The Laplace Transform (LT)

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CTFT

(Continuous-Time) Fourier transform is extremely useful for studying signal and LTI systems. Dirichlet (sufficient) conditions for CTFT exists:

1. $x(t)$ is absolutely integrable
2. finite number of extrema ... finite interval ...
3. finite number of finite discontinuity ... finite interval ...

However, not all signals have CTFT!

Try to find a transform which is more general than CTFT, and can be applied to larger class of signals.

Laplace Transform

Eigen-function e^{st}: $H(s) \equiv \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(s)e^{st}$$

Laplace transform (LT) of $x(t)$: complex $s = \sigma + j\omega$

$$X(s) := \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{1}$$

CTFT eigen-function: $e^{j\omega t}$ ($s = j\omega$: pure imaginary)

$$\Longrightarrow \quad \begin{aligned} X(s) &= FT\{x(t)e^{-\sigma t}\}, \\ X(s)\Big|_{s=j\omega} &= FT\{x(t)\}. \end{aligned}$$

LT

Note: Definition in (1) is called Bilateral LT

Unilateral LT:

$$X(s) := \int_{0^-}^{\infty} x(t)e^{-st}dt$$

practical since usually we deal with right-sided signals

Right-sided signal: $x(t) = 0, \forall t < t_0$ for some finite t_0

Example

$x_1(t) = e^{-at}u(t)$, $a \in \mathbb{R}$

$$
\begin{aligned}
X_1(s) &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \\
&= \int_{0}^{\infty} e^{-(a+\sigma)t}e^{-j\omega t}dt \\
&= \frac{1}{a+\sigma+j\omega}, \quad a+\sigma > 0 \\
&= \frac{1}{a+s}, \quad Re(s) > -a
\end{aligned}
$$

Integral converges only when $Re(s) > -a$

Example

$x_2(t) = -e^{-at}u(-t)$, $a \in \mathbb{R}$

$$
\begin{aligned}
X_2(s) &= -\int_{-\infty}^{0} e^{-at} e^{-st} dt \\
&= -\int_{0}^{\infty} e^{(s+a)t} dt \\
&= \frac{1}{s+a}, \quad Re(s) < -a
\end{aligned}
$$

Same LT, different convergence region!

If $a \in \mathbb{C}$, then convergence region $Re(s) < Re(-a)$

Example

$$x_3(t) = e^{-2t}u(t) + e^{-t}\cos(3t)u(t)$$

Region of Convergence

Region of (conditional) Convergence (ROC): region of s for which

$$\int_{-\infty}^{\infty} x(t)e^{-st}dt \quad \text{converges}$$

Region of (absolute) Convergence (ROC): region of s for which

$$\int_{-\infty}^{\infty} |x(t)e^{-st}|dt \quad \text{converges}$$

Note for some $x(t)$, these two regions might be different.

If $x(t)e^{-\sigma t}$ satisfies the first condition in Dirichlet conditions, these two regions are identical. Usually we assume this holds.

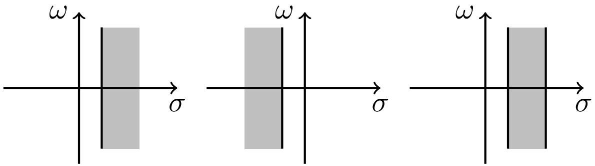
Properties of ROC

Property 1

ROC consists of strips in s-plane.

$s = \sigma + j\omega$:

$$\int_{-\infty}^{\infty} |x(t)e^{-st}|dt = \int_{-\infty}^{\infty} |x(t)|e^{-\sigma t}dt$$



The boundary $Re(s) = \sigma_0$ might be or not be in ROC

Properties of ROC

Polynomial $N(s)$: $N(s) = a_0 + a_1 s + \cdots a_n s^n$

Rational $X(s)$: ratio $N(s)/D(s)$ of two polynomials $N(s)$ and $D(s)$

Zero (for rational X): s such that $X(s) = 0$

Pole (for rational X): s such that $X(s) = \infty$

Property 2

ROC of rational X does not contain any pole.

