# Discrete-time Fourier Transform



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# Review of Continuous-Time Fourier Transform (CTFT)

- CTFT:

Fourier Spectrum
Or Spectrum

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$$

- Inverse CTFT:

$$x_a(t) = \int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} d\Omega$$

$$x_a(t) \overset{\text{CTFT}}{\longleftrightarrow} X_a(j\Omega)$$

# Fourier Series Representation

- Discrete case

# Fourier Series Representation

- Discrete case

# Discrete-time Fourier Transform

- From Fourier series to DTFT

# Discrete-time Fourier Transform

- From Fourier series to DTFT

# Discrete-Time Fourier Transform

- The frequency domain representation of discrete time sequence is the discrete-time Fourier transform (DTFT)

- This transform maps a time-domain sequence into a continuous function of the frequency variable $\omega$

# Discrete-Time Fourier Transform

- Discrete-Time Fourier Transform (DTFT)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$\omega$ is a continuous variable in the range of $-\infty < \omega < \infty$

- Inverse DTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Why one is sum and the other integral?

# Eigenfunctions for LTI Systems

- Complex exponential sequences are eigenfunctions of LTI systems

# Eigenfunctions for LTI Systems

- $H(e^{j\omega})$ is the frequency response of the system since it determines the change in complex magnitude of $e^{j\omega}$

# Example1

- Frequency response of the ideal delay system

# Example1

- An alternative method to determine the frequency response

# Example2

- Sinusoidal input

# Periodic Property of Frequency Response

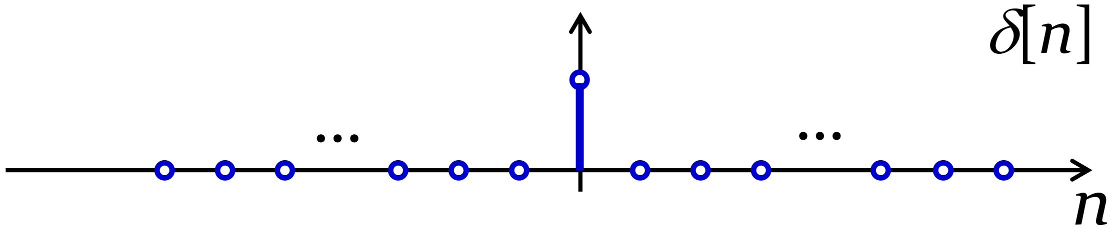
# Discrete-time Fourier Transform

- Verify x[n]=x[n]

# Discrete-time Fourier Transform

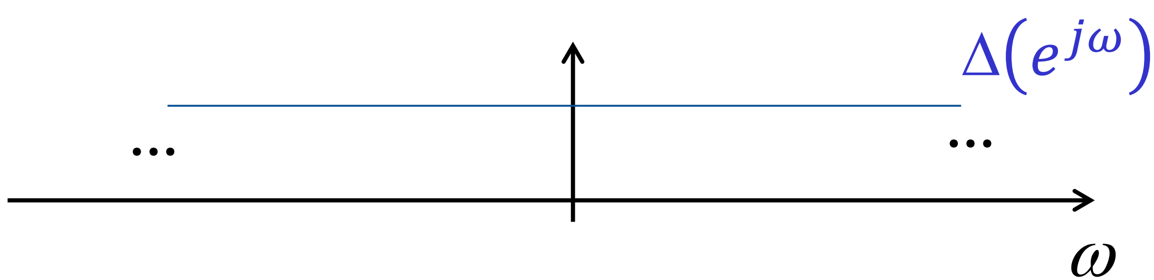
- Verify x[n]=x[n]

# Example 4




DTFT:

$$\Delta\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}\delta[n]e^{-j\omega n}=\sum_{n=0}e^{-j\omega n}=1$$



# Discrete-time Fourier Transform

- $X(e^{j\omega})$ is a complex function of the real variable $\omega$, and can be written as

$$X(e^{j\omega}) = X_{\mathrm{re}}(e^{j\omega}) + jX_{\mathrm{im}}(e^{j\omega})$$

- $X(e^{j\omega})$ can alternatively be expressed as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)}$$

where, $\theta(\omega) = \arg\{X(e^{j\omega})\}$

  - $|X(e^{j\omega})|$ is called the magnitude function
  - $\theta(\omega)$ is called the phase function

