Numerical Optimization

Lecture 9:   Integer Programming (II)

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本节内容

建模：

◆ Uncapacitated fixed-charge location problem
◆ Capacitated fixed-charge location problem

求解：

◆ Heuristics
◆ Lagrange Relaxation
◆ Subproblem Solution
◆ Lower Bound and Upper Bound
◆ Termination

Facility Location (FL)

◆ Facilities: warehouses, retailers, or other physical facilities

◆ Determine the number and locations of facilities

◆ Extended to many other public-sectors: bus stations, fire courses, telecommunications hubs, satellite orbits, bank account, and other items…

◆ Mathematical (subproblems) formulations are common to see in many other MILP

◆ Versions of FL: uncapacitated fixed-charge location, capacitated, multi-echelon, multi-product

Uncapacitated fixed-charge location problem (UFLP)

◆ **Problem Statement**: choose facility locations in order to minimize the total cost of building the facilities and transporting goods from facilities to customers

- two echelons: facility locations (warehouses/distribution centers (DC)) to serve fixed locations (customers)
- each potential DC location has a fixed cost to open, known
- transportation cost per unit of product from a DC to a customer, known
- single product
- DCs have no capacity restrictions

◆ **Objective**: to minimize the fixed cost and transportation costs

◆ **Decision Variables:** decide which DC serves each customers

◆ **Constraints**: every customer must be served by some open DC

Formulation

◆ **Sets:**

- $I =$ set of customers

- $J =$ set of potential facility locations

◆ **Parameters**:

- $h_i =$ annual demand of customer $i \in I$

- $c_{ij} =$ cost to transport one unit of demand from facility $j \in J$ to customer $i \in I$

- $f_j =$ fixed (annual) cost to open a facility at site $j \in J$

◆ **Decision Variables:**

- $x_j = 1$ if facility j is opened, 0 otherwise

- $y_{ij} =$ the fraction of customer i's demand that is served by facility j

UFLP

$$\text{minimize} \quad \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} c_{ij} h_i y_{ij} \quad \longrightarrow \quad \text{最优值记为} z^*$$

$$\text{subject to} \quad \sum_{j \in J} y_{ij} = 1 \qquad \forall i \in I$$

assignment constraint

$$y_{ij} \leq x_j \qquad \forall i \in I, \forall j \in J$$

linking constraint

$$x_j \in \{0,1\} \qquad \forall j \in J$$

$$y_{ij} \geq 0 \qquad \forall i \in I, \forall j \in J$$

alternative of linking constraint: $\quad \displaystyle\sum_{i \in I} y_{ij} \leq |I| x_j, \quad \forall j \in J$

Solution Methods: heuristics

◆ Greedy-add

- Starting with all facilities closed and open the single facility that can serve all customers with the smallest objective
- At each iteration, open the facility that gives the largest decrease in objective
- When open one facility, assign the nearest open facility to each customer
- Stop when no facility can be opened that will decrease the objective

◆ Greedy drop

- Starting with all facilities open and close single facility gives the largest decrease in objective

Solution Methods: heuristics

◆ Improvement heuristics: "swap" or "exchange" algorithm

- Starting with a feasible solution and attempt to improve it
- At each iteration, find a pair (j, k) of facilities with j open and k closed such that if j closed and k opened, the objective would decrease
- If such a pair can be found, the swap is made and the procedure continues
- If not, attempt to open a closed facility or close a open facility to decrease the objective

◆ All heuristics are proved to perform well in practice, meaning they return good solutions and execute quickly

Solution Methods: Lagrangian Relaxation

◆ Lagrangian relaxation is a standard technique for integer programming

◆ Basic idea is to remove a set of constraints to create a problem that's easier to solve than the original

◆ To yield a lower bound on the optimal value

◆ Any feasible solutions provides an upper bound on the optimal value

Which constraint to relax?