Properties of ROC

Property 3

If $x(t)$ is of finite duration and absolutely integrable, then ROC is the entire s-plane.

$$\int_a^b |x(t)e^{-st}|dt = \int_a^b |x(t)|e^{-\sigma t}dt \leq M_{a,b,\sigma} \int_a^b |x(t)|dt,$$

$$\text{where} \quad M_{a,b,\sigma} = \max_{t\in[a,b]} e^{-\sigma t} < +\infty$$

Properties of ROC

Right-sided signal: $x(t) = 0, \forall t < t_0$ for some finite t_0

Property 4

If $x(t)$ is right-sided, and if a line $Re(s) = \sigma_0$ is in ROC, then ROC contains all s such that $Re(s) \geq \sigma_0$ (right-half plane).

for $Re(s) = \sigma \geq \sigma_0$

$$\begin{aligned}\int_{t_0}^{\infty} |x(t)e^{-st}| dt = \int_{t_0}^{\infty} |x(t)| e^{-\sigma t} dt &= \int_{t_0}^{\infty} |x(t)| e^{-\sigma_0 t} e^{-(\sigma - \sigma_0)t} dt \\ &\leq e^{-(\sigma-\sigma_0)t_0} \int_{t_0}^{\infty} |x(t)| e^{-\sigma_0 t} dt \\ &< +\infty\end{aligned}$$

Properties of ROC

Left-sided signal: $x(t) = 0, \forall t > t_0$ for some finite t_0

Similarly

Property 5

If $x(t)$ is left-sided, and if a line $Re(s) = \sigma_0$ is in ROC, then ROC contains all s such that $Re(s) \le \sigma_0$ (left-half plane).

for $Re(s) = \sigma \le \sigma_0$

$$\begin{aligned}\int_{-\infty}^{t_0} |x(t)e^{-st}|dt = \int_{-\infty}^{t_0} |x(t)|e^{-\sigma t}dt &= \int_{-\infty}^{t_0} |x(t)|e^{-\sigma_0 t}e^{-(\sigma-\sigma_0)t}dt \\ &\le e^{-(\sigma-\sigma_0)t_0}\int_{-\infty}^{t_0} |x(t)|e^{-\sigma_0 t}dt \\ &< +\infty\end{aligned}$$

Properties of ROC

Two-sided signal: of infinite extent for both $t > 0$ and $t < 0$

Property 6

If $x(t)$ is two-sided, ROC is a strip (can be empty).

$$x(t) = x_R(t) + x_L(t),$$

$$\int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x_R(t)e^{-st}dt + \int_{-\infty}^{\infty} x_L(t)e^{-st}dt,$$

$$ROC = ROC_R \bigcap ROC_L$$

If $x(t)$ is two-sided, and if a line $Re(s) = \sigma_0$ is in ROC, then ROC is a strip containing all s such that $Re(s) = \sigma_0$.

Properties of ROC

A signal must fall into one of the following (see Properties 3-6):

- of finite-length signals,
- right-sided signals,
- left-sided signals,
- two-sided signals.

Hence ROC must be a single strip:

- the entire s-plane,
- a right-half plane,
- a left-half plane,
- a single strip.

Example

$$x_4(t) = e^{-b|t|}$$

Properties of ROC

Rational $X(s)$, from Property 2, ROC does not contain any pole.

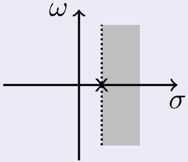
Property 7

Rational $X(s)$, ROC is bounded by poles or extends to infinity.

