Grover's Search Algorithm

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Introduction

This is a tutorial on the use of Quantum`Computing` *Mathematica* add-on to implement Grover's Algorithm. Grover's algorithm is a quantum algorithm for searching an unsorted database with N entries in O($N^{1/2}$) time and using O(log N) storage space (Grover L.K.: A fast quantum mechanical algorithm for database search, Proceedings, 28th Annual ACM Symposium on the Theory of Computing, (May 1996) p. 212).
Grover's algorithm can be generalized to "Quantum Counting" (Brassard G., P. Hoyer and Al. Tapp, arXiv:quant-ph/9805082) and "Quantum Amplitude Amplification and Estimation" (Brassard G., P. Hoyer, M. Mosca and A. Tapp, arXiv:quant-ph/0005055).

Load the Package

First load the Quantum`Computing` package. Write:
Needs["Quantum`Computing`"];
then press at the same time the keys SHIFT-ENTER to evaluate. *Mathematica* will load the package:

```
Needs["Quantum`Computing`"]
```

```
Quantum`Computing` Version 2.2.0. (June 2010)
A Mathematica package for Quantum Computing
  in Dirac bra-ket notation and plotting of quantum circuits
by José Luis Gómez-Muñoz

Execute SetComputingAliases[] in order to use
  the keyboard to enter quantum objects in Dirac's notation
SetComputingAliases[] must be executed again in each new notebook that is created
```

In order to use the keyboard to enter quantum objects write:
SetComputingAliases[ ];
then press at the same time the keys SHIFT-ENTER to evaluate. The semicolon prevents *Mathematica* from printing the help message. Remember that SetComputingAliases[ ] must be evaluated again in each new notebook:

```
SetComputingAliases[];
```

Initial State ("Database") $|s\rangle$

The initial state (the "database") $|s\rangle$ is a normalized, equally-weighted linear combination of all the kets that can be written with **nq** qubits.

```
nq = 4;
```

$$| s\rangle = \frac{1}{\sqrt{2^{nq}}} \sum_{j=0}^{2^{nq}-1} | j\rangle_{nq}$$

$$\frac{1}{4} \left(| 0_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}, 0_{\hat{4}}\rangle + | 0_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}, 1_{\hat{4}}\rangle + | 0_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}, 0_{\hat{4}}\rangle + | 0_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}, 1_{\hat{4}}\rangle + | 0_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}, 0_{\hat{4}}\rangle + | 0_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}, 1_{\hat{4}}\rangle + | 0_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}, 0_{\hat{4}}\rangle + | 0_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}, 1_{\hat{4}}\rangle + | 1_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}, 0_{\hat{4}}\rangle + | 1_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}, 1_{\hat{4}}\rangle + | 1_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}, 0_{\hat{4}}\rangle + | 1_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}, 1_{\hat{4}}\rangle + | 1_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}, 0_{\hat{4}}\rangle + | 1_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}, 1_{\hat{4}}\rangle + | 1_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}, 0_{\hat{4}}\rangle + | 1_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}, 1_{\hat{4}}\rangle \right)$$

This is the ket $| s\rangle$ in a more readable form. Each basis ket is suposed to be a "register" in the "database" $| s\rangle$:

```
TraditionalForm[ | s〉]
```

$$\frac{1}{4} (| 0000\rangle + | 0001\rangle + | 0010\rangle + | 0011\rangle + | 0100\rangle + | 0101\rangle + | 0110\rangle + | 0111\rangle + | 1000\rangle + | 1001\rangle + | 1010\rangle + | 1011\rangle + | 1100\rangle + | 1101\rangle + | 1110\rangle + | 1111\rangle)$$

Here we verify that the ket $| s\rangle$ is normalized. The norm template can be entered by pressing the keys [ESC]norm[ESC]

$$\| \, | s\rangle \|$$

```
1
```

Operators U_s and U_ω

Here we define operators U_s and U_ω that will be used in each iteration of the algorithm. Notice that *Mathematica* language can use the same symbol (u) for the two operators; *Mathematica* will only use "general" definitions (like the one for u[w_]) when "special" definitions (like the one for u[s]) do not apply. Definition for u[w_] is "general" because of the underscore _ after the w. Notice also the use of "SetDelayed" := in the general definition. All of these is standard *Mathematica* language. These definitions are using the values that were stored in **nq** and **| s〉** above in this document. Rember to press [SHIFT][ENTER] to evaluate and store the definitions in computer's memory:

```
u[s] = 2 | s〉 · 〈s | - 1;
u[w_] := 1 - 2 | w〉nq · 〈w |nq
```