• How easy the relaxed problem is to solve

• How tight the resulted lower bound is

• How many constraints are being relaxed

Lagrangian Relaxation (UFLP-LR$_\lambda$)

◆ Relaxing the assignment constraint

minimize
$$\sum_{j\in J} f_j x_j + \sum_{i\in I}\sum_{j\in J} c_{ij} h_i y_{ij} + \sum_{i\in I} \lambda_i \left(1 - \sum_{j\in J} y_{ij}\right)$$

$$= \sum_{j\in J} f_j x_j + \sum_{i\in I}\sum_{j\in J} (c_{ij} h_i - \lambda_i) y_{ij} + \sum_{i\in I} \lambda_i$$

subject to
$$\cancel{\sum_{j\in J} y_{ij} = 1 \qquad \forall i \in I}$$

$$y_{ij} \leq x_j \qquad \forall i \in I, \forall j \in J$$

Easy to solve!

$$x_j \in \{0,1\} \qquad \forall j \in J$$

$$y_{ij} \geq 0 \qquad \forall i \in I, \forall j \in J$$

Subproblem solution

◆ Relaxing the assignment constraint

$$\sum_{j\in J} f_j x_j + \sum_{i\in I}\sum_{j\in J} (c_{ij}h_i - \lambda_i)y_{ij} = \sum_{j\in J}\left[f_j x_j + \sum_{i\in I}(c_{ij}h_i - \lambda_i)y_{ij}\right]$$

◆ If $h_i c_{ij} - \lambda_i < 0$, set $y_{ij} = 1, \forall i \in I$, decrease in the objective

$$\beta_j := \sum_{i\in I}\min\{0, h_i c_{ij} - \lambda_i\}$$

◆ This will set $x_j = 1$, increase in the objective f_j; so would you do this?

◆ Let check the overall change:

$$\beta_j + f_j$$

Subproblem solution

◆ If there is a decrease, $\beta_j + f_j < 0$, set $x_j = 1$; otherwise, don't do this!

◆ So the subproblem solution is given by:

$$x_j = \begin{cases} 1, & \text{if } \beta_j + f_j < 0 \\ 0, & \text{otherwise} \end{cases}$$

$$y_{ij} = \begin{cases} 1, & \text{if } x_j = 1 \text{ and } h_i c_{ij} - \lambda_i < 0 \\ 0, & \text{otherwise} \end{cases}$$

◆ We will use z_{LR} to denote the optimal objective value of the Lagrangian relaxed problem. Then

$$z_{LR}(\lambda) = \sum_{j \in J} \min\{0, \beta_j + f_j\} + \sum_{i \in I} \lambda_i$$

Lower bound

◆ We have now solved the Lagrangian relaxation for given λ

◆ It turns out that for any λ, $z_{LR}(\lambda)$ is always a lower bound on the optimal value

Theorem: For any $\lambda \in \mathbb{R}^{|I|}$, $z_{LR}(\lambda) \leq z^*$.

Proof. Let (x, y) be a feasible solution for UFLP. Clearly it is feasible for the Lagrangian relaxed problem.

$$\sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} c_{ij} h_i y_{ij} + \sum_{i \in I} \lambda_i \underbrace{\left(1 - \sum_{j \in J} y_{ij} \right)}_{} = 0$$

Maximize the lower bound

◆ How do we choose the "best" λ?

$$\boxed{\text{maximize} \qquad z_{LR}(\lambda)}$$

$$\max_{\lambda} \left\{ \begin{array}{l} \textbf{minimize} \quad \displaystyle\sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} (c_{ij} h_i - \lambda_i) y_{ij} + \sum_{i \in I} \lambda_i \\[2em] \textbf{subject to} \quad \begin{array}{ll} y_{ij} \leq x_j & \forall i \in I, \ \forall j \in J \\[0.5em] x_j \in \{0,1\} & \forall j \in J \\[0.5em] y_{ij} \geq 0 & \forall i \in I, \ \forall j \in J \end{array} \end{array} \right.$$

◆ Let λ^* be the optimal multiplier. Let $z_{LR} = z_{LR}(\lambda^*)$

Which is the better bound?

◆ Let z_{LP} be the LP relaxation of UFLP; z_{LR} and z_{LP} which one is better?

Theorem: $z_{LP} \leq z_{LR}$.   This is a general result for MILP!!!