From Property 7 + Properties 4, 5, 6

Property 8.1

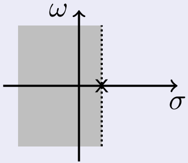
1. *If $x(t)$ right-sided and $X(s)$ rational, then ROC: the region to the right of the rightmost pole.*



Properties of ROC

Property 8.2

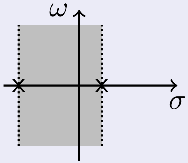
2. *If $x(t)$ left-sided and $X(s)$ rational, then ROC: the region to the left of the leftmost pole.*



Properties of ROC

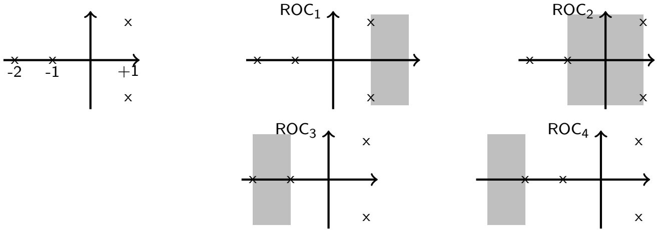
Property 8.3

3. *If $x(t)$ two-sided and $X(s)$ rational, then ROC: a strip between two consecutive poles*



ROC

Convergence Example: 4-pole rational $X(s)$ shown below, possible ROCs are:



ROC

$$X(s) = \frac{1}{s-2} + \frac{1}{s+3}$$

Given

$$ROC : -3 < Re(s) < 2$$

- Observe ROC is $\{-3 < Re(s)\} \cap \{Re(s) < 2\}$
- Therefore $x(t) = e^{-3t}u(t) - e^{2t}u(-t)$
- Q: Try inverting other two possibilities for ROC

Inverse LT

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$
$$= F\{x(t)e^{-\sigma t}\}$$

$$\implies x(t) = F^{-1}\{X(\sigma + j\omega)\} \cdot e^{\sigma t}$$
$$= e^{\sigma t} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$
$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}\frac{d(\sigma + j\omega)}{j}$$

$$\implies x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

Inverse LT

Again, this formal approach is more complex

Try to use partial-fraction expansion together with table of common functions for finding L^{-1}

Example

$X(s) = \frac{1}{(s+1)(s+2)}$

Learn Appendix A (partial-fraction expansion) by yourself in the O&W&N textbook.

Rational LT

$$X(s) = \frac{P(s)}{Q(s)}, \quad \text{simplest fraction,}$$

$$Q(s) = \prod_{i=1}^{I} (s - s_i)^{p_i}, \quad s_i\text{'s are distinct}$$

Then

$$X(s) = R(s) + \sum_{i=1}^{I} \sum_{k=1}^{p_i} \frac{C_{i,k}}{(s - s_i)^k},$$

where $R(s)$ is a polynomial of s, $deg(R) = deg(P) - deg(Q)$

Rational LT

How to find the expansion?

(1) By method of undetermined coefficients:

$$X(s) = \frac{4s}{(s+2)^2(s-4)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-4}$$

$$\begin{aligned}
s = 0, &\implies 0 = \frac{A}{2} + \frac{B}{4} - \frac{C}{4} \\
s = -1, &\implies \frac{4}{5} = A + B - \frac{C}{5} \\
s = 1, &\implies -\frac{4}{27} = \frac{A}{3} + \frac{B}{9} - \frac{C}{3} \\
&\implies A = -\frac{4}{9},\ B = \frac{4}{3},\ C = \frac{4}{9},
\end{aligned}$$

Rational LT

How to find the expansion?
(2) By limiting arguments:

$$\frac{4s}{(s+2)^2(s-4)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-4}$$

$$C = \lim_{s\to 4}(s-4)X(s) = \lim_{s\to 4}\frac{4s}{(s+2)^2} = \frac{4}{9},$$

$$B = \lim_{s\to -2}(s+2)^2X(s) = \lim_{s\to -2}\frac{4s}{s-4} = \frac{4}{3},$$

$$\text{Mth 1: } A = \lim_{s\to -2}\frac{d}{ds}(s+2)^2X(s) = \lim_{s\to -2}\frac{-16}{(s-4)^2} = -\frac{4}{9},$$

$$\text{Mth 2: } A = \lim_{s\to -2}(s+2)\left(X(s) - \frac{B}{(s+2)^2}\right) = \lim_{s\to -2}\frac{8}{3(s-4)} = -\frac{4}{9}.$$

