

SI231 - Matrix Computations, Fall 2020-21

Homework Set #2

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Acknowledgements:

- 1) Deadline: **2020-10-11 23:59:00**
 - 2) Submit your homework at **Gradescope**. Entry Code: **MY3XBJ**. Also, make sure that your gradescope account is your **school e-mail**. Homework #2 contains two parts, the theoretical part and the programming part.
 - 3) About the theoretical part:
 - (a) Submit your homework in **Homework 2** in gradescope. Make sure that you have assigned the correct pages for the problems in the outline.
 - (b) Your homework should be uploaded in the **PDF** format, and the naming format of the file is not specified.
 - (c) No handwritten homework is accepted. You need to use \LaTeX . (If you have difficulty in using \LaTeX , you are allowed to use **Word** for the first and the second homework to accommodate yourself.)
 - (d) Use the given template and give your solution in English. Solution in Chinese is not allowed.
 - 4) About the programming part:
 - (a) Submit your codes in **Homework 2 Programming part** in gradescope.
 - (b) Details of requirements in programming are listed in remarks of Problem 6, please read it carefully before you start to program.
 - 5) **No late submission is allowed.**
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I. GENERAL LINEAR SYSTEM

Problem 1 (6 points + 9 points)

Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ -2 & 4 & -6 & 0 \\ 3 & 1 & 14 & -1 \\ -1 & 7 & -5 & 3 \end{bmatrix} \in \mathbb{R}^{4 \times 4}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 1 & 1 \\ 2 & 2 & 2 & -1 \\ 5 & 5 & 2 & 3 \end{bmatrix} \in \mathbb{R}^{4 \times 4}$.

- 1) For \mathbf{A} and $\mathbf{b} = (-1, 2, 5, 3)^T \in \mathbb{R}^4$, find $\mathcal{N}(\mathbf{A})$, $\mathcal{R}(\mathbf{A})$, then solve $\mathbf{Ax} = \mathbf{b}$.
- 2) For \mathbf{B} and $\mathbf{b} = (1, 1, 1, 2)^T \in \mathbb{R}^4$, solve the linear equation system $\mathbf{Bx} = \mathbf{b}$ with Gauss Elimination, LU decomposition, and LU decomposition with partial pivoting, respectively. (Although not required, you are highly encouraged to write down your solution procedures in detail.)

Solution.

- 1) Please insert your solution here...
- 2)

II. UNDERSTANDING VARIOUS MATRIX DECOMPOSITIONS

Problem 2 (10 points)

Consider the following symmetric matrix $\mathbf{A} \in \mathbb{R}^{4 \times 4}$,

$$\mathbf{A} = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Give the LU decomposition of \mathbf{A} . Then describe under which conditions \mathbf{A} is nonsingular, according to the results of LU decomposition.

Solution. Please insert your solution here ...

Problem 3 (5 points + 10 points)

- 1) Consider a
- 3×3
- matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 5 & 1 \\ 1 & 1 & 8 \end{bmatrix},$$

find the LDM (also called LDU) decomposition of \mathbf{A} , i.e., factor \mathbf{A} as $\mathbf{A} = \mathbf{LDM}^T$ (or $\mathbf{A} = \mathbf{LDU}$), where $\mathbf{L} \in \mathbb{R}^{3 \times 3}$ is lower triangular with unit diagonal entries, $\mathbf{D} \in \mathbb{R}^{3 \times 3}$ is a diagonal matrix, and $\mathbf{M} \in \mathbb{R}^{3 \times 3}$ is lower triangular with unit diagonal entries ($\mathbf{U} \in \mathbb{R}^{3 \times 3}$ is upper triangular with unit diagonal entries).

- 2) Consider a
- 3×3
- matrix

$$\mathbf{B} = \begin{bmatrix} 8 & 1 & 1 \\ 1 & 5 & 1 \\ 4 & 2 & 2 \end{bmatrix},$$

find the UL decomposition of \mathbf{B} , i.e., factor \mathbf{B} as $\mathbf{B} = \mathbf{UL}$, where $\mathbf{U} \in \mathbb{R}^{3 \times 3}$ is upper triangular with unit diagonal entries and $\mathbf{L} \in \mathbb{R}^{3 \times 3}$ is lower triangular.

Hint: $\mathbf{B} = \mathbf{PAP}$, where \mathbf{P} is a unit anti-diagonal matrix ¹.

Solution.

- 1) Please insert your solution here...
- 2)

¹**Anti-diagonal matrix:** An anti-diagonal matrix is a square matrix where all the entries are zero except those on the diagonal going from the lower left corner to the upper right corner, known as the anti-diagonal. For example,

$$\text{adiag}(a_1, \dots, a_n) = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & \cdots & a_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

and consequently, unit anti-diagonal matrix means $\text{adiag}(1, \dots, 1)$, also known as the **exchange matrix** or the **permutation matrix**.

