Create New Quantum Gates

by José Luis Gómez-Muñoz
http://homepage.cem.itesm.mx/lgomez/quantum/
jose.luis.gomez@itesm.mx

Introduction

This is a tutorial on the use of Quantum`Computing` *Mathematica* add-on to create new quantum gates.

Load the Package

First load the Quantum`Computing` package. Write:
Needs["Quantum`Computing`"]
then press at the same time the keys SHIFT-ENTER to evaluate. *Mathematica* will load the package.

```
Needs["Quantum`Computing`"]
```

```
Quantum`Computing` Version 2.2.0. (July 2010)
A Mathematica package for Quantum Computing
  in Dirac bra-ket notation and plotting of quantum circuits
by José Luis Gómez-Muñoz

Execute SetComputingAliases[] in order to use
  the keyboard to enter quantum objects in Dirac's notation
SetComputingAliases[] must be executed again in each new notebook that is created
```

In order to use the keyboard to enter quantum objects write:
SetComputingAliases[ ];
then press at the same time the keys SHIFT-ENTER to evaluate. The semicolon prevents *Mathematica* from printing the help message. Remember that SetComputingAliases[ ] must be evaluated again in each new notebook:

```
SetComputingAliases[];
```

A New Quantum Gate of One Qubit without Explicit Dirac Expression

Use the command SetQuantumGate to specify that a name (**ga** in this example) is to be consider a Quantum Gate. The first argument is the name of the new quantum gate, and the second argument is the number of input/output qubits of this new gate:

```
SetQuantumGate[ga, 1]
```

```
The expression ga is a quantum gate of 1 arguments (qubits)
```

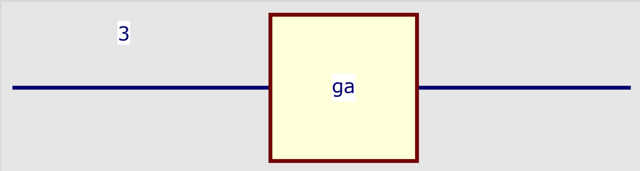
Press the keys [ESC]qg[ESC] in order to obtain the quantum gate template $\square_{\hat{\square}}$, press the [TAB] key to move between the empty place holders $\square$ and write the gate name (ga in this example, it is the name that was previously defined as a quantum gate) and also write the qubit number or label (3 in this example):

```
ga_3̂
```

```
ga_3̂
```

The new quantum gate that was defined above (with the command SetQuantumGate) can be included in the plot of quantum circuits:

```
QuantumPlot[ga_3̂]
```



The new quantum gate cannot be used with more neither less qubits than the number that was specified in SetQuantumGate, therefore next command gives an **error** because ga **cannot** be used as a two-qubits gate, and the calculation is **aborted**:

```
ga_{3̂,4̂}
```

SetQuantumGate::nqb :
Quantum gate ga was called with 2 argument(s):$\{\hat{3}, \hat{4}\}$. It must have a number na of arguments
such that 1<=na<=1

```
$Aborted
```

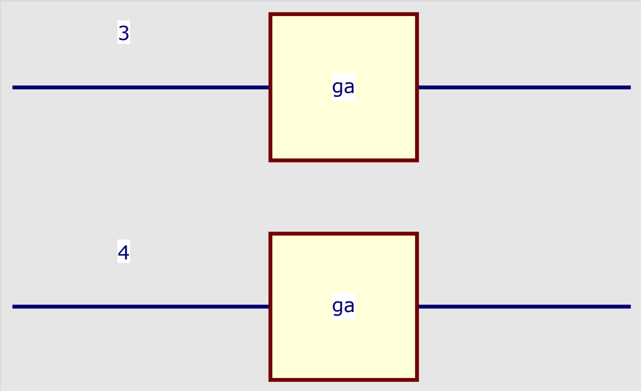
On the other hand, the notation $\text{ga}_{\{\hat{q1},\hat{q2}\}}$, **with curly brackets** in the subscript, is used for **two gates** of **one qubit** each. Therefore next input is accepted without errors. Press [ESC]qgg[ESC] for the "two gates of one qubit" template $\square_{\{\hat{\square},\hat{\square}\}}$

```
ga_{3̂,4̂}
```

```
ga_{3̂,4̂}
```

We can use the **two gates** of **one qubit** in a quantum circuit:

```
QuantumPlot[ga_{{3̂,4̂}}]
```



QuantumEvaluate transforms the notation $\mathrm{ga}_{\{\hat{3},\hat{4}\}}$ into the notation $\mathrm{ga}_{\hat{3}} \cdot \mathrm{ga}_{\hat{4}}$. Both mean the same: **two gates** of **one qubit**:

```
QuantumEvaluate[ga_{{3̂,4̂}}]
```