Properties of LT

Table 9.1

Property	Signal	LT	ROC
	$x(t)$, $x_1(t)$, $x_2(t)$	$X(s)$, $X_1(s)$, $X_2(s)$	R, R_1, R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	at least $R_1 \cap R_2$
Shifting in t (Time-shifting)	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in s	$e^{s_0 t}x(t)$	$X(s - s_0)$	$R + Re(s_0)$
Time t scaling (s scaling)	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{s}{a})$	$a \cdot R$
Time t reversal (s reversal)	$x(-t)$	$X(-s)$	$-R$
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution in t (Multiplication in s)	$x_1(t) * x_2(t)$	$X_1(s) \cdot X_2(s)$	at least $R_1 \cap R_2$
Differentiation in t	$\frac{d}{dt}x(t)$	$sX(s)$	at least R
Differentiation in s	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in t	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	at least $R \cap \{Re(s) > 0\}$

Initial- and Final-Value Theorem

If $x(t) = 0$ for $t < 0$ and $x(t)$ has no impulses or higher-order singularities at $t = 0$,

$$x(0^+) = \lim_{t\to 0^+} x(t) = \lim_{s\to\infty} sX(s)$$

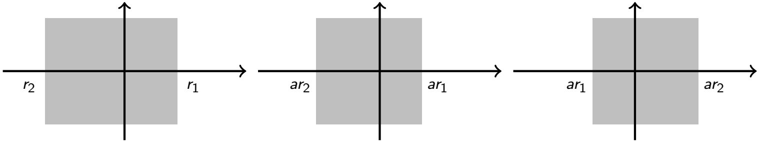
If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$,

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Properties of LT

ROC may be changed for some properties.
e.g. time scaling $x_1(t) = x(at) \longleftrightarrow \frac{1}{|a|}X(\frac{s}{a})$, ROC: $R_1 = aR$

ROC of $X(s)$ ROC of $X_1(s)$: $a = 0.8$, $a = -0.8$



Example

$$x(t) = te^{-at}u(t)$$

$$X(s) =$$

Properties of Unilateral LT

ROC for a unilateral LT must be a right-half plane. Hence, ROC is usually omitted.

$$ULT\{x(t)\} = LT\{x(t)u(t)\}$$

Example: $x(t) = e^{-a(t+1)}u(t+1)$

$LT : X(s) =$

$ULT : X(s) =$

Properties of Unilateral LT

Table 9.3

Property	Signal	LT
	$x(t),\ x_1(t),\ x_2(t)$	$X(s),\ X_1(s),\ X_2(s)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$
Shifting in s	$e^{s_0 t}x(t)$	$X(s - s_0)$
Time t scaling (s scaling)	$x(at),\ a > 0$	$\frac{1}{a}X(\frac{s}{a})$
Conjugation	$x^*(t)$	$X^*(s^*)$
Convolution in t ($x_1(t) = x_2(t) = 0 \quad \text{for} \quad t < 0$)	$x_1(t) * x_2(t)$	$X_1(s) \cdot X_2(s)$
Differentiation in t	$\frac{d}{dt}x(t)$	$sX(s) - x(0^-)$
Differentiation in s	$-tx(t)$	$\frac{d}{ds}X(s)$
Integration in t	$\int_{0^-}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$

Initial- and Final-Value Theorem

If $x(t) = 0$ for $t < 0$ and $x(t)$ has no impulses or higher-order singularities at $t = 0$,

$$x(0^+) = \lim_{t\to 0^+} x(t) = \lim_{s\to\infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$,

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Properties of Unilateral LT

$$\frac{d}{dt}x(t) \longleftrightarrow sX(s) - x(0^-)$$

$$\frac{d^n}{dt^n}x(t) \longleftrightarrow s^nX(s) - \sum_{r=0}^{n-1} s^{n-r-1}x^{(r)}(0^-)$$

Properties of Unilateral LT

Initial- and Final-Value Theorems: under proper conditions

$$x(0^+) = \lim_{s\to\infty} sX(s)$$
$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

These two theorems are useful to check whether your Unilateral LT or Inverse Unilateral LT is correct.

