Numerical Optimization

Lecture 5: Duality

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本节内容

- 对偶问题 – 基本概念
- 基本理论 – 重点&难点
  - 弱对偶定理及其推论
  - 强对偶定理及其与单纯形法的关系
  - 互补定理(最优解的充分必要条件)
- 对偶单纯形法 – 方法
  - 求解的问题类
  - 如何操作
  - 算法终止于什么情况

1. Duality（对偶理论）

对偶问题(The dual)

给定数据$A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$

- 原始问题 **(primal):**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

$x \in \mathbb{R}^n$ − 原始变量

- 对偶问题 **(dual):**

$$\begin{array}{ll} \text{maximize} & b^T \lambda \\ \text{subject to} & \lambda^T A \leq c^T \end{array}$$

$\lambda \in \mathbb{R}^m$ − 对偶变量

- ❖注意分清对偶变量(dual variable)，对偶问题(dual problem)，对偶目标值(dual value)，对偶间隙(duality gap)，对偶理论(duality)
- ❖对于线性规划，对偶是相互的，即对偶问题的对偶是原始问题
- ❖为了确定任一线性规划问题的对偶，可以利用对称形式的对偶对或者标准形的对偶对！

对偶问题：经济解释

配餐问题：确定食品数量，满足营养需求，花费最小?

参数：n种食品，m种营养成份

c_j－第j种食品的单价

a_{ij}－每单位第j种食品含第i种营养的数量

b_i－为了健康，每天最少要摄入的第i种营养的数量

变量：x_j－摄入第j种食品的数量

Primal：

$$
\begin{array}{ll}
\text{minimize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \\
& x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0
\end{array}
$$

Dual：－保健品公司－药剂师、营养丸、定价问题

$$
\begin{array}{ll}
\text{maximize} & \lambda_1b_1 + \lambda_2b_2 + \cdots + \lambda_mb_m \\
\text{subject to} & \lambda_1a_{11} + \cdots + \lambda_ma_{m1} \leq c_1 \\
& \lambda_1a_{12} + \cdots + \lambda_ma_{m2} \leq c_2 \\
& \vdots \\
& \lambda_1a_{1n} + \cdots + \lambda_ma_{mn} \leq c_n \\
& \lambda_i \geq 0, i = 1, \cdots, m.
\end{array}
$$

对偶问题：一般形式的对偶

给定数据 A, b, c；记 A 的第 i 行为 a^i，A 的第 j 列为 a_j

- 原始问题(primal)：

$$
\begin{array}{llll}
\text{minimize} & c^{\mathrm{T}} x & & \\
\text{subject to} & a^i x \geq b_i, & & i \in M_1 \\
 & a^i x \leq b_i, & & i \in M_2 \\
 & a^i x = b_i, & & i \in M_3 \\
 & x_j \geq 0, & & j \in N_1 \\
 & x_j \leq 0, & & j \in N_2 \\
 & x_j \text{ 无限制} & & j \in N_3
\end{array}
$$

- 对偶问题(dual)：

$$
\begin{array}{llll}
\text{maximize} & \lambda^{\mathrm{T}} b & & \\
\text{subject to} & \lambda_i \geq 0, & & i \in M_1 \\
 & \lambda_i \leq 0, & & i \in M_2 \\
 & \lambda_i \text{ 无限制}, & & i \in M_3 \\
 & \lambda^{\mathrm{T}} a_j \leq c_j, & & j \in N_1 \\
 & \lambda^{\mathrm{T}} a_j \geq c_j, & & j \in N_2 \\
 & \lambda^{\mathrm{T}} a_j = c_j, & & j \in N_3
\end{array}
$$

Typo in (4.1.2) [1].