Problem 4 (7 points + 6 points + 7 points + 5 points)

Given a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, suppose that the LDM (LDU) decomposition of \mathbf{A} exists, prove that

- 1) the LDM (LDU) decomposition of \mathbf{A} is *uniquely* determined;
- 2) if \mathbf{A} is a symmetric matrix, then its LDM (LDU) decomposition must be $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T$, which is called LDL (LDL^T) decomposition in this case;
- 3) \mathbf{A} is a symmetric and positive definite matrix if and only if its Cholesky decomposition exists (i.e., there exists a matrix $\mathbf{G} \in \mathbb{R}^{n \times n}$ such that $\mathbf{A} = \mathbf{G}\mathbf{G}^T$, where \mathbf{G} is lower triangular with *positive* diagonal entries);
- 4) if \mathbf{A} is a symmetric and positive definite matrix, then its Cholesky decomposition is *uniquely* determined.

Hints:

- 1) The existence of the LDM (LDU) decomposition implies the non-singularity of the matrix.
- 2) You can directly utilize the following lemmas,
 - the inverse (if it exists) of a lower (resp. upper) triangular matrix is also lower (resp. upper) triangular;
 - the product of two lower (resp. upper) triangular matrices is lower (resp. upper) triangular;
 - also, if such two lower (resp. upper) triangular matrices have unit diagonal entries, then their product also has unit diagonal entries.

Solution.

- 1) Please insert your solution here...
- 2)
- 3)
- 4)

Problem 5 (10 points + 5 points)

Consider matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ in the following form,

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & \dots & 0 & a_n & b_n \end{bmatrix},$$

where a_j , b_j , and c_j are non-zero entries. The matrix in such form is known as a **Tridiagonal Matrix** in the sense that it contains three diagonals.

- 1) LU decomposition is particularly efficient in the case of tridiagonal matrices. Find the LU decomposition of \mathbf{A} (derivation is expected) and try to complete the Algorithm 1.

Algorithm 1: LU decomposition for tridiagonal matrices

Input : Tridiagonal matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$.

Output: LU decomposition of \mathbf{A} .

1 Complete the algorithm here...

- 2) Consider symmetric tridiagonal matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix},$$

and give the LU decompositions and the LDL^T (also known as the LDL) decompositions of \mathbf{A} and \mathbf{B} respectively.

Solution.

- 1) Please insert your solution here...
- 2)

III. PROGRAMMING

Problem 6 (5 points + 15 points)

In this problem, we explore the efficiency of the LU method together with the classical linear system solvers we have learnt in linear algebra.

- 1) Derive the complexity of the LU decomposition. Particularly, how many flops does the LU decomposition require? The corresponding pseudo code (in **Matlab**) is provided as follows:

```

1 function [L,U]= Naive_lu(A)
2     n = size(A,1)
3     L = eye(n)
4     U = A
5     for k=1:n-1
6         for j=k+1:n
7             L(j,k)=U(j,k)/U(k,k)
8             U(j,k:n)=U(j,k:n)-L(j,k)*U(k,k:n)
9         end
10    end
11    for k=2:n
12        U(k,1:k-1)=0
13    end
14 end

```

- 2) **Programming part:** Randomly generate a non-singular matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ and a vector $\mathbf{b} \in \mathbb{R}^{n \times 1}$, then program the following methods to solve $\mathbf{Ax} = \mathbf{b}$:

- **The inverse method:** Use the inverse of \mathbf{A} to solve the problem, which can be written as,

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

- **Cramer rule:** Suppose $\mathbf{x} = [x_1, \dots, x_n]^T$, and we denote $\mathbf{A}_{-i}(\mathbf{b})$ the matrix that we replace the i -th column of \mathbf{A} with \mathbf{b} . Then we have

$$x_i = \frac{\det(\mathbf{A}_{-i}(\mathbf{b}))}{\det(\mathbf{A})}, i = 1, \dots, n.$$

- **Gauss Elimination:** We perform row operations on the augmented matrix $[\mathbf{A}|\mathbf{b}]$, and use back substitution to obtain the solution \mathbf{x} .
- **LU decomposition.** We first find the LU decomposition of \mathbf{A} , then we solve $\mathbf{Ly} = \mathbf{b}$ and $\mathbf{Ux} = \mathbf{y}$.

In your homework, you are required to submit the time-consuming plot (**one figure**) of given methods against the size of matrix \mathbf{A} (i.e., n), where $n = 100, 150, \dots, 1000$ (You can try larger n and see what will happen, but be careful with the memory use of your PC!).

Remarks: (Important!)

