3.5 Triangular factorizations and canonical forms

If a linear system $Ax = b$ has a nonsingular triangular (0.9.3) coefficient matrix $A \in M_n$, computation of the unique solution x is remarkably easy. If, for example, $A = [a_{ij}]$ is upper triangular and nonsingular, then all $a_{ii} \neq 0$ and one can employ *back substitution*: $a_{nn}x_n = b_n$ determines x_n; $a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$ then determines x_{n-1} since x_n is known and $a_{n-1,n-1} \neq 0$; proceeding in the same fashion upward through successive rows of A, one determines $x_{n-2}, x_{n-3}, \ldots, x_2, x_1$.

Exercise. Describe *forward substitution* as a solution technique for $Ax = b$ if $A \in M_n$ is nonsingular and lower triangular.

If $A \in M_n$ is not triangular, one can still use forward and back substitution to solve $Ax = b$ provided that A is nonsingular and can be factored as $A = LU$, in which L is lower triangular and U is upper triangular: First use forward substitution to solve $Ly = b$, and then use back substitution to solve $Ux = y$.

Definition 3.5.1. *Let $A \in M_n$. A presentation $A = LU$, in which $L \in M_n$ is lower triangular and $U \in M_n$ is upper triangular, is called an LU* factorization *of A.*

Exercise. Explain why $A \in M_n$ has an LU factorization in which L (respectively, U) is nonsingular if and only if it has an LU factorization in which L (respectively, U) is unit lower (respectively, unit upper) triangular. *Hint*: If L is nonsingular, write $L = L'D$, in which L' is unit lower triangular and D is diagonal.

Lemma 3.5.2. *Let $A \in M_n$ and suppose that $A = LU$ is an LU factorization. For any block 2-by-2 partition*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

with $A_{11}, L_{11}, U_{11} \in M_k$ and $k \leq n$, we have $A_{11} = L_{11}U_{11}$. Consequently, each leading principal submatrix of A has an LU factorization in which the factors are the corresponding leading principal submatrices of L and U.

Theorem 3.5.3. *Let $A \in M_n$ be given. Then*

(a) A has an LU factorization in which L is nonsingular if and only if A has the row inclusion property*: For each $i = 1, ..., n-1$, $A[\{i+1; 1, ..., i\}]$ is a linear combination of the rows of $A[\{1, ..., i\}]$*

(b) A has an LU factorization in which U is nonsingular if and only if A has the column inclusion property*: For each $j = 1, ..., n-1$, $A[\{1, ..., j; j+1\}]$ is a linear combination of the columns of $A[\{1, ..., j\}]$*

Proof. If $A = LU$, then $A[\{1, ..., i+1\}] = L[\{1, ..., i+1\}]U[\{1, ..., i+1\}]$. Thus, to verify the necessity of the row inclusion property, it suffices to take $i = k = n-1$ in the partitioned presentation given in (3.5.2). Since L is nonsingular and triangular, L_{11} is also nonsingular, and we have $A_{21} = L_{21}U_{11} = L_{21}L_{11}^{-1}L_{11}U_{11} = \left(L_{21}L_{11}^{-1}\right)A_{11}$, which verifies the row inclusion property.

Conversely, if A has the row inclusion property, we may construct inductively an LU factorization with nonsingular L as follows (the cases $n = 1, 2$ are easily verified): Suppose that $A_{11} = L_{11}U_{11}$, L_{11} is nonsingular, and the row vector A_{21} is a linear combination of the rows of A_{11}. Then there is a vector y such that $A_{21} = y^T A_{11} = y^T L_{11}U_{11}$, and we may take $U_{12} = L_{11}^{-1}A_{12}$, $L_{21} = y^T L_{11}$, $L_{22} = 1$, and $U_{22} = A_{22} - L_{21}U_{12}$ to obtain an LU factorization of A in which L is nonsingular.

The assertions about the column inclusion property follow from considering an LU factorization of A^T. □

Exercise. Consider the matrix $J_n \in M_n$, all of whose entries are 1. Find an LU factorization of J_n in which L is unit lower triangular. With this factorization in hand, $J_n = J_n^T = U^T L^T$ is an LU factorization of J_n with a unit upper triangular factor.

Exercise. Show that the row inclusion property is equivalent to the following formally stronger property: For each $i = 1, ..., n-1$, every row of $A[\{i+1, ..., n\}; \{1, ..., i\}]$ is a linear combination of the rows of $A[\{1, ..., i\}]$. What is the corresponding statement for column inclusion?

