SI231 - Matrix Computations, Fall 2020-21

Homework Set #1

Name: Tao Huang **Major:** Undergraduate in CS

Student No.: 2018533172 **E-mail:** huangtao1@shanghaitech.edu.cn

I. Orthogonality

1) **Solution:**

a) Since $\mathcal{N}(A)$ is orthogonal complement to $\mathcal{R}(A^T)$, then we have $\mathcal{N}(A) \cap \mathcal{R}(A^T) = \vec{0}, \dim(\mathcal{N}(A)) + \dim(\mathcal{R}(A^T)) = n$, which is equivalent to $\mathcal{N}(A) + \mathcal{R}(A^T) = \mathbb{R}^n$. Simply by the definition of direct sum, we have

$$\mathcal{N}(A) \oplus \mathcal{R}(A^T) = \mathbb{R}^n$$

Q.E.D.

b) Let k_1, k_2 denote the number of linear independent columns ,$\vec{a}_{k_1 i}, \vec{a}_{k_2 i}$,in A and B respectively. Then for $\forall \hat{a}_i, \hat{b}_i$, they can be represented by the linear combination of $\vec{a}_{k_1 i}, \vec{a}_{k_2 i}$

$$\vec{a}_i = \sum_{j=1}^{k_1} \lambda_{ij} \vec{a}_{k_1 i}$$
$$\vec{b}_i = \sum_{j=1}^{k_2} \lambda_{ij} \vec{b}_{k_2 i}$$

For $\forall \vec{c}_i$ in matrix A+B, we can also represented $\vec{c}_i$ in the same way

$$\begin{aligned}\vec{c}_i &= \vec{a}_i + \vec{b}_i \\ &= \sum_{j=1}^{k_1} \lambda_{ij} \vec{a}_{k_1 i} + \sum_{j=1}^{k_2} \lambda_{ij} \vec{b}_{k_2 i} \\ &= \sum_{j=1}^{k_3} \gamma_{ij} c_{k_3 j}\end{aligned}$$

Denote S_1, S_2 as the column space of A, B. By the properties of subspaces

$$k_3 = \dim(S_1 \cup S_2) \leq \dim(S_1) + \dim(S_2) = rank(A) + rank(B)$$

This equality induce that

$$rank(A + B) = k_3 \leq Rank(A) + Rank(B)$$

Q.E.D.

c) First we show that rank(AB)≤rank(A). By the matrix multiplication

$$AB = [Ab_1, Ab_2, ..., Ab_p] \triangleq [c_1, ..., c_p]$$

Then we find $\forall \vec{c}_i = \sum_{j=1}^{rank(A)} \lambda_{ij}\vec{a}'_j$ is a linear combination of linear independent columns of A. Thus $\forall \vec{x} = \sum_{j}^{rank(AB)} \gamma_j \vec{c}_j$ can also be represented by the linear combination of $\vec{a}'_j s$. This implies

$$rank(AB) \leq rank(A)$$

Secondly, we campain the same induction to B, finding the similar conclusion as above

$$rank(AB) \leq rank(B)$$

So, here we have proved the first part of the proposition that

$$rank(AB) \leq \min\{rank(A), rank(B)\}$$

When rank(AB) = n

- if rank(A)=n, and rank(B)$\geq$ n. Then rank(B) must equal to n beacause rank(B)$\leq \min\{n, p\}$. This implies rank(B)=n. So A has full-column rank and B has full-row rank.
- if rank(B)=n, and rank(A)$\geq$ n. Then rank(A) must equal to n beacause rank(A)$\leq \min\{m, n\}$. This implies rank(A)=n. So A has full-column rank and B has full-row rank.

So we have proved the second part of proposition that rank(AB)=n only when A has full-column rank and B has full-row rank. Q.E.D.

d) Firstly, we show that $\mathcal{R}(A|B) \subseteq \mathcal{R}(A) + \mathcal{R}(B)$. For $\forall \vec{x} \in \mathcal{R}(A|B)$

$$\begin{aligned}
\vec{x} &= \sum_{j=1}^{n+p} \lambda_j \vec{c}_j = \sum_{j=1}^{n} \lambda_j \vec{c}_j + \sum_{j=n+1}^{n+p} \lambda_j \vec{c}_j \\
&= \sum_{j=1}^{n} \lambda_j \vec{a}_j + \sum_{j=n+1}^{n+p} \beta_j \vec{b}_j \\
&\subseteq \mathcal{R}(A) + \mathcal{R}(B)
\end{aligned}$$

