{n-i-2j}\,Q(\alpha)^j,$$

which coincides with the expression given by (3.6)*. Thus the proof for the first case is finished.*

2. *If α is a root of Q with multiplicity one, the particular solution is of the form*

$$y_p(t) = t^{\alpha}\sum_{i=0}^{n} c_i\,(\log t)^{i+1}\,,$$

where $c_n = \frac{A}{(n+1)Q'(\alpha)}$ and $c_i = -\frac{i+2}{Q'(\alpha)}c_{i+1}$, for $i = 0, 1, 2, ..., n-1$. We want to prove that we can write the coefficients c_i in the general way presented in (3.7)*.*

As before, we use backwards mathematical induction. Starting with the base case and taking into account (3.7)*, we have*

$$c_n = (-1)^0\,\frac{n!}{(n+1)!}\,\frac{A}{Q'(\alpha)} = \frac{A}{(n+1)Q'(\alpha)},$$

which coincides with the expression given by Theorem 3.1. Thus, the base case is verified. To prove the induction step, for $i = 0, 1, 2, .., n-1$, we assume that c_{i+1} is given by (3.7) *and we want to prove that c_i is also given by* (3.7)*.*

From Theorem 3.1, we know that $c_i = -\frac{i+2}{Q'(\alpha)}c_{i+1}$, for $i = 0, 1, 2, ..., n-1$. Plugging in c_i the expression of c_{i+1} given by (3.7)*, we obtain*

$$c_i = -\frac{i+2}{Q'(\alpha)}\,(-1)^{n-i-1}\,\frac{n!}{(i+2)!}\,\frac{A}{Q'(\alpha)^{n-i}} = (-1)^{n-i}\,\frac{n!}{(i+1)!}\,\frac{A}{Q'(\alpha)^{n-i+1}},$$

and therefore the induction step is proved. With this we conclude the proof.

■

A particular case of Theorem 3.2 is to consider that the non-homogeneous part of the second order Euler-Cauchy equation (3.5) is only a monomial. In fact, this can be seen as a particular case of Theorem 3.5 of Sabuwala & Leon (2011). In the following corollary we state this explicitly.

Corollary 3.1. *Consider the second order Euler-Cauchy equation,*

$$t^2y''(t) + aty'(t) + by(t) = At^{\alpha},$$

with $t > 0$, $a, b \in \mathbb{R}$, $A \in \mathbb{R}\setminus\{0\}$ and $\alpha \in \mathbb{R}$.

The particular solution is of the form $y_p(t) = c\ t^{\alpha}(\log t)^r$, where $c = \frac{A}{Q^{(r)}(\alpha)}$,[1] *with r*[2] *being the multiplicity of α as a root of* (3.3)*.*

Proof of Corollary 3.1. *The result is straightforward considering $n = 0$ in Theorem 3.2.* ■

Finally, we can generalize the Theorem 3.2 in the following corollary.

[1] $Q^{(r)}(\beta)$ is the derivative of order r of Q w.r.t. β. In particular, if $r = 0$ we consider that $Q^{(r)}(\beta)$ is exactly $Q(\beta)$.
[2] r can take the values $0, 1$ or 2. We consider $r = 0$ when α is not a root of (3.3).

Corollary 3.2. *Consider the second order Euler-Cauchy equation,*

$$t^2y''(t) + aty'(t) + by(t) = \sum_{k=1}^{m} A_k t^{\alpha_k} (\log t)^{n_k}, \tag{3.8}$$

with $t > 0$, $a, b \in \mathbb{R}$, $A_k \in \mathbb{R} \setminus \{0\}$, $\alpha_k \in \mathbb{R}$, $n_k \in \mathbb{N}_0$, for $k = 1, 2, ..., m$, with $m \in \mathbb{N}$.

The particular solution of (3.8) is of the form $y_p(t) = \sum_{k=1}^{m} y_{p_k}(t)$ [3], where $y_{p_k}(t)$ is the solution of the equation

$$t^2y_k''(t) + aty_k'(t) + by_k(t) = A_k t^{\alpha_k} (\log t)^{n_k},$$

which is presented in Theorem 3.2.

Proof of Corollary 3.2. *The result follows from the superposition principle.* ■

3.3 Examples

In this section, we apply the results derived in Theorem 3.2 to obtain a particular solution of second order Euler-Cauchy equations. Check that the proposed functions are indeed solutions of the considered equations is a simple matter of calculations, that we omit here.

- **Example 1**

  $$t^2y''(t) - 3ty'(t) + 4y(t) = 6t^2$$

  In this case the characteristic polynomial is

  $$Q(\beta) = \beta^2 - 4\beta + 4, \tag{3.9}$$

  so $\alpha = 2$ is a root of the characteristic equation (3.9) with multiplicity two. With $n = 0$ and $A = 6$, Theorem 3.2 gives

  $$y_p(t) = 3t^2 (\log t)^2.$$

- **Example 2**

  $$t^2y''(t) - 3ty'(t) + 4y(t) = 6t^3$$

  Like Example 1 the characteristic polynomial is given by (3.9) but $\alpha = 3$ is not a root of (3.9). With $n = 0$ and $A = 6$, Theorem 3.2 gives

  $$y_p(t) = 6t^3.$$

- **Example 3**

  $$t^2y''(t) - 4ty'(t) + 4y(t) = 4t (\log t)^2$$

  In this case, the characteristic polynomial is

  $$Q(\beta) = \beta^2 - 5\beta + 4, \tag{3.10}$$

[3] Note that y_p has at least m parcels and at most $m + \sum_{k=1}^{m} n_k$ parcels. When $\alpha_1 = \alpha_2 = ... = \alpha_m$ are roots of Q all with multiplicity two, y_p has m parcels. Otherwise, when none of the α_k (with $k = 1, 2, ..., m$) has multiplicity two, y_p has $m + \sum_{k=1}^{m} n_k$ parcels.

so $\alpha = 1$ is a root of the characteristic polynomial (3.10) with multiplicity one. With $n = 2$ and $A = 4$, Theorem 3.2 gives

$$y_p(t) = -\frac{4}{9}t\left(\frac{2}{3}\log t + (\log t)^2 + (\log t)^3\right).$$

- **Example 4**

$$t^2 y''(t) - 4ty'(t) + 4y(t) = 4t^2 (\log t)^2$$

Like Example 3 the characteristic polynomial is given by (3.10) but $\alpha = 2$ is not a root of (3.10). With $n = 2$ and $A = 4$, Theorem 3.2 gives

$$y_p(t) = t^2\left(-3 + 2\log t - 2(\log t)^2\right).$$

3.4 Conclusion

We provide the analytical solution to a non-homogeneous Euler-Cauchy equation of order two, and we exemplify its use. This equation is motivated by optimal stopping problems, where the uncertainty process is modelled by a geometric Brownian motion with jumps driven by an independent Poisson process.

In this equation the non-homogeneous part results from a polynomial function multiplied by a power function of logarithm function. Our result extends the one presented in Sabuwala & Leon (2011) for second-order Euler-Cauchy equations.

4

Technology adoption in a declining market

Contents

4.1 Introduction

In 2007 Apple launched the IPhone. Since then Apple's sales have been risen more than tenfold and its profits more than twentyfold. Ten years later the expectation is that IPhone's growth will slow. The reason is twofold. First, already 1.2 billion IPhones have been sold and most of these consumers do not need another one. Second, as time passes more firms introduce products being competitive to the IPhone. Hence, Apple needs another product innovation to keep up growth. To do so it spends \$10 billion a year on research and development, but it will be difficult to find another product with the universal appeal and fat margins of the IPhone [1].

In general, demand for existing products decreases over time due to the arrival of more exciting alternatives.[2] This induces that firms need to change their product portfolio over time, and thus have to innovate in order to keep on making profits. This paper has the aim to study optimal firm behavior in such a setting. To do so, we study a problem of an existing incumbent producing an established product of which demand declines over time. The firm has an option to innovate, where, due to technological progress, a newer technology can produce better products. The resulting higher demand leads to higher profits. As time passes the best available new technology that can be adopted by the firm improves. So, the longer the firm waits with investing, the better the technology is that the firm can acquire and the better the products are the firm can produce.

In such a scenario the firm has the necessity to innovate, because otherwise the declining demand of the existing product diminishes its revenue over time.[3] In evaluating its innovation option the firm faces the following tradeoff. Adopting soon means the firm soon gets rid of the existing technology with reducing revenues, while it attracts a newer technology with higher profits. Adopting late means that, on the one hand, the firm suffers for a long time from declining profits due to the demand decrease of the established product. On the other hand, later adoption implies that, due to technological progress, the firm can attract a still better new technology with which the firm can obtain higher profits than when it adopted a new technology sooner.[4]

Existing works like Balcer & Lippman (1984), Farzin *et al.* (1998), Huisman (2001), and Hagspiel *et al.* (2015) consider similar innovation problems (see Huisman (2001) and Hoppe (1999) for an extensive survey about decision theoretic models of technology adoption), but they do not consider the for us important characteristic of declining demand for the existing product. As a result we obtain that the time to innovate can be governed by two different causes. First, like in Farzin *et al.* (1998), a firm innovates right at the moment of arrival of a far better technology, the use of which enables the firm to produce products with much higher demand, leading to a considerable profit increase. Second,

[1]Source: *The New Old Thing - Apple and the IPhone*, The Economist, July 1st, 2017.

[2]An example, among many others (like, e.g., the arrival of LCD television sets that influenced demand of CRT television sets and the replacement in the semiconductor industry of 200mm wafer plants by 300mm wafer plants (see Cho & McCardle (2009))) is the introduction of solid state drives as an alternative for hard disk drives for data storage in computers. Before the current transition to solid state drives, the computer storage market has in the past decades gone through significant innovations from 14-inch, via 8-inch and 5.25-inch to 3.5-inch drives (see Kwon (2010)).

[3]In fact, in the computer data storage industry (see footnote 1), Western Digital (producer of hard disk drives) announced in October 2015 that it plans to acquire SanDisk (producer of solid state drives) (https://www.sandisk.com/about/media-center/press-releases/2015/western-digital-announces-acquisition-of-sandisk).

[4]In the computer storage industry of footnote 1, the 8-inch drives were eventually superseded by 5.25-inch drives, which are currently replaced by solid state drives (Kwon (2010)).

the fact that demand for the existing product declines over time implies that the firm's revenue gets lower and lower as long as it does not innovate. For this reason it could be optimal for the firm to adopt a new technology a time lag after its introduction.

The latter result is as such not new in the literature, but what is new is that it is caused by declining demand for the existing product. To exemplify, first consider Balcer & Lippman (1984) that also shows that as time passes without new technological advantages, it may become profitable to purchase an existing technology that is superior to the one in place at the firm even though it was not profitable to do so in the past. However, in that paper this is caused by the fact that the discovery time was not memoryless. Hagspiel *et al.* (2015) show that changing arrival rates over time of new technologies can result in firms adopting a new technology at a later point in time than when it was available for the first time. McCardle (1985) argues that such a time lag can be explained by the uncertainty regarding the profit potential of a new technology. Doraszelski (2004) who distinguishes between innovations and improvements, concludes that the possibility of further improvements gives the firm an incentive to delay the adoption of a new innovative technology until it is sufficiently advanced.

Unlike the just mentioned contributions, Kwon (2010) has in common with our paper that it also considers a firm with a declining profit stream over time. However, Kwon (2010), and also Hagspiel *et al.* (2016), that extends Kwon (2010) by considering capacity optimization, does not consider a sequence of new technologies arriving over time. Instead, it analyzes whether to exercise a single innovation opportunity. In addition, the firm also has an option to exit the industry, which exists before and after the investment. Matomaki (2013) generalizes the work of Kwon (2010) by considering different stochastic processes representing profit uncertainty. Strategic interactions in a declining industry are studied by Fine & Li (1986) and Murto (2004).

The described product innovation problem is attacked as follows in this paper. As in Farzin *et al.* (1998), technological progress is modeled as a Poisson process, where the level of the frontier technology jumps up at unknown points in time. Demand for the existing product decreases over time, resulting in a reduction of the associated profit with a fixed rate. At the moment the firm adopts the new technology it replaces the old product by a new, technologically more advanced product. The revenue obtained from selling the new product is deterministic and increasing in the level of the adopted technology. We obtain a threshold level for the technology that needs to be reached in order for the firm to invest optimally. The threshold level is increasing in the profit level of the established product, i.e. the firm delays the product innovation if the established product market is more profitable, which makes sense.

We then present two examples for a specific functional form for the profit flow in the new market and derive comparative statics. First, we assume a concave profit function. Second, we allow the revenue of the new product to be stochastic. Particularly we impose that this revenue is governed by a geometric Brownian motion (GBM) process. *We find that a positive trend on the new market accelerates the adoption process, while a negative trend delays it.* Remarkable is that the uncertainty parameter does not influence the investment threshold level if the revenue is linearly dependent on the stochastic variable. Otherwise, the standard result holds that larger uncertainty accelerates investment.

The paper is organized as follows. Section 4.2 introduces the model and analyzes its robustness with respect to the assumptions about the revenue in the new market. Section 4.5 concludes.

4.2 Model

We consider an incumbent firm currently producing an established product. As time passes, more consumers already bought the product reducing the existing consumer base in case the good is durable, and consumers get access to better alternatives in an evolving economy, shifting their demand away from the established product. For these reasons profits earned on the established product market decrease over time. The firm has been active in this market for some time, and we, therefore, assume that it has a perfect foresight about the future demand of the established product. Thus, the profit flow of the firm at time t is deterministic and equals $\pi_0(X_t) = z_0 X_t$, with $z_0 > 0$. The declining nature of the established market is captured by process $\boldsymbol{X} = \{X_t : t \geq 0\}$, where

$$dX_t = \alpha X_t dt,$$

with $X_0 = x_0$ and $\alpha < 0$.

Facing a declining profit stream, the firm has an incentive to update its product portfolio. To do so it has to perform a product innovation by adopting a new, more advanced technology. More significant technological improvements allow to produce products of higher quality. The adoption of the new technology, thus, boosts the firm's revenues, as it is able to attract more consumers towards the upgraded product.

The development of technologies over time is governed by an uncertain process, which is exogenous to the firm. Similar to Huisman (2001) and Farzin *et al.* (1998), the state of the technological progress is given by a compound Poisson process, $\boldsymbol{\theta} = \{\theta_t : t \geq 0\}$. We may express $\theta_t = \theta_0 + uN_t$, where θ_0 denotes the state of technology at the initial point in time, $u > 0$ is the jump size and $\{N_t, t \geq 0\}$ follows a homogeneous Poisson process with rate $\lambda > 0$. This formulation implies that new technologies arrive at rate λ, and each arrival increases the technology level by u. Note that process $\boldsymbol{\theta}$ is non-decreasing over time, which reflects the non-declining nature of the technological progress.

Essentially, the firm has two reasons to innovate. The first reason is that the established product market profit has reduced too much so that keeping on to produce this established product is not economically viable for the firm. The second reason is that over time alternative technologies have been invented that are much more profitable than staying in the established product market. Translated to our model, the first reason is equivalent to a low value of X, whereas the second reason implies a high value of θ. We conclude that innovating is optimal for low values of X, and high values of θ, while the firm should keep on being active on the established product market when X is high and θ is low.

The objective of the firm is, thus, to decide how long is it willing to stay in the established market before innovating, or in other words to determine the optimal investment timing in the new technology. We assume that upon investment the firm abandons the established market, and replaces the old product with the new one. For instance, in the two-period model of Levinthal & Purohit (1989) it is established that replacing the existing version of the product with an upgrade gives higher profits than joint production.

Formally, the value of the firm, $F(\theta, x)$, is the solution of the following optimal stopping time problem

$$\sup_{\tau} \mathbb{E}\left[\int_0^{\tau} \pi_0(X_s)e^{-rs}ds + \left\{\int_{\tau}^{+\infty} \pi_1(\theta_\tau)e^{-rs}ds - Ie^{-r\tau}\right\}\chi_{\{\tau<+\infty\}}\middle|\theta_0=\theta, X_0=x\right], \tag{4.1}$$

where τ denotes the investment timing, r denotes the discount rate, $\pi_1(\theta)$ is the profit flow in the new market, and $I > 0$ is the sunk investment cost. Moreover, χ_A represents the indicator function of set A.

In this setting the firm faces a trade-off between early adoption and the significance of the technological improvement. In particular, waiting for a better technology comes at a cost of operating longer with lower profits. By investing at a certain technology level, the firm commits to the corresponding quality of the new product. Therefore, θ stays constant upon investment. If the firm decides to innovate at the current level of θ, it earns a profit flow of $\pi_1(\theta)$. Adding it up and discounting gives a total discounted profit stream $\frac{\pi_1(\theta)}{r}$. Since innovating requires an investment outlay of I, this results in the following value for investing instantaneously

$$V(\theta) = \frac{\pi_1(\theta)}{r} - I. \tag{4.2}$$

Following Huisman (2001), we assume that π_1 is an increasing and concave function of θ, with $\pi_1(0) = 0$ and $\lim_{\theta\to+\infty} \pi_1(\theta) = +\infty$. This entails that $V(\theta)$ is also increasing and concave.

The corresponding Hamilton-Jacobi-Bellman equation (HJB) for the optimization problem (4.1) is given by

$$\min\{rF(\theta, x) - \mathcal{L}F(\theta, x), F(\theta, x) - V(\theta)\} = 0, \tag{4.3}$$

where the infinitesimal generator is defined as follows

$$\mathcal{L}f(\theta, x) = \pi_0(x) + \alpha x \frac{\partial f(\theta, x)}{\partial x} + \lambda\left[f(\theta+u, x) - f(\theta, x)\right].$$

Let the set $\mathcal{C} := \{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : F(\theta, x) > V(\theta)\}$ denote the *continuation region*, and $\mathcal{S} := \mathbb{R}^+ \times \mathbb{R}^+ \setminus \mathcal{C} = \{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : F(\theta, x) = V(\theta)\}$ denote the *stopping region*. Then the optimal investment timing, denoted by τ^*, is given by

$$\tau^* = \inf\{t > 0 : (\theta_t, X_t) \notin \mathcal{C}\}.$$

Proposition 4.1. *The boundary that separates the continuation and stopping regions is defined as follows*

$$\partial\mathcal{S} = \left\{(\theta, x) : \theta \geq \bar{\theta} \wedge x = b(\theta)\right\},$$

where

$$b(\theta) = \frac{(r+\lambda)V(\theta) - \lambda V(\theta+u)}{z_0}, \quad \textit{with} \quad \theta > \bar{\theta}, \tag{4.4}$$

and $\bar{\theta}$ is implicitly defined by $(r+\lambda)V(\bar{\theta}) - \lambda V(\bar{\theta}+u) = 0$. Moreover, b is an increasing function of θ.

Proof of Proposition 4.1. *See Appendix 4.A for the proof.* ■

In fact, Expression (4.4) can be seen as the result of equating the value of investing immediately and the value of investing in the next instant, i.e.

$$\frac{\pi_0(x)}{r+\lambda} + \frac{\lambda}{r+\lambda}V(\theta+u) = V(\theta).$$

It reflects that when the established market declines to the level of $b(\theta)$, the firm is indifferent between innovating, which gives the payoff $V(\theta)$, and waiting for the next technology arrival, which results in extra expected revenue on the established product market, $\frac{\pi_0(x)}{r+\lambda}$, and the discounted payoff of innovating after the next jump, which equals $\frac{\lambda}{r+\lambda}V(\theta+u)$.

From Proposition 4.1, we conclude that $b(\theta)$ is an upward sloping curve in the (θ, x) −plane. It is important to realize that adoption of the new technology does not happen only due to a technology arrival, which corresponds to a horizontal jump in the (θ, x) −plane. It can also happen that the existing revenue for the established product becomes so low that innovating is optimal. This is reflected by the decrease in x over time, which corresponds to a vertical movement in the (θ, x) −plane, such that innovating takes place at the moment the $b(\theta)$ −curve is hit from above. These two possibilities are graphically illustrated in Figure 4.1.

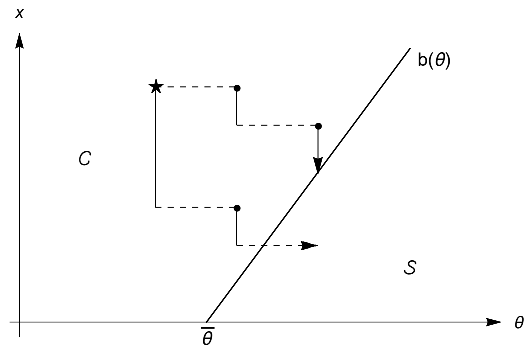


Figure 4.1: Illustration of the two possible ways of adopting: at the arrival of a new technology (left-right horizontal crossing of threshold curve) or after sufficient decrease of the profitability of the current market (downward vertical crossing of threshold curve).

In Figure 4.1 the initial level of (θ, x) is marked by a star $(\star)$. The solid lines correspond to the profit decline in the established market, whereas the dashed lines illustrate the technology arrivals. Evidently, the threshold curve can be reached in two ways. In the first scenario, the additional decline in the market is necessary for the investment to be optimal after two technology arrivals, and $b(\theta)$ is hit from above. In the second scenario, innovating is optimal immediately after two technology arrivals and $b(\theta)$ is crossed from the left.

Proposition 4.2 establishes the value of the firm.

Proposition 4.2. *Let* $n(\theta, x) = \left\lceil \frac{b^{-1}(\theta)-\theta}{u} \right\rceil$*, where for* $k \geq 0$*,* $\lceil k \rceil = \min\{n \in \mathbb{N} : n \geq k\}$*. Then the* *optimal value of the firm is given by* [5]

$$
\begin{aligned}
F(\theta, x) \quad = \quad & \left(\frac{\lambda}{r+\lambda}\right)^{n(\theta,x)} V\left(\theta + n(\theta,x)u\right) + \frac{z_0 x}{r-\alpha}\left[1 - \left(\frac{\lambda}{r+\lambda-\alpha}\right)^{n(\theta,x)}\right] \\
& + z_0 x \sum_{k=0}^{n(\theta,x)-1} \left\{ \left[\frac{x}{b(\theta+ku)}\right]^{\frac{r+\lambda}{\alpha}-1} \lambda^k \times \right. \\
& \left. \sum_{m=0}^{k} \frac{1}{m!\,(-\alpha)^m} \left[\frac{1}{(r+\lambda)^{k-m+1}} - \frac{1}{(r+\lambda-\alpha)^{k-m+1}}\right] \left[\ln\left[\frac{x}{b(\theta+ku)}\right]\right]^m \right\} \chi_{\{\theta > \bar{\theta} - ku\}}.
\end{aligned}
\tag{4.5}
$$

Proof of Proposition 4.2. *See Appendix 4.A for the proof.* ■

In Proposition 4.2, the number of arrivals of new technologies until it is optimal to innovate equals $n(\theta, x)$, given that the current technology is θ and the profitability of the current market is x. Hence, in the stopping region $n(\theta, x) = 0$, and the value is equal to $V(\theta)$. In the continuation region, the value function $F(\theta, x)$ consists of three parts. The first term in (4.5) can be interpreted as the expected discounted value of the future technology adoption. Here the fraction $\frac{\lambda}{r+\lambda}$ stands for the stochastic discount factor under a Poisson process (Huisman (2001, p.46)), i.e. it represents the discounted time between two consecutive jumps.

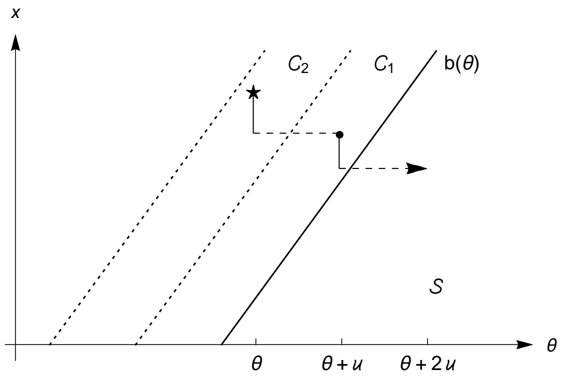
The second term in (4.5) represents what the firm earns on sales of the established product until it innovates. Here $\frac{z_0 x}{r-\alpha}$ stands for the discounted revenue stream if the firm were active on the established product market forever. However, after the firm innovates, it discontinues this activity. Therefore, we need to subtract the amount $\left(\frac{\lambda}{r+\lambda-\alpha}\right)^{n(\theta,x)} \frac{z_0 x}{r-\alpha}$. The denominator $r + \lambda - \alpha$ makes sure that the resulting expected revenue stream is discounted (r), and it is corrected for the fact that the revenue stream lasts up until the innovation time (λ), and that the revenue decreases over time with rate $-\alpha$ due to the declining demand of the established product.

The third term in (4.5) accounts for the fact that the innovation can occur not only due to the technology jump, but also by the decline in the established market. In order to illustrate this, consider the scenario when the current demand in the established market and the technology level are such that the innovation will always be optimal after two jumps. Let $\mathcal{C}_n$ denote the subset of the continuation region where stopping is optimal after n jumps in θ, i.e. if $(\theta, x) \in \mathcal{C}_n$ then $(\theta + nu, x) \in \mathcal{S}$. Thus, in the region $\mathcal{C}_2$ we can simplify the value function in (4.5) as follows

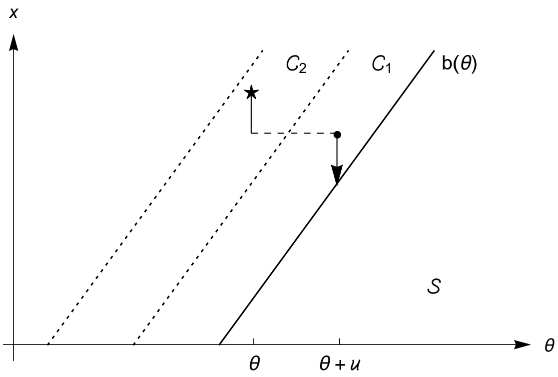
$$
\begin{aligned}
& \left(\frac{\lambda}{r+\lambda}\right)^2 V(\theta + 2u) + \frac{z_0 x}{r-\alpha}\left[1 - \left(\frac{\lambda}{r+\lambda-\alpha}\right)^2\right] \\
& + \left\{ z_0 b(\theta) \left[\frac{x}{b(\theta)}\right]^{\frac{r+\lambda}{\alpha}} \left[\frac{1}{r+\lambda} - \frac{1}{r+\lambda-\alpha}\right] \right\} \chi_{\{\theta > \bar{\theta}\}} \\
& + \left\{ z_0 b(\theta+u) \left[\frac{x}{b(\theta+u)}\right]^{\frac{r+\lambda}{\alpha}} \left[\lambda \left[\frac{1}{(r+\lambda)^2} - \frac{1}{(r+\lambda-\alpha)^2}\right] \right.\right. \\
& \left.\left. - \frac{\lambda}{\alpha}\left[\frac{1}{r+\lambda} - \frac{1}{r+\lambda-\alpha}\right] \ln\left[\frac{x}{b(\theta+u)}\right]\right]\right\} \chi_{\{\theta > \bar{\theta} - u\}}.
\end{aligned}
\tag{4.6}
$$

Figure 4.2 shows the four alternative ways the stopping region can be reached from an initial level of $(\theta, x) \in \mathcal{C}_2$.

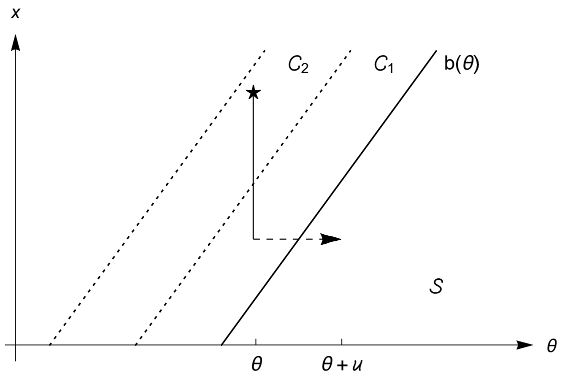
[5] We assume that if $n(\theta, x) = 0$ the third term (corresponding to the sum in k) in (4.5) is equal to zero. Also, when $(\theta, x) \in \mathcal{S}$, we have $F(\theta, x) = V(\theta)$, which is equivalent to consider $n(\theta, x) = 0$ in (4.5)



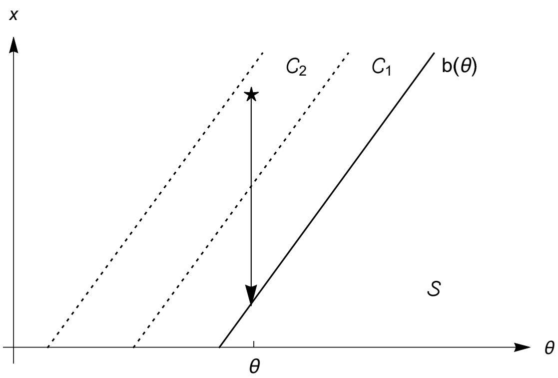
(a) Stopping region is reached by two consecutive jumps.



(b) Stopping region is reached by a decline in the established market after one jump.



(c) Stopping region is reached by one jump after a decline in the established market.



(d) Stopping region is reached by a decline in the established market.

Figure 4.2: Four different ways of reaching the stopping region from an initial level of $(\theta, x) \in \mathcal{C}_2$.

The first two terms in (4.6) capture the case when the technology level $\theta + 2u$ is reached after two jumps, as depicted in Figure 4.2(a). The last three terms in (4.6) correct for the fact that in certain scenarios the demand in the established market may decline enough for the firm to be willing to adopt a lower technology level than $\theta + 2u$. In particular, the firm might end up adopting a technology level, $\theta{+}u$, if the market declines enough before the second jump takes place to trigger the investment. In this case the stopping region can be reached in two different ways. The first is illustrated in Figure 4.2(b), where the first technology arrival happens relatively early. This brings the firm in the region one jump away from adopting, $\mathcal{C}_1$, where the further decline in the established market triggers the investment. This situation is captured by the last correction term in (4.6). The second possibility, when the jump occurs relatively late, is shown in Figure 4.2(c). In this case the decline in the established market brings the firm to the region $\mathcal{C}_1$, whereas the first technology arrival triggers the investment. This scenario is accounted for by the second correction term in (4.6). Finally, the firm may eventually adopt the current level of technology, θ, if the market declines even further before any jump occurs, as it is shown in Figure 4.2(d). The first term in (4.6) corrects for that. Equation (4.5) generalizes expression (4.6) , such that $n\,(\theta, x) = 2$.

The following remark highlight that the function $F(\theta, x)$ in (4.5) is well-behaved.

Remark 4.1. *The value of the firm $F(\theta, x)$ is continuous in both arguments and smooth in x.*

4.3 Example

In Section 4.2 we showed that it is possible to derive the analytical solution of the optimal stopping problem for a general expression for the profit flow in the new market. In this subsection we analyze a specific example for a functional form of $\pi_1(\theta)$, which is frequently used in the literature (Huisman, 2001).

First, let us consider a profit flow expressed by $\pi_1(\theta) = z_1\theta^\beta$ with $0 < \beta \leq 1$ (note that for $\beta = 1$ the profit flow is linear in θ). Plugging this expression into (4.2) and (4.4), we obtain that threshold curve $b(\theta)$ is given by the following equation

$$b(\theta) = \frac{1}{z_0}\left[z_1\frac{(r+\lambda)\theta^\beta - \lambda(\theta+u)^\beta}{r} - rI\right]. \tag{4.7}$$

From the expression of the threshold curve (4.7) the following comparative statics results are derived.

