Lecture 2
Linear Time-Invariant System

Contents

LTI Systems

- Linear Time-Invariant (LTI) System – a system satisfying the linearity and time-invariant properties
- LTI systems are mathematically easy to analyze and characterize, and consequently easy to design
- Highly useful signal processing algorithms have been developed utilizing this class of systems

Why LTI System?

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

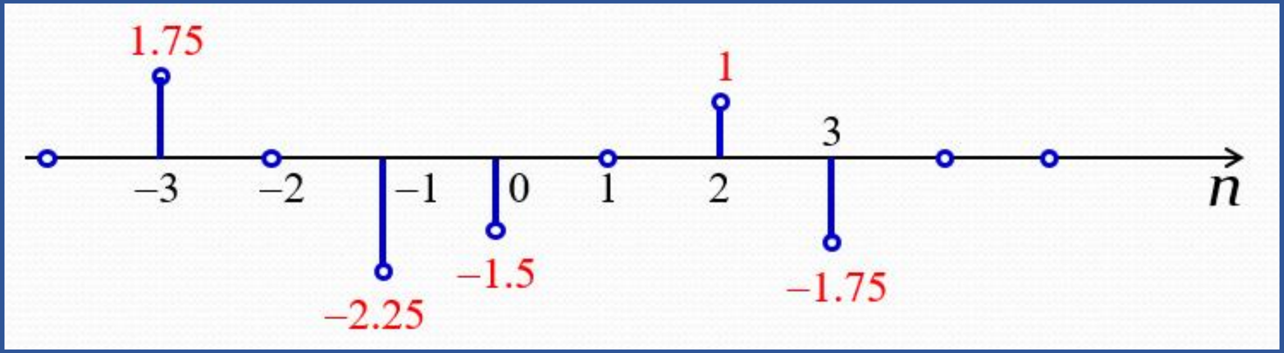
$$\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n-k]$$

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow x[k]h[n-k]$$

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Discrete LTI Systems: Arbitrary Sequence

- An arbitrary sequence can be represented as the weighted sum of shifted unit impulses



$$x[n]=1.75\delta[n+3]-2.25\delta[n+1]-1.5\delta[n]+\delta[n-2]-1.75\delta[n-3]$$

- A general form

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

Sifting property of $\delta[n]$

Impulse Response

- The response of a system to a unit impulse sequence $\{\delta[n]\}$ is called impulse response, denoted by $\{h[n]\}$

Impulse Response

- The impulse response of a system

$$y[n] = a_1 x[n] + a_2 x[n-1] + a_3 x[n-2] + a_4 x[n-3]$$

is obtained by setting $x[n] = \delta[n]$, resulting in

$$h[n] = a_1 \delta[n] + a_2 \delta[n-1] + a_3 \delta[n-2] + a_4 \delta[n-3]$$

- The impulse response is a finite length sequence of length 4, given by

$$\{h[n]\} = \{a_1, a_2, a_3, a_4\}$$

Example

- The impulse response of $y[n] = \sum_{k=-\infty}^{n} x[k]$

By setting $x[n] = \delta[n]$, we have

$$h(n) = \sum_{k=-\infty}^{n} \delta[k]$$

which is precisely the unit step sequence

Example

- The impulse response of factor-of-2 interpolator

$$y[n] = x_u[n-1] + \frac{1}{2}(x_u[n-2] + x_u[n])$$

is obtained by setting $x_u[n] = \delta[n]$, resulting in

$$h[n] = \delta[n-1] + \frac{1}{2}(\delta[n-2] + \delta[n])$$

- The impulse response is thus a finite length sequence of length 3 given by

$$h[n] = \{\underset{\uparrow}{0.5}, 1, 0.5\}$$

Time-Domain Characterization

- Input-output relationship – An LTI discrete system is completely characterized by its impulse response

- In other words, knowing the impulse response one can compute the output of the system for an arbitrary input

"DNA of LTI"

Output of LTI Systems

- Let $h[n]$ denote the impulse response of a discrete LTI system, compute the output $y[n]$ for the input:

$$x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2] + 0.75\delta[n-5]$$

Output of LTI Systems

- Since the system is time-invariant, we have

Input		Output
$\delta[n+2]$	$\rightarrow$	
$\delta[n-1]$	$\rightarrow$	
$\delta[n-2]$	$\rightarrow$	
$\delta[n-5]$	$\rightarrow$	

Then $\boldsymbol{x[n-n_0] \rightarrow y[n-n_0]}$

Output of LTI Systems

- Likewise, since the system is linear, we have

Input		Output
$0.5\delta[n+2]$	$\rightarrow$	
$1.5\delta[n-1]$	$\rightarrow$	
$\delta[n-2]$	$\rightarrow$	
$0.75\delta[n-5]$	$\rightarrow$	

- Hence, because of the linear property, we get

$$y[n] = 0.5h[n+2] + 1.5h[n-1] - h[n-2] + 0.75h[n-5]$$

Output of LTI Systems

- Recall, an arbitrary input $x[n]$ can be expressed as a linear combination of shifted unit impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- The response of a LTI system to $x[k]\delta[n-k]$ will be $x[k]h[n-k]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Impulse Response: DNA of LTI

- The impulse response $h[n]$ completely characterizes an LTI system

"DNA of LTI"

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n-k]$$

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow x[k]h[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Impulse Response: DNA of LTI

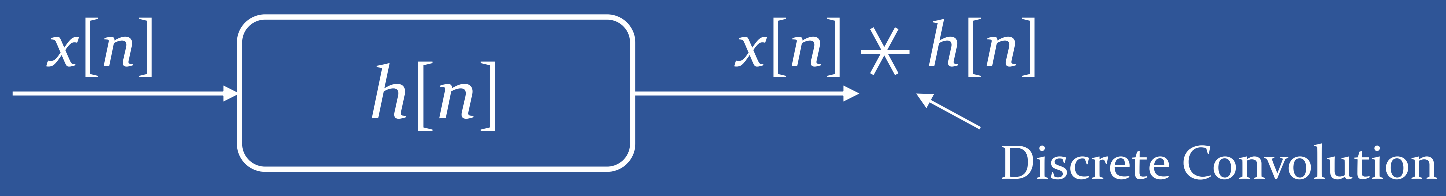
- Mathematically,

$$y[n] = \mathrm{LTI}\{x[n]\} = \mathrm{LTI}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\mathrm{LTI}\{\delta[n-k]\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The Discrete Convolution



$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

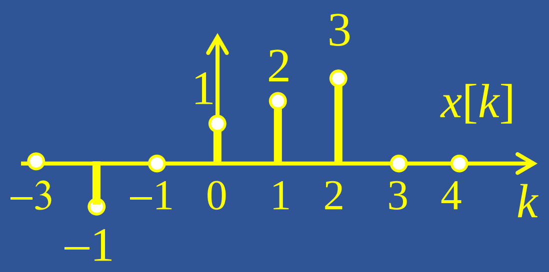
$$= \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n] * x[n]$$

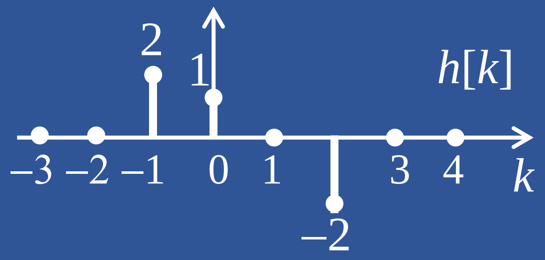
The above summation is defined to be the convolution of the sequences $x[n]$ and $h[n]$ and represented compactly as

$$y[n] = x[n] * h[n]$$

Discrete Convolution: An Example

Compute the convolution of $x[n]$ and $h[n]$

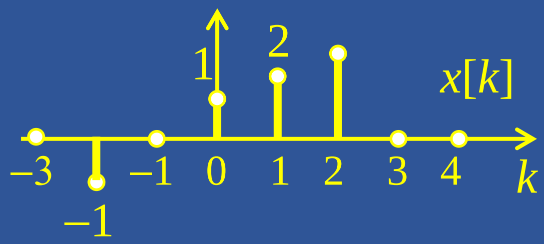


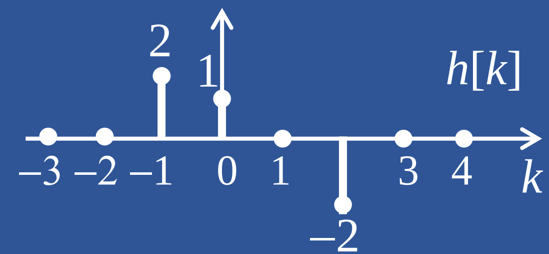


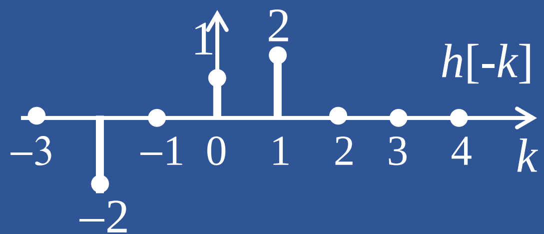
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k]$$

