The z-transform (ZT)

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z-transform

Remember the eigen-function for D-T LTI System:

$$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = H(z)z^n$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

z-transform (ZT):

$$x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z) \equiv \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad \text{(Bilateral)}$$

z-transform

In general: $z = r \cdot e^{j\omega}$ (polar form)

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]r^{-n}e^{-j\omega n}$$

$$\implies X(z) = FT\{x[n]r^{-n}\}$$

$$X(z)|_{z=e^{j\omega}} = FT\{x[n]\}$$

ZT is a generalization of DTFT

z-transform

Note: For different r value, $X(z)$ may or may not converge.

ROC: The set of z such that $\sum_{n=-\infty}^{\infty} |x[n]z^n|$ converges

Example

Consider the sequence $x[n] = a^n u[n]$, derive its ZT.

Example

Consider the sequence $x[n] = -a^n u[-n-1]$, derive its ZT.

Example

$x[n] = 7(\frac{1}{3})^n u[n] - 6(\frac{1}{2})^n u[n]$

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\
&= 7 \sum_{n=-\infty}^{\infty} (\frac{1}{3})^n u[n] z^{-n} - 6 \sum_{n=-\infty}^{\infty} (\frac{1}{2})^n u[n] z^{-n} \\
&= 7 \sum_{n=0}^{\infty} (\frac{1}{3} z^{-1})^n - 6 \sum_{n=0}^{\infty} (\frac{1}{2} z^{-1})^n \\
&= \frac{7}{1 - \frac{1}{3} z^{-1}} - \frac{6}{1 - \frac{1}{2} z^{-1}} \qquad (*) \\
&= \frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}
\end{aligned}
$$

Example

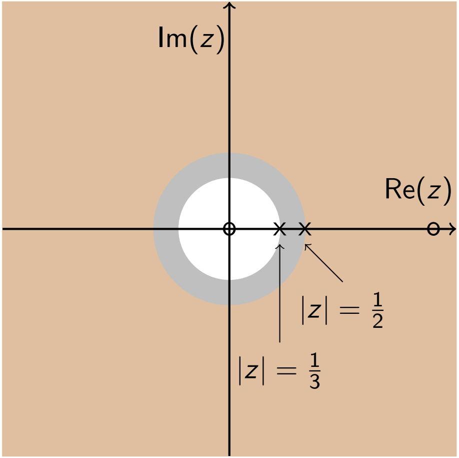
ROC: Both summations in $(*)$ have to converge

$$\implies \left|\frac{1}{3}z^{-1}\right| < 1 \quad \& \quad \left|\frac{1}{2}z^{-1}\right| < 1$$

$$\implies |z| > \frac{1}{3} \quad \& \quad |z| > \frac{1}{2}$$

$$\implies |z| > \frac{1}{2}$$

Example



o: Zero

$z = 0, \quad z = \frac{3}{2}$

x: Pole

$z = \frac{1}{3}, \quad z = \frac{1}{2}$

ROC: $|z| > \frac{1}{2}$

Example

Consider the sequence $x[n] = (\frac{1}{3})^n \sin(4\pi n)u[n]$, derive its ZT.

z-transform for LTI system and LCC difference equations

LTI system $x[n] * h[n] \xleftrightarrow{\mathcal{Z}} X(z) \cdot H(z)$

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = H(z)X(z)$$

A general method for solving difference equations: e.g.

$$
\begin{aligned}
& y[n] - ay[n-1] = \delta[n], \quad y[n] \text{ right-sided} \\
\Longrightarrow\ & Y(z) - az^{-1}Y(z) = 1 \\
\Longrightarrow\ & Y(z) = \frac{1}{1 - az^{-1}} \\
\Longrightarrow\ & Y(z) = 1 + az^{-1} + a^2z^{-2} + \cdots \\
\Longrightarrow\ & y[n] = a^n u[n]
\end{aligned}
$$