Secondly, we show that $\mathcal{R}(A) + \mathcal{R}(B) \subseteq \mathcal{R}(A|B)$. For $\forall \vec{x} \in \mathcal{R}(A) + \mathcal{R}(B)$

$$\begin{aligned}
\vec{x} &= \sum_{j=1}^{n} \lambda_j \vec{a}_j + \sum_{j=n+1}^{n+p} \beta_j \vec{b}_j \\
&= \sum_{j=1}^{n+p} c_j \\
&\subseteq \mathcal{R}(A|B)
\end{aligned}$$

where $\vec{c}_j$'s are the columns of $\mathcal{R}(A|B)$. So we have proved that $\mathcal{R}(A|B) = \mathcal{R}(A) + \mathcal{R}(B)$. Q.E.D.

e) Here I give my inductions through the concepts of vector space. Denote one basis of $\mathcal{R}(A)$ as A, one of $\mathcal{R}(B)$ as B, one of $\mathcal{R}(A|B)$ as C.

$$\begin{aligned}\dim(span\{C\}) &= \dim(span\{A\} + span\{B\})(\text{Conclusion from problem 4}) \\ &= \dim(\mathcal{R}(A)) + \dim(\mathcal{R}(B)) - dim(\mathcal{R}(A) \cap \mathcal{R}(B)) \\ &= rank(A) + rank(B) - dim(\mathcal{R}(A) \cap \mathcal{R}(B) \\ \Rightarrow rank(A|B) &= rank(A) + rank(B) - dim(\mathcal{R}(A) \cap \mathcal{R}(B)\end{aligned}$$

Q.E.D.

II. UNDERSTANDING SPAN, SUBSPACE

1) **Solution:**

a) Firstly, we show that $span\{\mathcal{S}\} \subseteq \mathcal{M}$. For $\forall \vec{x} \in span\{\mathcal{S}\}$

$$\vec{x} = \sum_{i=1}^{n} \lambda_i \vec{v}_i$$

By the definition of $\mathcal{V}$ that $\mathcal{V}$ denotes the subspace containing $\mathcal{S}$, for $\forall \mathcal{V}$ we have

$$\vec{x} = \sum_{i=1}^{n} \lambda_i \vec{v}_i \in \mathcal{V}$$

Thus, for the intersection of all $\mathcal{V}$ $\mathcal{M}$ we have

$$\vec{x} \in \cap_{\mathcal{S} \subseteq \mathcal{V}} \mathcal{V} = \mathcal{M}$$

Secondly, we show that $\mathcal{M} \subseteq span\{\mathcal{S}\}$. For $\forall \vec{x} \in \mathcal{M}$, since $span\{\mathcal{S}\}$ itself is a satisfied $\mathcal{M}$, it follows that $\forall \vec{x} \in \mathcal{M} \in span\{\mathcal{S}\}$. Proved.

In conclusion, $span\{\mathcal{S}\} \subseteq \mathcal{M}, \mathcal{M} \subseteq span\{\mathcal{S}\}$ implies that $span\{\mathcal{S}\} = \mathcal{M}$. Q.E.D.

III. BASIS, DIMENSION AND PROJECTION

1) **Solution:**

a) We can represent any desired polynomials with $x^0, x^1, ..., x^n$. So the dimension of this vector space is n+1.

b) Since $a_{ij} = a_{ji}$, $\forall 1 \leq i, j \leq n$, we n=only need to determine upper or lower triangular part of a symmetric matrix. So the dimension is $1 + 2 + ... + n = \frac{n^2+n}{2}$.