Duality Scheme

$$\min c^T x$$

$$Ax \left\{ \begin{array}{c} \geq \\ = \\ \leq \end{array} \right\} b$$

$$x \left\{ \begin{array}{c} \geq \\ = \\ \leq \end{array} \right\} 0$$

$$\max b^T y$$

$$A^T y \left\{ \begin{array}{c} \geq \\ = \\ \leq \end{array} \right\} c$$

$$y \left\{ \begin{array}{c} \geq \\ = \\ \leq \end{array} \right\} 0$$

Primal problem	**Dual problem**
minimize	maximize
Constraints	**Variables**
$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$	$y_i \geq 0$
$\sum_{j=1}^{n} a_{ij}x_j = b_i$	y_i is free
$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$	$y_i \leq 0$
Variables	**Constraints**
x_j is free	$a_j^T y = c_j$
$x_j \geq 0$	$a_j^T y \leq c_j$
$x_j \leq 0$	$a_j^T y \geq c_j$
$x_j = 0$	no constraint

对偶问题：例子

$$
\begin{array}{ll}
\text{minimize} & x_1 + 2x_2 + 3x_3 \\
\text{subject to} & -x_1 + 3x_2 = 5 \\
& 2x_1 - x_2 + 3x_3 \geq 6 \\
& x_3 \leq 4 \\
& x_1 \geq 0 \\
& x_2 \leq 0 \\
& x_3 \text{无限制}
\end{array}
$$

$$
\begin{array}{ll}
\text{maximize} & 5\lambda_1 + 6\lambda_2 + 4\lambda_3 \\
\text{subject to} & \lambda_1 \text{无限制} \\
& \lambda_2 \geq 0 \\
& \lambda_3 \leq 0 \\
& -\lambda_1 + 2\lambda_2 \leq 1 \\
& 3\lambda_1 - \lambda_2 \geq 2 \\
& 3\lambda_2 + \lambda_3 = 3
\end{array}
$$

弱对偶定理(Weak Duality)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \quad \longleftrightarrow \quad \begin{array}{ll} \text{maximize} & b^T \lambda \\ \text{subject to} & \lambda^T A \leq c^T \end{array}$$

弱对偶定理. 设$\boldsymbol{x}$和$\boldsymbol{\lambda}$分别是原问题和对偶问题的可行解，则$\boldsymbol{c}^T\boldsymbol{x} \geq \boldsymbol{\lambda}^T\boldsymbol{b}$. (定理4.1.1,[1])

推论1. 设$\boldsymbol{x}$和$\boldsymbol{\lambda}$分别是原始问题和对偶问题的可行解，如果$\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{\lambda}^T\boldsymbol{b}$. 设$\boldsymbol{x}$和$\boldsymbol{\lambda}$分别是原问题和对偶问题的最优解

推论2. 如果原始问题与对偶问题之一无界，则另一个问题没有可行解

强对偶定理(Strong Duality)

强对偶定理. 如果原始问题和对偶问题之一有解，则另一个问题也有解，且最优值相等.

原问题 \ 对偶问题	不可行	无(上)界	有最优解
不可行	√	√	×
无(下)界	√	×	×
有最优解	×	×	√

思考题：写出两阶段法中第一阶段的辅助问题的对偶问题？这个对偶问题有最优解吗？为什么？

对于一般形式的线性规划－－利用凸集分离定理证明！

与单纯形法的关系

如何由原始问题的解得到对偶问题的解?

定理. 设标准形线性规划问题有最优解 $\boldsymbol{x}^*$， $\boldsymbol{B}$ 是最优基本可行解对应的基，则

$$\boldsymbol{\lambda}^* = (\boldsymbol{c}_{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{B}^{-1})^T$$

是其对偶问题的最优解.

特例：当系数矩阵$\boldsymbol{A}$中有单位矩阵时，如何确定单纯形乘子?

例子：与单纯形法的关系

考虑问题

$$\begin{aligned} \text{minimize} \quad & -x_1 - 4x_2 - 3x_3 \\ \text{subject to} \quad & 2x_1 + 2x_2 + x_3 \le 4 \\ & x_1 + 2x_2 + 2x_3 \le 6 \\ & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{aligned}$$

对偶问题

$$\begin{aligned} \text{maximize} \quad & 4\lambda_1 + 6\lambda_2 \\ \text{subject to} \quad & 2\lambda_1 + \lambda_2 \le -1 \\ & 2\lambda_1 + 2\lambda_2 \le -4 \\ & \lambda_1 + 2\lambda_2 \le -3 \\ & \lambda_1 \le 0, \lambda_2 \le 0 \end{aligned}$$

引入松弛变量→标准形→利用单纯形法求解

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	2	$\boxed{2}$	1	1	0	4
	1	2	2	0	1	6
r^{T}	-1	-4	-3	0	0	0

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	2
	-1	0	$\boxed{1}$	-1	1	2
r^{T}	3	0	-1	2	0	8

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	$\frac{3}{2}$	1	0	1	$-\frac{1}{2}$	1
	-1	0	1	-1	1	2
r^{T}	2	0	0	1	1	10

why?