Proposition 4.3. *The threshold curve $b(\theta)$ is increasing in z_1, and decreasing in λ, z_0, I and u for a given value θ.*

Proof of Proposition 4.3. *See Appendix 4.A for the proof.* ■

The result in Proposition 4.3 implies that the firm will innovate later if profits on the established product market are higher. On the other hand, the firm will innovate sooner if the revenue from innovating is higher. Intuitively, waiting for the next technology arrival is more appealing if it is expected to occur sooner or when the technology arrival results in a higher increase of the technology level. If innovating is more expensive it will happen later. Interestingly, the threshold level is not affected by α. This is because it results in effects of order higher than one in dt so that they are too small to take into account.

Concerning the interest rate r there are conflicting effects. On the one hand, the firm innovates sooner when r increases, because the firm becomes more myopic. This implies that it is less inclined to wait for future technological breakthroughs and therefore wants to innovate sooner. This effect dominates for small r. On the other hand, the firm innovates late, because the net present value of the investment decreases with r.

4.4 Extension

A substantial simplification in our model is that the profit flow on the new market is deterministic. In reality, by entering a new market the firm is often exposed to uncertainty regarding the profitability of the product. They can be related to consumer behavior risk, risk of product malfunction or unsatisfactory performance, among others. In order to account for this, we now impose that the profit flow after investment is stochastic.

Similarly to the previous example, we start by assuming that at the moment the firm innovates, the instantaneous profit obtained by selling the new product equals to $z_1\theta_\tau^\beta$, with τ being the moment

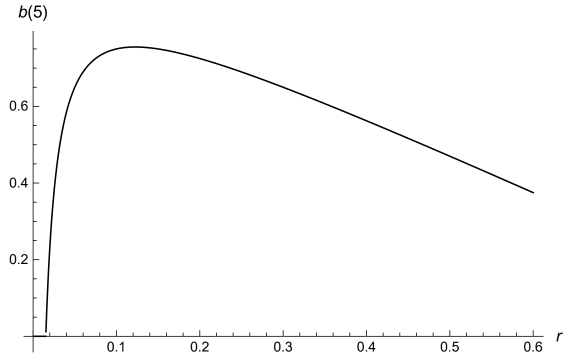


Figure 4.3: Example showing that the threshold curve $b(\theta)$ for a given $\theta = 5$ is first increasing in the discount rate r and then decreasing.
Parameter values used: $\lambda = 0.05$, $u = 0.5$, $I = 50$, $z_0 = 50$, $z_1 = 10$, $\beta = 1$, and $\theta = 5$.

of investment. Where in our benchmark model the instantaneous profit is fixed at this level, we now consider the situation that future profits are uncertain and follow a GBM. So, the profit flow of the new product now equals $z_1 Y_t^{\beta}$, where

$$dY_t = \mu Y_t dt + \sigma Y_t dW_t,$$

where μ denotes the drift, σ is the volatility, and $Y_0 = \theta_\tau$.

Applying Itô's lemma to determine the evolution of Y_t^{β}, we get

$$dY_t^{\beta} = \left(\mu\beta - \frac{1}{2}\beta(1-\beta)\sigma^2\right) Y_t^{\beta} dt + \sigma\beta Y_t^{\beta} dW_t.$$

Thus, the process Y_t^{β} also follows a GBM with drift $\mu\beta - \frac{1}{2}\beta(1-\beta)\sigma^2$ and volatility $\sigma\beta$. The value in the stopping region in this case is given by

$$V(\theta) = \mathbb{E}\left[\int_{t=0}^{+\infty} z_1 Y_t^{\beta} e^{-rt} dt - I \middle| Y_0 = \theta\right] = \frac{z_1 \theta^{\beta}}{r - \mu\beta + \frac{1}{2}\beta(1-\beta)\sigma^2} - I.$$

Note that the functional form of the value function is the same as for the benchmark model. The only difference is that the total discounted profit stream on the new market has been corrected for the trend parameter μ, and volatility parameter σ. The resulting expression for the threshold curve $b(\theta)$ is given by

$$b(\theta) = \frac{1}{z_0}\left[z_1 \frac{(r+\lambda)\theta^{\beta} - \lambda(\theta+u)^{\beta}}{r - \mu\beta + \frac{1}{2}\beta(1-\beta)\sigma^2} - rI\right]. \tag{4.8}$$

The comparative statics for (4.8) is derived in Proposition 4.4.

Proposition 4.4. *The threshold curve $b(\theta)$ is increasing in z_1 and μ, and decreasing in σ, λ, z_0, I and u for a given value θ.*

Proof of Proposition 4.4. *See Appendix 4.A for the proof.* ■

Compared to the deterministic model, the only change in the expression is that the drift μ and the volatility σ enter the expression. Thus, our earlier conclusions still hold for this model extension. In addition, we find that the firm invests earlier if the profit flow on the new market is expected to increase more, i.e. when μ is larger. In turn, the volatility parameter σ does not influence the investment decision if the profit flow is linear in the technology level ($\beta = 1$). For $0 < \beta < 1$, we find that a larger volatility decreases the threshold curve, implying that the firm delays investment. This is because in case of a concave profit function, a larger uncertainty parameter decreases the trend due to Jensen's inequality.

4.5 Conclusion

This paper studies the product innovation option of an incumbent. Initially the firm is active in selling its established product. However, due to the facts that, in case of durable goods, over time the consumer base declines because more consumers have already bought the product, and other firms introduce products that compete with the established product of the focal firm, the firm's profit associated with the established product decreases over time. For this reason the firm wants to change its product portfolio by innovating. Due to technological progress the firm is able to introduce a better product if it innovates later. Therefore, the firm faces the following trade-off. If it innovates early, it stops the profit decline associated with its established product early, but the adopted new product only incrementally improves the established one. If the firm innovates late, it is able to launch a product of much better quality, but at the same time it has to deal with a long period of declining demand of its established product. Depending on the realizations of the technological breakthroughs, the paper determines the firm's optimal profit innovation timing. We show that such an innovation can occur either right at the moment of a technological breakthrough, or some time after such an event. In the latter case the firm adopts the new product, because demand of the established product has reduced too much to stay active in the established product market. We further obtain an explicit expression for the value of the firm, reflecting a weighted average of all possible innovation patterns.

This paper provides a solid basis for interesting extensions that could be topics for future research. Here we think about determining the optimal production capacity associated with launching the new product, including the innovation strategy of competitors and how to optimally react to that, and to incorporate learning effects regarding the production processes of the different products.

Appendix 4.A Proofs of propositions

4.A.1 Proof of Propositions 4.1 and 4.2

The results presented in Propositions 4.1 and 4.2 are intrinsically related. Furthermore, some of them are obtained simultaneously. Then we decided to develop a joint proof. We also give some intuition about how we suspect about the form of the value function presented in Proposition 4.2.

Let us consider the set

$$U = \left\{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : rV(\theta) - \mathcal{L}V(\theta) < 0\right\}.$$

Note that $rV(\theta) - \mathcal{L}V(\theta) < 0 \Leftrightarrow (r + \lambda)V(\theta) - \lambda V(\theta + u) < \pi_0(x) \Leftrightarrow g(\theta) < \pi_0(x)$, where $g(\theta) = (r+\lambda)V(\theta) - \lambda V(\theta+u)$. Given that π_1 is a concave function then V is also a concave function, implying that V' is a decreasing function. So, $V'(\theta) > V'(\theta+u) > \frac{\lambda}{\lambda+r}V'(\theta+u)$, which implies that $g'(\theta) > 0$, i.e. g is an increasing function. Thus, given that $g(0) < 0$, $\lim_{\theta\to+\infty}\pi_1(\theta) = +\infty$ and $\pi_0(x) > 0$, for $x > 0$, we can say that U is of the form

$$U = \left\{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : \theta < h(x)\right\},$$

where h is implicitly defined as $g(\theta) < \pi_0(x) \Leftrightarrow \theta < h(x)$. Given that π_0 is an increasing function of x, then h is also an increasing function of x. In the limit case for $x = 0$, we have $\theta < h(0) \Leftrightarrow g(\theta) < \pi_0(0) = 0$. Let us define $\bar{\theta}$ such that $\bar{\theta} = h(0) \Leftrightarrow g(\bar{\theta}) = 0 \Leftrightarrow (r+\lambda)V(\bar{\theta}) - \lambda V(\bar{\theta}+u) = 0$. Given that g is an increasing function with $g(0) < 0$, we can guarantee that $\bar{\theta}$ exists and it is unique.

By Propositions 3.3 and 3.4 from Oksendal and Sulem (2007), we know that $U \subseteq \mathcal{C}$. In particular, as h is an increasing function of x, we notice that for $0 < \theta < \bar{\theta}$ we are always in the continuation region.

From the HJB Equation (4.3), we know that the value function in the stopping region is $F(\theta, x) = V(\theta)$ and in the continuation region satisfies the following differential equation

$$(r+\lambda)F(\theta, x) = \pi_0(x) + \alpha x \frac{\partial F(\theta, x)}{\partial x} + \lambda F(\theta+u, x),$$

which is equivalent to

$$\frac{\partial F(\theta, x)}{\partial x} - \frac{(r+\lambda)}{\alpha x}F(\theta, x) = -\frac{\lambda}{\alpha x}F(\theta+u, x) - \frac{z_0}{\alpha}. \tag{4.9}$$

The equation (4.9) is of the form $\frac{\partial y(x)}{\partial x} + P(x)y(x) = R(x)$, and it has a solution $y(x) = \frac{1}{u(x)}\int u(x)R(x)dx$, where $u(x) = \mathrm{e}^{\int P(x)dx}$. Then the solution of (4.9) can be written as

$$F(\theta, x) = x^{\frac{r+\lambda}{\alpha}} \int x^{-\frac{r+\lambda}{\alpha}} \left(-\frac{\lambda}{\alpha x}F(\theta+u, x) - \frac{z_0}{\alpha}\right) dx. \tag{4.10}$$

The value of $F(\theta+u, x)$ depends on which set $(\theta+u, x)$ belongs to, which in its turn depends on which set (θ, x) belongs to. This implies that we need to split the continuation region in different subsets.

Let $\mathcal{C}_n$ denote the subset of the continuation region where stopping is optimal after n jumps in θ, i.e. if $(\theta, x) \in \mathcal{C}_n$ then $(\theta+nu, x) \in \mathcal{S}$. Further, in region $\mathcal{C}_n$ we denote the value function by $f_n(\theta, x)$. Thus

$$F(\theta, x) = f_n(\theta, x)\chi_{\{(\theta,x)\in\mathcal{C}_n\}} + V(\theta)\chi_{\{(\theta,x)\in\mathcal{S}\}}.$$

In this problem the threshold is not only one point but a boundary, let us say $\theta^\star(x)$, for $x > 0$. In the boundary between continuation and stopping regions, value matching and smooth pasting on x should hold. In order to make the notation clearer, we will write the threshold in a different way. We assume that $\theta = \theta^\star(x) \Leftrightarrow x = {\theta^\star}^{-1}(\theta) = b(\theta)$. From now on we consider that the boundary between the continuation and stopping regions is given by $x = b(\theta)$, for $\theta \geq \bar{\theta}$ [6].

Using the new notation and taking into account the U definition, we can define the stopping region and the subsets of the continuation region as follows,

$$\mathcal{S} \;=\; \{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < x \leq b(\theta)\}$$

[6] We already showed that for $0 < \theta < \bar{\theta}$ we are always in the continuation region.

and, for $n \in \mathbb{N}$,

$$\mathcal{C}_n = \{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : b(\theta + nu) < x \leq b(\theta + (n-1)u)\}.$$

We can also write the value matching and smooth pasting conditions as follows

$$f_1\left(\theta, b(\theta)\right) = V(\theta) \quad \text{and} \quad \left.\frac{\partial f_1(\theta, x)}{\partial x}\right|_{x=b(\theta)} = \left.\frac{\partial V(\theta)}{\partial x}\right|_{x=b(\theta)} = 0 \tag{4.11}$$

Moreover, value matching between the different functions on the continuation region should also hold, i.e., for $n \in \mathbb{N}$ and $\theta \geq \bar{\theta} - nu$,

$$f_n\left(\theta, b(\theta, \theta + nu)\right) = f_{n+1}\left(\theta, b(\theta, \theta + nu)\right). \tag{4.12}$$

The first condition in (4.11) and condition (4.12) mean that function F is continuous everywhere.

Before on, we analyse the limit case $x = 0$. This means that there is no declining market but only the option to invest in a new technology, which is a well known problem. Taking into account Theorem 5.1, given that g is an increasing function with only one zero at $\bar{\theta}$, we conclude that the value function, for $0 < \theta < \bar{\theta}$, is $\left(\frac{\lambda}{r+\lambda}\right)^{n(0,\theta)} V\left(\theta + n(0,\theta)u\right)$, where $n(0,\theta) = \left\lceil \frac{\bar{\theta}-\theta}{u} \right\rceil$ with, for $k \geq 0$, $\lceil k \rceil = \min\{n \in \mathbb{N} : n \geq k\}$. Therefore,

$$\lim_{x \to 0^+} f_n(\theta, x) = \left(\frac{\lambda}{r+\lambda}\right)^n V\left(\theta + nu\right). \tag{4.13}$$

Let us consider $(\theta, x) \in \mathcal{C}_1$, so that $(\theta + u, x) \in \mathcal{S}$, meaning that $F\left(\theta + u, x\right) = V(\theta + u)$. From (4.10) we get

$$\begin{aligned} f_1\left(\theta, x\right) &= x^{\frac{r+\lambda}{\alpha}} \int x^{-\frac{r+\lambda}{\alpha}} \left(-\frac{\lambda}{\alpha x} V(\theta + u) - \frac{z_0}{\alpha}\right) dx \\ &= \frac{\lambda}{r+\lambda} V(\theta + u) + \frac{z_0 x}{r - \alpha}\left[1 - \frac{\lambda}{r + \lambda - \alpha}\right] + C_1(\theta) x^{\frac{r+\lambda}{\alpha}}, \end{aligned} \tag{4.14}$$

where $C_1(\theta)$ needs to be determined.

For $\theta > \bar{\theta}$, from the conditions presented in (4.11), we have

$$\frac{\lambda}{r+\lambda} V(\theta + u) + \frac{z_0 b(\theta)}{r - \alpha}\left[1 - \frac{1}{r + \lambda - \alpha}\right] + C_1(\theta) b(\theta)^{\frac{r+\lambda}{\alpha}} = V(\theta), \tag{4.15}$$

$$\frac{z_0}{r - \alpha}\left[1 - \frac{1}{r + \lambda - \alpha}\right] + \frac{r + \lambda}{\alpha} C_1(\theta) b(\theta)^{\frac{r+\lambda}{\alpha} - 1} = 0. \tag{4.16}$$

Solving the system of equations (4.15) and (4.16), we can derive the threshold curve:
$\left\{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : \theta > \bar{\theta} \wedge x = b(\theta)\right\}$, i.e., for $\theta > \bar{\theta}$,

$$\begin{aligned} C_1(\theta) &= \left[\frac{1}{r+\lambda} - \frac{1}{r + \lambda - \alpha}\right] z_0 b(\theta)^{-\frac{r+\lambda}{\alpha} + 1} \\ b(\theta) &= \frac{1}{z_0}\left[(r+\lambda)V(\theta) - \lambda V(\theta + u)\right], \end{aligned} \tag{4.17}$$

Note that $b(\theta) = \frac{1}{z_0} g(\theta)$, then $b(\bar{\theta}) = 0$ and b increases in θ, which finish the proof of Proposition 4.1.

In the rest of the proof, we derive the value function presented in Proposition 4.2. Here we have decided to prove the result providing first the intuition behind it. We consider that it helps to understand better the proof. Therefore, we start by deriving f_1. Then, using similar arguments, we derive f_2 and f_3, as well.

Regarding f_1, we start to note that, for $\theta > \bar{\theta}$, f_1 is given by (4.14), with $C_1(\theta)$ given by (4.17). For $\bar{\theta} - u < \theta \leq \bar{\theta}$, from f_1 definition and Condition (4.13), we have

$$\lim_{x \to 0^+} f_1(\theta, x) = \left(\frac{\lambda}{r+\lambda}\right) V(\theta + u) \Leftrightarrow C_1(\theta) \lim_{x \to 0+} x^{\frac{r+\lambda}{\alpha}} = 0.$$

Given that $\alpha < 0$, we have $\lim_{x \to 0+} x^{\frac{r+\lambda}{\alpha}} = +\infty$, implying that, for $\bar{\theta} - u < \theta \leq \bar{\theta}$, $C_1(\theta) = 0$.

Summarizing, for $(\theta, x) \in \mathcal{C}_1$, f_1 is given by

$$\frac{\lambda}{r+\lambda} V(\theta+u) + \frac{z_0 x}{r-\alpha}\left[1 - \frac{\lambda}{r+\lambda-\alpha}\right] + \left\{z_0 b(\theta)\left[\frac{x}{b(\theta)}\right]^{\frac{r+\lambda}{\alpha}}\left[\frac{1}{r+\lambda} - \frac{1}{r+\lambda-\alpha}\right]\right\}\chi_{\{\theta > \bar{\theta}\}}.$$

Now we move to the derivation of f_2, for which we use arguments similar to the ones presented for f_1 (and therefore we omit some remarks). Let us consider $(\theta, x) \in \mathcal{C}_2$, then $(\theta + u, x) \in \mathcal{C}_1$, meaning that $F(\theta + u, x) = f_1(\theta + u, x)$, and from (4.10), we get

$$\begin{aligned}
f_2(\theta, x) &= x^{\frac{r+\lambda}{\alpha}} \int x^{-\frac{r+\lambda}{\alpha}} \left(-\frac{\lambda}{\alpha x} f_1(\theta+u, x) - \frac{z_0}{\alpha}\right) dx \\
&= \left(\frac{\lambda}{\lambda+r}\right)^2 V(\theta+2u) + \frac{z_0 x}{r-\alpha}\left[1 - \left(\frac{\lambda}{r+\lambda-\alpha}\right)^2\right] \\
&+ \left\{z_0 b(\theta+u)\left[\frac{x}{b(\theta+u)}\right]^{\frac{r+\lambda}{\alpha}}\left(-\frac{\lambda}{\alpha}\right)\left[\frac{1}{r+\lambda} - \frac{1}{r+\lambda-\alpha}\right]\ln x\right\}\chi_{\{\theta > \bar{\theta}-u\}} \\
&+ C_2(\theta) x^{\frac{r+\lambda}{\alpha}},
\end{aligned}$$

where $C_2(\theta)$ needs to be determined.

For $\theta > \bar{\theta} - u$, from condition (4.12), $f_1(\theta, b(\theta+u)) = f_2(\theta, b(\theta+u))$, then

$$\begin{aligned}
& \frac{\lambda}{\lambda+r} V(\theta+u) + \frac{z_0 b(\theta+u)}{r-\alpha}\left[1 - \frac{\lambda}{r+\lambda-\alpha}\right] \\
& + \left\{z_0 b(\theta)\left[\frac{b(\theta+u)}{b(\theta)}\right]^{\frac{r+\lambda}{\alpha}}\left[\frac{1}{r+\lambda} - \frac{1}{r+\lambda-\alpha}\right]\right\}\chi_{\{\theta > \bar{\theta}\}} \\
= & \left(\frac{\lambda}{\lambda+r}\right)^2 V(\theta+2u) + \frac{z_0 b(\theta+u)}{r-\alpha}\left[1 - \left(\frac{\lambda}{r+\lambda-\alpha}\right)^2\right] \\
& + \left\{z_0 b(\theta+u)\left(-\frac{\lambda}{\alpha}\right)\left[\frac{1}{r+\lambda} - \frac{1}{r+\lambda-\alpha}\right]\ln b(\theta+u)\right\}\chi_{\{\theta > \bar{\theta}-u\}} + C_2(\theta) b(\theta+u)^{\frac{r+\lambda}{\alpha}}
\end{aligned}$$

from where

$$\begin{aligned}
C_2(\theta) \quad = \quad & \left\{z_0 b(\theta)^{-\frac{r+\lambda}{\alpha}+1}\left[\frac{1}{r+\lambda} - \frac{1}{r+\lambda-\alpha}\right]\right\}\chi_{\{\theta > \bar{\theta}\}} \\
& + \left\{z_0 b(\theta+u)^{-\frac{r+\lambda}{\alpha}+1}\left[\lambda\left[\frac{1}{(r+\lambda)^2} - \frac{1}{(r+\lambda-\alpha)^2}\right]\right.\right. \\
& \left.\left. + \frac{\lambda}{\alpha}\left[\frac{1}{r+\lambda} - \frac{1}{r+\lambda-\alpha}\right]\ln(b(\theta+u))\right]\right\}\chi_{\{\theta > \bar{\theta}-u\}},
\end{aligned}$$

where we use the fact that $(r+\lambda)V(\theta) - \lambda V(\theta+u) = z_0 b(\theta)$, for $\theta > \bar{\theta}$.

For $\bar{\theta} - 2u < \theta \leq \bar{\theta} - u$, from f_2 definition and condition (4.13), comes

$$\lim_{x \to 0^+} f_2(\theta, x) = \left(\frac{\lambda}{r+\lambda}\right)^2 V(\theta+2u) \Leftrightarrow C_2(\theta) \lim_{x \to 0+} x^{\frac{r+\lambda}{\alpha}} = 0,$$

then, in this case, $C_2(\theta) = 0$.

Thus, for $(\theta, x) \in \mathcal{C}_2$, the expression for f_2 is given by

$$\begin{aligned} f_2(\theta,x) &= \left(\frac{\lambda}{r+\lambda}\right)^2 V(\theta+2u) + \frac{z_0 x}{r-\alpha}\left[1-\left(\frac{\lambda}{r+\lambda-\alpha}\right)^2\right] \\ &+ \left\{ z_0 b(\theta)\left[\frac{x}{b(\theta)}\right]^{\frac{r+\lambda}{\alpha}}\left[\frac{1}{r+\lambda}-\frac{1}{r+\lambda-\alpha}\right]\right\}\chi_{\{\theta>\bar\theta\}} \\ &+ \left\{ z_0 b(\theta+u)\left[\frac{x}{b(\theta+u)}\right]^{\frac{r+\lambda}{\alpha}}\left[\lambda\left[\frac{1}{(r+\lambda)^2}-\frac{1}{(r+\lambda-\alpha)^2}\right]\right]\right. \\ &\left. -\frac{\lambda}{\alpha}\left[\frac{1}{r+\lambda}-\frac{1}{r+\lambda-\alpha}\right]\ln\left[\frac{x}{b(\theta+u)}\right]\right]\right\}\chi_{\{\theta>\bar\theta-u\}}. \end{aligned}$$

Let us consider $(\theta, x) \in \mathcal{C}_3$, then $(\theta+u, x) \in \mathcal{C}_2$, meaning that $F(\theta+u,x) = f_2(\theta+u,x)$, and from (4.10) we get

$$\begin{aligned} f_3(\theta,x) &= x^{\frac{r+\lambda}{\alpha}}\int x^{-\frac{r+\lambda}{\alpha}}\left(-\frac{\lambda}{\alpha x}f_2(\theta+u,x)-\frac{z_0}{\alpha}\right)dx \\ &= \left(\frac{\lambda}{r+\lambda}\right)^3 V(\theta+3u) + \frac{z_0 x}{r-\alpha}\left[1-\left(\frac{\lambda}{r+\lambda-\alpha}\right)^3\right] \\ &+ \left\{ z_0 b(\theta+u)\left[\frac{x}{b(\theta+u)}\right]^{\frac{r+\lambda}{\alpha}}\left(-\frac{\lambda}{\alpha}\right)\left[\frac{1}{r+\lambda}-\frac{1}{r+\lambda-\alpha}\right]\ln x\right\}\chi_{\{\theta>\bar\theta-u\}} \\ &+ \left\{ z_0 b(\theta+2u)\left[\frac{x}{b(\theta+2u)}\right]^{\frac{r+\lambda}{\alpha}}\left[-\frac{\lambda^2}{\alpha}\left[\frac{1}{(r+\lambda)^2}-\frac{1}{(r+\lambda-\alpha)^2}\right]\ln x\right.\right. \\ &\left.\left. +\frac{\lambda^2}{2\alpha^2}\left[\frac{1}{r+\lambda}-\frac{1}{r+\lambda-\alpha}\right]\left[(\ln x)^2-2\ln b(\theta+2u)\ln x\right]\right]\right\}\chi_{\{\theta>\bar\theta-2u\}} \\ &+ C_3(\theta)x^{\frac{r+\lambda}{\alpha}}, \end{aligned}$$

where $C_3(\theta)$ need to be determined.

For $\theta > \bar\theta - 2u$, from condition (4.12), $f_2(\theta, b(\theta+2u)) = f_3(\theta, b(\theta+2u))$, then

$$\begin{aligned} &\left(\frac{\lambda}{r+\lambda}\right)^2 V(\theta+2u) + \frac{z_0 b(\theta+2u)}{r-\alpha}\left[1-\left(\frac{\lambda}{r+\lambda-\alpha}\right)^2\right] \\ &+ \left\{ z_0 b(\theta)\left[\frac{b(\theta+2u)}{b(\theta)}\right]^{\frac{r+\lambda}{\alpha}}\left[\frac{1}{r+\lambda}-\frac{1}{r+\lambda-\alpha}\right]\right\}\chi_{\{\theta>\bar\theta\}} \\ &+ \left\{ z_0 b(\theta+u)\left[\frac{b(\theta+2u)}{b(\theta+u)}\right]^{\frac{r+\lambda}{\alpha}}\left[\lambda\left[\frac{1}{(r+\lambda)^2}-\frac{1}{(r+\lambda-\alpha)^2}\right]\right]\right. \\ &\left. -\frac{\lambda}{\alpha}\left[\frac{1}{r+\lambda}-\frac{1}{r+\lambda-\alpha}\right]\ln\left[\frac{b(\theta+2u)}{b(\theta+u)}\right]\right]\right\}\chi_{\{\theta>\bar\theta-u\}} \\ = &\left(\frac{\lambda}{r+\lambda}\right)^3 V(\theta+3u) + \frac{z_0 b(\theta+2u)}{r-\alpha}\left[1-\left(\frac{\lambda}{r+\lambda-\alpha}\right)^3\right] \\ &+ \left\{ z_0 b(\theta+u)\left[\frac{b(\theta+2u)}{b(\theta+u)}\right]^{\frac{r+\lambda}{\alpha}}\left(-\frac{\lambda}{\alpha}\right)\left[\frac{1}{r+\lambda}-\frac{1}{r+\lambda-\alpha}\right]\ln b(\theta+2u)\right\}\chi_{\{\theta>\bar\theta-u\}} \\ &+ \left\{ z_0 b(\theta+2u)\left[-\frac{\lambda^2}{\alpha}\left[\frac{1}{(r+\lambda)^2}-\frac{1}{(r+\lambda-\alpha)^2}\right]\ln x\right.\right. \\ &\left.\left. -\frac{\lambda^2}{2\alpha^2}\left[\frac{1}{r+\lambda}-\frac{1}{r+\lambda-\alpha}\right](\ln b(\theta+2u))^2\right]\right\}\chi_{\{\theta>\bar\theta-2u\}} \\ &+C_3(\theta)b(\theta+2u)^{\frac{r+\lambda}{\alpha}} \end{aligned}$$

from where

$$\begin{aligned}
C_3(\theta) &= \left\{z_0 b(\theta)^{-\frac{r+\lambda}{\alpha}+1}\left[\frac{1}{r+\lambda}-\frac{1}{r+\lambda-\alpha}\right]\right\}\chi_{\{\theta>\bar{\theta}\}} \\
&+\left\{z_0 b(\theta+u)^{-\frac{r+\lambda}{\alpha}+1}\left[\lambda\left[\frac{1}{(r+\lambda)^2}-\frac{1}{(r+\lambda-\alpha)^2}\right]\right.\right. \\
&\left.\left.+\frac{\lambda}{\alpha}\left[\frac{1}{r+\lambda}-\frac{1}{r+\lambda-\alpha}\right]\ln b(\theta+u)\right]\right\}\chi_{\{\theta>\bar{\theta}-u\}} \\
&+\left\{z_0 b(\theta+2u)^{-\frac{r+\lambda}{\alpha}+1}\left[\lambda^2\left[\frac{1}{(r+\lambda)^3}-\frac{1}{(r+\lambda-\alpha)^3}\right]\right.\right. \\
&+\frac{\lambda^2}{\alpha}\left[\frac{1}{(r+\lambda)^2}-\frac{1}{(r+\lambda-\alpha)^2}\right]\ln b(\theta+2u) \\
&\left.\left.+\frac{\lambda^2}{2\alpha^2}\left[\frac{1}{r+\lambda}-\frac{1}{r+\lambda-\alpha}\right](\ln b(\theta+2u))^2\right]\right\}\chi_{\{\theta>\bar{\theta}-2u\}}.
\end{aligned}$$

For $\bar{\theta}-3u<\theta\leq\bar{\theta}-2u$, from f_3 definition and condition (4.13), comes

$$\lim_{x\to 0^+} f_3(\theta,x)=\left(\frac{\lambda}{r+\lambda}\right)^3 V(\theta+3u)\Leftrightarrow C_3(\theta)\lim_{x\to 0+} x^{\frac{r+\lambda}{\alpha}}=0,$$

then, again we have $C_3(\theta)=0$.

For $(\theta,x)\in\mathcal{C}_3$, the expression for f_3 is given by

$$\begin{aligned}
f_3(\theta,x) &= \left(\frac{\lambda}{r+\lambda}\right)^3 V(\theta+3u)+\frac{z_0 x}{r-\alpha}\left[1-\left(\frac{\lambda}{r+\lambda-\alpha}\right)^3\right] \\
&+\left\{z_0 b(\theta)\left[\frac{x}{b(\theta)}\right]^{\frac{r+\lambda}{\alpha}}\left[\frac{1}{r+\lambda}-\frac{1}{r+\lambda-\alpha}\right]\right\}\chi_{\{\theta>\bar{\theta}\}} \\
&+\left\{z_0 b(\theta+u)\left[\frac{x}{b(\theta+u)}\right]^{\frac{r+\lambda}{\alpha}}\left[\lambda\left[\frac{1}{(r+\lambda)^2}-\frac{1}{(r+\lambda-\alpha)^2}\right]\right]\right. \\
&\left.-\frac{\lambda}{\alpha}\left[\frac{1}{r+\lambda}-\frac{1}{r+\lambda-\alpha}\right]\ln\left[\frac{x}{b(\theta+u)}\right]\right]\right\}\chi_{\{\theta>\bar{\theta}-u\}} \\
&+\left\{z_0 b(\theta+2u)\left[\frac{x}{b(\theta+2u)}\right]^{\frac{r+\lambda}{\alpha}}\left[\lambda^2\left[\frac{1}{(r+\lambda)^3}-\frac{1}{(r+\lambda-\alpha)^3}\right]\right.\right. \\
&-\frac{\lambda^2}{\alpha}\left[\frac{1}{(r+\lambda)^2}-\frac{1}{(r+\lambda-\alpha)^2}\right]\ln\left[\frac{x}{b(\theta+2u)}\right]\Bigg]\Bigg\} \\
&\left.\left.+\frac{\lambda^2}{2\alpha^2}\left[\frac{1}{r+\lambda}-\frac{1}{r+\lambda-\alpha}\right](\ln x-\ln b(\theta+2u))^2\right]\right\}\chi_{\{\theta>\bar{\theta}-2u\}}.
\end{aligned}$$

Now we are able to understand that (4.5) is a general expression for the cases presented for f_1, f_2 and f_3.