Unilateral z-transform

Note: Definition before is called Bilateral ZT

Unilateral ZT:

$$X(z) := \sum_{n=0}^{\infty} x[n]z^{-n}$$

written as

$$x[n] \xleftrightarrow{\mathcal{UZ}} X(z)$$

practical since usually we deal with right-sided signals and when we analyze the causal systems specified by LCC difference equations with nonzero initial conditions

z-transform

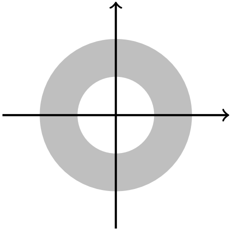
Different $x[n]$ may have the same ZT

$$u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-z^{-1}} \qquad \text{ROC: } |z| > 1$$

$$\begin{aligned} -u[-n-1] \xleftrightarrow{\mathcal{Z}} & -\sum_{n=-\infty}^{\infty} u[-n-1]z^{-n} \\ &= -\sum_{n=-\infty}^{-1} z^{-n} = -\sum_{n=1}^{\infty} z^{n} \\ &= -\frac{z}{1-z} = \frac{1}{1-z^{-1}} \qquad \text{ROC: } |z| < 1 \end{aligned}$$

Properties of ROC

1. ROC is a ring in the z-plane centered about origin.
   i.e. ROC is independent of ω



$$\sum_{n=-\infty}^{\infty} \left|x[n]z^{-n}\right| = \sum_{n=-\infty}^{\infty} |x[n]|\, r^{-n} < \infty$$

The inner boundary may extend inward to the origin and the outer boundary may extend outward to infinity.

Properties of ROC

2. ROC does not contain any pole

As with the Laplace transform, this property is simply a consequence of the fact that at a pole $X(z)$ is infinite and therefore, by definition, does not converge.

Properties of ROC

3. If $x[n]$ has finite duration, then ROC is the entire z-plane, except possibly $z = 0$ and/or $z = \infty$

Proof:

$$X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n}$$

If $X(z)$ contains negative power of z, then $X(0) = \infty$ $(N_2 > 0)$
If $X(z)$ contains positive power of z, then $X(\infty) = \infty$ $(N_1 < 0)$
For other values of z, summation always converge

E.g. Consider the ZT of $\delta[n]$, $\delta[n-1]$, and $\delta[n+1]$.

Example

Consider the signal $x[n] = a^n(u[n] - u[N-1-n])$ $(a > 0)$, and derive its ZT.

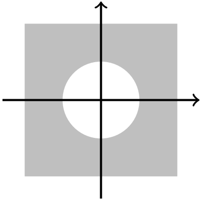
Properties of ROC

4. If $x[n]$ is right-sided, and if $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| \geq r_0$ will also be in the ROC.

$$X(z) = \sum_{n=N_1}^{\infty} x[n]z^{-n}$$

for $r_1 \geq r_0$

$$\left|\sum_{n=N_1}^{\infty} x[n]z^{-n}\right| = \left|\sum_{n=N_1}^{\infty} x[n](\frac{r_1}{r_0})^{-n}(r_0 e^{j\omega})^{-n}\right| \leq (\frac{r_1}{r_0})^{-N_1}\left|\sum_{n=N_1}^{\infty} x[n](r_0 e^{j\omega})^{-n}\right|$$

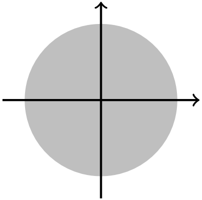


4′. If $x[n]$ is right-sided, then ROC takes the form: $c < |z| < \infty$. (ROC will include $z = \infty$, if $N_1 > 0$.)

