SI231 Matrix Computations
Lecture 9: Kronecker Product and Hadamard Product

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Lecture 9: Kronecker Product and Hadamard Product

- Kronecker product and properties
- vectorization
- Kronecker sum
- Khatri-Rao product and properties
- Hadamard product and properties

Motivating Problem: Matrix Equations

- Problem: given $\mathbf{A}$, $\mathbf{B}$, find an $\mathbf{X}$ such that

$$\mathbf{AX} = \mathbf{B}. \tag{$*$}$$

  - an easy problem; if $\mathbf{A}$ has full column rank and $(*)$ has a solution, the solution is merely $\mathbf{X} = \mathbf{A}^{\dagger}\mathbf{B}$.

- Question: but how about matrix equations like
  - $\mathbf{AX} + \mathbf{XB} = \mathbf{C}$,
  - $\mathbf{A}_1\mathbf{X}\mathbf{B}_1 + \mathbf{A}_2\mathbf{X}\mathbf{B}_2 = \mathbf{C}$,
  - $\mathbf{AX} + \mathbf{YB} = \mathbf{C}$, $\mathbf{X}, \mathbf{Y}$ both being unknown?

- such matrix equations can be tackled via matrix tools arising from the Kronecker product

Kronecker Product

The Kronecker product of $\mathbf{A} \in \mathbb{R}^{m_1 \times n_1}$ and $\mathbf{B} \in \mathbb{R}^{m_2 \times n_2}$ is defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & & a_{2n}\mathbf{B} \\ \vdots & & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix}.$$

- Kronecker product results in a block matrix.
- entries of $\mathbf{A} \otimes \mathbf{B}$ contain all possible products of entries in $\mathbf{A}$ with entries in $\mathbf{B}$
- Example: let $\mathbf{a} \in \mathbb{R}^{m_1}$, $\mathbf{b} \in \mathbb{R}^{m_2}$. By definition,

$$\mathbf{a} \otimes \mathbf{b} = \begin{bmatrix} a_1\mathbf{b} \\ a_2\mathbf{b} \\ \vdots \\ a_m\mathbf{b} \end{bmatrix}$$

- The outer product can be represented in Kronecker product

$$\mathbf{b}\mathbf{a}^T = [\ a_1\mathbf{b}, a_2\mathbf{b}, \dots, a_m\mathbf{b}\ ] = \mathbf{b} \otimes \mathbf{a}^T = \mathbf{a}^T \otimes \mathbf{b}.$$

- $\mathbf{a} \otimes \mathbf{b}$ is a column-by-column concatenation of the outer product $\mathbf{b}\mathbf{a}^T$.

Properties

Elementary properties:

1. $\mathbf{A} \otimes (\alpha \mathbf{B}) = (\alpha \mathbf{A}) \otimes \mathbf{B} = \alpha(\mathbf{A} \otimes \mathbf{B})$.
2. $(\mathbf{A}+\mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}$, $\quad \mathbf{A} \otimes (\mathbf{B}+\mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}$ (distributive)
3. $\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}$ (associativity).
4. $\mathbf{A} \otimes \mathbf{0} = \mathbf{0} \otimes \mathbf{A} = \mathbf{0}$.
5. $\mathbf{0}_{mn} = \mathbf{0}_m \otimes \mathbf{0}_n$, $\mathbf{I}_{mn} = \mathbf{I}_m \otimes \mathbf{I}_n$; $\mathbf{0}_n$ and $\mathbf{I}_n$ are $n \times n$ zero and identity matrices.
6. $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$, $(\mathbf{A} \otimes \mathbf{B})^H = \mathbf{A}^H \otimes \mathbf{B}^H$, $(\mathbf{A} \otimes \mathbf{B})^* = \mathbf{A}^* \otimes \mathbf{B}^*$.
7. (permutation equivalence) there exist perfect shuffle permutation matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ such that
$$\mathbf{P}_1(\mathbf{A} \otimes \mathbf{B})\mathbf{P}_2 = \mathbf{B} \otimes \mathbf{A}.$$

Note: Kronecker product is not commutative; i.e., $\mathbf{A} \otimes \mathbf{B} \neq \mathbf{B} \otimes \mathbf{A}$ in general. Property 6 above is a weak version of commutativity. If $\mathbf{A}$ and $\mathbf{B}$ are square, $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{B} \otimes \mathbf{A}$ are permutation similar, meaning that we can take $\mathbf{P}_1 = \mathbf{P}_2^T$.

