Quantum Random Walk: Naive Approach

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Introduction

The quantum random walker (Y. Aharonov, L. Davidovich, and N. Zagury "Quantum Random Walks" Phys. Rev. A 48, 1687 - 1690 (1993)) is made of a "coin" and a "walker", each one with its own state-space, which will make a composite system with the coin and the walker in entanglement. A unitary operator will be defined to "flip the coin", while another unitary operator will be defined to "move the walker" based on coin's result. The second operator produces entanglement between coin and walker.

Load the Package

First load the Quantum`Notation` package. Write:
Needs["Quantum`Notation`"];
then press at the same time the keys SHIFT-ENTER to evaluate. *Mathematica* will load the package:

```
Needs["Quantum`Notation`"]
```

```
Quantum`Notation` Version 2.2.0. (July 2010)
A Mathematica package for Quantum calculations in Dirac bra-ket notation
by José Luis Gómez-Muñoz

Execute SetQuantumAliases[] in order to use
  the keyboard to enter quantum objects in Dirac's notation
SetQuantumAliases[] must be executed again in each new
  notebook that is created, only one time per notebook.
```

In order to use the keyboard to enter quantum objects write:
SetQuantumAliases[ ];
then press at the same time the keys SHIFT-ENTER to evaluate. The semicolon prevents *Mathematica* from printing the help message. Remember that SetQuantumAliases[ ] must be evaluated again in each new notebook:

```
SetQuantumAliases[];
```

Quantum Random Walk in Dirac Notation

The coin C can have only two states, 0 and 1 (head and tail), while the walker P can be in any (integer) position. The coin is initially in state 0 and the wolker is initially at the origin 0:

$$|w[0]\rangle = |0_{\hat{c}}\rangle \otimes |0_{\hat{p}}\rangle$$

$$|0_{\hat{c}}, 0_{\hat{p}}\rangle$$

Here we define the "flipping coin" operator, which in this case is a Hadamard operator, but it could be any unitary operator that acts only on the coin C. The coin have only two states, 0 and 1 (head and tail). Notice the last sign is negative:

$$h = \frac{1}{\sqrt{2}} \left(|0_{\hat{c}}\rangle \cdot \langle 0_{\hat{c}}| + |1_{\hat{c}}\rangle \cdot \langle 0_{\hat{c}}| + |0_{\hat{c}}\rangle \cdot \langle 1_{\hat{c}}| - |1_{\hat{c}}\rangle \cdot \langle 1_{\hat{c}}| \right)$$

$$\frac{|0_{\hat{c}}\rangle \cdot \langle 0_{\hat{c}}| + |1_{\hat{c}}\rangle \cdot \langle 0_{\hat{c}}| + |0_{\hat{c}}\rangle \cdot \langle 1_{\hat{c}}| - |1_{\hat{c}}\rangle \cdot \langle 1_{\hat{c}}|}{\sqrt{2}}$$

Here we define the "move the walker" operator. Its interpretation is very simple, the walkers P moves to the left j-1 if the coin is in state 0, and it moves to the right j+1 if the coin is in state 1. This is one of the "naive" parts of this implementation, it looks nice because it is exactly the same way you would do it by hand, but it is not computationally efficient to define the operator this way.

$$s = |0_{\hat{c}}\rangle \cdot \langle 0_{\hat{c}}| \otimes \sum_{j=-\infty}^{\infty} \left(|(j-1)_{\hat{p}}\rangle \cdot \langle j_{\hat{p}}| \right) + |1_{\hat{c}}\rangle \cdot \langle 1_{\hat{c}}| \otimes \sum_{j=-\infty}^{\infty} \left(|(j+1)_{\hat{p}}\rangle \cdot \langle j_{\hat{p}}| \right)$$

$$\sum_{j=-\infty}^{\infty} |0_{\hat{c}}, (-1+j)_{\hat{p}}\rangle \cdot \langle 0_{\hat{c}}, j_{\hat{p}}| + \sum_{j=-\infty}^{\infty} |1_{\hat{c}}, (1+j)_{\hat{p}}\rangle \cdot \langle 1_{\hat{c}}, j_{\hat{p}}|$$