If $A \in M_n$, rank $A = k$, and det $A[\{1, ..., j\}] \neq 0$, $j = 1, ..., k$, then A has both the row inclusion and column inclusion properties. The following result follows from (3.5.3).

Corollary 3.5.4. *Suppose that $A \in M_n$ and rank $A = k$. If $A[\{1, \ldots, j\}]$ is nonsingular for all $j = 1, \ldots, k$, then A has an LU factorization. Furthermore, either factor may be chosen to be unit triangular; both L and U are nonsingular if and only if $k = n$, that is, if and only if A and all of its leading principal submatrices are nonsingular.*

Example 3.5.5. Not every matrix has an LU factorization. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ could be written as $A = LU = \begin{bmatrix} \ell_{11} & 0 \\ \ell_{21} & \ell_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$, then $l_{11}u_{11} = 0$ implies that one of L or U is singular; but $LU = A$ is nonsingular.

Exercise. Explain why a nonsingular matrix that has a singular leading principal submatrix cannot have an LU factorization.

Exercise. Verify that

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has an LU factorization even though A has neither the row nor column inclusion property. However, A is a principal submatrix of a 4-by-4 matrix

$$\hat{A} = \begin{bmatrix} A & e_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \hat{A}_{12} \\ \hat{A}_{21} & 0 \end{bmatrix}, \quad \hat{A}_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \hat{A}_{21} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

that *does not* have an LU factorization. Verify this by considering the block factorization in (3.5.2) with $k = 2$: $\hat{A}_{12} = L_{11}U_{12}$ implies that L_{11} is nonsingular, and hence $0 = L_{11}U_{11}$ implies that $U_{11} = 0$, which is inconsistent with $L_{21}U_{11} = \hat{A}_{21} \neq 0$.

Exercise. Consider $A = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 2-a \end{bmatrix}$ and explain why an LU factorization need not be unique even if the L is required to be unit lower triangular.

It is now clear that an LU factorization of a given matrix may or may not exist, and if it exists, it need not be unique. Much of the trouble arises from singularity, either of A or of its leading principal submatrices. Using the tools of (3.5.2) and (3.5.3), however, we can give a full description in the nonsingular case, and we can impose a normalization that makes the factorization unique.

Corollary 3.5.6 (LDU factorization). *Let $A = [a_{ij}] \in M_n$ be given.*

(a) Suppose that A is nonsingular. Then A has an LU factorization $A = LU$ if and only if $A[\{1, \ldots, i\}]$ is nonsingular for all $i = 1, \ldots, n$.
(b) Suppose that $A[\{1, \ldots, i\}]$ is nonsingular for all $i = 1, \ldots, n$. Then $A = LDU$, in which $L, D, U \in M_n$, L is unit lower triangular, U is unit upper triangular, $D = \operatorname{diag}(d_1, \ldots, d_n)$ is diagonal, $d_1 = a_{11}$, and

$$d_i = \det A\,[\{1, \ldots, i\}]/\det A\,[\{1, \ldots, i-1\}], \quad i = 2, \ldots, n$$

The factors L, U, and D are uniquely determined.

Exercise. Use (3.5.2), (3.5.3), and prior exercises to provide details for a proof of the preceding corollary.

Exercise. If $A \in M_n$ has an LU factorization with $L = [\ell_{ij}]$ and $U = [u_{ij}]$, show that $\ell_{11}u_{11} = \det[A\{1\}]$ and $\ell_{ii}u_{ii} \det A[\{1, \ldots, i-1\}] = \det A[\{1, \ldots, i\}]$, $i = 2, \ldots, n$.

Returning to the solution of the nonsingular linear system $Ax = b$, suppose that $A \in M_n$ cannot be factored as LU but can be factored as PLU, in which $P \in M_n$ is a permutation matrix and L and U are lower and upper triangular, respectively. This amounts to a reordering of the equations in the linear system prior to factorization. In this event, solution of $Ax = b$ is still quite simple via $Ly = P^T b$ and $Ux = y$. It is worth knowing that any $A \in M_n$ may be so factored and that L may be taken to be nonsingular. The solutions of $Ax = b$ are the same as those of $Ux = L^{-1}P^T b$.