Finally, we are in position to prove that the expression (4.5) is indeed the value function. In order to prove so, we need to prove that it is the solution of the optimal stopping problem (4.1), i.e., we need to prove that

$$rV(\theta)-\mathcal{L}V(\theta)\geq 0\ \wedge\ F(\theta,x)=V(\theta),\ \forall\,(\theta,x)\in\mathcal{S} \tag{4.18}$$

and

$$rF(\theta,x)-\mathcal{L}F(\theta,x)=0\ \wedge\ F(\theta,x)\geq V(\theta),\ \forall\,(\theta,x)\in\mathcal{C}.$$

Let us consider $(\theta,x)\in\mathcal{S}$, then $F(\theta,x)=V(\theta)$ and $F(\theta+u,x)=V(\theta+u)$. So,

$$rV(\theta)-\mathcal{L}V(\theta) = (r+\lambda)V(\theta)-\lambda V(\theta+u)-z_0 x=z_0\left(b(\theta)-x\right),$$

which is positive, given that in the stopping region we always have $0 < x < b(\theta)$. Thus, condition (4.18) is already verified.

Let us consider $(\theta, x) \in \mathcal{C}$ such that $n(\theta, x) = n$ and $n(\theta + u, x) = n - 1$, with $n \in \mathbb{N}$. We start realizing that

$$rF(\theta,x) - \mathcal{L}F(\theta,x) = 0 \Leftrightarrow \alpha x \frac{\partial F(\theta,x)}{\partial x} - (r+\lambda)F(\theta,x) + z_0 x = -\lambda F(\theta+u,x).$$

Calculations leads us to prove that $\alpha x \frac{\partial F(\theta,x)}{\partial x}$ is equivalent to

$$z_0 x \frac{\alpha}{r-\alpha}\left[1-\left(\frac{\lambda}{r+\lambda-\alpha}\right)^n\right] + (r+\lambda) z_0 x^{\frac{r+\lambda}{\alpha}} \sum_{k=0}^{n-1} \Big\{ b(\theta+ku)^{-\frac{r+\lambda}{\alpha}+1} \lambda^k \times$$
$$\sum_{m=0}^{k} \frac{1}{m!\,(-\alpha)^m}\left[\frac{1}{(r+\lambda)^{k-m+1}} - \frac{1}{(r+\lambda-\alpha)^{k-m+1}}\right]\left[\ln\left[\frac{x}{b(\theta+ku)}\right]\right]^m \Big\} \chi_{\{\theta>\bar{\theta}-ku\}}$$
$$+ z_0 x^{\frac{r+\lambda}{\alpha}} \alpha \sum_{k=1}^{n-1} \Big\{ b(\theta+ku)^{-\frac{r+\lambda}{\alpha}+1} \lambda^k \times$$
$$+ \sum_{m=1}^{k} \frac{1}{(m-1)!\,(-\alpha)^m}\left[\frac{1}{(r+\lambda)^{k-m+1}} - \frac{1}{(r+\lambda-\alpha)^{k-m+1}}\right]\left[\ln\left[\frac{x}{b(\theta+ku)}\right]\right]^{m-1} \Big\} \chi_{\{\theta>\bar{\theta}-ku\}}.$$

Based on previous calculations, we note that $\alpha x \frac{\partial F(\theta,x)}{\partial x} - (r+\lambda)F(\theta,x) + z_0 x$ can be written as

$$-(r+\lambda)\left(\frac{\lambda}{r+\lambda}\right)^n V(\theta+nu) - \frac{z_0 x}{r-\alpha}\left\{(r+\lambda-\alpha)\left[1-\left(\frac{\lambda}{r+\lambda-\alpha}\right)^n\right] - (r-\alpha)\right\}$$
$$- z_0 x^{\frac{r+\lambda}{\alpha}} \sum_{k=1}^{n-1} \Big\{ b(\theta+ku)^{-\frac{r+\lambda}{\alpha}+1} \lambda^k \times$$
$$\sum_{m=1}^{k} \frac{1}{(m-1)!\,(-\alpha)^{m-1}}\left[\frac{1}{(r+\lambda)^{k-m+1}} - \frac{1}{(r+\lambda-\alpha)^{k-m+1}}\right]\left[\ln\left[\frac{x}{b(\theta+ku)}\right]\right]^{m-1} \Big\} \chi_{\{\theta>\bar{\theta}-ku\}}.$$

Changing the variable in the first sum and simplifying the second term of the previous expression, we get

$$-\lambda\left(\frac{\lambda}{r+\lambda}\right)^{n-1} V(\theta+nu) - \lambda\frac{z_0 x}{r-\alpha}\left[1-\left(\frac{\lambda}{r+\lambda-\alpha}\right)^{n-1}\right]$$
$$- z_0 x^{\frac{r+\lambda}{\alpha}} \sum_{k=0}^{n-2} \Big\{ b(\theta+(k+1)u)^{-\frac{r+\lambda}{\alpha}+1} \lambda^{k+1} \sum_{m=1}^{k+1} \frac{1}{(m-1)!\,(-\alpha)^{m-1}} \times$$
$$\left[\frac{1}{(r+\lambda)^{k-m+2}} - \frac{1}{(r+\lambda-\alpha)^{k-m+2}}\right]\left[\ln\left[\frac{x}{b(\theta+(k+1)u)}\right]\right]^{m-1} \Big\} \chi_{\{\theta>\bar{\theta}-(k+1)u\}}.$$

Then, continuing with the simplification of the first two terms of the previous expression and changing the variable in the second sum, we obtain

$$-\lambda\Big\{\left(\frac{\lambda}{r+\lambda}\right)^{n-1} V((\theta+u)+(n-1)u) + \frac{z_0 x}{r-\alpha}\left[1-\left(\frac{\lambda}{r+\lambda-\alpha}\right)^{n-1}\right]$$
$$+ z_0 x^{\frac{r+\lambda}{\alpha}} \sum_{k=0}^{n-2} \Big\{ b((\theta+u)+ku)^{-\frac{r+\lambda}{\alpha}+1} \lambda^k \sum_{m=0}^{k} \frac{1}{m!\,(-\alpha)^m} \times$$
$$\left[\frac{1}{(r+\lambda)^{k-m+1}} - \frac{1}{(r+\lambda-\alpha)^{k-m+1}}\right]\left[\ln\left[\frac{x}{b((\theta+u)+ku)}\right]\right]^m \Big\} \chi_{\{\theta+u>\bar{\theta}-ku\}}\Big\},$$

which is exactly $-\lambda F(\theta+u,x)$.

Finally, it only remains to prove that $F(\theta,x) \geq V(\theta)$, $\forall\ (\theta,x) \in \mathcal{C}$.

Let us consider a fixed $\theta > 0$. From Theorem 5.1, for $0 < \theta \leq \bar{\theta}$,

$$\lim_{x\to 0^+} F(\theta,x) = \left(\frac{\lambda}{r+\lambda}\right)^{n(\theta,x)} V\left(\theta + n(\theta,x)u\right) \geq V(\theta).$$

Moreover, for $\theta > \bar{\theta}$, we have $F(\theta, b(\theta)) = V(\theta)$. If we prove that F is an increasing function on x, we conclude that $F(\theta,x) > V(\theta)$ for all $(\theta,x) \in \mathcal{C}$.

We start proving that, for a fixed $\theta > 0$, when $(\theta,x) \in \mathcal{C}$, $f_n(\theta,x)$ increases as a function of x, for all $n \in \mathbb{N}$. For that, we use an induction argument. Thus, consider initially f_1, which is given by expression

$$\frac{\lambda}{\lambda+r}V(\theta+u) + \frac{z_0 x}{r-\alpha}\left[1 - \frac{\lambda}{r+\lambda-\alpha}\right] + \left\{z_0 b(\theta)\left[\frac{x}{b(\theta)}\right]^{\frac{r+\lambda}{\alpha}}\left[\frac{1}{r+\lambda} - \frac{1}{r+\lambda-\alpha}\right]\right\}\chi_{\{\theta>\bar{\theta}\}}.$$

If $0 < \theta \leq \bar{\theta}$, since V is an increasing function of θ, it is straightforward that f_1 increases with x. For $\theta > \bar{\theta}$, to prove the monotony in x we need to use the first derivative, i.e.

$$\begin{aligned}
\frac{\partial f_1(\theta,x)}{\partial x} &= \frac{z_0}{r-\alpha}\left[1 - \frac{\lambda}{r+\lambda-\alpha}\right] + \left(\frac{r+\lambda}{\alpha}\right) z_0 \left[\frac{x}{b(\theta)}\right]^{\frac{r+\lambda}{\alpha}-1}\left[\frac{1}{r+\lambda} - \frac{1}{r+\lambda-\alpha}\right] \\
&= \frac{z_0}{r+\lambda-\alpha}\left[1 - \left(\frac{x}{b(\theta)}\right)^{\frac{r+\lambda}{\alpha}}\right].
\end{aligned}$$

We notice that, for $\theta > \bar{\theta}$, if $(\theta,x) \in \mathcal{C}$ this implies that $x > b(\theta)$, thus $\frac{\partial f_1(\theta,x)}{\partial x} > 0$. Then, for $(\theta,x) \in \mathcal{C}$, the function f_1 is always increasing in x.

Now, let us consider the function f_{n+1} given by, from (4.5),

$$\begin{aligned}
f_{n+1}(\theta,x) &= \left(\frac{\lambda}{r+\lambda}\right)^{n+1} V\left(\theta+(n+1)u\right) - \frac{z_0 x}{r-\alpha}\left[1 - \left(\frac{\lambda}{r+\lambda-\alpha}\right)^{n+1}\right] \\
&\quad + z_0 x^{\frac{r+\lambda}{\alpha}} \sum_{k=0}^{n}\Bigg\{ b(\theta+ku)^{-\frac{r+\lambda}{\alpha}+1}\lambda^k \times \\
&\quad \sum_{m=0}^{k}\frac{1}{m!\,(-\alpha)^m}\left[\frac{1}{(r+\lambda)^{k-m+1}} - \frac{1}{(r+\lambda-\alpha)^{k-m+1}}\right]\left[\ln\left[\frac{x}{b(\theta+ku)}\right]\right]^m\Bigg\}\chi_{\{\theta>\bar{\theta}-ku\}}.
\end{aligned}$$

Similarly to the case $n = 1$, for $0 < \theta \leq \bar{\theta} - u$, the function f_n increases with x. For $\theta > \bar{\theta} - u$ let us define $j_n(\theta,x) = f_{n+1}(\theta,x) - f_n(\theta,x)$. Doing some comprehensive calculations, we can write $j_n(\theta,x)$ as

$$\begin{aligned}
&z_0\lambda^n\Bigg\{\left[\frac{x}{(r+\lambda-\alpha)^{n+1}} - \frac{[(r+\lambda)V(\theta+nu) - \lambda V(\theta+(n+1)u)]}{(r+\lambda)^{n+1}}\right] + x^{\frac{r+\lambda}{\alpha}}\Bigg\{b(\theta+nu)^{-\frac{r+\lambda}{\alpha}+1}\times \\
&\sum_{m=0}^{n}\frac{1}{m!\,(-\alpha)^m}\left[\frac{1}{(r+\lambda)^{n-m+1}} - \frac{1}{(r+\lambda-\alpha)^{n-m+1}}\right]\left[\ln\left[\frac{x}{b(\theta+nu)}\right]\right]^m\Bigg\}\chi_{\{\theta>\bar{\theta}-nu\}}\Bigg\}.
\end{aligned}$$

Given that $\theta > \bar{\theta} - nu$ we have

$$\begin{aligned}
j_n(\theta,x) &= z_0\lambda^n\Bigg\{\left[\frac{x}{(r+\lambda-\alpha)^{n+1}} - \frac{b(\theta+nu)}{(r+\lambda)^{n+1}}\right] + x^{\frac{r+\lambda}{\alpha}} b(\theta+nu)^{-\frac{r+\lambda}{\alpha}+1}\times \\
&\quad \sum_{m=0}^{n}\frac{1}{m!\,(-\alpha)^m}\left[\frac{1}{(r+\lambda)^{n-m+1}} - \frac{1}{(r+\lambda-\alpha)^{n-m+1}}\right]\left[\ln\left[\frac{x}{b(\theta+nu)}\right]\right]^m\Bigg\}.
\end{aligned}$$

In order to set conclusions, we need to calculate the first and second derivatives, which are given by

$$\begin{aligned}\frac{\partial j_n(\theta,x)}{\partial x} &= z_0\lambda^n\left\{\frac{1}{(r+\lambda-\alpha)^{n+1}}\left[1-\left(\frac{x}{b(\theta+nu)}\right)^{\frac{r+\lambda}{\alpha}-1}\right]\right.\\ &\quad\left.-\left(\frac{x}{b(\theta+nu)}\right)^{\frac{r+\lambda}{\alpha}-1}\sum_{m=1}^{n}\frac{1}{m!\,(-\alpha)^m}\frac{1}{(r+\lambda-\alpha)^{n-m+1}}\left[\ln\left[\frac{x}{b(\theta+nu)}\right]\right]^m\right\}\end{aligned}$$

and

$$\frac{\partial^2 j_n(\theta,x)}{\partial x^2} = z_0\lambda^n b(\theta+nu)\left(\frac{x}{b(\theta+nu)}\right)^{\frac{r+\lambda}{\alpha}-2}\frac{1}{n!\,(-\alpha)^{n+1}}\left[\ln\left[\frac{x}{b(\theta+nu)}\right]\right]^n.$$

It is easy to check that $\left.\frac{\partial j_n(\theta,x)}{\partial x}\right|_{x=b(\theta+nu)}=0$ which, together with the second condition in (4.11), means that function $F(\theta,x)$ is smooth in x.

For a fixed $\theta>\bar{\theta}-nu$, we only want to study j_n for $x>b(\theta+nu)$. Since $x>b(\theta+nu)$ and $\alpha<0$, we have $\frac{\partial^2 j_n(\theta,x)}{\partial x^2}>0$. Thus, $\frac{\partial j_n(\theta,x)}{\partial x}$ increases as a function of x and, given that $\left.\frac{\partial j_n(\theta,x)}{\partial x}\right|_{x=b(\theta+nu)}=0$, we conclude that $\frac{\partial j_n(\theta,x)}{\partial x}$ is always positive for $x>b(\theta+nu)$. Then j_n also increases as a function of x.

Recall that $f_{n+1}(\theta,x)=f_n(\theta,x)+j_n(\theta,x)$ for $n\in\mathbb{N}$. We proved that f_1 increases as a function of x, for $x>b(\theta)$. Assuming that f_n increases as a function of x, for $x>b(\theta+nu)$ (induction assumption), we just proved that j_n also increases as a function of x. Then, by mathematical induction, f_{n+1} also increases as a function of x, for $x>b(\theta+nu)$.

At this point we know that each function f_n increases with x for a fixed $\theta>0$. In fact, given that $f_{n+1}(\theta,b(\theta+nu))=f_n(\theta,b(\theta+nu))$, this also means that the function F increases with x when $(\theta,x)\in\mathcal{C}$, which concludes the proof of Proposition 4.2.

■

4.A.2 Proof of Proposition 4.3

For $\theta>\bar{\theta}$, straightforward differentiation gives

$$\begin{aligned}\frac{\partial b(\theta)}{\partial z_1} &= \frac{(r+\lambda)\theta^\beta-\lambda(\theta+u)^\beta}{z_0 r}>0,\\ \frac{\partial b(\theta)}{\partial \lambda} &= z_1\left[\frac{\theta^\beta-(\theta+u)^\beta}{z_0 r}\right]<0,\\ \frac{\partial b(\theta)}{\partial z_0} &= -\frac{1}{z_0^2}\left[z_1\frac{(r+\lambda)\theta^\beta-\lambda(\theta+u)^\beta}{r}-rI\right]<0,\\ \frac{\partial b(\theta)}{\partial I} &= -\frac{r}{z_0}<0,\\ \frac{\partial b(\theta)}{\partial u} &= -\frac{\lambda z_1\beta(\theta+u)^{\beta-1}}{z_0 r}<0.\end{aligned}$$

■

4.A.3 Proof of Proposition 4.4

For $\theta > \bar{\theta}$, straightforward differentiation gives

$$\frac{\partial b(\theta)}{\partial \mu} = \frac{\beta z_1 \left[(r+\lambda)\theta^\beta - \lambda(\theta+u)^\beta\right]}{8z_0 r - \mu\beta + \frac{1}{2}\beta(1-\beta)\sigma^2]^2} > 0,$$
$$\frac{\partial b(\theta)}{\partial \sigma} = \frac{-\beta(1-\beta) z_1 \left[(r+\lambda)\theta^\beta - \lambda(\theta+u)^\beta\right]}{z_0[r - \mu\beta + \frac{1}{2}\beta(1-\beta)\sigma^2]^3} < 0.$$

The result for the other parameters is analogous to the proof of Proposition 4.3.

■

5

Capacity Optimization for Innovating Firms

Contents

5.1 Introduction

Technological progress implies that, as time passes, better products/technologies appear on the market. One of the questions is then: when should a firm invest in such products/technologies, and what should the firm do with the existing ones? Should the firm keep both products/technologies alive, or just use the most recent?

The following examples show that we may find different outcomes when it comes to the decision between add or replace. Shares of Target (TGT) were lower by nearly 5% in mid-morning trading on Wednesday 14th December, 2017, after the retailer announced that same-store sales for the holiday months of November and December decreased by 1.3%. While store sales were down 3% for the company, online sales were up over 30% (source: The Street). Similarly, Nordstrom shows a significant decline in its sales numbers, in opposition to online sales. Both companies are showing declining in-store sales numbers, since they have invested in in-sales with e-commerce. According to The Street, *the better Nordstrom's website becomes the less incentive you have to actually go to their stores. In other words, they are cannibalizing themselves*[1]. Thus these two firms are examples of firms that have invested large amounts in e-commerce, while keeping the tradicional commerce, and with a negative outcome.

Apple is an example of the exact opposite outcome. Since its early times, Apple has been producing both products (the established and the innovative one) for a while. Products like the iPhone, iPad, MacBook or iMac need to fight for their space, and thus everytime Apple realeses a new version or product, it runs the risk of cannibalizing its own products[2]. But so far this strategy has been showing to pay off.

These examples suggest that when a firm decides to innovate, creating a new product (as it is the case of Apple) or a new sales system (as the e-commerce), decisions about the existing ones are important for the firm's profit. But in spite of pros and cons, firms need to innovate in order to stay in business for a considerable amount of time. If firms stay with producing established products without performing any product or process innovations, their market share will decline and eventually they get out of business.

As we have seen from the previous examples, a firm should decide what to do with their existing production process after the firm has innovated. Essentially it can choose between replacing the established production process by the new one (single rollover), or keep on producing the established product so that it produces two products at the same time (dual rollover). The advantage of the latter is that the firm earns revenue from both markets, but, if the innovative product is a strategic substitute to the established product, the firm is competing with itself in the sense that growth of the innovative product market will attract consumers that at the same time leave the established product market, or the other way around. Also after initially choosing to simultaneously produce the established and the innovative product, after some time it can be optimal to stop taking the established product into

[1] Available at https://www.thestreet.com/story/13957574/1/target-s-holiday-sales-show-a-total-cannibalization-of-itself-more-squawk-from-jim-cramer.html

[2] Available at http://www.businessinsider.com/apple-on-cannibalization-2015-12

production due to the just described cannibalization effects.

Aim of this paper is to design a theoretical framework to analyze this problem. We start out with a firm producing an established product. The firm has an option to carry out a product innovation. To do so it has to adopt a new technology by which it can produce the innovative product. New technologies become available as time passes. Due to technological progress the quality of the newest available technology, and thus the quality of the innovative product that can be produced by this technology, increases over time, albeit in a stochastic way, because the firm does not know beforehand how fast technologies will develop. To capture this we impose that at discrete moments in time this quality jumps upward, the timing of which is modeled by a Poisson arrival process. The implication is that a later innovation enables the firm to produce a better innovative product, which will stimulate the innovative product demand and thus the innovative product revenue. So, typically the firm faces the tradeoff between innovating fast that enlarges its payoff soon but only by a small amount, or innovating later that leads to a larger payoff increase, the drawback being that the firm is stuck with producing the established product for a longer time.

While perfectly being aware of the size of the demand of the established product, the firm does not know beforehand how consumers will appreciate the innovative product and thus how demand of this product will develop over time. Therefore, we assume that demand of the new product is also stochastic, such that the output price satisfies a geometric Brownian motion (GBM, for short) process. A change in demand on the new market directly influences the size of the cannibalization effect on the established market, so therefore we impose that this cannibalization effect is also subject to the same GBM process.

Except from determining the optimal time to innovate, we also analyze the choice between the "add" and the "replace" option, where the replace option reflects the possibility that at the innovation time the firm stops producing the established product and begins producing the innovative product. The add option means that after innovating the firm produces both the established and the innovative product. After deciding to produce both products, the firm still has the option to stop producing the established product, which will boost demand of the innovative product.

Next to the timing of the innovation and the decision what to do with the established product, also the size of the innovation investment matters (see also Hagspiel *et al.* (2016)). A large investment enables the firm to install a large capacity so that it can produce a considerable amount of innovative products. This is good in case demand of the innovative product develops positively, but it is bad when demand develops worse. Then innovative product profits are low, and, in addition, putting a large amount of innovative products for sale cannibalizes the established product market to a large extent.

We now explain in what way we extend the existing literature. Farzin *et al.* (1998), Doraszelski (2001) and Doraszelski (2004) focus on the time to innovate where technological progress develops stochastically over time. The expected rate of new technologies arriving over time is constant, an assumption which is relaxed in Hagspiel *et al.* (2015). Cho & McCardle (2009) consider a firm simultaneously using two types of technologies and analyze the effect of their interdepencies on the timing of adopting upgrades. Smith & Ulu (2012) allow for uncertainty in future costs of adoption. Murto (2007) considers the effect of revenue uncertainty. Our paper also takes revenue uncertainty into account, but in addition to the just mentioned papers we explicitly analyze how to go further with the established product market after innovating and we also determine the production capacity size of the innovative product.

In Grenadier & Weiss (1997) different innovation strategies are outlined that take into account the established technology, but to innovate or not is just a yes-or-no decision. The new technology has given characteristics, so it is not taken into account that the newest technologies improve as time passes, as we do. Also the size of the innovation investment is optimally determined in our paper, while it is given in Grenadier & Weiss (1997).

Reinganum (1981), Fudenberg & Tirole (1985), and Milliou & Petrakis (2011) determine the optimal time to innovate in a framework where two firms have this innovation option, but the model is deterministic. Another difference with our work is that a process innovation that reduces costs is considered instead of a product innovation.

Huisman & Kort (2004) have a model with two firms that both can choose between adopting an existing technology immediately or wait for a given new technology that is better than the old one, which becomes available at some future unknown point in time. Our paper considers only one firm, but in addition we have that if a firm adopts later the corresponding technology is better due to technological progress, we determine the optimal size of the innovative investment, and explicitly consider whether and at what conditions to discontinue production of the established product.

Our work is closely related with Hagspiel *et al.* (2016) and Kwon (2010), in the sense that we also study the option to invest in a new product to boost the firm's profit. But here we consider some remarkable extensions. First, we assume that innovations occur according to a jump process, and therefore we have, besides the (stochastic) price, the state of technology. The former authors also address the option to invest in a new product, but there are no innovation events, being the price the only stochastic variable. Moreover, we include the option to produce both kinds of products until it is no longer optimal, and therefore the established product stops being produced. Finally, as considered in Hagspiel *et al.* (2016), we optimize with respect to capacity.

5.2 Model

We consider a firm that is currently producing an established product with capacity K_0. The firm produces up to capacity. The price for the established product satisfies

$$p_0 = \xi_0 - \alpha K_0,$$

assuming that $\xi_0 > \alpha K_0$, where ξ_0 is the maximum willingness to pay for the established product and $\alpha > 0$ is a constant parameter reflecting the sensitivity of the quantity with respect to the price. The instantaneous profit on the established product market equals

$$\pi_0 = (\xi_0 - \alpha K_0)K_0. \tag{5.1}$$

As the firm produces up to capacity and therefore, the variable costs are constant, we simplify notation by omitting these costs.

The firm has an option to innovate, i.e. to adopt a new technology by which it can produce a new innovative product. To do so the firm has to incur an investment cost. We consider that this cost is proportional to the capacity level of the new product, K_1, specifically the cost is equal to δK_1, with $\delta > 0$. For the new product we also assume that the firm produces up to capacity.

As in Huisman (2001) and Farzin *et al.* (1998), the state of the technology is given by a compound Poisson process, $\boldsymbol{\theta} = \{\theta_t : t \geq 0\}$. We may express θ_t as follows:

$$\theta_t = \theta_0 + uN_t$$

where θ_0 denotes the state of technology at the initial point in time, $u > 0$ is the jump size and $\{N_t, t \geq 0\}$ follows a homogeneous Poisson process with rate $\lambda > 0$. The later the firm adopts the higher quality the product has, so the higher the demand for this product will be.

We denote the time of adoption of a new technology by τ_1. When adopting, the firm has two options, either to add the new product to the product portfolio (for a certain time) or abolish the production of the established product. Moreover, the price of the new product is not known beforehand. It depends on a stochastic process, $\boldsymbol{X} = \{X_t, t \geq \tau_1\}$, that follows a GBM with drift μ and volatility $\sigma > 0$, with $r - \mu > 0$[3], where $r > 0$ is the (constant) interest rate. Typically for these new products the market is expected to be growing, therefore the drift usually is non-negative. In order to make sure that the price of the old market stays positive everywhere, we need to impose the additional assumption

$$r + \mu > \sigma^2. \tag{5.2}$$

In case the firm decides to replace the first product by the new one, at time τ_1, the price of the new product satisfies

$$p_1^R(X_t, \theta_{\tau_1}) = (\theta_{\tau_1} - \alpha K_1)X_t, \quad t \geq \tau_1.$$

Moreover, the investment cost is proportional to the quantity K_1 and equal to δK_1, with $\delta > 0$. On the other hand, if the new product is produced together with the established one, we need to formulate a demand system for the two products[4]:

$$\begin{aligned} p_0^A(X_t, \theta_{\tau_1}) &= \xi_0 - \alpha K_0 - \beta K_1 X_t, \qquad & t \geq \tau_1, \\ p_1^A(X_t, \theta_{\tau_1}) &= (\theta_{\tau_1} - \alpha K_1 - \beta K_0)X_t, \qquad & t \geq \tau_1. \end{aligned}$$

The new product is horizontally differentiated from the old one, where $\beta > 0$ represents the horizontal differentiation parameter. We assume β to be positive to reflect that the two products are substitutes. The upper bound of β is given by α ($\beta < \alpha$) meaning that it can never be the case that the quantity of the other product has a larger effect on the product price than the quantity of the product itself.

[3]This is a standard assumption to ensure that the optimal investment time is finite.

[4]The demand system can be derived from the following utility function

$$U = \xi_0 K_0 - \frac{1}{2}\alpha K_0^2 - \beta K_0 K_1 X + \theta_{\tau_1} K_1 X - \frac{1}{2}\alpha K_1^2 X - p_0 K_0 - p_1 K_1.$$

After adoption of the new technology also the old market becomes stochastic. This reflects that if the new product takes off, the demand of the old product will drop because consumers move to the new market. Eventually the firm will abandon the old product, which happens at the moment that the price of the established product falls too low. We denote the time of abandonment of the old product by τ_2.

The instantaneous profit function for the case that only the new product is produced is equal to

$$\pi_1^R(X_t, \theta_{\tau_1}) = (\theta_{\tau_1} - \alpha K_1)\, X_t K_1, \quad t \geq \tau_1,$$

while, in case that both products are produced, is given by

$$\pi_1^A(X_t, \theta_{\tau_1}) = (\xi_0 - \alpha K_0 - \beta K_1 X_t)\, K_0 + (\theta_{\tau_1} - \alpha K_1 - \beta K_0)\, X_t K_1, \quad t \geq \tau_1.$$

In the next section we solve the optimal stopping time problem, assuming that the capacity level of the new product, K_1, is given. Later we will also optimize w.r.t. K_1, and therefore the model derived in Section 5.3 is our benchmark model.

5.3 Benchmark model

In this section we derive the optimal decision regarding the following times:

i) when to invest in the new technology: τ_1;

ii) when to stop producing the first product: τ_2, with $\tau_2 \geq \tau_1$.

We assume that once the firm decides to invest in the new technology, the quantities of both products (K_0 and K_1) that will be produced are fixed. Moreover, we assume that the process $\boldsymbol{X}$, which influences the products prices, has initial value (at the moment the firm decides to invest in the new technology, τ_1) x, which is then a parameter of the problem. Furthermore, when the firm decides to invest in a new technology, the firm may decide if the new product replaces the original or if (and until when) the firm keeps producing both products.

The optimization problem is defined as follows

$$\begin{aligned} V_x(\theta) \quad &= \quad \sup_{\tau_1} \mathbb{E}\left[\int_0^{\tau_1} \pi_0 e^{-rs} ds + \left\{\sup_{\tau_2} \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \pi_1^A(X_s, \theta_{\tau_1}) e^{-rs} ds - \delta K_1 e^{-r\tau_1}\right.\right.\right. \\ &\quad \left.\left.\left.+\left\{\int_{\tau_2}^{+\infty} \pi_1^R(X_s, \theta_{\tau_1}) e^{-rs} ds\right\} \chi_{\{\tau_2<+\infty\}} \right| X_{\tau_1} = x\right]\right\} \chi_{\{\tau_1<+\infty\}} \left| \theta_0 = \theta \right], \qquad (5.3) \end{aligned}$$

for $\theta, x \in \mathbb{R}^+$, where $\chi_{\{A\}}$ represents the indicator function of set A. Note that we have indexed the value function by x as this depends explicitly on $X_{\tau_1} = x$, exogeneously fixed, which will play an important role in the sequel.