原问题最优解

$$x_1^* = 0$$
$$x_2^* = 1$$
$$x_3^* = 2$$

对偶问题最优解

$$\lambda_1^* = -1$$
$$\lambda_2^* = -1$$

互补 Complementary condition

Primal problem	Dual problem
$\min c^T x$	$\max b^T y$
$Ax = b$	$A^T y \leq c$
$x \geq 0$	

We need to solve:

$$Ax = b \qquad A^T y + s = c$$

$$x \geq 0 \qquad s \geq 0$$

$$\boxed{b^T y = c^T x}$$

The last condition $b^T y = c^T x$ can be written as

$$0 = c^T x - b^T y = c^T x - (Ax)^T y = (c - A^T y)^T x = s^T x$$

Because $s \geq 0$ and $x \geq 0$ the condition $s^T x = 0$ implies

$$\boxed{s_i x_i = 0 \qquad \forall i}$$

2. Interpretation of Duality

对偶的解释：线性系统

$$\min_x \quad c^T x \quad \text{s.t. } a_i^T x \le b_i, i = 1,\ldots,m\,.$$

考虑线性系统：

$$c^T x < v, \quad a_i^T x - b_i \le 0,\ i = 1,\ldots,m$$

如果无解，则v太小，是原问题最优值的下界

对于线性系统任意的解x，必满足对$\forall \lambda \ge 0$

$$c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i) < v$$

线性系统没有可行解

$$\Longleftrightarrow \quad \exists \lambda \ge 0, \text{ such that } \min_x L(x, \lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i) \ge v$$

对偶的解释：线性系统

$$\min_x \quad f(x) := c^T x \quad \text{s.t. } a_i^T x \le b_i, i = 1,\ldots,m\,.$$

找使得线性系统

$$c^T x < v, \quad a_i^T x - b_i \le 0,\ i = 1,\ldots,m$$

无解的最大的v，则v就是原问题的最优解

$$\text{无解} \iff v \le \min_x L(x,\lambda) = c^T x + \sum_{i=1}^m \lambda_i (a_i^T x - b_i) \text{ with } \lambda \ge 0$$

$$\text{选取最大的}v \implies v = \max_{\lambda \ge 0} \min_x L(x,\lambda) = c^T x + \sum_{i=1}^m \lambda_i (a_i^T x - b_i)$$

v是原问题最优解$\implies$ strong duality

$$g(\lambda) = \min_x L(x,\lambda) = c^T x + \sum_{i=1}^m \lambda_i (a_i^T x - b_i) \le f(x^*) \le f(x)$$

$\lambda \ge 0$以及feasible $x \implies$ weak duality

对偶的解释：惩罚函数

$$\min_x \quad c^T x \quad \text{s.t. } a_i^T x \le b_i, i = 1,\ldots,m\,.$$

$$\min_x \quad L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

注意暗含着λ的选值:

需要令$\lambda_i \ge 0$，避免得到$a_i^T x > b_i$

对偶的解释：惩罚函数

$$\min_x \quad c^T x \quad \text{s.t. } a_i^T x \le b_i, i = 1,\ldots,m\,.$$

$$\min_x \quad L(x,\lambda) = c^T x + \sum_{i=1}^m \lambda_i (a_i^T x - b_i)$$

- 给定λ，在x上极小化$L(x,\lambda)$；但这样未必能等价于原问题
- 如果最优的x^*和原问题最优解一致的话，必须保证原可行。故$a_i^T x - b_i > 0 \implies \lambda_i = +\infty$来促使原可行$\Longrightarrow$极大化约束的违反量
- 不极大化则可能由于$c^T x$成为主导，而得不到$a_i^T x - b_i \le 0$

