Qubit Measurements, Quantum Collapse and Density Operators

by José Luis Gómez-Muñoz
http://homepage.cem.itesm.mx/lgomez/quantum/
jose.luis.gomez@itesm.mx

Introduction

This is a tutorial on the use of Quantum`Computing` *Mathematica* add-on to simulate qubit measurements and wavefunction collapse

Load the Package

First load the Quantum`Computing` package. Write:
Needs["Quantum`Computing`"];
then press at the same time the keys SHIFT-ENTER to evaluate. *Mathematica* will load the package:

```
Needs["Quantum`Computing`"]
```

```
Quantum`Computing` Version 2.2.0. (July 2010)
A Mathematica package for Quantum Computing
  in Dirac bra-ket notation and plotting of quantum circuits
by José Luis Gómez-Muñoz

Execute SetComputingAliases[] in order to use
  the keyboard to enter quantum objects in Dirac's notation
SetComputingAliases[] must be executed again in each new notebook that is created
```

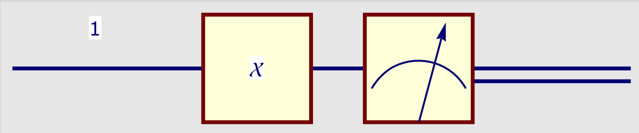
In order to use the keyboard to enter quantum objects write:
SetComputingAliases[ ];
then press at the same time the keys SHIFT-ENTER to evaluate. The semicolon prevents *Mathematica* from printing the help message. Remember that SetComputingAliases[ ] must be evaluated again in each new notebook:

```
SetComputingAliases[];
```

Plot of Measurement Meters in Quantum Circuits

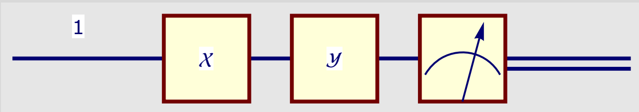
Next example shows the use of the command **QubitMeasurement** in order to plot measuring meters in quantum circuits. Remember that the $\chi_{\hat{\square}}$ template can be entered by pressing [ESC]xg[ESC], the qubit $\hat{\square}$ template can be entered pressing the keys [ESC]qb[ESC]:

$$\texttt{QuantumPlot}\left[\texttt{QubitMeasurement}\left[\mathcal{X}_{\hat{1}},\ \{\hat{1}\}\right]\right]$$



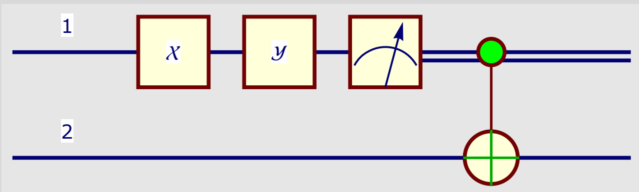
Here the measurement is taking place after two gates have been applied to the qubit. Remember that the infix "operator application" symbol (CenterDot ·) can be entered by pressing the keys [ESC]on[ESC]:

$$\texttt{QuantumPlot}\left[\texttt{QubitMeasurement}\left[\mathcal{Y}_{\hat{1}} \cdot \mathcal{X}_{\hat{1}},\ \{\hat{1}\}\right]\right]$$



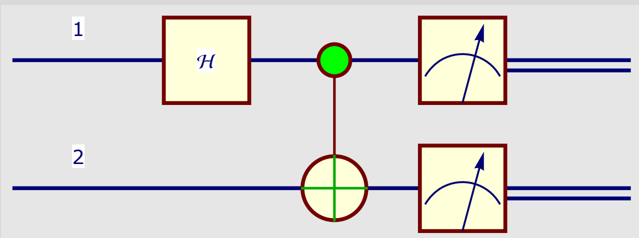
Next expression shows how to indicate that some gates are applied before the measurement and other gates are applied after the measurement:

$$\texttt{QuantumPlot}\left[\mathcal{C}^{\{\hat{1}\}}\left[\mathcal{NOT}_{\hat{2}}\right] \cdot \texttt{QubitMeasurement}\left[\mathcal{Y}_{\hat{1}} \cdot \mathcal{X}_{\hat{1}},\ \{\hat{1}\}\right]\right]$$