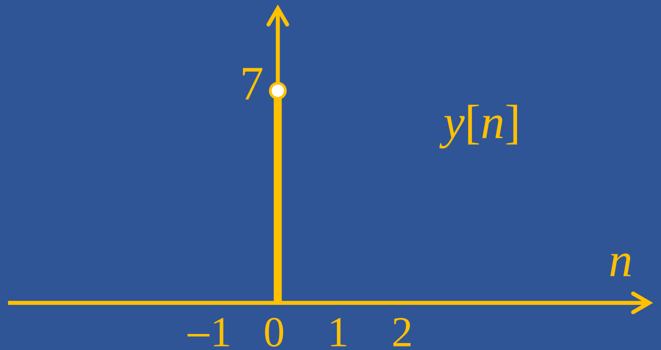
Discrete Convolution: An Example







$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

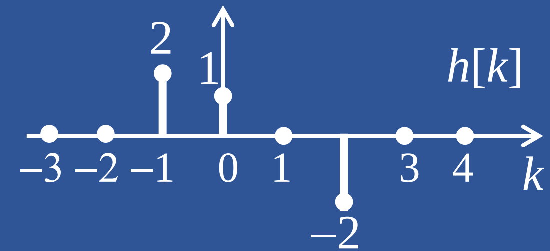


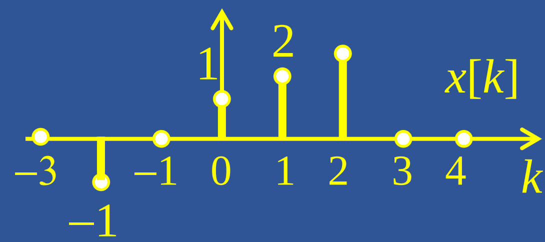
$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k]$$

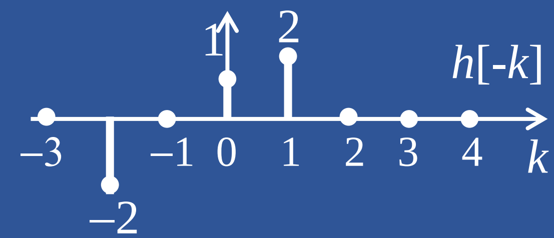
$$= x[-2]h[2] + x[-1]h[1] + x[0]h[0]$$

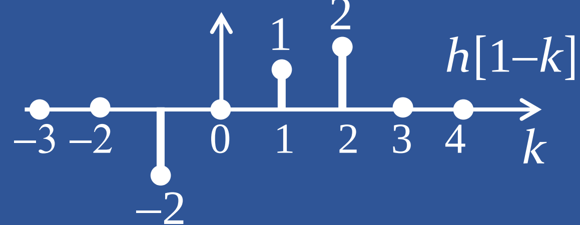
$$+x[1]h[-1] + x[2]h[-2]$$

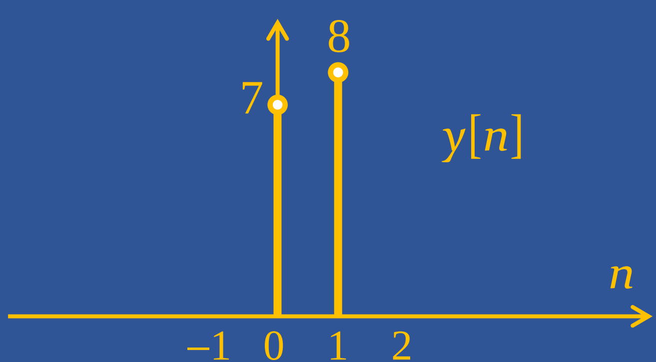
$$= -1 \times (-2) + 1 \times 1 + 2 \times 2 = 7$$











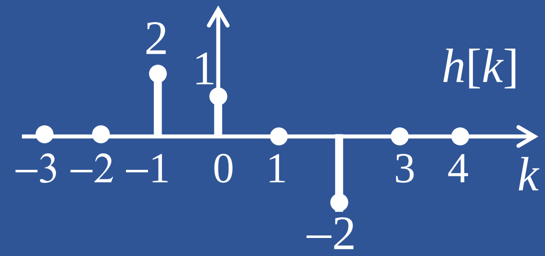
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

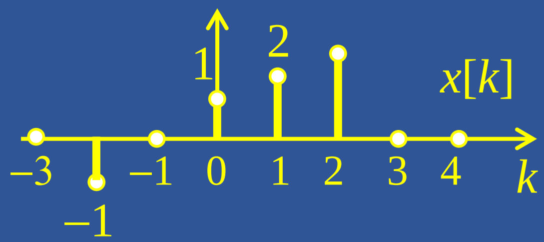
$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k]$$

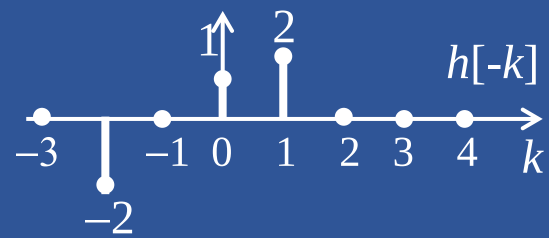
$$= x[-2]h[3] + x[-1]h[2] + x[0]h[1]$$

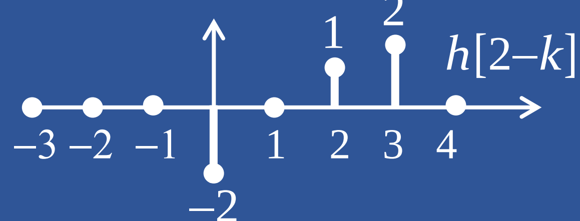
$$+x[1]h[0] + x[2]h[-1]$$

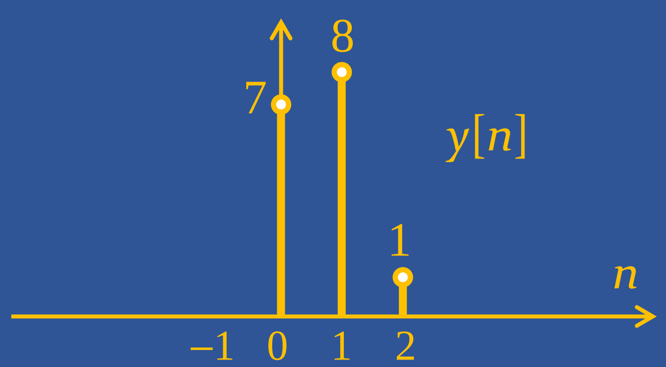
$$= 2\times1+3\times2 = 8$$







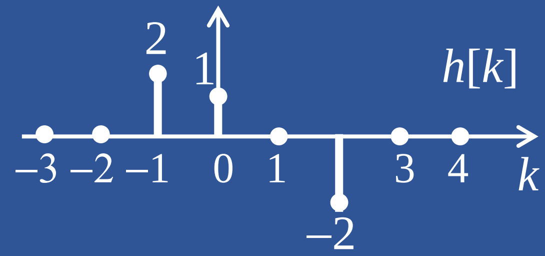


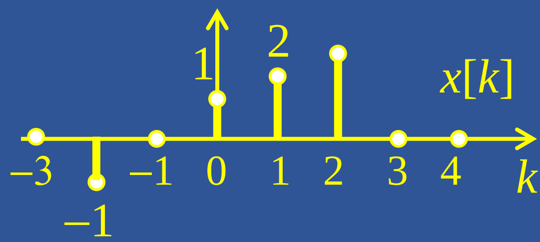


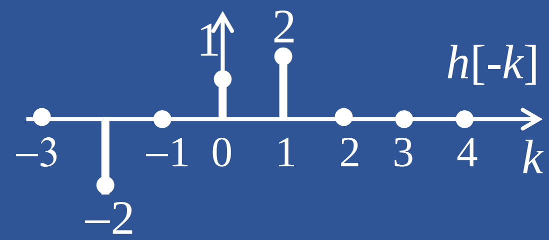
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

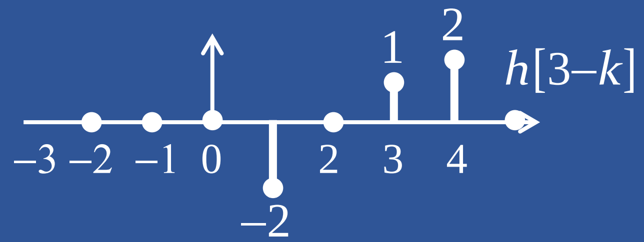
$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k]$$

$$= x[-2]h[4] + x[-1]h[3] + x[0]h[2]$$
$$+x[1]h[1] + x[2]h[0]$$
$$= 1\times(-2)+3\times1=1$$









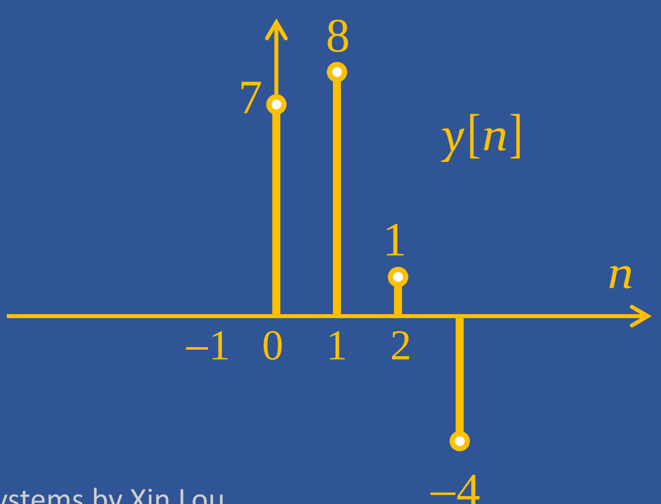
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

