Signals and Systems

娄鑫，louxin@shanghaitech.edu.cn
徐林，xulin1@shanghaitech.edu.cn
周勇，zhouyong@shanghaitech.edu.cn
赵子平, zhaoziping@shanghaitech.edu.cn

Reference Books

- Alan V. Oppenheim, Alan S. Willsky and S. Hanid Nawab, Signals and Systems, 2rd Edition

Topics

- ❑ Overview of Signals and Systems
- ❑ Linear-Time-Invariant Systems
- ❑ Fourier Series Representation of Periodic Signals
- ❑ The Continues-Time Fourier Transform
- ❑ The Discrete-Time Fourier Transform
- ❑ Sampling
- ❑ The Laplace Transform
- ❑ The Z-Transform

Assessment

- Homework: 20%
- Mid-term: 30%
- Final Exam: 50%

QQ Group

2020信号与系统

扫一扫二维码，加入群聊。

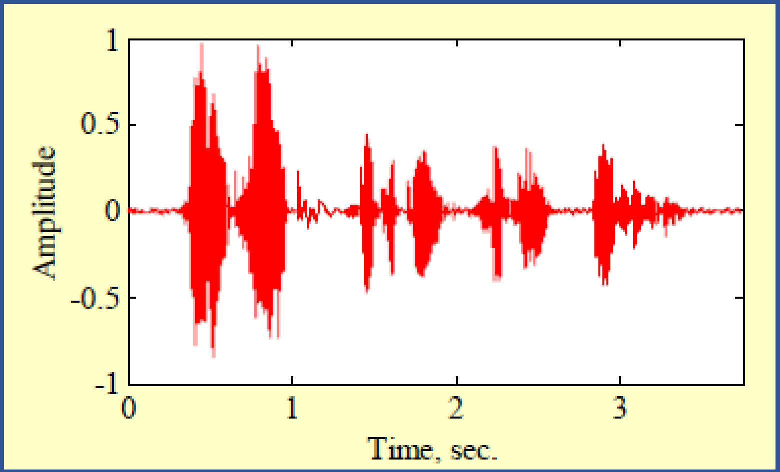
Lecture 1
Signals and Systems:
An Overview

Signals

- A signal is a function of independent variables such as time, distance, position, temperature, and pressure
- Example of typical signals
  - Sound
  - Image
  - Video

Examples of Typical Signals

- Sounds - represent air pressure as a function of time at a point in space



$f\,(t)$

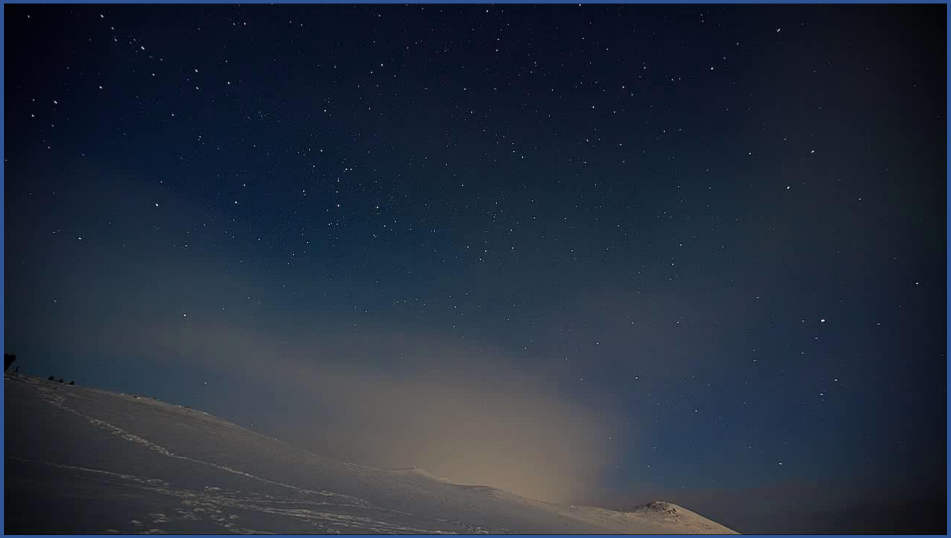
Examples of Typical Signals

- Grey-scale pictures - represent light intensity as a function of two spatial coordinates

$f(x,y)$

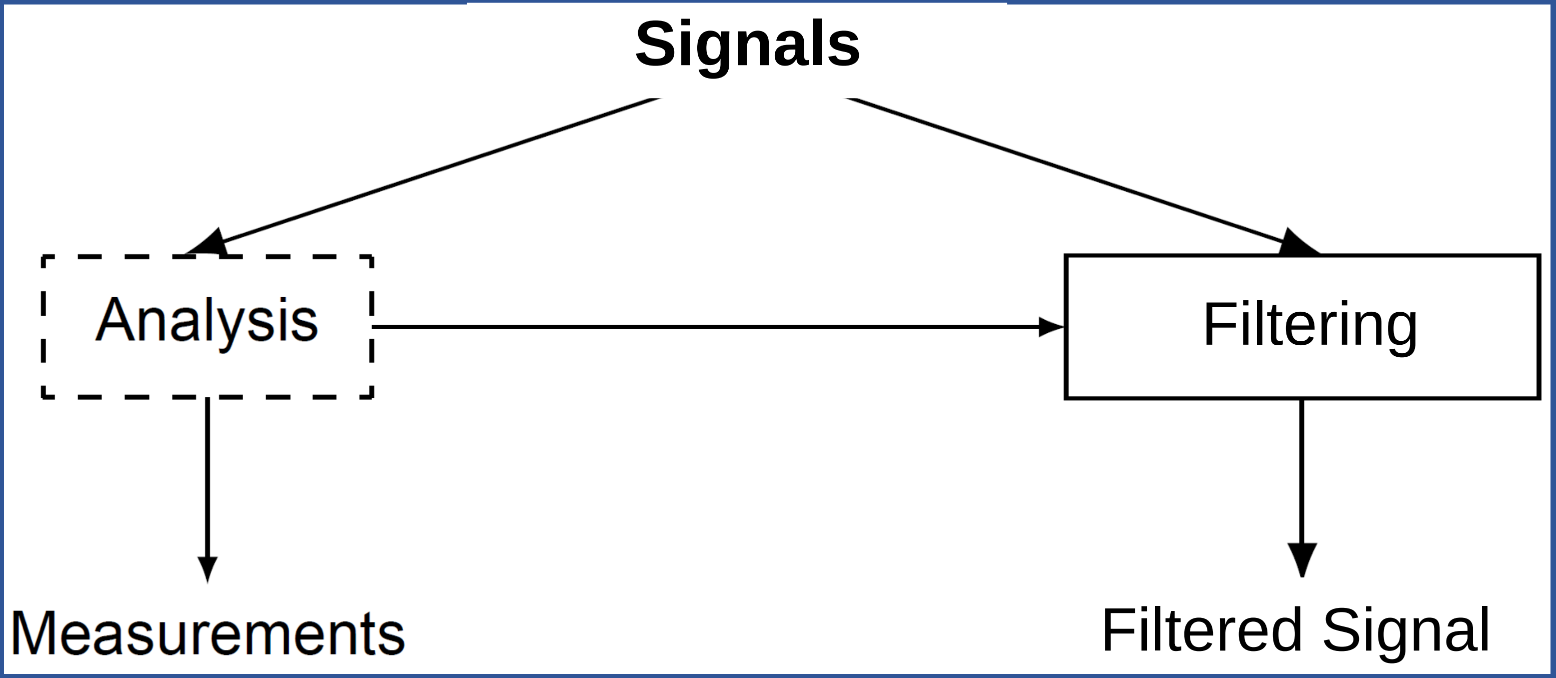
Examples of Typical Signals

- Videos - consists of a sequence of images, called frames, and is a function of 3 variables: 2 spatial coordinates and time



$$f(x, y, t)$$

The Objective of This Course



Signal Analysis

- This task deals with the measurement of signal properties
  - Spectrum analysis
    - frequency and/or phase
  - Speech recognition
  - Target detection and tracking
    - Radar

Signal Filtering

- This task deals with the transformation of signals. The systems that perform this task are called filters
  - Removal of unwanted background noise

Noise removal

- Original uncorrupted speech signal
- Impulse-noise-corrupted speech signal
- Median filtered version of the noisy signal

Noise removal

- Noise corrupted image and its noise-removed version

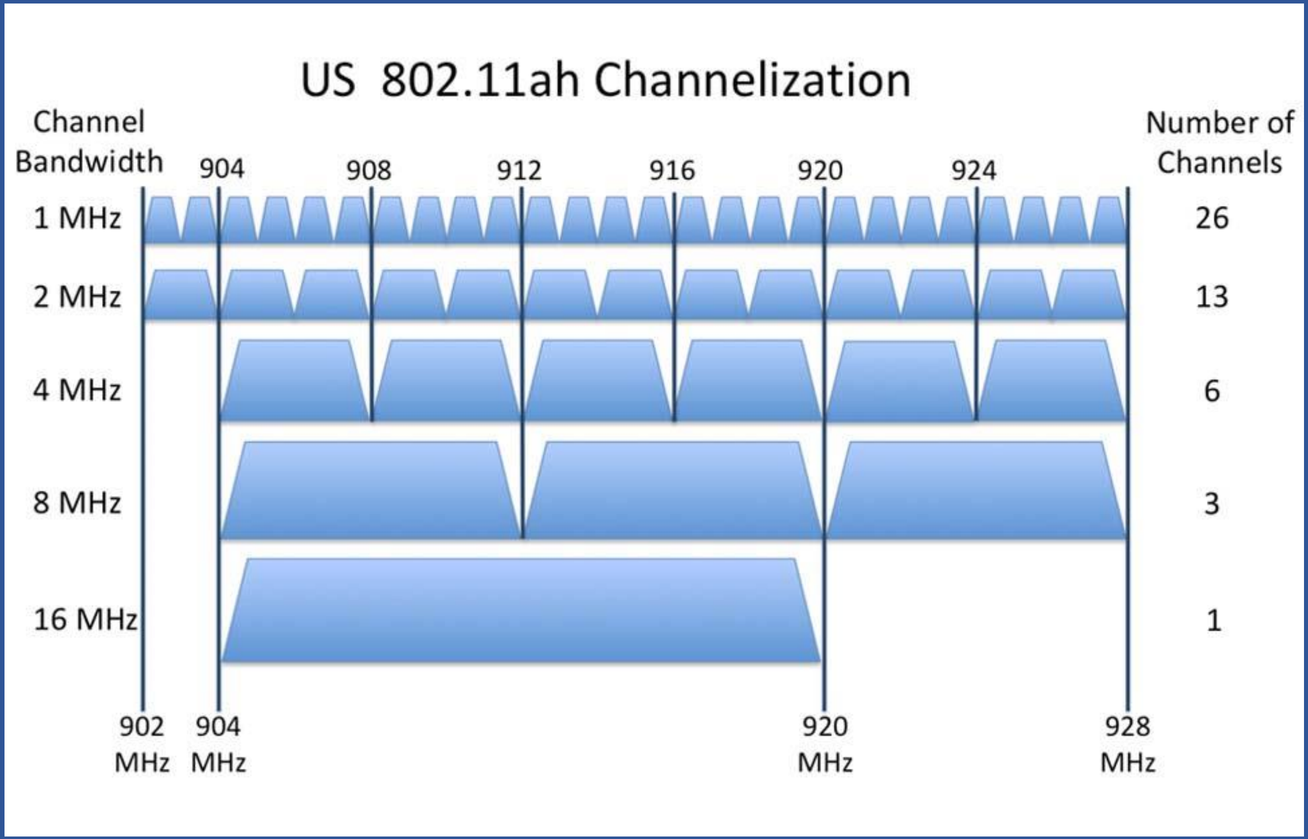
20% pixels corrupted with additive impulse noise

Noise-removed version

Signal Filtering

- This task deals with the transformation of signals. The systems that perform this task are called filters
  - Removal of unwanted background noise
  - Separation of frequency bands

Separation of frequency bands



Signal Filtering

- This task deals with the transformation of signals. The systems that perform this task are called filters
  - Removal of unwanted background noise
  - Separation of frequency bands
  - Removal of interference
  - Shaping of the signal spectrum

Representation of Signals

- In terms of basis functions in the domain of original independent variable
  - Time
  - Spatial, etc.