We can see the content of u[s] after evaluating the definitions above:

```
TraditionalForm[Expand[u[s]]]
```

$$\frac{1}{8}\,|0000\rangle\langle 0000| + \frac{1}{8}\,|0001\rangle\langle 0000| + \frac{1}{8}\,|0010\rangle\langle 0000| + \frac{1}{8}\,|0011\rangle\langle 0000| + \frac{1}{8}\,|0100\rangle\langle 0000| + \frac{1}{8}\,|0101\rangle\langle 0000| +$$

$$\frac{1}{8}\,|0110\rangle\langle 0000| + \frac{1}{8}\,|0111\rangle\langle 0000| + \frac{1}{8}\,|1000\rangle\langle 0000| + \frac{1}{8}\,|1001\rangle\langle 0000| + \frac{1}{8}\,|1010\rangle\langle 0000| + \frac{1}{8}\,|1011\rangle\langle 0000| +$$

$$\frac{1}{8}\,|1100\rangle\langle 0000| + \frac{1}{8}\,|1101\rangle\langle 0000| + \frac{1}{8}\,|1110\rangle\langle 0000| + \frac{1}{8}\,|1111\rangle\langle 0000| + \frac{1}{8}\,|0000\rangle\langle 0001| + \frac{1}{8}\,|0001\rangle\langle 0001| +$$

$$\frac{1}{8}\,|0010\rangle\langle 0001| + \frac{1}{8}\,|0011\rangle\langle 0001| + \frac{1}{8}\,|0100\rangle\langle 0001| + \frac{1}{8}\,|0101\rangle\langle 0001| + \frac{1}{8}\,|0110\rangle\langle 0001| + \frac{1}{8}\,|0111\rangle\langle 0001| +$$

$$\frac{1}{8}\,|1000\rangle\langle 0001| + \frac{1}{8}\,|1001\rangle\langle 0001| + \frac{1}{8}\,|1010\rangle\langle 0001| + \frac{1}{8}\,|1011\rangle\langle 0001| + \frac{1}{8}\,|1100\rangle\langle 0001| +$$

$$\frac{1}{8}\,|1101\rangle\langle 0001| + \frac{1}{8}\,|1110\rangle\langle 0001| + \frac{1}{8}\,|1111\rangle\langle 0001| + \frac{1}{8}\,|0000\rangle\langle 0010| + \frac{1}{8}\,|0001\rangle\langle 0010| +$$

$$\frac{1}{8}\,|0010\rangle\langle 0010| + \frac{1}{8}\,|0011\rangle\langle 0010| + \frac{1}{8}\,|0100\rangle\langle 0010| + \frac{1}{8}\,|0101\rangle\langle 0010| + \frac{1}{8}\,|0110\rangle\langle 0010| +$$

$$\frac{1}{8}\,|0111\rangle\langle 0010| + \frac{1}{8}\,|1000\rangle\langle 0010| + \frac{1}{8}\,|1001\rangle\langle 0010| + \frac{1}{8}\,|1010\rangle\langle 0010| + \frac{1}{8}\,|1011\rangle\langle 0010| +$$

$$\frac{1}{8}\,|1100\rangle\langle 0010| + \frac{1}{8}\,|1101\rangle\langle 0010| + \frac{1}{8}\,|1110\rangle\langle 0010| + \frac{1}{8}\,|1111\rangle\langle 0010| + \frac{1}{8}\,|0000\rangle\langle 0011| +$$

$$\frac{1}{8}\,|0001\rangle\langle 0011| + \frac{1}{8}\,|0010\rangle\langle 0011| + \frac{1}{8}\,|0011\rangle\langle 0011| + \frac{1}{8}\,|0100\rangle\langle 0011| + \frac{1}{8}\,|0101\rangle\langle 0011| +$$

$$\frac{1}{8}\,|0110\rangle\langle 0011| + \frac{1}{8}\,|0111\rangle\langle 0011| + \frac{1}{8}\,|1000\rangle\langle 0011| + \frac{1}{8}\,|1001\rangle\langle 0011| + \frac{1}{8}\,|1010\rangle\langle 0011| +$$