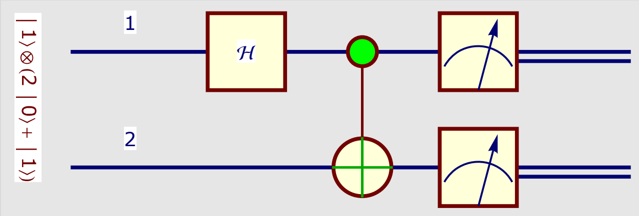
The second argument of **QubitMeasurement** is a list of all the qubits are going to be measured. Next example shows how to specify two measuring meters:

$$\texttt{QuantumPlot}\left[\texttt{QubitMeasurement}\left[\mathcal{C}^{\{\hat{1}\}}\left[\mathcal{NOT}_{\hat{2}}\right] \cdot \mathcal{H}_{\hat{1}},\ \left\{\hat{1},\ \hat{2}\right\}\right]\right]$$



Here it is specified that the circuit input is the ket $|11\rangle + 2\ |10\rangle$.

$$\texttt{QuantumPlot}\left[\texttt{QubitMeasurement}\left[\mathcal{C}^{\{\hat{1}\}}\left[\mathcal{NOT}_{\hat{2}}\right]\cdot\mathcal{H}_{\hat{1}}\cdot\left(\left|1_{\hat{1}},1_{\hat{2}}\right\rangle+2\left|1_{\hat{1}},0_{\hat{2}}\right\rangle\right),\{\hat{1},\hat{2}\}\right]\right]$$



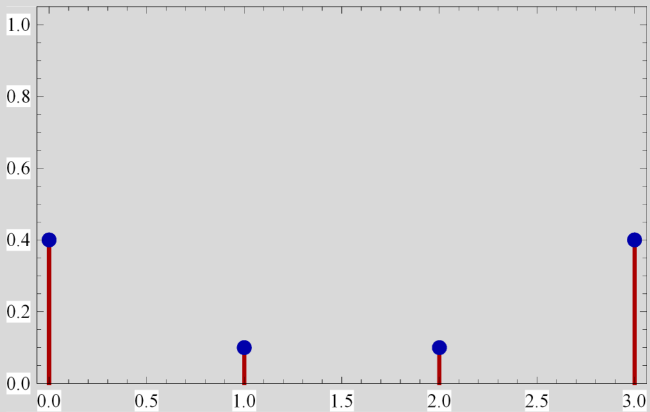
If the expression inside QubitMeasurement is (or evaluates to) a ket, then QuantumPlot can be replaced by QuantumEvaluate in order to obtain the possible measurement outcomes and the corresponding probabilities (the input ket is automatically normalized):

$$\texttt{QuantumEvaluate}\left[\texttt{QubitMeasurement}\left[\mathcal{C}^{\{\hat{1}\}}\left[\mathcal{NOT}_{\hat{2}}\right]\cdot\mathcal{H}_{\hat{1}}\cdot\left(\left|1_{\hat{1}},1_{\hat{2}}\right\rangle+2\left|1_{\hat{1}},0_{\hat{2}}\right\rangle\right),\{\hat{1},\hat{2}\}\right]\right]$$

Probability	Measurement	State
$\frac{2}{5}$	$\{\{0_{\hat{1}}, 0_{\hat{2}}\}\}$	$\left|0_{\hat{1}}\right\rangle \otimes \left|0_{\hat{2}}\right\rangle$
$\frac{1}{10}$	$\{\{0_{\hat{1}}, 1_{\hat{2}}\}\}$	$\left|0_{\hat{1}}\right\rangle \otimes \left|1_{\hat{2}}\right\rangle$
$\frac{1}{10}$	$\{\{1_{\hat{1}}, 0_{\hat{2}}\}\}$	$\left|1_{\hat{1}}\right\rangle \otimes - \left|0_{\hat{2}}\right\rangle$
$\frac{2}{5}$	$\{\{1_{\hat{1}}, 1_{\hat{2}}\}\}$	$\left|1_{\hat{1}}\right\rangle \otimes - \left|1_{\hat{2}}\right\rangle$
Probability	Measurement	State