Lemma 3.5.7. *Let $A \in M_k$ be nonsingular. Then there is a permutation matrix $P \in M_k$ such that* $\det(P^T A)[\{1, \ldots, j\}] \neq 0$, *$j = 1, \ldots, k$.*

Proof. The proof is by induction on k. If $k = 1$ or 2, the result is clear by inspection. Suppose that it is valid up to and including $k-1$. Consider a nonsingular $A \in M_k$ and delete its last column. The remaining $k-1$ columns are linearly independent and hence they contain $k-1$ linearly independent rows. Permute these rows to the first $k-1$ positions and apply the induction hypothesis to the nonsingular upper $(k-1)$-by-$(k-1)$ submatrix. This determines a desired overall permutation P, and $P^T A$ is nonsingular. □

The factorization in the following theorem is known as a PLU factorization; the factors need not be unique.

Theorem 3.5.8 (PLU factorization). *For each $A \in M_n$ there is a permutation matrix $P \in M_n$, a unit lower triangular $L \in M_n$, and an upper triangular $U \in M_n$ such that $A = PLU$.*

Proof. If we show that there is a permutation matrix Q such that QA has the row inclusion property, then (3.5.3) and the exercise following it ensure that $QA = LU$ with a unit lower triangular factor L, so $A = PLU$ for $P = Q^T$.

If A is nonsingular, the desired permutation matrix is guaranteed by (3.5.7).

If rank $A = k < n$, first permute the rows of A so that the first k rows are linearly independent. It follows that $A[\{i+1\};\{1, ..., i\}]$ is a linear combination of the rows of $A[\{1, ..., i\}]$, $i = k, ..., n-1$. If $A[\{1, ..., k\}]$ is nonsingular, apply (3.5.7) again to further permute the rows so that $A[\{1, ..., k\}]$, and thus A, has the row inclusion property. If rank $A[\{1, ..., k\}] = \ell < k$, treat it in the same way that we have just treated A, and obtain row inclusion for the indices $i = \ell, ..., n-1$. Continue in this manner until either the upper left block is 0, in which case we have row inclusion for all indices, or it is nonsingular, in which case one further permutation completes the argument. $\square$

Exercise. Show that each $A \in M_n$ may be factored as $A = LUP$, in which L is lower triangular, U is unit upper triangular, and P is a permutation matrix.

Exercise. For a given $X \in M_n$ and $k, \ell \in \{1, \ldots, n\}$, define

$$X_{[p,q]} = X[\{1, \ldots, p\}, \{1, \ldots, q\}] \tag{3.5.9}$$

Let $A = LBU$, in which $L, B, U \in M_n$, L is lower triangular, and U is upper triangular. Explain why $A_{[p,q]} = L_{[p,p]}B_{[p,q]}U_{[q,q]}$ for all $p, q \in \{1, \ldots, n\}$. If L and U are nonsingular, explain why

$$\operatorname{rank} A_{[p,q]} = \operatorname{rank} B_{[p,q]} \text{ for all } p, q \in \{1, \ldots, n\} \tag{3.5.10}$$

The following theorem, like the preceding one, describes a particular triangular factorization (the LPU factorization) that is valid for every square complex matrix. Uniqueness of the P factor in the nonsingular case has important consequences.

Theorem 3.5.11 (LPU factorization). *For each $A \in M_n$ there is a permutation matrix $P \in M_n$, a unit lower triangular $L \in M_n$, and an upper triangular $U \in M_n$ such that $A = LPU$. Moreover, the factor P is uniquely determined if A is nonsingular.*

Proof. Construct inductively permutations $\pi_1, \ldots, \pi_n$ of the integers $1, \ldots, n$ as follows: Let $A^{(0)} = [a_{ij}^{(0)}] = A$ and define the index set $\mathcal{I}_1 = \{i \in \{1, \ldots, n\} : a_{i1}^{(0)} \neq 0\}$. If $\mathcal{I}_1$ is nonempty, let π_1 be the smallest integer in $\mathcal{I}_1$; otherwise, let π_1 be any $i \in \{1, \ldots, n\}$ and proceed to the next step. If $\mathcal{I}_1$ is nonempty, use type 3 elementary row operations based on row π_1 of $A^{(0)}$ to eliminate all the nonzero entries $a_{i,1}^{(0)}$ in column 1 of $A^{(0)}$ other than $a_{\pi_1,1}^{(0)}$ (for all such entries, $i > \pi_1$); denote the resulting matrix by $A^{(1)}$. Observe that $A^{(1)} = \mathsf{L}_1 A^{(0)}$ for some unit lower triangular matrix L_1 (0.3.3).