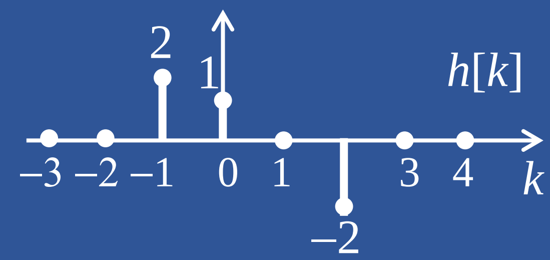
$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k]$$

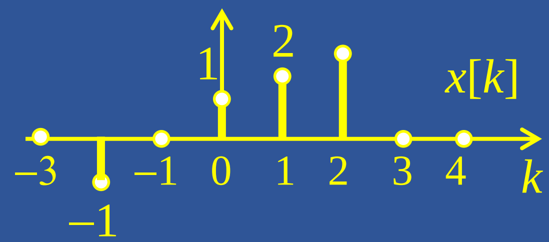
$$= x[-2]h[5] + x[-1]h[4] + x[0]h[3]$$

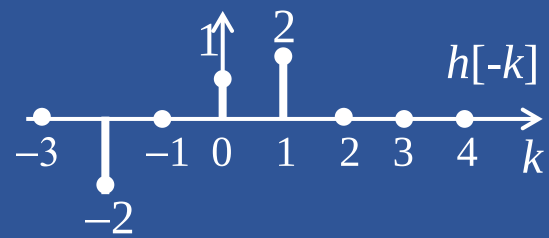
$$+x[1]h[2] + x[2]h[1]$$

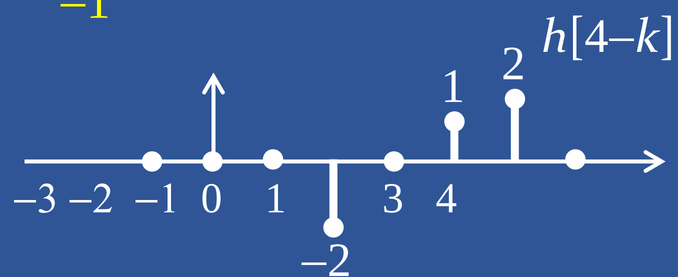
$$= 2\times(-2) = -4$$











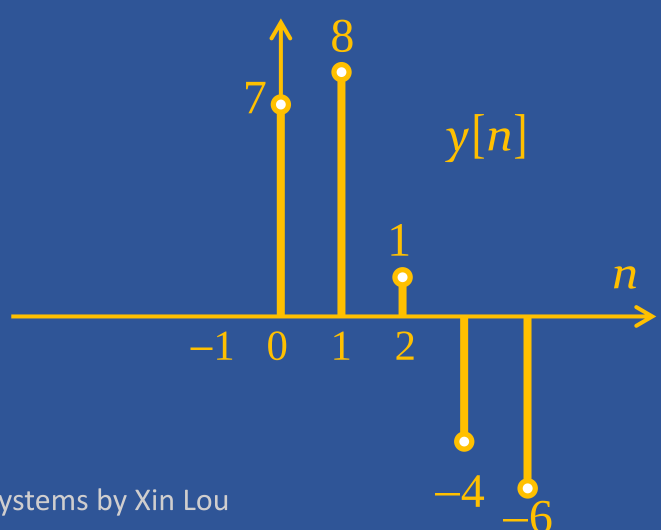
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

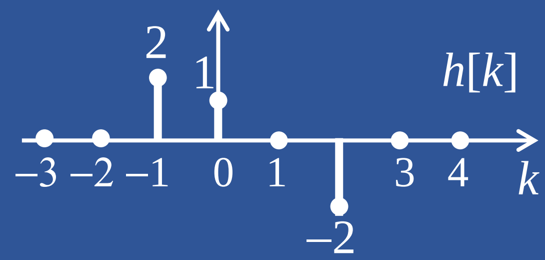
$$y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k]$$

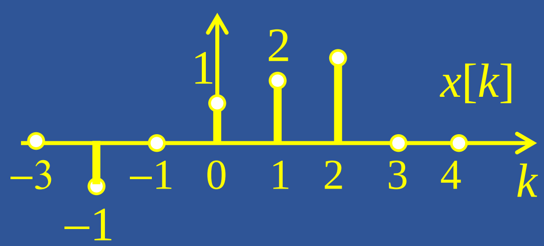
$$= x[-2]h[6] + x[-1]h[5] + x[0]h[4]$$

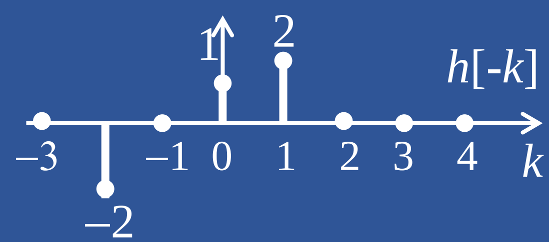
$$+x[1]h[3] + x[2]h[2]$$

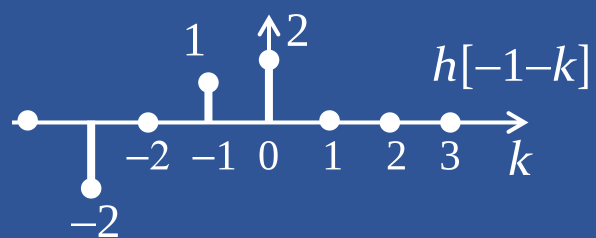
$$= 3\times(-2) = -6$$











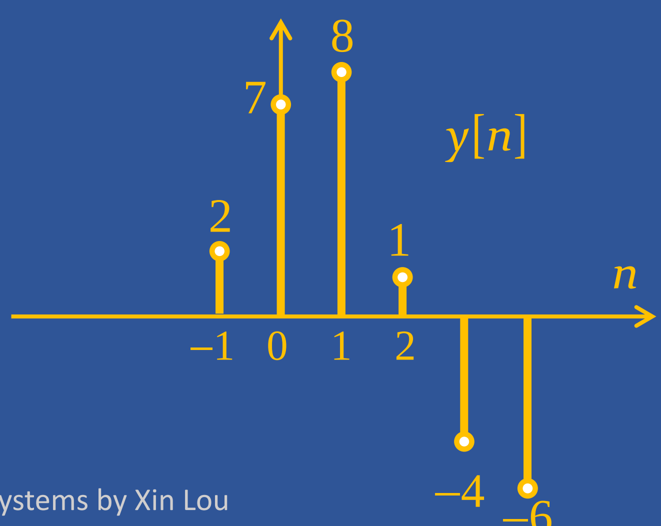
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

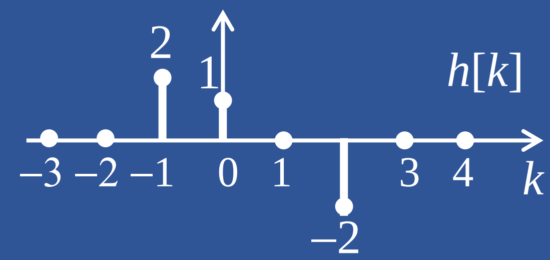
$$y[-1] = \sum_{k=-\infty}^{\infty} x[k]h[-1-k]$$