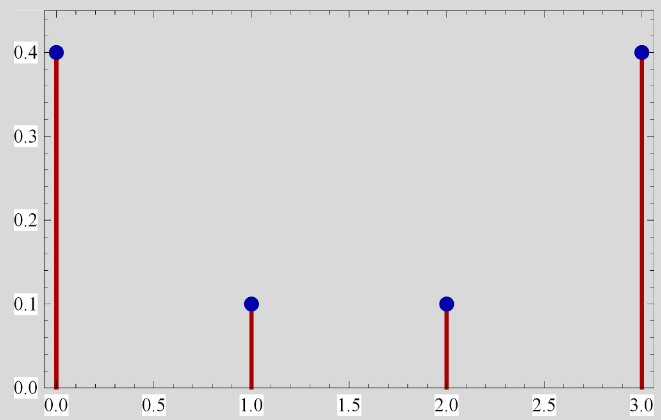
If the calculation above is "wrapped" with a QuantumPlot[] command, then we obtain a plot of the probabilities for each measurement, where each measurement outcome is interpreted as a binary number. For example, for two qubits we have $\{0_{\hat{1}}, 0_{\hat{2}}\}\to 0$, $\{0_{\hat{1}}, 1_{\hat{2}}\}\to 1$, $\{1_{\hat{1}}, 0_{\hat{2}}\}\to 2$, $\{1_{\hat{1}}, 1_{\hat{2}}\}\to 3$. Place the pointer over each disk in order to see the number and its corresponding probability:

$$\texttt{QuantumPlot}\left[\texttt{QuantumEvaluate}\left[\texttt{QubitMeasurement}\left[\mathcal{C}^{\{\hat{1}\}}\left[\mathcal{NOT}_{\hat{2}}\right]\cdot\mathcal{H}_{\hat{1}}\cdot\left(\left|1_{\hat{1}},1_{\hat{2}}\right\rangle+2\left|1_{\hat{1}},0_{\hat{2}}\right\rangle\right),\{\hat{1},\hat{2}\}\right]\right]\right]$$



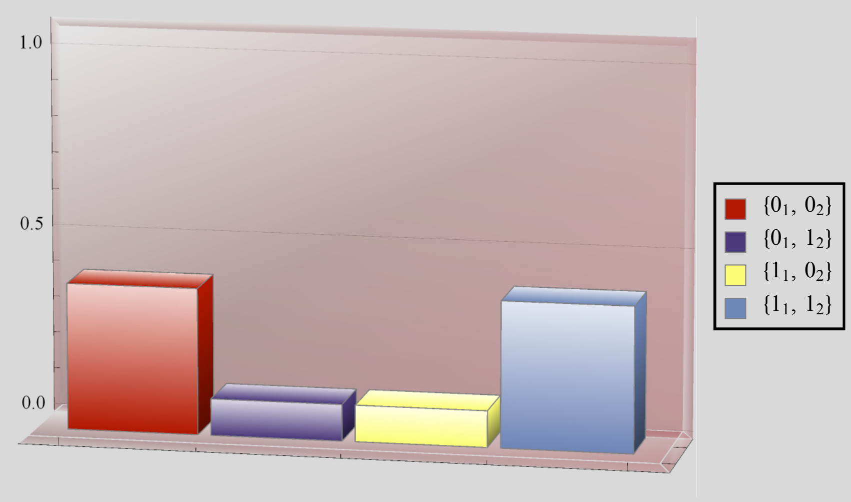
You can use options to improve the graph. For example, PlotRange, as shown below:

```
QuantumPlot[
 QuantumEvaluate[QubitMeasurement[C^{1̂}[NOT_2̂] · H_1̂ · ( | 1_1̂, 1_2̂⟩ + 2 | 1_1̂, 0_2̂⟩), {1̂, 2̂}]],
 PlotRange → {0, 0.45}]
```