$$\frac{1}{8}\,|1011\rangle\langle 0011| + \frac{1}{8}\,|1100\rangle\langle 0011| + \frac{1}{8}\,|1101\rangle\langle 0011| + \frac{1}{8}\,|1110\rangle\langle 0011| + \frac{1}{8}\,|1111\rangle\langle 0011| +$$

$$\frac{1}{8}\,|0000\rangle\langle 0100| + \frac{1}{8}\,|0001\rangle\langle 0100| + \frac{1}{8}\,|0010\rangle\langle 0100| + \frac{1}{8}\,|0011\rangle\langle 0100| + \frac{1}{8}\,|0100\rangle\langle 0100| +$$

$$\frac{1}{8}\,|0101\rangle\langle 0100| + \frac{1}{8}\,|0110\rangle\langle 0100| + \frac{1}{8}\,|0111\rangle\langle 0100| + \frac{1}{8}\,|1000\rangle\langle 0100| + \frac{1}{8}\,|1001\rangle\langle 0100| +$$

$$\frac{1}{8}\,|1010\rangle\langle 0100| + \frac{1}{8}\,|1011\rangle\langle 0100| + \frac{1}{8}\,|1100\rangle\langle 0100| + \frac{1}{8}\,|1101\rangle\langle 0100| + \frac{1}{8}\,|1110\rangle\langle 0100| +$$

$$\frac{1}{8}\,|1111\rangle\langle 0100| + \frac{1}{8}\,|0000\rangle\langle 0101| + \frac{1}{8}\,|0001\rangle\langle 0101| + \frac{1}{8}\,|0010\rangle\langle 0101| + \frac{1}{8}\,|0011\rangle\langle 0101| +$$

$$\frac{1}{8}\,|0100\rangle\langle 0101| + \frac{1}{8}\,|0101\rangle\langle 0101| + \frac{1}{8}\,|0110\rangle\langle 0101| + \frac{1}{8}\,|0111\rangle\langle 0101| + \frac{1}{8}\,|1000\rangle\langle 0101| +$$

$$\frac{1}{8}\,|1001\rangle\langle 0101| + \frac{1}{8}\,|1010\rangle\langle 0101| + \frac{1}{8}\,|1011\rangle\langle 0101| + \frac{1}{8}\,|1100\rangle\langle 0101| + \frac{1}{8}\,|1101\rangle\langle 0101| +$$

$\frac{1}{8}\,|1110\rangle\langle 0101| + \frac{1}{8}\,|1111\rangle\langle 0101| + \frac{1}{8}\,|0000\rangle\langle 0110| + \frac{1}{8}\,|0001\rangle\langle 0110| + \frac{1}{8}\,|0010\rangle\langle 0110| +$

$\frac{1}{8}\,|0011\rangle\langle 0110| + \frac{1}{8}\,|0100\rangle\langle 0110| + \frac{1}{8}\,|0101\rangle\langle 0110| + \frac{1}{8}\,|0110\rangle\langle 0110| + \frac{1}{8}\,|0111\rangle\langle 0110| +$

$\frac{1}{8}\,|1000\rangle\langle 0110| + \frac{1}{8}\,|1001\rangle\langle 0110| + \frac{1}{8}\,|1010\rangle\langle 0110| + \frac{1}{8}\,|1011\rangle\langle 0110| + \frac{1}{8}\,|1100\rangle\langle 0110| +$

$\frac{1}{8}\,|1101\rangle\langle 0110| + \frac{1}{8}\,|1110\rangle\langle 0110| + \frac{1}{8}\,|1111\rangle\langle 0110| + \frac{1}{8}\,|0000\rangle\langle 0111| + \frac{1}{8}\,|0001\rangle\langle 0111| +$

$\frac{1}{8}\,|0010\rangle\langle 0111| + \frac{1}{8}\,|0011\rangle\langle 0111| + \frac{1}{8}\,|0100\rangle\langle 0111| + \frac{1}{8}\,|0101\rangle\langle 0111| + \frac{1}{8}\,|0110\rangle\langle 0111| +$

$\frac{1}{8}\,|0111\rangle\langle 0111| + \frac{1}{8}\,|1000\rangle\langle 0111| + \frac{1}{8}\,|1001\rangle\langle 0111| + \frac{1}{8}\,|1010\rangle\langle 0111| + \frac{1}{8}\,|1011\rangle\langle 0111| +$

