SI140 Discussion 04

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1 Monty Hall Problem & Further

Recall in the class the classical Monty Hall problem (you should have been familiar with it and able to solve and explain), the most important thing in this part is not only to remember the answer, but also to have a clear picture of the whole process and deeply understand the inside reason. First understand the scenario, then do careful calculation.

Problem 1. (**Monty Hall 1.0: the Original Monty Hall Problem**) On the game show Let's Make a Deal, hosted by Monty Hall, a contestant chooses one of three closed doors, two of which have a goat behind them and one of which has a car. Monty, who knows where the car is, then opens one of the two remaining doors. The door he opens always has a goat behind it (he never reveals the car!). If he has a choice, then he picks a door at random with equal probabilities. Monty then offers the contestant the option of switching to the other unopened door. If the contestant's goal is to get the car, should she switch doors?

Solution 1. Yes. $P(\text{winning}|\text{switching}) = \frac{2}{3}$.

Exercise 1. (**Three Prisoners Problem**) Three prisoners, A, B and C, are in separate cells and sentenced to death. The governor has selected one of them at random to be pardoned. The warden knows which one is pardoned, but is not allowed to tell. Prisoner A begs the warden to let him know the identity of one of the others who is going to be executed, "If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin to decide whether to name B or C." The warden tells A that B is to be executed. Prisoner A is pleased because he believes that his probability of surviving has gone up from $1/3$ to $1/2$, as it is now between him and C. Prisoner A secretly tells C the news, who is also pleased, because he reasons that A still has a chance of $1/3$ to be the pardoned one, but his chance has gone up to $2/3$. What is the correct answer?

Remark 1. This exercise is indeed identical to the classical Monty Hall. The answer itself does not count here. But you are expected to realize the isomorphism between the two problems and quickly determine the independence relationship in the problem. (That is, no matter what the warden tells A, no information will be provided concerning whether A will be executed, it only makes differences for B and C.)

Problem 2. (**Monty Hall 2.0: Generalization to Any Number of Doors and Cars**) On the game show Let's Make a Deal, hosted by Monty Hall, a contestant chooses *one* of d closed doors, $d-c$ of which have a goat behind them and c of which has a car. Monty, who knows where the cars are, then opens o of the $d-1$ remaining doors. The doors he opens always have a goat behind it (he never reveals the car!). If he has a choice, then he picks doors at random with equal probabilities. Monty then offers the contestant the option of switching to another unopened door. If the contestant's goal is to get a car, should she switch? What is the probability of winning by switching?

Solution 2. Yes. $P(\text{winning}|\text{switching}) = \frac{c}{d}\frac{d-1}{d-1-o}$. $\frac{c}{d}$ is the original probability that one of the remaining doors containing a car. $\frac{d-1}{d-1-o}$ is the scaling factor after Monty's revealing.

Problem 3. (**Monty Hall 3.0: Heterogeneous Doors**) Suppose the car is not placed randomly behind the three doors with equivalent probability $\frac{1}{3}$. Let us now suppose that Monty's strategies remain the same to the classical Monty Hall problem, but the car is place behind door 1 with probability p_1, behind door 2 with probability p_2, and behind door 3 with probability p_3, i.e. $p_1 + p_2 + p_3 = 1$. Without loss of generality, $p_1 \leq p_2 \leq p_3$. Now what strategy should you follow?

Solution 3. Assume that the contestant opens door i, the probability that the contestant is correct can be figured out using Bayes' theorem and the law of total probability,

$$\begin{aligned} P\left(C_i \mid M_j\right) &= \frac{P\left(C_i\right) P\left(M_j \mid C_i\right)}{P\left(M_j\right)} \\ &= \frac{p_i \cdot \frac{1}{2}}{p_i\left(\frac{1}{2}\right) + \left(1 - p_i\right)\left(\frac{1}{2}\right)} \\ &= p_i \end{aligned}$$

That is, the probability of winning if the contestant does not switch depends on the door that they initially choose, it increases from door to door, $p_1 \leq p_2 \leq p_3$ so the contestant has advantage if they choose door 3. That means that the probability of winning if the contestant switches is $1 - p_1 \geq 1 - p_2 \geq 1 - p_3$. This implies that if the contestant switches, they have better advantage of winning if they initially choose door 1 and then switch.