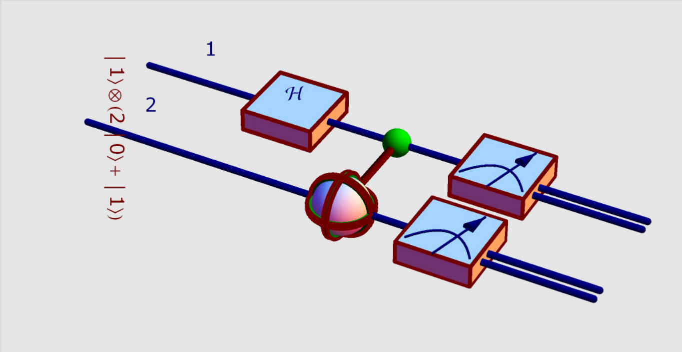
A plot of the probabilities in 3D is obtained if QuantumPlot3D is used:

```
QuantumPlot3D[
 QuantumEvaluate[QubitMeasurement[C^{1̂}[NOT_2̂] · H_1̂ · ( | 1_1̂, 1_2̂⟩ + 2 | 1_1̂, 0_2̂⟩), {1̂, 2̂}]]]
```



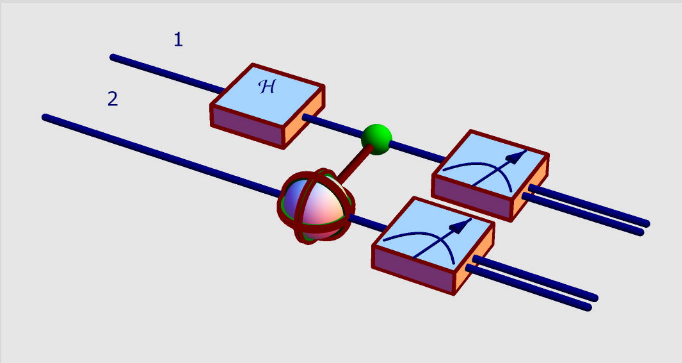
Remember that if QuantumEvaluate is not included, then QuantumPlot and QuantumPlot3D give the circuit instead of the probabilities:

$$\texttt{QuantumPlot3D}\left[\texttt{QubitMeasurement}\left[\mathcal{C}^{\{\hat{1}\}}\left[\mathcal{NOT}_{\hat{2}}\right]\cdot\mathcal{H}_{\hat{1}}\cdot\left(\left|1_{\hat{1}},1_{\hat{2}}\right\rangle+2\left|1_{\hat{1}},0_{\hat{2}}\right\rangle\right),\left\{\hat{1},\hat{2}\right\}\right]\right]$$



QunatumPlot3D output usually looks better without the input ket:

$$\texttt{QuantumPlot3D}\left[\texttt{QubitMeasurement}\left[\mathcal{C}^{\{\hat{1}\}}\left[\mathcal{NOT}_{\hat{2}}\right]\cdot\mathcal{H}_{\hat{1}},\left\{\hat{1},\hat{2}\right\}\right]\right]$$



Measurement of Qubits and Density Operators

Next example shows how to get the outcomes of measuring a qubit. The $\left|0_{\hat{\square}}\right\rangle$ and $\left|1_{\hat{\square}}\right\rangle$ templates can be obtained by pressing the key [ESC]qket0[ESC] and [ESC]qket1[ESC]; the $\hat{\square}$ template can be obtained by pressing the keys [ESC]qb[ESC]:

$$\texttt{QuantumEvaluate}\left[\texttt{QubitMeasurement}\left[\left|0_{\hat{1}}\right\rangle+\left|1_{\hat{1}}\right\rangle,\{\hat{1}\}\right]\right]$$

Probability	Measurement	State
$\frac{1}{2}$	$\{\{0_{\hat{1}}\}\}$	$\otimes\left|0_{\hat{1}}\right\rangle$
$\frac{1}{2}$	$\{\{1_{\hat{1}}\}\}$	$\otimes\left|1_{\hat{1}}\right\rangle$
Probability	Measurement	State