Suppose that $2 \leq k \leq n$ and that $\pi_1, \ldots, \pi_{k-1}$ and $A^{(k-1)} = [a_{ij}^{(k-1)}]$ have been constructed. Let $\mathcal{I}_k = \{i \in \{1, \ldots, n\} : i \neq \pi_1, \ldots, \pi_{k-1}$ and $a_{ik}^{(k-1)} \neq 0\}$. If $\mathcal{I}_k$ is nonempty, let π_k be the smallest integer in $\mathcal{I}_k$; otherwise, let π_k be any $i \in \{1, \ldots, n\}$ such that $i \neq \pi_1, \ldots, \pi_{k-1}$ and proceed to the next step. If $\mathcal{I}_k$ is nonempty, use type 3 elementary row operations based on row π_k of $A^{(k-1)}$ to eliminate every nonzero entry $a_{i,k}^{(k-1)}$ in column k of $A^{(k-1)}$ below the entry $a_{\pi_k,k}^{(k-1)}$ (for all such entries, $i > \pi_k$); denote the resulting matrix by $A^{(k)}$. Observe that these eliminations do not change any entries in columns $1, \ldots, k-1$ of $A^{(k-1)}$ (because $a_{\pi_k,j}^{(k-1)} = 0$ for $j = 1, \ldots, k-1$) and that $A^{(k)} = \mathsf{L}_k A^{(k-1)}$ for some unit lower triangular matrix L_k.

After n steps, our construction produces a permutation $\pi_1, \ldots, \pi_n$ of the integers $1, \ldots, n$ and a matrix $A^{(n)} = [a_{ij}^{(n)}] = \mathsf{L}A$, in which $\mathsf{L} = \mathsf{L}_n \cdots \mathsf{L}_1$ is unit lower triangular. Moreover, $a_{ij}^{(n)} = 0$ whenever $i > \pi_j$ or $i < \pi_j$ and $i \notin \{\pi_1, \ldots, \pi_{j-1}\}$. Let $L = \mathsf{L}^{-1}$ so that $A = LA^{(n)}$ and L is unit lower triangular. Let $P = [p_{ij}] \in M_n$ in which $p_{\pi_j,j} = 1$ for $j = 1, \ldots, n$ and all other entries are zero. Then P is a permutation matrix, $P^T A^{(n)} = U$ is upper triangular, and $A = LPU$.

If A is nonsingular, then both L and U are nonsingular. The preceding exercise ensures that rank $A_{[p,q]} =$ rank $P_{[p,q]}$ for all $p, q \in \{1, \ldots, n\}$, and these ranks uniquely determine the permutation matrix P (see (3.5.P11)). □

Exercise. Explain how the construction in the preceding proof ensures that $P^T A^{(n)}$ is upper triangular.

Definition 3.5.12. *Matrices $A, B \in M_n$ are said to be* triangularly equivalent *if there are nonsingular matrices $L, U \in M_n$ such that L is lower triangular, U is upper triangular, and $A = LBU$.*

Exercise. Verify that triangular equivalence is an equivalence relation on M_n.

Theorem 3.5.11 provides a canonical form for triangular equivalence of nonsingular matrices. The canonical matrices are the permutation matrices; the set of ranks of submatrices described in (3.5.10) is a complete set of invariants.

Theorem 3.5.13. *Let $A, B \in M_n$ be nonsingular. The following are equivalent:*

(a) There is a unique permutation matrix $P \in M_n$ such that both A and B are triangularly equivalent to P.
(b) A and B are triangularly equivalent.
(c) The rank equalities (3.5.10) are satisfied.

Proof. The implication (a) ⇒ (b) is clear, and the implication (b) ⇒ (c) is the content of the exercise preceding (3.5.11). If $A = L_1 P U_1$ and $B = L_2 P' U_2$ are LPU factorizations and if the hypothesis (c) is assumed, then (using the notation in (3.5.9)) rank $P_{[p,q]} =$ rank $A_{[p,q]} =$ rank $B_{[p,q]} =$ rank $P'_{[p,q]}$ for all $p, q \in \{1, \ldots, n\}$. Problem 3.5.P11 ensures that $P = P'$, which implies (a). □

Exercise. Let $A, P \in M_n$. Suppose that P is a permutation matrix and that all the main diagonal entries of A are ones. Explain why all the main diagonal entries of $P^T AP$ are ones.