Properties of ROC

5. If $x[n]$ is left-sided, and if $|z| = r_0$ is in the ROC, then all nonzero values of z for which $|z| \leq r_0$ will also be in the ROC.

$$X(z) = \sum_{n=-\infty}^{N_2} x[n]z^{-n}$$

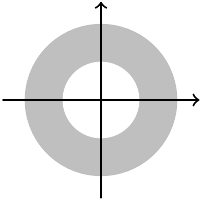


5′. If $x[n]$ is left-sided, then ROC takes the form: $0 < |z| < c$. (ROC will include $z = 0$, if $N_2 \leq 0$.)

Properties of ROC

6. If $x[n]$ is two-sided, and if $|z| = r_0$ is in ROC, then ROC is a ring that includes $|z| = r_0$.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{N_0} x[n]z^{-n} + \sum_{n=N_0}^{\infty} x[n]z^{-n}$$



6′. If $x[n]$ is two-sided, then ROC takes the form: $c_1 < |z| < c_2$

Example

Consider the signal $x[n] = b^{|n|}$ $(b > 0)$, and derive its ZT.

Properties of ROC

Rational $X(z)$ + Property 2

7. If $X(z)$ is rational, then ROC is bounded by poles or extends to infinity.

From Property 7 + Properties 4, 5, 6

8.1 If $x[n]$ is right-sided and $X(z)$ is rational, then ROC is outside the outermost finite pole (may not include $z = \infty$).

    Especially, if $x[n]$ is causal ($N_1 \geq 0$), the ROC contains $z = \infty$.

Properties of ROC

8.2 If $x[n]$ is left-sided and $X(z)$ is rational, then ROC is inside the innermost nonzero pole (may not include $z = 0$).

Especially, if $x[n]$ is anticausal $(N_2 \leq 0)$, the ROC contains $z = 0$.

8.3 If $x[n]$ is two-sided and $X(z)$ is rational, then ROC is a ring between two consecutive poles.

Example

Consider all the ROCs that can be associated with

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}$$

Inverse z-transform

Can we use F^{-1} to obtain Z^{-1}? Consider:

$$
\begin{aligned}
X(z) &= X(re^{j\omega}) = F\{x[n]r^{-n}\} \\
\implies x[n]r^{-n} &= F^{-1}\{X(re^{j\omega})\} \\
\implies x[n] &= r^n F^{-1}\{X(re^{j\omega})\} \\
&= r^n \cdot \frac{1}{2\pi}\int_0^{2\pi} X(re^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})(re^{j\omega})^n d\omega
\end{aligned}
$$

Inverse z-transform

Note that $X(re^{j\omega}) \cdot (re^{j\omega})^n$ is a function of both "r" & "ω".

However, the integration is only respect to ω: an integration along a circle contour $z = re^{j\omega}$ in ROC, with a fixed r, and ω varying over a 2π interval.

By changing of variable, $dz = jre^{j\omega} d\omega$ or $d\omega = (\frac{1}{j})z^{-1}dz$:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(z) z^n d\omega$$
$$= \frac{1}{2\pi j} \oint_{|z|=r} X(z) z^{n-1} dz$$

Inverse z-transform

$\oint$ integration around a counter-clockwise (CCW) closed circular contour centered at the origin with radius r

Remark: The formal inverse z-transform equation requires contour integration in the complex plane

Alternative: Try to use partial-fraction expansion & ZT pairs table:
For rational $X(z)$, express $X(z) = X_1(z) + X_2(z) + \ldots$
in which X_1, X_2, ... have known ZT pairs

Partial Fraction Expansion

A rational ZT can be expressed as

$$X(z) = \frac{P(z)}{Q(z)}, \quad \text{simplest fraction,}$$

$$Q(z) = \prod_{i=1}^{I} (1 - a_i z^{-1})^{p_i}, \quad a_i\text{'s are distinct}$$

Then

$$X(z) = \text{polynomial}(z^{-1}) + \sum_{i=1}^{I} \sum_{k=1}^{p_i} \frac{C_{i,k}}{(1 - a_i z^{-1})^k}$$

Learn Appendix A (partial-fraction expansion) by yourself in the O&W&N textbook.