# DTFT

- $X(e^{j\omega})$ is a complex function of the real variable $\omega$

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$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$

where, $\theta(\omega) = \arg\{X(e^{j\omega})\}$

  - $\left|X(e^{j\omega})\right|$ is called the magnitude function
  - $\theta(\omega)$ is called the phase function
  - $\left|X(e^{j\omega})\right|$ and $\theta(\omega)$ are also called magnitude and phase spectra

# Example 5

- For a causal sequence $x[n] = \alpha^n \mu[n], |\alpha| < 1$
  Its DTFT is given by

# Example 5

- For a causal sequence $x[n] = \alpha^n \mu[n]$, $|\alpha|<1$
  Its DTFT is given by

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n = \frac{1}{1-\alpha e^{-j\omega}}$$

as $\left|\alpha e^{-j\omega}\right| = |\alpha|<1$

Recall: $1 + q + q^2 + \cdots + q^{\infty} = \dfrac{1}{1-q}$ for $|q| < 1$

# Example 6

- Find the inverse DTFT of

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$$

# Example 6

- Find the inverse DTFT of

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

- A: $h_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right) = \boxed{\frac{\sin \omega_c n}{\pi n}}$$

# Symmetry Property of DTFT

- For a real sequence $x[n]$, $\left|X\left(e^{j\omega}\right)\right|$ and $X_{\mathrm{re}}\left(e^{j\omega}\right)$ are even functions of $\omega$, $\theta(\omega)$ and $X_{\mathrm{im}}\left(e^{j\omega}\right)$ are odd functions of $\omega$

- Proof:

# Symmetry Property of DTFT

- For a real sequence $x[n]$, $|X(e^{j\omega})|$ and $X_{\text{re}}(e^{j\omega})$ are even functions of $\omega$, $\theta(\omega)$ and $X_{\text{im}}(e^{j\omega})$ are odd functions of $\omega$
- Proof: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$$X(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n} = \left\{\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right\}^{*}$$

$$= X^{*}(e^{j\omega})$$

- $|X(e^{j\omega})| = |X(e^{-j\omega})|$ and $\theta(\omega) = -\theta(-\omega)$

# Periodic Property of DTFT

- $X(e^{j\omega}) = X(e^{j(\omega+2k\pi)})$, i.e.,

  $|X(e^{j\omega})|e^{j[\theta(\omega)+2k\pi]} = |X(e^{j\omega})|e^{j\theta(\omega)}$

  for any integer $k$

- Proof:

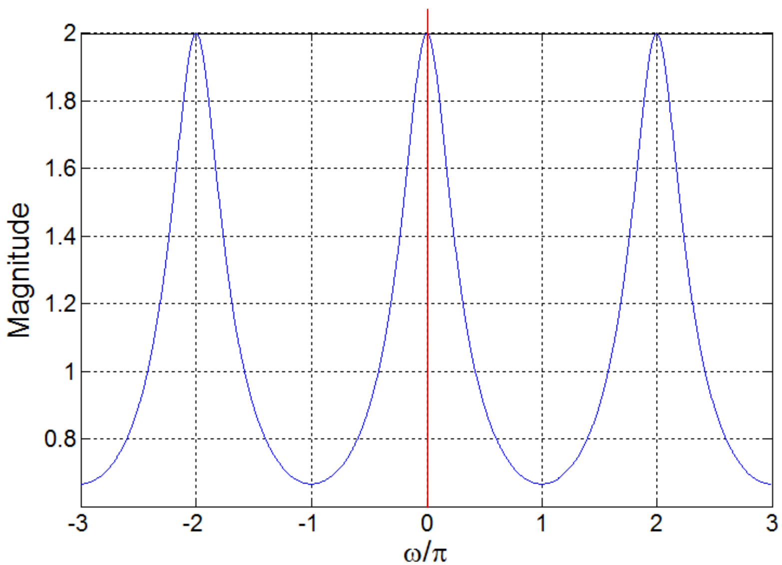
$$X(e^{j(\omega+2k\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2k\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j\omega})$$