$\mathrm{ga}_{\hat{3}} \cdot \mathrm{ga}_{\hat{4}}$

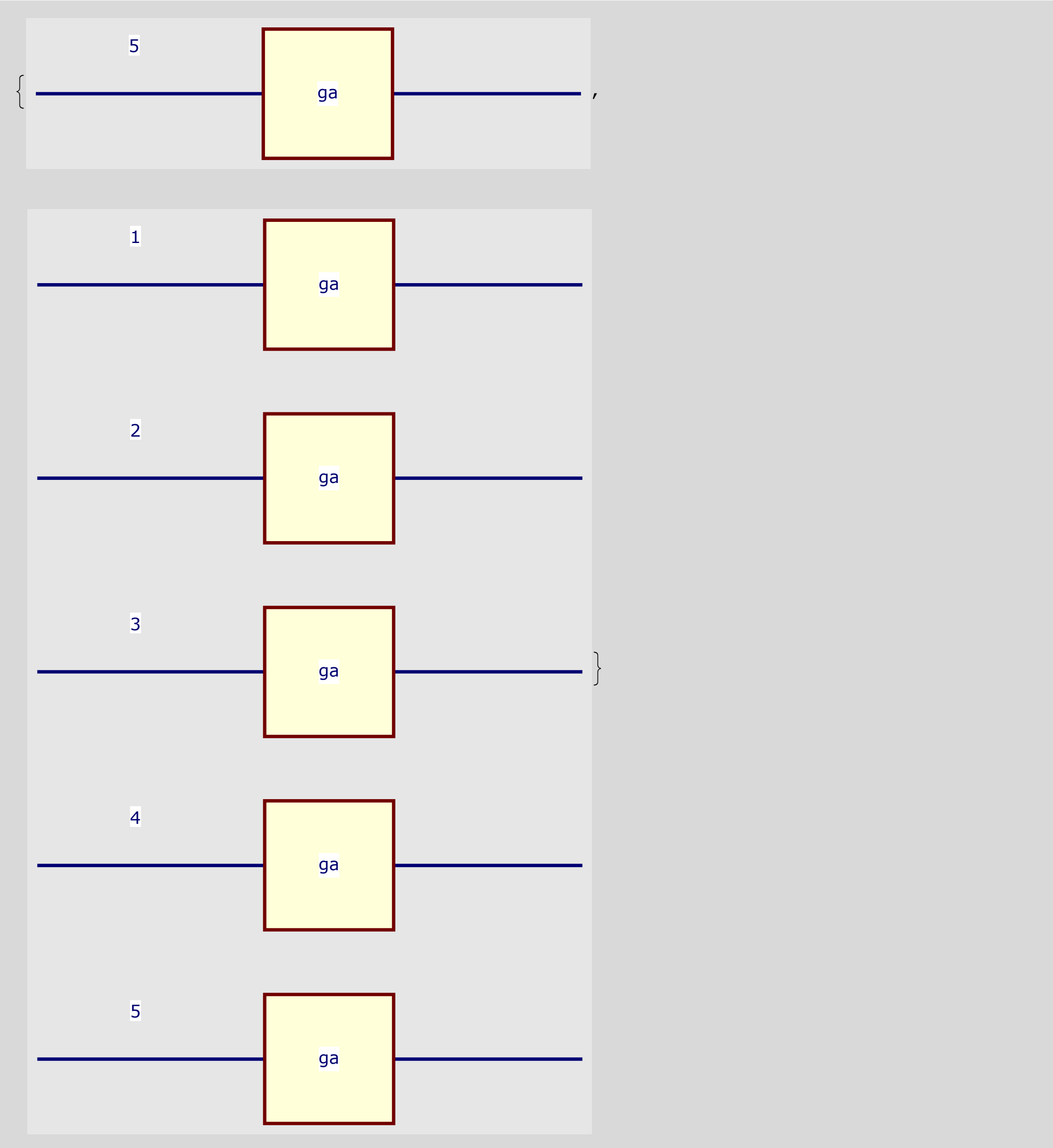
Use the [ESC]qn[ESC] template $\square_{\square}$ in order to easily write a large numbre of gates:

```
ga_5
```

$\mathrm{ga}_{\{\hat{1},\hat{2},\hat{3},\hat{4},\hat{5}\}}$

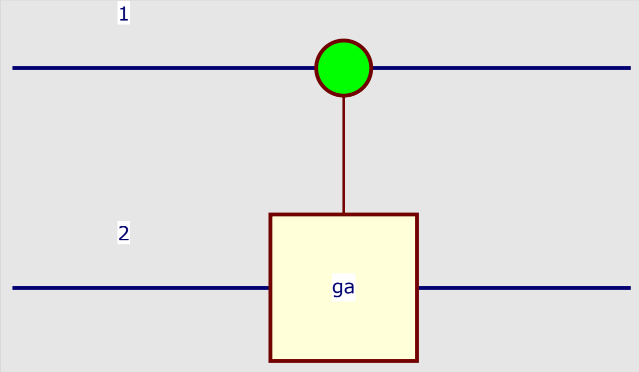
Notice that is very different the interpretation of $\mathrm{ga}_{\hat{5}}$ from ga_{5} (it is very different [ESC]qg[ESC] from [ESC]qn[ESC])

$\{\text{QuantumPlot}[\text{ga}_{\hat{5}}], \text{QuantumPlot}[\text{ga}_5]\}$



This is a "controlled" version of the new gate. Press [ESC]cgate[ESC] for the $C^{\{\hat{\square}\}}[\square]$ template.

```
QuantumPlot[C^{1̂}[ga_2̂]]
```



This is the matrix representation of the "controlled" version of the new gate. Press [ESC]cgate[ESC] for the $C^{\{\hat{\square}\}}[\square]$ template:

```
QuantumMatrixForm[C^{1̂}[ga_2̂]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \langle 0_{\hat{2}} | \cdot ga_{\hat{2}} \cdot | 0_{\hat{2}} \rangle & \langle 0_{\hat{2}} | \cdot ga_{\hat{2}} \cdot | 1_{\hat{2}} \rangle \\ 0 & 0 & \langle 1_{\hat{2}} | \cdot ga_{\hat{2}} \cdot | 0_{\hat{2}} \rangle & \langle 1_{\hat{2}} | \cdot ga_{\hat{2}} \cdot | 1_{\hat{2}} \rangle \end{pmatrix}$$

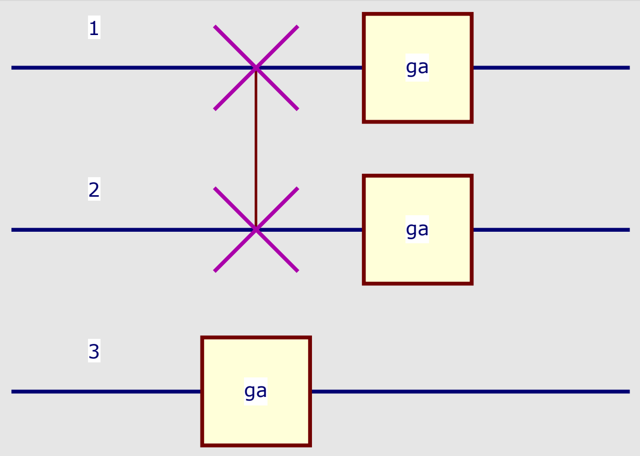
This is the truth table of the "controlled" version of the new gate. Press [ESC]cgate[ESC] for the $C^{\{\hat{\square}\}}[\square]$ template:

```
QuantumTableForm[C^{1̂}[ga_2̂]]
```

	Input	Output
0	$\mid 0_{\hat{1}}, 0_{\hat{2}} \rangle$	$\mid 0_{\hat{1}}, 0_{\hat{2}} \rangle$
1	$\mid 0_{\hat{1}}, 1_{\hat{2}} \rangle$	$\mid 0_{\hat{1}}, 1_{\hat{2}} \rangle$
2	$\mid 1_{\hat{1}}, 0_{\hat{2}} \rangle$	$\mid 1_{\hat{1}} \rangle \cdot ga_{\hat{2}} \cdot \mid 0_{\hat{2}} \rangle$
3	$\mid 1_{\hat{1}}, 1_{\hat{2}} \rangle$	$\mid 1_{\hat{1}} \rangle \cdot ga_{\hat{2}} \cdot \mid 1_{\hat{2}} \rangle$

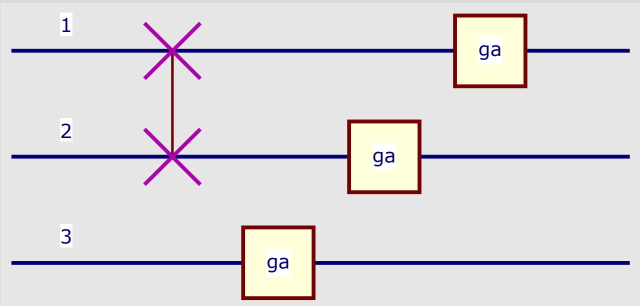
Notice that in the following circuit the three gates ga **are not** in the same column:

```
QuantumPlot[(⊗_{j=1}^{3} ga_ĵ) · SWAP_{1̂,2̂}]
```



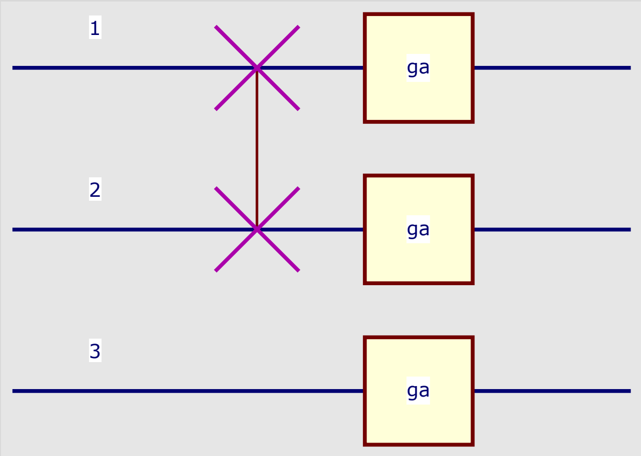
Notice that in the following circuit the three gates ga **are not** in the same column:

```
QuantumPlot[(⊗_{j=1}^{3} ga_ĵ) · SWAP_{1̂,2̂}, QuantumGateShifting → False]
```



We can plot the three gates in the same column using the $\square_{\{\hat{\square},\hat{\square},\hat{\square}\}}$ template, which can be entered by pressing the keys [ESC]qggg[ESC] (one q because they are one-qubit gates, three g because there are three gates):

```
QuantumPlot[ga_{1̂,2̂,3̂} · SWAP_{1̂,2̂}]
```



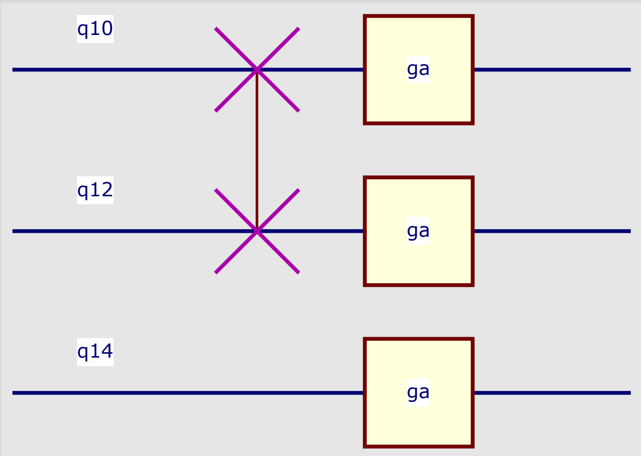
QuantumEvaluate transforms $\mathrm{ga}_{\{\hat{1},\hat{2},\hat{3}\}}$ into $\mathrm{ga}_{\hat{1}} \cdot \mathrm{ga}_{\hat{2}} \cdot \mathrm{ga}_{\hat{3}}$. Both expressions have the same meaning, but the first one makes QuantumPlot use the same column for the three gates, as was shown above.

```
QuantumEvaluate[ga_{1̂,2̂,3̂}]
```

$\mathrm{ga}_{\hat{1}} \cdot \mathrm{ga}_{\hat{2}} \cdot \mathrm{ga}_{\hat{3}}$

The "quantum register" notation [ESC]qn[ESC] gives more flexibility in the naming of the qubits:

```
QuantumPlot[ga_Register[10,14,2,"q"] · SWAP_Register[10,12,2,"q"]]
```



The notation $\otimes_{j=1}^{3} \mathrm{ga}_{\hat{j}}$ is transformed to $\mathrm{ga}_{\hat{1}} \cdot \mathrm{ga}_{\hat{2}} \cdot \mathrm{ga}_{\hat{3}}$ **without** having to use QuantumEvaluate:

```
⊗_{j=1}^{3} ga_ĵ
```

```
ga_1̂ · ga_2̂ · ga_3̂
```

Remember that [ESC]qn[ESC] is very different from [ESC]qg[ESC]

```
{ga_5, ga_5̂}
```

```
{ga_{1̂,2̂,3̂,4̂,5̂}, ga_5̂}
```

A New Quantum Gate of One Qubit with Explicit Dirac Expression

The third (optional) argument of SetQuantumGate is the function that will be applied to the qubit(s) when the gate is inside the QuantumEvaluate command:

```
SetQuantumGate[gb, 1,
 Function[{x},
  1/2 √3 |0_x̂〉·〈0_x̂| + 1/2 |1_x̂〉·〈0_x̂| - 1/2 |0_x̂〉·〈1_x̂| + 1/2 √3 |1_x̂〉·〈1_x̂|]]
```

```
The expression gb is a quantum gate of 1 arguments (qubits)
```

The commands above defined **gb** as a quantum gate with a specific Dirac expression. Here we can see the gate applied to the qubit $\hat{5}$

```
QuantumEvaluate[gb_5̂]
```

$$\frac{1}{2}\sqrt{3}\ \left|0_{\hat{5}}\right\rangle\cdot\left\langle 0_{\hat{5}}\right| + \frac{1}{2}\ \left|1_{\hat{5}}\right\rangle\cdot\left\langle 0_{\hat{5}}\right| - \frac{1}{2}\ \left|0_{\hat{5}}\right\rangle\cdot\left\langle 1_{\hat{5}}\right| + \frac{1}{2}\sqrt{3}\ \left|1_{\hat{5}}\right\rangle\cdot\left\langle 1_{\hat{5}}\right|$$

Here we can see the gate applied to the qubit $\hat{m}$