Our final theorem concerns triangular equivalence via unit triangular matrices. It uses the facts that (a) the inverse of a unit lower triangular matrix is unit lower triangular, and (b) a product of unit lower triangular matrices is unit lower triangular, with corresponding assertions about unit upper triangular matrices.

Theorem 3.5.14 ($LPDU$ factorization). *For each nonsingular $A \in M_n$ there is a unique permutation matrix P, a unique nonsingular diagonal matrix D, a unit lower triangular matrix L, and a unit upper triangular matrix U such that $A = LPDU$.*

Proof. Theorem 3.5.11 ensures that there is a unit lower triangular matrix L, a unique permutation matrix P, and a nonsingular upper triangular matrix U' such that $A = LPU'$. Let D denote the diagonal matrix whose respective diagonal entries are the same as those of U', that is, $D = \operatorname{diag}(\operatorname{diag}(U'))$, and let $U = D^{-1}U'$. Then U is unit upper triangular and $A = LPDU$. Suppose that D_2 is a diagonal matrix such that $A = L_2PD_2U_2$, in which L_2 is unit lower triangular and U_2 is unit triangular. Then

$$(P^T(L_2^{-1}L)P)D = D_2(U_2U^{-1}) \tag{3.5.15}$$

The main diagonal entries of U_2U^{-1} and $L_2^{-1}L$ are, of course, all ones; the preceding exercise ensures that the main diagonal entries of $P^T(L_2^{-1}L)P$ are also all ones. Thus, the main diagonal of the left-hand side of (3.5.15) is the same as that of D, while the main diagonal of the right-hand side of (3.5.15) is the same as that of D_2. It follows that $D = D_2$. □

Problems

3.5.P1 We have discussed the factorization $A = LU$, in which L is lower triangular and U is upper triangular. Discuss a parallel theory of $A = UL$ factorization, noting that the factors may be different.

3.5.P2 Describe how $Ax = b$ may be solved if A is presented as $A = QR$, in which Q is unitary and R is upper triangular (2.1.14).

3.5.P3 Matrices $A, B \in M_n$ are said to be *unit triangularly equivalent* if $A = LBU$ for some unit lower triangular matrix L and some unit upper triangular matrix U. Explain why (a) unit triangular equivalence is an equivalence relation on both M_n and $GL(n, \mathbf{C})$; (b) if $P, D, P', D' \in M_n$, P and P' are permutation matrices, and D and D' are nonsingular diagonal matrices, then $PD = P'D'$ if and only if $P = P'$ and $D = D'$; (c) each nonsingular matrix in M_n is unit triangularly equivalent to a unique generalized permutation matrix (0.9.5); (d) two generalized permutation matrices in M_n are unit triangularly equivalent if and only if they are identical; (e) the n-by-n generalized permutation matrices are a set of canonical matrices for the equivalence relation of unit triangular equivalence on $GL(n, \mathbf{C})$.

3.5.P4 If the leading principal minors of $A \in M_n$ are all nonzero, describe how an LU factorization of A may be obtained by using type 3 elementary row operations to zero out entries below the diagonal.

3.5.P5 (Lanczos tridiagonalization algorithm) Let $A \in M_n$ and $x \in \mathbf{C}^n$ be given. Define $X = [x \ Ax \ A^2x \ \ldots \ A^{n-1}x]$. The columns of X are said to form a *Krylov sequence*. Assume that X is nonsingular. (a) Show that $X^{-1}AX$ is a companion matrix (3.3.12) for the characteristic polynomial of A. (b) If $R \in M_n$ is any given nonsingular upper triangular matrix and $S = XR$, show that $S^{-1}AS$ is in upper Hessenberg form. (c) Let $y \in \mathbf{C}^n$ and define $Y = [y \ A^*y \ (A^*)^2y \ \ldots \ (A^*)^{n-1}y]$. Suppose that Y is nonsingular and that Y^*X can be written as LDU, in which L is lower triangular and U is upper triangular and nonsingular, and D is diagonal and nonsingular. Show that there exist nonsingular upper triangular matrices R and T such that $(XR)^{-1} = T^*Y^*$ and such that T^*Y^*AXR is tridiagonal and similar to A. (d) If $A \in M_n$ is Hermitian, use these ideas to describe an algorithm that produces a tridiagonal Hermitian matrix that is similar to A.

3.5.P6 Explain why the n, n entry of a given matrix in M_n has no influence on whether it has an LU factorization, or has one with L nonsingular, or has one with U nonsingular.