$\frac{1}{8}\,|1100\rangle\langle 0111| + \frac{1}{8}\,|1101\rangle\langle 0111| + \frac{1}{8}\,|1110\rangle\langle 0111| + \frac{1}{8}\,|1111\rangle\langle 0111| + \frac{1}{8}\,|0000\rangle\langle 1000| +$

$\frac{1}{8}\,|0001\rangle\langle 1000| + \frac{1}{8}\,|0010\rangle\langle 1000| + \frac{1}{8}\,|0011\rangle\langle 1000| + \frac{1}{8}\,|0100\rangle\langle 1000| + \frac{1}{8}\,|0101\rangle\langle 1000| + \frac{1}{8}\,|0110\rangle\langle 1000| +$

$\frac{1}{8}\,|0111\rangle\langle 1000| + \frac{1}{8}\,|1000\rangle\langle 1000| + \frac{1}{8}\,|1001\rangle\langle 1000| + \frac{1}{8}\,|1010\rangle\langle 1000| + \frac{1}{8}\,|1011\rangle\langle 1000| +$

$\frac{1}{8}\,|1100\rangle\langle 1000| + \frac{1}{8}\,|1101\rangle\langle 1000| + \frac{1}{8}\,|1110\rangle\langle 1000| + \frac{1}{8}\,|1111\rangle\langle 1000| + \frac{1}{8}\,|0000\rangle\langle 1001| +$

$\frac{1}{8}\,|0001\rangle\langle 1001| + \frac{1}{8}\,|0010\rangle\langle 1001| + \frac{1}{8}\,|0011\rangle\langle 1001| + \frac{1}{8}\,|0100\rangle\langle 1001| + \frac{1}{8}\,|0101\rangle\langle 1001| +$

$\frac{1}{8}\,|0110\rangle\langle 1001| + \frac{1}{8}\,|0111\rangle\langle 1001| + \frac{1}{8}\,|1000\rangle\langle 1001| + \frac{1}{8}\,|1001\rangle\langle 1001| + \frac{1}{8}\,|1010\rangle\langle 1001| +$

$\frac{1}{8}\,|1011\rangle\langle 1001| + \frac{1}{8}\,|1100\rangle\langle 1001| + \frac{1}{8}\,|1101\rangle\langle 1001| + \frac{1}{8}\,|1110\rangle\langle 1001| + \frac{1}{8}\,|1111\rangle\langle 1001| +$

$\frac{1}{8}\,|0000\rangle\langle 1010| + \frac{1}{8}\,|0001\rangle\langle 1010| + \frac{1}{8}\,|0010\rangle\langle 1010| + \frac{1}{8}\,|0011\rangle\langle 1010| + \frac{1}{8}\,|0100\rangle\langle 1010| +$

$\frac{1}{8}\,|0101\rangle\langle 1010| + \frac{1}{8}\,|0110\rangle\langle 1010| + \frac{1}{8}\,|0111\rangle\langle 1010| + \frac{1}{8}\,|1000\rangle\langle 1010| + \frac{1}{8}\,|1001\rangle\langle 1010| +$

$\frac{1}{8}\,|1010\rangle\langle 1010| + \frac{1}{8}\,|1011\rangle\langle 1010| + \frac{1}{8}\,|1100\rangle\langle 1010| + \frac{1}{8}\,|1101\rangle\langle 1010| + \frac{1}{8}\,|1110\rangle\langle 1010| + \frac{1}{8}\,|1111\rangle\langle 1010| +$

$\frac{1}{8}\,|0000\rangle\langle 1011| + \frac{1}{8}\,|0001\rangle\langle 1011| + \frac{1}{8}\,|0010\rangle\langle 1011| + \frac{1}{8}\,|0011\rangle\langle 1011| + \frac{1}{8}\,|0100\rangle\langle 1011| +$

$\frac{1}{8}\,|0101\rangle\langle 1011| + \frac{1}{8}\,|0110\rangle\langle 1011| + \frac{1}{8}\,|0111\rangle\langle 1011| + \frac{1}{8}\,|1000\rangle\langle 1011| + \frac{1}{8}\,|1001\rangle\langle 1011| +$