This is the evolution of the composite system of coin and walker: flip the coin with the operator **h** and then move the walker with operator **s**. The algebraic command Expand is necessary so that the calculation is performed and each intermidiate state is obtained as a linear superposition of basis states. This process is repeated "steps" times. This is a very naive implementation, therefore this evaluation can take several minutes in your computer. No output will be produced, however the evolution of the system is calculted and stored in $|w[0]\rangle$, $|w[1]\rangle$... $|w[20]\rangle$

```
steps = 20;
Do[ | w[k]〉 = Expand[s · (h ·  | w[k - 1]〉)],
 {k, 1, steps}]
```

Each state of the composite system coin-walker is stored. For example this is the state at the third step:

$$|w[3]\rangle$$

$$\frac{|0_{\hat{c}}, (-3)_{\hat{p}}\rangle}{2\sqrt{2}} + \frac{|0_{\hat{c}}, (-1)_{\hat{p}}\rangle}{\sqrt{2}} - \frac{|0_{\hat{c}}, 1_{\hat{p}}\rangle}{2\sqrt{2}} + \frac{|1_{\hat{c}}, (-1)_{\hat{p}}\rangle}{2\sqrt{2}} + \frac{|1_{\hat{c}}, 3_{\hat{p}}\rangle}{2\sqrt{2}}$$

And this is the state at the fifth state:

$$| w[5] \rangle$$

$$\frac{|0_{\hat{c}}, (-5)_{\hat{p}}\rangle}{4\sqrt{2}} + \frac{|0_{\hat{c}}, (-3)_{\hat{p}}\rangle}{\sqrt{2}} - \frac{|0_{\hat{c}}, 3_{\hat{p}}\rangle}{4\sqrt{2}} + \frac{|1_{\hat{c}}, (-3)_{\hat{p}}\rangle}{4\sqrt{2}} + \frac{|1_{\hat{c}}, (-1)_{\hat{p}}\rangle}{2\sqrt{2}} - \frac{|1_{\hat{c}}, 1_{\hat{p}}\rangle}{2\sqrt{2}} + \frac{|1_{\hat{c}}, 3_{\hat{p}}\rangle}{2\sqrt{2}} + \frac{|1_{\hat{c}}, 5_{\hat{p}}\rangle}{4\sqrt{2}}$$

Here we define the position projector for each position of the walker:

$$\text{pp}[\text{j\_}] := |0_{\hat{c}}, j_{\hat{p}}\rangle \cdot \langle 0_{\hat{c}}, j_{\hat{p}}| + |1_{\hat{c}}, j_{\hat{p}}\rangle \cdot \langle 1_{\hat{c}}, j_{\hat{p}}|$$

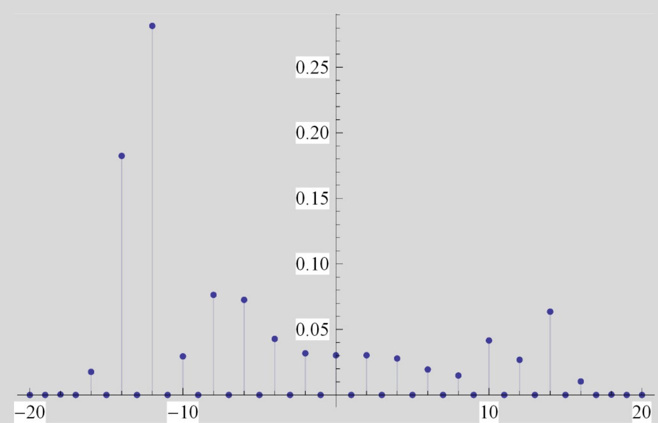
Here we calculate the probabilities for each walker position. This is another "naive" part of this implementation: It is not computationally efficient. This evaluation can take several seconds in your computer:

```
probabilities = Table[{j, ⟨w[steps] | · pp[j] · | w[steps]⟩}, {j, -steps, steps}]
```