Each row in the table represents a measurement outcome. The first column gives the probability of the corresponding outcome. The second column gives the measurement result, and the third column gives the new, collapsed state after the measurement. Remember that the $\left|\square_{\hat{\square}},\square_{\hat{\square}}\right\rangle$ template can be entered pressing the keys [ESC]qqket[ESC]:

```
QuantumEvaluate[QubitMeasurement[|0₁̂, 0₂̂⟩ + 2 |0₁̂, 1₂̂⟩ + 3 |1₁̂, 0₂̂⟩ + 4 |1₁̂, 1₂̂⟩, {2̂}]]
```

Probability	Measurement	State
$\frac{1}{3}$	$\{\{0_{\hat{2}}\}\}$	$(\lvert 0_{\hat{1}}\rangle + 3\ \lvert 1_{\hat{1}}\rangle) \otimes \frac{\lvert 0_{\hat{2}}\rangle}{\sqrt{10}}$
$\frac{2}{3}$	$\{\{1_{\hat{2}}\}\}$	$(\lvert 0_{\hat{1}}\rangle + 2\ \lvert 1_{\hat{1}}\rangle) \otimes \frac{\lvert 1_{\hat{2}}\rangle}{\sqrt{5}}$
Probability	Measurement	State

Same measurement as above, however this time the kets are **not** factorized in the answer:

```
QuantumEvaluate[QubitMeasurement[|0₁̂, 0₂̂⟩ + 2 |0₁̂, 1₂̂⟩ + 3 |1₁̂, 0₂̂⟩ + 4 |1₁̂, 1₂̂⟩,
  {2̂}, FactorKet → False]]
```

Probability	Measurement	State
$\frac{1}{3}$	$\{\{0_{\hat{2}}\}\}$	$\frac{\lvert 0_{\hat{1}}, 0_{\hat{2}}\rangle + 3\ \lvert 1_{\hat{1}}, 0_{\hat{2}}\rangle}{\sqrt{10}}$
$\frac{2}{3}$	$\{\{1_{\hat{2}}\}\}$	$\frac{\lvert 0_{\hat{1}}, 1_{\hat{2}}\rangle + 2\ \lvert 1_{\hat{1}}, 1_{\hat{2}}\rangle}{\sqrt{5}}$
Probability	Measurement	State

We can obtain the density operator that represents the outcome of the measurement. Notice that the operator is stored in the variable **qd**:

```
qd = QuantumDensityOperator[
  QubitMeasurement[|0₁̂, 0₂̂⟩ + 2 |0₁̂, 1₂̂⟩ + 3 |1₁̂, 0₂̂⟩ + 4 |1₁̂, 1₂̂⟩, {2̂}]]
```

$$\frac{1}{30}\ \lvert 0_{\hat{1}}, 0_{\hat{2}}\rangle \cdot \langle 0_{\hat{1}}, 0_{\hat{2}}\rvert + \frac{1}{10}\ \lvert 1_{\hat{1}}, 0_{\hat{2}}\rangle \cdot \langle 0_{\hat{1}}, 0_{\hat{2}}\rvert +$$
$$\frac{2}{15}\ \lvert 0_{\hat{1}}, 1_{\hat{2}}\rangle \cdot \langle 0_{\hat{1}}, 1_{\hat{2}}\rvert + \frac{4}{15}\ \lvert 1_{\hat{1}}, 1_{\hat{2}}\rangle \cdot \langle 0_{\hat{1}}, 1_{\hat{2}}\rvert + \frac{1}{10}\ \lvert 0_{\hat{1}}, 0_{\hat{2}}\rangle \cdot \langle 1_{\hat{1}}, 0_{\hat{2}}\rvert +$$
$$\frac{3}{10}\ \lvert 1_{\hat{1}}, 0_{\hat{2}}\rangle \cdot \langle 1_{\hat{1}}, 0_{\hat{2}}\rvert + \frac{4}{15}\ \lvert 0_{\hat{1}}, 1_{\hat{2}}\rangle \cdot \langle 1_{\hat{1}}, 1_{\hat{2}}\rvert + \frac{8}{15}\ \lvert 1_{\hat{1}}, 1_{\hat{2}}\rangle \cdot \langle 1_{\hat{1}}, 1_{\hat{2}}\rvert$$