Example

$$\delta(t) \longleftrightarrow 1, \quad \text{all } s$$

$$\sin(\omega_0 t)u(t) \longleftrightarrow \frac{\omega_0}{s^2+\omega_0^2}, \quad Re\{s\} > 0$$

$$e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}, \quad Re\{s\} > -a$$

Example

E.g. for the initial and final-value theorem

$$x(t) = e^{-2t}u(t) + e^{-t}\cos(3t)u(t)$$

$$L(x(t)) = X(s) = \frac{2s^2 + 5s + 12}{(s^2 + 2s + 10)(s + 2)}, \quad Rs\{s\} > -1$$

$$x(0^+) = 2; \quad \lim_{s \to \infty} sX(s) = 2$$

$$\lim_{t \to \infty} x(t) = 0; \quad \lim_{s \to 0} sX(s) = 0$$

Properties of LT

Multiplication in $t \to$ convolution in s:

$$x_1(t)x_2(t) \longleftrightarrow \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X_1(r)X_2(s-r)dr$$

$$\begin{aligned}
\int_{-\infty}^{\infty} x_1(t)x_2(t)e^{-st}dt &= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X_1(r)e^{rt}dr\right)x_2(t)e^{-st}dt \\
&= \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X_1(r)\left(\int_{-\infty}^{\infty} x_2(t)e^{-(s-r)t}dt\right)dr \\
&= \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X_1(r)X_2(s-r)dr
\end{aligned}$$

Integration in s:

$$\frac{1}{t}x(t) \longleftrightarrow \int_{s}^{\infty} X(s)ds$$

Some LT Pairs

Table 9.2

	signal	LT	ROC
(1)	$\delta(t)$	1	all s
(2)	$u(t)$	s^{-1}	$Re(s) > 0$
(3)	$-u(-t)$	s^{-1}	$Re(s) < 0$
(4)	$\frac{t^{n-1}}{(n-1)!}u(t)$	s^{-n}	$Re(s) > 0$
(5)	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	s^{-n}	$Re(s) < 0$
(6)	$e^{-at}u(t)$	$(s+a)^{-1}$	$Re(s+a) > 0$
(7)	$-e^{-at}u(-t)$	$(s+a)^{-1}$	$Re(s+a) < 0$
(8)	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$(s+a)^{-n}$	$Re(s+a) > 0$

Some LT Pairs

	signal	LT	ROC
(9)	$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$(s+a)^{-n}$	$Re(s+a)<0$
(10)	$\delta(t-T)$	e^{-sT}	All s
(11)	$\cos(\omega_0 t)u(t)$	$s(s^2+\omega_0^2)^{-1}$	$Re(s)>0$
(12)	$\sin(\omega_0 t)u(t)$	$\omega_0(s^2+\omega_0^2)^{-1}$	$Re(s)>0$
(13)	$\cos(\omega_0 t)e^{-at}u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$Re(s+a)>0$
(14)	$\sin(\omega_0 t)e^{-at}u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$Re(s+a)>0$
(15)	$u_n(t)=\frac{d^n}{dt^n}\delta(t)$	s^n	All s
(16)	$u_{-n}(t)=\underbrace{u(t)*\cdots*u(t)}_{n \text{ times}}$	s^{-n}	$Re(s)>0$

LTI System and System Function

For an LTI system, $y(t) = h(t) * x(t)$.

For an LTI system, $Y(s) = H(s) \cdot X(s)$.

The Laplace transform $H(s)$ is commonly referred to as the system function or the transfer function.