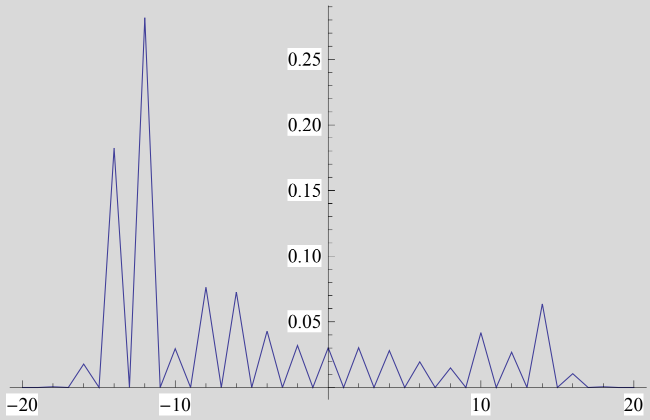
$$\left\{\left\{-20, \frac{1}{1\,048\,576}\right\}, \{-19, 0\}, \left\{-18, \frac{181}{524\,288}\right\}, \{-17, 0\}, \left\{-16, \frac{9257}{524\,288}\right\}, \{-15, 0\}, \left\{-14, \frac{95\,617}{524\,288}\right\}, \{-13, 0\}, \left\{-12, \frac{295\,265}{1\,048\,576}\right\}, \{-11, 0\}, \left\{-10, \frac{965}{32\,768}\right\}, \{-9, 0\}, \left\{-8, \frac{2501}{32\,768}\right\}, \{-7, 0\}, \left\{-6, \frac{2377}{32\,768}\right\}, \{-5, 0\}, \left\{-4, \frac{11\,221}{262\,144}\right\}, \{-3, 0\}, \left\{-2, \frac{4165}{131\,072}\right\}, \{-1, 0\}, \left\{0, \frac{3969}{131\,072}\right\}, \{1, 0\}, \left\{2, \frac{3969}{131\,072}\right\}, \{3, 0\}, \left\{4, \frac{7301}{262\,144}\right\}, \{5, 0\}, \left\{6, \frac{637}{32\,768}\right\}, \{7, 0\}, \left\{8, \frac{485}{32\,768}\right\}, \{9, 0\}, \left\{10, \frac{1361}{32\,768}\right\}, \{11, 0\}, \left\{12, \frac{28\,097}{1\,048\,576}\right\}, \{13, 0\}, \left\{14, \frac{33\,317}{524\,288}\right\}, \{15, 0\}, \left\{16, \frac{5417}{524\,288}\right\}, \{17, 0\}, \left\{18, \frac{145}{524\,288}\right\}, \{19, 0\}, \left\{20, \frac{1}{1\,048\,576}\right\}\right\}$$

Here is a plot of the probabilities for each position of the walker.

```
ListPlot[probabilities, PlotRange → All, Filling → Axis]
```



```
ListPlot[probabilities, PlotRange → All, Joined → True]
```



Another way to obtain the probabilities is using the Quantum *Mathematica* command QuantumMeasurement

```
qm = QuantumMeasurement[ | w[steps]〉, {p̂}, FactorKet → False]
```