故得到对偶问题：

$$\max_{\lambda \ge 0} \ \min_x \quad L(x,\lambda) = c^T x + \sum_{i=1}^m \lambda_i (a_i^T x - b_i)$$

对偶的解释：惩罚函数

得到Primal问题(min-max)：

$$\min_x f(x) = \min_x \max_\lambda L(x,\lambda) = c^T x + \sum_{i=1}^m \lambda_i (a_i^T x - b_i)$$

$$f(x)\begin{cases} = +\infty & \text{if } \exists i, a_i^T x > b_i \\ < +\infty & \text{if } \forall i, a_i^T x \le b_i \end{cases}$$

$\Longrightarrow$ We don't care about this case.

$\Longrightarrow$ This implies primal feasibility.

得到对偶问题(max-min)：

$$\max_{\lambda\ge 0} g(\lambda) = \max_{\lambda\ge 0} \min_x L(x,\lambda) = c^T x + \sum_{i=1}^m \lambda_i (a_i^T x - b_i)$$

$$g(\lambda)\begin{cases} = -\infty & \text{if } c + \sum_{i=1}^m \lambda_i a_i \ne 0 \\ > -\infty & \text{if } c + \sum_{i=1}^m \lambda_i a_i = 0 \end{cases}$$

$\Longrightarrow$ We don't care about this case.

$\Longrightarrow$ This implies dual feasibility.

对偶的解释：惩罚函数

Weak duality: for any primal-dual feasible (x, λ)

$$f(x) = \max_{\lambda \geq 0} \ L(x, \lambda) \geq L(x, \lambda) \geq \min_{x} L(x, \lambda) = g(\lambda)$$

对偶的解释：惩罚函数

Strong duality: based on previous discussion, for primal optimal x^* (it must be primal feasible) and dual optimal λ^*(it must be dual feasible)

$$f(x^*) = \min_{x} \max_{\lambda \geq 0} L(x,\lambda) = c^T x^* + \sum_{i=1}^{m} \lambda_i(x^*)(a_i^T x^* - b_i) = c^T x^*$$

$\lambda_i(x^*)$ is the maximizer of $L(x^*,\lambda)$, and so must be dual feasible

$$g(\lambda^*) = \max_{\lambda \geq 0} \min_{x} L(x,\lambda) = c^T x(\lambda^*) + \sum_{i=1}^{m} \lambda_i^*(a_i^T x(\lambda^*) - b_i) = c^T x(\lambda^*)$$

$x(\lambda^*)$ is the minimizer of $L(x,\lambda^*)$, and so must be primal feasible

$$f(x^*) = \max_{\lambda \geq 0} \; L(x^*,\lambda) = g(\lambda(x^*)) \geq g(\lambda) \implies g(\lambda(x^*)) = g(\lambda^*)$$

$$g(\lambda^*) = \min_{x} \; L(x,\lambda^*) = f(x(\lambda^*)) \leq f(x) \implies f(x(\lambda^*)) = f(x^*)$$

Saddle point: primal-dual (x^*, λ^*) so that for any $x, \lambda \geq 0$

$$L(x, \lambda^*) \geq L(x^*, \lambda^*) \geq L(x^*, \lambda)$$

$$\Longrightarrow \quad L(x^*, \lambda^*) \geq \max_{\lambda \geq 0} L(x^*, \lambda)$$

x^* must be feasible, otherwise the max = $+\infty$

In fact, we have $L(x^*, \lambda^*) = \max_{\lambda \geq 0} L(x^*, \lambda)$. Therefore,

$$a_i^T x^* - b_i < 0 \implies \lambda_i^* = 0 \implies \sum_{i=1}^{m} \lambda_i^* (a_i^T x^* - b_i) = 0$$

$$c^T x^* = L(x^*, \lambda^*) \leq \min_x L(x, \lambda^*) \leq \min_{x \text{ feas.}} L(x, \lambda^*) \leq \min_{x \text{ feas.}} c^T x \leq c^T x^*$$