3.5.P7 Show that $C_n = [1/\max\{i, j\}] \in M_n(\mathbf{R})$ has an LU decomposition of the form $C_n = L_n L_n^T$, in which the entries of the lower triangular matrix L_n are $\ell_{ij} = 1/\max\{i, j\}$ for $i \geq j$. Conclude that $\det L_n = (1/n!)^2$.

3.5.P8 Show that the condition "$A[\{1, \ldots, j\}]$ is nonsingular for all $j = 1, \ldots, n$" in (3.5.6) may be replaced with the condition "$A[\{j, \ldots, n\}]$ is nonsingular for all $j = 1, \ldots, n$."

3.5.P9 Let $A \in M_n(\mathbf{R})$ be the symmetric tridiagonal matrix (0.9.10) with all main diagonal entries equal to $+2$ and all entries in the first superdiagonal and subdiagonal equal to -1. Consider

$$L = \begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & \ddots & & \\ & & \ddots & 1 & \\ & & & -\frac{n-1}{n} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -1 & & & \\ & \frac{3}{2} & -1 & & \\ & & \ddots & \ddots & \\ & & & \frac{n}{n-1} & -1 \\ & & & & \frac{n+1}{n} \end{bmatrix}$$

Show that $A = LU$ and $\det A = n + 1$. The eigenvalues of A are $\lambda_k = 4\sin^2 \frac{k\pi}{2(n+1)}$, $k = 1, \ldots, n$ (see (1.4.P17)). Notice that $\lambda_1(A) \to 0$ and $\lambda_n(A) \to 4$ as $n \to \infty$, and $\det A = \lambda_1 \cdots \lambda_n \to \infty$.

3.5.P10 Suppose that $A \in M_n$ is symmetric and that all its leading principal submatrices are nonsingular. Show that there is a nonsingular lower triangular L such that $A = LL^T$, that is, A has an LU factorization in which $U = L^T$.

3.5.P11 Consider a permutation matrix $P = [p_{ij}] \in M_n$ corresponding to a permutation $\pi_1, \ldots, \pi_n$ of $1, \ldots, n$, that is, $p_{\pi_j, j} = 1$ for $j = 1, \ldots, n$ and all other entries are zero. Use the notation in (3.5.9) and define $\operatorname{rank} P_{[\ell,0]} = 0$, $\ell = 1, \ldots, n$. Show that $\pi_j = \min\{k \in \{1, \ldots, n\} : \operatorname{rank} P_{[k,j]} = \operatorname{rank} P_{[k,j-1]} + 1\}$, $j = 1, \ldots, n$. Conclude that the n^2 numbers $\operatorname{rank} P_{[k,j]}$, $k, j \in \{1, \ldots, n\}$, uniquely determine P.

3.5.P12 Let $P \in M_n$ be a permutation matrix, partitioned as $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$, so $P^{-1} = \begin{bmatrix} P_{11}^T & P_{12}^T \\ P_{21}^T & P_{22}^T \end{bmatrix}$. Provide details for the following argument to prove the law of complementary nullities (0.7.5) for P: nullity P_{11} = number of zero columns in P_{11} = number of ones in P_{21} = number of zero rows in P_{22} = nullity P_{22}^T.

3.5.P13 Provide details for the following approach to the law of complementary nullities (0.7.5), which deduces the general case from the (easy) permutation matrix case via an LPU factorization. (a) Let $A \in M_n$ be nonsingular. Partition A and A^{-1} conformally as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$. The law of complementary nullities asserts that nullity A_{11} = nullity B_{22}; this is what we seek to prove. (b) Let $A = LPU$ be an LPU factorization, so $A^{-1} = U^{-1}P^TL^{-1}$ is an LPU factorization. The permutation matrix factors in both factorizations are uniquely determined. Partition P as in the preceding problem, conformal to the partition of A. (c) nullity A_{11} = nullity P_{11} = nullity P_{22}^T = nullity B_{22}.

Further Readings. Problem 3.5.P5 is adapted from [Ste], where additional information about numerical applications of LU factorizations may be found. Our discussion of LPU and $LPDU$ factorizations and triangular equivalence is adapted from L. Elsner, On some algebraic problems in connection with general eigenvalue algorithms, *Linear Algebra Appl*. 26 (1979) 123–138, which also discusses *lower triangular congruence* ($A = LBL^T$, in which L is lower triangular and nonsingular) of symmetric or skew-symmetric matrices.