Numerical Optimization, 2020 Fall

Homework 4

2020 年 10 月 15 日

1. 请按要求写出下列线性规划问题对应的对偶问题.

(1) 请参考 Lecture 5 提供的原-对偶表格法, 写出如下问题对应的对偶问题并使用**图解法**求解. [15pts]

$$
\begin{aligned}
\min \quad & 12x_1 + 8x_2 + 16x_3 + 12x_4 \\
\text{s.t.} \quad & -2x_1 - x_2 - 4x_3 \le -2 \\
& -2x_1 - 2x_2 - 4x_4 \le -3 \\
& x_i \ge 0, \quad i = 1, \cdots, 4.
\end{aligned} \tag{1}
$$

(2) 请使用 Lagrange 方法写出如下问题对应的对偶问题. [10pts]

$$
\begin{aligned}
\min \quad & x_1 - x_2 \\
\text{s.t.} \quad & 2x_1 + 3x_2 - x_3 + x_4 \le 0 \\
& 3x_1 + x_2 + 4x_3 - 2x_4 \ge 3 \\
& -x_1 - x_2 + 2x_3 + x_4 = 6 \\
& x_1 \le 0 \\
& x_2, x_3 \ge 0.
\end{aligned} \tag{2}
$$

Solution:

(1). By the primal-dual relations summarized in our class, the dual is shown as the following maximization problem

$$
\begin{aligned}
\max \quad & -2\lambda_1 - 3\lambda_2 \\
\text{s.t.} \quad & -2\lambda_1 - 2\lambda_2 \le 12 \\
& -\lambda_1 - 2\lambda_2 \le 8 \\
& -4\lambda_1 \le 16 \\
& -4\lambda_2 \le 12 \\
& \lambda_1 \le 0 \\
& \lambda_2 \le 0.
\end{aligned} \tag{3}
$$

As shown in Fig. 1, we therefore have $\boldsymbol{\lambda}^* = (-4, -2)^\top$ as the optimal solution of (3).

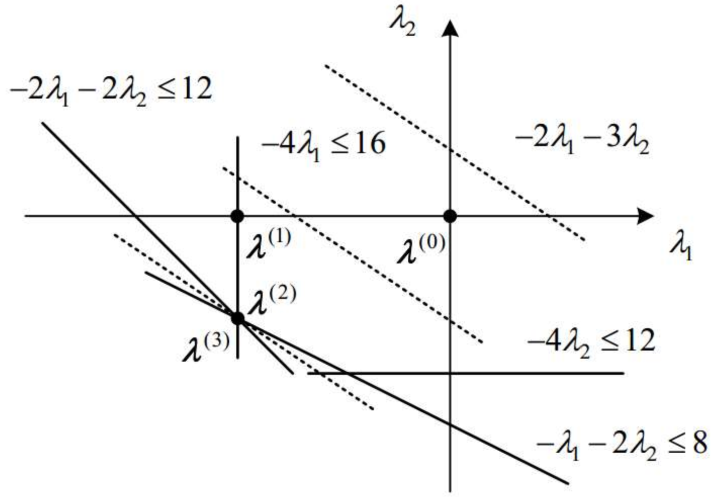


图 1: The feasible region of (3).

(2) First of all, the Lagrangian is defined as

$$\begin{aligned}\mathcal{L}(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu}) := {} & x_1 - x_2 + \lambda_1(2x_1+3x_2-x_3+x_4) - \lambda_2(3x_1+x_2+4x_3-2x_4-3) + \lambda_3(-x_1-x_2+2x_3+x_4-6) \\ & + \mu_1 x_1 - \mu_2 x_2 - \mu_3 x_3 \\ = {} & (1+2\lambda_1-3\lambda_2-\lambda_3+\mu_1)x_1 + (-1+3\lambda_1-\lambda_2-\lambda_3-\mu_2)x_2 + (-\lambda_1-4\lambda_2+2\lambda_3-\mu_3)x_3 \\ & + (\lambda_1+2\lambda_2+\lambda_3)x_4 + 3\lambda_2 - 6\lambda_3,\end{aligned} \tag{4}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^3$ and $\boldsymbol{\mu} \in \mathbb{R}^3$ are the multipliers. Then, we have the dual objective, which reads

$$g(\boldsymbol{\lambda},\boldsymbol{\mu}) = \min_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu}). \tag{5}$$

Finally, since we only have interests in the case that $g(\boldsymbol{\lambda},\boldsymbol{\mu}) > -\infty$, this implies all the coefficients in front of each primal variable should be set as 0. Hence,

$$\begin{array}{ll} \max & 3\lambda_2 - 6\lambda_3 \\ \text{s.t.} & -2\lambda_1 + 3\lambda_2 + \lambda_3 \ge 1 \\ & 3\lambda_1 - \lambda_2 - \lambda_3 \ge 1 \\ & -\lambda_1 - 4\lambda_2 + 2\lambda_3 \ge 0 \\ & \lambda_1 + 2\lambda_2 + \lambda_3 = 0 \\ & \lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \text{ free .} \end{array} \tag{6}$$

In (6), μ_1, μ_2 and μ_3 can be eliminated in the constraints of dual problem since they are all nonnegitve scalars.