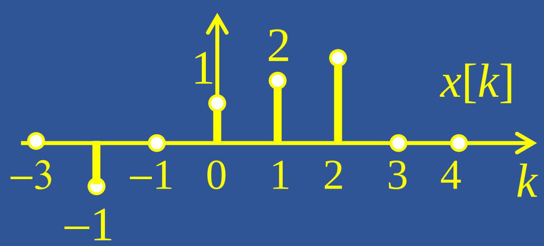
$$= x[-2]h[1] + x[-1]h[0] + x[0]h[-1]$$

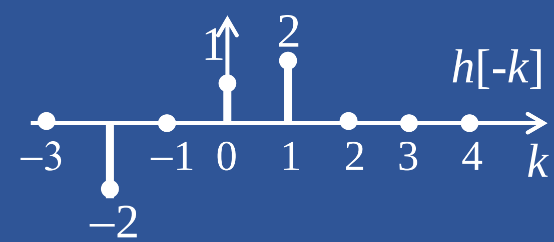
$$+x[1]h[-2] + x[2]h[-3]$$

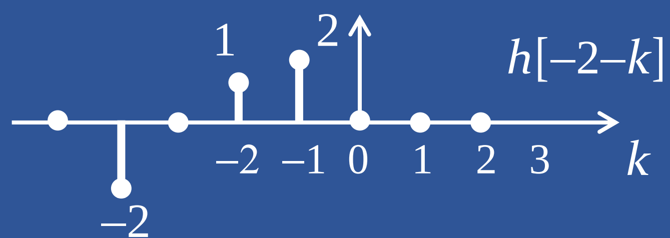
$$= 1 \times 2 = 2$$

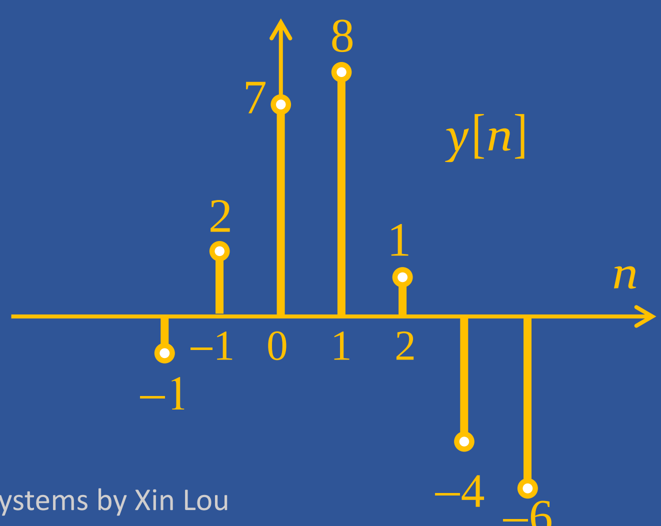








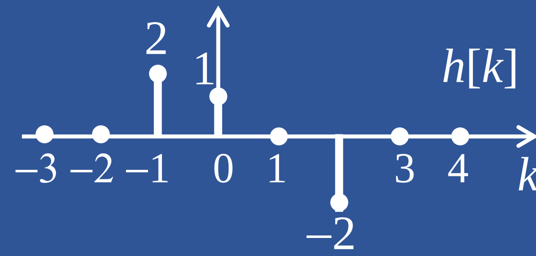


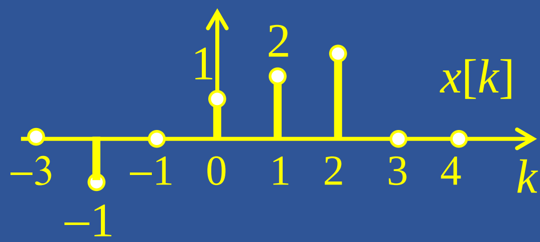


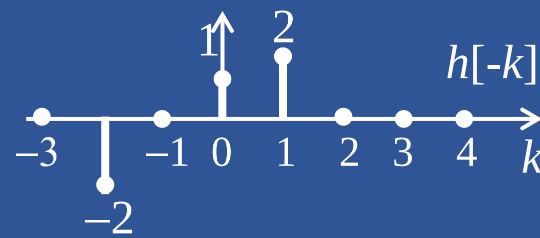
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

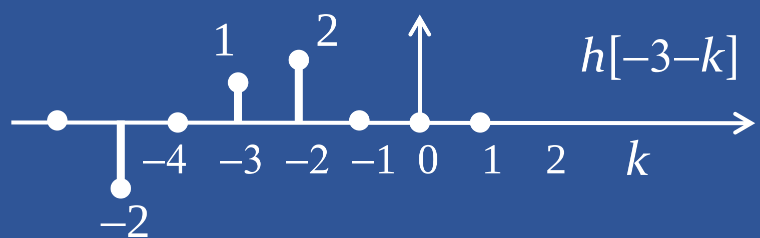
$$y[-2] = \sum_{k=-\infty}^{\infty} x[k]h[-2-k]$$

$$= x[-2]h[0] + x[-1]h[-1] + x[0]h[-2]$$
$$+x[1]h[-3] + x[2]h[-4]$$
$$= -1 \times 1 = -1$$

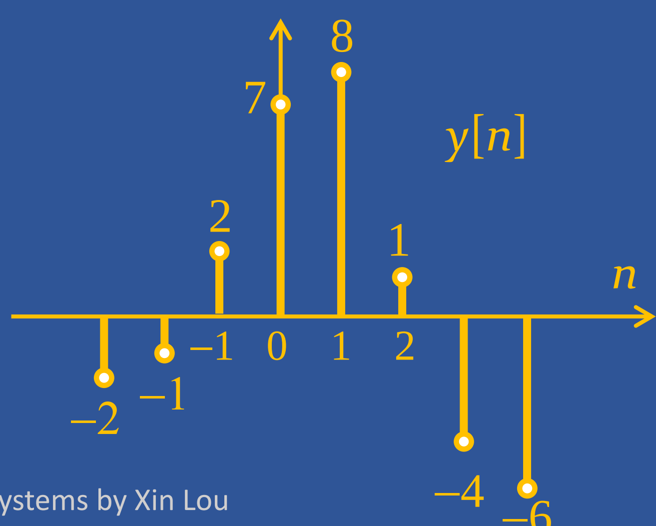








$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



$$y[-3] = \sum_{k=-\infty}^{\infty} x[k]h[-3-k]$$

$$= x[-2]h[-1] + x[-1]h[-2] + x[0]h[-3]$$

$$+x[1]h[-4] + x[2]h[-5]$$

$$= -1 \times 2 = -2$$

Computation of Discrete Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Fold $h[k]$ with respect to the origin to obtain $h[-k]$
- Shift to right (if $n \geq 0$) or to the left (if $n<0$) by $|n|$ samples
- Compute the products of the corresponding samples of sequences $h[n-k]$ and $x[k]$
- Sum the all products to obtain $y[n]$
- Fold, shift, product, and sum

Computation of Convolution

- The sum of indices of each sample product inside the convolution sum is equal to the index of the sample being computed

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[-3] = \sum_{k=-\infty}^{\infty} x[k]h[-3-k]$$

$$= x[-2]h[-1] + x[-1]h[-2] + x[0]h[-3] + x[1]h[-4] + x[2]h[-5]$$

- If the lengths of the two sequences are M and N, then the sequence generated by the convolution is of length $M+N-1$

Example

- If $x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$

$$\text{i.e., } y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- How about $x[n] \longrightarrow \boxed{h[n-m]} \longrightarrow y_1[n]$

$$y_1[n] = x[n] * h[n\text{-}m] = \sum_{k=-\infty}^{\infty} x[k]h[n-m-k]$$

$$= y[n-m]$$

Properties of Convolution

- Sequence shifting is equivalent to convolve with a shifted impulse

$$x[n] * \delta[n-d] = x[n-d]$$

Continuous LTI Systems: Arbitrary Signals

- In the discrete case, we have

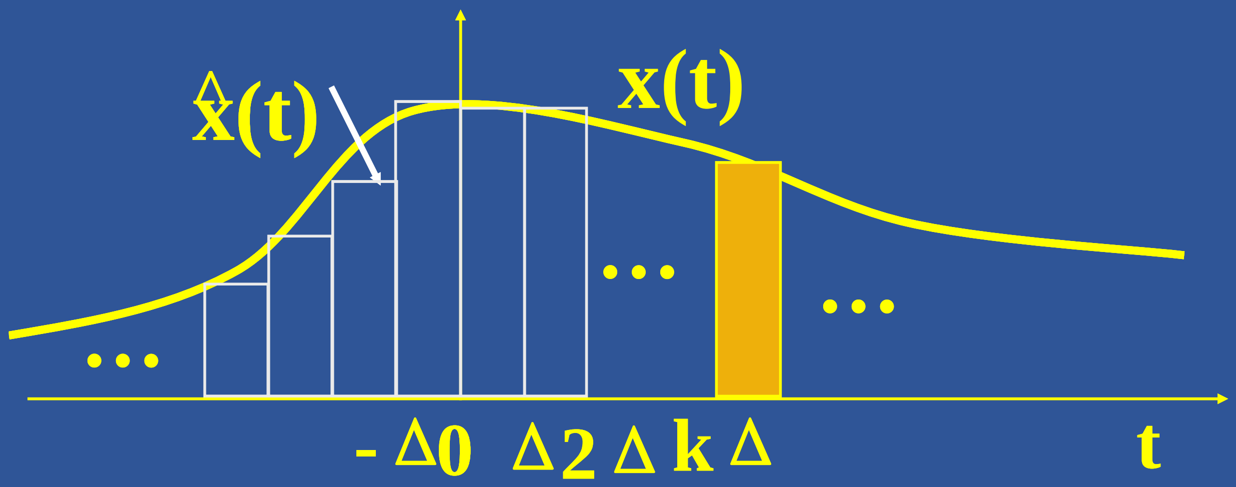
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- In the continuous case, we also have

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

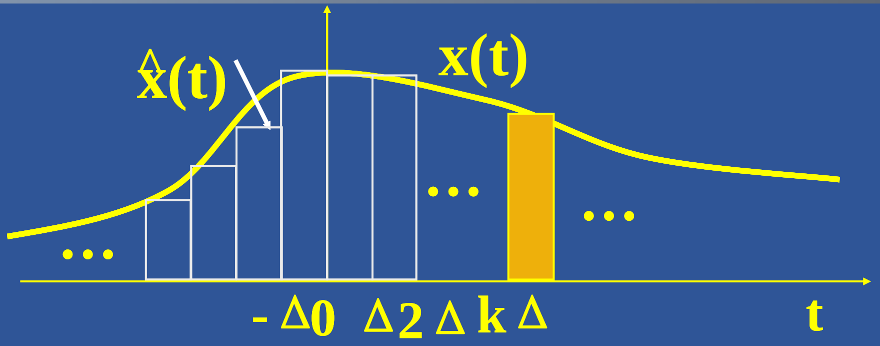
Continuous LTI Systems: Arbitrary Signals

- Consider a pulse or "staircase" approximation, $\hat{x}(t)$ to a continuous signal $x(t)$.



$$\delta_{\Delta}(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & otherwise \end{cases}$$