- In terms of basis functions in a transform domain
  - Fourier Transform
  - Laplace Transform
  - *Z*-Transform, etc.

Classification of Signals

- Continuous vs. Discrete
  - Depends on the independent variable
- Real-valued vs. Complex-valued
  - Depends on the function defining the signal
- 1-D signal vs. *M*-D signal
  - 1 independent variable or *M* independent variables
- Stationary vs. Non-stationary
- etc.

Classification of Signals

- The speech signal is an example of a 1-D signal
  - The independent variable is time
- The image signal is an example of a 2-D signal
  - The 2 independent variables are the 2 spatial variables
- The color image signal is composed of three 2-D signals representing the three primary colors: red, green and blue (RGB)
  - 3-channel 2D signal

RGB Image

- The 3 color components of a color image

R

G

B

RGB Image

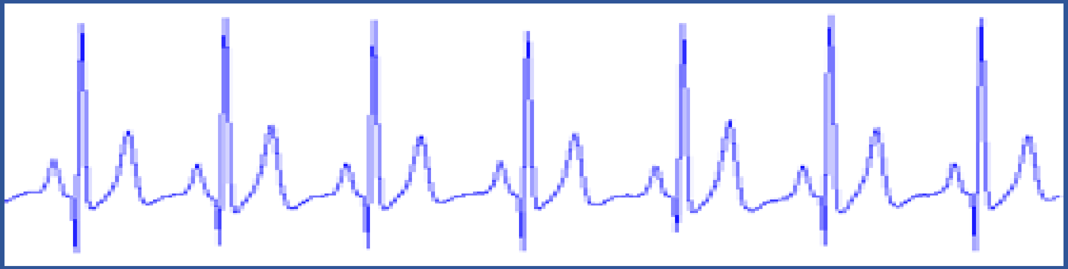
- The full color image obtained by displaying the previous 3 color components

Video Signals

- Black-and-white video signal is an example of a 3-D signal
  - 2 spatial variables and time
- Color video signal is a 3-channel 3-D signal
  - Red channel
  - Green channel
  - Blue channel

Characterization of Signals

- The value of a signal at a specific value of the independent variable is called amplitude
- The variation of the amplitude as a function of the independent variable is called waveform
- Let's consider 1-D signal

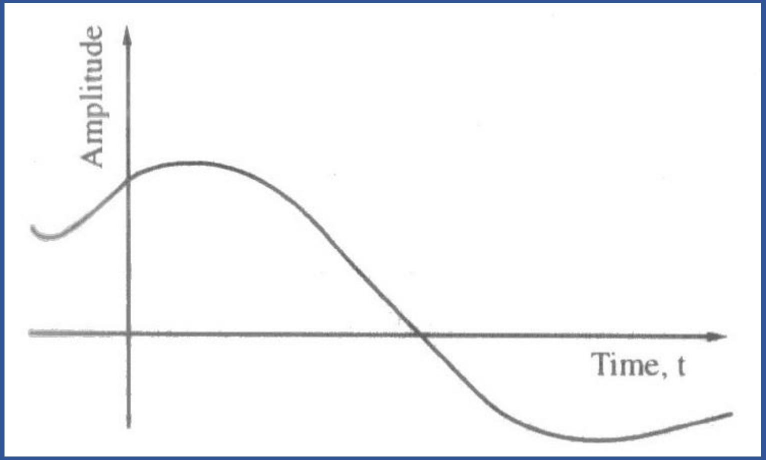


Continuous and Discrete Signals

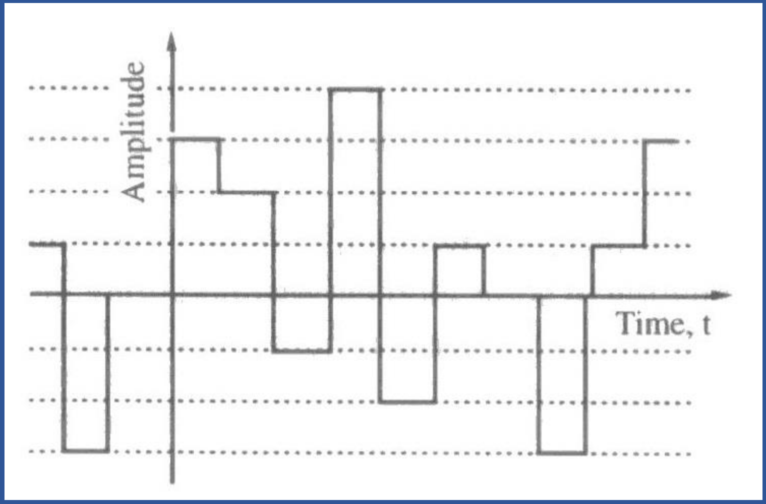
- If the independent variable is continuous, the signal is called a continuous-time signal
  - A continuous-time signal is defined at every instant of time
- If the independent variable is discrete, the signal is called a discrete-time signal
  - A discrete-time signal is defined at discrete instants of time, i.e., it is a sequence of numbers
  - The signal is not defined in between the time instants

Analog Signal

- Analog signal
  - Continuous-time signal with continuous-valued amplitude
  - A speech signal is an example of an analog signal

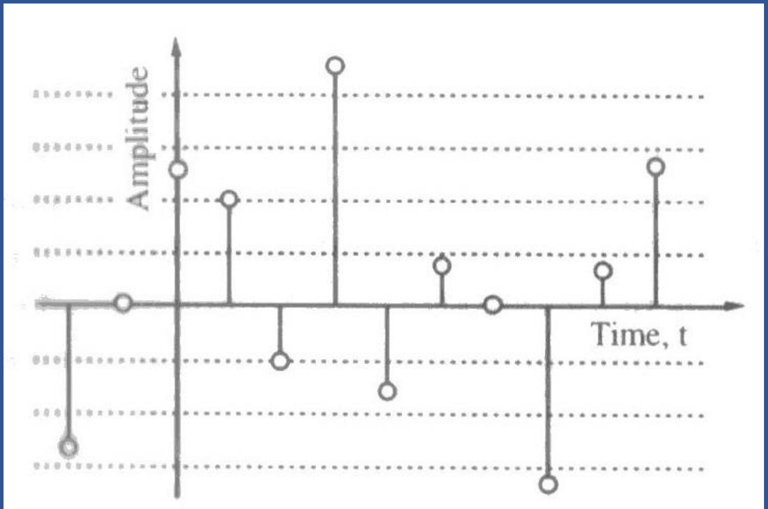


- Quantized boxcar signal
  - Continuous-time signal with discrete-valued amplitude
  - Occurs in digital electronic circuits

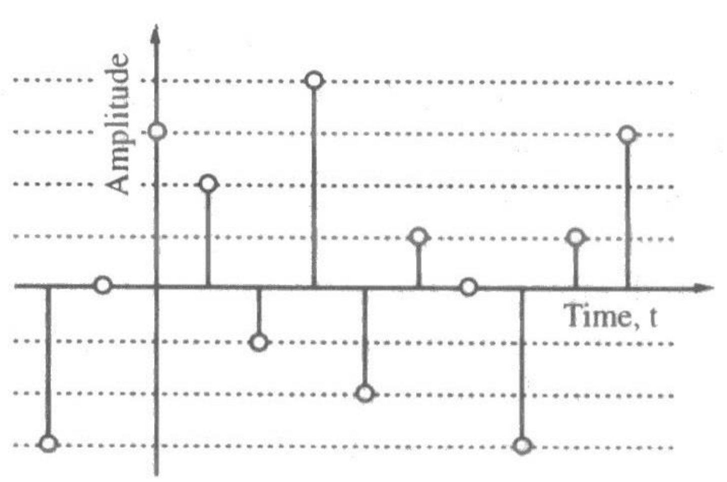


Digital Signal

- ❑ Sampled-data signal
  - ➢ Discrete-time signal with continuous-valued amplitude
  - ➢ The amplitude of the signal may be any value



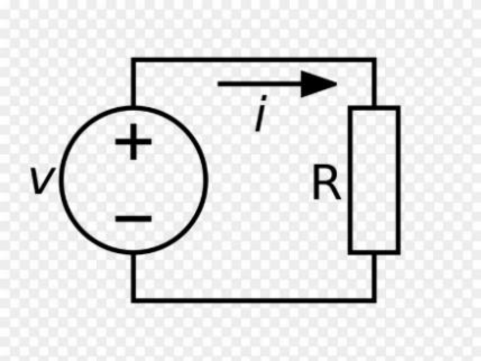
- ❑ Digital signal
  - ➢ Discrete-time signal with discrete-valued amplitude
  - ➢ A digital signal is a quantized sampled-data signal



Power & Energy

- A resistor R with $v(t)$ and $i(t)$, the instantaneous power is

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$



- The total energy over the time interval $t_1 \le t \le t_2$ is

$$E_R = \int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}V^2(t)dt$$

- The average power over this interval is

$$P_R = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)\ \ dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

Power & Energy

- Similarly for any $x(t)$ or any $x[n]$, the total energy is defined as

$$E = \int_{t_1}^{t_2} |x(t)|^2 dt \qquad t_1 \le t \le t_2$$

Continuous-time

$$E = \sum_{n=n_1}^{n_2} |x[n]|^2 \qquad n_1 \le n \le n_2$$

Discrete-time

- Similarly , the total power is defined as

$$P = E/(t_2 - t_1)$$

Continuous-time

$$P = E/(n_2 - n_1 + 1)$$

Discrete-time

Power & Energy

- Over infinite time interval $-\infty \le t \le \infty$ or $-\infty \le n \le \infty$

$$E_\infty = \lim_{\tau \to \infty} \int_{-\tau}^{\tau} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$ Continuous-time

$$E_\infty = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$ Discrete-time

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$ Continuous-time

$$P_\infty = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$ Discrete-time

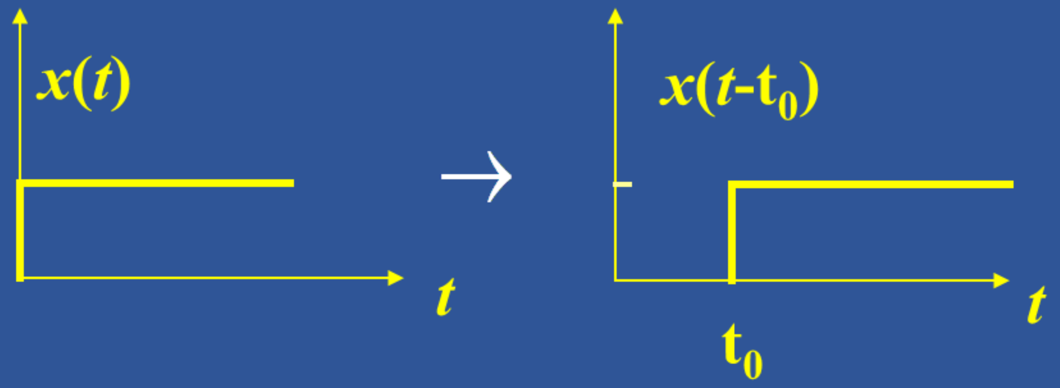
Three Classes of Signals

- Energy signal $E_\infty < \infty$ $P_\infty = 0$
- Power signal $E_\infty = \infty$ $P_\infty < \infty$
- Infinite energy & power signal $E_\infty \to \infty$ $P_\infty \to \infty$