Saddle Point $\Longrightarrow$ Primal-Dual feasible + Complemtarity

$$\text{Saddle Point} \iff \text{Primal-dual optimal}$$

$$\left.\begin{array}{l}\text{Primal feasible}\\ \text{Dual feasible}\\ \text{Duality Gap = Primal}-\text{dual}=0\end{array}\right\} \iff \text{Primal-dual optimal}$$

$$\left.\begin{array}{l}\text{Primal feasible}\\ \text{Dual feasible}\\ \text{Complementarity}\end{array}\right\} \iff \text{Primal-dual optimal}$$

Complementary: 指的是对于inactive的约束，其乘子为0

当$a_i^T x^* < b_i$时，$\lambda_i^* = 0$

$$L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

$$L(x^*,\lambda) = c^T x^* + \sum_{i=1}^{m} \lambda_i (a_i^T x^* - b_i) \le f(x^*) = L(x^*,\lambda^*) \le L(x,\lambda^*)$$

Primal-Dual feasible + Complementarity $\Longrightarrow$ Saddle Point

$$\min \; b^T x \quad \text{s.t. } A^T x \le c$$

The Lagrangian is

$L(x,\lambda) = b^T x + \lambda^T (A^T x - c)$, with $\lambda \ge 0$

Let's determine λ, using the dual problem

The dual objective is

$$g(\lambda,\mu) = \min_x L(x,\lambda) = b^T x + \lambda^T (A^T x - c)$$
$$= \min_x (b + A\lambda)^T x + \lambda^T c$$

Notice that we want to maximize $g(\lambda,\mu)$. So we only care about the case $g(\lambda,\mu) > -\infty$, which means $b + A\lambda = 0$

$$\Longrightarrow \quad \max \; c^T b \quad \text{s.t. } A\lambda = -b, \; \lambda \ge 0$$

$$\min \ c^T x \quad \text{s.t. } Ax = b,\ x \geq 0 \qquad \Longrightarrow \quad Ax \geq b,\ Ax \leq b$$

The Lagrangian is

$$L(x, \lambda) = c^T x + u^T(-Ax + b) + w^T(Ax - b) - v^T x, \text{with } u, w, v \geq 0$$

$$= c^T x - (u - w)^T(Ax - b) - v^T x$$

$$= c^T x - \lambda^T(Ax - b) - v^T x, \quad \text{with } \lambda \text{ free}, v \geq 0$$

The dual objective is

$$g(\lambda, v) = \min_x L(x, \lambda, v) = \min_x (c - \lambda^T A - v)^T x + \lambda^T b$$

Notice that we want to maximize $g(\lambda, \mu)$. So we only care about the case $g(\lambda, \mu) > -\infty$, which means $c - \lambda^T A - v = 0$

$$\max \lambda^T b \quad \text{s.t. } A^T \lambda \leq c,\ \lambda \geq 0$$

$$\begin{aligned} \textbf{minimize} \quad & -x_1 - 4x_2 - 3x_3 \\ \textbf{subject to} \quad & 2x_1 + 2x_2 + x_3 = 4 \\ & x_1 + 2x_2 + 2x_3 \le 6 \\ & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{aligned}$$

$$\begin{aligned} L(x,\lambda) &= -x_1 - 4x_2 - 3x_3 + \lambda_1(2x_1 + 2x_2 + x_3 - 4) \\ &\quad + \lambda_2(x_1 + 2x_2 + 2x_3 - 6) - \mu_1 x_1 - \mu_2 x_2 - \mu_3 x_3 \\ &= \boxed{(-1 + 2\lambda_1 + \lambda_2 - \mu_1)} x_1 + \boxed{(-4 + 2\lambda_1 + 2\lambda_2 - \mu_2)} x_2 \\ &\quad + \boxed{(-3 + \lambda_1 + 2\lambda_2 - \mu_3)} x_3 \boxed{- 4\lambda_1 - 6\lambda_2} \\ &\quad \lambda_1 \text{ is free }, \lambda_2 \ge 0,\ \mu_1 \ge 0,\ \mu_2 \ge 0,\ \mu_3 \ge 0 \end{aligned}$$

再写出对偶问题：

3. Dual Simplex Method
（对偶单纯形法）

对偶单纯形法：对偶可行基本解

$$\begin{aligned} &\text{maximize} \quad b^T\lambda \\ &\text{subject to} \quad \lambda^T A \le c^T \end{aligned}$$

如果知道哪些是active constraints，问题该如何求解?