Probability	Measurement	State
$\frac{1}{1\,048\,576}$	$\{\{(-20)_{\hat{p}}\}\}$	$\left\|0_{\hat{c}}, (-20)_{\hat{p}}\right\rangle$
$\frac{181}{524\,288}$	$\{\{(-18)_{\hat{p}}\}\}$	$\frac{19\left\|0_{\hat{c}}, (-18)_{\hat{p}}\right\rangle+\left\|1_{\hat{c}}, (-18)_{\hat{p}}\right\rangle}{\sqrt{362}}$
$\frac{9257}{524\,288}$	$\{\{(-16)_{\hat{p}}\}\}$	$\frac{135\left\|0_{\hat{c}}, (-16)_{\hat{p}}\right\rangle+17\left\|1_{\hat{c}}, (-16)_{\hat{p}}\right\rangle}{\sqrt{18\,514}}$
$\frac{95\,617}{524\,288}$	$\{\{(-14)_{\hat{p}}\}\}$	$\frac{425\left\|0_{\hat{c}}, (-14)_{\hat{p}}\right\rangle+103\left\|1_{\hat{c}}, (-14)_{\hat{p}}\right\rangle}{\sqrt{191\,234}}$
$\frac{295\,265}{1\,048\,576}$	$\{\{(-12)_{\hat{p}}\}\}$	$\frac{484\left\|0_{\hat{c}}, (-12)_{\hat{p}}\right\rangle+247\left\|1_{\hat{c}}, (-12)_{\hat{p}}\right\rangle}{\sqrt{295\,265}}$
$\frac{965}{32\,768}$	$\{\{(-10)_{\hat{p}}\}\}$	$\frac{-33\left\|0_{\hat{c}}, (-10)_{\hat{p}}\right\rangle+29\left\|1_{\hat{c}}, (-10)_{\hat{p}}\right\rangle}{\sqrt{1930}}$
$\frac{2501}{32\,768}$	$\{\{(-8)_{\hat{p}}\}\}$	$-\frac{49\left\|0_{\hat{c}}, (-8)_{\hat{p}}\right\rangle+51\left\|1_{\hat{c}}, (-8)_{\hat{p}}\right\rangle}{\sqrt{5002}}$
$\frac{2377}{32\,768}$	$\{\{(-6)_{\hat{p}}\}\}$	$\frac{65\left\|0_{\hat{c}}, (-6)_{\hat{p}}\right\rangle+23\left\|1_{\hat{c}}, (-6)_{\hat{p}}\right\rangle}{\sqrt{4754}}$
$\frac{11\,221}{262\,144}$	$\{\{(-4)_{\hat{p}}\}\}$	$\frac{-15\left\|0_{\hat{c}}, (-4)_{\hat{p}}\right\rangle+2\left\|1_{\hat{c}}, (-4)_{\hat{p}}\right\rangle}{\sqrt{229}}$
$\frac{4165}{131\,072}$	$\{\{(-2)_{\hat{p}}\}\}$	$\frac{11\left\|0_{\hat{c}}, (-2)_{\hat{p}}\right\rangle-7\left\|1_{\hat{c}}, (-2)_{\hat{p}}\right\rangle}{\sqrt{170}}$
$\frac{3969}{131\,072}$	$\{\{0_{\hat{p}}\}\}$	$\frac{-\left\|0_{\hat{c}}, 0_{\hat{p}}\right\rangle+\left\|1_{\hat{c}}, 0_{\hat{p}}\right\rangle}{\sqrt{2}}$
$\frac{3969}{131\,072}$	$\{\{2_{\hat{p}}\}\}$	$\frac{\left\|0_{\hat{c}}, 2_{\hat{p}}\right\rangle-\left\|1_{\hat{c}}, 2_{\hat{p}}\right\rangle}{\sqrt{2}}$
$\frac{7301}{262\,144}$	$\{\{4_{\hat{p}}\}\}$	$\frac{-10\left\|0_{\hat{c}}, 4_{\hat{p}}\right\rangle+7\left\|1_{\hat{c}}, 4_{\hat{p}}\right\rangle}{\sqrt{149}}$
$\frac{637}{32\,768}$	$\{\{6_{\hat{p}}\}\}$	$\frac{5\left\|0_{\hat{c}}, 6_{\hat{p}}\right\rangle-\left\|1_{\hat{c}}, 6_{\hat{p}}\right\rangle}{\sqrt{26}}$
$\frac{485}{32\,768}$	$\{\{8_{\hat{p}}\}\}$	$-\frac{21\left\|0_{\hat{c}}, 8_{\hat{p}}\right\rangle+23\left\|1_{\hat{c}}, 8_{\hat{p}}\right\rangle}{\sqrt{970}}$
$\frac{1361}{32\,768}$	$\{\{10_{\hat{p}}\}\}$	$\frac{-11\left\|0_{\hat{c}}, 10_{\hat{p}}\right\rangle+51\left\|1_{\hat{c}}, 10_{\hat{p}}\right\rangle}{\sqrt{2722}}$
$\frac{28\,097}{1\,048\,576}$	$\{\{12_{\hat{p}}\}\}$	$\frac{121\left\|0_{\hat{c}}, 12_{\hat{p}}\right\rangle-116\left\|1_{\hat{c}}, 12_{\hat{p}}\right\rangle}{\sqrt{28\,097}}$
$\frac{33\,317}{524\,288}$	$\{\{14_{\hat{p}}\}\}$	$\frac{75\left\|0_{\hat{c}}, 14_{\hat{p}}\right\rangle-247\left\|1_{\hat{c}}, 14_{\hat{p}}\right\rangle}{\sqrt{66\,634}}$
$\frac{5417}{524\,288}$	$\{\{16_{\hat{p}}\}\}$	$\frac{15\left\|0_{\hat{c}}, 16_{\hat{p}}\right\rangle-103\left\|1_{\hat{c}}, 16_{\hat{p}}\right\rangle}{\sqrt{10\,834}}$
$\frac{145}{524\,288}$	$\{\{18_{\hat{p}}\}\}$	$\frac{\left\|0_{\hat{c}}, 18_{\hat{p}}\right\rangle-17\left\|1_{\hat{c}}, 18_{\hat{p}}\right\rangle}{\sqrt{290}}$
$\frac{1}{1\,048\,576}$	$\{\{20_{\hat{p}}\}\}$	$-\left\|1_{\hat{c}}, 20_{\hat{p}}\right\rangle$
Probability	Measurement	State