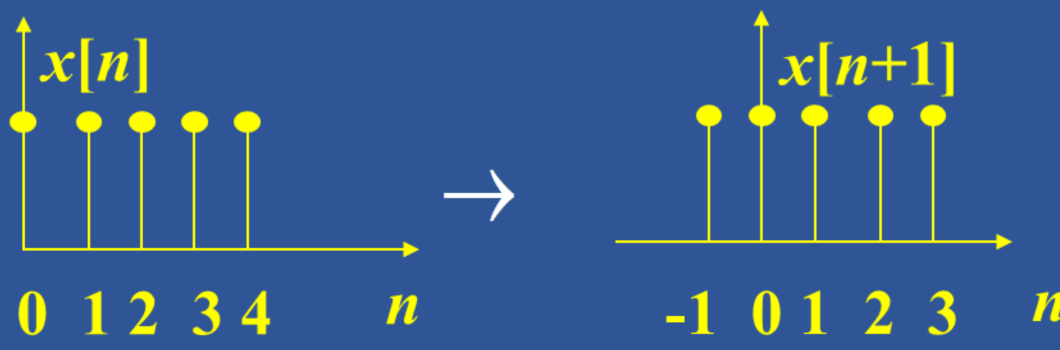
Transformations of Independent Variable

- Time shifting

$$x(t) \to x(t - t_0)$$



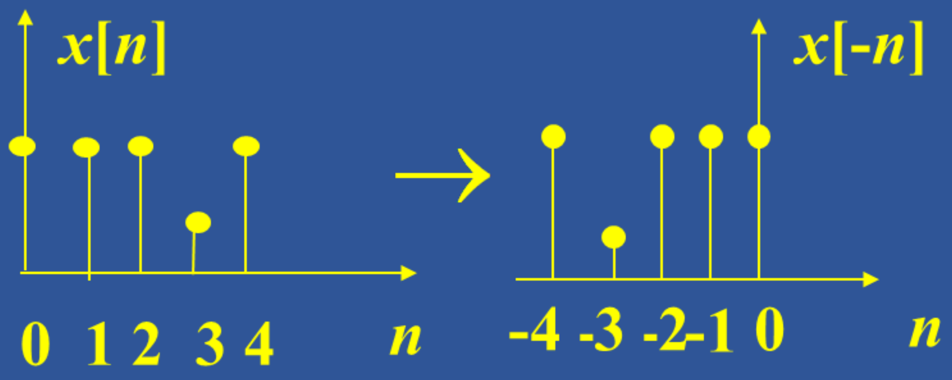
$$x[n] \to x[n - n_0]$$



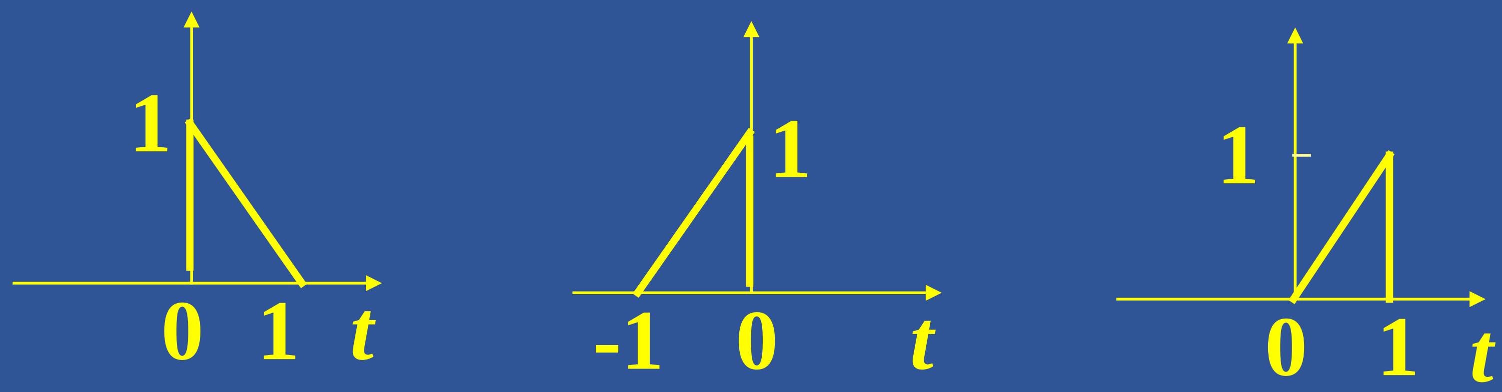
Transformations of Independent Variable

- Time reversal

$$x[n] \rightarrow x[-n]$$



$$x(t) \rightarrow x(-t) \rightarrow x(1-t)$$



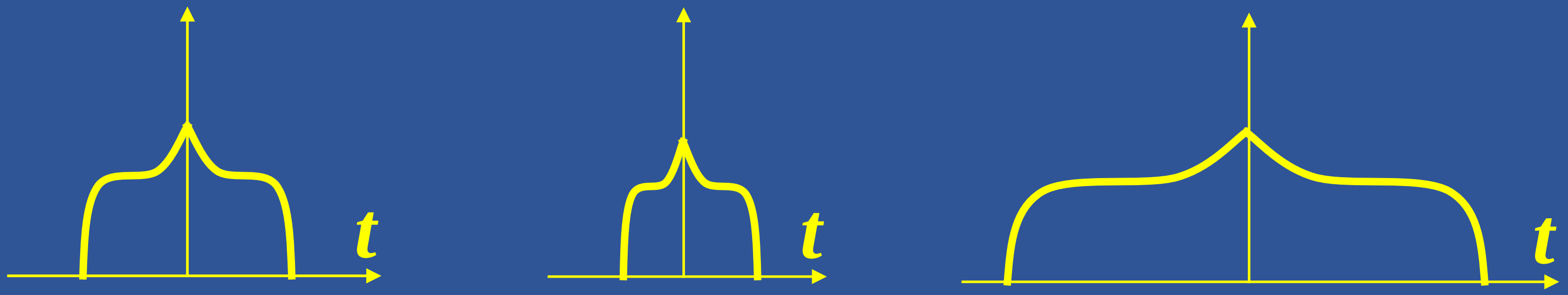
Transformations of Independent Variable

- Time scaling

$x(t) \rightarrow x(2t)$ compressed

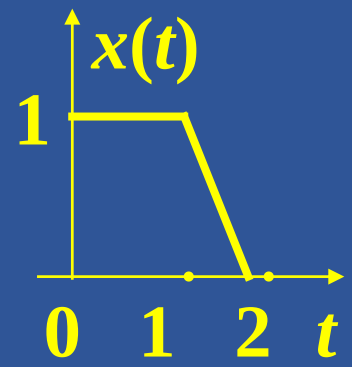
$x(t) \rightarrow x(t/2)$ stretched

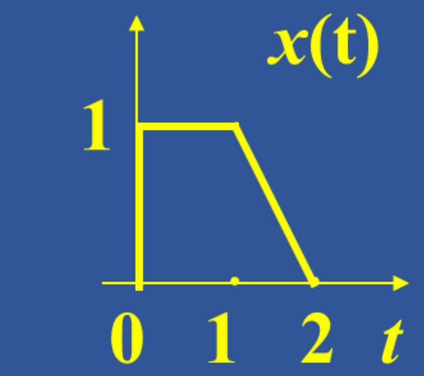
$x(t) \rightarrow x(2t) \rightarrow x(t/2)$

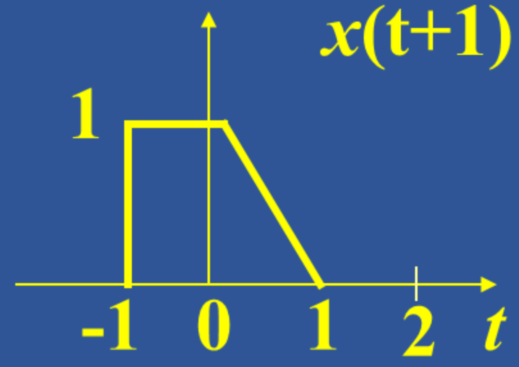


Transformations of Independent Variable

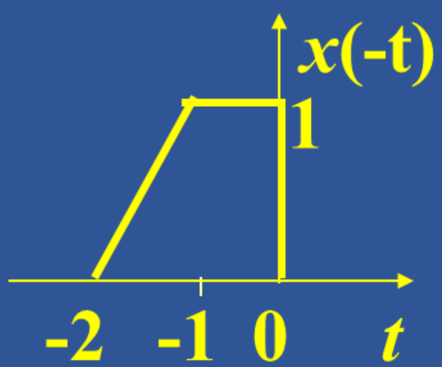
- Let $x(t) \rightarrow x(\alpha t + \beta)$
  - If $|\alpha| > 1$ compressed
  - If $|\alpha| < 1$ stretched
  - If $\alpha < 0$ reversed
  - If $\beta \neq 0$ shifted
- Example: Given the signal $x(t)$, to illustrate
  - $x(t+1)$
  - $x(-t+1)$
  - $x(3t/2)$
  - $x(3t/2 + 1)$

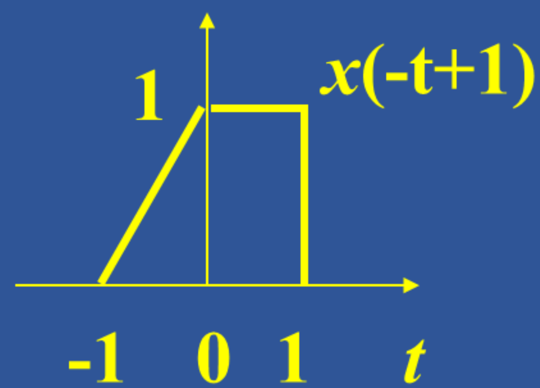




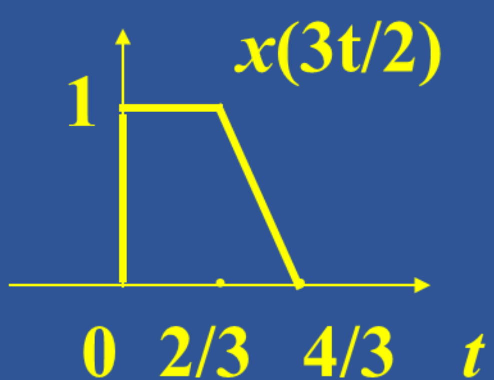


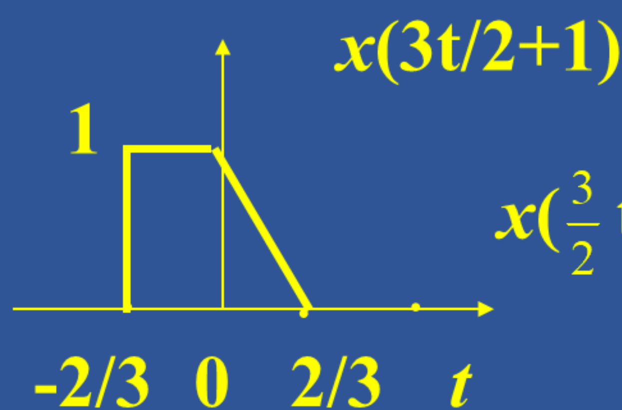
- $x(t+1)$
- $x(-t+1)$
- $x(3t/2)$
- $x(3t/2+1)$