More Properties

Property 9.1 (mixed product rule)**.**

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}),$$

for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ of appropriate matrix dimensions. It mixes the ordinary matrix product and the Kronecker product.

Some properties from Property 9.1:

1. if $\mathbf{A} \in \mathbb{R}^{m \times m}$ and $\mathbf{B} \in \mathbb{R}^{n \times n}$ are nonsingular, then
$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$$
   – proof: $(\mathbf{A}^{-1} \otimes \mathbf{B}^{-1})(\mathbf{A} \otimes \mathbf{B}) = (\mathbf{A}^{-1}\mathbf{A}) \otimes (\mathbf{B}^{-1}\mathbf{B}) = \mathbf{I}_m \otimes \mathbf{I}_n = \mathbf{I}_{mn}$.
   – It also holds by replacing the inverse by the pseudoinverse.

2. $\mathbf{A} \otimes \mathbf{B} = (\mathbf{I}_{m_1} \otimes \mathbf{B})(\mathbf{A} \otimes \mathbf{I}_{n_2}) = (\mathbf{A} \otimes \mathbf{I}_{m_2})(\mathbf{I}_{n_1} \otimes \mathbf{B})$

3. if $\mathbf{Q}_1$, $\mathbf{Q}_2$ are semi-orthogonal, then $\mathbf{Q}_1 \otimes \mathbf{Q}_2$ is semi-orthogonal.
   – proof: $(\mathbf{Q}_1 \otimes \mathbf{Q}_2)^T(\mathbf{Q}_1 \otimes \mathbf{Q}_2) = (\mathbf{Q}_1^T \otimes \mathbf{Q}_2^T)(\mathbf{Q}_1 \otimes \mathbf{Q}_2) = (\mathbf{Q}_1^T\mathbf{Q}_1) \otimes (\mathbf{Q}_1^T\mathbf{Q}_1) = \mathbf{I}$.

Example: Hadamard Matrix

Consider an 2×2 orthogonal matrix

$$\mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

From $\mathbf{H}_2$, construct a 4×4 matrix

$$\mathbf{H}_4 = \mathbf{H}_2 \otimes \mathbf{H}_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

and inductively, $\mathbf{H}_n = \mathbf{H}_{n/2} \otimes \mathbf{H}_{n/2}$ for any n that is a power of 2.

- is $\mathbf{H}_4$ orthogonal? Yes, because $\mathbf{H}_4\mathbf{H}_4^T = (\mathbf{H}_2 \otimes \mathbf{H}_2)(\mathbf{H}_2^T \otimes \mathbf{H}_2^T) = (\mathbf{H}_2\mathbf{H}_2^T \otimes \mathbf{H}_2\mathbf{H}_2^T) = \mathbf{I}$.
- for the same reason, any $\mathbf{H}_n$ is orthogonal

Kronecker Product and Eigenvalues

There is a direct correspondence between the eigen-equations of $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{A}$, $\mathbf{B}$.

Theorem 9.1. Let $\mathbf{A} \in \mathbb{R}^{m \times m}$, $\mathbf{B} \in \mathbb{R}^{n \times n}$. Let $\{\lambda_i, \mathbf{x}_i\}_{i=1}^m$ be the set of m eigen-pairs of $\mathbf{A}$, and let $\{\mu_i, \mathbf{y}_i\}_{i=1}^n$ be the set of n eigen-pairs of $\mathbf{B}$. The set of mn eigen-pairs of $\mathbf{A} \otimes \mathbf{B}$ is given by

$$\{\lambda_i \mu_j, \mathbf{x}_i \otimes \mathbf{y}_j\}_{i=1,\ldots,m,\ j=1,\ldots,n}$$

Properties arising from Theorem 9.1 (for square $\mathbf{A}, \mathbf{B}$):

1. $\det(\mathbf{A} \otimes \mathbf{B}) = [\det(\mathbf{A})]^n [\det(\mathbf{B})]^m = \det(\mathbf{B} \otimes \mathbf{A})$.
   - $\mathbf{A} \otimes \mathbf{B}$ is nonsingular if and only if both $\mathbf{A}$ and $\mathbf{B}$ are nonsingular.
2. $\mathrm{tr}(\mathbf{A} \otimes \mathbf{B}) = \mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) = \mathrm{tr}(\mathbf{B} \otimes \mathbf{A})$.
   - as a consequence, $\|\mathbf{A} \otimes \mathbf{B}\|_F = \|\mathbf{A}\|_F \|\mathbf{B}\|_F$.
3. if $\mathbf{A}$ and $\mathbf{B}$ are PSD (PD), then $\mathbf{A} \otimes \mathbf{B}$ is PSD (PD).