Example

Find the sequence $x[n]$ corresponding to the following z-transform

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}, \qquad \frac{1}{4} < |z| < \frac{1}{3}$$

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}$$

$$\implies X_1(Z) = \frac{1}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4}$$

$$X_2(Z) = \frac{2}{1 - \frac{1}{3}z^{-1}}, \qquad |z| < \frac{1}{3}$$

Example

$$\implies x_1[n] = (\frac{1}{4})^n u[n]$$

$$x_2[n] = -2(\frac{1}{3})^n u[-n-1]$$

$$\implies x[n] = (\frac{1}{4})^n u[n] - 2(\frac{1}{3})^n u[-n-1]$$

Q: Find the sequence $x[n]$ when the ROC is given by
ROC: $|z| < 1/4$
ROC: $|z| > 1/3$

Power-series Expansion (Long Division Method)

Find the sequence $x[n]$ corresponding to the following z-transform

$$X(z) = 4z^2 + 2 + 3z^{-1}, \quad 0 < |z| < \infty$$

Just follow the definition of the z-transform. Especially useful for nonrational $X(z)$.

Properties of ZT

1. Linearity

$$ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{Z}} aX_1(z) + bX_2(z)$$

ROC <u>at least</u> $R_1 \cap R_2$

ROC equals $R_1 \cap R_2$ if there is no pole-zero cancellation

2. Time-shifting

$$x[n - n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0} X(z)$$

ROC: R possibly add or delete zero/∞

Properties of ZT

3. Scaling in z-domain

$$z_0^n x[n] \xleftrightarrow{\mathcal{Z}} X(\frac{z}{z_0}), \qquad \text{ROC: } |z_0|R$$

Specifically,

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{Z}} X(e^{-j\omega_0} z), \qquad \text{ROC: } R$$

$$r_0^n x[n] \xleftrightarrow{\mathcal{Z}} X(r_0^{-1} z), \qquad \text{ROC: } r_0 R$$

Example: if $X(z) = \dfrac{1}{1 - az^{-1}}$, ROC: $|z| > |a|$

then $z_0^n x[n] \xleftrightarrow{\mathcal{Z}} X(\frac{z}{z_0}) = \dfrac{1}{1 - a(\frac{z}{z_0})^{-1}} = \dfrac{1}{1 - az_0 z^{-1}}$

ROC is $|z| > |z_0||a|$

Properties of ZT

4. Time-reversal

$$x[-n] \xleftrightarrow{\mathcal{Z}} X(\frac{1}{z}), \qquad \text{ROC: } \frac{1}{R}$$

Example: Given

$$u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1 - z^{-1}} \qquad \text{ROC: } |z| > 1$$

Find the z-transform of $-u[-n-1]$.

Properties of ZT

5. Time-expansion

$$x_{(k)}[n] := \begin{cases} x[n/k], & \text{if } n \text{ is multiple of } k \\ 0, & \text{otherwise} \end{cases}$$

$$x_{(k)}[n] \xleftrightarrow{\mathcal{Z}} X(z^k), \qquad \text{ROC: } R^{1/k}$$

Proof:

$$X_{(k)}(z) = \sum_{n=-\infty}^{\infty} x_{(k)}[n] z^{-n}$$

$$= \sum_{n=km,\ m=-\infty}^{\infty} x[n/k] z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} x[m](z^k)^{-m} = X(z^k)$$

Properties of ZT

6. Conjugation

$$x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*), \qquad \text{ROC: } R$$

Especially, when $x[n]$ is real, we have $X(z) = X^*(z^*)$.