$\frac{1}{8}\,|1010\rangle\langle 1011| + \frac{1}{8}\,|1011\rangle\langle 1011| + \frac{1}{8}\,|1100\rangle\langle 1011| + \frac{1}{8}\,|1101\rangle\langle 1011| + \frac{1}{8}\,|1110\rangle\langle 1011| +$

$\frac{1}{8}\,|1111\rangle\langle1011| + \frac{1}{8}\,|0000\rangle\langle1100| + \frac{1}{8}\,|0001\rangle\langle1100| + \frac{1}{8}\,|0010\rangle\langle1100| + \frac{1}{8}\,|0011\rangle\langle1100| +$
$\frac{1}{8}\,|0100\rangle\langle1100| + \frac{1}{8}\,|0101\rangle\langle1100| + \frac{1}{8}\,|0110\rangle\langle1100| + \frac{1}{8}\,|0111\rangle\langle1100| + \frac{1}{8}\,|1000\rangle\langle1100| +$
$\frac{1}{8}\,|1001\rangle\langle1100| + \frac{1}{8}\,|1010\rangle\langle1100| + \frac{1}{8}\,|1011\rangle\langle1100| + \frac{1}{8}\,|1100\rangle\langle1100| + \frac{1}{8}\,|1101\rangle\langle1100| +$
$\frac{1}{8}\,|1110\rangle\langle1100| + \frac{1}{8}\,|1111\rangle\langle1100| + \frac{1}{8}\,|0000\rangle\langle1101| + \frac{1}{8}\,|0001\rangle\langle1101| + \frac{1}{8}\,|0010\rangle\langle1101| +$
$\frac{1}{8}\,|0011\rangle\langle1101| + \frac{1}{8}\,|0100\rangle\langle1101| + \frac{1}{8}\,|0101\rangle\langle1101| + \frac{1}{8}\,|0110\rangle\langle1101| + \frac{1}{8}\,|0111\rangle\langle1101| +$
$\frac{1}{8}\,|1000\rangle\langle1101| + \frac{1}{8}\,|1001\rangle\langle1101| + \frac{1}{8}\,|1010\rangle\langle1101| + \frac{1}{8}\,|1011\rangle\langle1101| + \frac{1}{8}\,|1100\rangle\langle1101| +$
$\frac{1}{8}\,|1101\rangle\langle1101| + \frac{1}{8}\,|1110\rangle\langle1101| + \frac{1}{8}\,|1111\rangle\langle1101| + \frac{1}{8}\,|0000\rangle\langle1110| + \frac{1}{8}\,|0001\rangle\langle1110| +$
$\frac{1}{8}\,|0010\rangle\langle1110| + \frac{1}{8}\,|0011\rangle\langle1110| + \frac{1}{8}\,|0100\rangle\langle1110| + \frac{1}{8}\,|0101\rangle\langle1110| + \frac{1}{8}\,|0110\rangle\langle1110| +$
$\frac{1}{8}\,|0111\rangle\langle1110| + \frac{1}{8}\,|1000\rangle\langle1110| + \frac{1}{8}\,|1001\rangle\langle1110| + \frac{1}{8}\,|1010\rangle\langle1110| + \frac{1}{8}\,|1011\rangle\langle1110| +$
$\frac{1}{8}\,|1100\rangle\langle1110| + \frac{1}{8}\,|1101\rangle\langle1110| + \frac{1}{8}\,|1110\rangle\langle1110| + \frac{1}{8}\,|1111\rangle\langle1110| + \frac{1}{8}\,|0000\rangle\langle1111| +$
$\frac{1}{8}\,|0001\rangle\langle1111| + \frac{1}{8}\,|0010\rangle\langle1111| + \frac{1}{8}\,|0011\rangle\langle1111| + \frac{1}{8}\,|0100\rangle\langle1111| + \frac{1}{8}\,|0101\rangle\langle1111| +$
$\frac{1}{8}\,|0110\rangle\langle1111| + \frac{1}{8}\,|0111\rangle\langle1111| + \frac{1}{8}\,|1000\rangle\langle1111| + \frac{1}{8}\,|1001\rangle\langle1111| + \frac{1}{8}\,|1010\rangle\langle1111| +$
$\frac{1}{8}\,|1011\rangle\langle1111| + \frac{1}{8}\,|1100\rangle\langle1111| + \frac{1}{8}\,|1101\rangle\langle1111| + \frac{1}{8}\,|1110\rangle\langle1111| + \frac{1}{8}\,|1111\rangle\langle1111| - 1$