Kronecker Product and Matrix Decompositions

Property 9.2. Kronecker product of two upper (lower) triangular matrices is again upper (lower) triangular.

GIven $\mathbf{A} \in \mathbb{R}^{m_1 \times m_1}$ and $\mathbf{B} \in \mathbb{R}^{m_2 \times m_2}$

1. if $\mathbf{A}, \mathbf{B}$ are nonsingular with LU factorizations with partial pivoting given by $\mathbf{P}_A \mathbf{A} = \mathbf{L}_A \mathbf{U}_A$ and $\mathbf{P}_B \mathbf{B} = \mathbf{L}_B \mathbf{U}_B$, respectively, then
$$\mathbf{A} \otimes \mathbf{B} = (\mathbf{P}_A^T \mathbf{L}_A \mathbf{U}_A) \otimes (\mathbf{P}_B^T \mathbf{L}_B \mathbf{U}_B) = (\mathbf{P}_A \otimes \mathbf{P}_B)^T (\mathbf{L}_A \otimes \mathbf{L}_B)(\mathbf{U}_A \otimes \mathbf{U}_B)$$

2. if $\mathbf{A}, \mathbf{B}$ are positive (semi)definite with Cholesky factorizations given by $\mathbf{A} = \mathbf{G}_A \mathbf{G}_A^T$ and $\mathbf{B} = \mathbf{G}_B \mathbf{G}_B^T$, respectively, then
$$\mathbf{A} \otimes \mathbf{B} = (\mathbf{G}_A \mathbf{G}_A^T) \otimes (\mathbf{G}_B \mathbf{G}_B^T) = (\mathbf{G}_A \otimes \mathbf{G}_B)(\mathbf{G}_A \otimes \mathbf{G}_B)^T$$

3. if $\mathbf{A}, \mathbf{B}$ have Schur factorizations given by $\mathbf{A} = \mathbf{U}_A \mathbf{T}_A \mathbf{U}_A^H$ and $\mathbf{B} = \mathbf{U}_B \mathbf{T}_B \mathbf{U}_B^H$, respectively, then
$$\mathbf{A} \otimes \mathbf{B} = (\mathbf{U}_A \mathbf{T}_A \mathbf{U}_A^H) \otimes (\mathbf{U}_B \mathbf{T}_B \mathbf{U}_B^H) = (\mathbf{U}_A \otimes \mathbf{U}_B)(\mathbf{T}_A \otimes \mathbf{T}_B)(\mathbf{U}_A \otimes \mathbf{U}_B)^H$$
(This result can be used to prove the eigenvalue properties in the last slides.)

Kronecker Product and Matrix Decompositions

GIven $\mathbf{A} \in \mathbb{R}^{m_1 \times n_1}$ and $\mathbf{B} \in \mathbb{R}^{m_2 \times n_2}$

1. if $\mathbf{A}, \mathbf{B}$ ($m_1 \geq n_1$ and $m_2 \geq n_2$) are full rank with QR factorizations given by $\mathbf{A} = \mathbf{Q}_A \mathbf{R}_A$ and $\mathbf{B} = \mathbf{Q}_B \mathbf{R}_B$, respectively, then

$$\mathbf{A} \otimes \mathbf{B} = (\mathbf{Q}_A \mathbf{R}_A) \otimes (\mathbf{Q}_B \mathbf{R}_B) = (\mathbf{Q}_A \otimes \mathbf{Q}_B)(\mathbf{R}_A \otimes \mathbf{R}_B)$$