You can extract the list of probabilities using standard *Mathematica* notation. Remember that the measurement results were stored in the variable qm:

```
qm[[1, All, 1]]
```

$$\left\{\frac{1}{1048576}, \frac{181}{524288}, \frac{9257}{524288}, \frac{95617}{524288}, \frac{295265}{1048576}, \frac{965}{32768}, \frac{2501}{32768}, \frac{2377}{32768}, \frac{11221}{262144}, \frac{4165}{131072}, \frac{3969}{131072}, \frac{3969}{131072}, \frac{7301}{262144}, \frac{637}{32768}, \frac{485}{32768}, \frac{1361}{32768}, \frac{28097}{1048576}, \frac{33317}{524288}, \frac{5417}{524288}, \frac{145}{524288}, \frac{1}{1048576}\right\}$$

The standard *Mathematica* command N[] gives the numerical values of the probabilities:

```
N[qm]
```

Probability	Measurement	State
9.53674×10^{-7}	$\{\{(-20)_{\hat{p}}\}\}$	$\lvert 0_{\hat{c}}, (-20)_{\hat{p}}\rangle$
0.00034523	$\{\{(-18)_{\hat{p}}\}\}$	$0.0525588\,(19.\,\lvert 0_{\hat{c}}, (-18)_{\hat{p}}\rangle + \lvert 1_{\hat{c}}, (-18)_{\hat{p}}\rangle)$
0.0176563	$\{\{(-16)_{\hat{p}}\}\}$	$0.00734937\,(135.\,\lvert 0_{\hat{c}}, (-16)_{\hat{p}}\rangle + 17.\,\lvert 1_{\hat{c}}, (-16)_{\hat{p}}\rangle)$
0.182375	$\{\{(-14)_{\hat{p}}\}\}$	$0.00228674\,(425.\,\lvert 0_{\hat{c}}, (-14)_{\hat{p}}\rangle + 103.\,\lvert 1_{\hat{c}}, (-14)_{\hat{p}}\rangle)$
0.281587	$\{\{(-12)_{\hat{p}}\}\}$	$0.00184032\,(484.\,\lvert 0_{\hat{c}}, (-12)_{\hat{p}}\rangle + 247.\,\lvert 1_{\hat{c}}, (-12)_{\hat{p}}\rangle)$
0.0294495	$\{\{(-10)_{\hat{p}}\}\}$	$0.0227626\,(-33.\,\lvert 0_{\hat{c}}, (-10)_{\hat{p}}\rangle + 29.\,\lvert 1_{\hat{c}}, (-10)_{\hat{p}}\rangle)$
0.0763245	$\{\{(-8)_{\hat{p}}\}\}$	$-0.0141393\,(49.\,\lvert 0_{\hat{c}}, (-8)_{\hat{p}}\rangle + 51.\,\lvert 1_{\hat{c}}, (-8)_{\hat{p}}\rangle)$
0.0725403	$\{\{(-6)_{\hat{p}}\}\}$	$0.0145034\,(65.\,\lvert 0_{\hat{c}}, (-6)_{\hat{p}}\rangle + 23.\,\lvert 1_{\hat{c}}, (-6)_{\hat{p}}\rangle)$