$$x(-t+1)=x[-(t-1)]$$

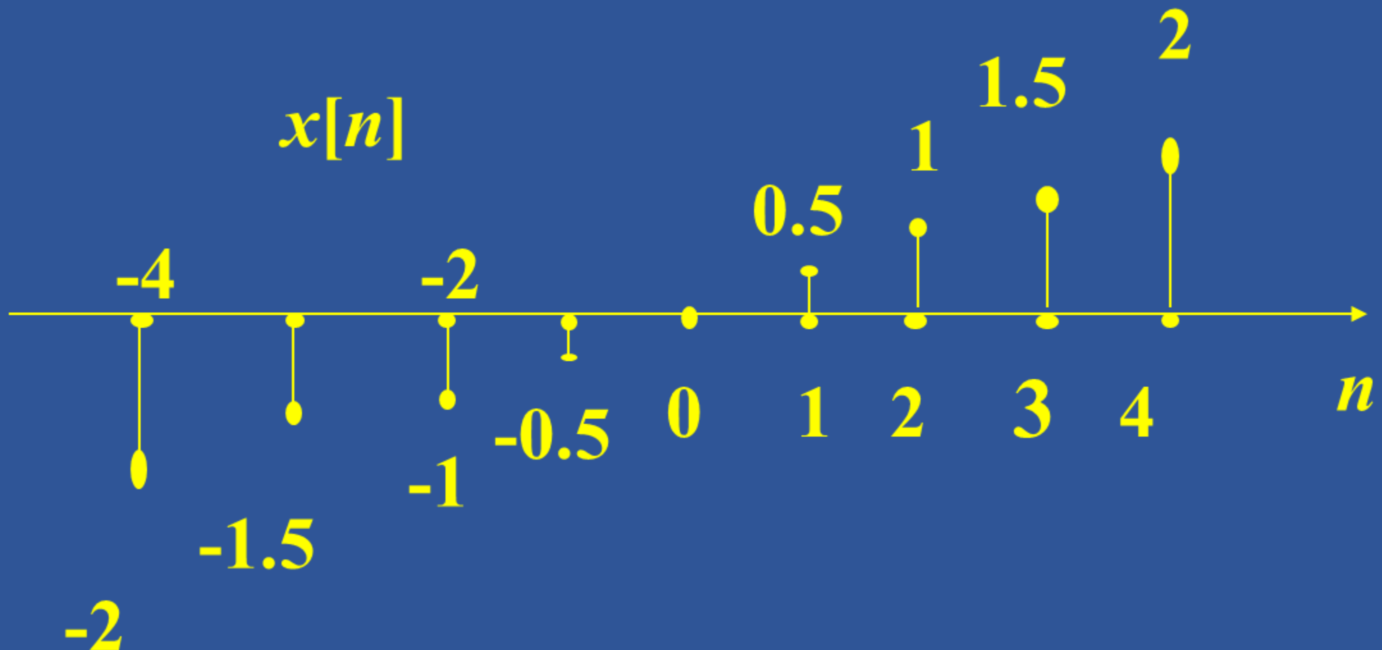


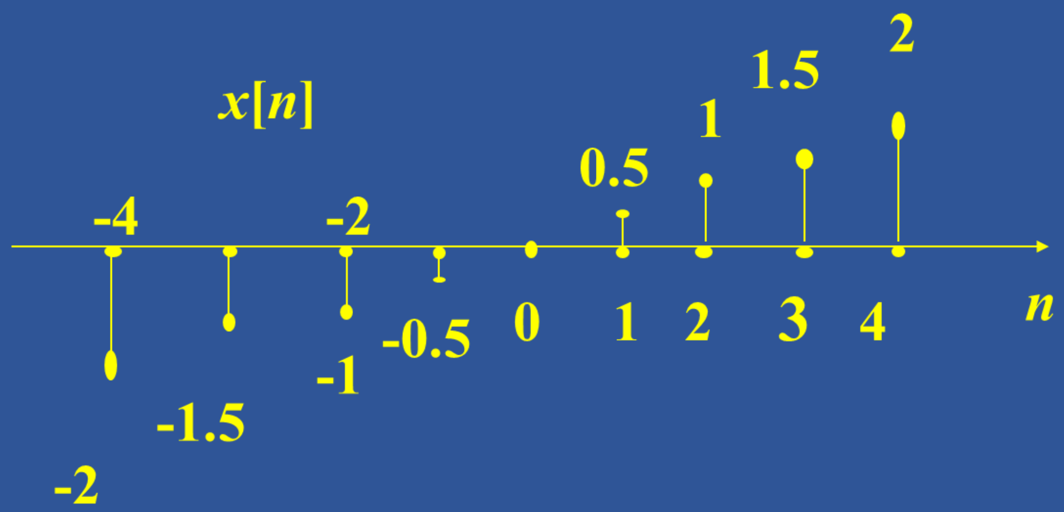


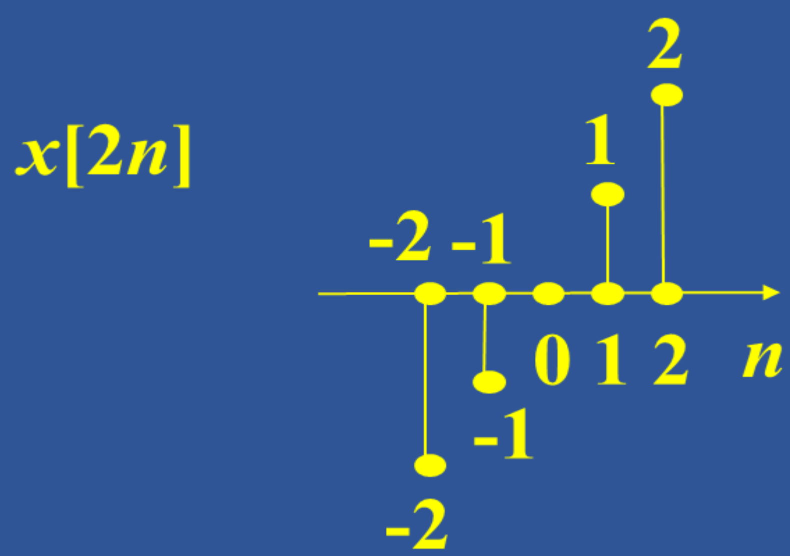
$$x(\tfrac{3}{2}t+1)=x[\tfrac{3}{2}(t+2/3)]$$

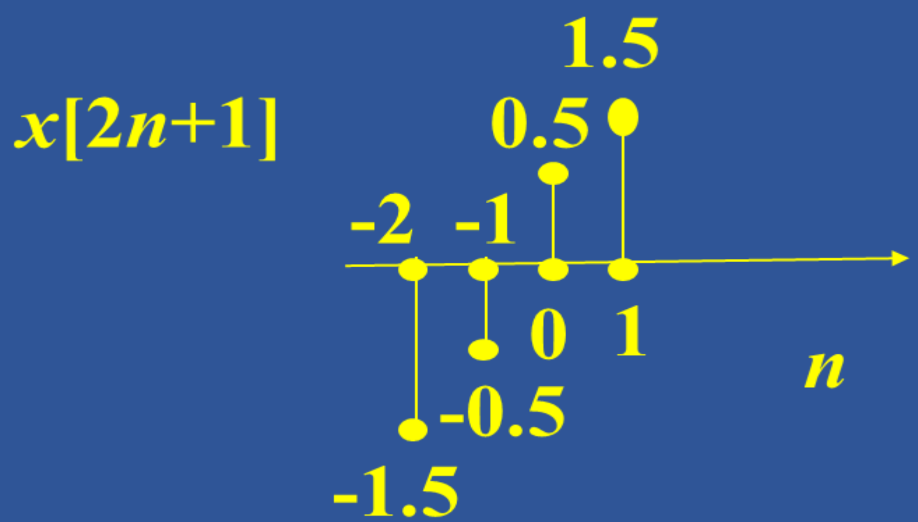
Transformations of Independent Variable

- Example: A discrete signal $x[n]$ is shown below, sketch and label following signals:
  - $x[2n]$
  - $x[2n+1]$

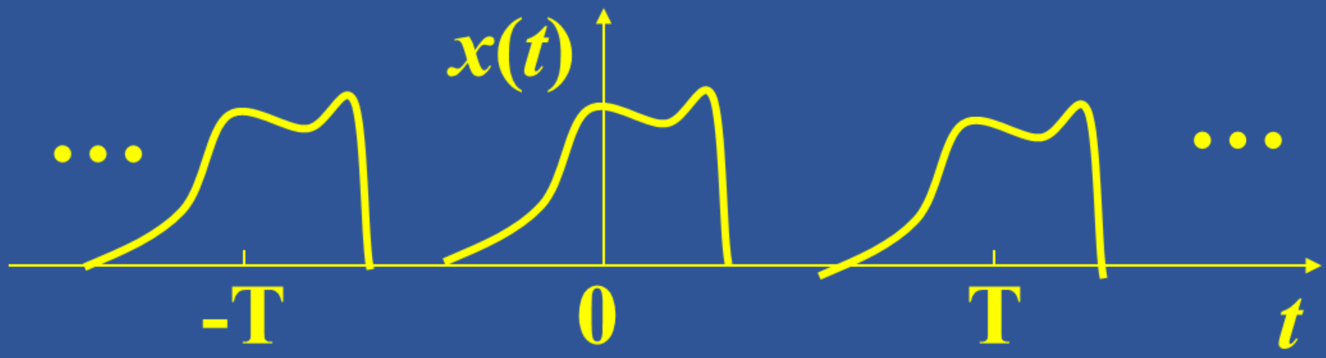



x[n]
2
1.5
1
0.5
-4
-2
-0.5
0
1
2
3
4
n
-1
-1.5
-2


x[2n]
2
1
-2 -1
0 1 2
n
-1
-2


x[2n+1]
1.5
0.5
-2 -1
0 1
n
-0.5
-1.5

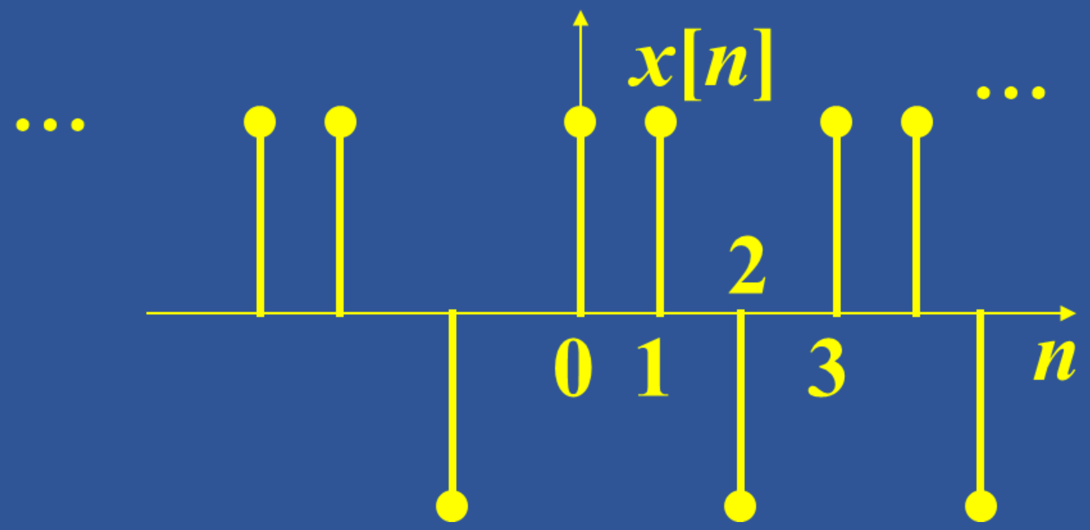
Periodic Signals



- Continuous-time: $x(t)=x(t+\mathrm{T})$ for all t
- Fundamental period
  - The smallest positive value of T for which $\boldsymbol{x(t)=x(t+\mathrm{T})}$ holds