2) **Solution:**

a) **Rotations.**

- All rotation matrix in $\mathbb{R}^2$

$$\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}, \quad \forall 0 \leq \theta \leq 2\pi$$

- Based on the rotations, we compute Rx directly

$$Rx = [\cos\frac{5}{6}\pi, \sin\frac{5}{6}\pi]^T = [-\frac{\sqrt{3}}{2}, \frac{1}{2}]^T$$

b) **Reflections.**

First we compute Hx

$$Hx = x - 2uu^T x$$

Secondly we show that $Qx = QHx$

$$\begin{aligned} Qx &= x - uu^T x \\ QHx &= x - 2uu^T x - uu^T x + 2uu^T x \\ &= x - uu^T x \\ &= Qx \end{aligned}$$

Finally, we show that $||x - Qx||_2 = ||Hx - QHx||_2$

$$\begin{aligned} ||x - Qx||_2 &= ||uu^T x||_2 \\ ||Hx - QHx||_2 &= ||x - 2uu^T x - x + uu^T x||_2 \\ &= ||uu^T x||_2 \\ &= ||x - Qx||_2 \end{aligned}$$

Then we conclude that Hx is a reflection of x with respect to $\mathcal{H}_u$. Q.E.D.

IV. DIRECT SUM

1) **Solution:**

a)
- Firstly, we show that $span\{B_1\} + span\{B_2\} = span\{B_1 \cup B_2\} = V$. Since $B_1 \cup B_2 = B$, we have $span\{B_1\} + span\{B_2\} = spanB = V$.
- Secondly, we show that $span\{B_1\} \cap span\{B_2\} = \vec{0}$. Since $B_1 \cap B_2$ and they are subsets of B, vectors of B_1, B_2 are linear independent of each other. Assume there exist any vector $\vec{x} \in spanB_1, x \in spanB_2$, then

$$\vec{x} = \sum_{i=1}^{m} \lambda_i \vec{b_i} = \sum_{i=1}^{n-m} \lambda_i \vec{b}_i$$

This implies that

$$\sum_{i=1}^{m} \lambda_i \vec{b_i} - \sum_{i=1}^{n-m} \lambda_i \vec{b}_i = \sum_{i=1}^{n} \lambda_i \vec{b_i} = 0$$

We determine that $\lambda_i = 0, \forall i$ according to the property of basis. This means

$$span\{B_1\} \cap span\{B_2\} = \vec{0}$$

By the definition of direct sum, $V = span\{B_1\} \oplus span\{B_2\}$. Q.E.D.

2) **Solution:**

a) Let B be a basis of V, B_1 be the basis of S. Then let $B_2 = B/B$, $\mathcal{T} = span\{B\}$. Then we have

$$B = B_1 \cup B_2, \qquad B_1 \cap B_2 = \phi$$

By the conclusion from **Problem 1**, we have $V = span\{B_1\} \oplus span\{B_2\} = S \oplus \mathcal{T}$. Q.E.D

V. UNDERSTANDING THE MATRIX NORM

1) **Solution:**

a) the matrix 1-norm

$$\begin{aligned}
\|A\|_1 &= \max_{\|x\|_1=1} \|Ax\|_1 \\
&= \max_{\|x\|_1=1} \|x_1\vec{a}_1 + ... + x_n\vec{a}_n\|_1 \\
&\leq \max_{\|x\|_1=1} (\|x_1\vec{a}_1\|_1 + ... + \|x_n\vec{a}_n\|_1) \\
&= \max_{\|x\|_1=1} (|x_1|\,\|\vec{a}_1\|_1 + ... + |x_n|\,\|\vec{a}_n\|_1) \\
&\leq \max_{\|x\|_1=1} \left[(|x_1| + ... + |x|_n) \max_j \|\vec{a}_j\|_1\right] \\
&= \max_j \|\vec{a}_j\|_1 \\
&= \max_j \sum_i^m |a_{ij}|
\end{aligned}$$

The equalities holds only when $x_{j^\star} = 1$ where $j^\star$ corresponds to the column with largest absolute column sum. So the inequalities can be rewritten as equality

$$\|A\|_1 = \max_j \sum_i^m |a_{ij}|$$

Q.E.D.

b) the matrix ∞-norm

$$\begin{aligned}
\|A\|_\infty &= \max_{\|x\|_\infty=1} \|Ax\|_\infty \\
&= \max_{\|x\|_\infty=1} \left\|[\vec{a}_1^T x, ..., \vec{a}_m^T x]^T\right\|_\infty \quad (\vec{a}_j's \text{ are row vectors}) \\
&= \max_{\|x\|_\infty=1} \max_j |\vec{a}_j^T x| \\
&\leq \max_{\|x\|_\infty=1} \max_j \left(\left\|\vec{a}_j^T\right\|_1 \|x\|_\infty\right) \quad (Holder's Inequalities) \\
&= \max_j \left\|\vec{a}_j^T\right\|_1 \\
&= \max_i \sum_j^n |a_{ij}|
\end{aligned}$$

The euqalities holds only when $x_j = 1$ if $a_{i^\star j} \geq 0$, $x_j = -1$ if $a_{i^\star j} < 0$, where $i^\star$ corresponds to the row with largest absolute row sum. So the inequalities can be rewritten as equality

$$\|A\|_\infty = \max_i \sum_j^n |a_{ij}|$$

Q.E.D.