2. 考虑如下的两阶段法中第一阶段的辅助问题

$$\begin{array}{ll} \min\limits_{\substack{\boldsymbol{x}\in\mathbb{R}^n \\ \boldsymbol{y}\in\mathbb{R}^m}} & \sum\limits_{i=1}^{m} y_i \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0}, \boldsymbol{y} \ge \boldsymbol{0} \end{array} \tag{7}$$

其中 $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{c} \in \mathbb{R}^n$ 和 $\boldsymbol{b} \in \mathbb{R}^m$ 给定.

(1) 写出问题 (7) 的对偶问题. [15pts]

(2) 对于上述问题 (1) 中得到的对偶问题, 请问它有最优解吗? 请给出充分的理由. [15pts]

Solution:

(1) We use the Lagrange method to write out the dual problem. First of all, the Lagrangian is defined as

$$\begin{aligned}\mathcal{L}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) : &= \boldsymbol{e}^\top \boldsymbol{y} + \boldsymbol{\alpha}^\top(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}) - \boldsymbol{\beta}^\top \boldsymbol{x} - \boldsymbol{\gamma}^\top \boldsymbol{y} \\ &= (\boldsymbol{e} - \boldsymbol{\alpha} - \boldsymbol{\gamma})^\top \boldsymbol{y} - (\boldsymbol{A}^\top \boldsymbol{\alpha} + \boldsymbol{\beta})^\top \boldsymbol{x} + \boldsymbol{\alpha}^\top \boldsymbol{b},\end{aligned} \tag{8}$$

where $\boldsymbol{e} \in \mathbb{R}^m$ denotes the vector whose elements are all 1s, $\boldsymbol{\alpha} \in \mathbb{R}^m_+$, $\boldsymbol{\beta} \in \mathbb{R}^n_+$ and $\boldsymbol{\gamma} \in \mathbb{R}^n_+$ are multipliers. Then, the dual objective is

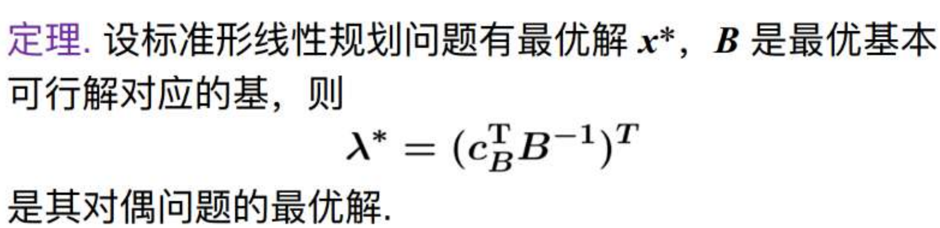
$$g(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \min_{\boldsymbol{x},\boldsymbol{y}} \mathcal{L}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \min_{\boldsymbol{x},\boldsymbol{y}}\ (\boldsymbol{e} - \boldsymbol{\alpha} - \boldsymbol{\gamma})^\top \boldsymbol{y} - (\boldsymbol{A}^\top \boldsymbol{\alpha} + \boldsymbol{\beta})^\top \boldsymbol{x} + \boldsymbol{\alpha}^\top \boldsymbol{b}. \tag{9}$$

Note that we are merely interested in the case that $g(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) > -\infty$. This means $\boldsymbol{e} - \boldsymbol{\alpha} - \boldsymbol{\gamma} = \boldsymbol{0}$ and $\boldsymbol{A}^\top \boldsymbol{\alpha} - \boldsymbol{\beta} = \boldsymbol{0}$. Therefore, we have the dual problem as follows

$$\begin{aligned} \max \quad & \boldsymbol{\alpha}^\top \boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{A}^\top \boldsymbol{\alpha} \le \boldsymbol{0} \\ & \boldsymbol{\alpha} \le \boldsymbol{e} \\ & \boldsymbol{\alpha} \text{ free }. \end{aligned} \tag{10}$$

(2) Yes, (10) has an optimal solution. It should be noticed that 0 is served as a lower bound of the primal problem (7), and hence, we must have an optimal solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ of (7), which can be obtained by applying the simplex algorithm. On the other hand, by strong duality theorem, the dual problem (10) must have an optimal solution, say $\hat{\boldsymbol{\alpha}}$. and moreover, we have $\hat{\boldsymbol{\alpha}}^\top \boldsymbol{b} = \boldsymbol{e}^\top \hat{\boldsymbol{y}}$.