Periodic Signals

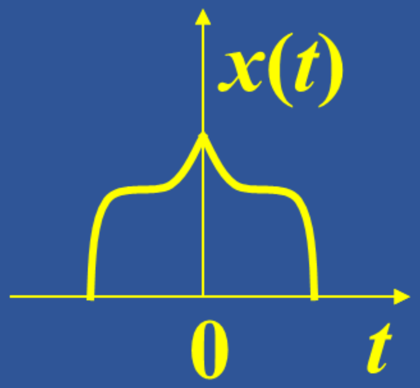
- Discrete-time: $x[n]=x[n+N]$ **for all n**



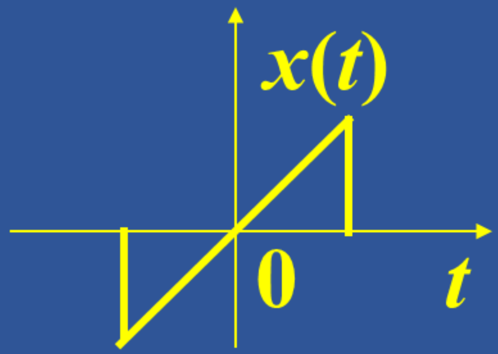
- **Fundamental period**
  - The smallest positive value of N for which $x[n]=x[n+N]$ holds

Even and Odd Signals

- Even signal
  - $x(-t)=x(t)$ $\quad$ $x[-n]=x[n]$



- Odd signal
  - $x(-t)= -x(t)$ $\quad$ $x[-n]= -x[n]$



Even and Odd Signals

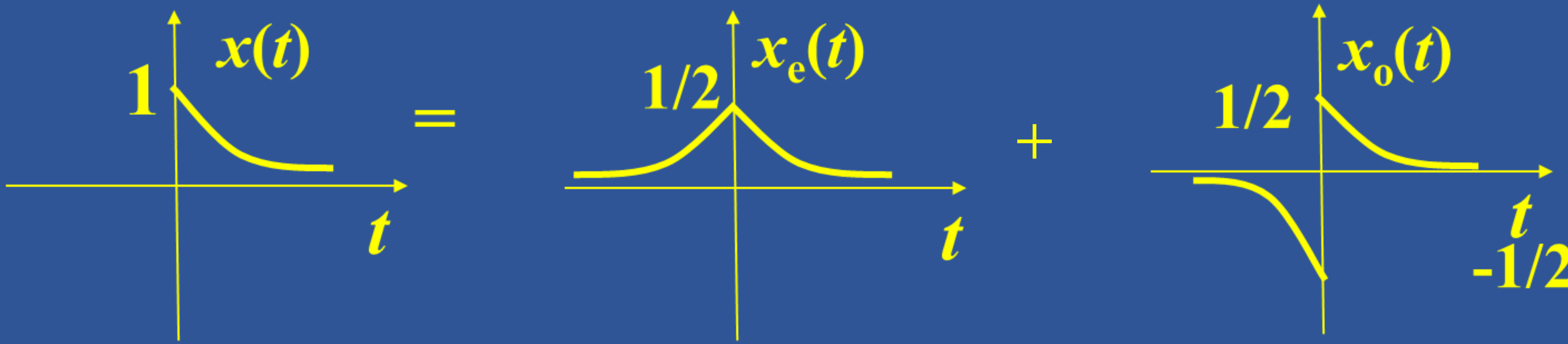
- Any signal can be broken into a sum of two signals
  - One even and one odd

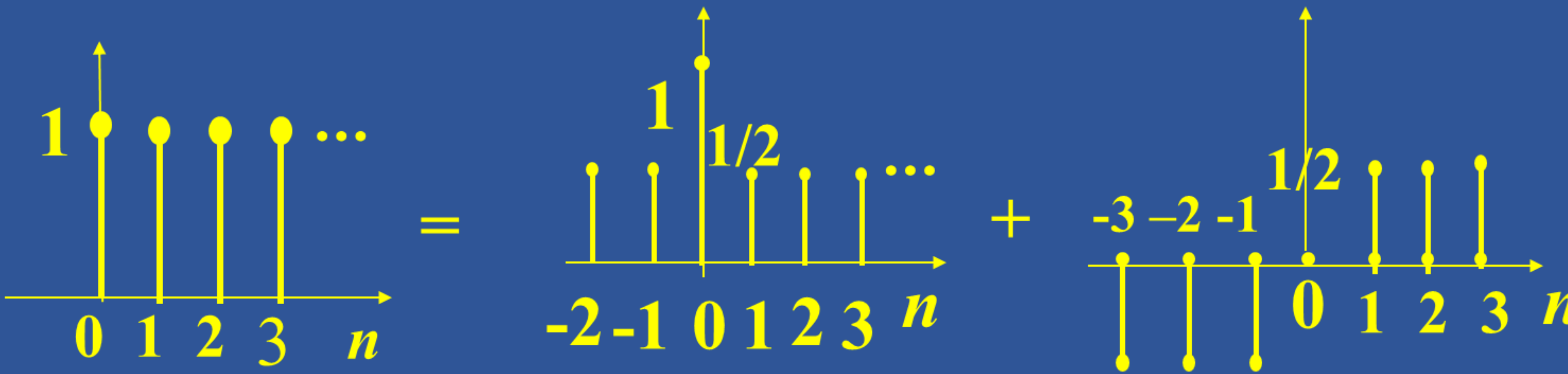
$$x(t) = x_e(t) + x_o(t)$$

$$x_o(t) = O_d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

$$x_e(t) = E_v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

$$x_o(t) = O_d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)] \qquad x_e(t) = E_v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$





General Complex Signals

- c and a are complex numbers

$$x(t) = ce^{at} \qquad c = |c|e^{j\theta} \quad a = r + j\omega_0$$

$$ce^{at} = |c|e^{j\theta}e^{(r+j\omega_0)t} = |c|e^{rt}e^{j(\omega_0 + \theta)}$$

$$= |c|e^{rt}\cos(\omega_0 t + \theta) + j|c|e^{rt}\sin(\omega_0 t + \theta)$$

Exponential Signals

- Real exponential signal $x(t) = ce^{at}$
  - c and a are real
  - $a>0$, as $t\uparrow$, $|x(t)|\uparrow$
  - $a<0$, as $t\uparrow$, $|x(t)|\downarrow$
  - $a=0$, $|x(t)|$ is constant

- Periodic exponential signals $x(t) = e^{j\omega_0 t}$
  - a is purely imaginary
  - Fundamental period?

$$T_0 = \frac{2\pi}{|\omega_0|}$$

Sinusoidal Signals

$$x(t) = A\cos(\omega_0 t + \phi)$$

- Closely related to complex exponential signals
- How?

Exponential and Sinusoidal Signals

- $e^{j\omega_0 t}$ and $A\cos(\omega_0 t + \phi)$

$$E_{period} = \int_0^{T_0} \left| e^{j\omega_0 t} \right|^2 dt = \int_0^{T_0} 1 dt = T_0$$

$$p_{period} = \frac{1}{T_0} E_{period} = 1$$

  - Total energy? Infinite
  - Average power? Finite

Examples- Periodic or Not?

$$(1)\ x_1(t) = je^{j10t} \qquad \omega_0 = 10,\ T_0 = \frac{2\pi}{10} = \frac{\pi}{5}$$

$$(2)\ x_2(t) = e^{(-1+j)t} \qquad \text{Aperiodic}$$

$$(3)\ x_3(t) = 2\cos(3t + \frac{\pi}{4}) \qquad \omega_0 = 3,\ T_0 = \frac{2\pi}{3}$$

$$(4)\ x(t) = 2\cos(3t + \frac{\pi}{4}) + 3\cos(2t - \frac{\pi}{6})$$

$$T_{01} = \frac{2\pi}{3}, \quad T_{02} = \pi \qquad T_0 = SCM(T_{01}, T_{02}) = 2\pi$$

Discrete Complex Exponential

$$x[n] = c\,a^{n} \qquad c = |c|\,e^{j\theta} \qquad a = |a|\,e^{j\omega_0}$$

- Real exponential signals: c and a are real
  - $|a|>1$, as $n\uparrow$, $|x[n]|$ exponentially $\uparrow$
  - $|a|<1$, as $n\uparrow$, $|x[n]|$ exponentially $\downarrow$
  - $a>0$, all values of $x[n]$ have same sign
  - $a<0$, then the sign of $x[n]$ alternates
  - $a=0$, $x[n]=0$ for all n

Periodicity Properties

- Three definite difference between $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$

1) $e^{j(\omega_0+2\pi)t} = e^{j2\pi t} \cdot e^{j\omega_0 t} \neq e^{j\omega_0 t}$

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n} \cdot e^{j\omega_0 n} = e^{j\omega_0 n} \quad (\because \quad e^{j2\pi n} = 1 \ )$$

Periodicity Properties

2) For $e^{j\omega_0 t}$, the larger the magnitude of ω_0, the higher is the rate of oscillation

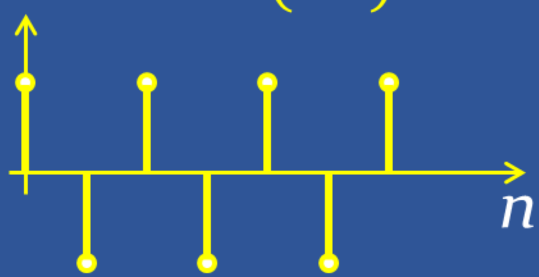
   For $e^{j\omega_0 n}$, the low-frequency have values of ω_0 near 0, 2π, and any other even multiple of π, while the high frequency are located near $\omega_0 \approx \pm\pi$ and other odd multiples of π

Example

- Q: Which one is a higher frequency signal?

$$\omega_0 = \pi \qquad \text{or} \qquad \omega_0 = \frac{3}{2}\pi$$

$$\cos(\pi n)$$



$$\cos\left(\frac{3\pi n}{2}\right) = \cos\left(\frac{\pi n}{2}\right)$$

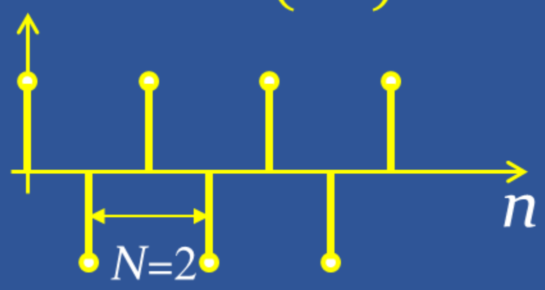
Example

- Q: Which one is a higher frequency signal?