```
QuantumEvaluate[gb_m̂]
```

$$\frac{1}{2}\sqrt{3}\ \left|0_{\hat{m}}\right\rangle\cdot\left\langle 0_{\hat{m}}\right| + \frac{1}{2}\ \left|1_{\hat{m}}\right\rangle\cdot\left\langle 0_{\hat{m}}\right| - \frac{1}{2}\ \left|0_{\hat{m}}\right\rangle\cdot\left\langle 1_{\hat{m}}\right| + \frac{1}{2}\sqrt{3}\ \left|1_{\hat{m}}\right\rangle\cdot\left\langle 1_{\hat{m}}\right|$$

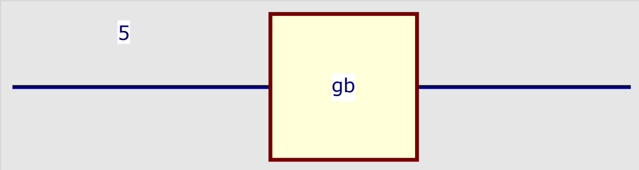
If the gate is **not** inside QuantumEvaluate, then it will **not** be transformed to its Dirac expression:

```
gb_5̂
```

```
gb_5̂
```

The new quantum gate that was defined above can be included in the plot of quantum circuits:

```
QuantumPlot[gb_5̂]
```



This is the truth table of the new quantum gate that was defined above:

```
QuantumTableForm[gb_5̂]
```

	Input	Output
0	$\vert 0_{\hat{5}}\rangle$	$\frac{1}{2}\sqrt{3}\ \vert 0_{\hat{5}}\rangle + \frac{1}{2}\ \vert 1_{\hat{5}}\rangle$
1	$\vert 1_{\hat{5}}\rangle$	$-\frac{1}{2}\ \vert 0_{\hat{5}}\rangle + \frac{1}{2}\sqrt{3}\ \vert 1_{\hat{5}}\rangle$

This is the matrix representation of the new quantum gate that was defined above:

```
QuantumMatrixForm[gb_5̂]
```

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

This is the new gate in terms of Pauli operators:

```
PauliExpand[gb_5̂]
```

$$\frac{1}{2}\sqrt{3}\ \sigma_{0,\hat{5}} - \frac{1}{2}\, i\, \sigma_{y,\hat{5}}$$

A New Quantum Gate of Two Qubits With Explicit Dirac Expression

The third (optional) argument of SetQuantumGate is the function that will be applied to the qubit(s) when the gate is inside the QuantumEvaluate command. The second argument (the number 2) indicates that this quantum gate takes as input two qubits, therefore the third argument must be a function of two qubits:

$$\texttt{SetQuantumGate}\Big[\texttt{gc, 2,}\ \texttt{Function}\Big[\{\texttt{q1, q2}\},\ \vert 0_{\hat{q1}}, 0_{\hat{q2}}\rangle \cdot \langle 0_{\hat{q1}}, 0_{\hat{q2}}\vert - \vert 0_{\hat{q1}}, 1_{\hat{q2}}\rangle \cdot \langle 0_{\hat{q1}}, 1_{\hat{q2}}\vert + \vert 1_{\hat{q1}}, 1_{\hat{q2}}\rangle \cdot \langle 1_{\hat{q1}}, 0_{\hat{q2}}\vert - \vert 1_{\hat{q1}}, 0_{\hat{q2}}\rangle \cdot \langle 1_{\hat{q1}}, 1_{\hat{q2}}\vert\Big]\Big]$$

```
The expression gc is a quantum gate of 2 arguments (qubits)
```

The new quantum gate cannot be used with more neither less qubits than the number that was specified in SetQuantumGate, therefore next command gives an **error** because gc **cannot** be used as a one-qubit gate, and the calculation is **aborted**:

```
gc_9̂
```

SetQuantumGate::nqb :
Quantum gate gc was called with 1 argument(s):{$\hat{9}$}. It must have a number na of arguments such that 2<=na<=2

```
$Aborted
```

The new quantum gate gc was defined as a two-qubit gate. Press [ESC]qqg[ESC] for the two-qubits-one-gate template:

```
gc_{5̂,9̂}
```

```
gc_{5̂,9̂}
```

The commands above (at the top of this section) defined **gc** as a quantum gate with a specific Dirac expression. Here we can see the gate applied to the qubits $\hat{5}$, $\hat{9}$

```
QuantumEvaluate[gc_{5̂,9̂}]
```

$$|0_{\hat{5}}, 0_{\hat{9}}\rangle \cdot \langle 0_{\hat{5}}, 0_{\hat{9}}| - |0_{\hat{5}}, 1_{\hat{9}}\rangle \cdot \langle 0_{\hat{5}}, 1_{\hat{9}}| + |1_{\hat{5}}, 1_{\hat{9}}\rangle \cdot \langle 1_{\hat{5}}, 0_{\hat{9}}| - |1_{\hat{5}}, 0_{\hat{9}}\rangle \cdot \langle 1_{\hat{5}}, 1_{\hat{9}}|$$

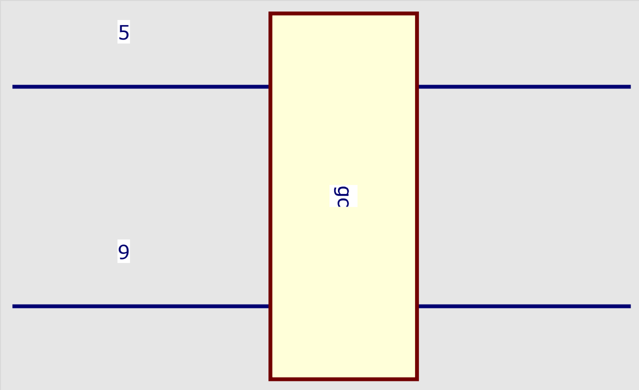
This is the truth table of the new quantum gate that was defined above:

```
QuantumTableForm[gc_{5̂,9̂}]
```

	Input	Output
0	$\vert 0_{\hat{5}}, 0_{\hat{9}}\rangle$	$\vert 0_{\hat{5}}, 0_{\hat{9}}\rangle$
1	$\vert 0_{\hat{5}}, 1_{\hat{9}}\rangle$	$-\vert 0_{\hat{5}}, 1_{\hat{9}}\rangle$
2	$\vert 1_{\hat{5}}, 0_{\hat{9}}\rangle$	$\vert 1_{\hat{5}}, 1_{\hat{9}}\rangle$
3	$\vert 1_{\hat{5}}, 1_{\hat{9}}\rangle$	$-\vert 1_{\hat{5}}, 0_{\hat{9}}\rangle$

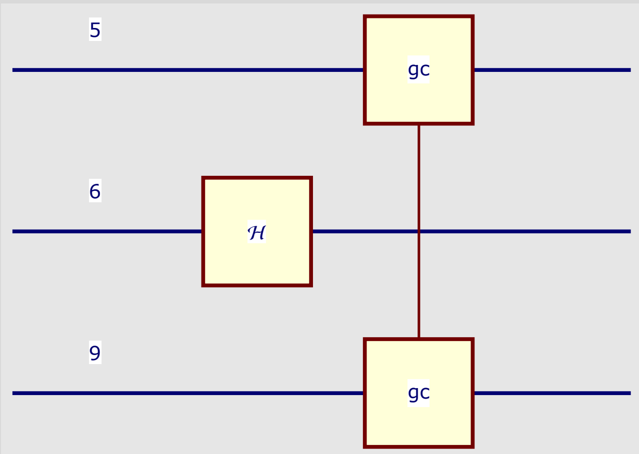
The new quantum gate that was defined above can be included in the plot of quantum circuits:

```
QuantumPlot[gc_{5̂,9̂}]
```



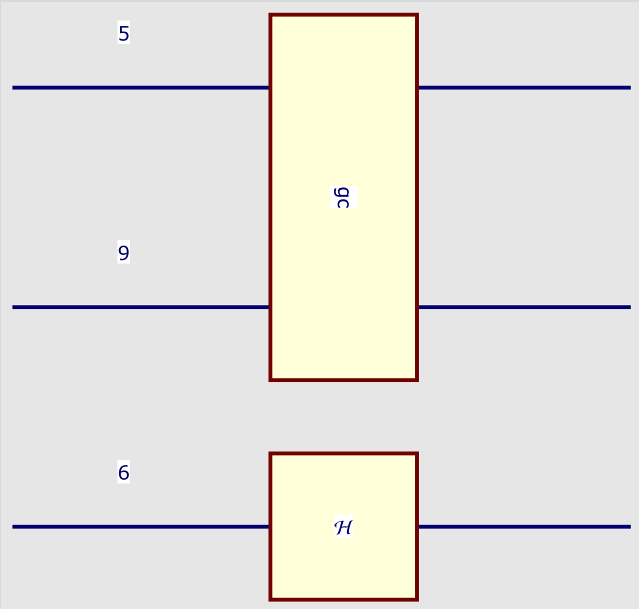
The gate is separated in two blocks with the default qubit order. A vertical line is used to indicate that the two bocks represent a single two-qubits gate

```
QuantumPlot[gc_{5̂,9̂} · ℋ_{6̂}]
```



The option QubitList can be used to specify another order for the qubits, so that the gate gc can be plotted as a single block:

```
QuantumPlot[gc_{5̂,9̂} · ℋ_{6̂}, QubitList → {5̂, 9̂, 6̂}]
```



A New Quantum Gate With a Number of Qubits Inside a Range

The second argument of SetQuantumGate can be a list of two integers, specifying that the number of qubits for this gate must be within the interval spanned by those integers. If such kind gate is defined with the third (optional) argument of SetQuantumGate, then it is expected that the third argument will be a Function using the standard *Mathematica* notation for an arbotrary sequence of arguments: ## (SlotSequence).

```
SetQuantumGate[ge, {2, 4}, Function[ℋ_{##} · X_{##}]]
```

```
The expression ge is a quantum gate
  with a number na of arguments (qubits) such that 2≤na≤4
```

The quantum gate **ge** that was defined above **cannot** be used with only one qubit, therefore we obtain an error message and the calculation is **aborted** if we try to use it with only one qubit:

```
ge_{1̂}
```

SetQuantumGate::nqb :
Quantum gate ge was called with 1 argument(s):{1̂}. It must have a number na of arguments such that 2<=na<=4

```
$Aborted
```

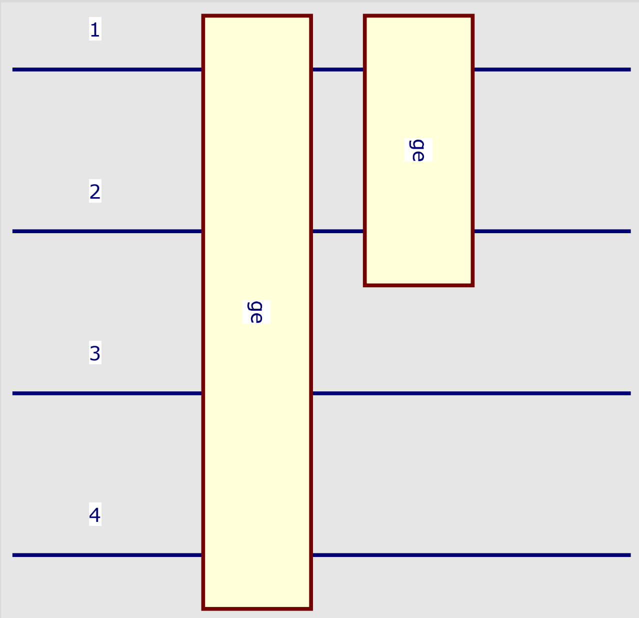
The quantum gate **ge** can be used with 2, 3 or 4 qubits. Press [ESC]qqg[ESC] for the $\square_{\hat{\square},\hat{\square}}$ template:

```
ge1̂,2̂
```

```
ge1̂,2̂
```

In order to write the quantum gate with **four qubits**, first write with three [ESC]qqqg[ESC] $\square_{\hat{\square},\hat{\square},\hat{\square}}$, then place the cursor to the right of the last qubit, and write another qubit pressing [ESC]qb[ESC] $\hat{\square}$

```
QuantumPlot[ge1̂,2̂ · ge1̂,2̂,3̂,4̂]
```



This is the truth table of $ge_{\hat{1},\hat{2},\hat{3}}$. This command can take several seconds in your computer to evaluate

```
QuantumTableForm[ ge1̂,2̂,3̂]
```

	Input	Output
0	$\mid 0_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}\rangle$	$\frac{\mid 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \mid 1$
1	$\mid 0_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}\rangle$	$\frac{\mid 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \mid 1$
2	$\mid 0_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}\rangle$	$\frac{\mid 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \mid 1$
3	$\mid 0_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}\rangle$	$\frac{\mid 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \mid 1$
4	$\mid 1_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}\rangle$	$\frac{\mid 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \mid 1$
5	$\mid 1_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}\rangle$	$\frac{\mid 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \mid 1$
6	$\mid 1_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}\rangle$	$\frac{\mid 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\mid 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \mid 1$
7	$\mid 1_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}\rangle$	$\frac{\mid 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\mid 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \mid 1$

This is the representation of the gate in terms of Pauli operators. This calculation can take several seconds in your computer:

```
PauliExpand[ge_{1̂,2̂,3̂}]
```

$$\frac{\sigma_{0,\hat{1}}\cdot\sigma_{0,\hat{2}}\cdot\sigma_{0,\hat{3}}}{2\sqrt{2}}+\frac{i\,\sigma_{y,\hat{1}}\cdot\sigma_{0,\hat{2}}\cdot\sigma_{0,\hat{3}}}{2\sqrt{2}}+\frac{i\,\sigma_{0,\hat{1}}\cdot\sigma_{y,\hat{2}}\cdot\sigma_{0,\hat{3}}}{2\sqrt{2}}-\frac{\sigma_{y,\hat{1}}\cdot\sigma_{y,\hat{2}}\cdot\sigma_{0,\hat{3}}}{2\sqrt{2}}+\frac{i\,\sigma_{0,\hat{1}}\cdot\sigma_{0,\hat{2}}\cdot\sigma_{y,\hat{3}}}{2\sqrt{2}}-\frac{\sigma_{y,\hat{1}}\cdot\sigma_{0,\hat{2}}\cdot\sigma_{y,\hat{3}}}{2\sqrt{2}}-\frac{\sigma_{0,\hat{1}}\cdot\sigma_{y,\hat{2}}\cdot\sigma_{y,\hat{3}}}{2\sqrt{2}}-\frac{i\,\sigma_{y,\hat{1}}\cdot\sigma_{y,\hat{2}}\cdot\sigma_{y,\hat{3}}}{2\sqrt{2}}$$

A New Quantum Gate With a Parameter

Nested **Function[]** commands can be used in order to define a parametric quantum gate, as shown in the example below. The outer **Function[]** handles the dependence on the qubits, while the inner **Function[]** handles the dependence on the parameter:

```
SetQuantumGate[rot, 1,
 Function[{q1},
  Function[{θ},
   Cos[θ] |0_q1̂⟩ · ⟨0_q1̂| - Sin[θ] |0_q1̂⟩ · ⟨1_q1̂| +
    Sin[θ] |1_q1̂⟩ · ⟨0_q1̂| + Cos[θ] |1_q1̂⟩ · ⟨1_q1̂| ]]]
```

```
The expression rot is a quantum gate of 1 arguments (qubits)
```

The "parametric quantum gate" template $\square_{\hat{\square}}[\square]$ can be entered by pressing the keys [ESC]pqg[ESC]. This is the truth table of the parametric quantum gate that was defined above, when the parameter is $\pi/6$ and it is applied to the qubit with label $\hat{3}$:

```
QuantumTableForm[rot_3̂[π / 6]]
```

	Input	Output
0	$\lvert 0_{\hat{3}}\rangle$	$\frac{1}{2}\sqrt{3}\ \lvert 0_{\hat{3}}\rangle+\frac{1}{2}\ \lvert 1_{\hat{3}}\rangle$
1	$\lvert 1_{\hat{3}}\rangle$	$-\frac{1}{2}\ \lvert 0_{\hat{3}}\rangle+\frac{1}{2}\sqrt{3}\ \lvert 1_{\hat{3}}\rangle$

The "parametric quantum gate" template $\square_{\hat{\square}}[\square]$ can be entered by pressing the keys [ESC]pqg[ESC]. This is the truth table of the parametric quantum gate that was defined above, when the parameter is $\pi/12$ and it is applied to the qubit with label $\hat{3}$:

```
QuantumTableForm[rot_3̂[π / 12]]
```

	Input	Output
0	$\lvert 0_{\hat{3}}\rangle$	$\left(\frac{\sqrt{\frac{3}{2}}}{2}+\frac{1}{2\sqrt{2}}\right)\lvert 0_{\hat{3}}\rangle+\left(\frac{\sqrt{\frac{3}{2}}}{2}-\frac{1}{2\sqrt{2}}\right)\lvert 1_{\hat{3}}\rangle$
1	$\lvert 1_{\hat{3}}\rangle$	$\left(-\frac{\sqrt{\frac{3}{2}}}{2}+\frac{1}{2\sqrt{2}}\right)\lvert 0_{\hat{3}}\rangle+\left(\frac{\sqrt{\frac{3}{2}}}{2}+\frac{1}{2\sqrt{2}}\right)\lvert 1_{\hat{3}}\rangle$

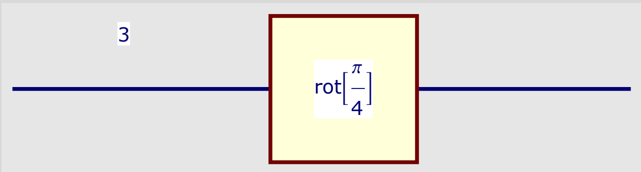
The "parametric quantum gate" template $\square_{\hat{\square}}[\square]$ can be entered by pressing the keys [ESC]pqg[ESC]. This is the truth table of the parametric quantum gate that was defined above, when the parameter is $\pi/4$ and it is applied to the qubit with label $\hat{3}$:

```
QuantumTableForm[rot₃̂[π / 4]]
```

	Input	Output
0	$\left\| 0_{\hat{3}} \right\rangle$	$\frac{\left\| 0_{\hat{3}} \right\rangle}{\sqrt{2}} + \frac{\left\| 1_{\hat{3}} \right\rangle}{\sqrt{2}}$
1	$\left\| 1_{\hat{3}} \right\rangle$	$-\frac{\left\| 0_{\hat{3}} \right\rangle}{\sqrt{2}} + \frac{\left\| 1_{\hat{3}} \right\rangle}{\sqrt{2}}$

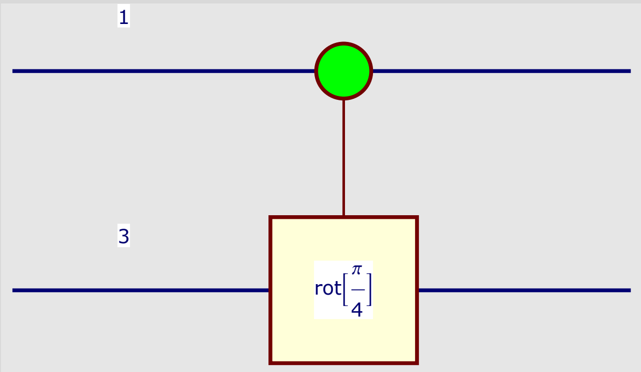
The parametric quantum gate that was defined above can be included in quantum circuits:

```
QuantumPlot[rot₃̂[π / 4]]
```



This is a "controlled version" of the new parametric quantum gate. Press [ESC]cgate[ESC] for the $C^{\{\hat{\square}\}}[\square]$ template, and [ESC]pqg[ESC] for the $\square_{\hat{\square}}[\square]$ template:

```
QuantumPlot[C^{1̂}[rot₃̂[π / 4]]]
```



This is the truth table for a "controlled version" of the new parametric quantum gate. Press [ESC]cgate[ESC] for the $C^{\{\hat{\square}\}}[\square]$ template, and [ESC]pqg[ESC] for the $\square_{\hat{\square}}[\square]$ template:

```
QuantumTableForm[C^{1̂}[rot_3̂[π / 4]]]
```

	Input	Output
0	$\vert 0_{\hat{1}}, 0_{\hat{3}}\rangle$	$\vert 0_{\hat{1}}, 0_{\hat{3}}\rangle$
1	$\vert 0_{\hat{1}}, 1_{\hat{3}}\rangle$	$\vert 0_{\hat{1}}, 1_{\hat{3}}\rangle$
2	$\vert 1_{\hat{1}}, 0_{\hat{3}}\rangle$	$\frac{\vert 1_{\hat{1}},0_{\hat{3}}\rangle}{\sqrt{2}} + \frac{\vert 1_{\hat{1}},1_{\hat{3}}\rangle}{\sqrt{2}}$
3	$\vert 1_{\hat{1}}, 1_{\hat{3}}\rangle$	$-\frac{\vert 1_{\hat{1}},0_{\hat{3}}\rangle}{\sqrt{2}} + \frac{\vert 1_{\hat{1}},1_{\hat{3}}\rangle}{\sqrt{2}}$

This is a "tensor power" of the parametric quantum gate. Notice that **each gate** in the answer is applied to a **different qubit**. Press [ESC]tpow[ESC] for the $(\square)^{\otimes\square}$ template, and [ESC]pqg[ESC] for the $\square_{\hat{\square}}[\square]$ template:

```
(rot_1̂[π / 4])^⊗3
```

$$\mathrm{rot}_{\hat{1}}\left[\frac{\pi}{4}\right] \cdot \mathrm{rot}_{\hat{2}}\left[\frac{\pi}{4}\right] \cdot \mathrm{rot}_{\hat{3}}\left[\frac{\pi}{4}\right]$$

This is the truth table of a **"tensor power"** of the parametric quantum gate. Notice that there are **three different qubits** in the answer. Press [ESC]tpow[ESC] for the $(\square)^{\otimes\square}$ template, and [ESC]pqg[ESC] for the $\square_{\hat{\square}}[\square]$ template:

```
QuantumTableForm[(rot_1̂[π / 4])^⊗3]
```

	Input	Output
0	$\vert 0_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}\rangle$	$\frac{\vert 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \vert 1$
1	$\vert 0_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}\rangle$	$-\frac{\vert 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} +$
2	$\vert 0_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}\rangle$	$-\frac{\vert 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} +$
3	$\vert 0_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}\rangle$	$\frac{\vert 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \vert 1$
4	$\vert 1_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}\rangle$	$-\frac{\vert 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} +$
5	$\vert 1_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}\rangle$	$\frac{\vert 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \vert 1$
6	$\vert 1_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}\rangle$	$\frac{\vert 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \vert 1$
7	$\vert 1_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}\rangle$	$-\frac{\vert 0_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 0_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 0_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 0_{\hat{1}},1_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} + \frac{\vert 1_{\hat{1}},0_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 1_{\hat{1}},0_{\hat{2}},1_{\hat{3}}\rangle}{2\sqrt{2}} - \frac{\vert 1_{\hat{1}},1_{\hat{2}},0_{\hat{3}}\rangle}{2\sqrt{2}} +$

Compare the following truth table with the previous one.
The difference is that below we have a **"normal power"** [ESC]po[ESC]; it means the repeated application of the gate to the **same qubit**:

$$\texttt{QuantumTableForm}\left[\left(\texttt{rot}_{\hat{1}}[\pi/4]\right)^3\right]$$

	Input	Output
0	$\lvert 0_{\hat{1}}\rangle$	$-\frac{\lvert 0_{\hat{1}}\rangle}{\sqrt{2}}+\frac{\lvert 1_{\hat{1}}\rangle}{\sqrt{2}}$
1	$\lvert 1_{\hat{1}}\rangle$	$-\frac{\lvert 0_{\hat{1}}\rangle}{\sqrt{2}}-\frac{\lvert 1_{\hat{1}}\rangle}{\sqrt{2}}$

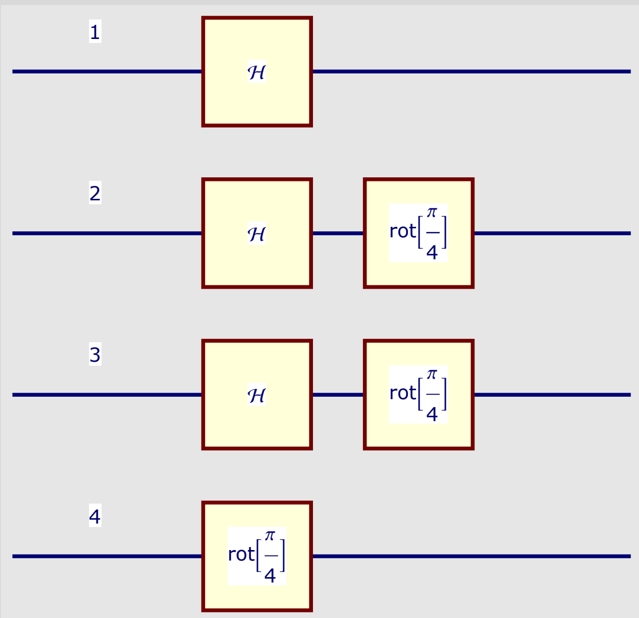
This is a tensor product of an expression that includes the new parametric gate. Press [ESC]tprod[ESC] for the $\bigotimes_{\square=\square}^{\square}\square$ template, [ESC]pqg[ESC] for the $\square_{\hat{\square}}[\square]$ template, [ESC]hg[ESC] for the $\mathcal{H}_{\hat{\square}}$ template and [ESC]on[ESC] for the centerdot · symbol:

$$\bigotimes_{j=1}^{3}\left(\texttt{rot}_{\widehat{j+1}}[\pi/4]\cdot\mathcal{H}_{\hat{j}}\right)$$

$$\texttt{rot}_{\hat{2}}\left[\frac{\pi}{4}\right]\cdot\mathcal{H}_{\hat{1}}\cdot\texttt{rot}_{\hat{3}}\left[\frac{\pi}{4}\right]\cdot\mathcal{H}_{\hat{2}}\cdot\texttt{rot}_{\hat{4}}\left[\frac{\pi}{4}\right]\cdot\mathcal{H}_{\hat{3}}$$

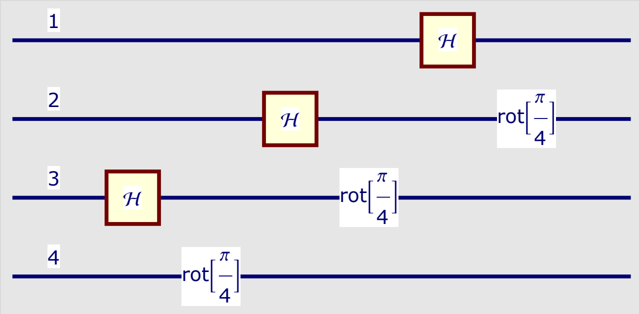
This is the quantum circuit for the expression. Notice that the three parametric gates are **not** in the same column:

$$\texttt{QuantumPlot}\left[\bigotimes_{j=1}^{3}\left(\texttt{rot}_{\widehat{j+1}}[\pi/4]\cdot\mathcal{H}_{\hat{j}}\right)\right]$$



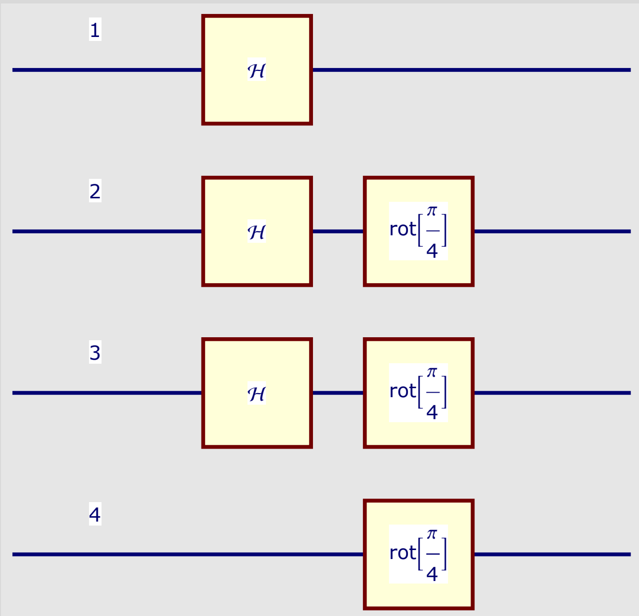
The circuit looks more symetrical with the option QuantumGateShifting → False, however the three parametric gates are **not** in the same column:

$$\text{QuantumPlot}\left[\bigotimes_{j=1}^{3}\left(\text{rot}_{\widehat{j+1}}[\pi / 4] \cdot \mathcal{H}_{\hat{j}}\right),\ \text{QuantumGateShifting} \rightarrow \text{False}\right]$$



Finally a version of the circuit where the three parametric gates **are** in the same column. Notice that the tensor product template is not used anymore. Instead, press the keys [ESC]pqggg[ESC] for the the $\square_{\{\hat{\square},\hat{\square},\hat{\square}\}}[\square]$ template and [ESC]hggg[ESC] for the $\mathcal{H}_{\{\hat{\square},\hat{\square},\hat{\square}\}}$ template:

$$\text{QuantumPlot}\left[\text{rot}_{\{\hat{2},\hat{3},\hat{4}\}}[\pi / 4] \cdot \mathcal{H}_{\{\hat{1},\hat{2},\hat{3}\}}\right]$$



These two notations represent the same set of gates, the difference is whether the gates will or will not be plotted in the same column in a quantum circuit:

$$\text{QuantumEvaluate}\left[\text{rot}_{\{\hat{1},\hat{2},\hat{3}\}}[\pi / 4]\right] == \text{QuantumEvaluate}\left[\bigotimes_{j=1}^{3}\text{rot}_{\hat{j}}[\pi / 4]\right]$$

```
True
```

This is the Dirac expression for our new parametric quantum gate, with parameter $\pi/6$:

```
QuantumEvaluate[rot_3̂[π / 6]]
```

$\frac{1}{2}\sqrt{3}\ |0_{\hat{3}}\rangle\cdot\langle 0_{\hat{3}}| + \frac{1}{2}\ |1_{\hat{3}}\rangle\cdot\langle 0_{\hat{3}}| - \frac{1}{2}\ |0_{\hat{3}}\rangle\cdot\langle 1_{\hat{3}}| + \frac{1}{2}\sqrt{3}\ |1_{\hat{3}}\rangle\cdot\langle 1_{\hat{3}}|$

This is the matrix representation of our new parametric quantum gate, with parameter π/6:

```
QuantumMatrixForm[rot_3̂[π / 6]]
```

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

This is the matrix representation of the hermitian conjugate of our new parametric quantum gate, with parameter π/6. Press [ESC]her[ESC] for the $(\square)^{\dagger}$ template:

```
QuantumMatrixForm[(rot_3̂[π / 6])†]
```

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

This is the Pauli expansion of our new parametric quantum gate, with parameter π/6::

```
PauliExpand[rot_3̂[π / 6]]
```

$\frac{1}{2}\sqrt{3}\ \sigma_{0,\hat{3}} - \frac{1}{2}\, i\, \sigma_{y,\hat{3}}$

If we try yo get the expansion for an arbitrary value of the parameter, in this case t, we get an error, because if t is a complex number, then gate would be nonunitary:

```
PauliExpand[rot_3̂[t]]
```

PauliExpand::nonunit : PauliExpand can only expand unitary operators
The expression $rot_{\hat{3}}$[t] is or might be nonunitary for some complex values of its parameters.

```
$Aborted
```

A simple way to indicate that the parameter is always real is to write **Re[t]** instead of **t**. We obtain the correct expansion without any error:

```
PauliExpand[rot_3̂[Re[t]]]
```

$\text{Cos[Re[t]]}\ \sigma_{0,\hat{3}} - i\ \text{Sin[Re[t]]}\ \sigma_{y,\hat{3}}$

by José Luis Gómez-Muñoz
http://homepage.cem.itesm.mx/lgomez/quantum/
jose.luis.gomez@itesm.mx