2. $X_i \sim \text{Bern}(p=.2)$

$S_n = \sum_{i=1}^{n} X_i \qquad \theta = P(S_n > a), \; n = 40, \; a = 16$

By Cramer's we get that $\frac{1}{n}\log P(S_n > a) = -I(a)$

where $I(a) = \sup_\theta (\theta a - \psi(\theta))$

so we pick θ^* s.t. $a = \psi'(\theta^*)$

From example 9u we know that for $E[\theta]$ and S_n that

$$\varphi(\theta) = \prod_{i=1}^{n} P_i e^\theta + 1 - P_i$$

$$= (Pe^\theta + 1 - P)^n \quad \text{when } P_i = p \; \forall i$$

so $\psi(\theta) = n\log(Pe^\theta + 1 - p)$

and $\psi'(\theta) = \dfrac{nPe^\theta}{Pe^\theta + 1 - P}$ so we pick θ^* s.t. $a = \dfrac{nPe^{\theta^*}}{Pe^{\theta^*} + 1 - P}$

so $\theta^* = \ln\left(\dfrac{a(1-p)}{p(n-a)}\right)$.

rather than simulate $I(S_n > a)$ directly by generating $X_i \sim \text{Bern}(p)$

we instead simulate $I(S_n > a)\varphi(\theta^*)e^{-\theta^* S_n}$ where $X_i \sim \text{Bern}(e^{\theta^*})$

in our problem $\theta^* = \ln\left(\frac{8}{3}\right)$, $\varphi(\theta^*) = (.2e^{\theta^*} + .8)^{40}$, $e^{-\theta^* S_n} = \left(\frac{3}{8}\right)^{S_n}$

$= \left(\frac{4}{3}\right)^{40}$

Raw E: 0.002887. Tilted E: 0.002943

Raw Var: 2.9e-05. Tilted Var: 0.0

3.

By Siegmund we choose θ^* s.t. $\psi(\theta^*) = 0$, $\varphi(\theta^*) = 1$

For the normal dist $\psi(\theta) = \theta\mu + \frac{1}{2}\sigma^2\theta^2$

$$\text{so} \quad \theta^*\mu + \frac{1}{2}\theta^2\sigma^2 = 0$$

$$\theta^* = \frac{-2\mu}{\sigma^2}$$

Thus to find $P(\tau(X) < \infty)$, $\tau(X) = \inf\{k : S_n > X\}$, $S_n = \sum_{i=1}^{n} X_i$,

where $X_i \sim N(\mu, \sigma^2)$ we can generate the X_i with

$X_i \sim N(\mu + \theta^*\sigma^2, \sigma^2)$ and find

$$p(\tau(X) < \infty) = \frac{1}{n}\sum_{i=1}^{n} e^{-\theta^* X} e^{-\theta^* B^{(i)}(X)} \quad \text{where } B(X) = S_{\tau(X)} - X$$

the following algorithm suffices to simulate $P(\tau(X) < \infty) = P$

```
E = []   X = 10   θ* = .4
B = {}   μ = -.1  σ² = .5
N ∈ ℝ
for i in range(N)
    S_τ(X) = α ~ N(μ + θ*σ², σ²)
    while S_τ(X) ≤ X
        S_τ(X) += α ~ N(μ + θ*σ², σ²)
    B[i] = S_τ(X) - X
    E.append( e^(-θ* X) · 1/|B| · Σ_{k=1}^{|B|} e^(-θ* B[k]) )
return mean(E), var(E)    "P"
```

E: 0.015574615063278889, Var: 1.1624724272507568e-09
95% CI: (0.015574612068376651, 0.015574618058181127)

4.

Let $\Delta_n = S_n - T_{n+1}$, then $E[\Delta] < 0$ and $D_n = (D_{n-1} + \Delta_{n-1})^+$ follow

$$D_n \overset{d}{=} \max_{k \le n} \sum_{i=1}^{k} \Delta_i = M_n, \text{ so } P(D_n > d) \overset{d}{=} P(D > d) \text{ in steady state}$$

and $P(D > d) = P(M > d) = P(\tau(d) < \infty)$, $\tau(d) = \inf\{n : R_n > d\}$

where $R_n = \sum_{i=1}^{n} \Delta_i$, so

$$P(D > d) = e^{-\theta^* d} E_{\theta^*}[e^{-\theta^* B(d)}], \quad B(d) = \sum_{i=1}^{\tau(d)} \Delta_i - d$$

thus we pick a θ^* s.t. $E[e^{\theta^* \Delta}] = 1$

b/c
$$E[e^{\theta^* \Delta}] = E[e^{\theta^*(S-T)}]$$
$$= E[e^{\theta^* S}] E[e^{-\theta^* T}]$$

since $T \sim \exp(\lambda)$, $E[e^{-\theta^* T}] = \frac{\lambda}{\lambda - \theta^*}$

Since S is erlang-k

$$\psi(\theta) = \int_0^\infty e^{-\theta t} \frac{\mu^t t^{k-1} e^{-\mu t}}{(k-1)!} dt = I_k$$

$$\text{so} \quad I_1 = \mu \int_0^\infty e^{-(\theta+\mu)t} dt = (\mu)\left(-\frac{e^{-(\theta+\mu)t}}{\theta+\mu}\Big|_0^\infty\right) = \frac{\mu}{\theta+\mu}$$

$$\text{and} \quad I_{k+1} = \frac{\mu^{k+1}}{k!}\left(t^{k-1} e^{-(\theta+\mu)t}\Big|_0^\infty + \frac{\mu}{\theta+\mu}\int_0^\infty t^{k-1} e^{-(\theta+\mu)t} dt\right)$$

$$= \frac{\mu}{\theta+\mu}\int_0^\infty e^{-\theta t} \frac{\mu^k t^{k-1} e^{-\mu t}}{(k-1)!} dt$$

$$= \frac{\mu}{\theta+\mu} I_k$$

so $\psi(\theta) = \left(\frac{\mu}{\mu+\theta}\right)^k$ for erlang dist.

4. cont.

So then

$$E[e^{\theta^* \Delta}] = \left(\frac{\lambda}{\lambda-\theta^*}\right)\left(\frac{\mu}{\mu+\theta^*}\right)^k = 1, \quad k=2$$

since $\mu = 3$, $\lambda = 1$ we get

$$\left(\frac{1}{1-\theta^*}\right)\left(\frac{3}{3+\theta^*}\right)^2 = 1$$

and $3 = (3+\theta^*)\sqrt{1-\theta^*} \implies \theta^* \approx .697224$

We can simulate $P(\Delta > d) = e^{-\theta^* d} E_{\theta^*}[e^{-\theta^* B(d)}]$, $B(d) = \sum_{i=1}^{T(d)} \Delta_i - d$

by using $T_i \sim \exp(\lambda - \theta^*)$, $S_i \sim$ erlang-2 $(\mu - \theta^*)$

with the following algorithm

$E = [\,]$ $d = 10$ $\theta^* = .697224$

$B = \{\}$ $\mu = 3$ $\lambda = 1$ $N \in \mathbb{R}$

for k in range (N)

  gen $S_i \sim$ erlang-2$(\mu - \theta^*)$, $T_i \sim \exp(\lambda - \theta^*)$

  set $\Delta = 0$

  while $\Delta \leq d$

    $\Delta \pm S_i - T_i$

  $B[i] = \Delta - d$

  E.append$\left(e^{-\theta^* d} \frac{1}{|B|} \sum_{i=1}^{|B|} e^{-\theta^* B[i]}\right)$

return mean(E)

E: 0.005086344730696122