VI. UNDERSTANDING THE HOLDER INEUQALITY

1) **Solution:**

a) Since the $\ln x$ is a concave function, by the Jensen's inequality

$$\lambda \ln \alpha + (1-\lambda) \ln \beta \leq \ln(\lambda\alpha + (1-\lambda)\beta)$$

Take the exponential of each side, we complete the proof

$$\alpha^\lambda \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta$$

Q.E.D.

b) Apply the inequality of part(a), we obtain

$$\begin{aligned}\sum_{i=1}^n |\hat{x}_i \hat{y}_i| &= \sum_{i=1}^n |\frac{x_i}{\|x\|_p} \cdot \frac{y_i}{\|y\|_q}| \\ &= \sum_{i=1}^n |\frac{(x_i^p)^{\frac{1}{p}}}{\sum_{i=1}^n (|x|^p)^{\frac{1}{p}}} \cdot \frac{(y_i^q)^{\frac{1}{q}}}{\sum_{i=1}^n (|y|^q)^{\frac{1}{q}}}| \\ &\leq \frac{1}{p} \sum_{i=1}^n \frac{|x_i|^p}{\sum_{i=1}^n |x|^p} + \frac{1}{q} \frac{|y_i|^q}{\sum_{i=1}^n |y|^q} \\ &= \frac{1}{p} \sum_{i=1}^n |\hat{x}_i|^p + \frac{1}{q} \sum_{i=1}^n |\hat{y}_i|^q \\ &= 1\end{aligned}$$

Q.E.D.

c) With the above results

$$\begin{aligned}\sum_{i=1}^n |\hat{x}_i \hat{y}_i| = \sum_{i=1}^n |\frac{(x_i^p)^{\frac{1}{p}}}{\sum_{i=1}^n (|x|^p)^{\frac{1}{p}}} \cdot \frac{(y_i^q)^{\frac{1}{q}}}{\sum_{i=1}^n (|y|^q)^{\frac{1}{q}}}| \leq 1 \\ \Leftrightarrow \frac{\sum_{i=1}^n |\hat{x}_i \hat{y}_i|}{\sum_{i=1}^n (|x|^p)^{\frac{1}{p}} \cdot \sum_{i=1}^n (|y|^q)^{\frac{1}{q}}} \leq 1 \\ \Leftrightarrow \frac{|x^T y|}{\|x\|_p, \|y\|_q} \leq 1 \\ \Leftrightarrow |x^T y| \leq \|x\|_p \|y\|_q\end{aligned}$$

Q.E.D.

d)

$$\begin{aligned}
(||x||_p)^p &= \sum_{i=1}^{n}(x_i+y_i)^p \\
&= \sum_{i=1}^{n} x_i(x_i+y_i)^{p-1} + \sum_{i=1}^{n} y_i(x_i+y_i)^{q-1} \\
&\leq (\sum_{i=1}^{n} x_i^p)^{\frac{1}{p}}(\sum_{i=1}^{n}(x_i+y_i)^{(p-1)q})^{\frac{1}{q}} + (\sum_{i=1}^{n} y_i^p)^{\frac{1}{q}}(\sum_{i=1}^{n}(x_i+y_i)^{(p-1)q})^{\frac{1}{q}} \\
&= (||x||_p + ||y||_p)(\sum_{i=1}^{n}(x_i+y_i)^p)^{1-\frac{1}{p}}
\end{aligned}$$

This implies that

$$\begin{aligned}
(\sum_{i=1}^{n}(x_i+y_i)^p)^{1-1+\frac{1}{p}} &\leq ||x||_p + ||y||_p \\
\Leftrightarrow ||x+y||_p &\leq ||x||_p + ||y||_p
\end{aligned}$$

Q.E.D.