2. if $\mathbf{A}, \mathbf{B}$ have rank r_A and r_B, respectively and have SVDs given by $\mathbf{A} = \mathbf{U}_A \mathbf{\Sigma}_A \mathbf{V}_A^T$ and $\mathbf{B} = \mathbf{U}_B \mathbf{\Sigma}_B \mathbf{V}_B^T$, respectively, then

$$\mathbf{A} \otimes \mathbf{B} = (\mathbf{U}_A \mathbf{\Sigma}_A \mathbf{V}_A^T) \otimes (\mathbf{U}_B \mathbf{\Sigma}_B \mathbf{V}_B^T) = (\mathbf{U}_A \otimes \mathbf{U}_B)(\mathbf{\Sigma}_A \otimes \mathbf{\Sigma}_B)(\mathbf{V}_A \otimes \mathbf{V}_B)^T$$

   - the singular values of $\mathbf{A} \otimes \mathbf{B}$ are the $r_A r_B$ possible positive products of singular values of $\mathbf{A}$ and $\mathbf{B}$ (counting multiplicities)
   - $\mathrm{rank}(\mathbf{A} \otimes \mathbf{B}) = \mathrm{rank}(\mathbf{A})\mathrm{rank}(\mathbf{B}) = r_A r_B = \mathrm{rank}(\mathbf{B} \otimes \mathbf{A})$.
   - This implies that $\mathbf{A} \otimes \mathbf{B}$ is nonsingular if and only if both $\mathbf{A}$ and $\mathbf{B}$ are nonsingular.

Vectorization

The vectorization of $\mathbf{A} \in \mathbb{R}^{m \times n}$ is defined as

$$\mathrm{vec}(\mathbf{A}) = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix},$$

i.e., we stack the columns of a matrix to form a column vector.

Elementary properties:

1. $\mathrm{vec}(\mathbf{A}^T) = \mathbf{P}\mathrm{vec}(\mathbf{A})$ with $\mathbf{P}$ a permutation matrix
2. $\mathrm{vec}(\mathbf{A} + \mathbf{B}) = \mathrm{vec}(\mathbf{A}) + \mathrm{vec}(\mathbf{B})$
3. $\mathrm{tr}(\mathbf{A}^T\mathbf{B}) = \mathrm{vec}(\mathbf{A})^T\mathrm{vec}(\mathbf{B})$
4. $\mathrm{vec}(\mathbf{A} * \mathbf{B}) = \mathrm{vec}(\mathbf{A}) * \mathrm{vec}(\mathbf{B})$
5. $\mathbf{a} \otimes \mathbf{b} = \mathrm{vec}(\mathbf{b}\mathbf{a}^T)$

Vectorization

Property 9.3. $\text{vec}(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A})\text{vec}(\mathbf{B})$.

- As a direct result,

$$\text{vec}(\mathbf{AB}) = (\mathbf{I} \otimes \mathbf{A})\text{vec}(\mathbf{B}) = (\mathbf{B}^T \otimes \mathbf{I})\text{vec}(\mathbf{A})$$

  and

$$\text{vec}(\mathbf{ABC}) = (\mathbf{I} \otimes \mathbf{AB})\text{vec}(\mathbf{C}) = (\mathbf{C}^T\mathbf{B}^T \otimes \mathbf{I})\text{vec}(\mathbf{A})$$

- Combine the above property and the ones on the last page, we have

$$\begin{aligned}\text{tr}(\mathbf{ABC}) &= \text{vec}(\mathbf{A}^T)^T(\mathbf{I} \otimes \mathbf{B})\text{vec}(\mathbf{C}) = \text{vec}(\mathbf{A}^T)^T(\mathbf{C}^T \otimes \mathbf{I})\text{vec}(\mathbf{B}) \\ &= \text{vec}(\mathbf{B}^T)^T(\mathbf{A} \otimes \mathbf{I})\text{vec}(\mathbf{C}) = \dots\end{aligned}$$

- more generally,

$$\begin{aligned}\text{tr}(\mathbf{ABCD}) &= \text{vec}(\mathbf{A}^T)^T(\mathbf{D}^T \otimes \mathbf{B})\text{vec}(\mathbf{C}) = \text{vec}(\mathbf{D}^T)^T(\mathbf{C}^T \otimes \mathbf{A})\text{vec}(\mathbf{B}) \\ &= \text{vec}(\mathbf{D})^T(\mathbf{A} \otimes \mathbf{C}^T)\text{vec}(\mathbf{B}^T) = \dots\end{aligned}$$