定义 假设$\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$是$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$的基本解. 如果$\boldsymbol{\lambda}^T = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}$是对偶问题的可行解，即$(\boldsymbol{r}^T =)\ \boldsymbol{c}^T - \boldsymbol{\lambda}^T\boldsymbol{A} \ge \boldsymbol{0}$. 则称 $\boldsymbol{x}$ 是标准形问题的对偶可行基本解.

初始对偶可行基本解 → 新的对偶可行基本解 → ……迭代…… → “原始可行＋对偶可行”的基本解！

基本解(互补性)＋可行＋对偶可行＝最优解

对偶单纯形法：

设对偶可行基本解 $\boldsymbol{x}$ 对应的基$\boldsymbol{B} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_p, \ldots, \boldsymbol{a}_m]$

不妨设$\bar{b}_p < 0$；此外还假设$\boldsymbol{\lambda}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}$非退化，即

$$\begin{aligned} (r_j =) c_j - \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{a}_j &= 0, \quad j = 1, 2, \cdots, m \\ (r_j =) c_j - \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{a}_j &> 0, \quad j = m+1, \cdots, n \end{aligned}$$

对偶问题的非退化极点是$\mathbb{R}^m$中恰好m个超平面的交点！

回忆修正单纯形法

令$\hat{\boldsymbol{\lambda}}^T = \boldsymbol{\lambda}^T - \epsilon \boldsymbol{u}_p$, 其中$\boldsymbol{u}_p$是$\boldsymbol{B}^{-1}$的第$p$行

于是有$\hat{\boldsymbol{\lambda}}^T \boldsymbol{b} = \boldsymbol{\lambda}^T \boldsymbol{b} - \epsilon \bar{\boldsymbol{b}}_p$ 对偶目标函数值

目的：找新的$\hat{\lambda}$使m个等式中的某个与其他$n-m$个不等式中的某个角色互换(即$\hat{\lambda}$是对偶问题的与λ相邻的极点)，同时使对偶问题的目标函数值增大！

对偶单纯形法

更新：

$$\hat{r}_j = c_j - \lambda^T a_j = r_j + \epsilon u_p^T a_j$$

$$\hat{r}_p = \epsilon$$

$$\hat{r}_j = 0, \quad j = 1, 2, \ldots, m, j \neq p$$

$$\hat{r}_j = r_j + \epsilon y_{pj}, \quad j = m+1, m+2, \ldots, n.$$

选$\hat{\epsilon}$保证dual feasible

$$\hat{\epsilon} = \frac{r_q}{-y_{pq}} = \min\left\{\frac{r_j}{-y_{pj}} : y_{pj} < 0, j = 1, \cdots, n\right\}$$

出基变量：取负值的基变量(Infeasible Primal Variable)

进基变量：其p行的负元素，且取到最小正比值

对偶单纯形法：计算步骤

步0 给定对偶可行基本解对应的单纯形表.

步1 若对每个 i 都有$\bar{b}_i \geq 0$，停；当前**DFBS**是最优的.

步2 选取 p 满足$\bar{b}_p < 0$，这时，第 p 个基变量出基.

步3 若$(y_{p1}, y_{p2}, \ldots, y_{pn}) \geq 0$，问题无可行解；否则，选 q 满足

$$\hat{\epsilon} = \frac{r_q}{-y_{pq}} = \min\left\{\frac{r_j}{-y_{pj}} : y_{pj} < 0, j = 1, \cdots, n\right\}$$

步4 以 y_{pq} 为转轴元进行转轴，更新单纯形表，返步1.