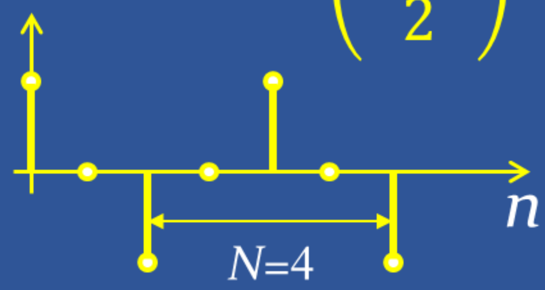
$$\omega_0 = \pi \qquad \text{or} \qquad \omega_0 = \frac{3}{2}\pi$$

A: $\omega_0 = \pi$

$$\cos(\pi n)$$

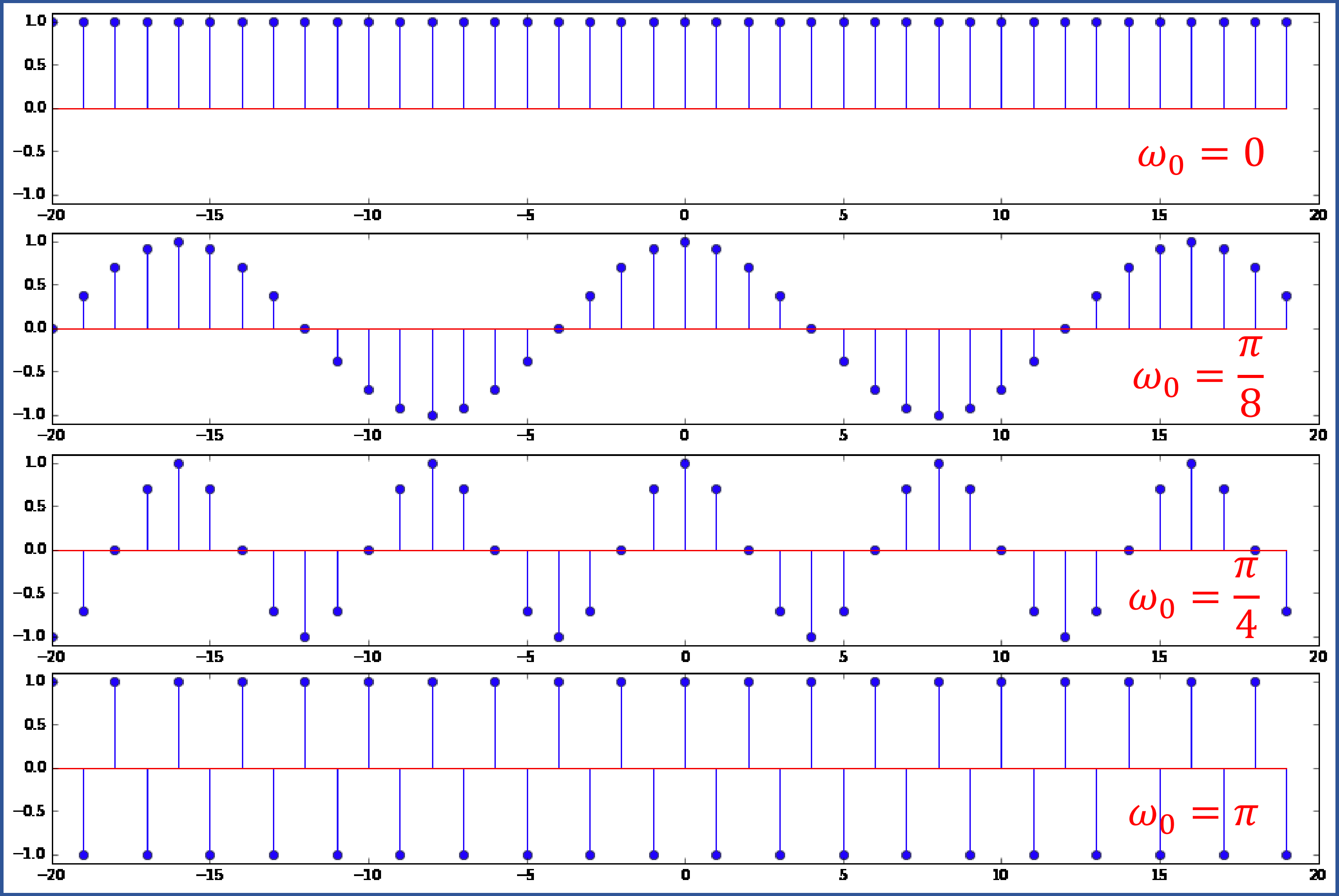


$$\cos\left(\frac{3\pi n}{2}\right) = \cos\left(\frac{\pi n}{2}\right)$$



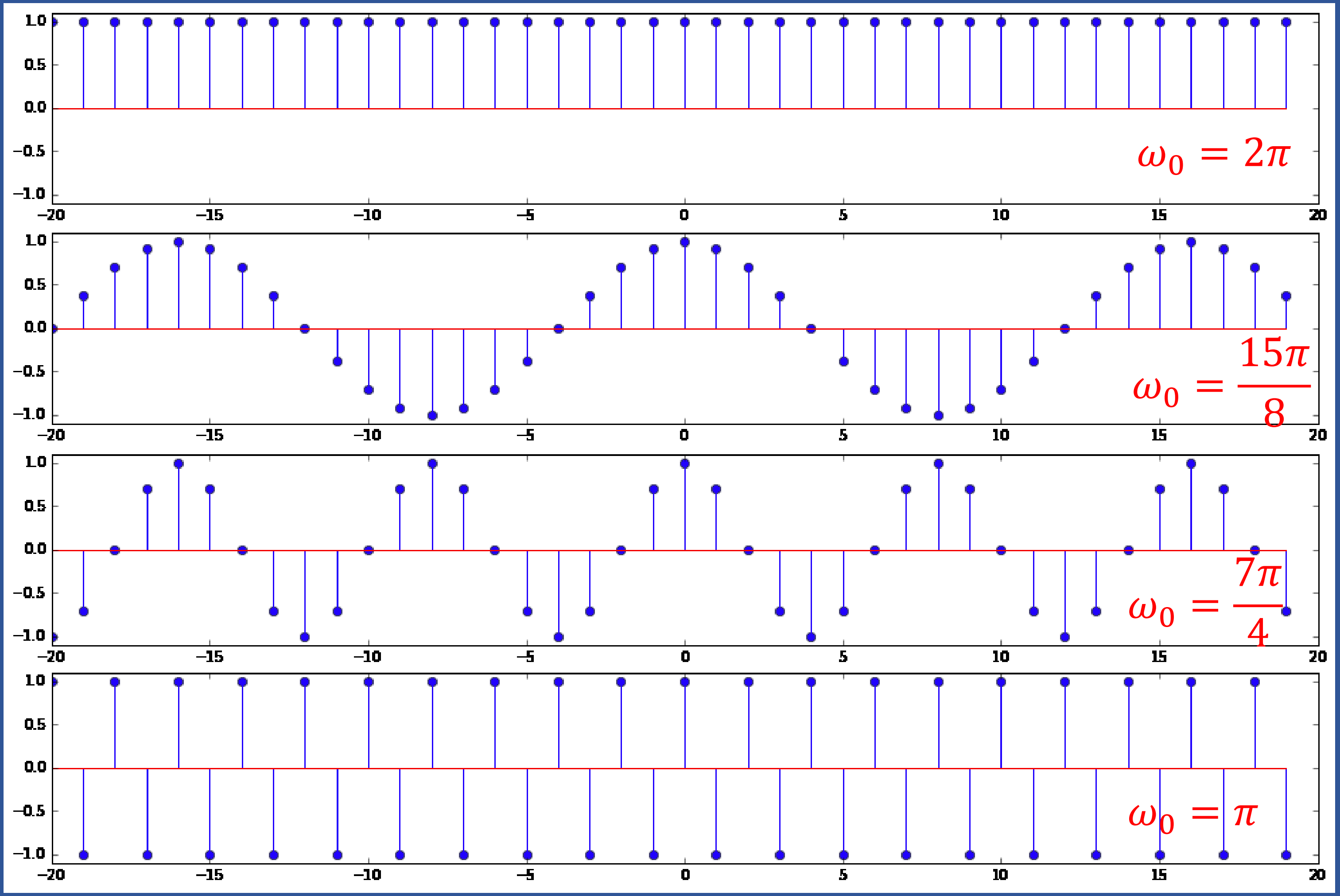
Example

$\cos(\omega_0 n)$



Example

$\cos(\omega_0 n)$



Periodicity Properties

3) $e^{j\omega_0 t}$ is periodic for any value of ω_0, while $e^{j\omega_0 n}$ may not be periodic for any ω_0

Examples:

e^{j3t} Periodic $T_0 = 2\pi / 3$

e^{j3n} Aperiodic

Periodicity Properties

- In order for $e^{j\omega_0 n}$ to be periodic with N>0, must

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n} \Rightarrow e^{j\omega_0 N} = 1$$

$$\Rightarrow \omega_0 N = 2\pi m$$

$$\Rightarrow \frac{\omega_0}{2\pi} = \frac{m}{N}$$ Rational

$$\Rightarrow N = \frac{2\pi m}{\omega_0}$$

Examples

$$N = \frac{2\pi m}{\omega_0}$$

- $x[n]=\cos(2\pi n/12)$
  - periodic $N=12$
- $x[n]=\cos(8\pi n/31)$
  - periodic $N=31$
- $x[n]=\cos(n/6)$
  - aperiodic
- $x[n]=\exp(j(2\pi/3)n)+\exp(j(3\pi/4)n)$
  - Periodic, $N=24$

Periodicity Properties

- In the continuous case, all of the harmonically related complex exponentials $e^{jk\omega_0 t}\big|_{\omega_0=2\pi/T}$ for k=0, ±1, ±2…, are distinct

- This is not true in discrete case

Why?

if $k_1 = k_2 + mN$ $\qquad e^{jk_1(2\pi/N)n} = e^{jk_2(2\pi/N)n}$

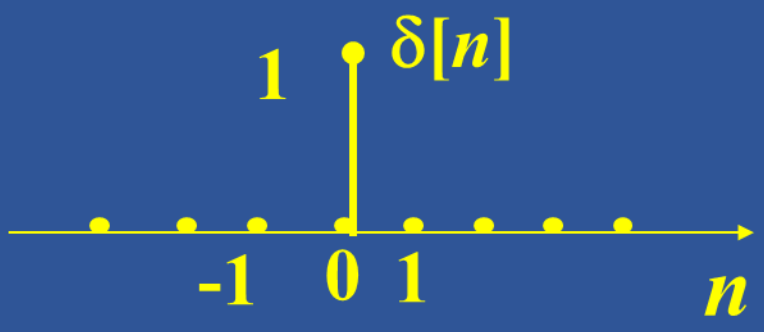
Periodicity Properties

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any ω_0	Only if $\omega_0=2\pi m/N$ for some integers $N>0$ and m
fundamental frequency ω_0	ω_0/m
fundamental period $2\pi/\omega_0$	$N=m(2\pi/\omega_0)$