Proof Sketch of Property 9.3

- write

$$\mathbf{X} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}\mathbf{e}_i\mathbf{e}_j^T$$

- by letting $\mathbf{a}_i$ be the ith column of $\mathbf{A}$ and $\mathbf{b}_j$ the jth row of $\mathbf{B}$,

$$\mathrm{vec}(\mathbf{AXB}) = \mathrm{vec}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}\mathbf{A}\mathbf{e}_i\mathbf{e}_j^T\mathbf{B}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}\mathrm{vec}(\mathbf{a}_i\mathbf{b}_j^T).$$

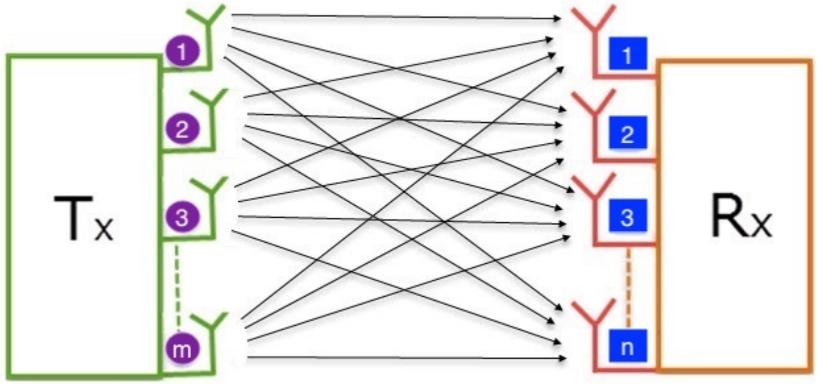
- by noting

$$\mathrm{vec}(\mathbf{a}_i\mathbf{b}_j^T) = \mathrm{vec}([\ \mathbf{a}_i b_{j1}, \ldots, \mathbf{a}_i b_{j,q}\ ]) = \begin{bmatrix} b_{j1}\mathbf{a}_i \\ \vdots \\ b_{j,q}\mathbf{a}_i \end{bmatrix} = \mathbf{b}_j \otimes \mathbf{a}_i$$

we get $\mathrm{vec}(\mathbf{AXB}) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}\mathbf{b}_j \otimes \mathbf{a}_i = (\mathbf{B}^T \otimes \mathbf{A})\mathrm{vec}(\mathbf{X})$.

Example: Space-Time Block Coding

- Let m and n be the no. of Tx and Rx antennas in a MIMO system. Let T be the code length.



- Signal model:

$$\mathbf{Y} = \mathbf{H}\mathbf{C}(\mathbf{s}) + \mathbf{V}$$

where $\mathbf{Y} \in \mathbb{C}^{m \times T}$ is the received code matrix, $\mathbf{H} \in \mathbb{C}^{m \times n}$ is the channel matrix, and $\mathbf{C}(\mathbf{s}) \in \mathbb{C}^{n \times T}$ is the transmitted space-time block code (STBC).

- The Tx STBC has a linear dispersion structure

$$\mathbf{C}(\mathbf{s}) = \sum_{k=1}^{K} \mathbf{X}_k s_k$$

where where $\mathbf{X}_k \in \mathbb{C}^{n \times T}$ are the basis matrices.

Example: Space-Time Block Coding

- Our aim is to estimate $\mathbf{s}$ from $\mathbf{Y}$.
- Vectorizing the signal model yields

$$\text{vec}(\mathbf{Y}) = (\mathbf{I}_T \otimes \mathbf{H})\text{vec}(\mathbf{C}(\mathbf{s})) + \text{vec}(\mathbf{V})$$

- Moreover,

$$\begin{aligned}\text{vec}(\mathbf{C}(\mathbf{s})) &= \sum_{k=1}^{K} \text{vec}(\mathbf{X}_k) s_k \\ &= \underbrace{\left[\ \text{vec}(\mathbf{X}_1), \ldots, \text{vec}(\mathbf{X}_K)\ \right]}_{\bar{\mathbf{X}}} \mathbf{s}\end{aligned}$$

- Hence, we obtain a familiar linear model:

$$\text{vec}(\mathbf{Y}) = (\mathbf{I}_T \otimes \mathbf{H})\bar{\mathbf{X}}\mathbf{s} + \text{vec}(\mathbf{V})$$

which allows us to use LS to estimate $\mathbf{s}$.