0.0428047	$\{\{(-4)_{\hat{p}}\}\}$	$0.0660819\,(-15.\,\lvert 0_{\hat{c}}, (-4)_{\hat{p}}\rangle + 2.\,\lvert 1_{\hat{c}}, (-4)_{\hat{p}}\rangle)$
0.0317764	$\{\{(-2)_{\hat{p}}\}\}$	$0.0766965\,(11.\,\lvert 0_{\hat{c}}, (-2)_{\hat{p}}\rangle - 7.\,\lvert 1_{\hat{c}}, (-2)_{\hat{p}}\rangle)$
0.0302811	$\{\{0_{\hat{p}}\}\}$	$0.707107\,(-1.\,\lvert 0_{\hat{c}}, 0_{\hat{p}}\rangle + \lvert 1_{\hat{c}}, 0_{\hat{p}}\rangle)$
0.0302811	$\{\{2_{\hat{p}}\}\}$	$0.707107\,(\lvert 0_{\hat{c}}, 2_{\hat{p}}\rangle - 1.\,\lvert 1_{\hat{c}}, 2_{\hat{p}}\rangle)$
0.0278511	$\{\{4_{\hat{p}}\}\}$	$0.0819232\,(-10.\,\lvert 0_{\hat{c}}, 4_{\hat{p}}\rangle + 7.\,\lvert 1_{\hat{c}}, 4_{\hat{p}}\rangle)$
0.0194397	$\{\{6_{\hat{p}}\}\}$	$0.196116\,(5.\,\lvert 0_{\hat{c}}, 6_{\hat{p}}\rangle - 1.\,\lvert 1_{\hat{c}}, 6_{\hat{p}}\rangle)$
0.014801	$\{\{8_{\hat{p}}\}\}$	$-0.0321081\,(21.\,\lvert 0_{\hat{c}}, 8_{\hat{p}}\rangle + 23.\,\lvert 1_{\hat{c}}, 8_{\hat{p}}\rangle)$
0.0415344	$\{\{10_{\hat{p}}\}\}$	$0.0191671\,(-11.\,\lvert 0_{\hat{c}}, 10_{\hat{p}}\rangle + 51.\,\lvert 1_{\hat{c}}, 10_{\hat{p}}\rangle)$
0.0267954	$\{\{12_{\hat{p}}\}\}$	$0.00596582\,(121.\,\lvert 0_{\hat{c}}, 12_{\hat{p}}\rangle - 116.\,\lvert 1_{\hat{c}}, 12_{\hat{p}}\rangle)$
0.0635471	$\{\{14_{\hat{p}}\}\}$	$0.00387393\,(75.\,\lvert 0_{\hat{c}}, 14_{\hat{p}}\rangle - 247.\,\lvert 1_{\hat{c}}, 14_{\hat{p}}\rangle)$
0.0103321	$\{\{16_{\hat{p}}\}\}$	$0.00960739\,(15.\,\lvert 0_{\hat{c}}, 16_{\hat{p}}\rangle - 103.\,\lvert 1_{\hat{c}}, 16_{\hat{p}}\rangle)$
0.000276566	$\{\{18_{\hat{p}}\}\}$	$0.058722\,(\lvert 0_{\hat{c}}, 18_{\hat{p}}\rangle - 17.\,\lvert 1_{\hat{c}}, 18_{\hat{p}}\rangle)$
9.53674×10^{-7}	$\{\{20_{\hat{p}}\}\}$	$-1.\,\lvert 1_{\hat{c}}, 20_{\hat{p}}\rangle$
Probability	Measurement	State

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