Example: Consider the zero-pole plot of $x[n] = (\frac{1}{3})^n \sin(4\pi n)u[n]$.

$$X(z) = \frac{\frac{1}{s\sqrt{2}}z}{(z - \frac{1}{3}e^{\frac{j\pi}{4}})(z - \frac{1}{3}e^{-\frac{j\pi}{4}})}$$

Properties of ZT

7. Convolution

$$x_1[n] * x_2[n] \xleftrightarrow{\mathcal{Z}} X_1(z) \cdot X_2(z), \quad \text{ROC } \underline{\text{at least}}\ R_1 \cap R_2$$

Example: First-difference $x[n] - x[n-1] = (\delta[n] - \delta[n-1]) * x[n]$

$$x[n] - x[n-1] \xleftrightarrow{\mathcal{Z}} (1 - z^{-1}) \cdot X(z)$$

ROC contains intersection of R and $|z| > 0$

Example: Accumulation/summation $\sum_{k=-\infty}^{n} x[k] = u[n] * x[n]$

$$\sum_{k=-\infty}^{n} x[k] \xleftrightarrow{\mathcal{Z}} U(z) \cdot X(z) = \frac{1}{1 - z^{-1}} \cdot X(z)$$

ROC contains intersection of R and $|z| > 1$

Properties of ZT

8. Differentiation in the z-domain

$$nx[n] \xleftrightarrow{\mathcal{Z}} -z\frac{d}{dz}X(z), \qquad \text{ROC: } R$$

Example: Derive the inverse z-transform of

$$X(z) = \frac{az^{-1}}{(1 - az^{-1})^2} \quad \text{ROC:} |z| > |a|$$

Properties of ZT

9. Initial-Value Theorem

   If $x[n] = 0,\ n < 0$, then $x[0] = \lim_{z\to\infty} X(z)$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} x[n]z^{-n}$$

As $z \to \infty$, $z^{-n} \to 0$ for $n > 0$, whereas for $n = 0$, $z^{-n} = 1$. $\square$

Remark: For a causal sequence with finite $x[0]$, $\lim_{z\to\infty} X(z)$ is finite. It can be used to check whether your Unilateral ZT or Inverse Unilateral ZT is correct.

Unilateral ZT

ROC for a unilateral ZT must be a exterior of a circle. Hence, ROC is usually omitted.
$UZT\{x[n]\} = ZT\{x[n]u[n]\}$

Example: $x[n] = a^{n+1}u[n+1]$
ZT: $X(z) =$

UZT: $X(z) =$

Unilateral ZT

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

Note: inverse UZT provides information about $x[n]$ only for $n \geq 0$

Properties of Unilateral ZT

Table 10.3

Property	Signal	Unilateral ZT
	$x[n],\ x_1[n],\ x_2[n]$	$X(z),\ X_1(z),\ X_2(z)$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$
Time delay	$x[n-1]$	$z^{-1}X(z) + x[-1]$
Time advance	$x[n+1]$	$zX(z) - zx[0]$
z scaling	$z_0^n x[n]$	$X(\frac{z}{z_0})$
	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$
	$r_0^n x[n]$	$X(r_0^{-1}z)$
Time expansion	$x_k[n]$	$X(z^k)$
Conjugation	$x^*[n]$	$X^*(z^*)$
Convolution ($x_1[n] = x_2[n] = 0 \quad \text{for} \quad n < 0$)	$x_1[n] * x_2[n]$	$X_1(z) \cdot X_2(z)$
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z) - x[-1]$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\frac{1}{1-z^{-1}}X(z)$
Differentiation in z	$nx[n]$	$-z\frac{dX(z)}{dz}$

Initial-Value Theorem

$x[0] = \lim_{z\to\infty} X(z)$

Properties of Unilateral ZT

$$x[n-1] \longleftrightarrow z^{-1}X(z) + x[-1]$$

$$x[n-2] \longleftrightarrow z^{-2}X(z) + x[-2] + x[-1]z^{-1}$$

$$\vdots$$

$$x[n-m] \longleftrightarrow z^{-m}\left[X(z) + \sum_{k=-m}^{-1} x[k]z^{-k}\right] = z^{-m}X(z) + \sum_{k=-m}^{-1} x[k]z^{-m-k}$$