Discrete Unit Impulse & Unit Step

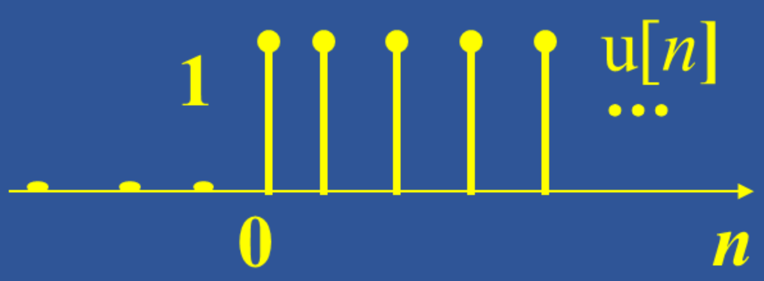
- Unit impulse **(unit sample)** is defined as

$$\delta[n] = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases}$$



- Unit step is defined as

$$u[n] = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases}$$



Discrete Unit Impulse & Unit Step

- The impulse is the first difference of the step

$$\delta[n] = u[n] - u[n-1]$$

- Conversely, the step is the running sum of unit sample

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

$$\text{or} \qquad u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

Sampling Property of Unit Impulse

- Sampling property

$$x[n]\delta[n] = x[0]\delta[n] = x[0]$$

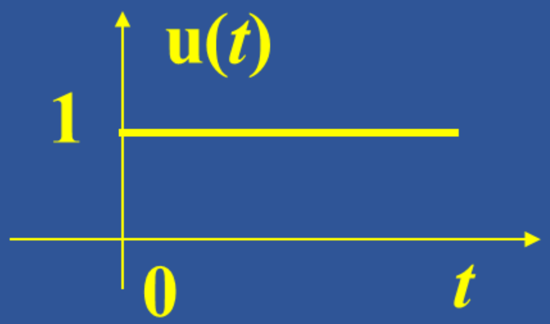
More generally

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0] = x[n_0]$$

Continuous Unit Step & Unit Impulse

- Unit step

$$u(t) = \begin{cases} 0, & t < o \\ 1, & t > 0 \end{cases}$$

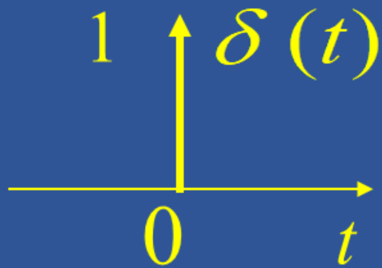


- The continuous unit step $u(t)$ is the running integral of unit impulse $\delta(t)$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\mathrm{d}\tau$$

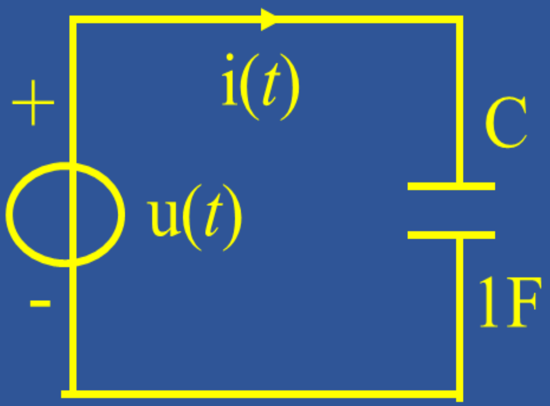
- $\delta(t)$ the first derivative of $u(t)$

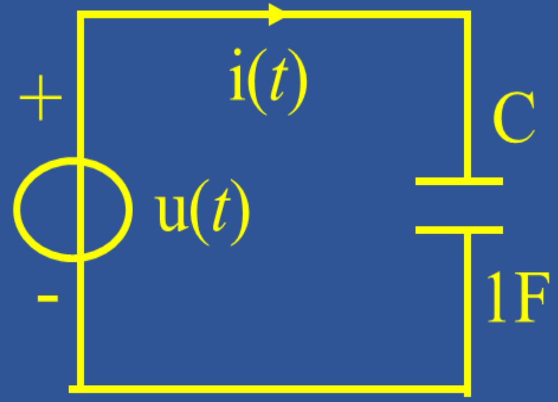
$$\delta(t) = \frac{du(t)}{dt}$$

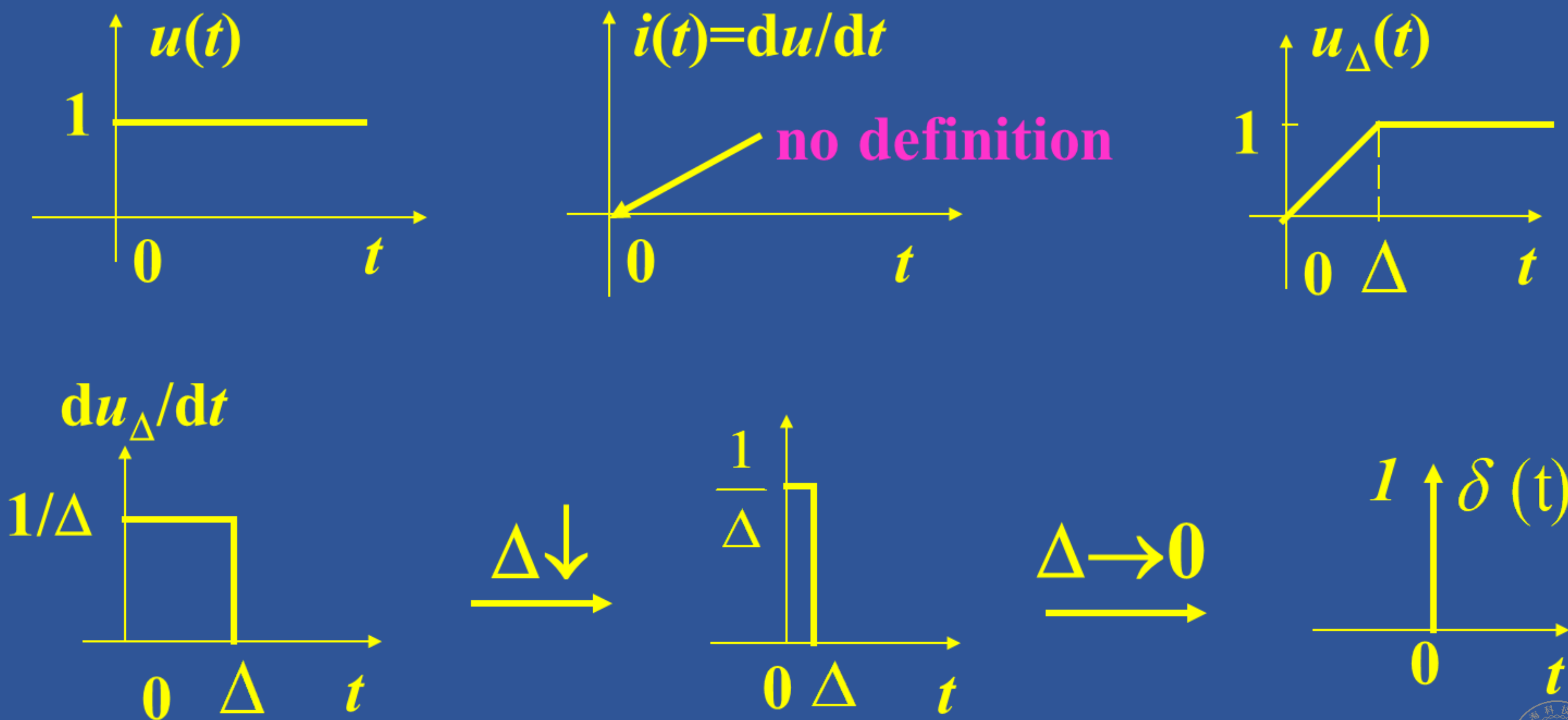


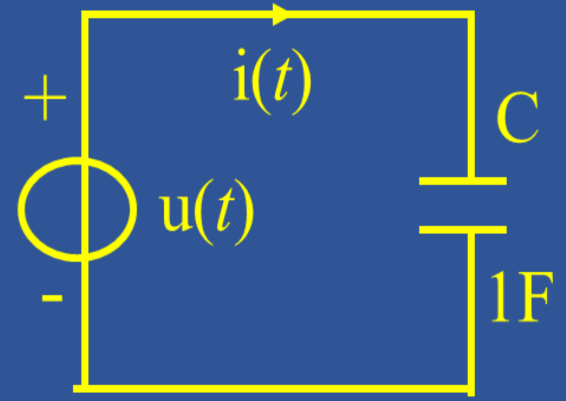
Continuous Unit Impulse

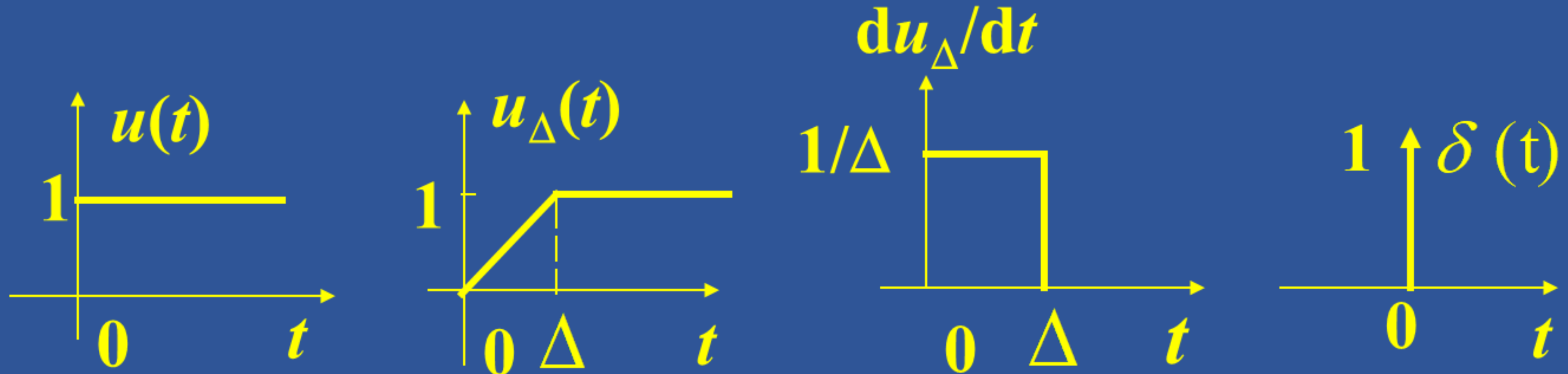
- How we can get $\delta(t)$?




i(t)
C
+
u(t)
-
1F


u(t)
1
0
t
i(t)=du/dt
no definition
0
t
u_Δ(t)
1
0
Δ
t
du_Δ/dt
1/Δ
0
Δ
t
Δ↓
1
Δ
0
Δ
t
Δ→0
1
δ (t)
0
t





$$\because u(t) = \lim_{\Delta\to 0} u_{\Delta}(t)$$

$$\therefore \frac{du(t)}{dt} = \lim_{\Delta\to 0}\frac{du_{\Delta}(t)}{dt} == \delta(t)$$

Sampling Property

- As with $\delta[n]$, $\delta(t)$ also has a very important sampling property

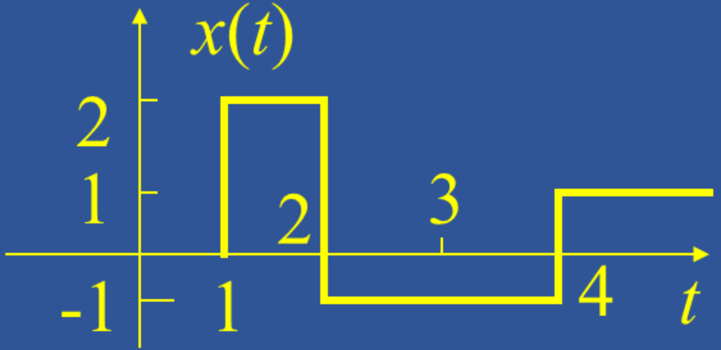
$$x(t)\delta(0) = x(0)\delta(t)$$

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

Example

(1) Calculate and sketch the $x'(t)$;

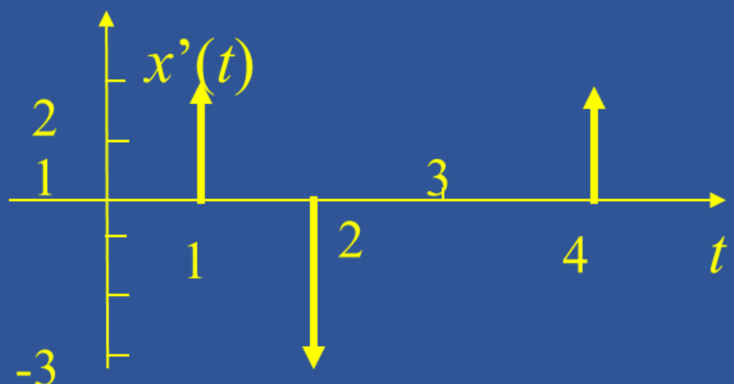
(2) Recover $x(t)$ from $x'(t)$.