Kronecker Sum

- Problem (Sylvester equation): given $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{m \times m}$, $\mathbf{C} \in \mathbb{R}^{n \times m}$, solve

$$\mathbf{AX} + \mathbf{XB} = \mathbf{C} \tag{$*$}$$

  with respect to $\mathbf{X} \in \mathbb{R}^{n \times m}$.

- the above problem is a linear system. By vectorizing $(*)$, we get

$$(\mathbf{I}_m \otimes \mathbf{A})\mathrm{vec}(\mathbf{X}) + (\mathbf{B}^T \otimes \mathbf{I}_n)\mathrm{vec}(\mathbf{X}) = \mathrm{vec}(\mathbf{C})$$

- the Kronecker sum of $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{B} \in \mathbb{R}^{m \times m}$ is

$$\mathbf{A} \oplus \mathbf{B} = (\mathbf{I}_m \otimes \mathbf{A}) + (\mathbf{B} \otimes \mathbf{I}_n).$$

  – if a unique solution to $(*)$ is desired, we wish to know conditions under which $\mathbf{A} \oplus \mathbf{B}^T$ is nonsingular

Kronecker Sum

Theorem 9.2. Let $\{\lambda_i, \mathbf{x}_i\}_{i=1}^n$ be the set of n eigen-pairs of $\mathbf{A}$, and let $\{\mu_i, \mathbf{y}_i\}_{i=1}^m$ be the set of m eigen-pairs of $\mathbf{B}$. The set of mn eigen-pairs of $\mathbf{A} \oplus \mathbf{B}$ is given by

$$\{\lambda_i + \mu_j, \mathbf{y}_j \otimes \mathbf{x}_i\}_{i=1,\ldots,n,\ j=1,\ldots,m}$$

Theorem 9.3. The matrix equations

$$\mathbf{AX} + \mathbf{XB} = \mathbf{C}$$

has a unique solution for every given $\mathbf{C}$ if and only if

$$\lambda_i \neq -\mu_j, \qquad \text{for all } i, j,$$

where $\{\lambda_i\}_{i=1}^n$ and $\{\mu_i\}_{i=1}^m$ are the set of eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, resp.

- idea behind Theorem 9.3: if $\lambda_i = -\mu_j$ for some i, j, then from Theorem 9.2 there exists a zero eigenvalue for $\mathbf{A} \oplus \mathbf{B}$ and also $\mathbf{A} \oplus \mathbf{B}^T$.

Kronecker Sum

- Consider

$$\mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} = \mathbf{C},$$

  which is called the Lyapunov equations.

- from Theorem 9.3, the Lyapunov equations admit a unique solution if

$$\lambda_i \neq -\lambda_j, \qquad \text{for all } i,\ j.$$

- if $\mathbf{A}$ is PD such that $\lambda_i > 0$ for all i, the Lyapunov equations always have a unique solution.

- The generalized Lyapunov equations $\mathbf{A}_1\mathbf{X}\mathbf{B}_1 + \mathbf{A}_2\mathbf{X}\mathbf{B}_2 = \mathbf{C}$ and the commutativity equation $\mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{A}$ can also be solved based on similar idea.

Khatri-Rao Product

The Khatri-Rao product of $\mathbf{A} \in \mathbb{R}^{m_1 \times n}$ and $\mathbf{B} \in \mathbb{R}^{m_2 \times n}$ is defined as

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{b}_1 & a_{12}\mathbf{b}_2 & \dots & a_{1n}\mathbf{b}_n \\ a_{21}\mathbf{b}_1 & a_{22}\mathbf{b}_2 & & a_{2n}\mathbf{b}_n \\ \vdots & & \ddots & \vdots \\ a_{m1}\mathbf{b}_1 & a_{m2}\mathbf{b}_2 & \dots & a_{mn}\mathbf{b}_n \end{bmatrix},$$

or, equivalently,

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} \mathbf{a}_1 \otimes \mathbf{b}_1 & \mathbf{a}_2 \otimes \mathbf{b}_2 & \dots & \mathbf{a}_n \otimes \mathbf{b}_n \end{bmatrix}.$$