Primal MILP

$$\text{minimize} \quad cx$$
$$\text{subject to} \quad Ax = b$$
$$Dx \leq e$$
$$x \geq 0 \text{ and integer}$$

$$\text{minimize} \quad cx + \lambda(Ax - b)$$
$$\text{subject to} \quad Dx \leq e$$
$$x \geq 0 \text{ and integer}$$

Lagrangian Relaxation

$$z_{\mathrm{LR}} = \max_{\lambda} \left\{ \min_{x} \; cx + \lambda(Ax - b) \,\middle|\, Dx \le e, x \ge 0 \text{ and integer} \right\}$$

$$\ge \max_{\lambda} \left\{ \min_{x} \; cx + \lambda(Ax - b) \,\middle|\, Dx \le e, x \ge 0 \right\}$$

$$= \max_{\lambda} \left\{ \min_{x} \; (c + \lambda A)x - \lambda b \,\middle|\, Dx \le e, x \ge 0 \right\}$$

$$= \max_{\lambda} \left\{ \max_{\mu} \; \mu e - \lambda b \,\middle|\, \mu D \le c + \lambda A, \mu \le 0 \right\}$$

$$= \max_{\lambda,\mu} \left\{ \mu e - \lambda b \,\middle|\, \mu D \le c + \lambda A, \mu \le 0 \right\}$$

$$= \max_{\lambda,\mu} \left\{ \mu e - \lambda b \,\middle|\, \mu D - \lambda A \le c, \mu \le 0 \right\}$$

$$= \min_{y} \left\{ cy \,\middle|\, Ay = b, Dy \le e, y \ge 0 \right\}$$

$$= z_{\mathrm{LP}}$$

Which is the better lower bound?

◆ Generally, $z_{LP} < z^*$, so where in the gap does z_{LR} fall?

◆ An IP is said to have the *integrality property* if its LP relaxation naturally has an all-integer solution

Theorem: Let (P) be an integer program and (P-LR$_\lambda$) its Lagrangian subproblem for a given λ. If (P-LR$_\lambda$) has the integrality property for all λ, then

$$z_{LP} = z_{LR}$$

◆ We know the Lagrangian relaxation of UFLP must have integer solutions

Corollary: For the UFLP, $z_{LP} = z_{LR}$

Upper bound

- For UFLP, we have $z_{LR}(\lambda) \leq z_{LR} = z_{LP} \leq z^* \leq z(x, y)$

- How can we find a "good" upper bound? Any feasible solution yields an upper bound, but which one is good?

- Any heuristic method mentioned previously would work.

- But we would like to convert a solution to (UFLP-LR$_\lambda$)—how?

- If the solution to (UFLP-LR$_\lambda$) is feasible, then we're lucky!!

- Remember it's infeasible since the linking constraint is violated.

$$\exists i \in I, \quad \text{s.t.} \quad \sum_{j \in J} y_{ij} \neq 1$$

- This mean the i-th customer is assigned to 0 or more than 1 facility

- Remedy!

Updating the multipliers

- What makes a good value of λ_i? It should be chosen to entice

$$\sum_{j \in J} y_{ij} = 1 \qquad \forall i \in I$$

- On the objective, it appears

$$\lambda_i \left(1 - \sum_{j \in J} y_{ij} \right)$$

- If $\sum_{j \in J} y_{ij} = 0$, then λ_i is too small; it should be increased

- If $\sum_{j \in J} y_{ij} > 1$, then λ_i is too large; it should be decreased

- If $\sum_{j \in J} y_{ij} = 1$, then λ_i is just right; it should not be changed

$$\lambda_i^{n+1} \leftarrow \lambda_i^n + t^n \left(1 - \sum_{j \in J} y_{ij} \right)$$

Initialization and Termination

◆ Initialization: choose λ according to

- Set $\lambda_i = 0$, for all i
- Set it to some random number
- Set it according to some other ad-hoc rule

◆ Terminate if one of the following happens:

- The upper bound and lower bound are less than some pre-specified tolerance, say 0.1%, either in absolute or percentage terms
- A certain number of iterations, say 1200, have passed
- The displacement $|\lambda^{n+1} - \lambda^n|$ is smaller than pre-specified tolerate

Branch and Bound

- ◆ If the Lagrangian procedure stops because the 2nd or 3rd criterion, there is no guarantee that the solution found is optimal

- ◆ If we stop and accept the best feasible solution we found without a guarantee of optimality, this means Lagrangian is treated as a heuristic

- ◆ Switching to branch and bound for an accurate solution

- ◆ At each node of the branch-and-bound tree, fixing x_j = 1 or 0 for branching. Then solve a Lagrangian relaxation, instead of an LP relaxation

Relaxation for Inequalities

◆ Generally inequality or equality constraints $c(x) \leq , \geq , =$

- For $\leq$ constraints, λ is restricted to be *non-positive*.

- For $\geq$ constraints, λ is restricted to be *non-negative*.