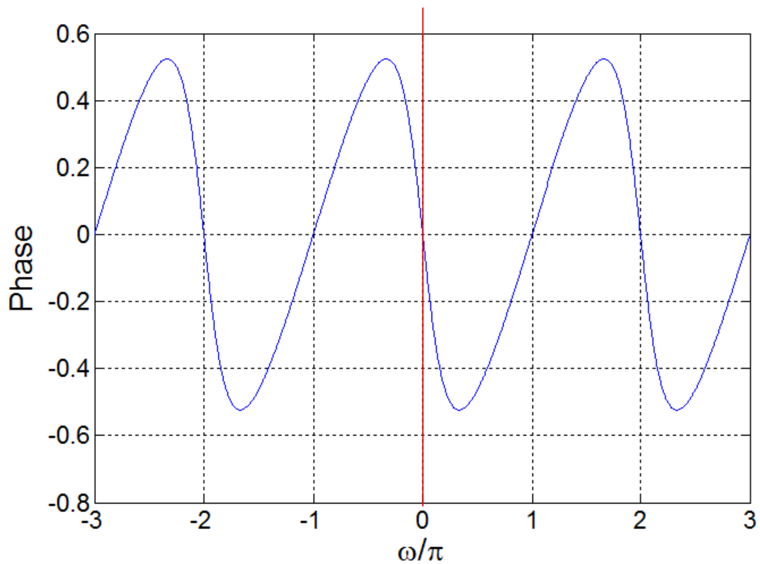
# Example 7

- The magnitude and phase of DTFT of

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$




$|X(e^{j\omega})| = |X(e^{-j\omega})|$




$\theta(\omega) = -\theta(-\omega)$



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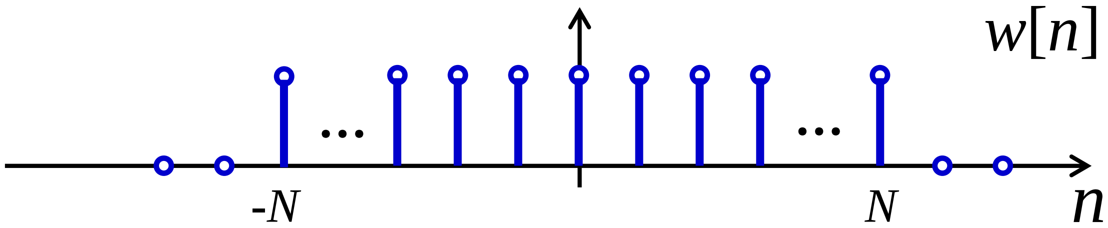
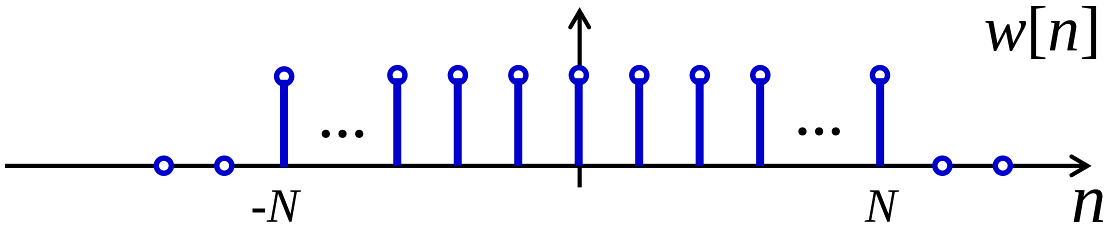
# Example 8




"Window"

DTFT:

# Example 8




"Window"

DTFT:

$$W\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} w[n]e^{-j\omega k} = \sum_{k=-N}^{N} e^{-j\omega k}$$

$$= e^{-j\omega N}\left(1 + e^{j\omega} + e^{j2\omega} + \cdots + e^{j2N\omega}\right)$$

Recall: $1 + q + q^2 + \cdots + q^M = \frac{1 - q^{M+1}}{1 - q}$ $\quad q = e^{j\omega}$, $M = 2N$



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# Example 8

$$W\left(e^{j\omega}\right) = e^{-j\omega N}\left(1 + e^{j\omega} + e^{j2\omega} + \cdots + e^{j2N\omega}\right)$$

$$= e^{-j\omega N}\,\frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$

$$= \frac{e^{-j\omega N} - e^{j\omega N}e^{j\omega}}{1 - e^{j\omega}} \quad \boxed{\times \frac{e^{-j\frac{\omega}{2}}}{e^{-j\frac{\omega}{2}}}}$$