It follows from simple manipulations that we can re-write the value function V as

$$\begin{aligned} V_x(\theta) \quad &= \quad \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{\theta_0=\theta}\left[\left\{\sup_{\tau_2} \mathbb{E}^{X_{\tau_1}=x}\left[\int_{\tau_1}^{\tau_2} \left(\pi_1^A(X_s, \theta_{\tau_1}) - \pi_0\right) e^{-rs} ds - \delta K_1 e^{-r\tau_1}\right.\right.\right. \\ &\quad \left.\left.\left.+\left\{\int_{\tau_2}^{+\infty} \left(\pi_1^R(X_s, \theta_{\tau_1}) - \pi_0\right) e^{-rs} ds\right\} \chi_{\{\tau_2<+\infty\}}\right]\right\} \chi_{\{\tau_1<+\infty\}}\right] \end{aligned}$$

where, in order to ease the notation, $\mathbb{E}^{\theta_0=\theta}[...]$ and $\mathbb{E}^{X_{\tau_1}=x}[...]$ denote the conditional expectations $\mathbb{E}[...|\theta_0=\theta]$ and $\mathbb{E}[...|X_{\tau_1}=x]$, respectively. Applying a change of variable, we obtain

$$\begin{aligned} V_x(\theta) &= \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau_1}\left\{\sup_{\tau_2} \mathbb{E}^{X_{\tau_1}=x}\left[\int_0^{\tau_2-\tau_1} \left(\pi_1^A(X_{\tau_1+s},\theta_{\tau_1})-\pi_0\right)e^{-rs}ds \;-\delta K_1\right.\right.\right.\\ &\left.\left.\left.+\left\{\int_{\tau_2-\tau_1}^{+\infty}\left(\pi_1^R(X_{\tau_1+s},\theta_{\tau_1})-\pi_0\right)e^{-rs}ds\right\}\chi_{\{\tau_2<+\infty\}}\right]\right\}\chi_{\{\tau_1<+\infty\}}\right], \end{aligned}$$

from which, denoting by τ the time the firm is producing both products (i.e. $\tau=\tau_2-\tau_1$), we can write $V_x(\theta)$ equivalently

$$\begin{aligned} &\frac{\pi_0}{r} + \sup_{\tau_1}\mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau_1}\left\{\mathbb{E}^{X_{\tau_1}=x}\left[\int_0^{+\infty}\left(\pi_1^A(X_{\tau_1+s},\theta_{\tau_1})-\pi_0\right)e^{-rs}ds\right]-\delta K_1\right.\right. &(5.4)\\ &\left.\left.+\sup_{\tau}\mathbb{E}^{X_{\tau_1}=x}\left[\left\{\int_{\tau}^{+\infty}\left(\pi_1^R(X_{\tau_1+s},\theta_{\tau_1})-\pi_1^A(X_{\tau_1+s},\theta_{\tau_1})\right)e^{-rs}ds\right\}\chi_{\{\tau<+\infty\}}\right]\right\}\chi_{\{\tau_1<+\infty\}}\right]. \end{aligned}$$

We treat the two integrals separately. Firstly, we get

$$\mathbb{E}^{X_{\tau_1}=x}\left[\int_0^{+\infty}\left(\pi_1^A(X_{\tau_1+s},\theta_{\tau_1})-\pi_0\right)e^{-rs}ds\right] = \frac{(\theta_{\tau_1}-\alpha K_1-2\beta K_0)\,K_1 x}{r-\mu}. \tag{5.5}$$

We have used Fubini's theorem (see Hildebrandt (1963)) and the fact that the GBM has stationary increments. Besides that, the integral convergence is guaranteed by the initial assumption $r>\mu$.

Regarding the second integral, one note in view of the strong Markov property of the GBM (see Karlin (2014)) that $\{(X_t|X_{\tau_1}=x), t\geq\tau_1\} \stackrel{d}{=} \{(X_t|X_0=x), t\geq 0\}$, and by Fubini's theorem, it follows that $\mathbb{E}^{X_{\tau_1}=x}\left[\int_\tau^{+\infty}\left(\pi_1^R(X_{\tau_1+s},\theta_{\tau_1})-\pi_1^A(X_{\tau_1+s},\theta_{\tau_1})\right)e^{-rs}ds\right]$ is equal to

$$\mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{2\beta K_0 K_1 X_\tau}{r-\mu}-\frac{\pi_0}{r}\right)\right]. \tag{5.6}$$

Plugging (5.5) and (5.6) into (5.4) we arrive to the following expression for the value of the firm

$$V_x(\theta) = \frac{\pi_0}{r}+\sup_{\tau_1}\mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau_1}\rho_x(\theta_{\tau_1})\chi_{\{\tau_1<+\infty\}}\right] \tag{5.7}$$

with

$$\rho_x(\theta) = F(x) + \frac{\left[(\theta-\alpha K_1-2\beta K_0)\,x-\epsilon\right]K_1}{r-\mu}$$

where

$$\epsilon = \delta(r-\mu) \tag{5.8}$$

and

$$F(x) = \sup_{\tau}\mathbb{E}^{X_0=x}\left[e^{-r\tau}g(X_\tau)\chi_{\{\tau<+\infty\}}\right] \quad \text{and} \quad g(x)=\frac{2\beta K_0K_1}{r-\mu}x-\frac{\pi_0}{r}. \tag{5.9}$$

In fact, the optimization problem (5.3) can be seen as two optimization problems that need to be solved: one related with the optimal investment time in the new product (τ_1), and other related with the time from which the firm produces only the innovative product ($\tau_2=\tau_1+\tau$). Note that F, defined in (5.9), is the value function for a standard investment problem.

In the following proposition we present the solution to problem (5.9).

Proposition 5.1. *The solution of the problem presented in* (5.9) *is given by*

$$F(x) = \begin{cases} ax^{d_1} & 0<x<x^\star \\ \frac{2\beta K_0K_1}{r-\mu}x-\frac{\pi_0}{r} & x\geq x^\star \end{cases}$$

for all $x > 0$, where

$$x^\star = \frac{K_b}{K_1} \tag{5.10}$$

$$a = \frac{\pi_0}{r(d_1-1)} x^{\star - d_1} = \left[\frac{2\beta K_0 K_1}{d_1(r-\mu)}\right]^{d_1} \left[\frac{\pi_0}{r(d_1-1)}\right]^{1-d_1} \tag{5.11}$$

with

$$K_b = \frac{d_1}{2(d_1-1)} \frac{(r-\mu)\pi_0}{r\beta K_0} \quad \text{and} \quad d_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1. \tag{5.12}$$

Proof of Proposition 5.1. *See Appendix 5.A.1 for the proof.* ∎

Using the expression derived for F in Proposition 5.1, we can rewrite

$$\rho_x(\theta) = \rho_x^A(\theta)\chi_{\{0<x<x^\star\}} + \rho_x^R(\theta)\chi_{\{x \geq x^\star\}} \tag{5.13}$$

with

$$\rho_x^A(\theta) = a x^{d_1} + \frac{\left[(\theta - \alpha K_1 - 2\beta K_0)\, x - \epsilon\right] K_1}{r-\mu} \tag{5.14}$$

$$\rho_x^R(\theta) = \frac{\left[(\theta - \alpha K_1)\, x - \epsilon\right] K_1}{r-\mu} - \frac{\pi_0}{r}. \tag{5.15}$$

Next we present a general Theorem that is the basis for Propositions 5.2 and 5.3. We highlight that this theorem is general enough to be applied in many other stopping time problems.

Theorem 5.1. *Let us consider the optimal stopping problem $G(\theta) = \sup_\tau \mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau} g(\theta_\tau)\chi_{\{\tau<+\infty\}}\right]$, where $\boldsymbol{\theta} = \{\theta_t : t > 0\}$ is a compound Poisson process, with rate $\lambda > 0$ and jump size $u > 0$, and g is a continuous function. Let us also assume that*

$$\exists\,!\, \theta^\star > 0 \;:\; h(\theta) > 0 \Leftrightarrow \theta > \theta^\star, \tag{5.16}$$

where $h(\theta) = (r+\lambda)g(\theta) - \lambda g(\theta+u)$.

Then, the solution of the problem is given by

$$G(\theta) = \begin{cases} \left(\frac{\lambda}{\lambda+r}\right)^{n(\theta)} g(\theta + n(\theta)u) & 0 < \theta < \theta^\star \\ g(\theta) & \theta \geq \theta^\star \end{cases} \tag{5.17}$$

with $n(\theta) = \left\lceil \frac{\theta^\star - \theta}{u} \right\rceil$, where, for $k \geq 0$, $\lceil k \rceil = \min\{n \in \mathbb{N} : n \geq k\}$.

Proof of Theorem 5.1. *See Appendix 5.A.2 for the proof.* ∎

Now we are in position to present the solution of the optimization problem (5.7), which is done in Proposition 5.2.

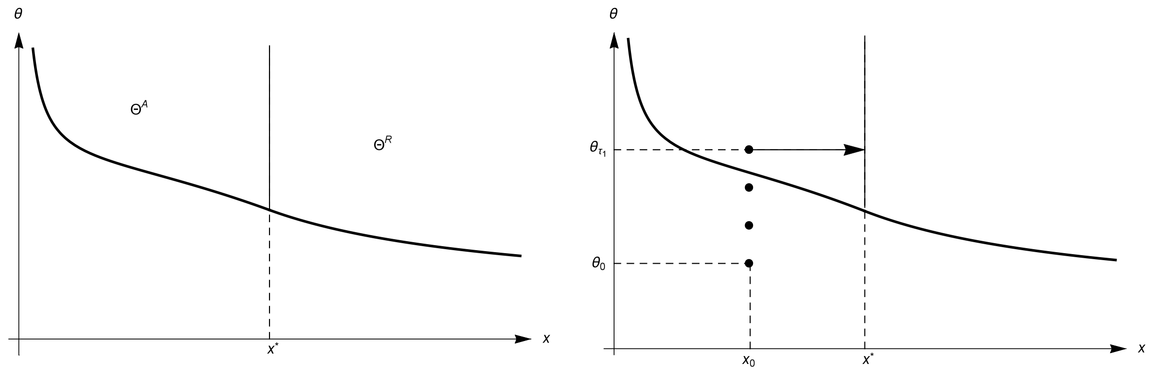
Proposition 5.2. *The solution of the optimal stopping problem* (5.7) *is given by*

$$V_x(\theta) = \frac{\pi_0}{r} + \begin{cases} \left(\frac{\lambda}{\lambda+r}\right)^{n_x(\theta)} \rho_x(\theta + n_x(\theta)\, u) & 0 < \theta < \theta_x^\star \\ \rho_x(\theta) & \theta \geq \theta_x^\star \end{cases} \tag{5.18}$$

for all $x > 0$ and $\theta > 0$, where ρ_x is defined at (5.13) *and*

$$n_x(\theta) = \left\lceil \frac{\theta_x^\star - \theta}{u} \right\rceil \tag{5.19}$$

$$\theta_x^\star = \upsilon^A(x)\chi_{\{0<x<x^\star\}} + \upsilon^R(x)\chi_{\{x\geq x^\star\}} \tag{5.20}$$



(a) Plan division. **(b)** Possible movements.

Figure 5.1: Benchmark case.

with

$$\upsilon^A(x) = \frac{\epsilon}{x} + \frac{\lambda u}{r} + \alpha K_1 + 2\beta K_0 - \frac{a(r-\mu)x^{d_1-1}}{K_1} \quad \textit{and} \quad \upsilon^R(x) = \frac{\epsilon}{x} + \frac{\lambda u}{r} + \alpha K_1 + \frac{(r-\mu)\pi_0}{rK_1 x}$$

where π_0*,* ϵ*,* x^**,* a *and* d_1 *are defined in* (5.1)*,* (5.8)*,* (5.10)*,* (5.11) *and* (5.12)*, respectively.*

Proof of Proposition 5.2. *See Appendix 5.A.3 for the proof.* ■

The firm invests in a certain technology level θ_{τ_1}, given the initial value x_0. Depending on the value of x_0, it will either produce both products ($x_0 < x^\star$) or only produce the new one ($x_0 > x^\star$). Then, we define the following sets (see Figure 5.1(a)) [5],

$$\begin{aligned} \Theta^A &= \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < x < x^\star \wedge \theta > \upsilon^A(x)\right\} \\ \Theta^R &= \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : x \geq x^\star \ \wedge \ \theta > \upsilon^R(x)\right\}. \end{aligned}$$

The set Θ^A represents the cases the firm produces both products upon investment, whereas Θ^R is when the firm decides to replace the original product by the innovative.

In Figure 5.1(a) we represent these sets and in Figure 5.1(b) we illustrate a possible behaviour of the firm. This figure should be read as follows: given that the firm is currently producing according to the technology level θ_0 and the initial value of the process $\boldsymbol{X}$ is $x_0 < x^\star$, the firm waits for new technology improvements. When the level of the technology hits or exceeds the threshold $\theta^\star$, the firm undertakes the investment. From time τ_1 on, the process $\boldsymbol{X}$, with initial value x evolves and when it hits the level $x^\star$, the firm abandons the first product and produces only the second.

5.4 Capacity optimization model

In this section we assume that the firm can optimally choose the invested capacity K_1. Therefore, regarding the optimization problem defined in (5.3), we also need to take into account the maximization

[5]The relative position between the two curves, given by the functions υ^A and υ^R, is described in Appendix 5.E.

w.r.t. K_1, i.e.,

$$\begin{aligned}
V_x(\theta) &= \sup_{\tau_1} \mathbb{E}\left[\int_0^{\tau_1} \pi_0 e^{-rs} ds + \max_{K_1}\left\{\sup_{\tau_2}\mathbb{E}\left[\int_{\tau_1}^{\tau_2} \pi_1^A(X_s,\theta_{\tau_1})e^{-rs}ds - \delta K_1 e^{-r\tau_1}\right.\right.\right.\\
&\quad \left.\left.\left. + \left\{\int_{\tau_2}^{+\infty} \pi_1^R(X_s,\theta_{\tau_1})e^{-rs}ds\right\}\chi_{\{\tau_2<+\infty\}}\right| X_{\tau_1}=x\right]\right\}\chi_{\{\tau_1<+\infty\}}\Bigg|\,\theta_0=\theta\Bigg].
\end{aligned}$$

Using similar arguments to the ones used in the previous section, we arrive to the following expression for the value of the firm

$$V_x(\theta) = \frac{\pi_0}{r} + \sup_{\tau_1}\mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau_1}\max_{K_1}\rho_x(K_1,\theta_{\tau_1})\chi_{\{\tau_1<+\infty\}}\right], \tag{5.21}$$

with ρ given by the same expression as in (5.13), but now we write explicitly K_1 as an argument of the function. In fact, in this section we change slightly the notation to emphasize the dependency of the profit function ρ in the capacity K_1. In particular, we change the way to present the profit function in the add and replace cases, in order to highlight this dependency. So, concerning the Expressions (5.14) and (5.15), we use the following representations,

$$\begin{aligned}
\rho_x^A(K_1,\theta) &= c(x)K_1^{d_1} - \frac{\alpha x}{r-\mu}K_1^2 + \frac{(\theta-2\beta K_0)x-\epsilon}{r-\mu}K_1 \\
\rho_x^R(K_1,\theta) &= -\frac{\alpha x}{r-\mu}K_1^2 + \frac{(\theta x-\epsilon)}{r-\mu}K_1 - \frac{\pi_0}{r}
\end{aligned} \tag{5.22}$$

where

$$c(x) = \frac{\Pi_0}{r(d_1-1)}\left(\frac{x}{K_b}\right)^{d_1}.$$

Moreover

$$\rho_x(K_1,\theta) = \rho_x^A(K_1,\theta)\chi_{\left\{0\le K_1<\frac{K_b}{x}\right\}} + \rho_x^R(K_1,\theta)\chi_{\left\{K_1\ge\frac{K_b}{x}\right\}}.$$

We remark that, for each choice of x and θ in $\mathbb{R}^+$, ρ is a continuous function in K_1, with $\rho_x^A\left(\frac{K_b}{x},\theta\right) = \rho_x^R\left(\frac{K_b}{x},\theta\right) = \frac{K_b}{r-\mu}\left(\theta - \frac{\epsilon+\alpha K_b}{x}\right) - \frac{\Pi_0}{r}$. Furthermore,

$$\rho_x\left(\frac{K_b}{x},\theta\right) > 0 \Leftrightarrow \theta > \frac{\epsilon}{x} + \frac{\alpha K_b}{x} + \frac{2(d_1-1)\beta K_0}{d_1}. \tag{5.23}$$

This condition will play an important role in the rest of the derivations, as we will see later on. We also underline that, given $x,\theta\in\mathbb{R}^+$, if $\rho_x(K_1,\theta)\le 0$ for all $K_1\in\mathbb{R}_0^+$, then $max_{K_1}\rho_x(K_1,\theta)=0$, and therefore it is never optimal to invest.[6]

We start by solving the maximization problem with respect to K_1. As ρ is the sum of two functions with non-overlapping domains, we first maximize each one of them (ρ^A and ρ^R) and afterwards we choose their maximum, i.e.,

$$\begin{aligned}
\max_{K_1}\rho_x(K_1,\theta) &= \max_{K_1}\left[\rho_x^A(K_1,\theta)\chi_{\left\{0\le K_1<\frac{K_b}{x}\right\}} + \rho_x^R(K_1,\theta)\chi_{\left\{K_1\ge\frac{K_b}{x}\right\}}\right] \\
&= \max\left\{\max_{K_1}\rho_x^A(K_1,\theta)\chi_{\left\{0\le K_1<\frac{K_b}{x}\right\}}, \max_{K_1}\rho_x^R(K_1,\theta)\chi_{\left\{K_1\ge\frac{K_b}{x}\right\}}\right\}.
\end{aligned}$$

In Appendix 5.B and 5.C we study the functions ρ_x^R and ρ_x^A , respectively. For the first function, we are able to derive all results analytically. However, for the second function, we are able to do so only for some cases, in particular for $d_1=2$. Furthermore, in Appendix 5.D we prove the results concerning the maximum of ρ. As this maximum depends on the value of d_1, we need to consider the following cases:

[6]This argument is often used along the paper to rule out some cases.

- For $d_1 = 2$,

$$\arg\max_{K_1} \rho_x(K_1, \theta) = \begin{cases} K_1^A(x,\theta) & \text{if} \quad (x,\theta) \in \Psi_1^A \\ K_1^R(x,\theta) & \text{if} \quad (x,\theta) \in \Psi_2^R(2) \end{cases} \tag{5.24}$$

where $K_1^A(x,\theta) = \frac{[(\theta - 2\beta K_0)x - \epsilon]\omega(2)}{2\alpha x(\omega(2) - x)}$, $K_1^R(x,\theta) = \frac{\theta x - \epsilon}{2\alpha x}$,

$$\Psi_1^A = \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < x < \omega(2) \ \wedge \ 2\beta K_0 < \theta - \frac{\epsilon}{x} \leq \frac{2\alpha K_b(2)}{x}\right\},$$

$$\Psi_2^R(d_1) = \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : \theta > \frac{\epsilon}{x} + 2\max\left\{\frac{\alpha K_b(d_1)}{x}, \sqrt{\frac{\alpha \Pi_0(r-\mu)}{rx}}\right\}\right\},$$

with $\omega(d_1) = \frac{d_1}{2(d_1-1)} \frac{\alpha K_b(d_1)}{\beta K_0}$ and $K_b(d_1) = \frac{d_1}{2(d_1-1)} \frac{(r-\mu)\pi_0}{r\beta K_0}$ [7].

- For $1 < d_1 < 2$,

$$\arg\max_{K_1} \rho_x(K_1, \theta) = \begin{cases} \phi_2(x,\theta) & \text{if} \quad (x,\theta) \in \Psi_1^{A_m}(d_1) \\ K_1^R(x,\theta) & \text{if} \quad (x,\theta) \in \Psi_2^R(d_1) \end{cases}$$

where $\phi_2(x,\theta)$ is the largest root of the first order derivative of ρ_x^A w.r.t. K_1,

$$\Psi_1^{A_m}(d_1) = \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < x < \omega(d_1) \ \wedge \ \vartheta_m(x) < \theta \leq \frac{\epsilon}{x} + \frac{2\alpha K_b(d_1)}{x}\right\}$$

and ϑ_m is implicitly defined as $\rho_x^A(\phi_2(x, \vartheta_m(x)), \vartheta_m(x)) = 0$.

- For $d_1 > 2$,

$$\arg\max_{K_1} \rho_x(K_1, \theta) = \begin{cases} \phi_1(x,\theta) & \text{if} \quad (x,\theta) \in \Upsilon_1^{A_M}(d_1) \\ K_1^R(x,\theta) & \text{if} \quad (x,\theta) \in \Upsilon_2^{R_M}(d_1) \end{cases}$$

where $\phi_1(x,\theta)$ is the smallest root of the first order derivative of ρ_x^A w.r.t. K_1,

$$\Upsilon_1^{A_M}(d_1) = \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < x < \omega(2) \ \wedge \ \frac{\epsilon}{x} + 2\beta K_0 < \theta \leq \upsilon_M(x)\right\}$$

$$\Upsilon_2^{R_M}(d_1) = \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : \theta > \upsilon_M(x)\right\}$$

with υ_M being implicitly defined by $\rho_x^R\left(\frac{K_1^R(x,\upsilon_M(x))}{x}, \upsilon_M(x)\right) = \rho_x^A\left(\phi_1\left(x, \upsilon_M(x)\right), \upsilon_M(x)\right)$.

In Figure 5.2 we present a sketch of the add and replace regions for different cases of d_1. In Figures 5.2(a) and 5.2(c), we use dotted lines to indicate that these values were obtained numerically. Only for case $d_1 = 2$, represented in Figure 5.10, we can provide analytical results.

As we can see from the figure, only the case $d_1 = 2$ can be analytically studied, as in the other cases we can only obtain numerical values for ρ_x (since we do not have an explicit expressions for $\phi_1(x,\theta)$ and $\phi_2(x,\theta)$).

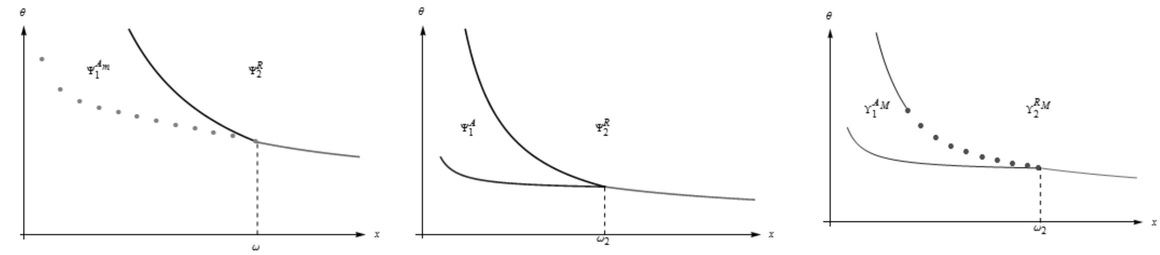
Now we are in position to present the solution for the optimization problem defined in (5.21). We start by providing the solution for the case $d_1 = 2$, which can be analytically derived. Afterwards, we discuss the solution for other values of d_1.

- Case $d_1 = 2$

Taking into account (5.24), we can rewrite the optimization problem (5.21) as

$$V_x(\theta) = \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{\theta_0 = \theta}\left[e^{-r\tau_1}\psi_x(\theta_{\tau_1})\chi_{\{\tau_1 < +\infty\}}\right] \tag{5.25}$$

[7] This is the same definition as (5.12) but only highlighting the dependence on d_1.



(a) $1 < d_1 < 2$ **(b)** $d_1 = 2$ **(c)** $d_1 > 2$

Figure 5.2: Plan division for ρ.

where

$$\psi_x(\theta) = \psi_x^A(\theta)\chi_{\left\{(x,\theta)\in\Psi_1^A\right\}} + \psi_x^R(\theta)\chi_{\left\{(x,\theta)\in\Psi_2^R\right\}}, \tag{5.26}$$

with

$$\begin{aligned}
\psi_x^A(\theta) &= \rho_x^A(K_1^A(x,\theta),\theta) = \frac{\left[(\theta - 2\beta K_0)\,x - \epsilon\right]^2 \omega(2)}{4\alpha(r-\mu)x\,(\omega(2) - x)} \\
\psi_x^R(\theta) &= \rho_x^R(K_1^R(x,\theta),\theta) = \frac{(\theta x - \epsilon)^2}{4\alpha(r-\mu)x} - \frac{\pi_0}{r}.
\end{aligned}$$

The solution of the problem is given by the next proposition.

Proposition 5.3. *The solution of the optimal stopping problem* (5.25) *is given by*

$$V_x(\theta) = \frac{\pi_0}{r} + \begin{cases} \left(\frac{\lambda}{\lambda+r}\right)^{n_x(\theta)} \psi_x(\theta + n_x(\theta)u) & 0 < \theta < \theta_x^* \\ \psi_x(\theta) & \theta \geq \theta_x^\star \end{cases} \tag{5.27}$$

for all $\theta > 0$ and $x > 0$, where n_x and ψ_x are defined at (5.19) *and* (5.26)*, respectively, and*

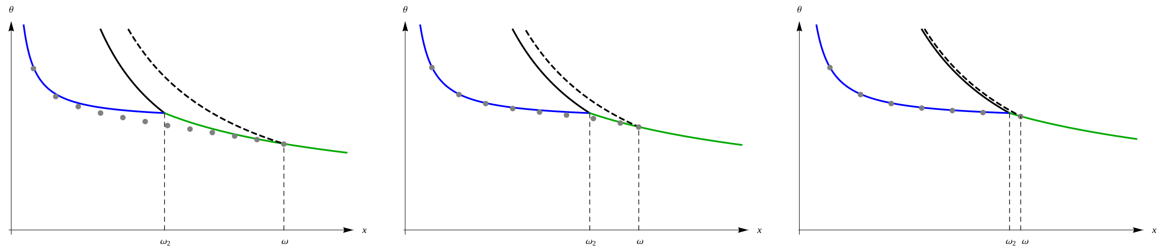
$$\theta_x^\star = \vartheta_1(x)\chi_{\{0<x<\xi_1\}} + \vartheta_2(x)\chi_{\{\xi_1\leq x<\xi_2\}} + \vartheta_3(x)\chi_{\{x\geq\xi_2\}}$$

with

$$\begin{aligned}
\vartheta_1(x) &= \frac{\epsilon}{x} + \frac{u}{r}\left[\lambda + \sqrt{\lambda(\lambda+r)}\right] + 2\beta K_0 \\
\vartheta_2(x) &= \frac{\epsilon}{x} + \frac{u\lambda(\omega(2)-x) + 2\beta K_0(r+\lambda)\omega(2)}{r\omega(2) + \lambda x} \\
&\quad + \frac{\sqrt{\lambda\omega(2)(\omega(2)-x)x\left[r(2\beta K_0)^2(x-\omega(2)) + u(r+\lambda)(4\beta K_0 + u)x\right]}}{x(r\omega(2)+\lambda x)} \\
\vartheta_3(x) &= \frac{\epsilon}{x} + \frac{u\lambda}{r} + \sqrt{\left(\frac{u}{r}\right)^2 \lambda(\lambda+r) + (2\beta K_0)^2\frac{\omega(2)}{x}} \\
\xi_1 &= \frac{2\beta K_0 r\omega(2)}{2\beta K_0 r + u\left[r + \lambda + \sqrt{\lambda(r+\lambda)}\right]} \\
\xi_2 &= \frac{2\beta K_0\omega(2)}{\lambda u^2}\left[\sqrt{(r\beta K_0 + \lambda u)^2 + \lambda r u^2} - (r\beta K_0 + \lambda u)\right]
\end{aligned}$$

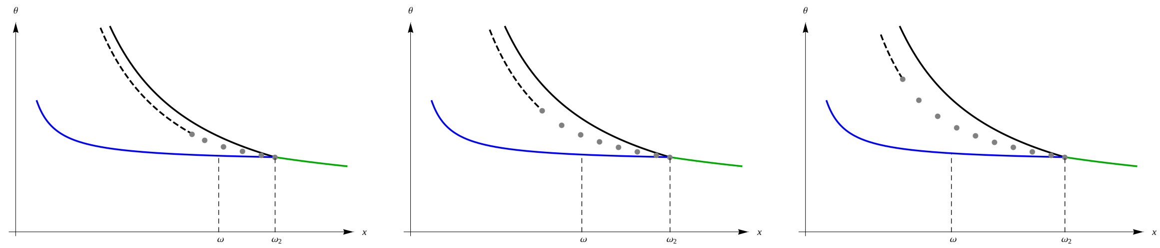
where π_0 and ϵ are defined in (5.1), (5.8)*, respectively.*

Proof of Proposition 5.3. *See Appendix 5.A.4 for the proof.* ■



(a) $d_1 = 1.6$ **(b)** $d_1 = 1.8$ **(c)** $d_1 = 1.95$

Figure 5.3: Comparison between case $1 < d_1 < 2$ (dashed line and dots) and case $d_1 = 2$ (bold lines).



(a) $d_1 = 2.3$ **(b)** $d_1 = 2.6$ **(c)** $d_1 = 3$

Figure 5.4: Comparison between case $d_1 > 2$ (dashed line and dots) and case $d_1 = 2$ (bold lines).

- Case $d_1 \neq 2$

We have already pointed out that when $d_1 \neq 2$ we are not able to have an explicit solution, neither for the value function nor for the threshold. However, we can understand how the structure of the solution is and relate it with the structure of the solution for $d_1 = 2$.

In the previous sections we did not emphasize the dependence in d_1 for many parameters, in order to keep notation simple. However in this section, we need to explicitly state the dependency of this parameters on d_1.

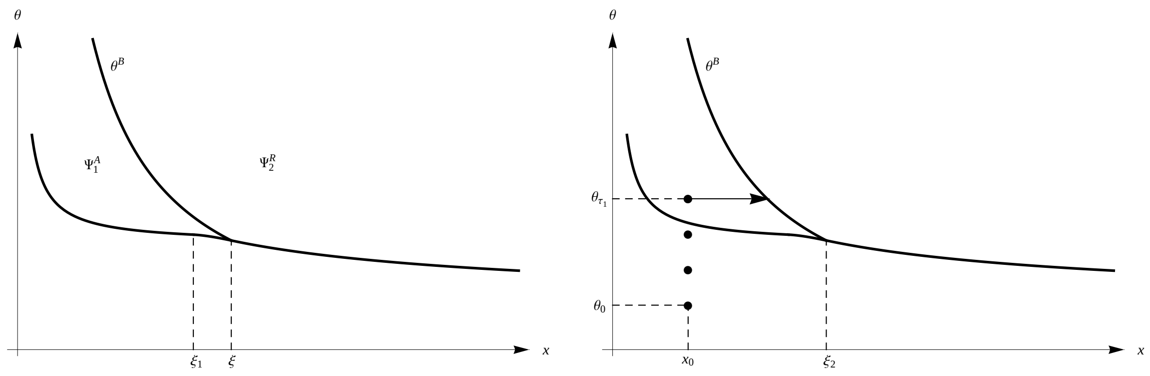
From the expressions of K_b and ω, as functions of d_1, we notice that they are continuous, in particular for $d_1 = 2$. We expect that the sets $\Upsilon_1^{A_M}(d_1), \Psi_1^A, \Psi_1^{A_m}(d_1)$ and $\Psi_2^R(d_1), \Psi_2^R(2), \Upsilon_2^{R_M}(d_1)$ move continuously with d_1, and that on the limit case of $d_1 \to 2$, they collapse in the corresponding sets for $d_1 = 2$. This behaviour is precisely illustrated in Figures 5.3 and 5.4, where we plot the relevant sets for $d_1 \to 2^-$ and $d_1 \to 2^+$, respectively. Moreover, as $\lim_{d_1 \to 2} \omega(d_1) = \omega(2)$, the area corresponding to the add/replace case decreases/increases with d_1.

In Figures 5.3 and 5.4 we highlight the evolution of the structure of the plan division. From there, we realize that $\Upsilon_1^{A_M}(d_1) \subset \Psi_1^A \subset \Psi_1^{A_m}(d_1)$ and $\Psi_2^R(d_1) \subset \Psi_2^R(2) \subset \Upsilon_2^{R_M}(d_1)$.

We stress that the add/replace boundary in the capacity optimization case is a curve instead of a point, which is given by the expression

$$\theta_x^B = \frac{\epsilon}{x} + \frac{2\alpha K_b(2)}{x}. \tag{5.28}$$

We note that the add/replace boundary has an interesting features, $x \times \theta_x^B$ is always constant, i.e,



(a) Plan division.

(b) Possible movements.

Figure 5.5: Capacity optimization case ($d_1 = 2$).

it is an iso-curve.