- Coding languages are restricted, but do not use any built-in function. For example, do not use Matlab functions such as A/b , $\text{inv}(A)$ or $\text{lu}(A)$. Otherwise, your results will contradict the complexity analysis, and your scores will be discounted. You can implement the simplest version of these methods by yourself.
- When handing in your homework in gradescope, package all your codes into `your_student_id+hw2_code.zip` and upload. In the package, you also need to include a file named `README.txt/md` to clearly identify the function of each file.
- Make sure that your codes can run and are consistent with your solutions.

Solution.

- 1) Please insert your solution here...
- 2)

IV. ROUND OFF ERROR

Problem 7 (Bonus Problem: 10 points + 8 points + 2 points)

Given a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, consider the roundoff error in the process of solving $\mathbf{Ax} = \mathbf{b}$ by Gaussian elimination in three stages:

1. Decompose \mathbf{A} into \mathbf{LU} , in a machine with roundoff error \mathbf{E} , $\bar{\mathbf{L}}$ and $\bar{\mathbf{U}}$ are computed instead, i.e.,

$$\mathbf{A} + \mathbf{E} = \bar{\mathbf{L}}\bar{\mathbf{U}}.$$

2. Solving $\mathbf{Ly} = \mathbf{b}$, numerically with roundoff error $\delta\bar{\mathbf{L}}$, $\hat{\mathbf{y}} = \mathbf{y} + \delta\mathbf{y}$ are computed instead, i.e.,

$$(\bar{\mathbf{L}} + \delta\bar{\mathbf{L}})(\mathbf{y} + \delta\mathbf{y}) = \mathbf{b}.$$

3. Solving $\mathbf{Ux} = \mathbf{y}$, numerically with roundoff error $\delta\bar{\mathbf{U}}$, $\hat{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$ are computed instead, i.e.,

$$(\bar{\mathbf{U}} + \delta\bar{\mathbf{U}})(\mathbf{x} + \delta\mathbf{x}) = \hat{\mathbf{y}}.$$

Finally, we can get the computed solution $\hat{\mathbf{x}}$ and

$$\begin{aligned} \mathbf{b} &= (\bar{\mathbf{L}} + \delta\bar{\mathbf{L}})(\bar{\mathbf{U}} + \delta\bar{\mathbf{U}})(\mathbf{x} + \delta\mathbf{x}) \\ &= (\mathbf{A} + \delta\mathbf{A})(\mathbf{x} + \delta\mathbf{x}). \end{aligned}$$

- 1) Prove that the relative error of \mathbf{x} has an upper bound as follows,

$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|}{\|\mathbf{x}\|} = \frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \frac{1}{1 - \kappa(\mathbf{A}) \frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}} \kappa(\mathbf{A}) \frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|},$$

where $\kappa(\mathbf{A}) = \|\mathbf{A}\| \|\mathbf{A}^{-1}\|$ denotes the condition number of matrix \mathbf{A} (Suppose \mathbf{A} and $\mathbf{A} + \delta\mathbf{A}$ are nonsingular and $\|\mathbf{A}^{-1}\| \|\delta\mathbf{A}\| < 1$), and $\|\cdot\|$ can be any norm.

Hint: The following equation might be useful,

$$\|(\mathbf{I} - \mathbf{B})^{-1}\| = \left\| \sum_{k=0}^{\infty} \mathbf{B}^k \right\| \leq \sum_{k=0}^{\infty} \|\mathbf{B}\|^k \leq \frac{1}{1 - \|\mathbf{B}\|}.$$

where $\mathbf{I} - \mathbf{B}$ is nonsingular and $\lim_{n \rightarrow \infty} \mathbf{B}^n = \mathbf{0}$.

- 2) Consider a linear system $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 10^{-10} & 10^{-10} \\ 1 & 10^{-10} & 10^{-10} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2(1 + 10^{-10}) \\ -10^{-10} \\ 10^{-10} \end{bmatrix}$$

find the solution \mathbf{x} , and calculate the condition number of \mathbf{A} with the matrix infinite norm², i.e. $\kappa_{\infty}(\mathbf{A}) = \|\mathbf{A}\|_{\infty} \|\mathbf{A}^{-1}\|_{\infty}$. Suppose $|\delta\mathbf{A}| < 10^{-18} |\mathbf{A}|$ ³, use $\kappa_{\infty}(\mathbf{A})$ to verify that

$$\|\delta\mathbf{x}\| < 10^{-7} \|\mathbf{x}\|.$$

²If $\mathbf{A} \in \mathbb{R}^{n \times n}$, then the matrix infinite norm is $\|\mathbf{A}\|_{\infty} = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{i,j}|$.

³ $|\mathbf{A}| \leq |\mathbf{B}|$ means each element in \mathbf{A} is relative smaller to the corresponding element of \mathbf{A} .

- 3) Discuss what you have observed from the previous 2 questions. What are the main factors that influence the relative error of the computed solution? Does the ill-conditioned matrix (i.e. the condition number is large) always lead to a large error of the solution?

Solution.

- 1) Please insert your solution here...
- 2)
- 3)