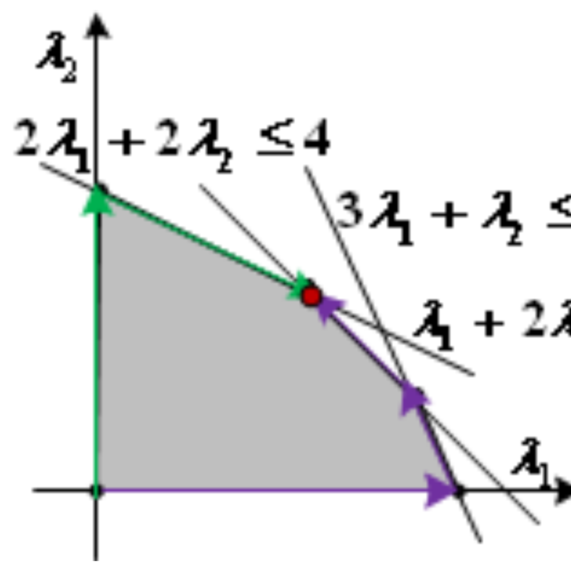
对偶单纯形法：例子

$$\begin{array}{ll} \text{minimize} & 3x_1 + 4x_2 + 5x_3 \\ \text{subject to} & x_1 + 2x_2 + 3x_3 \geq 5 \\ & 2x_1 + 2x_2 + x_3 \geq 6 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{array}$$

1) 写出对偶问题并用图解法求解。
2) 用对偶单纯形法求解所给问题。

解. 对偶问题为

$$\begin{array}{ll} \text{maximize} & 5\lambda_1 + 6\lambda_2 \\ \text{subject to} & \lambda_1 + 2\lambda_2 \leq 3 \\ & 2\lambda_1 + 2\lambda_2 \leq 4 \\ & 3\lambda_1 + \lambda_2 \leq 5 \\ & \lambda_1 \geq 0, \lambda_2 \geq 0 \end{array}$$



对偶单纯形法：例子

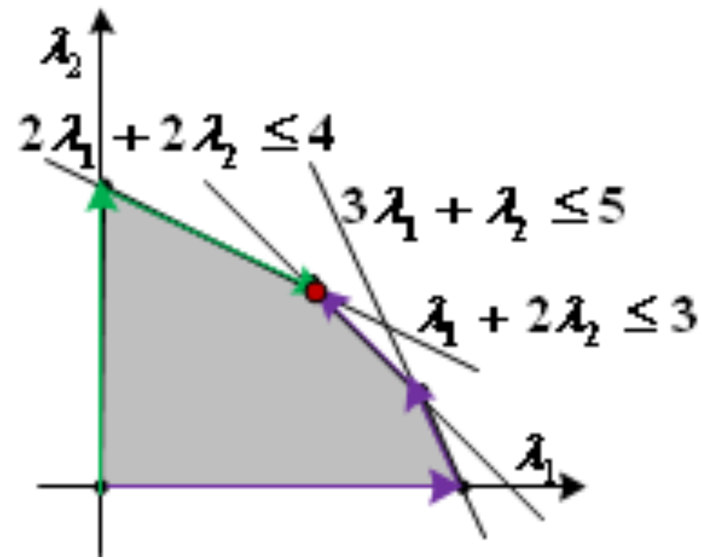
$$
\begin{array}{ll}
\textbf{minimize} & 3x_1 + 4x_2 + 5x_3 \\
\textbf{subject to} & x_1 + 2x_2 + 3x_3 \geq 5 \\
& 2x_1 + 2x_2 + x_3 \geq 6 \\
& x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\end{array}
$$

引入盈余变量；并给等式两边同乘−1；得初始表格/第一张单纯形表

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	-1	-2	-3	1	0	-5
	$\boxed{-2}$	-2	-1	0	1	-6
r^{T}	3	4	5	0	0	0

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	0	$\boxed{-1}$	$-\frac{5}{2}$	1	$-\frac{1}{2}$	-2
	1	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	3
r^T	0	1	$\frac{7}{2}$	0	$\frac{3}{2}$	-9

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	0	1	$\frac{5}{2}$	-1	$\frac{1}{2}$	2
	1	0	-2	1	-1	1
r^T	0	0	1	1	1	-11



最优解：$x^* = (1, 2, 0)^T$

单纯形乘子的迭代为$(0,0) \to (0,\frac{3}{2}) \to (1,1)$

若第一步让x_4出基，单纯形乘子的迭代为

$$(0,0) \to (\frac{5}{3},0) \to (\frac{3}{2},\frac{1}{2}) \to (1,1)$$

对偶单纯形法：收敛性

定理. 如果标准形线性规划问题的任意的对偶可行基本解所对应的非基变量的相对费用系数大于零，则对偶单纯形法在有限步内终止.