Furthermore, compared with the benchmark case the add/replace boundary is not a single point (does not depend on θ - vertical line) but is a curve (do depend on θ). As a consequence, for the benchmark case, the decision between add and replace depends only on the initial value x_0. On the other hand, for the capacity optimization case, this decision depends both on the initial value x_0 and the sample path of the innovation process.

We also notice that in this case the add/replace boundary is decreasing in x. If the firm invests for a larger θ, it will invest more, i.e. K_1 is larger. This increases the cannibalization effect, and thus makes replace more attractive than add. For this reason the firm changes from add to replace sooner when θ is larger. It is clear that when K_1 is given, this effect does not occur, so that the add/replace boundary is independent of θ.

In Figure 5.5(b) we depict one of the possible behaviour of the firm to achieve the stopping region. The interpretation of this figure is similar to the one presented for Figure 5.1(b).

5.5 Comparative statics

In this section we study the behaviour of the add/replace boundaries and decision thresholds with the different parameters. We note that in the benchmark case the boundary between the add and replace is a point, $x^\star$, whereas in the capacity optimization is a curve, θ_x^B. Therefore, we expect this latter case to be more challenging to analyse.

We start by studying the benchmark case, for which we provide results either for the add/replace boundary as well as for the investment threshold. For the capacity optimization the analysis is not so complete due to the reasons previously pointed out.

5.5.1 Benchmark model

We start studying how the add/replace boundary, defined in (5.10), $x^\star$, changes with all parameters.

Proposition 5.4. *The add/replace boundary, $x^\star$, decreases with $\alpha, \beta, K_0, K_1, \mu$ and r; increases with ξ_0 and σ; and it is constant with δ, λ and u.*

Proof of Proposition 5.4. *See Appendix 5.G.1 for the proof.*

Now we present interpretations for the results shown in Proposition 5.4.

The add/replace boundary is decreasing in α due to the fact that the firm looses less revenue on the old market for a higher α while the cannibalization effect stays the same. Regarding to the cannibalization parameter, β, we notice that the stronger the cannibalization effect the less attractive it is for the firm to produce both products. Relatively to the drift, if μ increases the innovative product market becomes more attractive and therefore, the firm has more incentive to increase the instantaneous profit of the innovative product by getting rid of the cannibalization effect. In respect of the interest rate, the higher r the less impact the profits made in the far future have. Therefore, the option value to replace in the future when first adding the product is of less importance. In all previous cases replace gets more attractive relative to add.

Concerning to the capacities, the add/replace boundary is also decreasing with both of them. The higher the capacity of the innovative product, K_1, the larger the cannibalization effect. This hurts the profit of add so that replace becomes more attractive. Relatively to the capacity of the old product, K_0, three effects can be distinguished. Due to an increase of K_0, the $x^\star$ decreases because the output price on the old market becomes lower and because of the increased cannibalization effect. On the other hand, a higher K_0 leads to a larger quantity on the old market and this has a positive effect on the $x^\star$. It turns out that the latter effect cancels against the cannibalization effect. Hence, $x^\star$ decreases because the market price of the old product becomes lower.

The add/replace boundary is increasing with demand intercept for the old product and the volatility. For the first case, the higher the demand intercept for the old product, ξ_0, the higher the value of the old product and therefore, the firm is more hesitant to replace it so that $x^\star$ gets larger. For the second case, if uncertainty goes up, it is known from real options theory that it is optimal to delay irreversible decision. In this case it means that the firm wants to delay leaving the old market. This implies that the add region gets larger and the eventual switch to replace will occur for a larger value of x (i.e. later). Therefore, $x^\star$ increases with σ.

In the following proposition we present the behaviour of the investment threshold, defined in (5.20), $\theta_x^\star$, with the relevant parameters. Before that, we highlight that v^A and v^R are both decreasing functions of x, which implies that $\theta_x^\star$ is also a decreasing function of x. This result is important because the comparative statics in $\theta_x^\star$ involves to compare two curves and not only two points.

Proposition 5.5. *The investment threshold, $\theta_x^\star$, decreases with μ; increases with δ, ξ_0, λ and u; decreases/increases with σ/β in the add region and stays constant in the replace region; and does not have a monotonic behaviour with K_0, K_1, α and r.*

Proof of Proposition 5.5. *See Appendix 5.G.1 for the proof.*

Hereafter, we present interpretations to the results presented in Proposition 5.5.

Starting with the drift, μ, we highlight that the larger the growth of the innovative product market, the more attractive the market is and therefore, the firm invests sooner.

Moving to the parameters with which the investment threshold is always increasing. Regarding the unit investment cost δ, we notice that the higher the costs for a given capacity K_1 the higher the technology level needs to be for the firm to justify investment. For ξ_0, the investment threshold is increasing with it, as the old market is more profitable for a higher ξ_0 and therefore, the firm waits for a higher technology level to justify investment. With regard to λ and u, $\theta^\star$ increases as well because it pays more for the firm to wait for the next technology jump if this is effected to arrive sooner and/or when this jump is larger in size.

The investment threshold is non-constant with the volatility of the innovative product market and the cannibalization parameter only in the add region. It decreases with σ given that the firm stays in the old market for a given time upon investment. This is due to the fact that upon investing the firm gains the option to eventually abolish the old market. As expected the parameter β will only affect the investment threshold when the firm will produce both on the old as well as on the innovative product market right after the investment. Since this parameter enhances the cannibalization effect, a larger β makes investment in that case less attractive.

The investment threshold does not have a monotonic behaviour with the remaining parameters. In some cases there are several effects driven the movements, which makes the interpretation very complex or not understandable. Hence, in this case we omit an explanation for the effect of the interest rate, r.

In general the firm invests later for a higher α because the firm has to wait for a higher technology level to justify the capacity K_1. However, three other effects can be identified that in total could result in investing earlier when α goes up. The first effect is what we refer to as the *option effect*. It explains that the option to replace after having added the innovative product in the first place, is smaller for higher α. The second effect has to do with the revenue before the firm invests. The higher the α the smaller the revenue on the old market on which the firm solely produces before the investment. We refer to this as the *opportunity cost effect*. In Proposition 5.4 we have established that the add/replace boundary $x^\star$ decreases with α. So if α goes up it could happen that the firm changes from an add to a replace strategy. This implies that no cannibalization takes place anymore and therefore, the firm will invest earlier. This is the third effect which we call the *cannibalization effect*.

Also for K_1, in general the firm invests later for a higher values because the investment cost is higher. However, four other effects can be distinguished that sometimes can lead to an earlier investment when K_1 goes up. The first effect is the *price effect* indicating that for a larger K_1 the price of the innovative product is lower. Second, there is the *scale effect* which refers to the impact that losing the old market counts less if K_1 is larger. Also in this we have an *option effect*. For larger K_1 it becomes more attractive to change from add to replace. Therefore, the corresponding option value is larger which affects the investment decision. Finally, we have the *cannibalization effect*, which has the same explanation given for α.

The effect of K_0 on the investment threshold is less straightforward from an intuitive point of view. The overall result that $\theta^\star$ can increase as well as decrease when K_0 goes up. Three effects can be distinguished here. The first effect is the *cannibalization effect* which is larger for a larger value of K_0. The second effect is the *option effect.* It is clear that K_0 affects the revenue on the old market. Therefore, it will also affect the value of switching from add to replace. The third effect comes from the fact that the revenue before the investment depends on K_0. So this is the *opportunity cost effect.*

5.5.2 Capacity optimization model

First of all, we note that the add/replace boundary, θ_x^B, defined in (5.28), is a decreasing function of x. In the following proposition we study how θ_x^B changes with the parameters.

Proposition 5.6. *The add/replace boundary, θ^B, decreases with β, K_0 and μ; increases with δ, ξ_0 and σ; it is constant with λ and u; and does not have a monotonic behaviour with α and r, concretely,*

- *for $\alpha_1 < \alpha_2$, if $\alpha_1 + \alpha_2 < \frac{\xi_0}{K_0}$ then $\theta^B(x;\alpha_1) < \theta^B(x;\alpha_2)$, otherwise $\theta^B(x;\alpha_1) > \theta^B(x;\alpha_2)$;*
- *for $r_1 < r_2$, if $\delta(r_2 - r_1) + \frac{\alpha\pi_0}{\beta K_0}\left[\frac{(r_2-\mu)d_1(r_2)}{r_2(d_1(r_2)-1)} - \frac{(r_1-\mu)d_1(r_1)}{r_1(d_1(r_1)-1)}\right] > 0$ then $\theta^B(x;r_1) < \theta^B(x;r_2)$, otherwise $\theta^B(x;r_1) > \theta^B(x;r_2)$.*

Proof of Proposition 5.6. *See Appendix 5.G.2 for the proof.*

As we observe comparing Propositions 5.4 and 5.6, optimizing the capacity in the new market does not affect how β, K_0, μ, ξ_0 and σ influence the add/replace boundary. Therefore, the interpretation we gave for the benchmark model carries over for this case.

Contrary to the benchmark case, now the add/replace boundary increases with δ. Taking as an example the case $d_1 = 2$, when δ goes up, the capacity that the firm invests in the new market decreases, implying a larger cannibalization effect. Then add becomes more attractive compared to replace.

From the benchmark case we know that if α goes up the replace region becomes bigger. This was because the price on the old market is lower. However, in the case of capacity optimization, this effect is not monotonic anymore. The reason is that there is now a contradictory effect. If the α goes up this has a negative effect on the capacity of the innovative product. This reduces the cannibalization effect which makes add more attractive.

In the benchmark case the replace region became bigger when the discount rate increased. Now the total effect is non-monotonic. In fact too many interacting effects play a role to draw a straightforward conclusion about the comparative statics here.

In the following proposition we show how the capacity chosen in the investment moment changes with x.

Proposition 5.7. *In the replace case, the optimal capacity K_1^R, increases with x. In the add case, for $d_1 = 2$, the optimal capacity K_1^A, increases with x.*

Proof of Proposition 5.7. *See Appendix 5.G.2 for the proof.*

From Proposition 5.7, we know that K_1 increases with x. This is because revenue of the new product increases with x. So, for a larger x increasing K_1 with one unit will increase revenue more, while investment costs are unaffected. Therefore, the firm will invest more in K_1.

5.6 Conclusion

This paper studies a setting where the firm, currently operating, has the option to invest in a more innovative technology, in order to boost its profits. Moreover, the firm may decide not only about the timing, but also about the capacity, and if the new product replaces the old one, or simply is added.

The fact that the firm needs to decide not only about the timing but also about the capacity size and the number of products kept alive, introduces more interactions between the involved parameters. Therefore the effect of the parameters is sometimes unexpected.

We established the following interesting finding. After innovating the firm has the choice to solely produce the new product, or for some time simultaneously producing the established and the new product, while it will abolish the old product once demand on the new market has grown enough. In the last case the chosen capacity is some weighted average between a lower level associated with producing both products and a higher level when just the new product is produced. In this way the firm anticipates the later abolishment of the old product by choosing a larger capacity for producing the innovative product.

Also, contrary to the standard investment problem, the investment threshold is not a single point but instead it is a curve, as the investment decision depends not only on price but also on the innovation level. In case we do not consider capacity optimization, we are able to derive, in a closed form, the optimal decision with respect to the timing and the add/replace decision.

A natural extension of this work is to assume that before investment the price is also stochastic, evolving accordingly to a GBM with negative drift.

Appendix 5.A Proofs of the optimal stopping problems

5.A.1 Proof of Proposition 5.1

We want to solve the optimal stopping time (5.9), i.e.

$$F(x) = \sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau} g(X_\tau)\chi_{\{\tau<+\infty\}}\right],$$

where

$$g(x) = \frac{2\beta K_0 K_1}{r-\mu}x - \frac{\pi_0}{r}.$$

The corresponding Hamilton-Jacobi-Bellman (HJB, for short) equation for the optimization problem is given by

$$\min\{rF(x) - \mathcal{L}_X F(x), F(x) - g(x)\} = 0$$

where $\mathcal{L}_X$ is the infinitesimal generator of the process $\boldsymbol{X}$, i.e.

$$\mathcal{L}_X f(x) = \frac{\sigma^2}{2}x^2 f''(x) + \mu x f'(x).$$

By construction, in the stopping region we trivially have $F(x) = g(x)$. Moreover, the continuation region, hereby denoted by $\mathcal{C}_X$, must contain the following set (see Øksendal (2014) for details)

$$\mathcal{U}_X = \{u > 0 : rg(u) - \mathcal{L}_X g(u) < 0\} = \left(0, \frac{\pi_0}{2\beta K_0 K_1}\right)$$

which implies that $\mathcal{C}_X = (0, x^\star)$, where $x^\star$ still needs to be derived, such that $x^\star \geq \frac{\pi_0}{2\beta K_0 K_1}$.

In the continuation region the function F must satisfy the left hand side of the HJB equation. Let us define ζ as the solution of the equation $r\zeta(x) - \mathcal{L}_X\zeta(x) = 0$, that is

$$\frac{\sigma^2}{2}x^2\zeta''(x) + \mu x\zeta'(x) - r\zeta(x) = 0. \tag{5.29}$$

The differential Equation (5.29) is a Cauchy-Euler equation, which means that the solution is given by

$$\zeta(x) = ax^{d_1} + bx^{d_2},$$

where d_1 and d_2 are the positive and negative solutions, respectively, of the quadratic equation

$$\frac{\sigma^2}{2}d(d-1) + \mu d - r = 0, \tag{5.30}$$

given by

$$d_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \qquad \text{and} \qquad d_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}.$$

Given that $r > \mu$, it follows that $d_1 > 1$ and $d_2 < 0$.

We note that this optimization problem is in fact a special case of the case studied by Guerra *et al.* (2016). The profit function g is a non-decreasing function of polynomial type, as considered by the referred authors. Then, one of the boundary conditions is that the solution for $x = 0$ needs to be zero, i.e. $\lim_{x\to 0^+}\zeta(x) = 0$. Therefore, we must have $b = 0$, and thus $\zeta(x) = ax^{d_1}$.

As the value function needs to be continuous and smooth in all its domain and, in particular, in x^*, then it follows from the smooth pasting conditions, $F(x^\star) = g(x^\star)$ and $F'(x) = F'(x)|_{x=x^\star}$ (for more details see Øksendal (2014)), that $x^\star$ and a are as presented in Expressions (5.10) and (5.11), respectively. The remain of the proof is to check that $F(x) = \zeta(x)\chi_{\{0<x<x^\star\}} + g(x)\chi_{\{x\geq x^\star\}}$ is indeed the solution of the HJB equation. For that we need to prove that:

- in the stopping region $rg(x) - \mathcal{L}_X g(x) \geq 0$.

  As $rg(x) - \mathcal{L}_X g(x) = 2\beta K_0 K_1 x - \pi_0$, it will be positive if and only if $x > \frac{\pi_0}{2\beta K_0 K_1}$. The result holds in case $x^\star > \frac{\pi_0}{2\beta K_0 K_1}$, which, in view of the expression for $x^\star$, is equivalent to

  $$\frac{r-\mu}{r}\frac{d_1}{d_1 - 1} > 1 \Leftrightarrow r - \mu d_1 > 0.$$

  Recalling the d_1 definition, which comes from the Equation (5.30), we have

  $$r - \mu d_1 = \frac{\sigma^2}{2}d_1(d_1 - 1),$$

  which is always positive. Thus $x^\star > \frac{\pi_0}{2\beta K_0 K_1}$ and therefore $rg(x) - \mathcal{L}_X g(x) \geq 0$ for $x \geq x^\star$.

- in the continuation region $\zeta(x) \geq g(x)$.

  By construction g is tangent with ζ at point x^*. Moreover $g(0) = -\frac{\pi_0}{r} < 0$, $\zeta(0) = 0$ and ζ is a convex function (because $a > 0$ and $d_1 > 1$). Therefore, by Boyd & Vandenberghe (2004), ζ must be above all its tangents, and thus, in particular, is above g.

Therefore, we conclude that F is indeed the solution of the HJB equation, which ends the proof.

■

5.A.2 Proof of Theorem 5.1

We want to solve the problem $G(\theta) = \sup_\tau \mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau}g(\theta_\tau)\chi_{\{\tau<+\infty\}}\right]$, for which the HJB equation is given by

$$\min\{rG(\theta) - \mathcal{L}_\theta G(\theta), G(\theta) - g(\theta)\} = 0,$$

with $\mathcal{L}_\theta\, l(\theta) = \lambda\,[l(\theta+u) - l(\theta)]$ being the infinitesimal generator of the compounded Poisson process $\boldsymbol{\theta}$. In the continuation region we should have $G(\theta) = \frac{\lambda}{\lambda+r}G(\theta+u)$ and $G(x) \geq g(x)$. In the stopping region we should have $G(\theta) = g(\theta)$ and $rg(\theta) - \mathcal{L}_\theta g(\theta) \geq 0$. Considering the function $h(\theta) = (r+\lambda)g(\theta) - \lambda g(\theta+u)$, which is continuous, and taking into account the Condition (5.16), we realize that $h(\theta^\star) = 0 \Leftrightarrow \frac{\lambda}{\lambda+r}G(\theta^\star+u) = g(\theta^\star)$, i.e. $\theta^\star$ is exactly the threshold between the continuation and the stopping regions. Given Condition (5.16), we propose that the continuation region stands for lower values of θ and the stopping region stands for higher values of θ. Therefore, we may write

$$G(\theta) = \begin{cases} \frac{\lambda}{\lambda+r}G(\theta+u) & 0 < \theta < \theta^\star \\ g(\theta) & \theta \geq \theta^\star \end{cases}$$

Using a backwards iterative reasoning, we can conclude that for θ such that $\theta^\star - nu \leq \theta < \theta^\star - (n-1)u$, with $n \in \mathbb{N}$ such that $\theta^\star - nu \geq 0$, we have

$$G(\theta) = \left(\frac{\lambda}{\lambda+r}\right)^n g(\theta+nu).$$

This expression can be written in a more compact way as presented in (5.17).

To finish the proof, it remains to check $rg(\theta) - \mathcal{L}_\theta g(\theta) \geq 0$ when $\theta \geq \theta^\star$, and $G(\theta) \geq g(\theta)$ when $0 < \theta < \theta^\star$. The first part comes instantly from Condition (5.16). For the second part, we must prove that $\left(\frac{\lambda}{\lambda+r}\right)^n g(\theta+nu) \geq g(\theta)$ for $\theta^\star - nu \leq \theta \leq \theta^\star - (n-1)u$, with $n \in \mathbb{N}$ such that $\theta^\star - nu \geq 0$. In order to prove it, we start realizing that

$$\left(\frac{\lambda}{\lambda+r}\right)^{n-1} g(\theta+(n-1)u) \leq \left(\frac{\lambda}{\lambda+r}\right)^n g(\theta+nu) \Leftrightarrow h\left(\theta+(n-1)u\right) \leq 0 \Leftrightarrow \theta \leq \theta^\star - (n-1)u.$$

This imply that, for a specific $n \in \mathbb{N}$ and $\theta > 0$, such that $\theta \leq \theta^\star - (n-1)u < ... < \theta^\star - 2u < \theta^\star - u < \theta^\star$, we have

$$g(\theta) \leq \left(\frac{\lambda}{\lambda+r}\right) g(\theta+u) \leq \left(\frac{\lambda}{\lambda+r}\right)^2 g(\theta+2u) \leq ... \leq \left(\frac{\lambda}{\lambda+r}\right)^n g(\theta+nu).$$

So, we proved that $\left(\frac{\lambda}{\lambda+r}\right)^n g(\theta+nu) \geq g(\theta)$ for $\theta^\star - nu \leq \theta \leq \theta^\star - (n-1)u$.

Therefore, we conclude that function G, given by (5.17), is indeed the solution of the HJB equation, which ends the proof.

■

5.A.3 Proof of Proposition 5.2

We want to solve the problem (5.7), which can be rewritten as

$$V_x(\theta) = \frac{\pi_0}{r} + \sup_\tau \mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau}\rho_x\left(\theta_\tau\right)\chi_{\{\tau<+\infty\}}\right],$$

where

$$\begin{aligned}\rho_x(\theta) &= \left\{\frac{\left[\left(\theta-\alpha K_1-2\beta K_0\right)x-\epsilon\right]K_1}{r-\mu}+ax^{d_1}\right\}\chi_{\{0<x<x^\star\}}\\ &\quad+\left\{\frac{\left(\theta-\alpha K_1\right)K_1x}{r-\mu}-\delta K_1-\frac{\pi_0}{r}\right\}\chi_{\{x\geq x^\star\}}\\ &= \frac{\left[\left(\theta-\alpha K_1\right)x-\epsilon\right]K_1}{r-\mu}+\left\{ax^{d_1}-\frac{2\beta K_0K_1x}{r-\mu}\right\}\chi_{\{0<x<x^\star\}}-\frac{\pi_0}{r}\chi_{\{x\geq x^\star\}}\end{aligned}$$

with $x^\star, a$ and d_1 defined in (5.10), (5.11) and (5.12), respectively.

For each $x>0$, the optimal stopping problem $G_x(\theta)=\sup_\tau \mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau}\rho_x(\theta_\tau)\chi_{\{\tau<+\infty\}}\right]$ is of the same type as that presented in Theorem 5.1. Given that ρ_x is a continuous function, we only need to prove that Condition (5.16) holds, in order to have the solution.

Let us consider the function $h_x(\theta)=(r+\lambda)\rho_x(\theta)-\lambda\rho_x(\theta+u)$, which can be written as

$$h_x(\theta) = \frac{\left[\left[r\left(\theta-\alpha K_1\right)-\lambda u\right]x-r\epsilon\right]K_1}{r-\mu}+r\left\{ax^{d_1}-\frac{2\beta K_0K_1x}{r-\mu}\right\}\chi_{\{0<x<x^\star\}}-\pi_0\chi_{\{x\geq x^\star\}}.$$

Notice that, for a fixed x, h_x is an increasing linear function in θ, with zero at

$$\gamma_x=\frac{\epsilon}{x}+\frac{\lambda u}{r}+\alpha K_1+\left\{2\beta K_0-\frac{a(r-\mu)}{K_1}x^{d_1-1}\right\}\chi_{\{0<x<x^\star\}}+\frac{(r-\mu)\pi_0}{rK_1x}\chi_{\{x\geq x^\star\}}.$$

Obviously, if $x\geq x^\star$ we certainly have $\gamma_x>0$. If $0<x<x^\star$ then, in view of the definitions of a and $x^\star$, it follows that $2\beta K_0K_1-a(r-\mu)x^{d_1-1}>2\beta K_0K_1-a(r-\mu)x^{\star d_1-1}=\frac{2\beta K_0K_1(d_1-1)}{d_1}>0$. Therefore $\gamma_x>0$ for all $x>0$. Hence $\theta^\star_x=\gamma_x$ is the only zero of h_x, and $h_x(\theta)>0\Leftrightarrow\theta>\theta^\star$, which means that Condition (5.16) holds.

By Theorem 5.1, we conclude that function V_x, given by (5.18), is indeed the solution of the optimal stopping problem (5.7).

■

5.A.4 Proof of Proposition 5.3

We want to solve the problem (5.25), which is equivalent to solve the optimal stopping problem $G_x\left(\theta\right)=\sup_{\tau_1}\mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau_1}\psi_x(\theta_{\tau_1})\chi_{\{\tau_1<+\infty\}}\right]$.

In view of Theorem 5.1, given that ψ_x is a continuous function, we just need to prove that Condition (5.16) holds for $h_x(\theta)=(r+\lambda)\psi_x(\theta)-\lambda\psi_x(\theta+u)$. Thus, the solution of the problem (5.25) is the one presented in (5.27).

Let us recall that problem (5.25) refers to the case $d_1=2$. For this specific choose, $\omega_2=\frac{\alpha(r-\mu)\pi_0}{r(\beta K_0)^2}$ and

$$\psi_x(\theta)=\psi_x^A(\theta)\chi_{\left\{(x,\theta)\in\Psi_1^A\right\}}+\psi_x^R(\theta)\chi_{\left\{(x,\theta)\in\Psi_2^R\right\}}$$

with

$$\begin{aligned}\Psi_1^A &= \left\{(x,\theta)\in\mathbb{R}^+\times\mathbb{R}^+ : 0<x<\omega_2\ \wedge\ 2\beta K_0<\theta-\frac{\epsilon}{x}\leq 2\beta K_0\frac{\omega_2}{x}\right\};\\ \Psi_2^R &= \left\{(x,\theta)\in\mathbb{R}^+\times\mathbb{R}^+ : \theta>\frac{\epsilon}{x}+2\beta K_0\max\left\{\frac{\omega_2}{x},\sqrt{\frac{\omega_2}{x}}\right\}\right\}.\end{aligned}$$

In view of the definition of h_x, Ψ^A and Ψ^R, it follows that h_x can be written as

$$h_x(\theta) = \begin{cases} -\lambda\psi_x^A(\theta+u) & (x,\theta) \notin \Psi_1^A \wedge (x,\theta+u) \in \Psi_1^A \\ h_x^1(\theta) = (r+\lambda)\psi_x^A(\theta) - \lambda\psi_x^A(\theta+u) & (x,\theta) \in \Psi_1^A \wedge (x,\theta+u) \in \Psi_1^A \\ h_x^2(\theta) = (r+\lambda)\psi_x^A(\theta) - \lambda\psi_x^R(\theta+u) & (x,\theta) \in \Psi_1^A \wedge (x,\theta+u) \in \Psi_2^R \\ h_x^3(\theta) = (r+\lambda)\psi_x^R(\theta) - \lambda\psi_x^R(\theta+u) & (x,\theta) \in \Psi_2^R \wedge (x,\theta+u) \in \Psi_2^R \\ -\lambda\psi_x^R(\theta+u) & (x,\theta) \notin \Psi_2^R \wedge (x,\theta+u) \in \Psi_2^R \\ 0 & otherwise \end{cases}$$

We start exploring how is the h_x function for different choices of x. If $0 < x < \omega_2$, we can have two different expressions for h_x, depending if the condition $2\beta K_0 \frac{\omega_2}{x} - 2\beta K_0 < u$ holds or not. So, if $0 < x < \frac{2\beta K_0\omega_2}{2\beta K_0+u}$, we have $h_x(\theta) = -\lambda\psi_x^A(\theta+u)\chi_{\left\{\frac{\epsilon}{x}+2\beta K_0-u<\theta\leq\frac{\epsilon}{x}+2\beta K_0\right\}} + h_x^1(\theta)\chi_{\left\{\frac{\epsilon}{x}+2\beta K_0<\theta\leq\frac{\epsilon}{x}+2\beta K_0\frac{\omega_2}{x}-u\right\}} + h_x^2(\theta)\chi_{\left\{\frac{\epsilon}{x}+2\beta K_0\frac{\omega_2}{x}-u<\theta\leq\frac{\epsilon}{x}+2\beta K_0\frac{\omega_2}{x}\right\}} + h_x^3(\theta)\chi_{\left\{\theta>\frac{\epsilon}{x}+2\beta K_0\frac{\omega_2}{x}\right\}}$. Also, for $\frac{2\beta K_0\omega_2}{2\beta K_0+u} \leq x < \omega_2$, we have $h_x(\theta) = -\lambda\psi_x^A(\theta+u)\chi_{\left\{\frac{\epsilon}{x}+2\beta K_0-u<\theta\leq\frac{\epsilon}{x}+2\beta K_0\right\}} + h_x^2(\theta)\chi_{\left\{\frac{\epsilon}{x}+2\beta K_0<\theta\leq\frac{\epsilon}{x}+2\beta K_0\frac{\omega_2}{x}\right\}} + h_x^3(\theta)\chi_{\left\{\theta>\frac{\epsilon}{x}+2\beta K_0\frac{\omega_2}{x}\right\}}$. Finally, if $x \geq \omega_2$, we have

$$h_x(\theta) = -\lambda\psi_x^R(\theta+u)\chi_{\left\{\frac{\epsilon}{x}+2\beta K_0\sqrt{\frac{\omega_2}{x}}-u<\theta\leq\frac{\epsilon}{x}+2\beta K_0\sqrt{\frac{\omega_2}{x}}\right\}} + h_x^3(\theta)\chi_{\left\{\theta>\frac{\epsilon}{x}+2\beta K_0\sqrt{\frac{\omega_2}{x}}\right\}}.$$

We notice that $h_x(\theta) = 0$ when $\theta \leq \frac{\epsilon}{x} + 2\beta K_0 \min\left\{1, \sqrt{\frac{\omega_2}{x}}\right\} - u$. We also highlight that $h_x(\theta) < 0$ for $\frac{\epsilon}{x} + 2\beta K_0 \min\left\{1, \sqrt{\frac{\omega_2}{x}}\right\} - u < \theta \leq \frac{\epsilon}{x} + 2\beta K_0 \min\left\{1, \sqrt{\frac{\omega_2}{x}}\right\}$. On one hand, taking into account that ψ_x^A is a convex quadratic function, with an unique zero at $\frac{\epsilon}{x} + 2\beta K_0$, we conclude that $\psi_x^A(\theta) > 0$ for all $x > \frac{\epsilon}{x} + 2\beta K_0$. Therefore, when $0 < x < \omega_2$, $h_x(\theta) < 0$ for $\frac{\epsilon}{x} + 2\beta K_0 - u < \theta \leq \frac{\epsilon}{x} + 2\beta K_0$. On the other hand, as ψ_x^R is a convex quadratic function, with $\psi_x^R\left(\frac{\epsilon}{x}\right) < 0$ and $\psi_x^R\left(\frac{\epsilon}{x} + 2\beta K_0\sqrt{\frac{\omega_2}{x}}\right) = 0$, we conclude that $\psi_x^R(\theta) > 0$ for all $\theta > \frac{\epsilon}{x} + 2\beta K_0\sqrt{\frac{\omega_2}{x}}$. Hence, when $x \geq \omega_2$, $h_x(\theta) < 0$ for $\frac{\epsilon}{x} + 2\beta K_0\sqrt{\frac{\omega_2}{x}} - u < \theta \leq \frac{\epsilon}{x} + 2\beta K_0\sqrt{\frac{\omega_2}{x}}$.

Now, to ease the explanation we investigate the functions h_x^1, h_x^2 and h_x^3 separately. Before proceeding, we reinforce that the functions h_x^1 and h_x^2 are considered only when $0 < x < \frac{2\beta K_0\omega_2}{2\beta K_0+u} < \omega_2$ and $0 < x < \omega_2$, respectively.

- $h_x^1(\theta) = \frac{\omega_2\left\{r[(\theta-2\beta K_0)x-\epsilon]^2-2u\lambda x[(\theta-2\beta K_0)x-\epsilon]-\lambda(ux)^2\right\}}{4\alpha(r-\mu)x(\omega_2-x)}$ is a convex quadratic function (when $x < \omega_2$), its minimizer occurs at $\frac{\epsilon}{x} + \frac{u\lambda}{r} + 2\beta K_0$ and its largest zero is $\vartheta_1(x) = \frac{\epsilon}{x} + \frac{u}{r}\left[\lambda + \sqrt{\lambda(\lambda+r)}\right] + 2\beta K_0$. Also, $h_x^1\left(\frac{\epsilon}{x} + 2\beta K_0\right) < 0$ when $x < \omega_2$, meaning that $\vartheta_1(x)$ is the only zero of h_x^1 that can also be a zero of h_x. Further, we only have $\vartheta_1(x)$ as a zero of h_x if $\vartheta_1(x) < \frac{\epsilon}{x} + 2\beta K_0\frac{\omega_2}{x} - u \Leftrightarrow x < \xi_1$, where $\xi_1 = \frac{2r\beta K_0\omega_2}{2r\beta K_0+u\left[r+\lambda+\sqrt{\lambda(r+\lambda)}\right]}$[8].

