EE152: HW4

Fourier series

1. **Let $x[n]$ and $y[n]$ be periodic signals with common period N, and let**

$$z[n] = \sum_{r=<N>} x[r]y[n-r]$$

be their period convolution.

(a) **Show that $z[n]$ is also periodic with period N.**

(b) **Verify that if a_k, b_k and c_k are the Fourier coefficients of $x[n]$, $y[n]$ and $z[n]$ respectively, then**

$$c_k = Na_kb_k$$

(c) **Let**

$$x[n] = \sin(\frac{3\pi n}{4})$$

and

$$y[n] = \begin{cases} 1, & 0 \leq n \leq 3 \\ 0, & 4 \leq n \leq 7 \end{cases}$$

be two signals that are periodic with period 8. Find the Fourier series representation for the periodic convolution of these signals.

Solution:

(a) $z[n] = \sum_{r=<N>} x[r]y[n-r]$ **so we can get** $z[n+N] = \sum_{r=<N>} x[r]y[n+N-r]$, **since y is a periodic signal with period N, $y[n+N-r] = y[n-r]$ holds.**
In this way ,we can easily get

$$z[n+N] = \sum_{r=<N>} x[r]y[n+N-r] = \sum_{r=<N>} x[r]y[n-r] = z[n]$$

, so that$z[n]$ is also periodic with period N.

(b)

$$\begin{aligned}
\sum_{r=<N>} x[r]y[n-r] &= \sum_{r=<N>}\sum_{k=<N>} a_k e^{jk\omega_0 r}\sum_{l=<N>} b_l e^{jl\omega_0(n-r)} \\
&= \sum_{r=<N>}\sum_{k=<N>} a_k b_l \sum_{l=<N>} e^{jl\omega_0(n-r)} e^{jk\omega_0 r} \\
&= \sum_{r=<N>}\sum_{k=<N>} a_k b_l \underbrace{\sum_{l=<N>} e^{j(k-l)\omega_0 r}}_{N\delta[k-l]} e^{jl\omega_0 n} \\
&= \sum_{r=<N>}\sum_{k=<N>} N a_k b_l \delta[k-l] e^{jl\omega_0 n} \\
&= \sum_{k=<N>} N a_k b_k e^{jl\omega_0 n}
\end{aligned}$$

In this way we proved $c_k = N a_k b_k$.

(c) $x[n] = \sin(\frac{3\pi n}{4}) = \frac{e^{j\frac{3\pi n}{4}} - e^{-j\frac{3\pi n}{4}}}{2j}$**, here the period of this signal is 8 so we can easily get $a_3 = -a_{-3} = \frac{1}{2j}$, here $-a_{-3} = -a_5$ as a a Fourier Series of a discrete time signal.**
So we only need to evaluate b_3 and b_5 for signal $y[n]$.

$$b_k = \frac{1}{8}\sum_{n=<N>} y[n]e^{-jk\frac{2\pi n}{8}}$$

so these two coefficients cane be easily get by this formula.

$$b_3 = \frac{1}{8}\sum_{n=0}^{3} e^{-j3\frac{2\pi n}{8}} = \frac{1}{4}\frac{1}{1-e^{-j\frac{3\pi}{4}}}$$

$$b_5 = \frac{1}{8}\sum_{n=0}^{3} e^{-j5\frac{2\pi n}{8}} = \frac{1}{4}\frac{1}{1-e^{-j\frac{3\pi}{4}}}$$

From $0 \le k \le 7$, $c_3 = 8a_3b_3$ and $c_5 = 8a_5b_5$.

2. Let $x[n]$ be a periodic signal with period $N = 8$ and Fourier series coefficients $a_k = -a_{k-4}$. A signal

$$y[n] = (\frac{1+(-1)^n}{2})x[n-1]$$

with period $N = 8$ is generated. Denoting the Fourier series coefficients of $y[n]$ by b_k, find a function $f[k]$ such that

$$b_k = f[k]a_k$$

.

