Sampling (Ch. 7)

Exercise 1
Consider the discrete-time sequence $x[n] = \cos\left[\frac{n\pi}{8}\right]$. Find two different continuous-time signals that would produce this sequence when sampled at a frequency of $f_s = 10$ kHz.

Answer:

A continuous-time sinusoid

$$x_a(t) = \cos(\omega_0 t) = \cos(2\pi f_0 t)$$

that is sampled with a sampling frequency of f_s results in the discrete-time sequence

$$x[n] = x_a(nT_s) = \cos(2\pi \frac{f_0}{f_s} n)$$

However, note that for any integer k,

$$\cos(2\pi \frac{f_0}{f_s} n) = \cos(2\pi \frac{f_0 + kf_s}{f_s} n)$$

Therefore, any sinusoid with a frequency

$$f = f_0 + kf_s$$

will produce the same sequence of samples $x[n]$ when sampled with a sampling frequency f_s. With $x[n] = \cos(n\pi/8)$, we want

$$2\pi \frac{f_0}{f_s} = \frac{\pi}{8}$$

or

$$f_0 = \frac{1}{16} f_s = 625Hz$$

Therefore, two signals that produce the given sequence are

$$x_1(t) = \cos(1250\pi t)$$

and

$$x_2(t) = \cos(21250\pi t)$$

Exercise 2
A continuous-time signal $x_a(t)$ is to be filtered to remove frequency components in the range 5 kHz $\leq f \leq$ 10 kHz. The maximum frequency present in $x_a(t)$ is 20 kHz. The filtering is to be done by sampling $x_a(t)$, filtering the sampled signal, and reconstructing an analog signal using an ideal D/C converter. Find the minimum sampling frequency that may be used to avoid aliasing, and for this minimum sampling rate, find the frequency response of the ideal digital filter $H(e^{j\theta})$ that will remove the desired frequencies from $x_a(t)$.

Answer:

Because the highest frequency in $x_a(t)$ is 20 [kHz], the minimum sampling frequency to avoid aliasing is $f_s = 40[kHz]$. The relationship between the continuous frequency variable ω and the discrete frequency variable θ is given by

$$\theta = \omega T_s$$

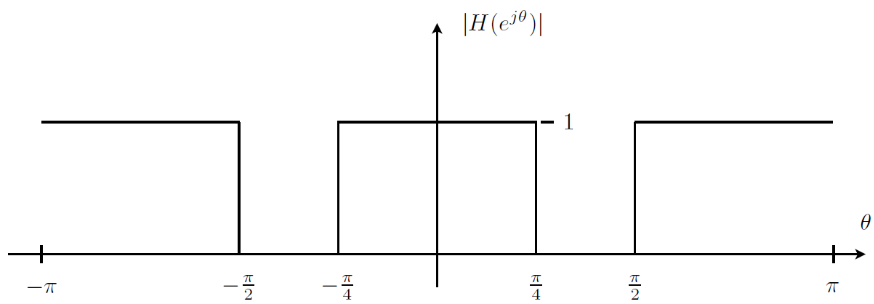
or

$$\theta = 2\pi \frac{f}{f_s}$$

Therefore, the frequency range $5[kHz] \leq f \leq 10[kHz]$ corresponds to a digital frequency range

$$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

and the desired digital filter is a bandstop filter that has a frequency response as illustrated in the figure below.

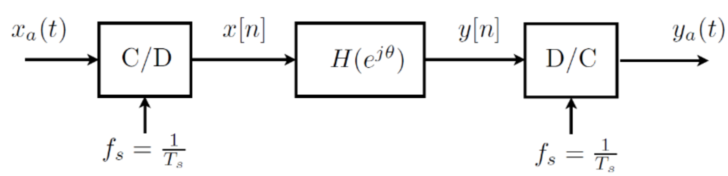


Exercise 3

Consider the system in the figure below for implementing a continuous-time system in terms of a discrete-time system. Assume that the input to the C/D converter is bandlimited to $\omega_0 = \omega_s/2$ and that the impulse response of the discrete-time system is

$$h[n] = \delta[n] - 0.9\delta[n-1]$$

Find the overall frequency response of this system.