  Summing up, for $0 < x < \xi_1$, we have $h_x^1(\theta) < 0$ if $\frac{\epsilon}{x} + 2\beta K_0 < \theta < \vartheta_1(x)$ and $h_x^1(\theta) > 0$ if $\vartheta_1(x) < \theta \leq \frac{\epsilon}{x} + 2\beta K_0\frac{\omega_2}{x} - u$; for $\xi_1 \leq x < \frac{2\beta K_0\omega_2}{2\beta K_0+u}$, we have $h_x^1(\theta) < 0$ if $\frac{\epsilon}{x} + 2\beta K_0 < \theta \leq \frac{\epsilon}{x} + 2\beta K_0\frac{\omega_2}{x} - u$.

- $h_x^2(\theta)$, which is given by
  $\frac{r\left\{(r\omega_2+\lambda x)(\theta x-\epsilon)^2-2[2\beta K_0(r+\lambda)\omega_2+u\lambda(\omega_2-x)]x(\theta x-\epsilon)+(2\beta K_0)^2\omega_2 x[(r+\lambda)x+\lambda(\omega_2-x))]-\lambda(\omega_2-x)(ux)^2\right\}}{4\alpha r(r-\mu)x(\omega_2-x)}$,
  is a convex quadratic function (when $x < \omega_2$), with minimizer equals to $\frac{\epsilon}{x} + \frac{2\beta K_0(r+\lambda)\omega_2+u\lambda(\omega_2-x)}{r\omega_2+\lambda x}$ and, when $x > \frac{r(2\beta K_0)^2\omega_2}{r(2\beta K_0)^2+u(r+\lambda)(4\beta K_0+u)}$, h_x^2 has two zeros and the largest is $\vartheta_2(x) = \frac{\epsilon}{x} +$

[8] Note that $\xi_1 < \frac{2\beta K_0\omega_2}{2\beta K_0+u}$.

$\frac{2\beta K_0(r+\lambda)\omega_2+u\lambda(\omega_2-x)}{r\omega_2+\lambda x}+\frac{\sqrt{\lambda\omega_2(\omega_2-x)x[r(2\beta K_0)^2(x-\omega_2)+u(r+\lambda)(4\beta K_0+u)x]}}{x(r\omega_2+\lambda x)}$. In addition, it is straightforward to prove that the minimizer of h_x^2 is lower than $\frac{\epsilon}{x}+2\beta K_0\frac{\omega_2}{x}-u$ only if $x<\frac{2\beta K_0 r\omega_2}{2\beta K_0 r+u(r+\lambda)}$[9]. This means that, for $0<x<\frac{2\beta K_0 r\omega_2}{2\beta K_0 r+u(r+\lambda)}$, the function h_x^2 is strictly increasing when $\frac{\epsilon}{x}+2\beta K_0\frac{\omega_2}{x}-u<\theta\leq\frac{\epsilon}{x}+2\beta K_0\frac{\omega_2}{x}$.

- $h_x^3(\theta)=\frac{r(\theta x-\epsilon)^2-2u\lambda x(\theta x-\epsilon)-\left[(2\beta K_0)^2 rx+(\lambda x)^2\right]}{4\alpha(r-\mu)x}$ is a convex quadratic function, with minimizer $\frac{\epsilon}{x}+\frac{u\lambda}{r}$ and its largest zero is $\vartheta_3(x)=\frac{\epsilon}{x}+\frac{u\lambda}{r}+\sqrt{\left(\frac{u}{r}\right)^2\lambda(\lambda+r)+(2\beta K_0)^2\frac{\omega_2}{x}}$. Given that $h_x^3\left(\frac{\epsilon}{x}\right)<0$, then $\vartheta_3(x)$ is the only zero of h_x^3 that can also be a zero of h_x. Moreover, when $0<x<\omega_2$, $\vartheta_3(x)$ is a zero of h_x only if $\vartheta_3(x)>\frac{\epsilon}{x}+2\beta K_0\frac{\omega_2}{x}\Leftrightarrow x>\xi_2$, with $\xi_2=\frac{2\beta K_0\omega_2\left[\sqrt{(r\beta K_0+\lambda u)^2+\lambda r u^2}-(r\beta K_0+\lambda u)\right]}{\lambda u^2}$[10]. For $x>\omega_2$, as $h_x^3\left(\frac{\epsilon}{x}+2\beta K_0\sqrt{\frac{\omega_2}{x}}\right)<0$, $\vartheta_3(x)$ is the only zero of h_x.

  Summarizing, for $0<x\leq\xi_2$, we have $h_x^3(\theta)>0$ if $\theta>\frac{\epsilon}{x}+2\beta K_0\frac{\omega_2}{x}$; for $x>\xi_2$, we have $h_x^3(\theta)<0$ if $\frac{\epsilon}{x}+2\beta K_0\max\left\{\frac{\omega_2}{x},\sqrt{\frac{\omega_2}{x}}\right\}<\theta<\vartheta_3(x)$ and $h_x^3(\theta)>0$ if $\theta>\vartheta_3(x)$.

Joining all the previous arguments, we can conclude that, for $\theta>\frac{\epsilon}{x}+2\beta K_0\max\left\{1,\sqrt{\frac{\omega_2}{x}}\right\}-u$, the h_x function has only one zero given by $\theta_x^\star=\vartheta_1(x)\chi_{\{0<x<\xi_1\}}+\vartheta_2(x)\chi_{\{\xi_1\leq x<\xi_2\}}+\vartheta_3(x)\chi_{\{x\geq\xi_2\}}$. Furthermore, $h_x(\theta)>0\Leftrightarrow\theta>\theta^\star$, which concludes the proof.

■

5.A.5 Discussion of (5.2)

As $p_0^A(x,\theta)$ is decreasing in x, we just need to prove the price of the old product at the add/replace boundary is positive (which is $x=x^\star$ in the benchmark case and $\theta(x)=\theta^B(x)$ in the capacity optimization case). For both the benchmark and the capacity optimization models, at the add/replace boundary the price is given by $\xi_0-\alpha K_0-\beta K_b$, which is positive if (5.2) holds.[11]

Appendix 5.B Maximization of ρ^R

We stress that in following two sections we assume that x and θ are given and we only study the behaviour of the functions w.r.t. K_1.

Trivial calculations show that the maximum of ρ^R w.r.t. K_1 is attained at

$$K_1^R(x,\theta)=\frac{\theta x-\epsilon}{2\alpha x}, \tag{5.31}$$

which is positive if and only if $\theta>\frac{\epsilon}{x}$. In addition, in order to ensure that the maximum is positive, we also need to have

$$\theta>\frac{\epsilon}{x}+2\sqrt{\frac{\alpha\Pi_0(r-\mu)}{rx}}. \tag{5.32}$$

In view of the restriction $K_1\geq\frac{K_b}{x}$, we notice that

$$\frac{K_b}{x}<K_1^R(x,\theta)\Leftrightarrow\theta>\frac{\epsilon}{x}+\frac{2\alpha K_b}{x}. \tag{5.33}$$

[9]Note that $\xi_1<\frac{2\beta K_0 r\omega_2}{2\beta K_0 r+u(r+\lambda)}<\frac{2\beta K_0\omega_2}{2\beta K_0+u}$.

[10]After some comprehensive calculus, one can prove that $\xi_1<\xi_2<\omega_2$.

[11]We note that $\pi_1^A(x,\theta_x^\star)-\pi_0=\beta K_0 K_1 x+p_1^A(x,\theta_x^\star)K_1>0$, then it follows that $p_1^A(x,\theta_x^\star)>\beta K_0 x$ and therefore it is positive.

Therefore, $arg\max_{K_1} \rho_x^R(K_1,\theta)\chi_{\left\{K_1 \geq \frac{K_b}{x}\right\}}$ is either equal to $\frac{K_b}{x}$ or to $K_1^R(x,\theta)$ (depending on the relative position of the curves defined by Conditions (5.23), (5.32) and (5.33)), which leads to, respectively, $\theta = \frac{\epsilon}{x} + \frac{\alpha K_b}{x} + \frac{2(d_1-1)\beta K_0}{d_1}$, $\theta = \frac{\epsilon}{x} + 2\sqrt{\frac{\alpha \Pi_0 (r-\mu)}{rx}}$ and $\theta = \frac{\epsilon}{x} + \frac{2\alpha K_b}{x}$.[12]

Henceforward we will often use the notation

$$\omega = \frac{\alpha r K_b^2}{(r-\mu)\pi_0} = \frac{d_1}{2(d_1-1)} \frac{\alpha K_b}{\beta K_0}. \tag{5.34}$$

Now, considering the following notation

$$\begin{aligned}
\Psi_1^R &= \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < x < \omega \wedge \frac{\alpha K_b}{x} + \frac{2(d_1-1)\beta K_0}{d_1} < \theta - \frac{\epsilon}{x} \leq \frac{2\alpha K_b}{x}\right\} \\
\Psi_2^R &= \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : \theta > \frac{\epsilon}{x} + 2\max\left\{\frac{\alpha K_b}{x}, \sqrt{\frac{\alpha \Pi_0 (r-\mu)}{rx}}\right\}\right\}
\end{aligned}$$

and using the fact that ρ^R is a quadratic function of K_1, we conclude that

$$\arg\max_{K_1} \rho_x^R(K_1,\theta)\chi_{\left\{K_1 \geq \frac{K_b}{x}\right\}} = \begin{cases} \frac{K_b}{x} & \text{if} \quad (x,\theta) \in \Psi_1^R \\ K_1^R(x,\theta) & \text{if} \quad (x,\theta) \in \Psi_2^R \end{cases}. \tag{5.35}$$

Appendix 5.C Maximization of ρ^A

When the new product is added and the firm produces both, its profit, ρ^A, depends in d_1 (see expression (5.22)). Here we analyse separately the cases $d_1 = 2$ and $d_1 \neq 2$, as in the case $d_1 = 2$ one can derive analytical results, in opposition to the case $d_1 \neq 2$. Contrary to the replace case, now K_1 is bounded above by $\frac{K_b}{x}$.

5.C.1 Case $d_1 = 2$

In this case, ρ^A is given by

$$\rho_x^A(K_1,\theta) = K_1 \left[\frac{\pi_0}{rK_b^2} x(x-\omega)K_1 + \frac{(\theta - 2\beta K_0)x - \epsilon}{r-\mu}\right].$$

We need to distinguish the following cases for x:

- if $x = \omega$ then ρ^A is a linear function in K_1, which is positive and increasing if and only if $\theta > \frac{\epsilon}{x} + 2\beta K_0$. In this case, the maximum occurs when $K_1 = \frac{K_b}{x}$.
- if $x \neq \omega$ then ρ^A is a quadratic function, with

$$K_1^A(x,\theta) = \frac{[(\theta - 2\beta K_0)x - \epsilon]\,\omega}{2\alpha x(\omega - x)} \tag{5.36}$$

  being the zero of the first derivative. Therefore, one needs to distinguish between the cases $x < \omega$ (ρ^A is a concave function) and $x > \omega$ (ρ^A is a convex function). So,

  - if $0 < x < \omega$, given that $\rho^A(0,\theta) = 0$, we merely want to consider the situations where $K_1^A(x,\theta) > 0$, or equivalently

$$\theta > \frac{\epsilon}{x} + 2\beta K_0. \tag{5.37}$$

[12] In Appendix 5.E we check the relative position of the curves defined by Conditions (5.23), (5.32) and (5.33), which are shown in Figure 5.9.

We still need to check the relative position of $K_1^A(x,\theta)$ and $\frac{K_b}{x}$, which is given by

$$\frac{K_b}{x} < K_1^A(x,\theta) \Leftrightarrow \theta > \frac{\epsilon}{x} + \frac{2\alpha K_b}{x}. \tag{5.38}$$

Thus, if $K_1^A < \frac{K_b}{x}$, then K_1^A is indeed the maximizer of ρ^A in the considered domain, otherwise its maximizer is $\frac{K_b}{x}$.

– if $x > \omega$ and, as $\rho^A(0,\theta) = 0$, then $\frac{K_b}{x}$ is the maximizer of ρ^A in the considered domain, as long as $\rho_x^A\left(\frac{K_b}{x},\theta\right) > 0$, which is given by Condition (5.23).

Therefore,

$$\arg\max_{K_1} \rho_x^A(K_1,\theta)\chi_{\left\{0<K_1\leq\frac{K_b}{x}\right\}} = \begin{cases} K_1^A(x,\theta) & \text{if} \quad (x,\theta)\in\Psi_1^A \\ \frac{K_b}{x} & \text{if} \quad (x,\theta)\in\Psi_2^A \end{cases} \tag{5.39}$$

where the regions Ψ_1^A and Ψ_2^A [13] are defined as

$$\begin{aligned} \Psi_1^A &= \left\{(x,\theta)\in\mathbb{R}^+\times\mathbb{R}^+ : 0<x<\omega \ \wedge \ 2\beta K_0 < \theta - \frac{\epsilon}{x} \leq \frac{2\alpha K_b}{x}\right\}, \\ \Psi_2^A &= \left\{(x,\theta)\in\mathbb{R}^+\times\mathbb{R}^+ : \theta > \frac{\epsilon}{x} + 2\max\left\{\frac{\alpha K_b}{x}, \beta K_0\right\}\right\}. \end{aligned}$$

5.C.2 Case $d_1 \neq 2$

Recalling that

$$\rho_x^A(K_1,\theta) = \left[c(x)K_1^{d_1} - \frac{\alpha x}{r-\mu}{K_1}^2 + \frac{(\theta-2\beta K_0)x-\epsilon}{r-\mu}K_1\right],$$

we compare the first and second order derivatives (for simplicity, we omit the dependence in x and θ in the notation), in order to study the monotonicity of this function,

$$\begin{aligned} \nabla_1(K_1) &\equiv \frac{d\rho_x^A(K_1,\theta)}{dK_1} = c(x)d_1K_1^{d_1-1} - \frac{2\alpha x}{r-\mu}K_1 + \frac{(\theta-2\beta K_0)\,x-\epsilon}{r-\mu}, \\ \nabla_2(K_1) &\equiv \frac{d^2\rho_x^A(K_1,\theta)}{dK_1^2} = c(x)d_1(d_1-1)K_1^{d_1-2} - \frac{2\alpha x}{r-\mu}. \end{aligned}$$

Note that ∇_2 has an unique zero and it is such that

$$\lim_{K_1\to 0^+}\nabla_2(K_1) = \begin{cases} +\infty & \text{if} \quad 1<d_1<2 \\ -\frac{2\alpha x}{r-\mu} & \text{if} \quad d_1>2 \end{cases} \quad \text{and} \quad \lim_{K_1\to+\infty}\nabla_2(K_1) = \begin{cases} -\frac{2\alpha x}{r-\mu} & \text{if} \quad 1<d_1<2 \\ +\infty & \text{if} \quad d_1>2 \end{cases}$$

Indeed, ∇_2 is a monotonic function in K_1 - decreasing if $1<d_1<2$ and increasing if $d_1>2$. On the one hand, this implies that ρ^A has an unique inflection point at $\zeta(x) = K_b\left(\frac{2\omega}{d_1}\right)^{\frac{1}{d_1-2}} x^{\frac{d_1-1}{2-d_1}}$ - changing from convex to concave, if $1<d_1<2$, and from concave to convex, otherwise. On the other hand, it also implies that, if $1<d_1<2$ (respectively, $d_1>2$), ∇_1 increases in K_1 (respectively, decreases) from 0 to $\zeta(x)$ and decreases (respectively, increases) afterwards. Thus, $\zeta(x)$ is a maximizer of ∇_1, if $1<d_1<2$, and a minimizer, if $d_1>2$. Furthermore, $\nabla_1(0) = \frac{(\theta-2\beta K_0)x-\epsilon}{r-\mu}$ and

$$\lim_{K_1\to+\infty}\nabla_1(K_1) = \begin{cases} -\infty & \text{if} \quad 1<d_1<2 \\ +\infty & \text{if} \quad d_1>2 \end{cases}$$

[13] In Appendix 5.E we check the relative position of the curves defined by Conditions (5.23), (5.37) and (5.38), which are shown in Figure 5.10.

So, ∇_1 depends also on the value of d_1 and the signs of $\nabla_1(0)$ and $\nabla_1(\zeta(x))$. Simple calculations lead to the following relations

$$\nabla_1(0) > 0 \quad \Leftrightarrow \quad \theta > \frac{\epsilon}{x} + 2\beta K_0, \tag{5.40}$$

$$\nabla_1(\zeta(x)) > 0 \quad \Leftrightarrow \quad \theta > \frac{\epsilon}{x} + 2\beta K_0 + 2\,\alpha\,K_b\,\left(\frac{2\omega}{d_1}\right)^{\frac{1}{d_1-2}}\;\frac{d_1-2}{d_1-1}\;x^{\frac{d_1-1}{2-d_1}}. \tag{5.41}$$

We highlight that the Conditions (5.40) and (5.41) are mutually exclusive/inclusive in the following sense: if $1 < d_1 < 2$ then $\nabla_1(0) > 0 \Rightarrow \nabla_1(\zeta(x)) > 0$; if $d_1 > 2$ then $\nabla_1(\zeta(x)) > 0 \Rightarrow \nabla_1(0) > 0$. Whether (5.40) and (5.41) hold or not, will determine the behaviour of ρ^A. Namely, it will imply that ρ^A has none, one or two extreme points, some of them candidates to become maximizers. In Figures 5.6 and 5.7 we plot some of the possibilities for ρ^A as a function of K_1, for $1 < d_1 < 2$, and $d_1 > 2$, respectively. In order to plot these figures we also take into account that $\rho^A_x(0,\theta) = 0$ and

$$\lim_{K_1 \to +\infty} \rho^A_x(K_1,\theta) = \begin{cases} -\infty & \text{if} \quad 1 < d_1 < 2 \\ +\infty & \text{if} \quad d_1 > 2 \end{cases}$$

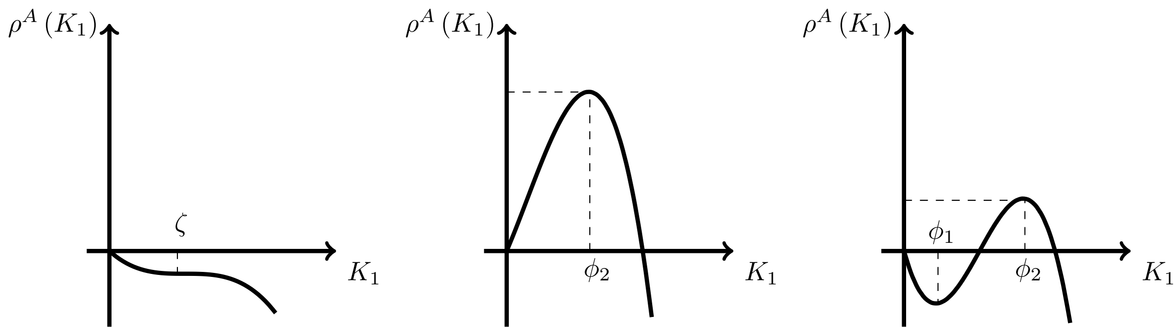
Moreover, let us consider the set $\Theta = \left\{K \in \mathbb{R}^+_0 : \nabla_1(K) = 0\right\}$. If $\#\Theta = 2$ we define the two points where the extremes of ρ^A are attained as

$$\phi_1(x,\theta) = \min\left\{K \in \mathbb{R}^+_0 : \nabla_1(K) = 0\right\} \quad \text{and} \quad \phi_2(x,\theta) = \max\left\{K \in \mathbb{R}^+_0 : \nabla_1(K) = 0\right\}. \tag{5.42}$$

If $\#\Theta = 1$, ρ^A has only one extreme and we define

$$\phi_1(x,\theta) = 0 \quad \text{and} \quad \phi_2(x,\theta) = \left\{K \in \mathbb{R}^+_0 : \nabla_1(K) = 0\right\}.$$

Finally, when $\#\Theta = 0$, ρ^A has no extremes and $\phi_1(x,\theta)$ and $\phi_2(x,\theta)$ are not defined (see Figures 5.6 and 5.7).

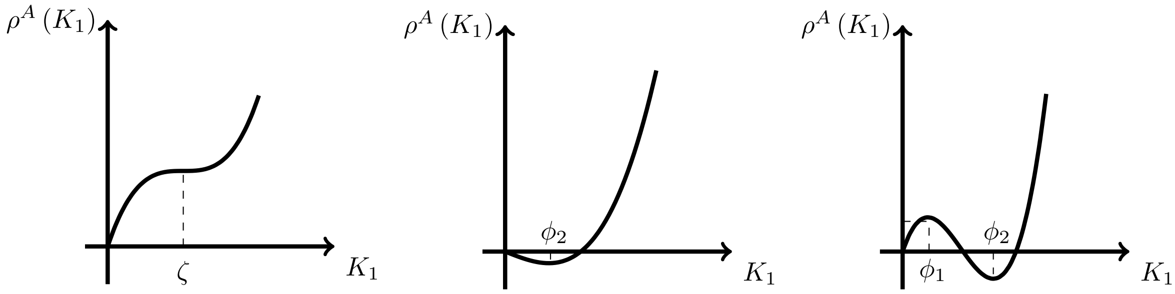


(a) $\nabla_1(\zeta(x)) < 0$ **(b)** $\nabla_1(0) > 0$ **(c)** $\nabla_1(0) < 0 \wedge \nabla_1(\zeta(x)) > 0$

Figure 5.6: Examples of ρ^A graphical representation if $1 < d_1 < 2$.

In order to maximize ρ^A w.r.t. K_1, we need to consider different situations:

- If $1 < d_1 < 2$, as can be seen in Figure 5.6, we are only interested in cases where $\nabla_1(\zeta(x)) > 0$. In this case, the candidate to be the maximizer is $\phi_2(x,\theta)$. As we only study the function ρ^A for $0 \leq K_1 < \frac{K_b}{x}$, we need to take into account what is the relative position of $\frac{K_b}{x}$ and



(a) $\nabla_1(\zeta(x)) > 0$ **(b)** $\nabla_1(0) < 0$ **(c)** $\nabla_1(0) > 0 \wedge \nabla_1(\zeta(x)) < 0$

Figure 5.7: Examples of ρ^A graphical representation if $d_1 > 2$.

$\phi_2(x,\theta)$. If $\frac{K_b}{x} > \phi_2(x,\theta)$[14], whenever $\rho_x^A(\phi_2(x,\theta),\theta) > 0$[15], the maximizer of ρ^A is $\phi_2(x,\theta)$. If $\frac{K_b}{x} \leq \phi_2(x,\theta)$, then we merely want to consider the cases where $\rho_x(\frac{K_b}{x},\theta) > 0$, wherein the maximizer is $\frac{K_b}{x}$.

- If $d_1 > 2$, we have two substantially different cases (see Figure 5.7):
  - for $\nabla_1(\zeta(x)) \geq 0$ the maximizer is $\frac{K_b}{x}$; for $\nabla_1(0) \leq 0$, the maximizer is also $\frac{K_b}{x}$, as long as $\rho_x(\frac{K_b}{x},\theta) > 0$.
  - for $\nabla_1(0) > 0 \wedge \nabla_1(\zeta(x)) < 0$, the candidate to be the maximizer is $\phi_1(x,\theta)$. One still needs to check the relative position of $\frac{K_b}{x}$ and $\phi_1(x,\theta)$. Indeed, the maximizer is $\frac{K_b}{x}$ if $\frac{K_b}{x} \leq \phi_1(x)$ or $\frac{K_b}{x} > \phi_1(x)$ and $\rho_x(\frac{K_b}{x},\theta) \geq \rho_x^A(\phi_1(x,\theta),\theta)$; otherwise the maximizer is $\phi_1(x,\theta)$.

As we have seen, the signal of the functions ρ, ∇_1 and ∇_2 at the point $\frac{K_b}{x}$ play an important role. The first is already shown in Condition (5.23). Besides that, straightforward calculus lead us to

$$\nabla_1\left(\frac{K_b}{x}\right) > 0 \Leftrightarrow \theta > \frac{\epsilon}{x} + \frac{2\alpha K_b}{x} \tag{5.43}$$

and

$$\nabla_2\left(\frac{K_b}{x}\right) > 0 \Leftrightarrow x > \frac{2\omega}{d_1}, \tag{5.44}$$

with $\frac{2}{d_1} \leq 1 \Leftrightarrow d_1 \geq 2$.[16]

Regarding the case $1 < d_1 < 2$, it could happen that $\rho_x^A(\phi_2(x,\theta),\theta) < 0$. Thus one needs to check what conditions lead to

$$\rho_x^A(\phi_2(x,\theta),\theta) = 0. \tag{5.45}$$

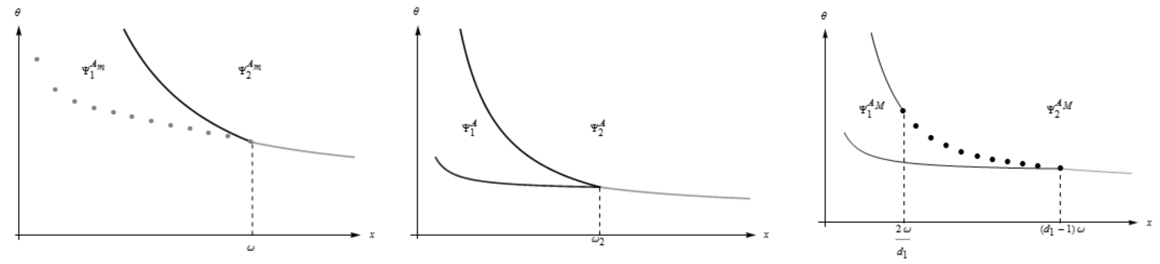
Similarly, for $d_1 > 2$, we need to check in which cases $\rho_x\left(\frac{K_b}{x},\theta\right) > \rho_x^A\left(\phi_1\left(x,\theta\right),\theta\right)$, which leads to

$$\rho_x\left(\frac{K_b}{x},\theta\right) = \rho_x^A\left(\phi_1\left(x,\theta\right),\theta\right). \tag{5.46}$$

[14] For $1 < d_1 < 2$, we have $\frac{K_b}{x} > \phi_2(x,\theta) \Leftrightarrow \nabla_1\left(\frac{K_b}{x}\right) < 0 \wedge \nabla_2\left(\frac{K_b}{x}\right) < 0$.

[15] Note that in Figure 5.6 (c), we have only plotted one of the possible situations. However, we can have $\rho_x^A(\phi_2(x,\theta),\theta) < 0$, which implies that ρ^A is always negative as a function of K_1.

[16] In Appendix 5.E, we check the relative position of the curves given by Conditions (5.23), (5.40), (5.41), (5.43) and (5.44), which are shown in Figure 5.11.



(a) $1 < d_1 < 2$ **(b)** $d_1 = 2$ **(c)** $d_1 > 2$

Figure 5.8: Plan division for ρ^A.

Checking these conditions is challenging, as in general one does not have even explicit expressions for $\phi_2(x, \theta)$. Nevertheless, we are able to provide some inside results, that we explain in the rest of the section[17].

For that purpose, we define the following sets in $\mathbb{R}^+ \times \mathbb{R}^+$. For notation reasons, we user the letter m for the case $1 < d_1 < 2$, and the letter M for the case $d_1 > 2$.

$$
\begin{aligned}
\Psi_1^{A_m} &= \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < x < \omega \ \wedge \ \vartheta_m(x) < \theta \leq \frac{\epsilon}{x} + \frac{2\alpha K_b}{x}\right\} \\
\Psi_2^{A_m} &= \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : \theta > \frac{\epsilon}{x} + \frac{\alpha K_b}{x} + \max\left\{\frac{\alpha K_b}{x}, \frac{2(d_1-1)\beta K_0}{d_1}\right\}\right\} \\
\Psi_1^{A_M} &= \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < x < (d_1-1)\omega \ \wedge \ \frac{\epsilon}{x} + 2\beta K_0 < \theta \leq \vartheta_M(x)\right\} \\
\Psi_2^{A_M} &= \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : \theta > \vartheta_M(x)\right\}
\end{aligned}
$$

where ϑ_M and ϑ_m are implicitly defined as

$$
\begin{aligned}
&\rho_x\left(\frac{K_b}{x}, \vartheta_M(x)\right) = \rho_x^A\left(\phi_1\left(x, \vartheta_M(x)\right), \vartheta_M(x)\right) \\
&\rho_x^A(\phi_2(x, \vartheta_m(x)), \vartheta_m(x)) = 0,
\end{aligned}
$$

respectively[18]. Finally, we are able to write, for $1 < d_1 < 2$,

$$
\arg\max_{K_1} \rho_x^A(K_1,\theta)\chi_{\left\{0<K_1\leq\frac{K_b}{x}\right\}} = \begin{cases} \phi_2(x,\theta) & \text{if} \quad (x,\theta) \in \Psi_1^{A_m} \\ \frac{K_b}{x} & \text{if} \quad (x,\theta) \in \Psi_2^{A_m} \end{cases} \tag{5.47}
$$

and, for $d_1 > 2$,

$$
\arg\max_{K_1} \rho_x^A(K_1,\theta)\chi_{\left\{0<K_1\leq\frac{K_b}{x}\right\}} = \begin{cases} \phi_1(x,\theta) & \text{if} \quad (x,\theta) \in \Psi_1^{A_M} \\ \frac{K_b}{x} & \text{if} \quad (x,\theta) \in \Psi_2^{A_M} \end{cases}
$$

In Figure 5.8 we present an illustration of the sets $\Psi_i^{A_m}$, Ψ_i^{A} and $\Psi_i^{A_M}$, with $i = 1, 2$. For the three corresponding cases, we use also some of the inequalities derived in Appendix 5.F. Moreover, for the cases (a) and (c) the dot points correspond to ϑ_m and ϑ_M, respectively, which can be only found numerically.

[17] In Appendix 5.F, we relate the Conditions (5.45) and (5.46) with the previous Conditions (5.23), (5.40), (5.41), (5.43) and (5.44), which are shown in Figure 5.12.

[18] Note that ϑ_M has an explicit expression in some domains: $\vartheta_M(x) = \frac{\epsilon}{x} + \frac{2\alpha K_b}{x}$ if $0 < x < \frac{2\omega}{d_1}$ and $\vartheta_M(x) = \frac{\epsilon}{x} + \frac{\alpha K_b}{x} + \frac{2(d_1-1)\beta K_0}{d_1}$ if $x > (d_1-1)\omega$ (see Appendix 5.F for details).