Solution: $a_k = -a_{k-4} \longrightarrow -a_k = a_{k-4} \longrightarrow \underbrace{-a_{k+4} = a_k = -a_{k-4}}_{will\ be\ used\ later}$ **so** $a_{k+4} = a_{k-4} \longrightarrow a_k = a_{k+8}$

$$\begin{aligned} y[n] &= [\frac{1+(-1)^n}{2}]x[n-1] \\ &= [\frac{1}{2} + \frac{1}{2}\cos(n\pi)]x[n-1] \\ &= [\frac{1}{2} + \frac{1}{4}(e^{j\pi n} + e^{-j\pi n})]x[n-1] \end{aligned}$$

(1)

$$\frac{1}{2}x[n-1] \xrightarrow{FS} \frac{1}{2}a_k e^{-jk\frac{2\pi}{8}}$$

(2)
$$\begin{aligned} \frac{1}{4}e^{j\pi n}x[n-1] &\xrightarrow{FS} \frac{1}{4}a_{k-4}e^{-j(k-4)\frac{2\pi}{8}} \\ &= \frac{1}{4}a_k(e^{-j\frac{2\pi k}{8}}) \end{aligned}$$

(3)
$$\begin{aligned} \frac{1}{4}e^{-j\pi n}x[n-1] &\xrightarrow{FS} \frac{1}{4}a_{k+4}e^{-j(k+4)\frac{2\pi}{8}} \\ &= \frac{1}{4}a_k(e^{-j\frac{2\pi k}{8}}) \end{aligned}$$

From (1) (2) (3) we have $y[n] \xrightarrow{FS} a_k e^{-j\frac{2\pi k}{8}}$**, so** $f[k] = e^{-j\frac{2\pi k}{8}}$

3. $[Hint]$**(Another form of Fourier series)**

$$a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)dt \qquad \omega = \frac{2\pi}{T}$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)\cos(n\omega t)dt$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)\sin(n\omega t)dt$$

$$u(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega t) + \sum_{n=1}^{\infty} b_n\sin(n\omega t)$$

A voltage $u(t)$ **is applied to a resistor of one ohm.**

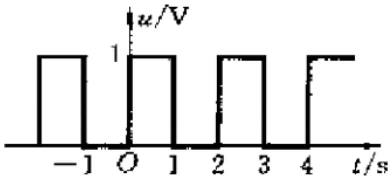

Figure 1: Voltage $u(t)$

(a) Find the trigonometric Fourier series for $u(t)$.

(b) Use the result of (a) and $u(\frac{1}{2})$ to find the sum of the following infinite series

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

.

(c) Find the average power of the above resistors.

(d) Use the result of (c) to find the sum of the following infinite series

$$S = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

Solution:

(a)

$$a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)dt = \int_0^1 1dt = 1$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)\cos(n\omega t)dt = \int_0^1 \cos(n\omega t)dt = 0, \; n = 1, 2, 3, \dots$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)\sin(n\omega t)dt = \int_0^1 \sin(n\omega t)dt = \frac{1-\cos(n\omega)}{n\omega}, \; n = 1, 2, 3, \dots$$

$$\begin{aligned} u(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(n\pi t) + \sum_{n=1}^{\infty} b_n\sin(n\pi t) \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-\cos(n\pi)}{n\pi}\sin(n\pi t), \; n = 1, 2, 3 \dots \end{aligned}$$

(b)

$$u(\frac{1}{2}) = 1 = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi}\sin(\frac{n\pi}{2})$$

so we have:

$$\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi}\sin(\frac{n\pi}{2}) = \frac{1}{2}$$

which means

$$2(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots)=\frac{\pi}{2}$$

in the end

$$S=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$$

(c)

$$P=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)dt=\frac{1}{2}\int_0^1 1dt=\frac{1}{2}$$

(d) From Parseval's relation:

$$p=\frac{1}{2}=(\frac{1}{2})^2+\frac{1}{2}\sum_{n=1}^{\infty}[\frac{1-\cos(n\pi)}{n\pi}]^2=\frac{1}{4}+\frac{1}{2}\sum_{n=1}^{\infty}[\frac{1-(-1)^n}{n\pi}]^2$$

which means that

$$\sum_{n=1}^{\infty}[\frac{1-(-1)^n}{n}]^2=\frac{1}{2}\pi^2$$

$\Rightarrow$

$$4(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots)=\frac{1}{2}\pi^2$$

in the end

$$S=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}$$

Continuous time Fourier transform

1. **A causal and stable LTI system S has the frequency response**

$$H(j\omega)=\frac{j\omega+4}{6-\omega^2+5j\omega}$$

   (a) **Determine a differential equation relating the input $x(t)$ and the output $y(t)$ of the system.**