This is the matrix (in *Mathematica* notation) representing the operator **qd** that was defined above:

```
QuantumMatrix[qd]
```

$$\left\{\left\{\frac{1}{30}, 0, \frac{1}{10}, 0\right\}, \left\{0, \frac{2}{15}, 0, \frac{4}{15}\right\}, \left\{\frac{1}{10}, 0, \frac{3}{10}, 0\right\}, \left\{0, \frac{4}{15}, 0, \frac{8}{15}\right\}\right\}$$

Remember that QuantumMatrix if for calculations and QuantumMatrixForm is for display:

```
QuantumMatrixForm[qd]
```

$$\begin{pmatrix} \frac{1}{30} & 0 & \frac{1}{10} & 0 \\ 0 & \frac{2}{15} & 0 & \frac{4}{15} \\ \frac{1}{10} & 0 & \frac{3}{10} & 0 \\ 0 & \frac{4}{15} & 0 & \frac{8}{15} \end{pmatrix}$$

Several qubits can be measured at the same time, as specified by the second argument of **QubitMeasurement**:

$$\text{QuantumEvaluate}\Big[\text{QubitMeasurement}\Big[\,|0_{\hat{1}}, 0_{\hat{2}}\rangle + 2\,|0_{\hat{1}}, 1_{\hat{2}}\rangle + 3\,|1_{\hat{1}}, 0_{\hat{2}}\rangle + 4\,|1_{\hat{1}}, 1_{\hat{2}}\rangle, \{\hat{1}, \hat{2}\}\Big]\Big]$$

Probability	Measurement	State
$\frac{1}{30}$	$\{\{0_{\hat{1}}, 0_{\hat{2}}\}\}$	$\|0_{\hat{1}}\rangle \otimes \|0_{\hat{2}}\rangle$
$\frac{2}{15}$	$\{\{0_{\hat{1}}, 1_{\hat{2}}\}\}$	$\|0_{\hat{1}}\rangle \otimes \|1_{\hat{2}}\rangle$
$\frac{3}{10}$	$\{\{1_{\hat{1}}, 0_{\hat{2}}\}\}$	$\|1_{\hat{1}}\rangle \otimes \|0_{\hat{2}}\rangle$
$\frac{8}{15}$	$\{\{1_{\hat{1}}, 1_{\hat{2}}\}\}$	$\|1_{\hat{1}}\rangle \otimes \|1_{\hat{2}}\rangle$
Probability	Measurement	State

We can obtain the density operator that represents the outcome of the measurement:

$$\text{QuantumDensityOperator}\Big[\text{QubitMeasurement}\Big[\,|0_{\hat{1}}, 0_{\hat{2}}\rangle + 2\,|0_{\hat{1}}, 1_{\hat{2}}\rangle + 3\,|1_{\hat{1}}, 0_{\hat{2}}\rangle + 4\,|1_{\hat{1}}, 1_{\hat{2}}\rangle, \{\hat{1}, \hat{2}\}\Big]\Big]$$

$$\frac{1}{30}\,|0_{\hat{1}}, 0_{\hat{2}}\rangle \cdot \langle 0_{\hat{1}}, 0_{\hat{2}}| + \frac{2}{15}\,|0_{\hat{1}}, 1_{\hat{2}}\rangle \cdot \langle 0_{\hat{1}}, 1_{\hat{2}}| + \frac{3}{10}\,|1_{\hat{1}}, 0_{\hat{2}}\rangle \cdot \langle 1_{\hat{1}}, 0_{\hat{2}}| + \frac{8}{15}\,|1_{\hat{1}}, 1_{\hat{2}}\rangle \cdot \langle 1_{\hat{1}}, 1_{\hat{2}}|$$