Properties of Unilateral ZT

$$x[n+1] \longleftrightarrow zX(z) - zx[0]$$

$$x[n+m] \longleftrightarrow z^m \left[X(z) - \sum_{k=0}^{m-1} x[k]z^{-k}\right]$$

Some Common ZT Pairs

- Right-sided signal, ROC is $|z| > a$
  e.g. $x[n] = u[n]$
- Left-sided signal, ROC is $|z| < b$
  e.g. $x[n] = u[-n-1]$

Some Common ZT Pairs

Table 10.2

signal	z-transform	ROC
(1) $\delta[n]$	1	all z
(2) $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
(3) $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
(4) $\delta[n-m]$	z^{-m}	$z \neq 0$ (for $m > 0$) $z \neq \infty$ (for $m < 0$)
(5) $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
(6) $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$

Some Common ZT Pairs

signal	z-transform	ROC
(7) $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
(8) $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
(9) $\cos(\omega_0 n) \cdot u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
(10) $\sin(\omega_0 n) \cdot u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
(11) $r^n \cos(\omega_0 n) \cdot u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
(12) $r^n \sin(\omega_0 n) \cdot u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

LTI System and System Function

LTI system with $h[n]$, the input & output are related by

$$y[n] = h[n] * x[n]$$
$$Y(z) = H(z)X(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$H(z)$ is called the **system function** or **transfer function** of the system

Note:
(1) Eigen-function $x[n] = z^n \to y[n] = H(z)z^n$
(2) $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$ **frequency response** of the system

LTI System and System Function

Causality:

LTI system with $h[n]$: Causal $\iff$ $h[n] = 0, \forall n < 0$

$\implies$ $h[n]$ is right-sided and $H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$

LTI system with $H(z)$: Causal $\implies$ ROC is the exterior of a circle
LTI system with $H(z)$: Causal $\iff$ ROC is the exterior of a circle, including ∞

LTI system with Rational $H(z)$: Causal $\iff$ (a) ROC is the exterior of a circle outside the outermost pole & **(b)** the order of numerator $\leq$ the order of denominator in $H(z)$ expressed as a ratio of polynomials in z.

Similar results follow for anticausal systems.

Example

Example 1:

$$H(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}, \quad \rightarrow \text{non-causal}$$

Example 2:

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}, \quad |z| > 2, \quad \rightarrow \text{causal}$$

$$h[n] = \left[\left(\frac{1}{2} \right)^n + 2^n \right] u[n], \quad \rightarrow \text{causal}$$

LTI System and System Function

Stability:

LTI system with $h[n]$: Stable $\iff$ $h[n]$ absolutely summable, i.e., $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$\implies$ DTFT of $h[n]$ exists

LTI system with $H(z)$: Stable $\iff$ ROC includes unit circle ($|z| = r = 1$)

LTI System and System Function

Example:

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}$$

(1) ROC: $|z| > 2 \to$ causal, non-stable

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[n]$$

(2) ROC: $\frac{1}{2} < |z| < 2 \to$ non-causal, stable

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1]$$

(3) ROC: $|z| < \frac{1}{2} \to$ non-causal, non-stable

$$h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - 2^n u[-n-1]$$

LTI System and System Function

Inference:

LTI system with Rational $H(z)$: Causal + Stable $\iff$ all poles are within the unit circle in the z-plane

(all poles have magnitude smaller than 1)

Example 10.27

System: LTI + Causal + Stable + Rational $H(z)$.
$H(z)$ contains a pole $z = \frac{1}{2}$, a zero somewhere on unit circle, other poles and zeros are unknown.

The followings are true, false, or insufficient to determine?