- it is a column-wise Kronecker product

Properties

Elementary properties:

1. $\mathbf{A} \odot (\alpha \mathbf{B}) = (\alpha \mathbf{A}) \odot \mathbf{B}$.
2. $(\mathbf{A}+\mathbf{B}) \odot \mathbf{C} = \mathbf{A} \odot \mathbf{C} + \mathbf{B} \odot \mathbf{C}, \quad \mathbf{A} \odot (\mathbf{B}+\mathbf{C}) = \mathbf{A} \odot \mathbf{B} + \mathbf{A} \odot \mathbf{C}$ (distributive)
3. $\mathbf{A} \odot (\mathbf{B} \odot \mathbf{C}) = (\mathbf{A} \odot \mathbf{B}) \odot \mathbf{C}$ (associativity).
4. $\mathbf{A} \odot \mathbf{0} = \mathbf{0} \odot \mathbf{A} = \mathbf{0}$.
5. (permutation equivalence) there exist permutation matrices $\mathbf{P}$ such that

$$\mathbf{P}(\mathbf{A} \odot \mathbf{B}) = \mathbf{B} \odot \mathbf{A}.$$

Property 9.4. (mixed-product property)

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \odot \mathbf{D}) = \mathbf{AC} \odot \mathbf{BD}$$

Property 9.5. if $\mathbf{X}$ is diagonal and given by $\mathbf{X} = \mathrm{Diag}(\mathbf{x})$,

$$\mathrm{vec}(\mathbf{AXB}) = \mathrm{vec}(\mathbf{A}\mathrm{Diag}(\mathbf{x})\mathbf{B}) = (\mathbf{B}^T \odot \mathbf{A})\mathbf{x}.$$

Although defined based on the Kronecker product, the Khatri-Rao product does not have many nice properties.

Hadamard Product

The Hadamard product (elementwise product, Schur product) of $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$ is defined as

$$\mathbf{A} * \mathbf{B} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & \dots & a_{1n}b_{1n} \\ a_{21}b_{21} & a_{22}b_{22} & & a_{2n}b_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1}b_{m1} & a_{m2}b_{m2} & \dots & a_{mn}b_{mn} \end{bmatrix},$$

or simply, $[\mathbf{A} * \mathbf{B}]_{ij} = [\mathbf{A}]_{ij}[\mathbf{B}]_{ij} = a_{ij}b_{ij}$.

- Hadamard product operates on matrices of the same dimension.
- **Fact:** $\mathbf{A} * \mathbf{B}$ is a principal submatrix of $\mathbf{A} \otimes \mathbf{B}$.

Properties

Elementary properties:

1. $\mathbf{A} * (\alpha\mathbf{B}) = (\alpha\mathbf{A}) * \mathbf{B}$.
2. $\mathbf{A} * \mathbf{B} = \mathbf{B} * \mathbf{A}$ (commutative).
3. $(\mathbf{A} + \mathbf{B}) * \mathbf{C} = \mathbf{A} * \mathbf{C} + \mathbf{B} * \mathbf{C}$, $\quad \mathbf{A} * (\mathbf{B} + \mathbf{C}) = \mathbf{A} * \mathbf{B} + \mathbf{A} * \mathbf{C}$ (distributive)
4. $\mathbf{A} * (\mathbf{B} * \mathbf{C}) = (\mathbf{A} * \mathbf{B}) * \mathbf{C}$ (associativity).
5. $\mathbf{A} * \mathbf{0} = \mathbf{0} * \mathbf{A} = \mathbf{0}$.
6. $(\mathbf{A} * \mathbf{B})^T = \mathbf{A}^T * \mathbf{B}^T$, $(\mathbf{A} * \mathbf{B})^H = \mathbf{A}^H * \mathbf{B}^H$.

Property 9.6. Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$,

$$\mathbf{a} * \mathbf{b} = \mathrm{Diag}(\mathbf{a}) * \mathbf{b} = \mathrm{Diag}(\mathbf{b}) * \mathbf{a}.$$

Property 9.7. Given $\mathbf{A} \in \mathbb{R}^{m \times m}$ and $\mathbf{d} \in \mathbb{R}^m$,

$$\mathbf{A} * \mathrm{Diag}(\mathbf{d}) = \mathrm{Diag}(\mathbf{d}) * \mathbf{A} = \mathrm{Diag}(\mathrm{diag}(\mathbf{A}) * \mathbf{d}).$$

Specifically, when $\mathrm{Diag}(\mathbf{d}) = \mathbf{I}_m$ $(\mathbf{d} = \mathbf{1}_m)$, $\mathbf{A} * \mathbf{I}_m = \mathbf{I}_m * \mathbf{A} = \mathrm{Diag}(a_{11}, \ldots, a_{mm})$.