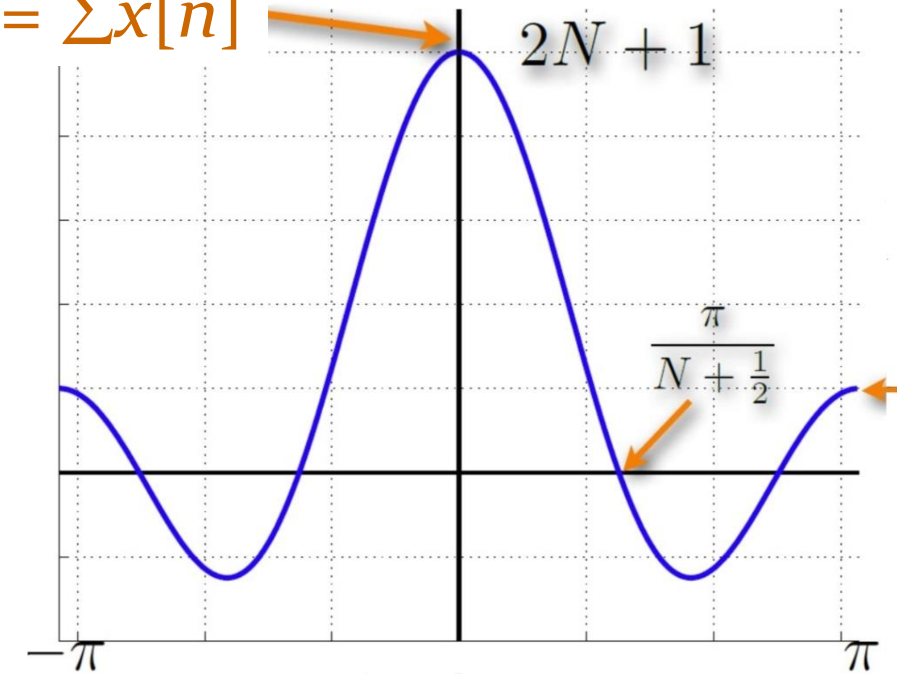
Periodic Sinc

$$= \frac{e^{-j\omega\left(N+\frac{1}{2}\right)} - e^{j\omega\left(N+\frac{1}{2}\right)}}{e^{-j\frac{\omega}{2}} - e^{j\frac{\omega}{2}}} = \frac{\sin\left(\omega\left(N+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$

# Example 8

$$W\left(e^{j\omega}\right) = \frac{\sin\left(\omega\left(N + \frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$ $\rightarrow 2N+1$ as $\omega \rightarrow 0$

Also = $\sum x[n]$




=1 if $N$ is even.
Why?

# Linearity & Periodicity

- Linearity:

$$g[n] \leftrightarrow G\left(e^{j\omega}\right) \text{ and } h[n] \leftrightarrow H\left(e^{j\omega}\right)$$

$$\alpha g[n] + \beta h[n] \leftrightarrow \alpha G\left(e^{j\omega}\right) + \beta H\left(e^{j\omega}\right)$$

- Periodicity: $X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$

# Time Reversal

- Time reversal:

Let $\quad x[n] \leftrightarrow X\left(e^{j\omega}\right)$

Then $\quad x[-n] \leftrightarrow X\left(e^{-j\omega}\right) = X^*\left(e^{j\omega}\right)$

if $x[n]$ is real

If $x[n]=x[-n]$ and $x[n]$ is real, then

$X\left(e^{j\omega}\right) = X^*\left(e^{j\omega}\right) \rightarrow X\left(e^{j\omega}\right)$ is real

- Q: Suppose $x[n] \leftrightarrow X(e^{j\omega})$, $x[n] \in \mathcal{Real}$

$$? \leftrightarrow \mathcal{Re}\{X(e^{j\omega})\}$$

- A: Decompose $x[n]$ to even and odd functions

$$x[n] = x_e[n] + x_o[n],$$

where

$$x_e[n] = \frac{1}{2}(x[n] + x[-n])$$

$$x_o[n] = \frac{1}{2}(x[n] - x[-n])$$

$$\boxed{x_e[n]} \leftrightarrow X_e(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) + X(e^{-j\omega})\right)$$

$$= \frac{1}{2}\left(X(e^{j\omega}) + X^*(e^{j\omega})\right) = \mathcal{Re}\{X(e^{j\omega})\}$$

$$x_o[n] \leftrightarrow X_o(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) - X^*(e^{-j\omega})\right) = j\mathcal{Im}\{X(e^{j\omega})\}$$