LTI System and System Function

Causality:

LTI system: Causal $h(t) = 0,\ t < 0$

LTI system with $H(s)$: Causal $\Rightarrow$ ROC of is a right-half plane.

Note: the converse statement is not true

e.g. $h(t) = e^{-(t+1)}u(t+1) \longleftrightarrow H(s) = \frac{e^s}{s+1},\ Re(s) > -1$, non-causal

LTI System and System Function

Causality:
e.g.

$$h(t) = e^{-t}u(t)$$
$$h(t) = e^{-|t|}$$

LTI system with Rational $H(s)$: Causal $\Leftrightarrow$ ROC to the right of the rightmost pole

Similar results follow for anticausal systems.

LTI System and System Function

Stability:

LTI system with $H(s)$: stable $\Leftrightarrow$ ROC includes $j\omega$-axis ($Re(s) = 0$).

Proof: stable, BIBO

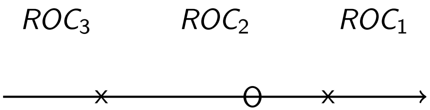
$$
\begin{aligned}
|y(t)| &= \left| \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \right| \quad \text{bounded for all bounded } x \\
&\leq \int_{-\infty}^{+\infty} |h(\tau)x(t-\tau)| \, d\tau \quad \text{for } |x(t)| < B \\
&\leq B \int_{-\infty}^{+\infty} |h(\tau)| \, d\tau
\end{aligned}
$$

$$
|y(t)| \implies \int_{-\infty}^{+\infty} |h(\tau)| d\tau \quad \text{exists}
$$

the other direction is trivial

Example

$$H(s) = \frac{s-1}{(s+1)(s-2)}$$



ROC_1: causal, not stable

ROC_2: not causal, stable

ROC_3: not causal, not stable

LTI System and System Function

LTI system with Rational $H(s)$: causal and stable $\Leftrightarrow$ all poles lie in the left-half of the s-plane

(all poles have negative real parts)

e.g.

$$h(t) = e^{-t}u(t)$$
$$h(t) = e^{2t}u(t)$$
$$h(t) = e^{-|t|}$$

Example

LTI + Stable + Causal system with impulse response $h(t)$ and system function $H(s)$. Suppose $H(s)$ is rational, contain a pole at $s = -2$, and does not have a zero at the origin. The location of all other poles and zeros is unknown.Determine whether each of the following statements is true, false, or insufficient to determine.

(a) $FT\{h(t)e^{3t}\}$ converges

(b) $\int_{-\infty}^{\infty} h(t)dt = 0$

(c) $t \cdot h(t)$ is the impulse response of a causal and stable system.

(d) $dh(t)/dt$ contains at least one pole in its LT.

(e) $h(t)$ has finite duration

(f) $H(s) = H(-s)$

(g) $\lim_{s \to \infty} H(s) = 2$

Example

Answer:

(a) False, $FT\{h(t)e^{3t}\} = H(s)|_{s=-3}$. But $s = -3$ is not in the ROC as ...
(b) False. The integration $= H(0) = 0$. But $H(s)$ does not have a zero at origin.
(c) True. $LT\{t \cdot h(t)\}$ has a ROC the same as that of $H(s)$. As $H(s)$'s ROC includes $j\omega$-axis (why?), the corresponding system is also stable. Since $h(t) = 0$ for $t < 0$ (why?), $th(t) = 0$ for $t < 0$. So the system is also causal.
(d) True. LT of $dh(t)/dt$ is $sH(s)$. So the original pole of $H(s)$ at $s = -2$ will not be cancelled by the multiplication of s. $\Longrightarrow$ $H(s)$ also has a pole at $s = -2$.
(f) False. ROC of the Laplace transform for a finite duration signal spans the whole s-plane.
(f) False. It implies $s = 2$ is also a pole. Then, $j\omega$-axis is not in the ROC (why?) and $H(s)$ cannot be a stable system.
(g) It cannot be ascertained. We need to know the order of the numerator and denominator of $H(s)$