Answer:

Assuming bandlimited inputs with $X_a(\omega) = 0$ for $|\omega| > \omega_s/2$, the output $Y_a(\omega)$ is related to the input $X_a(\omega)$ as follows:

$$Y_a(\omega) = H_a(\omega)X(\omega)$$

where

$$H_a(\omega) = \begin{cases} H(e^{j\omega T_s}) & |\omega| \le \frac{\pi}{T_s} \\ 0 & \text{otherwise} \end{cases}$$

Because the frequency response of the discrete-time system is

$$H(\theta) = 1 - 0.9e^{-j\theta}$$

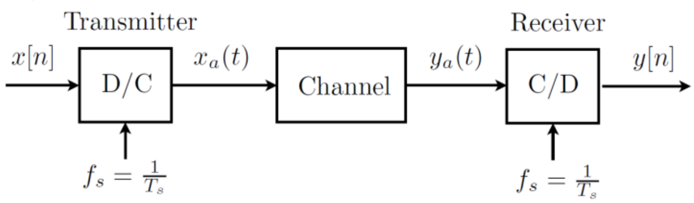
then

$$H_a(\omega) = \begin{cases} 1 - 0.9e^{-j\omega T_s} & |\omega| \le \frac{\pi}{T_s} \\ 0 & \text{otherwise} \end{cases}$$

Exercise 4

A digital sequence $x[n]$ is to be transmitted across a linear time-invariant bandlimited analog channel as illustrated in the figure below.

The transmitter is a D/C converter, and the receiver simply samples the received waveform $y_a(t)$:



$$y[n] = y_a(nT_s)$$

Assume that the channel may be modeled as an ideal low-pass filter with a cutoff frequency of 4 kHz:

$$G_a(\omega) = \begin{cases} 1 & |\omega| \le 2\pi(4000) \\ 0 & |\omega| > 2\pi(4000) \end{cases}$$

Assuming an ideal C/D and D/C, and perfect synchronization between the transmitter and receiver, what values of T_s (if any) will guarantee that $y[n] = x[n]$?

Answer:

The output of the D/C converter is a bandlimited signal $x_a(t)$ with a Fourier transform that is equal to zero for $|f| > f_s/2$. Because $x_a(t)$ is passed through a bandlimited channel that rejects all frequencies greater than 4 [kHz], in order for there to be no distortion at the receiver, it is necessary that

$$\frac{f_s}{2} < 4000$$

or

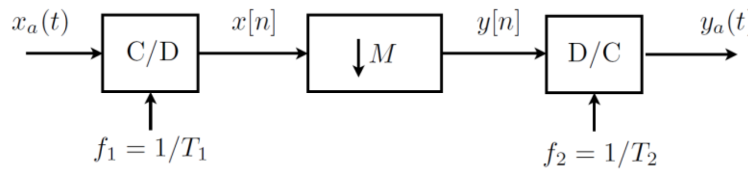
$$f_s < 8000$$

Thus, the C/D and D/C converters must operate at a rate less than 8 [kHz].

Exercise 5

Consider the system shown in the figure below.

Assume that the input is bandlimited, $X_a(\omega) = 0$ for $|\omega| > 2\pi \cdot 1000$.

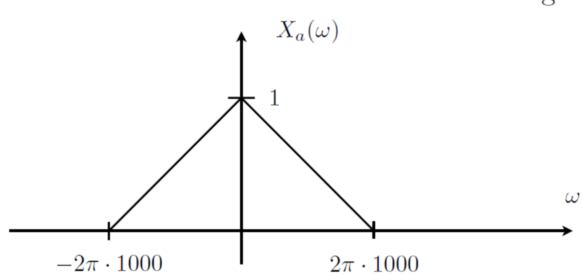


(a) What constraints must be placed on M, T_1, and T_2 in order for $y_a(t)$ to be equal to $x_a(t)$?

(b) If $f_1 = f_2 = 20$ kHz and $M = 4$, find an expression for $y_a(t)$ in terms of $x_a(t)$.