# Symmetry Relations

If $x[n]$ is real

| Sequence | Discrete-Time Fourier Transform |
|---|---|
| $x[n]$ | $X(e^{j\omega})$ |
| $x_{\mathrm{ev}}[n]$ | $X_{\mathrm{re}}(e^{j\omega})$ |
| $x_{\mathrm{od}}[n]$ | $jX_{\mathrm{im}}(e^{j\omega})$ |
| Conjugate Symmetric | $X(e^{j\omega}) = X^*(e^{-j\omega})$ |
| | $X_{\mathrm{re}}(e^{j\omega}) = X_{\mathrm{re}}(e^{-j\omega})$ |
| | $X_{\mathrm{im}}(e^{j\omega}) = -X_{\mathrm{im}}(e^{-j\omega})$ |
| | $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ |
| | $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$ |

# Symmetry Relations

If $x[n]$ is complex

| Sequence | Discrete-Time Fourier Transform |
|---|---|
| $x[n]$ | $X(e^{j\omega})$ |
| $x[-n]$ | $X(e^{-j\omega})$ |
| $x^*[-n]$ | $X^*(e^{j\omega})$ |
| $x^*[n]$ | $X^*(e^{-j\omega})$ |
| $x_{\mathrm{re}}[n]$ | $X_{\mathrm{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$ |
| $jx_{\mathrm{im}}[n]$ | $X_{\mathrm{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$ |
| $x_{\mathrm{cs}}[n]$ | $X_{\mathrm{re}}(e^{j\omega})$ |
| $x_{\mathrm{ca}}[n]$ | $jX_{\mathrm{im}}(e^{j\omega})$ |

# Time & Frequency Shifting

- Time and Frequency Shifting

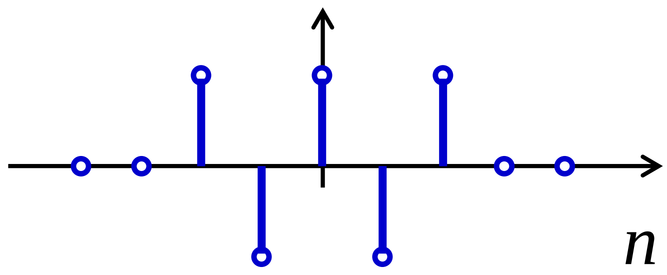
Let $\quad x[n] \leftrightarrow X\left(e^{j\omega}\right)$

Then $\quad x[n-n_d] \leftrightarrow e^{-j\omega n_d} X\left(e^{j\omega}\right)$

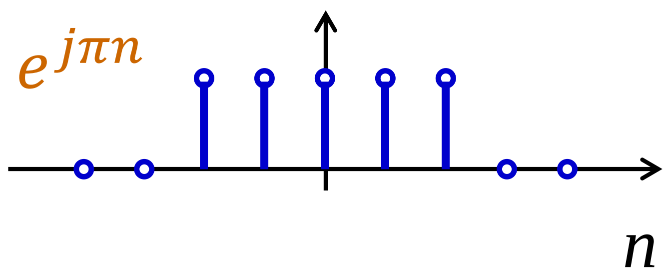
$$e^{j\omega_0 n} x[n] \leftrightarrow X\left(e^{j(\omega-\omega_0)}\right)$$

# Example 9

What is the DTFT of:

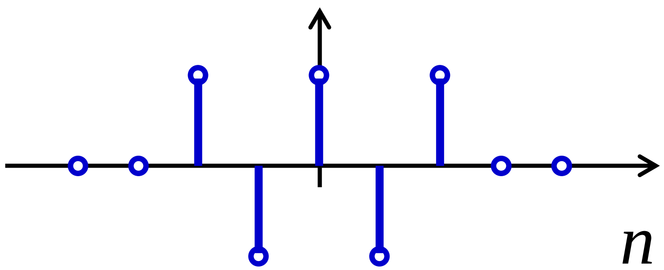



$\times\, e^{j\pi n}$

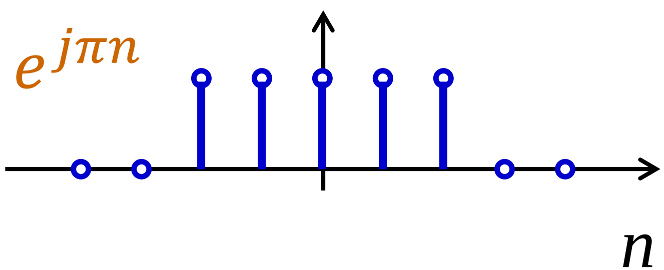


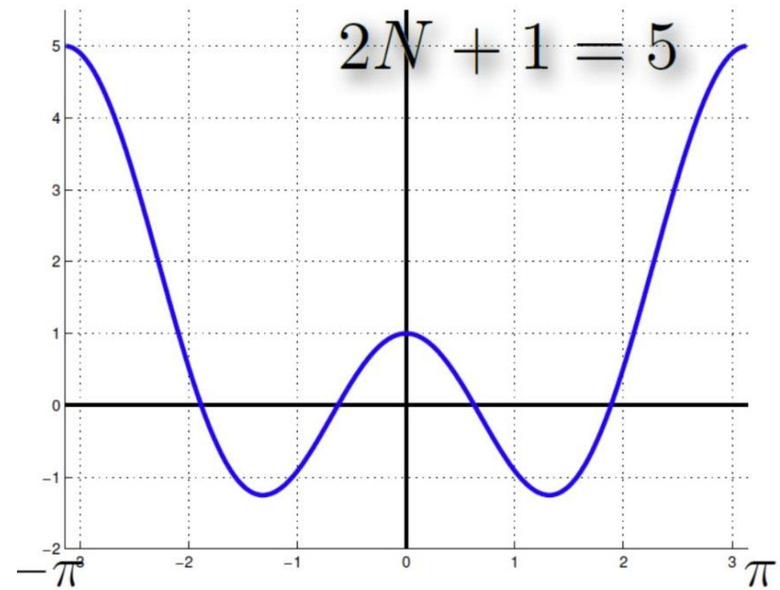
# Example 9

What is the DTFT of:




$\times\, e^{j\pi n}$




$$W\left(e^{j\omega}\right)=\frac{\sin\left((\omega-\pi)\left(N+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega-\pi}{2}\right)}$$

# Differentiation in Frequency

- Differentiation in frequency

Let $$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

Then $$nx[n] \leftrightarrow j\frac{dX\left(e^{j\omega}\right)}{d\omega}$$

# Differentiation in Frequency

- Differentiation in frequency

Let $x[n] \leftrightarrow X(e^{j\omega})$

Then $nx[n] \leftrightarrow j\dfrac{dX(e^{j\omega})}{d\omega}$

Proof: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Differentiate both side to get $\dfrac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} -jnx[n]e^{-j\omega n}$

Multiply both side by *j*, we get $j\dfrac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n}$

# Example 9

- Determine DTFT $Y(e^{j\omega})$ of

$$y[n] = (n+1)\alpha^n \mu[n], \qquad |\alpha| < 1$$

# Example 9

- Determine DTFT $Y(e^{j\omega})$ of

$$y[n] = (n+1)\alpha^n \mu[n], \qquad |\alpha| < 1$$

- Let $x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$
- We can therefore write

$$y[n] = nx[n] + x[n]$$

- From example 2, we have known that the DTFT of $x[n]$ is given by

$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

# Example 9

- Using the differentiation in frequency, we observe that DTFT of $nx[n]$ is given by,

$$j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

- Next, using linear property of DTFT, we arrive at

$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} + \frac{1}{1-\alpha e^{-j\omega}}$$

$$= \frac{1}{(1-\alpha e^{-j\omega})^2}$$

# Convolution & Modulation Theorem

## Convolution

Let $x[n] \leftrightarrow X\left(e^{j\omega}\right)$ and $h[n] \leftrightarrow H\left(e^{j\omega}\right)$