Continuous LTI Systems: Arbitrary Signals

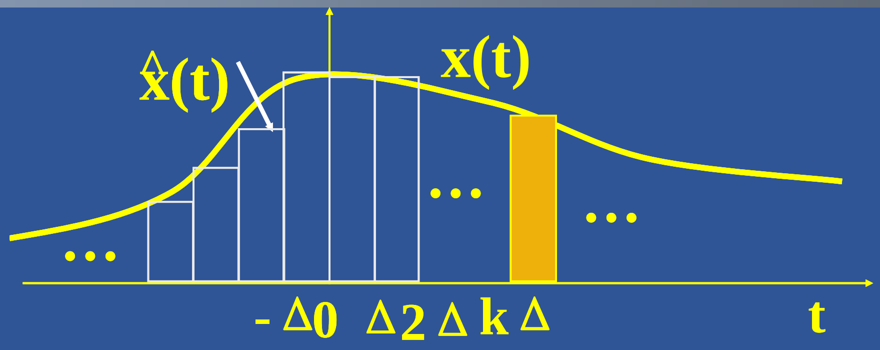


$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & otherwise \end{cases}$$

- Then , $\Delta\delta_\Delta(t) = 1$, the shade pulse is $x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$
- If we sum up all the shapes, we get

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

Continuous LTI Systems: Arbitrary Signals



$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

As $\Delta \to 0$, Δ becomes $d\tau$

$$\hat{x}(t) \to x(t) \qquad \sum_{k=-\infty}^{\infty} \to \int_{-\infty}^{\infty} \qquad k\Delta \to \tau$$

$$x(k\Delta) \to x(\tau) \qquad \delta_\Delta(t-k\Delta) \to \delta(t-\tau)$$

$$\sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \to \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Sifting property of $\delta(t)$

Example

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

- Let $x(t)=u(t)$,

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau$$

$$= \int_{0}^{\infty} \delta(t-\tau)d\tau$$

The Convolution Integral

- Let $h(t)$ be the response of LTI to $\delta(t)$

$$\delta(t) \xrightarrow{\text{LTI}} h(t) \quad \text{and} \quad x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Q: $x(t) \xrightarrow{\text{LTI}} y(t) = ?$

A: Time-Invariant property $\longrightarrow$ $\delta(t-\tau) \xrightarrow{\text{LTI}} h(t-\tau)$

$$x(t) \xrightarrow{\text{LTI}} y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Convolution integral

Computation of Convolution Integral

- Change time variables $x(t) \to x(\tau)$ $h(t) \to h(\tau)$ and reverse $h(\tau) \to h(-\tau)$
- Shift $h(-\tau) \to h(t-\tau)$, if $t \geq 0$ toward right, otherwise toward left
- Multiply $x(\tau) \cdot h(t-\tau)$
- Integral $\int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) d\tau$

Example

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) * \delta(t)$$

Q: $x(t) * \delta(t-1) = ?$

A: $x(t) * \delta(t-1)$

$$= \int_{-\infty}^{\infty} x(\tau)\delta(t-1-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\delta(t-(\tau+1))d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau'-1)\delta(t-\tau')d\tau'$$

$$= x(t-1) * \delta(t)$$

Q: $x(t) * \delta(t-2) = ?$

A: $x(t-2)$

Example

- Q: $x(t) = e^{-at}u(t),\ h(t) = u(t),\ \ a > 0$

  $x(t) * h(t) = ?$

- A: $y(t) = \int_{-\infty}^{\infty} e^{-a\tau} u(\tau) \cdot u(t-\tau) d\tau$

  For $t<0$ $\quad x(\tau) \cdot h(t-\tau) = 0 \ \Rightarrow\ y(t) = 0$

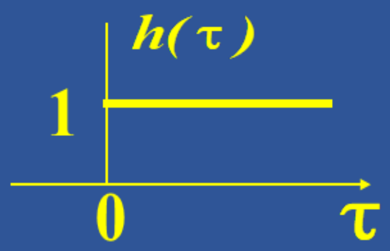
  For $t \geq 0$ $\quad y(t) = \int_{0}^{t} e^{-a\tau} d\tau$

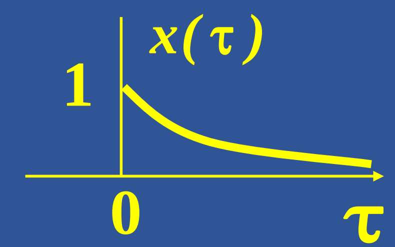
  $$= \frac{-1}{a} e^{-at} \Big|_{0}^{t} = \frac{1}{a}(1 - e^{-at})$$

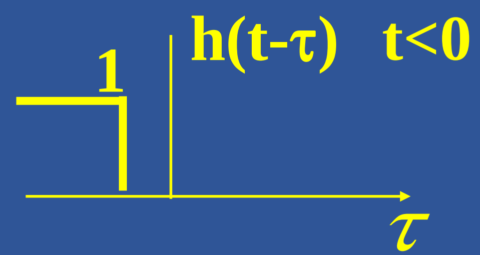
A Graphical Solution

- Q: $x(t) = e^{-at}u(t),\ h(t) = u(t),\ \ a > 0$

  $x(t) * h(t) = ?$

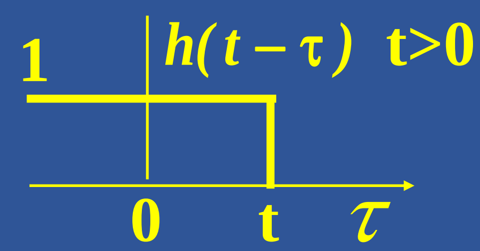






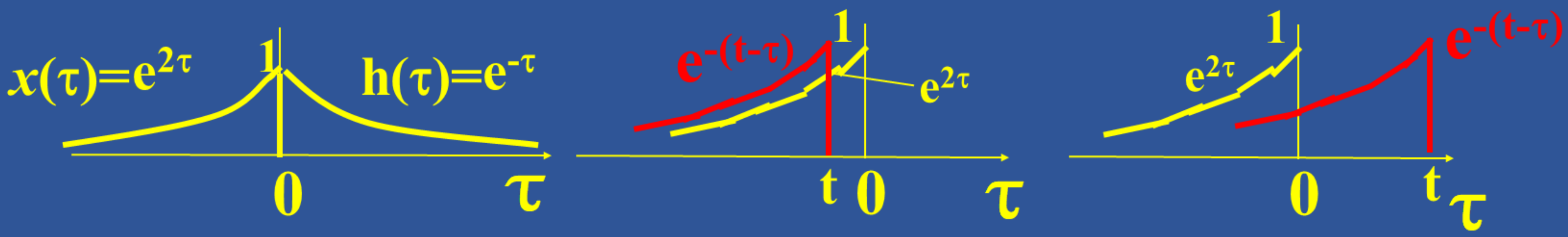
For $t \geq 0$ $y(t) = \int_0^t e^{-a\tau} d\tau$

$$= \frac{-1}{a} e^{-at} \Big|_0^t = \frac{1}{a}(1 - e^{-at})$$



Example

- Determine $y(t) = e^{2t}u(-t) * e^{-t}u(t)$



- A: $y(t) = \int_{-\infty}^{\infty} e^{2\tau}u(-\tau)\cdot e^{-(t-\tau)}u(t-\tau)d\tau$

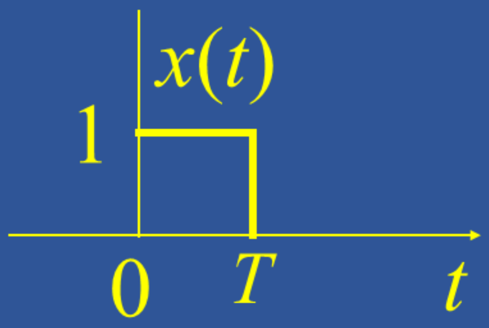
$$= \int_{-\infty}^{t} e^{2\tau}e^{-(t-\tau)}d\tau + \int_{-\infty}^{0} e^{-\tau}e^{-(t-\tau)}d\tau$$

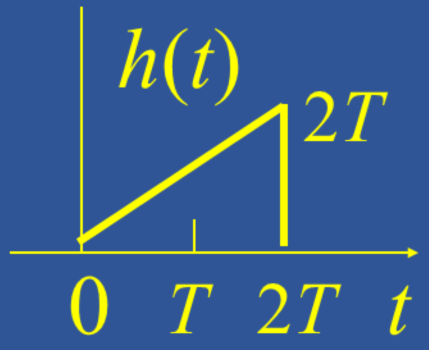
$(t<0)$ $(t>0)$

$$= \frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(t)$$

Example

$$x(t)=\begin{cases}1, & 0<t<T\\ 0, & \text{otherwise}\end{cases}\qquad h(t)=\begin{cases}t, & 0<t<2T\\ 0, & \text{otherwise}\end{cases}$$





Determine $y(t)=x(t)*h(t)$

Example

1) For $t<0$ and $t>3T$ $\quad y(t)=0$

2) For $0<t<T$

$$y(t)=\int_0^t x(\tau)h(t-\tau)d\tau=\int_0^t (t-\tau)d\tau$$

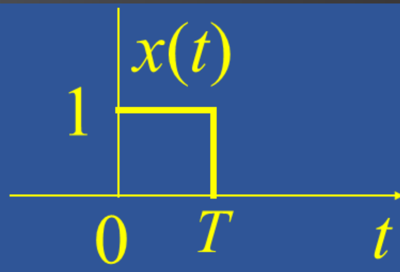
$$=t^2-\frac{1}{2}t^2=\frac{1}{2}t^2$$

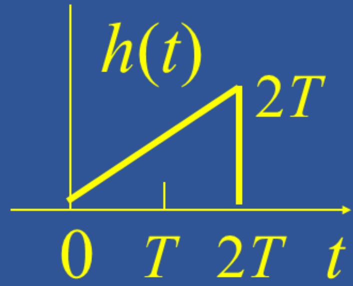
3) For $T<t<2T$

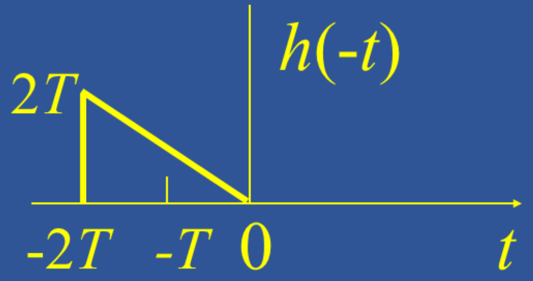
$$y(t)=\int_0^T (t-\tau)d\tau=Tt-\frac{1}{2}T^2$$