Solution:

(1) $x(t) = 2u(t-1) - 3u(t-2) + 2u(t-4)$

$$\therefore \quad x'(t) = 2\delta(t-1) - 3\delta(t-2) + 2\delta(t-4)$$



(2) $x(t) = \int_0^{\infty} x'(t)dt$

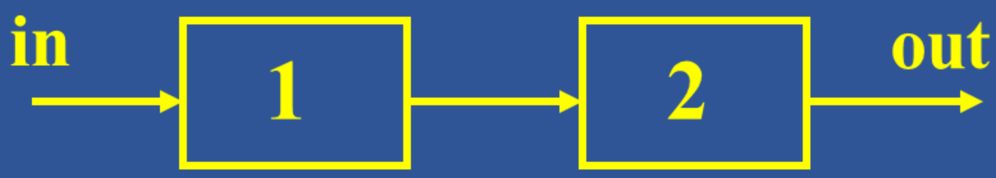
Continuous and Discrete Systems

- Input and output are continuous

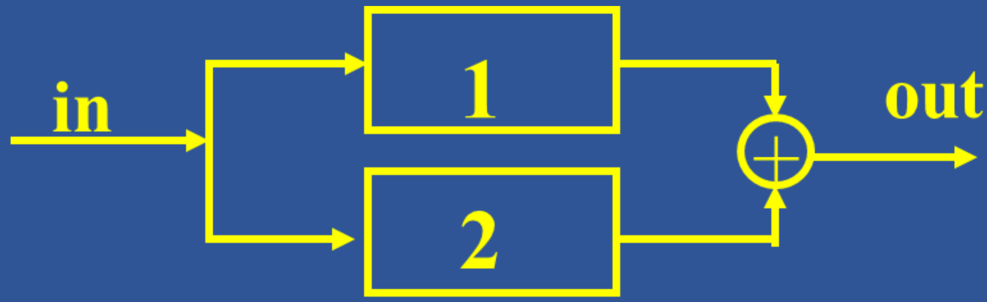
- Input and output are discrete

Interconnection of Systems

- Series (or cascade), parallel, feedback

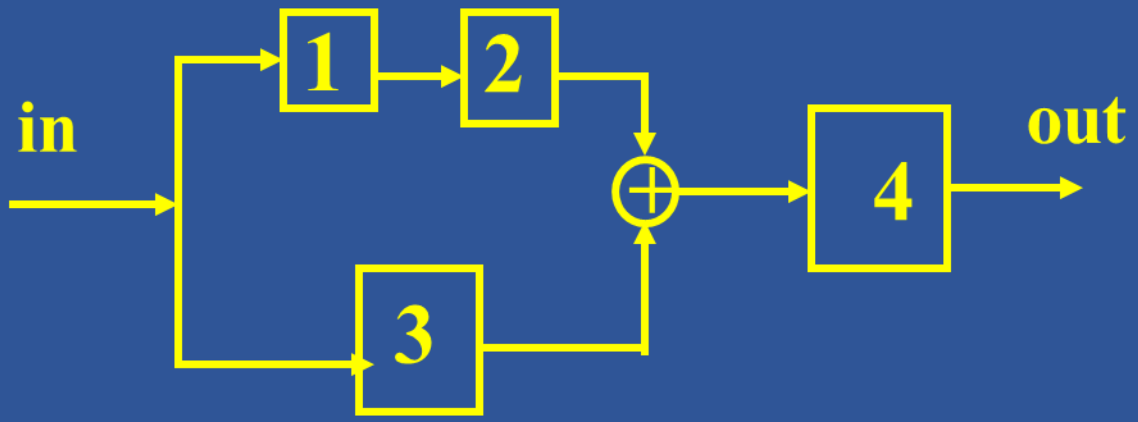


series

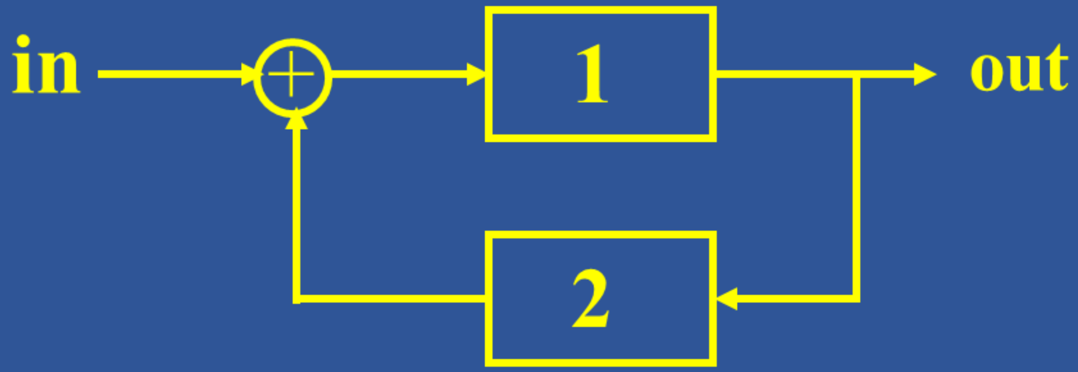


parallel

Interconnection of Systems



Series-parallel



Feedback

System Properties: Memory or Memoryless

- Memoryless system
  - Output is dependent only on the current input
- Examples:

$$y[n] = (2x[n] - x^2[n])^2$$

$$y(t) = x(t)$$

System Properties: Memory or Memoryless

- Memory system:
  - Output is dependent on the current and previous inputs

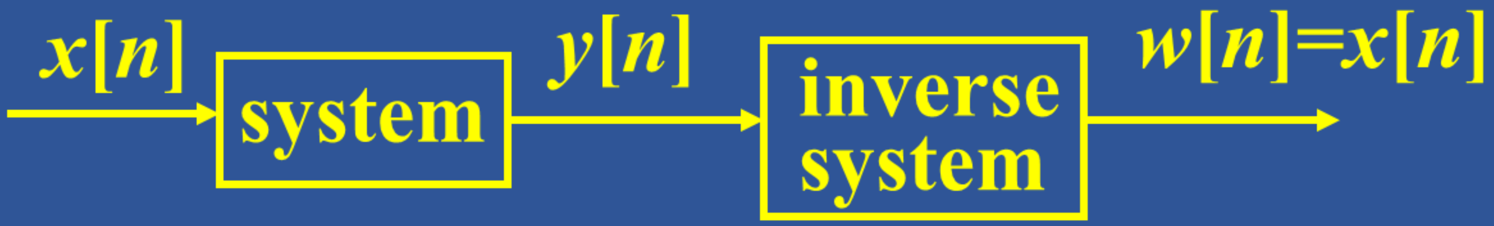
$$y[n] = \sum_{k=-\infty}^{n} x[k]$$ Accumulator

$$y[n] = x[n-1]$$ Delay

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$ Integrator

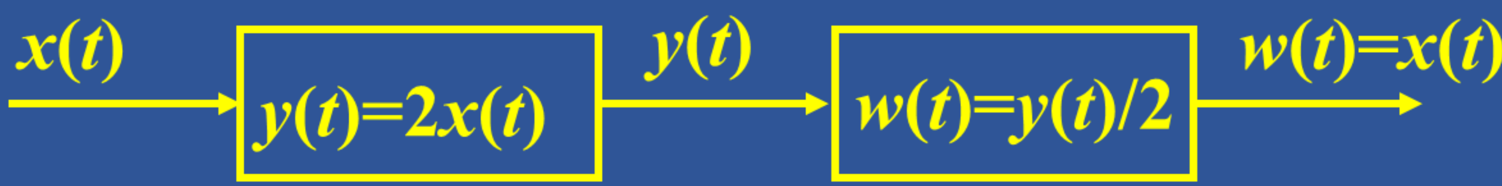
System Properties: Invertibility and Inverse System

- A system is said to be invertible if distinct inputs lead to distinct outputs.



Examples: Invertible Systems

$$y(t) = 2x(t) \qquad\qquad w(t) = y(t)/2$$



Examples: Invertible Systems

$$y[n]=\sum_{k=-\infty}^{n} x[k]$$ Accumulator

- The difference between two successive outputs is precisely the inputs

$$y[n]-y[n-1]=x[n]$$

$$x[n] \longrightarrow \boxed{y[n]=\sum_{k=-\infty}^{n} x[k]} \xrightarrow{y[n]} \boxed{w[n]=y[n]-y[n-1]} \longrightarrow w[n]=x[n]$$

Examples: Noninvertible Systems

$$y[n] = 0$$

All $x[n]$ leads to the same $y[n]$

$$y(t) = x^2(t)$$

Cannot determine the sign of the inputs

$$y(t) = \cos[x(t)]$$

$x(t)$ and $[x(t) + 2\pi k] \rightarrow$ same $y(t)$

System Properties: Causality

- Causal: the output at any time depends only on the inputs at the present time and in the past

$y[n] = (2x[n] - x^2[n])^2$	Causal
$y(t) = x(t)$	Causal
$y[n] = x[n] - x[n+1]$	Noncausal
$y(t) = x(t+1)$	Noncausal
$y[n] = x[-n]$	Noncausal

System Properties: Stability

- Stable: if the input to a system is bounded, then the output is also bounded

S_1: $y(t) = tx(t)$ Unstable

S_2: $y(t) = e^{x(t)}$ $|x(t)| < B \Rightarrow e^{-B} < |y(t)| < e^{B}$

Stable

System Properties: Time Invariance

- Time invariant: a time shift in the input signal results in an identical time shift in the output signal

If $x[n] \rightarrow y[n]$

Then $x[n-n_0] \rightarrow y[n-n_0]$

If $x(t) \rightarrow y(t)$

Then $x(t-t_0) \rightarrow y(t-t_0)$

Examples of time-varying system:

$$y[n] = nx[n]$$

$$y(t) = x(2t)$$

System Properties: Linearity

- For a system, if

$$x_1(t) \rightarrow y_1(t), \quad x_2(t) \rightarrow y_2(t)$$

$$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$$

then the system is linear

This is known as the superposition property (additivity and scaling or homogeneity)

Examples

$$y[n] = y[n-1] + x[n]$$

$$y(t) = \frac{1}{c}\int_{-\infty}^{t} x(\tau)\,d\tau$$

$$y(t) = tx(t)$$

Linear

$$y(t) = x^2(t)$$

$$y[n] = \Re e\{x[n]\}$$

$$y(t) = \sin[x(t)]$$

$$y[n] = 2x[n] + 3$$

Nonlinear