If $y[n] = x[n] \circledast h[n]$

Then $y[n] \leftrightarrow Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$

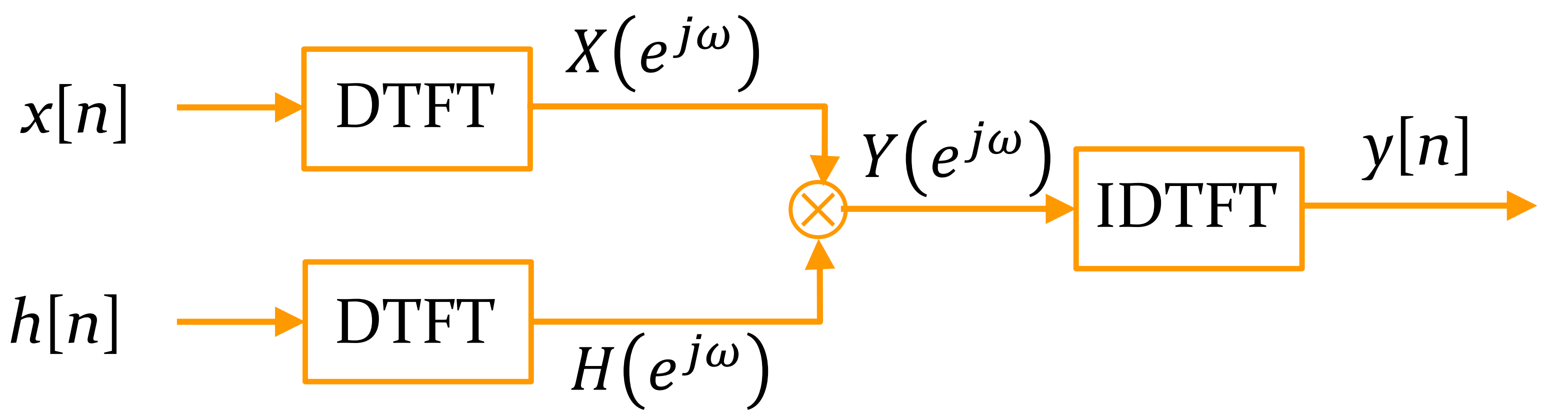
## Modulation

If $y[n] = x[n]h[n]$

Then $y[n] \leftrightarrow Y\left(e^{j\omega}\right) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X\left(e^{j\theta}\right)H\left(e^{j(\omega-\theta)}\right)d\theta$

# Linear Convolution Using DTFT

- Linear convolution $y[n]$ of the sequence $x[n]$ and $h[n]$ can be performed as follows:
  - Compute the DTFTs $X(e^{j\omega})$ and $H(e^{j\omega})$ of the sequences $x[n]$ and $h[n]$, respectively
  - Form DTFT $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
  - Compute the IDTFT $y[n]$ of $Y(e^{j\omega})$



# Parseval's Theorem

- Parseval's theorem

Let $x[n] \leftrightarrow X(e^{j\omega})$ $\quad h[n] \leftrightarrow H(e^{j\omega})$

Then $$\sum_{n=-\infty}^{\infty} x[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H^*(e^{j\omega})d\omega$$

$$\sum_{n=-\infty}^{\infty} x[n]h^*[n] = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})e^{-j\omega n}d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})\left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right)d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})\,X(e^{j\omega})d\omega$$

# Energy & Energy Density Spectrum

- Energy: $E_g = \sum_{-\infty}^{\infty} |x[n]|^2$
- According to Parseval's theorem, when $h[n] = x[n]$,

$$E_g = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]x^*[n] = \frac{1}{2\pi} \boxed{\int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2}$$

- Energy density spectrum:

$$S_{xx} = \int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2$$

# Energy Density Spectrum

- Example – Compute the energy of the sequence

$$h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n}, \qquad -\infty < n < \infty$$

- Here $$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_{LP}(e^{j\omega})\right|^2 d\omega$$

where $$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

- Therefore: $$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

## TABLE 2.2 FOURIER TRANSFORM THEOREMS

| Sequence | Fourier Transform |
|---|---|
| $x[n]$ | $X(e^{j\omega})$ |
| $y[n]$ | $Y(e^{j\omega})$ |
| 1. $ax[n] + by[n]$ | $aX(e^{j\omega}) + bY(e^{j\omega})$ |
| 2. $x[n-n_d]$ ($n_d$ an integer) | $e^{-j\omega n_d} X(e^{j\omega})$ |
| 3. $e^{j\omega_0 n} x[n]$ | $X(e^{j(\omega-\omega_0)})$ |
| 4. $x[-n]$ | $X(e^{-j\omega})$<br>$X^*(e^{j\omega})$ if $x[n]$ real. |
| 5. $nx[n]$ | $j\dfrac{dX(e^{j\omega})}{d\omega}$ |
| 6. $x[n] * y[n]$ | $X(e^{j\omega})Y(e^{j\omega})$ |
| 7. $x[n]y[n]$ | $\dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ |

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega}) d\omega$$