4) $2T<t<3T$

$$y(t)=\int_{t-2T}^{T}(t-\tau)d\tau=-\frac{1}{2}t^2+Tt+\frac{3}{2}T^2$$

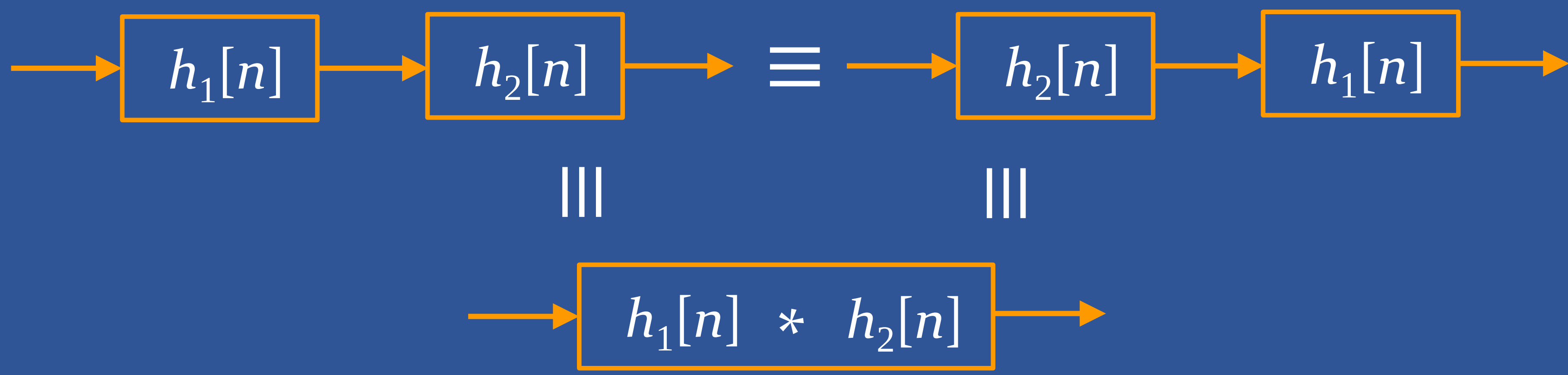






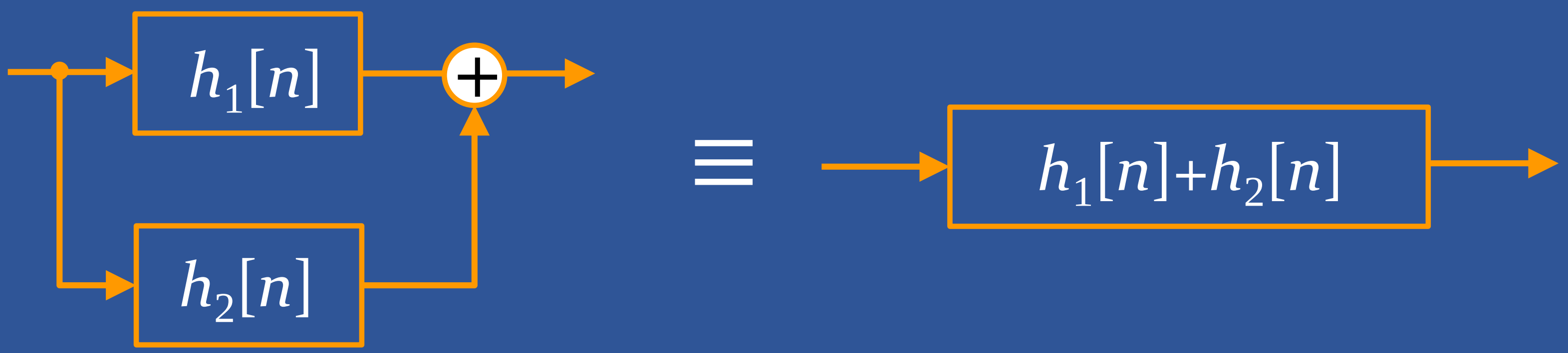
Simple Interconnection Schemes

- Cascade Connection

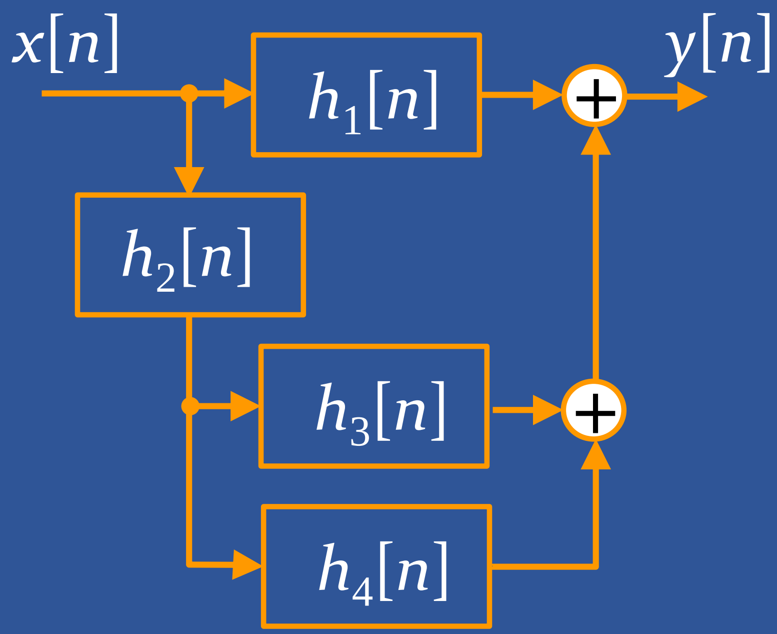


Simple Interconnection Schemes

- Parallel Connection



Analysis of Cascade and Parallel Connections



$$y[n] = x[n] \circledast h[n]$$

$$h[n] = h_1[n] + h_2[n] \circledast (h_3[n] + h_4[n])$$

$$h_1[n] = \delta[n] + \frac{1}{2}\delta[n-1]$$

$$h_2[n] = \frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1]$$

$$h_3[n] = 2\delta[n]$$

$$h_4[n] = -2\left(\frac{1}{2}\right)^n \mu[n]$$

$$\rightarrow \quad h[n] = \delta[n]$$

Properties of Convolution

- The commutative property

$$x[n] * h[n] = h[n] * x[n]$$

$$x(t) * h(t) = h(t) * x(t)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

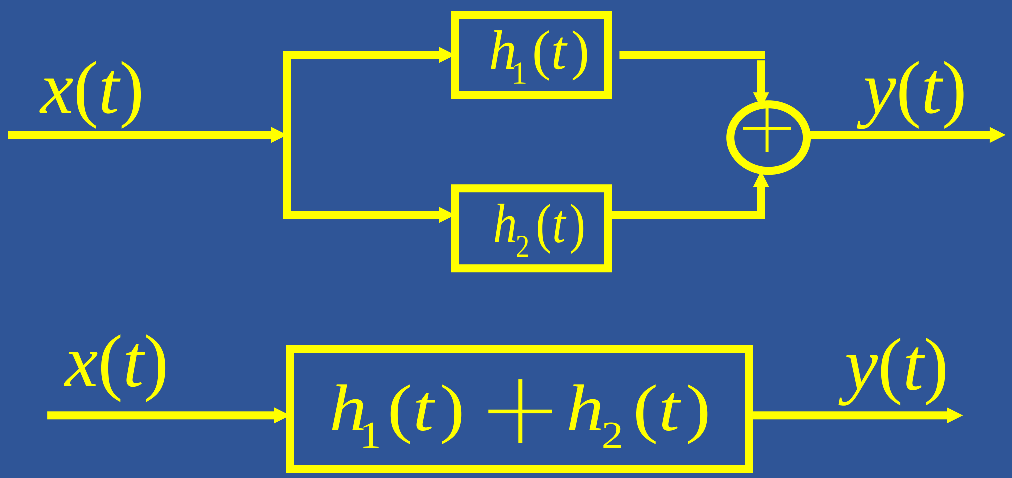
$$h(t) \longrightarrow \boxed{x(t)} \longrightarrow y(t)$$