On the other hand, u[7] uses the "general" definition u[w_]:

```
TraditionalForm[u[7]]
```

$1 - 2\,|0111\rangle\langle0111|$

Algorithm Iterations for Finding the "Register" $|\,w\rangle$

In order to "find" the register with label "w" **(in this example w=9)**, the operators U_w and U_s are applied to the ket $|\,s\rangle$; and this iteration is repeated IntegerPart$\left[\frac{\pi}{4\,\mathrm{ArcSin}\left(\frac{1}{\sqrt{N}}\right)}\right]$ times, where $N = 2^{\mathrm{nq}}$ is the number of "registers" in the "database".

The final state is stored in $|\,\mathtt{s[steps]}\rangle$ and shown as result of this calculation. This evaluation can take several seconds in yout computer.

```
w = 9;
steps = IntegerPart[π / (4 ArcSin[1 / √(2^nq)])];
 | s[0]⟩ = | s⟩;
Do[ | s[k]⟩ = Expand[u[s] · u[w] · | s[k - 1]⟩],
  {k, 1, steps}];
TraditionalForm[ | s[steps]⟩]
```

$$-\frac{13}{256}|0000\rangle-\frac{13}{256}|0001\rangle-\frac{13}{256}|0010\rangle-\frac{13}{256}|0011\rangle-\frac{13}{256}|0100\rangle-\frac{13}{256}|0101\rangle-\frac{13}{256}|0110\rangle-\frac{13}{256}|0111\rangle-\frac{13}{256}|1000\rangle+\frac{251}{256}|1001\rangle-\frac{13}{256}|1010\rangle-\frac{13}{256}|1011\rangle-\frac{13}{256}|1100\rangle-\frac{13}{256}|1101\rangle-\frac{13}{256}|1110\rangle-\frac{13}{256}|1111\rangle$$

Final Stage: Measurement

As the final stage, a measurement is performed in state that resulted from the iterations. It can be seen that all the measurement results have **small probabilty**, **except** the one that leads to the "register" (ket) with the desired label "w" **(in this example w=9 wich corresponds to** $\{1_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}, 1_{\hat{4}}\}$**)**:

```
qm = QuantumEvaluate[QubitMeasurement[ | s[steps]⟩, Table[q̂, {q, 1, nq}]]]
```