- 如果线性规划问题可以用对偶单纯形法求解，则计算结果只能是不可行或者有解！
- 如果线性规划问题可以用(原)单纯形法求解，则计算结果只能是无界或有解
- 两阶段法可以求解任一线性规划问题；
  - 第I阶段的结果为可行或者不可行两种；
  - 对于可行的，在第II阶段可得问题无界或有解！

对偶单纯形法：启动

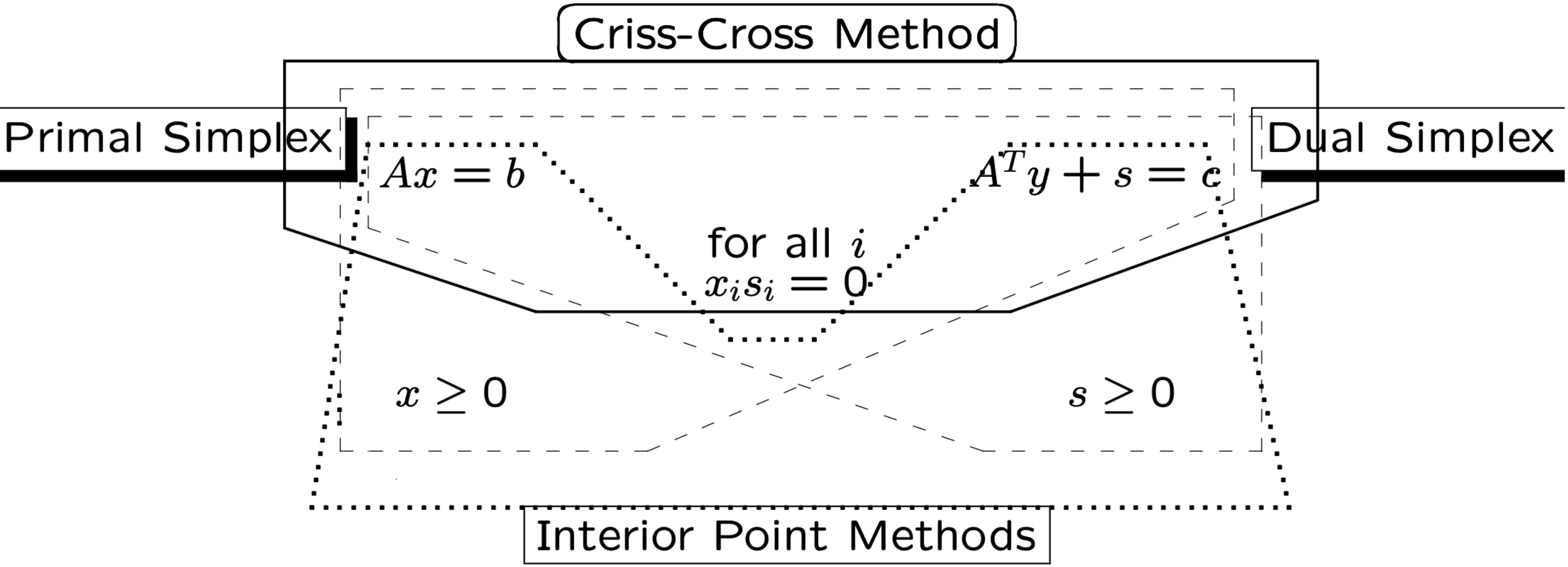
◎ 典型情况(有显然的对偶可行基本解)

⊙ “不等式约束” + “$\boldsymbol{x} \geq \mathbf{0}$” + “$\boldsymbol{c} \geq \mathbf{0}$” + “ **min** ”

本节的例题

◎ 一般情况 4.2.3节[1]

	单纯形法			对偶单纯形法		
	原始可行	对偶可行	互补性	原始可行	对偶可行	互补性
初始	√	×	√	×	√	√
迭代中	√	×	√	×	√	√
最优终止	√	√	√	√	√	√



All algorithms for LP keep a part of the optimality criteria valid while working towards the others.