Properties of Convolution

- The distributive property

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n]$$

$$x(t)*[h_1(t)+h_2(t)] = x(t)*h_1(t)+x(t)*h_2(t)$$

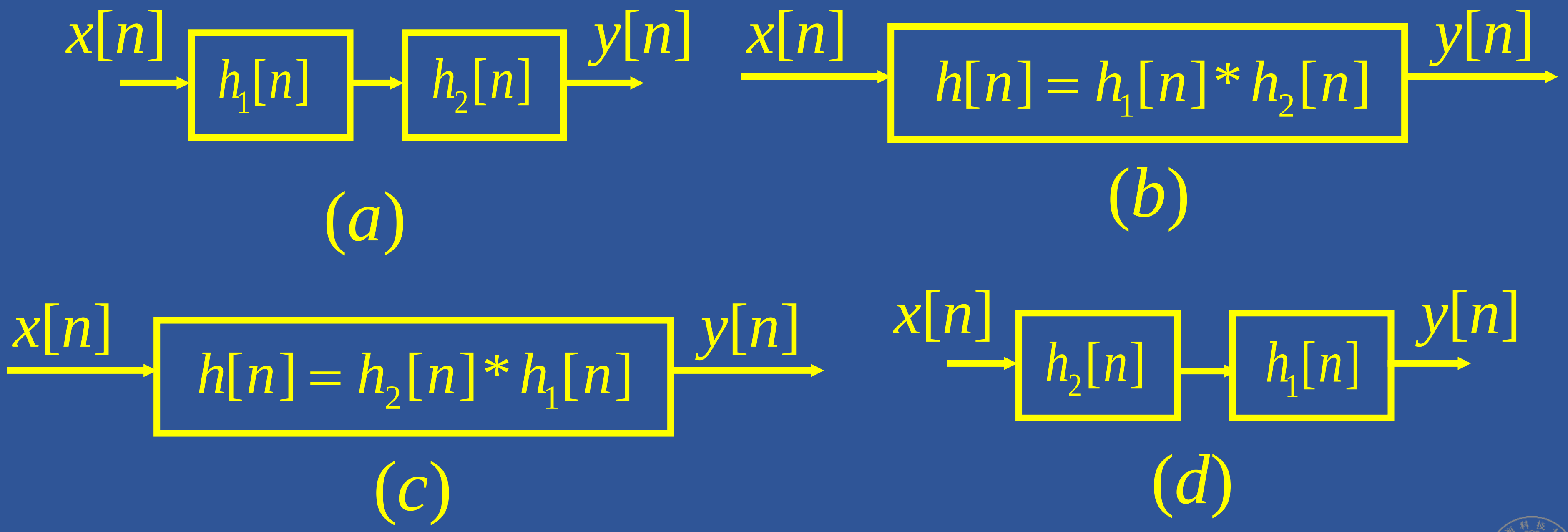


Properties of Convolution

- The associative property

$$x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n]$$

$$y[n] = x[n]*h_1[n]*h_2[n]$$



Properties of Convolution

- LTI systems with and without memory

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)$$

- For discrete system without memory:

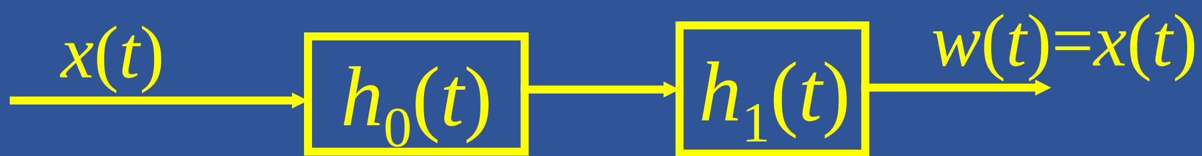
$h[n]=0$ for $n \neq 0$ $\quad h[n] = k\delta[n] \quad y[n] = kx[n] \quad$ Why?

- For continuous system without memory:

$h(t)=0$ for $t \neq 0$ $\quad h(t) = k\delta(t) \quad y(t) = kx(t)$

Properties of Convolution

- Invertibility of LTI system
  - If $h_0(t) * h_1(t) = \delta(t)$ then the system with $h_1(t)$ is the inverse of the system with $h_0(t)$



  - Similarly, if $h_0[n] * h_1[n] = \delta[n]$ then system of $h_1[n]$ is the inverse system of $h_0[n]$

Example

- Consider $\boldsymbol{h_0[n] = u[n]}$, determine $\boldsymbol{h_1[n]}$

$$\because h_0[n] * h_1[n] = u[n] * h_1[n] \overset{hold}{=} \delta[n]$$

$$\delta[n] = u[n] - u[n-1] = u[n] * (\delta[n] - \delta[n-1])$$

$$\therefore h_1[n] = \delta[n] - \delta[n-1]$$

Properties of Convolution

- Causality for LTI systems
  - If $h[n]=0$, for $n<0$ or $h(t)=0$, for $t<0$, then the system is causal

$$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]=\sum_{k=-\infty}^{-1} h[k]x[n-k]+\sum_{k=0}^{\infty} h[k]x[n-k]$$

$$y[n]=\sum_{k=-\infty}^{n} x[k]h[n-k]\overset{or}{=}\sum_{k=0}^{\infty} h[k]x[n-k]$$

$$y(t)=\int_{-\infty}^{t} x(\tau)h(t-\tau)\overset{or}{=}\int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

Example

- Causal LTI system
  - Accumulator: $y[n] = \sum_{l=-\infty}^{n} x[l]$

    $h[n] = \mu[n]$, a causal impulse response sequence

- Non causal LTI system
  - Factor-of-2 interpolator

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

$$h[n] = \{0.5, \underset{\uparrow}{1}, 0.5\}$$

a non causal impulse response sequence

Properties of Convolution

- Stability for LTI systems
  - A discrete LTI system is stable iff $h[n]$ is absolutely summable
  - A continuous LTI system is stable iff $h(t)$ is absolutely integrable

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$ absolutely summable

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$ absolutely integrable

Stability of LTI Systems

- Proof: "if" (Sufficient condition)

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \le \sum_{k=-\infty}^{\infty} |h[k]x[n-k]|$$

$$= \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$$

$$\le B_x$$

$$\le B_x \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Stability of LTI Systems

- Proof: "only if" (Necessary condition)

  suppose $\sum_{k=-\infty}^{\infty} |h(k)| = \infty$, show that there exists

  bounded $x[n]$ that gives unbounded $y[n]$

- Let $x[n] = \frac{h[-n]}{|h[-n]|} = \text{sign}\{h[-n]\}$

Assume $h[n]$ is real sequence

$\text{sign}\{h[-n]\} \longrightarrow \boxed{h[n]} \longrightarrow$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$y[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} \frac{h[k]h[k]}{|h[k]|} = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

Stability of LTI Systems

- The continuous case

$$\left|y(t)\right| = \left|\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right|$$

$$\leq \int_{-\infty}^{\infty} \left|h(\tau)\right|\left|x(t-\tau)\right|d\tau$$

$$\leq B_x \int_{-\infty}^{\infty} \left|h(\tau)\right|d\tau$$

$$< \infty$$

Example

- Consider an LTI discrete-time system with an impulse response $h[n] = \alpha^n \mu[n]$
- For this system:

$$S = \sum_{n=-\infty}^{\infty} |\alpha^n| \mu[n] = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|}, \text{if } |\alpha| < 1$$

- Therefore $S < \infty$ if $|\alpha| < 1$, i.e, the system is stable
- If $|\alpha| = 1$, the system is unstable

The Unit Step Response

- The unit step response, $s(t)$ or $s[n]$, corresponding to the output when $x[n]=u[n]$ or $x(t)=u(t)$

$$s[n]=u[n]*h[n]=\sum_{k=-\infty}^{\infty}h[k]u[n-k]=\sum_{k=-\infty}^{n}h[k]$$

$$h[n]=s[n]-s[n-1]$$

$$s(t)=u(t)*h(t)=\int_{-\infty}^{\infty}h(\tau)u(t-\tau)d\tau=\int_{-\infty}^{t}h(\tau)d\tau$$

$$h(t)=\frac{ds(t)}{dt}=s'(t)$$

CT-LTI System: Differential Equation

- A first-order example

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

CT-LTI System: Differential Equation

- General case

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k x(t)}{dt^k}$$

- A solution generally consist of a particular solution and a homogeneous solution, i.e.

$$y(t) = y_p(t) + y_h(t)$$

- Initial rest used as auxiliary condition, that is $x(t)=0$ for $t \le t_0$, i.e.,

$$y(t_0) = \frac{dy(t_0)}{dt} = \dots = \frac{d^{N-1} y(t_0)}{dt^{N-1}} = 0$$

DT-LTI System: Difference Equation

- General case

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- $y[n]$ can be computed recursively

$$a_0 y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]$$

- When a_0 is normalized to $a_0 = 1$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]$$

Example

- Consider $y[n]-\frac{1}{2}y[n-1]=x[n]$, determine $h[n]$=?
- A: the system is initially reset, i.e., $y[n]=0$ for $n\leq -1$, and let $x[n]=\delta[n]$

$$y[0]=x[0]+\frac{1}{2}y[-1]=1$$

$$y[1]=x[1]+\frac{1}{2}y[0]=\frac{1}{2}$$

$$y[2]=x[2]+\frac{1}{2}y[1]=(\frac{1}{2})^2$$

$$\vdots$$

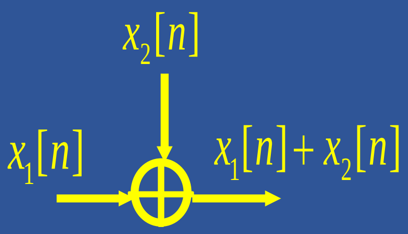
$$y[n]=x[n]+\frac{1}{2}y[n-1]=(\frac{1}{2})^n$$