Answer:

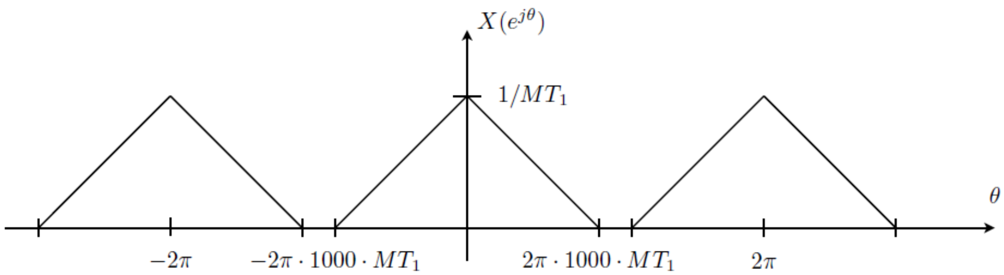
(a) Suppose that $x_a(t)$ has a Fourier transform as shown in the figure below. Because $y(n) =$



$x(Mn) = x_a(nMT_1)$, in order to prevent $x(n)$ from being aliased, it is necessary that

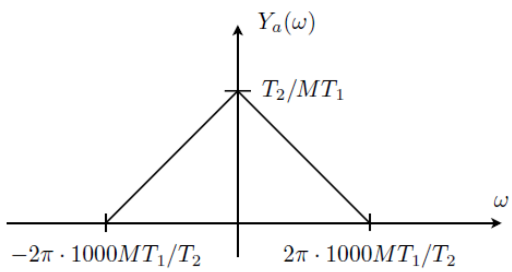
$$MT_1 < \frac{1}{2000}$$

If this constraint is satisfied, the output of the down-sampler has a FTD as shown below.



Going through the D/C converter produces signal $y_a(t)$, which has the Fourier transform shown below.

Therefore, in order for $y_a(t)$ to be equal to $x_a(t)$, we require that



1. $MT_1 \leq 1/2000$ in order to avoid aliasing.
2. $T_2 = MT_1$ to prevent frequency scaling.

(b) With $T_1 = T_2 = 1/20000$ and $M = 4$, note that

$$MT_1 = \frac{1}{5000} < \frac{1}{2000}$$

Therefore, there is no aliasing. Thus, as we see from the figure above,

$$Y_a(\omega) = \frac{1}{4} X_a\left(\frac{\omega}{4}\right)$$

or

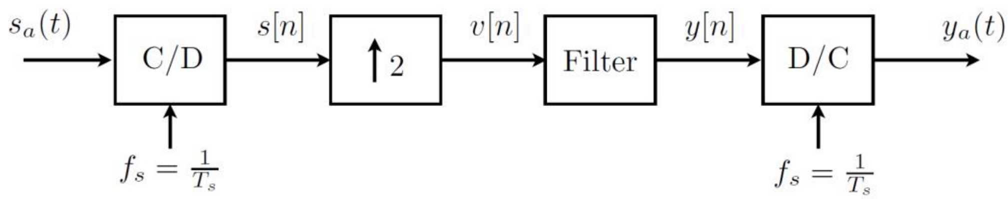
$$y_a(t) = x_a(4t)$$

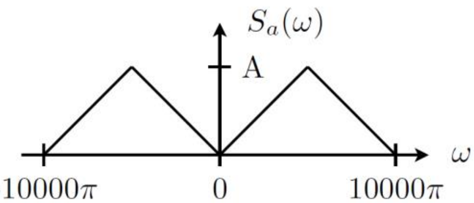
Exercise 6

Suppose that we would like to slow a segment of speech to one-half its normal speed. The speech signal $s_a(t)$ is assumed to have no energy outside of 5kHz, and is sampled at a rate of 10 kHz, yielding the sequence

$$s[n] = s_a(nT_s)$$

The following system is proposed to create the slowed-down speech signal. Assume that $S_a(\omega)$ is as shown in the following figure:





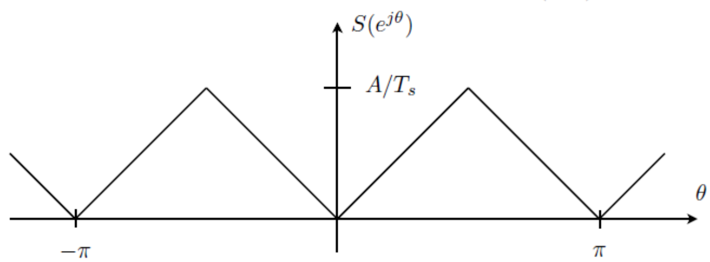
a) Find the spectrum of $v[n]$.
b) Suppose that the discrete-time filter is described by the difference equation:

$$y[n] = v[n] + \frac{1}{2}\left(v[n-1] + v[n+1]\right)$$

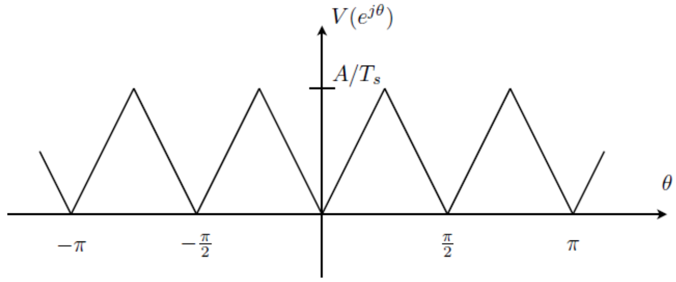
Find the frequency response of the filter and describe its effect on $v[n]$.
c) What is $Y_a(\omega)$ in terms of $X_a(\omega)$? Does $y_a(t)$ correspond to slowed-down speech?

Answer:

(a) Since $s_a(t)$ is sampled at the Nyquist rate, the FTD of the sampled speech signal, $s(n)$, is as follows:
Up-sampling by a factor of 2 scales the frequency axis of $S(e^{j\theta})$ by a factor of two as shown



below.
(b) The impulse response of the discrete-time filter is



$$h(n) = \frac{1}{2}\delta(n+1) + \delta(n) + \frac{1}{2}\delta(n-1)$$

which has a frequency response

$$H(e^{j\theta}) = 1 + \cos\theta$$

To see the effect of this filter on $v(n)$, note that due to the up-sampling, $v(n) = 0$ for n odd. Therefore, with

$$y(n) = v(n) + \frac{1}{2}v(n-1) + \frac{1}{2}v(n+1)$$

it follows that

$$y(n) = \begin{cases} v(n) & \text{n odd} \\ \frac{1}{2}v(n-1) + \frac{1}{2}v(n+1) & \text{n even} \end{cases}$$

Thus, the even-index values of $v(n)$ are unchanged, and the odd-index values are the average of the two neighboring values. As a result, $h(n)$ performs a linear interpolation between the values of $v(n)$.

(c) The output of the DC converter, $y_a(t)$, has a Fourier transform

$$Y_a(\omega) = \begin{cases} T_s Y(e^{j\omega T_s}) & |\omega| < \pi/T_s \\ 0 & \text{otherwise} \end{cases}$$

Since

$$Y(e^{j\theta}) = H(e^{j\theta})V(e^{j\theta}) = (1 + \cos\theta)V(e^{j\theta})$$

and

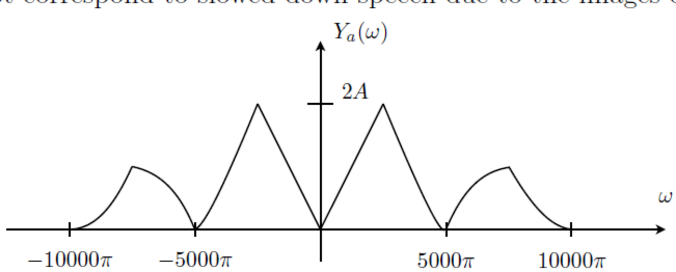
$$V(e^{j\theta}) = S(e^{j2\theta})$$

then

$$Y_a(\omega) = \begin{cases} T_s(1 + \cos\omega T_s)S(e^{j2\omega T_s}) & |\omega| < 10000\pi \\ 0 & \text{otherwise} \end{cases}$$

which is the product of $(1 + \cos\omega T_s)$ and $T_s S(e^{j2\omega T_s})$ as illustrated below.

Thus, $y_a(t)$ does not correspond to slowed-down speech due to the images of $s_a(t)$ that occur in



the frequency range $5000\pi < |\omega| < 10000\pi$ and the nonideal linear interpolator.