It can be easier to read the probablities in numeric form, using the standard *Mathematica* command N[]:

$$\text{N}\Big[\text{QuantumEvaluate}\Big[\text{QubitMeasurement}\Big[\,|0_{\hat{1}}, 0_{\hat{2}}\rangle + 2\,|0_{\hat{1}}, 1_{\hat{2}}\rangle + 3\,|1_{\hat{1}}, 0_{\hat{2}}\rangle + 4\,|1_{\hat{1}}, 1_{\hat{2}}\rangle, \{\hat{1}, \hat{2}\}\Big]\Big]\Big]$$

Probability	Measurement	State
0.0333333	$\{\{0_{\hat{1}}, 0_{\hat{2}}\}\}$	$\|0_{\hat{1}}\rangle \otimes \|0_{\hat{2}}\rangle$
0.133333	$\{\{0_{\hat{1}}, 1_{\hat{2}}\}\}$	$\|0_{\hat{1}}\rangle \otimes \|1_{\hat{2}}\rangle$
0.3	$\{\{1_{\hat{1}}, 0_{\hat{2}}\}\}$	$\|1_{\hat{1}}\rangle \otimes \|0_{\hat{2}}\rangle$
0.533333	$\{\{1_{\hat{1}}, 1_{\hat{2}}\}\}$	$\|1_{\hat{1}}\rangle \otimes \|1_{\hat{2}}\rangle$
Probability	Measurement	State

TraditionalForm[] gives a format closer to the one used in papers:

```
TraditionalForm[N[QuantumEvaluate[
    QubitMeasurement[ | 0₁̂, 0₂̂⟩ + 2 | 0₁̂, 1₂̂⟩ + 3 | 1₁̂, 0₂̂⟩ + 4 | 1₁̂, 1₂̂⟩, {1̂, 2̂}]]]]
```

Probability	Measurement	State
0.0333333	$(0_1 \;\; 0_2)$	$\vert 0\rangle \otimes \vert 0\rangle$
0.133333	$(0_1 \;\; 1_2)$	$\vert 0\rangle \otimes \vert 1\rangle$
0.3	$(1_1 \;\; 0_2)$	$\vert 1\rangle \otimes \vert 0\rangle$
0.533333	$(1_1 \;\; 1_2)$	$\vert 1\rangle \otimes \vert 1\rangle$
Probability	Measurement	State

Same as above, however this time kets are **not** factorized:

```
TraditionalForm[
 N[QuantumEvaluate[QubitMeasurement[ | 0₁̂, 0₂̂⟩ + 2 | 0₁̂, 1₂̂⟩ + 3 | 1₁̂, 0₂̂⟩ + 4 | 1₁̂, 1₂̂⟩,
     {1̂, 2̂}, FactorKet → False]]]]
```

Probability	Measurement	State
0.0333333	$(0_1 \;\; 0_2)$	$\vert 00\rangle$
0.133333	$(0_1 \;\; 1_2)$	$\vert 01\rangle$
0.3	$(1_1 \;\; 0_2)$	$\vert 10\rangle$
0.533333	$(1_1 \;\; 1_2)$	$\vert 11\rangle$
Probability	Measurement	State

The operators $\mathcal{H}_{\hat{\square}}$ [ESC]hg[ESC] and $C^{\{\hat{\square}\}}\left[\mathcal{NOT}_{\hat{\square}}\right]$ [ESC]cnot[ESC] generate entanglement when applied to the ket $\vert 0_{\hat{1}}, 1_{\hat{2}}\rangle$:

```
QuantumEvaluate[C^{1̂}[NOT₂̂] · H₁̂ · | 0₁̂, 1₂̂⟩]
```

$$\frac{\vert 0_{\hat{1}}, 1_{\hat{2}}\rangle}{\sqrt{2}} + \frac{\vert 1_{\hat{1}}, 0_{\hat{2}}\rangle}{\sqrt{2}}$$

In this case, measuring produces non-entangled final states in the computational basis:

```
QuantumEvaluate[ QubitMeasurement[C^{1̂}[NOT₂̂] · H₁̂ · | 0₁̂, 1₂̂⟩, {1̂}] ]
```