Example

Consider an LTI system for which the input $x(t)$ and output $y(t)$ satisfy the linear constant-coefficient differential equation

$$\frac{d}{dt}y(t) + 3y(t) = x(t)$$

LTI System Characterized by LCC Differential Eqn

LTI system characterized by Linear Constant-Coefficient (LCC) Differential Equation:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

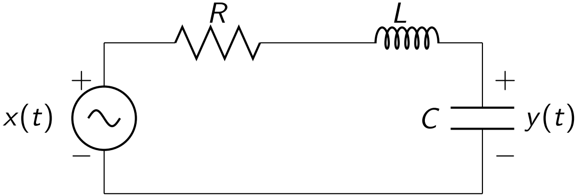
$$\xleftrightarrow{LT} \sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$\implies H(s) = \frac{\sum b_k s^k}{\sum a_k s^k}$$

$\implies$ $H(s)$ is rational for a system by LCC Differential Equation

Example

Voltage drops at input and output are $x(t)$ and $y(t)$ respectively



$$v_L = L\frac{d}{dt}i_L, \qquad i_C = C\frac{d}{dt}v_C$$

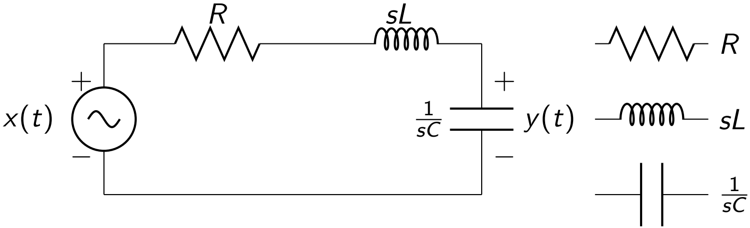
Kirchhoff's voltage law

$$x(t) = R \cdot C\frac{dy(t)}{dt} + L \cdot \frac{d}{dt}\left\{C\frac{dy(t)}{dt}\right\} + y(t) \implies x(t) = RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t)$$

$$X(s) = RCsY(s) + LCs^2Y(s) + Y(s) \implies H(s) = \frac{1/(LC)}{s^2 + (R/L)s + 1/(LC)}$$

Example

Other approach: using the impedance of R, L, and C



$$v_R(t) = Ri_R(t), \qquad v_L(t) = L\frac{d}{dt}i_L(t), \qquad i_C(t) = C\frac{d}{dt}v_C(t)$$

$$V_R(s) = RI_R(s), \qquad V_L(s) = sLI_L(s), \qquad V_C(s) = \frac{1}{sC}I_C(s)$$

$$\implies Y(s) = \frac{1/(sC)}{R + sL + 1/(sC)}X(s) \implies H(s) = \frac{1/(LC)}{s^2 + (R/L)s + 1/(LC)}$$

LTI System Characterized by LCC Differential Eqn

Note:
Only LCC Differential Equation is not complete to specify an LTI system.
Need extra information like causality, stability to find the ROC and consequently the impulse response.

Example

Suppose a causal LTI system is described by the LCC differential equation

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t)$$

together with the condition of initial rest.

Let the input to this system be $x(t) = \alpha u(t)$. Derive the output $y(t)$.

Example

Suppose a LTI system is described by the LCC differential equation

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t)$$

together with initial conditions $y(0^-) = \beta$ and $y'(0^-) = \gamma$.

Let the input to this system be $x(t) = \alpha u(t)$. Derive the output $y(t)$.