(a) $F\{(\frac{1}{2})^n h[n]\}$ converges
(b) $H(e^{j\omega}) = 0$ for some ω
(c) $h[n]$ has finite duration
(d) $h[n]$ is real
(e) $g[n] = n \cdot (h[n] * h[n])$ is the impulse response of a stable system

Example 10.27

Answer:

(a) $F\{(\frac{1}{2})^n h[n]\}$ converges?

$$F\{(\frac{1}{2})^n h[n]\} = \sum_n (\frac{1}{2})^n h[n] e^{-j\omega n} = \sum_n h[n](2e^{j\omega})^{-n}$$

equivalent to ROC contains $|z| = 2$.
True since ROC contains the area exterior to the unit circle for LTI + stable + causal

Example 10.27

(b) $H(e^{j\omega}) = 0$ for some ω?
True: Since there is a zero on unit circle, implies $H(z) = 0$ for some $z = e^{j\omega}$

(c) $h[n]$ has finite duration?
False. If true, ROC includes $|z| \in (0, \infty)$, whereas $z = \frac{1}{2}$ is a pole.

(d) $h[n]$ is real?
If true, $H(z) = H(z^*)^*$. Information is not sufficient

(e) system $g[n] = n \cdot (h[n] * h[n])$ is stable?
$G(z) = -z\frac{d}{dz}(H(z) \cdot H(z))$, ROC is at least R_H (actually equals, why?), includes unit circle, thus true

Example

Consider an LTI system for which the input x[n] and output y[n] satisfy the linear constant-coefficient difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$$

LTI System Characterized by LCC Difference Eqn

General form of an Nth order difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\stackrel{\mathcal{Z}}{\longleftrightarrow} \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Longrightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Note:
(1) $H(z)$ is rational for a system by LCC difference equation
(2) Only $H(z)$ is not enough to find $h[n]$. Need extra information (like causality, stability) to find ROC and then $h[n]$

Example

Given
(1) input $x_1[n] = (\frac{1}{6})^n u[n]$, and output:

$$y_1[n] = \left[a(\frac{1}{2})^n + 10(\frac{1}{3})^n \right] u[n]$$

(2) input $x_2[n] = (-1)^n$, and output

$$y_2[n] = \frac{7}{4}(-1)^n$$

Q: Find the system function $H(z)$, system properties, and the LCC difference equation

Example

Answer:
From (1), $X_1(z) = \frac{1}{1-\frac{1}{6}z^{-1}}$, $|z| > \frac{1}{6}$

$$Y_1(z) = \frac{a}{1-\frac{1}{2}z^{-1}} + \frac{10}{1-\frac{1}{3}z^{-1}} = \frac{(a+10) - (5+\frac{a}{3})z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}, \quad |z| > 1/2$$

then

$$H(z) = \frac{Y_1(z)}{X_1(z)}$$

From (2),

$$H(-1) = \frac{7}{4} = \frac{Y_1(-1)}{X_1(-1)}$$

$$\Longrightarrow \frac{7}{4} = H(-1) = \frac{(a+10+5+\frac{a}{3}) \cdot \frac{7}{6}}{\frac{3}{2} \cdot \frac{4}{3}}$$

$$\Longrightarrow a = -9$$

Example

$$\implies H(z) = \frac{Y_1(z)}{X_1(z)} = \frac{(1-2z^{-1})(1-\frac{1}{6}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{1-\frac{13}{6}z^{-1}+\frac{1}{3}z^{-2}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}} = \frac{z^2-\frac{13}{6}z+\frac{1}{3}}{z^2-\frac{5}{6}z+\frac{1}{6}}$$

Possible ROCs for $H(z)$: $|z| > \frac{1}{2}$, $\frac{1}{3} < |z| < \frac{1}{2}$, $|z| < \frac{1}{3}$

Since ROC of $Y_1(z)$ includes ROC of $X_1(z) \cap H(z)$,
$\implies$ ROC of $H(z)$ is $|z| > \frac{1}{2}$

$\implies$ the system is stable (includes $|z| = 1$) and casual (rational and exterior to the outermost pole & order of numerator $\leq$ the order of denominator in $H(z)$)

The system can be characterized by:

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{13}{6}x[n-1] + \frac{1}{3}x[n-2]$$

LTI System Characterized by LCC Difference Eqn

Suppose a causal LTI system is described by the LCC difference equation

$$y[n] + 3y[n-1] = x[n]$$

together with the condition of initial rest.