   (b) **Determine the impulse response $h(t)$.**

   Solution:

   (a) **We have $\frac{Y(j\omega)}{X(j\omega)}=\frac{j\omega+4}{6-\omega^2+5j\omega}$ Cross-multiplying and taking the inverse Fourier transform, we obtain**

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(y)=\frac{dx(t)}{dt}+4x(t)$$

   (b) $H(j\omega)=\frac{j\omega+4}{(2+j\omega)(3+j\omega)}$
   Step1:

$$H(j\omega)=\frac{k_1}{2+j\omega}+\frac{k_2}{3+j\omega}$$

   Step2:

$$k_1=H(j\omega)*(2+j\omega)|_{2+j\omega=0}=\frac{j\omega+4}{3+j\omega}|_{\omega=2j}=2$$

$$k_2 = H(j\omega) * (3 + j\omega)|_{3+j\omega=0} = \frac{j\omega + 4}{2 + j\omega}|_{\omega=3j} = -1$$

So $H(j\omega) = \frac{2}{2+j\omega} - \frac{1}{3+j\omega} \longrightarrow h(t) = 2e^{-2t} - e^{-3t}$

2. Find the Fourier transform of the following signal:

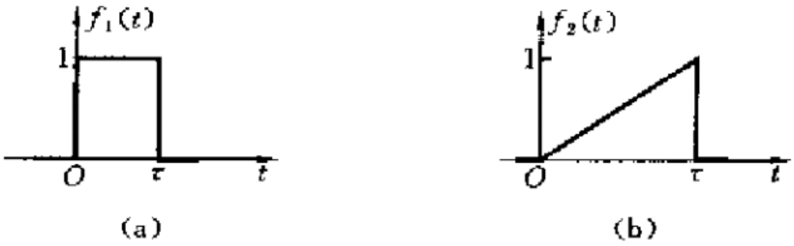

Figure 2: Signals

Solution:

(a)

$$\begin{aligned}
F_1(j\omega) &= \int_{-\infty}^{\infty} f_1(t)e^{-j\omega t}dt \\
&= \int_0^{\tau} e^{-j\omega t}dt \\
&= \frac{1 - e^{-j\omega\tau}}{j\omega} \\
&= \frac{(e^{\frac{j\omega\tau}{2}} - e^{\frac{-j\omega\tau}{2}})}{j\omega}e^{-\frac{j\omega\tau}{2}} \\
&= \tau\frac{e^{\frac{j\omega\tau}{2}} - e^{\frac{-j\omega\tau}{2}}}{2\frac{\omega\tau}{2}}e^{-\frac{j\omega\tau}{2}} \\
&= \tau\frac{\sin(\frac{j\omega\tau}{2})}{\frac{j\omega\tau}{2}}e^{-\frac{j\omega\tau}{2}} \\
&= \tau Sa(\frac{j\omega\tau}{2})e^{-\frac{j\omega\tau}{2}}
\end{aligned}$$

(b)

$$
\begin{aligned}
F_2(j\omega) &= \int_{-\infty}^{\infty} f_2(t)e^{-j\omega t}dt \\
&= \int_0^{\tau} \frac{1}{\tau} t e^{-j\omega t}dt \\
&= \frac{1}{\tau}\frac{1}{-j\omega}\int_0^{\tau} t d(e^{-j\omega t}) \\
&= -\frac{1}{j\omega\tau}[te^{-j\omega t}|_0^{\tau} - \int_0^{\tau} e^{-j\omega t}dt] \\
&= -\frac{1}{j\omega\tau}[\tau e^{-j\omega\tau} + \frac{1}{j\omega}(e^{-j\omega\tau}-1)] \\
&= \frac{j\omega\tau e^{-j\omega\tau} + e^{-j\omega\tau} - 1}{\omega^2\tau}
\end{aligned}
$$