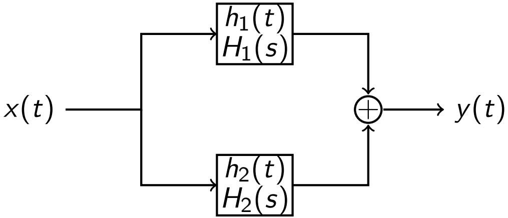
System Functions and Block Diagram Representations

- Parallel Interconnection:
  Consider the parallel interconnection of two systems

$$h(t) = h_1(t) + h_2(t)$$

Then from the linearity of LT,

$$H(s) = H_1(s) + H_2(s)$$



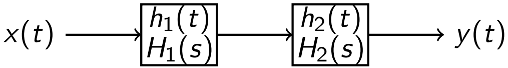
System Functions and Block Diagram Representations

- Series Interconnection:
  Similarly, the impulse response of the series interconnection of two systems is

$$h(t) = h_1(t) * h_2(t)$$

The resultant system function is then

$$H(s) = H_1(s)H_2(s)$$



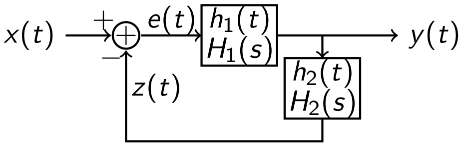
System Functions and Block Diagram Representations

- Feedback Inerconnection:
  The feedback interconnection of two systems is given as follows:

$$y(t) = h_1(t) * e(t), \quad e(t) = x(t) - z(t), \quad z(t) = h_2(t) * y(t),$$
$$Y(s) = H_1(s)E(s), \quad E(s) = X(s) - Z(s), \quad Z(s) = H_2(s)Y(s),$$

The resultant system function is then

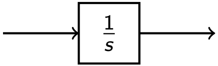
$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$



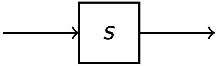
System Functions and Block Diagram Representations

Block Diagram Representation for Causal LTI System Described by Differential Equations and Rational System Function

- Integration:



- Differentiation:



Example

Consider a causal LTI system with system function:

$$H(s) = \frac{1}{s+3}$$

The input $x(t)$ and output $y(t)$ satisfy the following differential equation

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

Example

Consider a causal LTI system with system function:

$$H(s) = \frac{s+2}{s+3}$$

The input $x(t)$ and output $y(t)$ satisfy the following differential equation

$$\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Example

Consider a causal second-order system with system function:

$$H(s) = \frac{1}{s^2 + 3s + 2}$$

The input $x(t)$ and output $y(t)$ satisfy the following differential equation

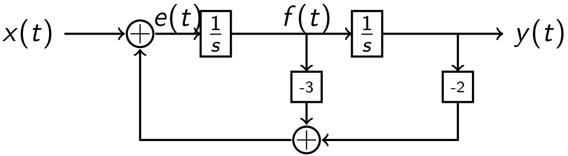
$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

Example

- Direct Form:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t), \quad f(t) = \frac{dy(t)}{dt}, \quad e(t) = \frac{df(t)}{dt} = \frac{d^2y(t)}{dt^2}$$

$$e(t) = -3f(t) - 2y(t) + x(t)$$

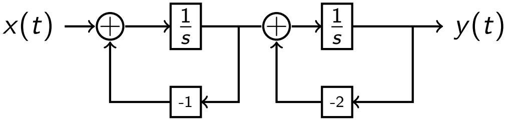


$$H(s) = \frac{1}{s^2 + 3s + 2}$$

Example

- Series Form:

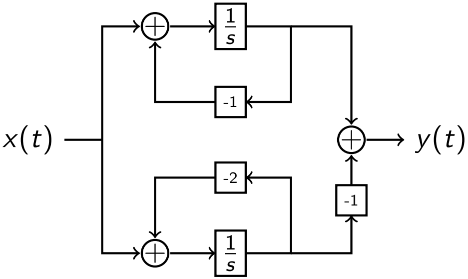
$$H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} \cdot \frac{1}{s+2}$$



Example

- Parallel Form:

$$H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2}$$



Example

Consider a causal second-order system with system function:

$$H(s) = \frac{2s^2 + 4s - 6}{s^2 + 3s + 2}$$

The input $x(t)$ and output $y(t)$ satisfy the following differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 2\frac{d^2x(t)}{dt^2} + 4\frac{dx(t)}{dt} - 6x(t)$$