Probability	Measurement	State
$\frac{1}{2}$	$\{\{0_{\hat{1}}\}\}$	$\vert 0_{\hat{1}}\rangle \otimes \vert 1_{\hat{2}}\rangle$
$\frac{1}{2}$	$\{\{1_{\hat{1}}\}\}$	$\vert 1_{\hat{1}}\rangle \otimes \vert 0_{\hat{2}}\rangle$
Probability	Measurement	State

However in this other case there is entanglement after the measurement. The $\vert \mathcal{B}_{00,\hat{\square},\hat{\square}}\rangle$ template can be entered by pressing the keys [ESC]k00[ESC]:

```
QuantumEvaluate[ QubitMeasurement[C^{1̂}[NOT_2̂] · H_1̂ · | B_{00,1̂,2̂}⟩, {1̂}] ]
```

Probability	Measurement	State
$\frac{1}{2}$	$\{\{0_{\hat{1}}\}\}$	$\vert 0_{\hat{1}}\rangle \otimes \left(\frac{\vert 0_{\hat{2}}\rangle}{\sqrt{2}} + \frac{\vert 1_{\hat{2}}\rangle}{\sqrt{2}}\right)$
$\frac{1}{2}$	$\{\{1_{\hat{1}}\}\}$	$\vert 1_{\hat{1}}\rangle \otimes \left(-\frac{\vert 0_{\hat{2}}\rangle}{\sqrt{2}} + \frac{\vert 1_{\hat{2}}\rangle}{\sqrt{2}}\right)$
Probability	Measurement	State

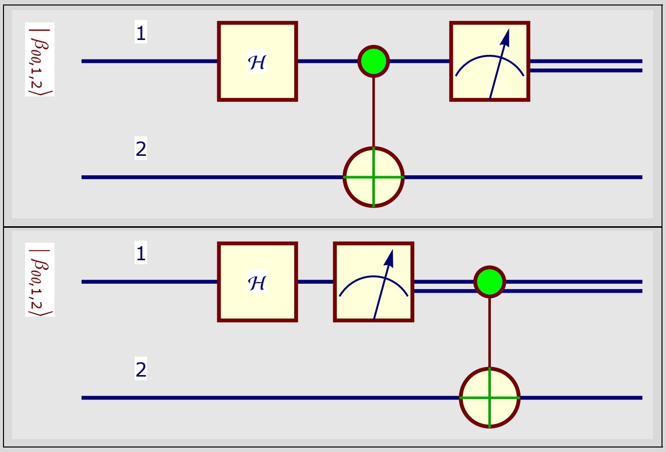
Controlled gates commute with measurements in their control qubit. This means we can measure first and apply the controlled gate later, and we get the same result as above:

```
QuantumEvaluate[ C^{1̂}[NOT_2̂] · QubitMeasurement[H_1̂ · | B_{00,1̂,2̂}⟩, {1̂}] ]
```

Probability	Measurement	State
$\frac{1}{2}$	$\{\{0_{\hat{1}}\}\}$	$\vert 0_{\hat{1}}\rangle \otimes \left(\frac{\vert 0_{\hat{2}}\rangle}{\sqrt{2}} + \frac{\vert 1_{\hat{2}}\rangle}{\sqrt{2}}\right)$
$\frac{1}{2}$	$\{\{1_{\hat{1}}\}\}$	$\vert 1_{\hat{1}}\rangle \otimes \left(-\frac{\vert 0_{\hat{2}}\rangle}{\sqrt{2}} + \frac{\vert 1_{\hat{2}}\rangle}{\sqrt{2}}\right)$
Probability	Measurement	State

These are the two equivalent circuits that were calculated above. They are equivalent because controlled-gates "commute" with measurements in their control qubit:

```
Grid[{{QuantumPlot[ QubitMeasurement[C^{1̂}[NOT_2̂] · H_1̂ · | B_{00,1̂,2̂}⟩, {1̂}] ]},
  {QuantumPlot[ C^{1̂}[NOT_2̂] · QubitMeasurement[H_1̂ · | B_{00,1̂,2̂}⟩, {1̂}] ]}}, Dividers → All]
```



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