Appendix 5.D Maximization of ρ

When $1 < d_1 \leq 2$ the calculations are straightforward. Indeed, for $d_1 = 2$, combining the results from Expressions (5.35) and (5.39) as well as noticing that $\Psi_1^R \subset \Psi_1^A$, $\Psi_2^A \subset \Psi_2^R$ and $\Psi_1^A \cap \Psi_2^R = \emptyset$, we obtain

$$\arg\max_{K_1} \rho_x(K_1, \theta) = \begin{cases} K_1^A(x,\theta) & \text{if} \quad (x,\theta) \in \Psi_1^A \\ K_1^R(x,\theta) & \text{if} \quad (x,\theta) \in \Psi_2^R \end{cases}$$

where $K_1^R(x,\theta)$ and $K_1^A(x,\theta)$ are defined in (5.31) and (5.36), respectively. Identically, for $1 < d_1 < 2$, combining the Expressions (5.35) and (5.47), and realizing that $\Psi_1^R \subset \Psi_1^{A_m}$, $\Psi_2^{A_m} \subset \Psi_2^R$ and $\Psi_1^{A_m} \cap \Psi_2^R = \emptyset$, we have

$$\arg\max_{K_1} \rho_x(K_1, \theta) = \begin{cases} \phi_2(x,\theta) & \text{if} \quad (x,\theta) \in \Psi_1^{A_m} \\ K_1^R(x,\theta) & \text{if} \quad (x,\theta) \in \Psi_2^R \end{cases}$$

where $\phi_2(x,\theta)$ is defined in (5.42), for which we do not have a closed expression. The case $d_1 > 2$ is more challenging due to the fact that we cannot draw conclusions only from the inclusion relationships between the sets[19]. In particular, we need to explore the position of the curve given by Condition (5.32) relatively to the curves given by Conditions (5.40) and (5.46)[20]. Furthermore, in the set $\Psi_1^{A_M} \cap \Psi_2^R$, we need to compare which has the largest value: $\rho_x^R(K_1^R(x,\theta),\theta)$ or $\rho_x^A(\phi_1(x,\theta),\theta)$. As before, we are not able to have an explicit expression for the condition

$$\rho_x^R(K_1^R(x,\theta),\theta) = \rho_x^A(\phi_1(x,\theta),\theta) \tag{5.48}$$

but we can show some intermediate results about it[21].

Finally, we define the new sets

$$\begin{aligned} \Upsilon_1^{A_M} &= \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < x < \omega_2 \ \wedge \ \frac{\epsilon}{x} + 2\beta K_0 < \theta \leq \upsilon_M(x)\right\} \\ \Upsilon_2^{R_M} &= \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : \theta > \upsilon_M(x)\right\} \end{aligned}$$

where υ_M is implicitly defined as $\rho_x^R\left(\frac{K_1^R(x,\upsilon_M(x))}{x}, \upsilon_M(x)\right) = \rho_x^A\left(\phi_1\left(x, \upsilon_M(x)\right), \upsilon_M(x)\right)$[22] and $\omega_2 = \frac{\alpha(r-\mu)\pi_0}{r(\beta K_0)^2}$[23]. Thus, we can write

$$\arg\max_{K_1} \rho_x(K_1, \theta) = \begin{cases} \phi_1(x,\theta) & \text{if} \quad (x,\theta) \in \Upsilon_1^{A_M} \\ K_1^R(x,\theta) & \text{if} \quad (x,\theta) \in \Upsilon_2^{R_M} \end{cases}$$

where $\phi_1(x,\theta)$ is defined in (5.42), for which we do not have a closed expression either.

Appendix 5.E Relative position between curves with explicit expression

In this section we want to explore the relative position between the curves defined along the text. To ease the presentation, we exhibit a table with all the conditions involved in the capacity optimization

[19] For $d_1 > 2$, we have $\Psi_1^R \subset \Psi_1^{A_M}$ and $\Psi_2^{A_M} \subset \Psi_2^R$ but $\Psi_1^{A_M} \cap \Psi_2^R \neq \emptyset$.

[20] In Appendix 5.E, we check the relative position of the curve given by Conditions (5.32) relatively to the curves given by Conditions (5.40) and (5.46).

[21] In Appendix 5.F, we relate the Condition (5.48) with the previous Conditions (5.23), (5.32), (5.40), (5.43), (5.44) and (5.46), which are shown in Figure 5.13.

[22] Note that υ_M has an explicit expression in some domains: $\upsilon_M(x) = \frac{\epsilon}{x} + \frac{2\alpha K_b}{x}$ if $0 < x < \frac{2\omega}{d_1}$ and $\upsilon_M(x) = \frac{\epsilon}{x} + 2\sqrt{\frac{\alpha\Pi_0(r-\mu)}{rx}}$ if $x > \omega_2$ (see Appendix 5.F for details).

[23] Note that ω_2 is, in fact, the value of ω, defined in (5.34), when $d_1 = 2$.

case.

Curve number	Conditions verified	Functions
(5.23)	$\rho_x\left(\frac{K_b}{x},\theta\right)=0$	$i_1(x)=\frac{\epsilon}{x}+\frac{\alpha K_b}{x}+\frac{2(d_1-1)\beta K_0}{d_1}$
(5.32)	$\rho_x^R(K_1^R(x,\theta),\theta)=0$	$i_2(x)=\frac{\epsilon}{x}+2\sqrt{\frac{\alpha\Pi_0(r-\mu)}{rx}}$
(5.33) (5.38) (5.43)	$K_1^R(x,\theta)=\frac{K_b}{x}$ $K_1^A(x,\theta)=\frac{K_b}{x}$ $\nabla_1\left(\frac{K_b}{x}\right)=0$	$i_3(x)=\frac{\epsilon}{x}+\frac{2\alpha K_b}{x}$
(5.37) (5.40)	$K_1^A(x,\theta)=0$ $\nabla_1(0)=0$	$i_4(x)=\frac{\epsilon}{x}+2\beta K_0$
(5.41)	$\nabla_1(\zeta(x))=0$	$i_5(x)=\frac{\epsilon}{x}+2\beta K_0+2\ \alpha\ K_b\ \left(\frac{2\omega}{d_1}\right)^{\frac{1}{d_1-2}}\ \frac{d_1-2}{d_1-1}\ x^{\frac{d_1-1}{2-d_1}}$

5.E.1 Benchmark model

We explore the relative position between the curves given by (5.14) and (5.15). In fact, we only need to relate $\frac{(r-\mu)\pi_0}{rK_1x}$ and $2\beta K_0-\frac{a(r-\mu)x^{d_1-1}}{K_1}$. Note that $2\beta K_0-\frac{a(r-\mu)x^{d_1-1}}{K_1}\leq\frac{(r-\mu)\pi_0}{rK_1x}\Leftrightarrow ax^{d_1}\geq\frac{2\beta K_0K_1x}{r-\mu}-\frac{\pi_0}{r}$, which we already show it is always true, in the proof of Proposion 5.1. Hence, the replace's curve is always above the add's curve, except at $x=x^\star$, in which they are tangent.

5.E.2 Capacity optimization model: ρ^R

We check the relative position of the curves defined by Conditions (5.23), (5.32) and (5.33). Firstly, we provide a result that will be used to prove the relation between Conditions (5.23) and (5.32).

Lemma 5.1. *For $x\in\mathbb{R}^+$, the following inequality holds*

$$2\sqrt{\frac{\alpha\Pi_0(r-\mu)}{rx}}\leq\frac{\alpha K_b}{x}+\frac{2(d_1-1)\beta K_0}{d_1}.$$

Proof of Lemma 5.1. *Let us consider the function $g(x)=\left[\frac{\alpha K_b}{x}+\frac{2(d_1-1)\beta K_0}{d_1}\right]^2-\frac{4\alpha\Pi_0(r-\mu)}{r\mu}$. We can rewrite $g(x)=\left(\frac{\alpha K_b}{x}\right)^2-\frac{2\alpha(r-\mu)\pi_0}{rx}+\left(\frac{2(d_1-1)\beta K_0}{d_1}\right)^2$. Computing the first and second derivatives, $g'(x)=-\frac{2(\alpha K_b)^2}{x^3}+\frac{2\alpha(r-\mu)\pi_0}{rx^2}$ and $g''(x)=\frac{6(\alpha K_b)^2}{x^4}$, we verify that g is a convex function, with only one minimum, attained at ω. As $g(\omega)=0$, we conclude that $g(x)\geq 0,\forall x>0$ and therefore the result holds.*

■

From Lemma 5.1 we conclude that the curve given by Condition (5.32) is always above the curve given by Condition (5.23), except for $x=\omega$, where both coincide. Next we analyze the relation among Conditions (5.32) and (5.33). It is straightforward to verify that, for $x>0$,

$$\frac{\alpha K_b}{x}\geq\sqrt{\frac{\alpha\Pi_0(r-\mu)}{rx}}\Leftrightarrow x\leq\omega.$$

Similarly, the relation between Conditions (5.23) and (5.33) can be obtained. For $x>0$,

$$\frac{2\alpha K_b}{x}\geq\frac{\alpha K_b}{x}+\frac{2(d_1-1)\beta K_0}{d_1}\Leftrightarrow x\leq\omega.$$

We call the attention that the three curves intersect at $x=\omega$, as it is shown in Figure 5.9.

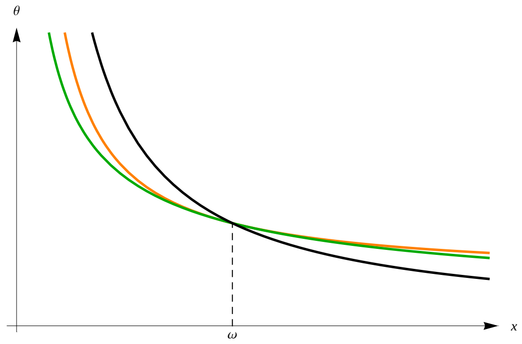


Figure 5.9: Plan division for ρ^R.
Orange curve: $\rho_x\left(\frac{K_b}{x}, \theta\right) = 0$; green curve: $\rho_x^R\left(K_1^R(x,\theta), \theta\right) = 0$; black curve: $\frac{K_b}{x} = K_1^R(x, \theta)$.

5.E.3 Capacity optimization model: ρ^A

For case $d_1 = 2$, we have three curves given by the Conditions (5.23), (5.37) and (5.38). Be aware that the curve given by Condition (5.38) is the same given by Condition (5.33). Thus, the relation between (5.23) and (5.38) was already explored. It remains to study the relation of (5.37) with the other two. Relating Condition (5.23) with Condition (5.37), we have

$$\frac{\alpha K_b}{x} + \frac{2(d_1 - 1)\beta K_0}{d_1} \geq 2\beta K_0 \Leftrightarrow x \leq (d_1 - 1)\omega,$$

where $d_1 - 1 \geq 1 \Leftrightarrow d_1 \geq 2$. Note that, when $d_1 = 2$, $(d_1 - 1)\omega$ coincides with ω. The relationship between Conditions (5.37) and (5.38) is such that

$$\frac{\alpha K_b}{x} \geq \beta K_0 \Leftrightarrow x \leq \frac{2(d_1 - 1)}{d_1}\omega.$$

We draw the attention to the fact that $\frac{2(d_1-1)}{d_1} \geq 1 \Leftrightarrow d_1 \geq 2$, meaning that, for $d_1 = 2$ the expression $\frac{2(d_1-1)}{d_1}\omega$ coincides with ω. Finally, if $d_1 = 2$, the three curves given by the Conditions (5.23), (5.37) and (5.38) intersect each other at $x = \omega$, as can be seen in Figure 5.10.

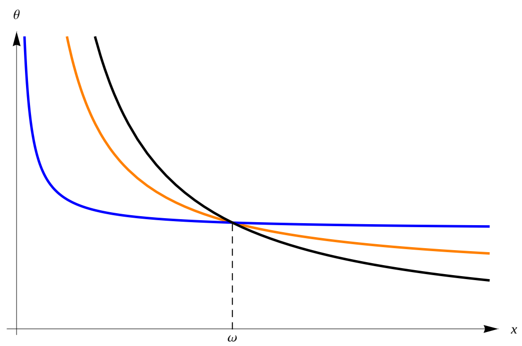


Figure 5.10: Plan division for ρ^A when $d_1 = 2$.
Orange curve: $\rho_x\left(\frac{K_b}{x}, \theta\right) = 0$; blue curve: $K_1^A(x, \theta) = 0$; black curve: $\frac{K_b}{x} = K_1^A(x, \theta)$.

For case $d_1 \neq 2$, we have four curves given by the Conditions (5.23), (5.40), (5.41) and (5.43). We notice that the only new curve is the one given by Condition (5.41). Indeed, Conditions (5.40) and

(5.37) coincide; and also Conditions (5.43), (5.38) and (5.33) are all the same. Then, we only need to explore the relation of (5.41) with each one of the others.

To relate Conditions (5.40) and (5.41), we highlight that $2\alpha K_b \left(\frac{2\omega}{d_1}\right)^{\frac{1}{d_1-2}} \frac{d_1-2}{d_1-1} x^{\frac{d_1-1}{2-d_1}} \geq 0 \Leftrightarrow d_1 \geq 2$. This means that if $1 < d_1 < 2$, $\nabla_1(0) > 0 \Rightarrow \nabla_1(\zeta(x)) > 0$ and if $d_1 > 2$, $\nabla_1(\zeta(x)) > 0 \Rightarrow \nabla_1(0) > 0$. Indeed, one curve is always above the other. Regarding the position of the curve given by Condition (5.41) relatively to the curves defined by Conditions (5.23) and (5.43) it is explained in the next lemmas.

Lemma 5.2. *For $x \in \mathbb{R}^+$, the following inequality holds*

$$\iota \left[2\beta K_0 + 2\,\alpha\, K_b \left(\frac{2\omega}{d_1}\right)^{\frac{1}{d_1-2}} \frac{d_1-2}{d_1-1}\, x^{\frac{d_1-1}{2-d_1}}\right] > \iota \left[\frac{\alpha K_b}{x} + \frac{2(d_1-1)\beta K_0}{d_1}\right]$$

where $\iota = -1$ if $1 < d_1 < 2$ and $\iota = 1$ if $d_1 > 2$.

Proof of Lemma 5.2. *Let us consider the function*

$$g(x) = \left[2\beta K_0 + 2\,\alpha\, K_b \left(\frac{2\omega}{d_1}\right)^{\frac{1}{d_1-2}} \frac{d_1-2}{d_1-1}\, x^{\frac{d_1-1}{2-d_1}}\right] - \left[\frac{\alpha K_b}{x} + \frac{2(d_1-1)\beta K_0}{d_1}\right],$$

which we can rewrite as follows

$$g(x) = \frac{\alpha\, K_b}{x}\left[2\left(\frac{2\omega}{d_1}\right)^{\frac{1}{d_1-2}} \frac{d_1-2}{d_1-1}\, x^{\frac{1}{2-d_1}} - 1\right] + \frac{2\beta K_0}{d_1}.$$

Computing the first derivative $g'(x) = -\frac{\alpha\, K_b}{x^2}\left[2\left(\frac{2\omega}{d_1}\right)^{\frac{1}{d_1-2}}\, x^{\frac{1}{2-d_1}} - 1\right]$ we verify that $g'(x) > 0 \Leftrightarrow \iota\, x > \iota\, 2^{d_1-1}\frac{\omega}{d_1}$. This implies that g has a minimum (respectively, maximum) at $x = 2^{d_1-1}\frac{\omega}{d_1}$ if $d_1 > 2$ (respectively, $1 < d_1 < 2$). Moreover, some calculations lead us to $g\left(2^{d_1-1}\frac{\omega}{d_1}\right) = \frac{\alpha K_b\left(2^{d_1-1}-d_1\right)}{\omega(d_1-1)2^{d_1-1}}$, which means that the sign of $g\left(2^{d_1-1}\frac{\omega}{d_1}\right)$ only depends on the sign of $2^{d_1-1} - d_1$. Thus $g\left(2^{d_1-1}\frac{\omega}{d_1}\right)$ is positive if $d_1 > 2$ and negative if $1 < d_1 < 2$. Therefore, for $x > 0$, the function g is always positive if $d_1 > 2$ and always negative if $1 < d_1 < 2$, which concludes the proof.

■

Lemma 5.3. *For $x \in \mathbb{R}^+$, the following inequality holds*

$$\iota \left[2\beta K_0 + 2\,\alpha\, K_b \left(\frac{2\omega}{d_1}\right)^{\frac{1}{d_1-2}} \frac{d_1-2}{d_1-1}\, x^{\frac{d_1-1}{2-d_1}}\right] \geq \iota \left[\frac{2\alpha K_b}{x}\right]$$

where $\iota = -1$ if $1 < d_1 < 2$ and $\iota = 1$ if $d_1 > 2$.

Proof of Lemma 5.3. *Let us consider the function*

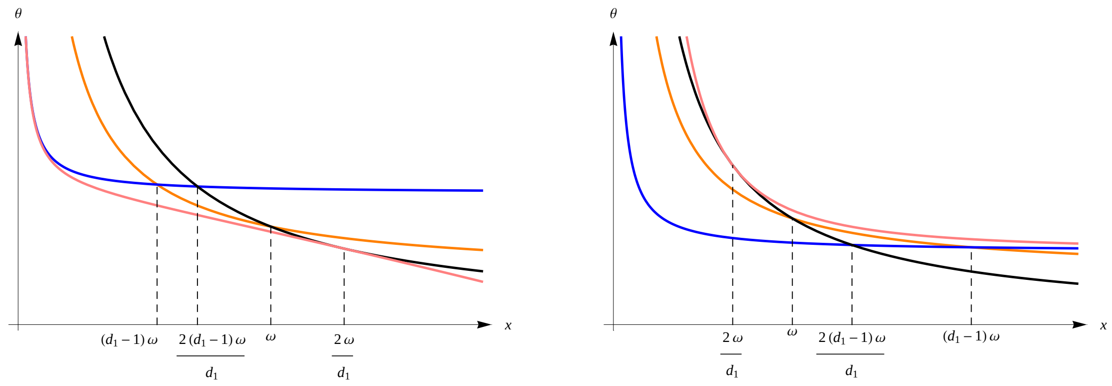
$$g(x) = 2\beta K_0 + 2\,\alpha\, K_b \left(\frac{2\omega}{d_1}\right)^{\frac{1}{d_1-2}} \frac{d_1-2}{d_1-1}\, x^{\frac{d_1-1}{2-d_1}} - \frac{2\alpha K_b}{x}.$$

We can rewrite

$$g(x) = \frac{2\alpha\, K_b}{x}\left[\left(\frac{2\omega}{d_1}\right)^{\frac{1}{d_1-2}} \frac{d_1-2}{d_1-1}\, x^{\frac{1}{2-d_1}} - 1\right] + 2\beta K_0.$$

Computing the first derivative $g'(x) = -\frac{2\alpha\, K_b}{x^2}\left[\left(\frac{2\omega}{d_1}\right)^{\frac{1}{d_1-2}}\, x^{\frac{1}{2-d_1}} - 1\right]$ we verify that $g'(x) > 0 \Leftrightarrow \iota\, x > \iota\, \frac{2\omega}{d_1}$. This implies that g has a minimum (respectively, maximum) at $x = \frac{2\omega}{d_1}$ if $d_1 > 2$ (respectively, $1 < d_1 < 2$). Moreover, as $g\left(\frac{2\omega}{d_1}\right) = 0$, it follows that $\iota\, g(x) \geq 0, \forall x > 0$, and therefore the result holds. ■

We present in Figure 5.11 the relationship between the four curves described before.



(a) $1 < d_1 < 2$ **(b)** $d_1 > 2$

Figure 5.11: Plan division for ρ^A.
Orange curve: $\rho_x\left(\frac{K_b}{x},\theta\right)=0$; blue curve: $\nabla(0)=0$; black curve: $\nabla_1\left(\frac{K_b}{x}\right)=0$; pink curve: $\nabla_1(\zeta(x))=0$.

5.E.4 Capacity optimization model: ρ

The relationship between the Conditions (5.32) and (5.40) is easily given by

$$\beta K_0 > \sqrt{\frac{\alpha(r-\mu)\Pi_0}{rx}} \Leftrightarrow x > \omega_2,$$

with $\omega(2)=\frac{\alpha(r-\mu)\pi_0}{r(\beta K_0)^2}$. It is straightforward to check that $\omega(2)=\left(\frac{2(d_1-1)}{d_1}\right)^2\omega$. Then, $\iota\left(\frac{2(d_1-1)}{d_1}\right)\omega \le \iota\,\omega(2)$, where $\iota=-1$ if $1<d_1<2$ and $\iota=1$ if $d_1\ge 2$, and $\omega(2)\le(d_1-1)\omega$, for $d_1>1$. Together with the fact that the curve given by Condition (5.32) is always below to the curve given by Condition (5.23), for $d_1>2$, the previous results imply that the curve given by Condition (5.32) is always below to the curve given by Condition (5.46).

Appendix 5.F Relative position for conditions with no explicit expression

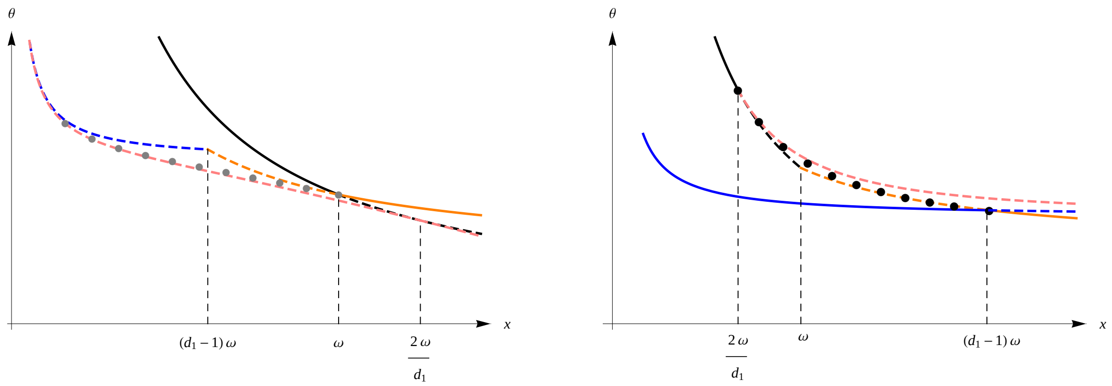
The first two sections explore the Conditions (5.45) and (5.46) and the correspondent graphical representations are shown in Figure 5.12. In the third section we inspect the Condition (5.48) which is presented in Figure 5.13.

5.F.1 $\rho_x^A(\phi_2(x,\theta),\theta)=0$

For $1<d_1<2$, we want to explore where the curve given by the condition $\rho_x^A(\phi_2(x,\theta),\theta)=0$, when $x<\omega$, is placed, relatively to the curves given by the Conditions (5.23), (5.40), (5.41) and (5.43).

When $\nabla_1(\zeta(x))<0$, $\phi_2(x,\theta)$ is not defined. Whenever $\nabla_1(\zeta(x))=0$, we have $\zeta(x)=\phi_2(x,\theta)$ and $\rho_x^A(\phi_2(x,\theta),\theta)=\rho_x^A(\zeta(x),\theta)<0$. Then, the new curve is always above the curve given by Condition (5.41). Also $\rho_x^A(\phi_2(x,\theta),\theta)<0 \Rightarrow \rho_x\left(\frac{K_b}{x},\theta\right)<0$ and $\nabla_1(0)>0\Rightarrow \rho_x^A(\phi_2(x,\theta),\theta)>0$, implying that the new curve is below the curves given by Conditions (5.23) and (5.40).

Further, for $x=\omega$, when we have the interception between the curves given by Conditions (5.23) and (5.43), we notice that $\rho_x\left(\frac{K_b}{x},\theta\right)=0$ and $\frac{K_b}{x}=\phi_2(x,\theta)$, implying that $\rho_x^A(\phi_2(x,\theta),\theta)=0$. Thus,



(a) $1 < d_1 < 2$ **(b)** $d_1 > 2$

Figure 5.12: Plan division for ρ^A.
Orange curve: $\rho_x\left(\frac{K_b}{x}, \theta\right) = 0$; blue curve: $\nabla_1(0) = 0$; black curve: $\nabla_1\left(\frac{K_b}{x}\right) = 0$; pink curve: $\nabla_1(\zeta(x)) = 0$; gray points: $\rho_x^A(\phi_2(x,\theta), \theta) = 0$; black points: $\rho_x\left(\frac{K_b}{x}, \theta\right) = \rho_x^A(\phi_1(x,\theta), \theta)$.

when $x = \omega$, the new curve coincides with the curve given by Conditions (5.23). Finally, we highlight that the curve given by Condition (5.43) does not play an important role in this case, as can be seen in Figure 5.12 (a).

To summarize, we proved that the curve defined in (5.45), when $x < \omega$, is above the curve given by Condition (5.41) and bellow the curves given by Conditions (5.23) and (5.40).

5.F.2 $\rho_x\left(\frac{K_b}{x}, \theta\right) = \rho_x^A\left(\phi_1\left(x, \theta\right), \theta\right)$

For $d_1 > 2$, we want to explore where the curve given by the condition $\rho_x\left(\frac{K_b}{x}, \theta\right) = \rho_x^A\left(\phi_1\left(x, \theta\right), \theta\right)$ is placed, relatively to the curves given by the Conditions (5.23), (5.40), (5.41) and (5.43).

We start noticing that if $\nabla_1(\zeta(x)) > 0$ then $\phi_1(x,\theta)$ is not defined. In the case that $\nabla_1(\zeta(x)) = 0$, we have $\phi_1(x,\theta) = 0$, which implies that $\rho_x\left(\frac{K_b}{x}, \theta\right) > \rho_x^A\left(\phi_1\left(x, \theta\right), \theta\right) = 0$. Thus, the new curve is bellow the curve given by Condition (5.41). We also have $\phi_1(x,\theta) = 0$ in case $\nabla_1(0) < 0$. Then for $x > (d_1 - 1)\omega$, the new curve and the curve given by Condition (5.23) coincide, as in this case $\rho_x\left(\frac{K_b}{x}, \theta\right) = 0 = \rho_x^A\left(\phi_1\left(x, \theta\right), \theta\right)$. Moreover, when $\nabla_1\left(\frac{K_b}{x}\right) = 0$ implies that $\frac{K_b}{x}$ coincides with $\phi_1(x,\theta)$ if $0 < x < \frac{2\omega}{d_1}$ and with $\phi_2(x,\theta)$ otherwise. Then, for $0 < x < \frac{2\omega}{d_1}$, the new curve coincides with the curve given by Condition (5.43). Further, for $x > \frac{2\omega}{d_1}$, when $\nabla_1\left(\frac{K_b}{x}\right) = 0$ we have $\rho_x\left(\frac{K_b}{x}, \theta\right) = \rho_x^A\left(\phi_2\left(x, \theta\right), \theta\right) < \rho_x^A\left(\phi_1\left(x, \theta\right), \theta\right)$, implying that the new curve is above the curve given by Condition (5.43). Also, given that $\rho_x^A\left(\phi_1\left(x, \theta\right), \theta\right) \geq 0$, when $\rho_x\left(\frac{K_b}{x}, \theta\right) < 0$ we have $\rho_x\left(\frac{K_b}{x}, \theta\right) < \rho_x^A\left(\phi_1\left(x, \theta\right), \theta\right)$, which implies that the new curve is above (or coincides) with the curve given by Condition (5.23). Finally, we note that Condition (5.40) does not play a role in this case, as it is shown in Figure 5.12 (b).

To sum up, we proved that the new curve coincides with the curve given by Condition (5.43), for $0 < x < \frac{2\omega}{d_1}$, and with (5.23), for $x > (d_1 - 1)\omega$, and for $\frac{2\omega}{d_1} < x < (d_1 - 1)\omega$, it is bellow (5.41) and above (5.23) and (5.43).

5.F.3 $\rho_x^R\left(K_1^R(x,\theta),\theta\right) = \rho_x^A\left(\phi_1\left(x,\theta\right),\theta\right)$

For $d_1 > 2$, we want to explore where the curve given by the condition $\rho_x^R\left(K_1^R(x,\theta),\theta\right) = \rho_x^A\left(\phi_1\left(x,\theta\right),\theta\right)$ is placed, relatively to the curves given by the Conditions (5.23), (5.32), (5.43) and (5.46).

When $\nabla_1(0) \leq 0$, we have $\phi_1(x,\theta) = 0$, so $\rho_x^A(\phi_1(x,\theta),\theta) = 0$. Then, for $x > \omega_2$, when $\rho_x^R\left(K_1^R(x,\theta),\theta\right) = 0$, we have $\rho_x^R\left(K_1^R(x,\theta),\theta\right) = \rho_x^A(\phi_1(x,\theta),\theta) = 0$, which means that the new curve coincides with a curve given by Condition (5.32). Furthermore, for $0 < x < \omega_2$, as for this interval $\rho_x^A\left(\phi_1\left(x,\theta\right),\theta\right) \geq 0$, the new curve is always above (or coincides) the curve given by Condition (5.32).

Considering $\nabla_1\left(\frac{K_b}{x}\right) = 0$, we have $\frac{K_b}{x} = K_1^R(x,\theta)$. We also have $\frac{K_b}{x} = \phi_1(x,\theta)$ when $0 < x < \frac{2\omega}{d_1}$ and $\frac{K_b}{x} = \phi_2(x,\theta)$ otherwise. Thus, for $0 < x < \frac{2\omega}{d_1}$, the new curve coincides with the curve given by Condition (5.43). Besides as $\rho_x^A\left(\phi_1\left(x,\theta\right),\theta\right) > \rho_x^A\left(\phi_2\left(x,\theta\right),\theta\right)$ for $x > \frac{2\omega}{d_1}$, the new curve is above the curve given by Condition (5.43).

For $\frac{2\omega}{d_1} < x < (d_1-1)\omega$, when $\rho_x\left(\frac{K_b}{x},\theta\right) = \rho_x^A\left(\phi_1\left(x,\theta\right),\theta\right)$, since the curve given by Condition (5.46) is above the curve given by Condition (5.43), we also have $\rho_x^R(K_1^R(x,\theta),\theta) \geq \rho_x(\frac{K_b}{x},\theta) = \rho_x^A\left(\phi_1\left(x,\theta\right),\theta\right)$. This means that the new curve is bellow the curve given by Condition (5.46).

In summary, as it is shown in Figure 5.13, for $0 < x < \frac{2\omega}{d_1}$, the new curve coincides with the curve given by Condition (5.43); and when $x > (d_1-1)\omega$ coincides with (5.32); for $\frac{2\omega}{d_1} < x < (d_1-1)\omega$, it is bellow (5.46) and above (5.32) and (5.43).

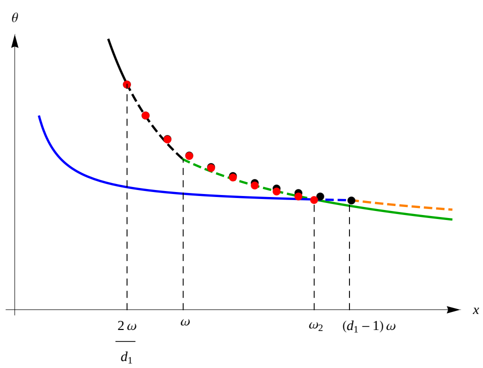


Figure 5.13: Plan division for ρ when $d_1 > 2$.
Green curve: $\rho_x^R\left(K_1^R(x,\theta),\theta\right) = 0$; orange curve: $\rho_x\left(\frac{K_b}{x},\theta\right) = 0$;
black curve: $\frac{K_b}{x} = K_1^R(x,\theta) \Leftrightarrow \nabla_1\left(\frac{K_b}{x}\right) = 0$; blue curve: $\nabla_1(0) = 0$;
black points: $\rho_x\left(\frac{K_b}{x},\theta\right) = \rho_x^A(\phi_1(x,\theta))$; red points: $\rho_x^R\left(K_1^R(x,\theta),\theta\right) = \rho_x^A\left(\phi_1\left(x,\theta\right),\theta\right)$.