- For $=$ constraints, λ is unrestricted in sign.

$$\lambda_i^{n+1} \leftarrow \lambda_i^n + t^n c(x^n)$$

$$\lambda_i^{n+1} \leftarrow \lambda_i^n + t^n \max\{c(x^n), 0\}$$

$$\lambda_i^{n+1} \leftarrow \lambda_i^n + t^n \min\{c(x^n), 0\}$$

How should we update the multipliers?

Alternate Relaxation

$$\text{minimize} \quad \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} h_i c_{ij} y_{ij} + \sum_{i \in I} \sum_{j \in J} \lambda_{ij} \left(x_j - y_{ij} \right)$$

$$= \sum_{j \in J} \left(\sum_{i \in I} \lambda_{ij} + f_j \right) x_j + \sum_{i \in I} \sum_{j \in J} (h_i c_{ij} - \lambda_{ij}) y_{ij}$$

$$\text{subject to} \quad \sum_{j \in J} y_{ij} = 1 \qquad \forall i \in I$$

$$x_j \in \{0, 1\} \qquad \forall j \in J$$

$$y_{ij} \geq 0 \qquad \forall i \in I, \forall j \in J$$

Alternate Relaxation

$(x\text{-problem})$ minimize $\displaystyle\sum_{j \in J}\left(\sum_{i \in I}\lambda_{ij} + f_j\right)x_j$

              subject to $x_j \in \{0, 1\} \qquad \forall j \in J$

$(y\text{-problem})$ minimize $\displaystyle\sum_{i \in I}\sum_{j \in J}(h_i c_{ij} - \lambda_{ij})y_{ij}$

              subject to $\displaystyle\sum_{j \in J} y_{ij} = 1 \qquad \forall i \in I$

                                   $y_{ij} \geq 0 \qquad \forall i \in I, \forall j \in J$

Capacitated fixed-charge location problem (CFLP)

minimize $\displaystyle\sum_{j\in J} f_j x_j + \sum_{i\in I}\sum_{j\in J} c_{ij} h_i y_{ij}$

subject to $\displaystyle\sum_{j\in J} y_{ij} = 1 \qquad \forall i \in I$

$\displaystyle\sum_{i\in I} h_i y_{ij} \le b_j \qquad \forall j \in J$

$y_{ij} \le x_j \qquad \forall i \in I, \forall j \in J$

$x_j \in \{0,1\} \qquad \forall j \in J$

$y_{ij} \ge 0 \qquad \forall i \in I, \forall j \in J$

Lagrangian Relaxation (CFLP-LR$_\lambda$)

minimize $\displaystyle\sum_{j\in J} f_j x_j + \sum_{i\in I}\sum_{j\in J} c_{ij} h_i y_{ij} + \sum_{i\in I} \lambda_i \left(1 - \sum_{j\in J} y_{ij}\right)$

$\displaystyle = \sum_{j\in J}\left[f_j x_j + \sum_{i\in I}(c_{ij}h_i - \lambda_i)y_{ij}\right] + \sum_{i\in I}\lambda_i$

subject to $\displaystyle\sum_{i\in I} h_i y_{ij} \leq b_j \qquad\qquad \forall j \in J$

$\qquad\qquad\qquad y_{ij} \leq x_j \qquad\qquad\quad \forall i \in I,\ \forall j \in J$

$\qquad\qquad\qquad x_j \in \{0,1\} \qquad\qquad \forall j \in J$

$\qquad\qquad\qquad y_{ij} \geq 0 \qquad\qquad\quad \forall i \in I,\ \forall j \in J$

Subproblem Solution

For each j, solve $\quad \beta_j = $

minimize $\quad \displaystyle\sum_{i \in I} a_i z_i$

0-1 Continuous Knapsack Problem

subject to $\quad \displaystyle\sum_{i \in I} h_i z_i \leq b$

$$0 \leq z_i \leq 1, \quad \forall i \in I$$

Here $a_i = h_i c_{ij} - \lambda_i,\; z_i = y_{ij},\; b = b_j$

Solution: following the order $\dfrac{a_1}{h_1} \leq \dfrac{a_2}{h_2} \leq \ldots \leq \dfrac{a_{|I|}}{h_{|I|}}$ to set

$z_1 = 1, z_2 = 1, \ldots$ until $z_m = 1$ such that $\displaystyle\sum_{i=1}^{m} h_i \leq b$