3. 如图 2 示. 请解释: 为什么该定理成立? (提示: 利用强对偶定理.) [15pts]


定理. 设标准形线性规划问题有最优解 $\boldsymbol{x}^*$, $\boldsymbol{B}$ 是最优基本可行解对应的基, 则

$$\boldsymbol{\lambda}^* = (\boldsymbol{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1})^T$$

是其对偶问题的最优解.

图 2: Lecture 5 第 11 页给出的定理.

Solution: First of all, we show that λ^* is feasible to the dual problem. It holds true that

$$(\boldsymbol{\lambda}^*)^\top \boldsymbol{A} = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}[\boldsymbol{B}\ \boldsymbol{N}] = \boldsymbol{c}_B^\top(\boldsymbol{B}^{-1}[\boldsymbol{B}\ \boldsymbol{N}]) = [\boldsymbol{c}_B^\top\ \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{N}] \le [\boldsymbol{c}_B^\top\ \boldsymbol{c}_N^\top] = \boldsymbol{c}^\top, \tag{11}$$

where the inequality holds owing to the optimality condition (i.e., the reduced costs are non-negative).

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	2	$\boxed{2}$	1	1	0	4
	1	2	2	0	1	6
r^{T}	-1	-4	-3	0	0	0

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	2
	-1	0	$\boxed{1}$	-1	1	2
r^{T}	3	0	-1	2	0	8

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	$\frac{3}{2}$	1	0	1	$-\frac{1}{2}$	1
	-1	0	1	-1	1	2
r^{T}	2	0	0	1	1	10

原问题
最优解

$x_1^* = 0$
$x_2^* = 1$
$x_3^* = 2$

对偶问题
最优解

$\lambda_1^* = -1$
$\lambda_2^* = -1$

why?

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图 3: Lecture 5 第 13 页给出的单纯型表示例.

On the other hand, $(\boldsymbol{\lambda}^*)^\top \boldsymbol{b} = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{c}_B^\top \boldsymbol{x}_B^* = \boldsymbol{c}^\top \boldsymbol{x}^*$, and the last equality holds because $\boldsymbol{B}$ is the optimal basis matrix. Then, by strong duality theorem, we know λ^* is optimal for the dual problem.

4. 如图 3 示. 请解释: 如果初始单纯型表格含有单位阵, 为什么转轴完成后对应的最下方的位置是最优乘子? (注意: 回答需要针对一般的线性规划问题转轴, 不能仅仅解释给出的例子.) [15pts]

Solution: Recall the expression of reduced cost $\boldsymbol{r}^\top = \boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A}$. For those nonbasic variables, we have $\boldsymbol{r}_{\boldsymbol{I}}^\top = \boldsymbol{c}_N^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{I} = \boldsymbol{0} - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1} = -\boldsymbol{c}_B^\top \boldsymbol{B}^{-1}$, which is exactly happen to be the expression of the optimal multiplier $-\boldsymbol{\lambda}^*$. In particular, in the provided instance, we have $\boldsymbol{r}^\top = -(\lambda_1^*, \lambda_2^*)^\top = (1,1)^\top$.

5. 证明: 线性规划问题求解等价于求解一个线性可行性问题.(提示: 请参考 Lecture 5 第 14 页.) [15pts]

Solution:

证明. Consider the following pairs of primal and dual problems:

$$\begin{aligned} \min\ & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.}\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned} \tag{P}$$

$$\begin{aligned} \max\ & \boldsymbol{b}^\top \boldsymbol{y} \\ \text{s.t.}\ & \boldsymbol{A}^\top \boldsymbol{y} = \boldsymbol{c} \end{aligned} \tag{D}$$

Let $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ denote the optimal solution pair of the concerned primal-dual linear programming problems. It must satisfy the feasibility conditions of (P) and (D). On the other hand, by the strong duality theorem, we have $\boldsymbol{c}^\top \boldsymbol{x}^* = \boldsymbol{b}^\top \boldsymbol{y}^*$. Therefore, it is equivalent to solving the following linear system

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b} \\ \boldsymbol{A}\boldsymbol{y} + \boldsymbol{s} &= \boldsymbol{c} \\ \boldsymbol{x} &\geq \boldsymbol{0} \\ \boldsymbol{s} &\geq \boldsymbol{0} \\ \boldsymbol{c}^\top \boldsymbol{x} &= \boldsymbol{b}^\top \boldsymbol{y}. \end{aligned} \tag{12}$$

□