Properties

Property 9.8. Given $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{a} \in \mathbb{R}^{m}$, and $\mathbf{b} \in \mathbb{R}^{n}$,

$$\left(\mathbf{a}\mathbf{b}^T\right) * \mathbf{A} = \mathrm{Diag}(\mathbf{a})\mathbf{A}\mathrm{Diag}(\mathbf{b}).$$

Property 9.9. Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$, $\mathbf{a} \in \mathbb{R}^{m}$, and $\mathbf{b} \in \mathbb{R}^{n}$,

$$\mathbf{a}^T(\mathbf{A} * \mathbf{B})\mathbf{b} = \mathrm{tr}(\mathrm{Diag}(\mathbf{a})\mathbf{A}\mathrm{Diag}(\mathbf{b})\mathbf{B}^T).$$

- In particular, $\mathbf{1}^T(\mathbf{A} * \mathbf{B})\mathbf{1} = \mathrm{tr}(\mathbf{A}\mathbf{B}^T)$ (sum of all elements is the trace of $\mathbf{A}\mathbf{B}^T$).
- For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times m}$, the row-sums of $\mathbf{A} * \mathbf{B}$ are the diagonal elements of $\mathbf{A}\mathbf{B}^T$.

Property 9.10. Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$, $\mathbf{d}_1 \in \mathbb{R}^{m}$, and $\mathbf{d}_2 \in \mathbb{R}^{n}$, and define $\mathbf{D}_1 = \mathrm{Diag}(\mathbf{d}_1)$ and $\mathbf{D}_2 = \mathrm{Diag}(\mathbf{d}_2)$,

$$(\mathbf{D}_1\mathbf{A})*(\mathbf{B}\mathbf{D}_2) = \mathbf{D}_1(\mathbf{A}*\mathbf{B})\mathbf{D}_2 = (\mathbf{D}_1\mathbf{A}\mathbf{D}_2)*\mathbf{B} = (\mathbf{A}\mathbf{D}_2)*(\mathbf{D}_1\mathbf{B}) = \mathbf{A}*(\mathbf{D}_1\mathbf{B}\mathbf{D}_2).$$

Properties

other properties:

- Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{m \times n}$, $\mathrm{tr}(\mathbf{A}^T(\mathbf{B} * \mathbf{C})) = \mathrm{tr}((\mathbf{A}^T * \mathbf{B}^T)\mathbf{C}))$.
- if $\mathbf{A}$ and $\mathbf{B}$ are PSD (PD), then $\mathbf{A} * \mathbf{B}$ is PSD (PD).
- Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times m}$, $\det(\mathbf{A} * \mathbf{B}) \geq \det(\mathbf{A}) \det(\mathbf{B})$.
- Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$, $\mathrm{rank}(\mathbf{A} * \mathbf{B}) \leq \mathrm{rank}(\mathbf{A})\mathrm{rank}(\mathbf{B})$
- $(\mathbf{A} \odot \mathbf{B})^T(\mathbf{A} \odot \mathbf{B}) = (\mathbf{A}^T\mathbf{A}) * (\mathbf{B}^T\mathbf{B})$
- $(\mathbf{A} \odot \mathbf{B})^\dagger = ((\mathbf{A}^T\mathbf{A}) * (\mathbf{B}^T\mathbf{B}))^\dagger(\mathbf{A} \odot \mathbf{B})^T$
- (mixed-product property)

$$(\mathbf{A} \otimes \mathbf{B}) * (\mathbf{C} \otimes \mathbf{D}) = (\mathbf{A} * \mathbf{C}) \otimes (\mathbf{B} * \mathbf{D})$$

and

$$(\mathbf{A} \odot \mathbf{B}) * (\mathbf{C} \odot \mathbf{D}) = (\mathbf{A} * \mathbf{C}) \odot (\mathbf{B} * \mathbf{D})$$