Suppose that the contestant chooses door 3, then Monty shows door 2, and the contestant decides to stick with door 3. Then they have a probability p_3 of winning. If the contestant had decided to switch, suppose that they have chosen door 1 initially, then they will have a probability of $1 - p_1$ of winning if they do switch. We again have to figure out the best approach for this problem. So far, we have seen that there has been an advantage to switching, so let us say that $1 - p_1 \geq p_3$. We know that

$$p_1 + p_2 + p_3 = 1$$

when $p_2 = 0$, then it is the case that $p_1 = 0$, so in this case, the probability of switching from door 1 still wins with probability 1, so there is not an advantage of switching or sticking. Now consider the case where $p_2 \neq 0$, so

$$\begin{aligned} p_1 + p_3 &< 1 \\ p_3 &< 1 - p_1 \end{aligned}$$

As we can see from the above inequality, it is strictly a greater chance of choosing door 1 and switching, than choosing door 3 and sticking.

Problem 4. (**Monty Hall 4.0: Unknown Car Location for Monty**) As before, Monty shows you three identical doors. One contains a car, the other two contain goats. You choose one of the doors but do not open it. This time, however, Monty does not know the location of the car. He randomly chooses one of the two doors different from your selection and opens it. The door turns out to conceal a goat. He now gives you the options either of sticking with your original door or switching to the other one. What should you do?

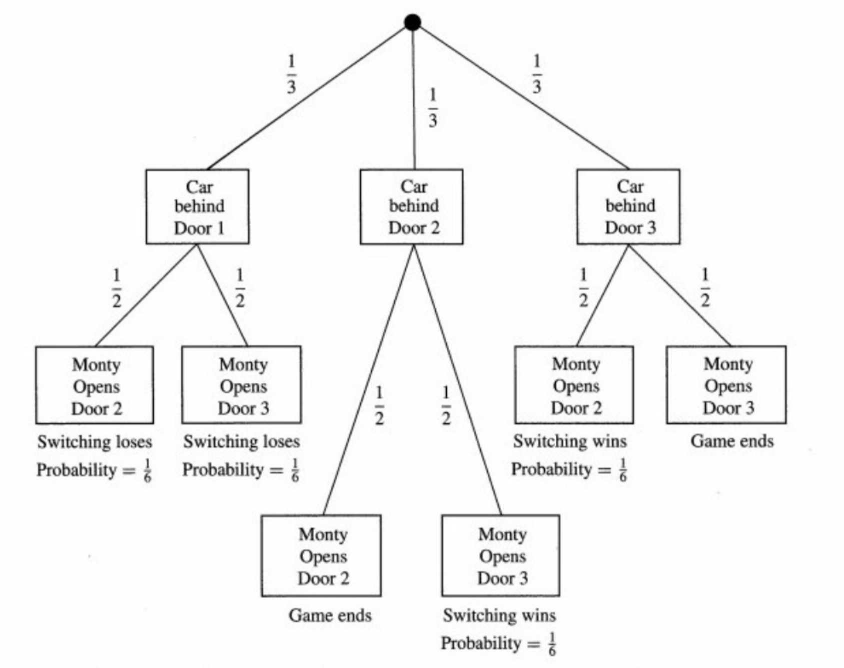

Fig. 1. Random

Solution 4. Both choices are the same. $P(\text{Winning}|\text{Switching}) = (1/6 + 1/6)/(1/6 + 1/6) = \frac{1}{2}$, as shown in figure 1.

Exercise 2. **Partially Known Car Location for Monty.** As before, you are shown three equally likely doors. You choose door one. Monty now points to door two but does not open it. Instead he merely tells you that it conceals a goat. You know that in those cases where the car really is behind door one, Monty chooses randomly between door two and door three. You also know that when the car is behind door two or door three, it is Monty's intention to identify the car's location, but that his assertions regarding the location of the car are only correct with probability p, What should you do now?

Problem 5. (**Monty Hall 5.0: Two players**) As usual, we are presented with three doors. This time, however, there is a second player in the game. Player one chooses a door, and then player two chooses a different door. If both have chosen goats, then Monty eliminates one of the players at random. If one has chosen the car, then the other player is eliminated. The surviving player knows the other has been eliminated, but does not know the reason for the elimination. After eliminating a player, Monty then opens that player's door and gives the surviving player the options of switching or sticking. What should the player do?

Solution 5. The following are the different scenarios to this version of the problem:

- Contestant 1 selects the car. Monty eliminates contestant 2. Switching loses.
- Contestant 2 selects the car. Monty eliminates contestant 1. Switching loses.

· Neither contestant selects the car. One contestant is eliminated at random. Switching wins.

We know that these different scenarios each happen $\frac{1}{3}$ of the time. Switching loses $\frac{2}{3}$ of the time. Therefore this is a case where switching is not advantageous. This makes sense because, thinking back to a single-contestant game, the only door that remained with $\frac{1}{3}$ probability of concealing the car is the door that cannot be opened by Monty. Therefore, the $\frac{2}{3}$ probability of concealing the car is for those doors that can be opened by Monty, mainly the door that the contestants choose. Therefore, the contestant will win $\frac{2}{3}$ of the time if they stick with their initial choice.