Appendix 5.G Proofs for comparative statics

In the comparative statics it is important to highlight the dependence of the functions in each parameter. For ease the notation, when we want to emphasize the dependency of one quantity (a, say) with one parameter (β, say) we simply write the dependency on that parameter, assuming the others constant ($a(\beta)$, say).

5.G.1 Benchmark model

We start studying how the parameter d_1, defined in (5.12), change with the parameters μ, σ and r.

Proposition 5.8. *The parameter d_1 decreases with μ and σ and increases with r.*

Proof of Proposition 5.8. *We look at d_1, defined in* (5.12),

$$d_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \nabla,$$

with $\nabla = \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$. It is straightforward to conclude that d_1 increases with r. It is also very simple to take conclusions with respect to μ and σ, considering the derivatives, namely,

$$\frac{\partial d_1(\mu)}{\partial \mu} = -\frac{d_1}{\sigma^2 \nabla} < 0 \quad \textit{and} \quad \frac{\partial d_1(\sigma)}{\partial \sigma} = \frac{2}{\sigma^3 \nabla}\left(\mu d_1 - r\right) < 0$$

because $r - \mu d_1 = \frac{\sigma^2}{2} d_1(d_1 - 1) > 0$. Thus, d_1 decreases with μ and σ. ■

Proof of Proposition 5.4 We want to investigate how the add/replace threshold, $x^\star$, defined on (5.10), evolves with the different parameters. We have $x^\star = \frac{K_b}{K_1}$, where $K_b = \frac{d_1}{2(d_1-1)} \frac{(r-\mu)\pi_0}{r\beta K_0}$. As K_b does not depend on δ, λ and u, neither does $x^\star$. Also K_b does not depend on K_1, but $x^\star$ decreases with it. Obviously, K_b decreases with β and $x^\star$, as well. Taking into account the π_0 definition, we easily conclude that K_b increases with ξ_0 and decreases with α and K_0, and therefore so does $x^\star$. To study the behavior of $x^\star$ with σ, we need to explore $\frac{d_1(\sigma)}{d_1(\sigma)-1} = 1 + \frac{1}{d_1(\sigma)-1}$, as a function of σ. Given that d_1 decreases with σ, then $\frac{d_1(\sigma)}{d_1(\sigma)-1}$ increases with σ, as well as K_b and, consequently, $x^\star$. The challenging cases are μ and r.

For μ, we need to study $\varrho_1(\mu) = (r-\mu)\frac{d_1(\mu)}{d_1(\mu)-1}$. Note that $\varrho_1'(\mu) = -\frac{d_1(\mu)[d_1(\mu)-1]+(r-\mu)\frac{\partial d_1(\mu)}{\partial \mu}}{[d_1(\mu)-1]^2}$, which can be simplified as $\varrho_1'(\mu) = -\frac{d_1(\mu)}{\sigma^2\nabla}\left[\sigma^2\nabla\left(d_1(\mu)-1\right) - (r-\mu)\right]$. After some calculations we end up with $\varrho_1'(\mu) = -\frac{d_1(\mu)}{\nabla}\left[\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2 + \frac{r+\mu}{\sigma^2} - \left(\frac{1}{2}+\frac{\mu}{\sigma^2}\right)\nabla\right]$, to which we can apply the conjugate and after some comprehensive calculations, we get $\varrho_1'(\mu) = -\frac{d_1(\mu)\left(\frac{r-\mu}{\sigma^2}\right)^2}{\nabla\left[\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{r+\mu}{\sigma^2}+\left(\frac{1}{2}+\frac{\mu}{\sigma^2}\right)\nabla\right]} < 0$. Therefore, we conclude that K_b and $x^\star$ decrease with μ.

For r we need to study $\varrho_2(r) = \frac{r-\mu}{r}\frac{d_1(r)}{d_1(r)-1}$. Note that $\varrho_2'(r) = \frac{\mu d_1(r)(d_1(r)-1) - r(r-\mu)\frac{\partial d_1(r)}{\partial r}}{[r(d_1(r)-1)]^2}$, which is equivalent to $\varrho_2'(r) = \frac{2\mu\nabla\left[r-\mu\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)-\mu\nabla\right]-r(r-\mu)}{\nabla[r\sigma(d_1(r)-1)]^2}$. Applying again the conjugate, we obtain $\varrho_2'(r) = -\frac{(r-\mu)^2}{\nabla[\sigma(d_1(r)-1)]^2\left[2\mu\nabla\left[r-\mu\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)\right]+\left[2\mu^2\left[\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}\right]+r(r-\mu)\right]\right]} < 0$. Then, K_b and $x^\star$ also decrease with r. ■

Proof of Proposition 5.5

We want to investigate how the decision threshold, $\theta_x^\star$, defined in (5.20), evolves with the different parameters. Let us start recalling the expressions

$$\begin{aligned} \upsilon^A(x) &= \frac{\epsilon}{x} + \frac{\lambda u}{r} + \alpha K_1 + 2\beta K_0 - \frac{a(r-\mu)x^{d_1-1}}{K_1} \\ \upsilon^R(x) &= \frac{\epsilon}{x} + \frac{\lambda u}{r} + \alpha K_1 + \frac{(r-\mu)\pi_0}{rK_1 x}. \end{aligned}$$

It is immediate to see that the derivatives of υ^A and υ^R in order to either δ, λ or u are equal and positive, and thus $\theta_x^\star$ increases with those parameters.

Note that we can rewrite a as follows $a = \left[\frac{r(d_1-1)}{\pi_0}\right]^{d_1-1}\left[\frac{2\beta K_0 K_1}{(r-\mu)d_1}\right]^{d_1}$, showing that a decreases with ξ_0 and increases with β. Then, $\frac{\partial \upsilon^A(x,\xi_0)}{\partial \xi_0} = -\frac{(r-\mu)x^{d_1(\xi_0)-1}}{K_1}\frac{\partial a(\xi_0)}{\partial \xi_0} > 0$ and $\frac{\partial \upsilon^R(x,\xi_0)}{\partial \xi_0} = \frac{(r-\mu)K_0}{rK_1 x} > 0$, implying that $\theta_x^\star$ increases with ξ_0. Now we notice that υ^R does not depend on either σ or β, thus for replacement the threshold $\theta_x^\star$ is the same for any σ or β. The proofs for those two parameters are very similar.

Regarding σ, let us consider $\sigma_1 < \sigma_2$. First we want to discover the sign of $\upsilon^A(x,\sigma_2) - \upsilon^A(x,\sigma_1) = -\frac{r-\mu}{K_1}\left[a(\sigma_2)x^{d_1(\sigma_2)-1} - a(\sigma_1)x^{d_1(\sigma_1)-1}\right]$. We have $\upsilon^A(x,\sigma_2) > \upsilon^A(x,\sigma_1) \Leftrightarrow x > \left[\frac{a(\sigma_2)}{a(\sigma_1)}\right]^{\frac{1}{d_1(\sigma_1)-d_1(\sigma_2)}}$, which means that the graphs of $\upsilon^A(x,\sigma_1)$ and $\upsilon^A(x,\sigma_2)$ intercept each other only once. We know that the graphs of $\upsilon^A(x,\sigma_1)$ and $\upsilon^A(x,\sigma_2)$ are tangent to the graph of $\upsilon^R(x)$, respectively, at $x^\star(\sigma_1)$ and $x^\star(\sigma_2)$, where we already proved that $x^\star(\sigma_1) < x^\star(\sigma_2)$. Given that $\upsilon^A(x^\star(\sigma_1),\sigma_1) = \upsilon^R(x^\star(\sigma_1)) > \upsilon^A(x^\star(\sigma_1),\sigma_2)$ and $\upsilon^A(x^\star(\sigma_2),\sigma_2) = \upsilon^R(x^\star(\sigma_2)) > \upsilon^A(x^\star(\sigma_2),\sigma_1)$, this implies that the graphs of $\upsilon^A(x,\sigma_1)$ and $\upsilon^A(x,\sigma_2)$ need to intercept each other between $x^\star(\sigma_1)$ and $x^\star(\sigma_2)$. Since they only intercept once, we conclude that $x^\star(\sigma_1) < \left[\frac{a(\sigma_2)}{a(\sigma_1)}\right]^{\frac{1}{d_1(\sigma_1)-d_1(\sigma_2)}} < x^\star(\sigma_2)$. We can conclude that $\theta_x^\star(\sigma_1) > \theta_x^\star(\sigma_2)$ for $0 < x < x^\star(\sigma_2)$ and $\theta_x^\star(\sigma_1) = \theta_x^\star(\sigma_2)$ for $x \geq x^\star(\sigma_2)$.

Concerning β, let us take $\beta_1 < \beta_2$, and study the sign of $\upsilon^A(x,\beta_2) - \upsilon^A(x,\beta_1) = 2(\beta_2-\beta_1)K_0 - \frac{(r-\mu)x^{d_1-1}}{K_1}[a(\beta_2) - a(\beta_1)]$. As in the previous case, the graphs of $\upsilon^A(x,\beta_1)$ and $\upsilon^A(x,\beta_2)$ only intercept each other once; also $\upsilon^A(x,\beta_2) > \upsilon^A(x,\beta_1) \Leftrightarrow x > \left[\frac{2(\beta_2-\beta_1)K_0K_1}{(r-\mu)[a(\beta_2)-a(\beta_1)]}\right]^{\frac{1}{d_1-1}}$[24]. Given that $\upsilon^A(x^\star(\beta_1),\beta_1) = \upsilon^R(x^\star(\beta_1)) > \upsilon^A(x^\star(\beta_1),\beta_2)$ and $\upsilon^A(x^\star(\beta_2),\beta_2) = \upsilon^R(x^\star(\beta_2)) > \upsilon^A(x^\star(\beta_2),\beta_1)$, and recalling that $x^\star(\beta_2) < x^\star(\beta_1)$, then the graphs of $\upsilon^A(x,\beta_1)$ and $\upsilon^A(x,\beta_2)$ need to intercept each other between $x^\star(\beta_2)$ and $x^\star(\beta_1)$. Since they only intercept once, we conclude that $x^\star(\beta_2) < \left[\frac{2(\beta_2-\beta_1)K_0K_1}{(r-\mu)[a(\beta_2)-a(\beta_1)]}\right]^{\frac{1}{d_1-1}} < x^\star(\beta_1)$. We can conclude that $\theta_x^\star(\beta_1) < \theta_x^\star(\beta_2)$ for $0 < x < x^\star(\beta_1)$ and $\theta_x^\star(\beta_1) = \theta_x^\star(\beta_2)$ for $x \geq x^\star(\beta_1)$.

The study of μ has some more complex details. The replace case is straightforward, as $\frac{\partial \upsilon^R(x,\mu)}{\partial \mu} = -\frac{1}{x}\left[\delta + \frac{\pi_0(\mu)}{rK_1}\right] < 0$. Let us consider $\upsilon^A(x,\mu_2) - \upsilon^A(x,\mu_1) = -\frac{1}{x}\nu(x)$, where $\nu(x) = \delta(\mu_2-\mu_1) + \frac{1}{K_1}\left[a(\mu_2)(r-\mu_2)x^{d_1(\mu_2)} - a(\mu_1)(r-\mu_1)x^{d_1(\mu_1)}\right]$, for $\mu_2 > \mu_1$. Given that $d_1(\mu_1) > d_1(\mu_2)$, we have $\nu(0) = \delta(\mu_2-\mu_1) > 0$ and $\lim_{x\to+\infty}\nu(x) = -\infty$. Considering the derivative

$$\nu'(x) = \frac{1}{K_1}\left[a(\mu_2)(r-\mu_2)d_1(\mu_2)x^{d_1(\mu_2)-1} - a(\mu_1)(r-\mu_1)d_1(\mu_1)x^{d_1(\mu_1)-1}\right],$$

we see that its sign is changing only once; indeed $\nu'(x) > 0 \Leftrightarrow x < \left[\frac{a(\mu_2)(r-\mu_2)d_1(\mu_2)}{a(\mu_1)(r-\mu_1)d_1(\mu_1)}\right]^{\frac{1}{d_1(\mu_1)-d_1(\mu_2)}}$. Then ν has only one zero, changing from positive to negative. Moreover, given that υ^R decreases on μ, we have $\upsilon^A(x^\star(\mu_2),\mu_1) > \upsilon^R(x^\star(\mu_2),\mu_1) > \upsilon^R(x^\star(\mu_2),\mu_2) = \upsilon^A(x^\star(\mu_2),\mu_2)$, which means that $\nu(x^\star(\mu_2)) > 0$. Thus, for $0 < x \leq x^\star(\mu_2)$, we have $\nu(x) > 0$, i.e. $\upsilon^A(x,\mu_2) < \upsilon^A(x,\mu_1)$. It remains to see that for $x^\star(\mu_2) < x \leq x^\star(\mu_1)$, $\upsilon^R(x,\mu_2) < \upsilon^R(x,\mu_1) \leq \upsilon^A(x,\mu_1)$. With this we conclude that $\theta^\star$ decreases with μ.

The behaviour of $\theta_x^\star$ with respect to K_0, K_1, α and r depends intrinsically on the way the functions υ^A and υ^R behave when one changes values of the parameters. As some of the arguments are similar for all these parameters, we propose to use y to denote one of the parameters under study.

[24]Given that a increases with β, we have $a(\beta_2) - a(\beta_1) > 0$.

In order to clarify the proof, we present a sketch of the basic idea. We start by recalling that $x^\star$ decreases with K_0, K_1, α and r, i.e., for $y_1 < y_2$, we have $x^\star(y_1) > x^\star(y_2)$. In all cases, the following functions have to be compared

$$\Phi_{AA}(x; y_1, y_2) = \upsilon^A(x, y_2) - \upsilon^A(x, y_1) \tag{5.49}$$

$$\Phi_{RR}(x; y_1, y_2) = \upsilon^R(x, y_2) - \upsilon^R(x, y_1) \tag{5.50}$$

$$\Phi_{RA}(x; y_1, y_2) = \upsilon^R(x, y_2) - \upsilon^A(x, y_1) \tag{5.51}$$

In Lemma 5.4 we derive the following expressions

$$\Phi_{AA}(x; y_1, y_2) = \begin{cases} \varsigma_1 - \varsigma_2 x^{d_1 - 1} & \text{if the parameter is } \alpha, K_0 \text{ or } K_1 \\ \frac{1}{x}\left[\varsigma_4 - \varsigma_5 x^{d_1(y_2)} + \varsigma_6 x^{d_1(y_1)}\right] - \varsigma_3 & \text{if the parameter is } r \end{cases} \tag{5.52}$$

where $\varsigma_i > 0$ for $i = 1, 2, 3, 4, 5, 6$. Furthermore,

$$\Phi_{RR}(x; y_1, y_2) = \varsigma_1 - \frac{\varsigma_2}{x} \tag{5.53}$$

where, for $i = 1, 2$, $\varsigma_i > 0$ for K_1 and α, and $\varsigma_i < 0$ for r. For K_0, $\varsigma_1 = 0$ and $\varsigma_2 > 0 \Leftrightarrow y_1 + y_2 < \frac{\xi_0}{\alpha}$. Finally,

$$\Phi_{RA}(x; y_1, y_2) = \frac{\varsigma_1}{x}\left[\varsigma_2 x^{d_1} + \varsigma_3\right] - \varsigma_4 \tag{5.54}$$

where $\varsigma_i > 0$ for $i = 1, 2, 3$ and $\varsigma_4 \in \mathbb{R}$.

Next we study the sign of these functions. When the parameter in study is α, K_0 or K_1, then $\Phi_{AA}(x; y_1, y_2) > 0 \Leftrightarrow x < \left(\frac{\varsigma_1}{\varsigma_2}\right)^{\frac{1}{d_1 - 1}} = i_1$.

For r, as $\lim_{x \to 0^+} \Phi_{AA}(x; y_1, y_2) = +\infty$ and $\lim_{x \to +\infty} \Phi_{AA}(x; y_1, y_2) = -\infty$, it follows that Φ_{AA} has at least one zero. So, here we just consider the case where we have exactly one zero, hereby denoted by i_1. Then $\Phi_{AA}(x; y_1, y_2) > 0 \Leftrightarrow x < i_1$.

For Φ_{RR} the following holds:

- for K_0, $\Phi_{RR}(x; y_1, y_2) > 0 \Leftrightarrow y_1 + y_2 < \frac{\xi_0}{\alpha}$;
- for K_1 and α, $\Phi_{RR}(x; y_1, y_2) > 0 \Leftrightarrow x > \frac{\xi_2}{\xi_1} = i_2$;
- for r, $\Phi_{RR}(x; y_1, y_2) > 0 \Leftrightarrow x < \frac{\xi_2}{\xi_1} = i_2$.

Regarding Φ_{RA}, given that $d_1 > 1$, then it follows that $\lim_{x \to 0^+} \Phi_{RA}(x; y_1, y_2) = +\infty$ and $\lim_{x \to +\infty} \Phi_{RA}(x; y_1, y_2) = +\infty$. We have $\Phi'_{RA}(x; y_1, y_2) = \frac{\varsigma_1}{x^2}\left[\varsigma_2(d_1 - 1)x^{d_1} - \varsigma_3\right]$, which has only one zero when $x = \left[\frac{\varsigma_3}{\varsigma_2(d_1 - 1)}\right]^{\frac{1}{d_1}}$. Thus Φ_{RA} is a convex function, that either it is always non negative or there are two points, j_1 and j_2, such that $\Phi_{RA}(x; y_1, y_2) < 0 \Leftrightarrow x \in (j_1, j_2)$. This fact only depends on the sign of $m = \Phi_{RA}\left(\left[\frac{\varsigma_3}{\varsigma_2(d_1 - 1)}\right]^{\frac{1}{d_1}}; y_1, y_2\right) = d_1 \varsigma_1 \varsigma_2^{\frac{1}{d_1}} \left[\frac{\varsigma_3}{d_1 - 1}\right]^{\frac{d_1 - 1}{d_1}} - \varsigma_4$. Summing up, if $m \geq 0$ then $\Phi_{RA}(x; y_1, y_2) \geq 0$ for all $x > 0$; if $m < 0$ then $\Phi_{RA}(x; y_1, y_2) < 0 \Leftrightarrow x \in (j_1, j_2)$.

In view of the behaviour of these functions, for α and K_1 we conclude that: if $m \geq 0$ then $\theta^\star(x, y_1) \leq \theta^\star(x, y_2)$; if $m < 0$ then $\theta^\star(x, y_1) \leq \theta^\star(x, y_2)$ when $0 < x \leq \varpi_1$ or $x \geq \varpi_2$ and $\theta^\star(x, y_1) > \theta^\star(x, y_2)$ when $\varpi_1 < x < \varpi_2$, where

- if $i_1 \leq x^\star(y_2)$ then $\varpi_1 = i_1$, otherwise $\varpi_1 = j_1$;

- if $i_2 \geq x^\star(y_1)$ then $\varpi_2 = i_2$, otherwise $\varpi_2 = j_2$.

For K_0, if $y_1 + y_2 \leq \frac{\xi_0}{\alpha}$ then $\theta^\star(x, y_1) \leq \theta^\star(x, y_2)$; otherwise $\theta^\star(x, y_1) \leq \theta^\star(x, y_2)$ when $0 < x \leq \varpi$ and $\theta^\star(x, y_1) > \theta^\star(x, y_2)$ when $x > \varpi$, where $\varpi = i_1$ if $i_1 \leq x^\star(y_2)$ and $\varpi = j_1$ if $i_1 > x^\star(y_2)$.

Finally, with respect to r, we have the following behaviours: $\theta^\star(x, y_1) \leq \theta^\star(x, y_2)$ when $0 < x \leq \varpi$ and $\theta^\star(x, y_1) > \theta^\star(x, y_2)$ when $x > \varpi$, where

- $\varpi = i_1$ if $m > 0$ and $j_1 < x^\star(y_2) < x^\star(y_1) < j_2$;
- $\varpi = j_1$ if $m > 0$ and $x^\star(y_2) < x^\star(y_1) < j_2$;
- $\varpi = i_2$ otherwise [25].

$\blacksquare$

Lemma 5.4. *For the parameters K_0, K_1, α and r, the functions defined in* (5.49), (5.50) *and* (5.51) *are given by* (5.52), (5.53) *and* (5.54), *respectively.*

Proof of Lemma 5.4.

- *For parameter K_0, let us consider two possible values, $K_{01} < K_{02}$.*

  i) $\Phi_{AA}(x; K_{01}, K_{02}) = 2\beta\,(K_{02} - K_{01}) - \frac{(r-\mu)[a(K_{02}) - a(K_{01})]}{K_1} x^{d_1 - 1}$.

  Note that $a\,(K_{02}) - a\,(K_{01}) > 0 \Leftrightarrow K_{01}\,[\xi_0 - \alpha K_{01}]^{1-d_1} < K_{02}\,[\xi_0 - \alpha K_{02}]^{1-d_1}$. *Considering the function* $\phi(k) = k\,(\xi_0 - \alpha k)^{1-d_1}$, *with* $k < \frac{\xi_0}{\alpha}$, *we have* $\phi'(k) = \frac{(\xi_0 - \alpha k) + \alpha k (d_1 - 1)}{(\xi_0 - \alpha k)^{d_1}} > 0$, *meaning that ϕ is an increasing function. Thus, it holds* $a\,(K_{02}) - a\,(K_{01}) > 0$.

  ii) $\Phi_{RR}(x; K_{01}, K_{02}) = \frac{(r-\mu)[\pi_0(K_{02}) - \pi_0(K_{01})]}{r K_1 x}$.

  Note that $\pi_0\,(K_{02}) - \pi_0\,(K_{01}) > 0 \Leftrightarrow K_{01} + K_{02} < \frac{\xi_0}{\alpha}$.

  iii) $\Phi_{RA}(x; K_{01}, K_{02}) = \frac{(r-\mu)}{r K_1 x}\left[r a\,(K_{01})\, x^{d_1} + \pi_0\,(K_{02})\right] - 2\beta K_{01}$.

- *For parameter K_1, let us consider two possible values, $K_{11} < K_{12}$.*

  i) $\Phi_{AA}(x; K_{11}, K_{12}) = \alpha\,(K_{12} - K_{11}) - \frac{(r-\mu)[K_{11} a(K_{12}) - K_{12} a(K_{11})]}{K_{11} K_{12}} x^{d_1 - 1}$.

  Note that $K_{11} a(K_{12}) - K_{12} a(K_{11}) > 0 \Leftrightarrow \left(\frac{K_{11}}{K_{12}}\right)^{d_1 - 1} < 1$, *which is true because $d_1 > 1$ and* $\frac{K_{11}}{K_{12}} < 1$.

  ii) $\Phi_{RR}(x; K_{11}, K_{12}) = (K_{12} - K_{11})\left[\alpha - \frac{(r-\mu)\pi_0}{r K_{11} K_{12} x}\right]$.

  iii) $\Phi_{RA}(x; K_{11}, K_{12}) = \frac{(r-\mu)}{x}\left[\frac{a(K_{11})}{K_{11}} x^{d_1} + \frac{\pi_0}{r K_{12}}\right] - [2\beta K_0 - \alpha\,(K_{12} - K_{11})]$.

- *For parameter α, let us consider two possible values, $\alpha_1 < \alpha_2$.*

  i) $\Phi_{AA}(x; \alpha_1, \alpha_2) = K_1\,(\alpha_2 - \alpha_1) - \frac{(r-\mu)[a(\alpha_2) - a(\alpha_1)]}{K_1} x^{d_1 - 1}$.

  Note that $a(\alpha_2) - a(\alpha_1) = \left[\frac{2\beta K_0 K_1}{(r-\mu) d_1}\right]^{d_1} [r(d_1 - 1)]^{d_1 - 1}\left[\pi_0(\alpha_2)^{1-d_1} - \pi_0(\alpha_1)^{1-d_1}\right]$ *and* $\pi_0(\alpha_2)^{1-d_1} - \pi_0(\alpha_1)^{1-d_1} > 0 \Leftrightarrow \left[\frac{\xi_0 - \alpha_1 K_0}{\xi_0 - \alpha_2 K_0}\right]^{d_1 - 1} > 1$, *which is always true because* $\frac{\xi_0 - \alpha_1 K_0}{\xi_0 - \alpha_2 K_0} > 1$ *and* $d_1 > 1$.

  ii) $\Phi_{RR}(x; \alpha_1, \alpha_2) = (\alpha_2 - \alpha_1)\left[K_1 - \frac{(r-\mu) {K_0}^2}{r K_1 x}\right]$.

[25] Note that we always have $x^\star(y_1) < j_2$

iii) $\Phi_{RA}(x;\alpha_1,\alpha_2)=\frac{(r-\mu)}{rK_1x}\left[ra(\alpha_1)x^{d_1}+\pi_0(\alpha_2)\right]-[2\beta K_0-K_1(\alpha_2-\alpha_1)]$.

- *For parameter r, let us consider two possible values, $r_1<r_2$.*

i) $\Phi_{AA}(x;r_1,r_2) = \frac{1}{x}\left[\delta(r_2-r_1)-\frac{1}{K_1}\left[(r_2-\mu)a(r_2)x^{d_1(r_2)-1}-(r_1-\mu)a(r_1)x^{d_1(r_1)-1}\right]\right]+\frac{\lambda u(r_2-r_1)}{r_1r_2}$.

ii) $\Phi_{RR}(x;r_1,r_2)=\frac{(r_2-r_1)}{r_1r_2}\left[-\lambda u+\frac{\delta K_1r_1r_2+\mu\pi_0}{K_1x}\right]$.

iii) $\Phi_{RA}(x;r_1,r_2)=\frac{1}{r_2K_1x}\left[r_2(r_1-\mu)a(r_1)x^{d_1(r_1)}+r_2(r_2-r_1)\delta K_1+(r_2-\mu)\pi_0\right]$
$-\left[2\beta K_0+\frac{\lambda u(r_2-r_1)}{r_1r_2}\right]$.

■

5.G.2 Capacity optimization model

Proof of Proposition 5.6 We want to investigate how θ_x^B, defined in (5.28), changes with the different parameters. It is easy to see that $\theta_x^B=\frac{\epsilon}{x}+\frac{2\alpha K_b}{x}$ does not depend on λ and u, increases with δ, and has the same monotony as K_b for σ,β,ξ_0 and K_0. It remains to study the μ,α and r cases. Note that θ_x^B, as a function of μ, can be written as $\frac{1}{x}\left[\delta(r-\mu)+\frac{\alpha\pi_0}{r\beta K_0}(r-\mu)\frac{d_1(\mu)}{d_1(\mu)-1}\right]$, where we have already proven that $(r-\mu)\frac{d_1(\mu)}{d_1(\mu)-1}$ is a decreasing function of μ. Then, we conclude that θ_x^B decreases with μ.

To investigate the α case, we need to study the function $\varrho_3(\alpha)=\alpha\left(\xi_0-\alpha K_0\right)$ for $0<\alpha<\frac{\xi_0}{K_0}$. Given that ϱ_3 is a concave function with zeros on 0 and $\frac{\xi_0}{K_0}$, then it is not monotonic. Let us consider $\alpha_1<\alpha_2$. Then $\varrho_3(\alpha_2)-\varrho_3(\alpha_1)>0\Leftrightarrow(\alpha_2-\alpha_1)\left[\xi_0-(\alpha_1+\alpha_2)K_0\right]>0\Leftrightarrow\alpha_1+\alpha_2<\frac{\xi_0}{K_0}$. Consequently, if $\alpha_1+\alpha_2<\frac{\xi_0}{K_0}$ the θ_x^B increases with α and decreases otherwise.

Regarding r case, note that θ_x^B can be written as $\frac{1}{x}(r-\mu)\left[\delta+\frac{\alpha\pi_0}{\beta K_0}\frac{d_1(r)}{r[d_1(r)-1]}\right]$. Considering the function $\varrho_4(r)=(r-\mu)\left[\delta+\frac{\alpha\pi_0}{\beta K_0}\frac{d_1(r)}{r[d_1(r))-1]}\right]$, we have[26]

$$\varrho_4(r_2)-\varrho_4(r_1)=\delta(r_2-r_1)+\frac{\alpha\pi_0}{\beta K_0}\left[\frac{(r_2-\mu)d_1(r_2)}{r_2(d_1(r_2)-1)}-\frac{(r_1-\mu)d_1(r_1)}{r_1(d_1(r_1)-1)}\right].$$

Then, if $\delta(r_2-r_1)+\frac{\alpha\pi_0}{\beta K_0}\left[\frac{(r_2-\mu)d_1(r_2)}{r_2(d_1(r_2)-1)}-\frac{(r_1-\mu)d_1(r_1)}{r_1(d_1(r_1)-1)}\right]>0$ the θ_x^B increases and decreases otherwise. ■

Proof of Proposition 5.7 It is straightforward to show that

$$\begin{aligned}\frac{\partial K_1^R(x,\theta)}{\partial x} &= \frac{\epsilon}{2\alpha x^2}>0;\\ \frac{\partial K_1^A(x,\theta)}{\partial x} &= \frac{\omega(2)\left\{\left[(\theta-2\beta K_0)x-\epsilon\right]x+\epsilon\left(\omega(2)-x\right)\right\}}{2\alpha\left(x(\omega(2)-x)\right)^2}>0.\end{aligned}$$

■

[26]Note that $\frac{(r_2-\mu)d_1(r_2)}{r_2(d_1(r_2)-1)}-\frac{(r_1-\mu)d_1(r_1)}{r_1(d_1(r_1)-1)}=\varrho_2(r_2)-\varrho_2(r_1)<0$ because ϱ_2 decreases with r.

6
Conclusions and further work

This thesis is build upon independent papers, thus the conclusions and the punchlines were already presented in each chapter. Here we only summarize the main findings of each chapter, dedicated to specific investment problems

In Chapter 2 it is interesting to look into the results obtained in the comparative statics. For some parameters the result is as expected, and similar to the ones discussed in earlier works (like the behaviour of the investment threshold with volatility of both the demand and the investment cost). In contrast, for instance, when we study the monotonicity of the threshold with respect to the expected value of the jumps of demand, the result depends on analytical conditions whose economical interpretation is not straightforward.

Regarding Chapter 4, we show that the innovation can occur either right at the moment of a technological breakthrough, or some time after such an event. In the latter case the firm adopts the new product, because demand of the established product has reduced too much to stay active in the established product market. We further obtain an explicit expression for the value of the firm, reflecting a weighted average of all possible innovation patterns.

Lastly, in Chapter 5, after innovating the firm has the choice to solely produce the new product, or for some time simultaneously producing the established and the new product, while it will abolish the old product once demand on the new market has grown enough. In the last case the chosen capacity is some weighted average between a lower level associated with producing both products, and a higher level when just the new product is produced. In this way the firm anticipates the later abolishment of the old product by choosing a larger capacity for producing the innovative product.

Now it remains to look at the possible extensions of each work and how they could be linked. Even though it is not very simple, we can try to find a particular solution for other order Euler-Cauchy equations, going further in Chapter 3. Using the knowledge gained in Chapter 3, we can generalize the work presented in Chapter 2, considering jumps in the opposite directions from the ones here presented. We can also extend the Chapter 4 in two ways. Firstly, allowing for the possibility of adding the new product to the established one, instead of only replacing (as it is considered in Chapter 5). Secondly, assuming that the established market does not exhibit a deterministic decline but a diffusion with negative drift. Finally, Chapter 5 can borrow some of the ideas of Chapter 4, in particular consider that the established market has a decline profit instead of a constant.

It is not that I am so smart,
it is just that I stay with the problems longer.

Albert Einstein

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