Impulse response

$$h[n]=(\frac{1}{2})^n u[n]$$

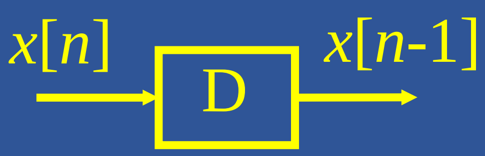
Block Diagrams

- Basic operations

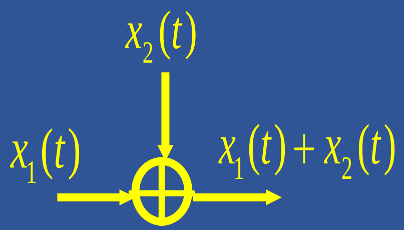


(a) An adder

(b) multiplication by a coefficient

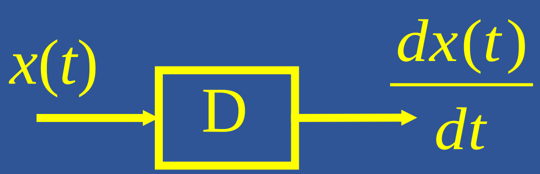


(c) a unit delay



(d) An adder

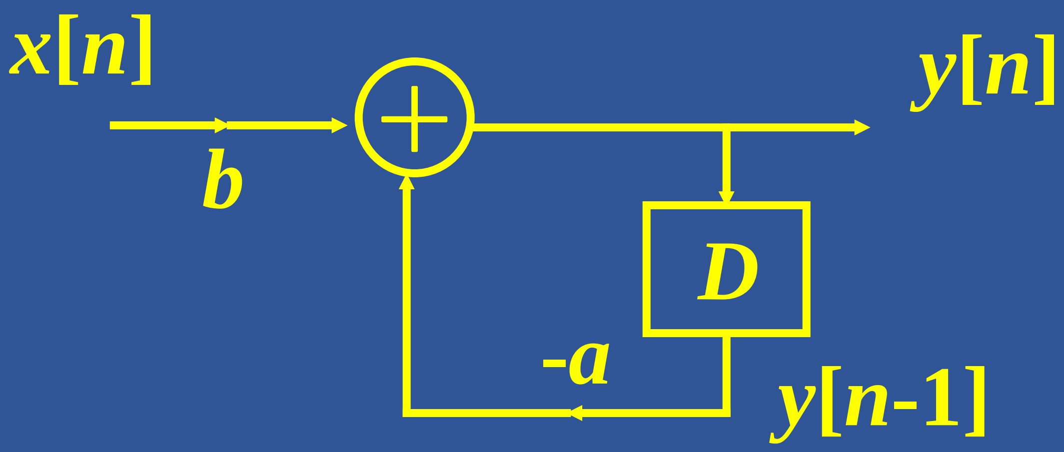
(e) multiplication by a coefficient



(f) a differentiator

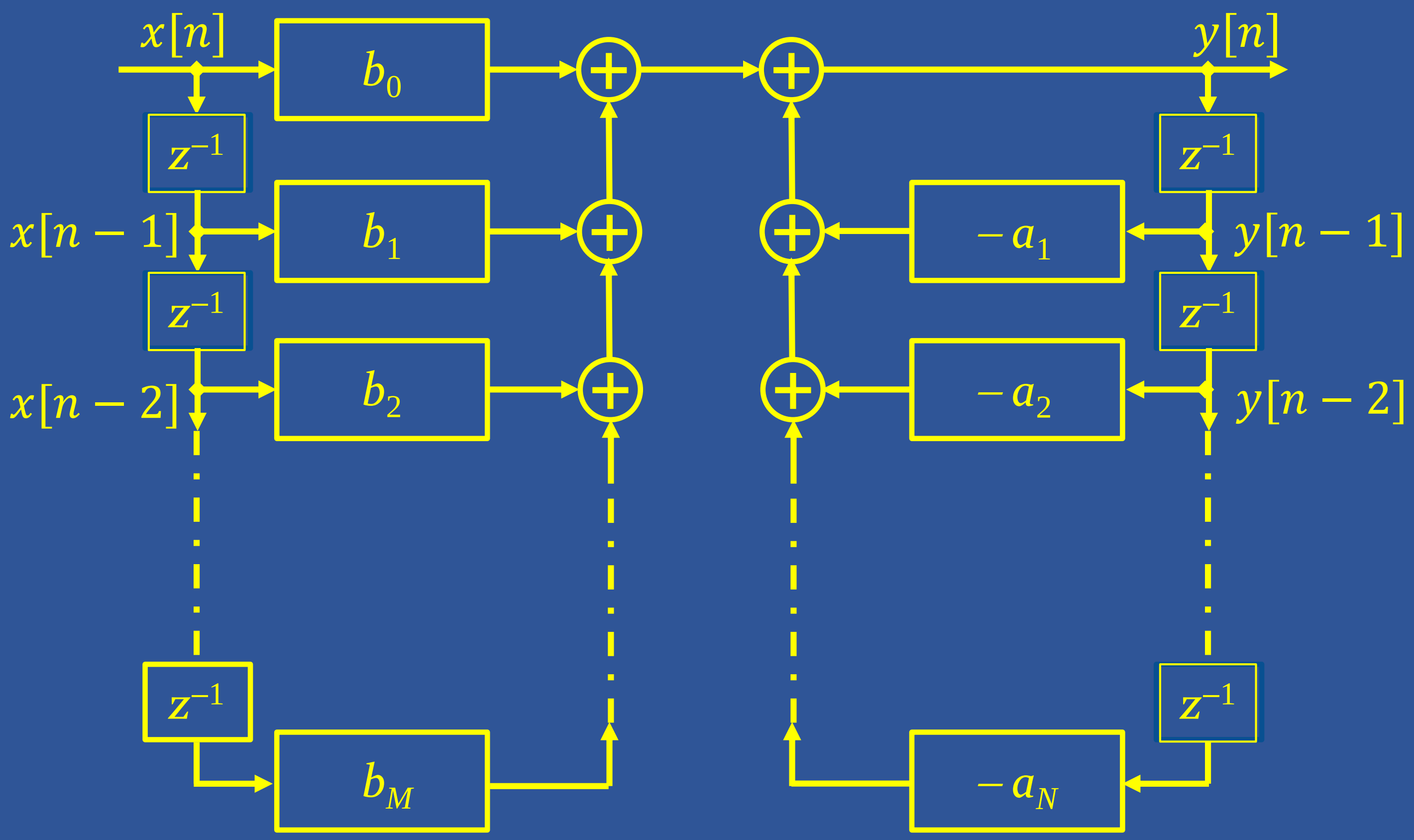
Block Diagrams: Example

- Block diagram for $y[n]+ay[n-1]=bx[n]$



Block Diagrams: Example

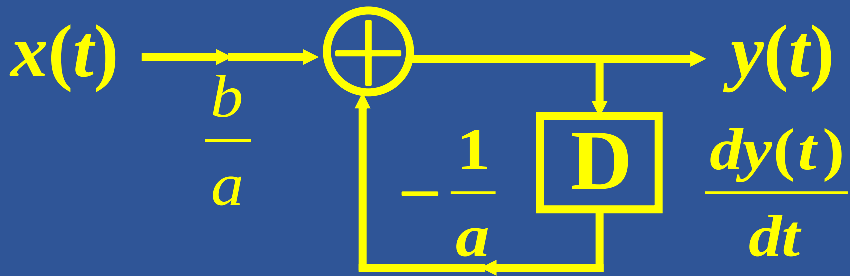
$$y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]$$



Block Diagrams: Example

- Block diagram for $\frac{dy(t)}{dt} + ay(t) = bx(t)$

- A: rewrite the equation to $y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$

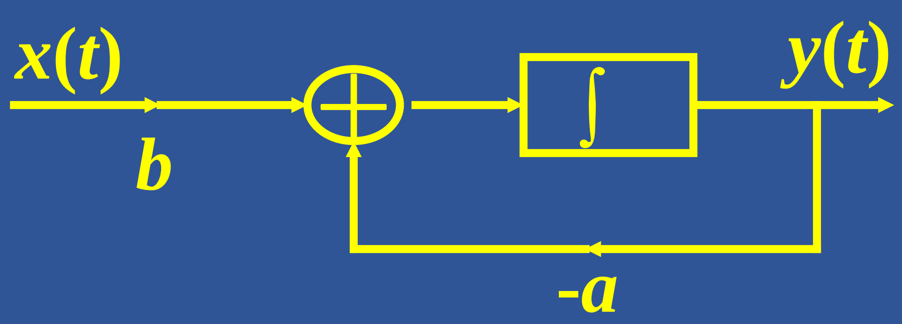


Another Implementation

- The system can be implemented using an integrator
- A: $\frac{dy(t)}{dt} + ay(t) = bx(t) \;\; \rightarrow \frac{dy(t)}{dt} = bx(t) - ay(t)$

integrating from $-\infty$ to t:

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)]d\tau$$



Classification of DT LTI System

- If $h[n]$ is of finite length, then it is known as a finite impulse response (FIR) system

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k]$$

  - Examples: moving averaged filter

$$y[n] = \sum_{k=0}^{4} \frac{1}{5} x[n-k]$$

Classification of DT LTI System

- If $h[n]$ is of infinite length, then it is know as a infinite impulse response (IIR) system

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

  - If causal (system causal, input sequence causal),

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k]$$

  - Example: accumulator

$$y[n] = \sum_{l=-\infty}^{n} x[l] = y[n-1] + x[n]$$