Let the input to this system be $x(t) = \alpha u[n]$. Derive the output $y[n]$.

LTI System Characterized by LCC Difference Eqn

Suppose a LTI system is described by the LCC difference equation

$$y[n] + 3y[n-1] = x[n]$$

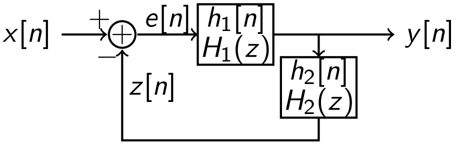
together with the initial condition $y[-1] = \beta$.

Let the input to this system be $x(t) = \alpha u[n]$. Derive the output $y[n]$.

System Functions and Block Diagram Representations

System functions for interconnections of LTI systems

- series interconnection $H(z) = H_1(z)H_2(z)$
- parallel interconnection $H(z) = H_1(z) + H_2(z)$
- feedback interconnection

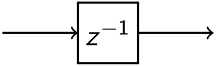


$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

System Functions and Block Diagram Representations

Block diagram for LTI characterized by LCC Difference Eqn
Three basic operations:

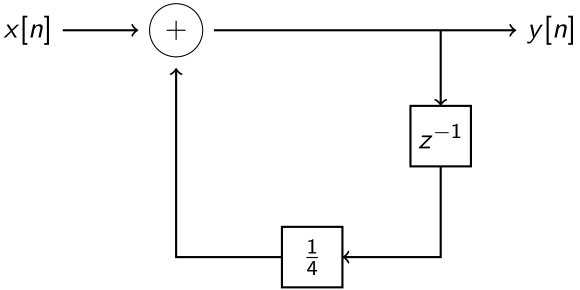
- addition
- multiplication by a coefficient
- unit delay (time-shifting $x[n] \to x[n-1]$)



Example

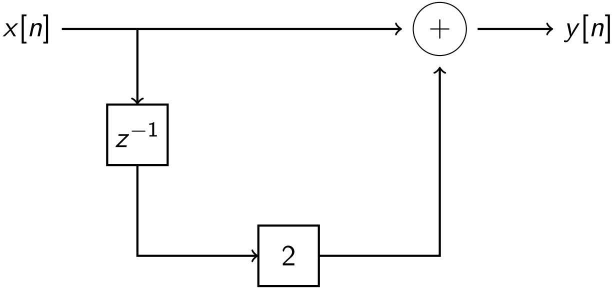
$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

$$y[n] - \frac{1}{4}y[n-1] = x[n]$$



Example

$$H(z) = 1 + 2z^{-1}$$
$$y[n] = x[n] + 2x[n-1]$$



Example

$$H(z) = \frac{1 + 2z^{-1}}{1 - \frac{1}{4}z^{-1}}$$

$$y[n] - \frac{1}{4}y[n-1] = x[n] + 2x[n-1]$$

Example

Consider the second-order system

$$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{1}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n]$$

direct form parallel form series form

Example

Consider the second-order system

$$H(z) = \frac{1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n] - \frac{7}{4}x[n-1] - \frac{1}{2}x[n-2]$$

Summary

- ZT and inverse ZT (using partial fraction expansion)
- ROC
- properties of ZT:
  linearity, time shifting, scaling in the z-domain, time reverse and expansion, conjugation, convolution, differentiation in z-domain, the initial-value theorem.
- common ZT pairs
- analysis and characterization of LTI system using ZT:
  causality, stability, LTI system characterized by LCC difference equations (to find $h[n]$ or $H(z)$)
- system function algebra and block diagram representations:
  system interconnections, block diagrams for LTI described by LCC difference equations