Probability	Measurement	State
$\frac{169}{65536}$	$\{\{0_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}, 0_{\hat{4}}\}\}$	$\|0_{\hat{1}}\rangle \otimes \|0_{\hat{2}}\rangle \otimes \|0_{\hat{3}}\rangle \otimes - \|0_{\hat{4}}\rangle$
$\frac{169}{65536}$	$\{\{0_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}, 1_{\hat{4}}\}\}$	$\|0_{\hat{1}}\rangle \otimes \|0_{\hat{2}}\rangle \otimes \|0_{\hat{3}}\rangle \otimes - \|1_{\hat{4}}\rangle$
$\frac{169}{65536}$	$\{\{0_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}, 0_{\hat{4}}\}\}$	$\|0_{\hat{1}}\rangle \otimes \|0_{\hat{2}}\rangle \otimes \|1_{\hat{3}}\rangle \otimes - \|0_{\hat{4}}\rangle$
$\frac{169}{65536}$	$\{\{0_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}, 1_{\hat{4}}\}\}$	$\|0_{\hat{1}}\rangle \otimes \|0_{\hat{2}}\rangle \otimes \|1_{\hat{3}}\rangle \otimes - \|1_{\hat{4}}\rangle$
$\frac{169}{65536}$	$\{\{0_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}, 0_{\hat{4}}\}\}$	$\|0_{\hat{1}}\rangle \otimes \|1_{\hat{2}}\rangle \otimes \|0_{\hat{3}}\rangle \otimes - \|0_{\hat{4}}\rangle$
$\frac{169}{65536}$	$\{\{0_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}, 1_{\hat{4}}\}\}$	$\|0_{\hat{1}}\rangle \otimes \|1_{\hat{2}}\rangle \otimes \|0_{\hat{3}}\rangle \otimes - \|1_{\hat{4}}\rangle$
$\frac{169}{65536}$	$\{\{0_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}, 0_{\hat{4}}\}\}$	$\|0_{\hat{1}}\rangle \otimes \|1_{\hat{2}}\rangle \otimes \|1_{\hat{3}}\rangle \otimes - \|0_{\hat{4}}\rangle$
$\frac{169}{65536}$	$\{\{0_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}, 1_{\hat{4}}\}\}$	$\|0_{\hat{1}}\rangle \otimes \|1_{\hat{2}}\rangle \otimes \|1_{\hat{3}}\rangle \otimes - \|1_{\hat{4}}\rangle$
$\frac{169}{65536}$	$\{\{1_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}, 0_{\hat{4}}\}\}$	$\|1_{\hat{1}}\rangle \otimes \|0_{\hat{2}}\rangle \otimes \|0_{\hat{3}}\rangle \otimes - \|0_{\hat{4}}\rangle$
$\frac{63001}{65536}$	$\{\{1_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}, 1_{\hat{4}}\}\}$	$\|1_{\hat{1}}\rangle \otimes \|0_{\hat{2}}\rangle \otimes \|0_{\hat{3}}\rangle \otimes \|1_{\hat{4}}\rangle$
$\frac{169}{65536}$	$\{\{1_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}, 0_{\hat{4}}\}\}$	$\|1_{\hat{1}}\rangle \otimes \|0_{\hat{2}}\rangle \otimes \|1_{\hat{3}}\rangle \otimes - \|0_{\hat{4}}\rangle$
$\frac{169}{65536}$	$\{\{1_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}, 1_{\hat{4}}\}\}$	$\|1_{\hat{1}}\rangle \otimes \|0_{\hat{2}}\rangle \otimes \|1_{\hat{3}}\rangle \otimes - \|1_{\hat{4}}\rangle$
$\frac{169}{65536}$	$\{\{1_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}, 0_{\hat{4}}\}\}$	$\|1_{\hat{1}}\rangle \otimes \|1_{\hat{2}}\rangle \otimes \|0_{\hat{3}}\rangle \otimes - \|0_{\hat{4}}\rangle$
$\frac{169}{65536}$	$\{\{1_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}, 1_{\hat{4}}\}\}$	$\|1_{\hat{1}}\rangle \otimes \|1_{\hat{2}}\rangle \otimes \|0_{\hat{3}}\rangle \otimes - \|1_{\hat{4}}\rangle$
$\frac{169}{65536}$	$\{\{1_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}, 0_{\hat{4}}\}\}$	$\|1_{\hat{1}}\rangle \otimes \|1_{\hat{2}}\rangle \otimes \|1_{\hat{3}}\rangle \otimes - \|0_{\hat{4}}\rangle$
$\frac{169}{65536}$	$\{\{1_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}, 1_{\hat{4}}\}\}$	$\|1_{\hat{1}}\rangle \otimes \|1_{\hat{2}}\rangle \otimes \|1_{\hat{3}}\rangle \otimes - \|1_{\hat{4}}\rangle$
Probability	Measurement	State

Measurement results were stored in the variable qm. The standard Mathematica command **N[]** can be used to obtain numerical values of the probabilities. It can be easily seen that a measurement will give (with high probability) the desired "register" $|w\rangle$ **(in this example w=9 wich corresponds to** $\{1_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}, 1_{\hat{4}}\}$**):**

```
N[qm]
```

Probability	Measurement	State
0.00257874	$\{\{0_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}, 0_{\hat{4}}\}\}$	$\lvert 0_{\hat{1}}\rangle \otimes \lvert 0_{\hat{2}}\rangle \otimes \lvert 0_{\hat{3}}\rangle \otimes (-1.\ \lvert 0_{\hat{4}}\rangle)$
0.00257874	$\{\{0_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}, 1_{\hat{4}}\}\}$	$\lvert 0_{\hat{1}}\rangle \otimes \lvert 0_{\hat{2}}\rangle \otimes \lvert 0_{\hat{3}}\rangle \otimes (-1.\ \lvert 1_{\hat{4}}\rangle)$
0.00257874	$\{\{0_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}, 0_{\hat{4}}\}\}$	$\lvert 0_{\hat{1}}\rangle \otimes \lvert 0_{\hat{2}}\rangle \otimes \lvert 1_{\hat{3}}\rangle \otimes (-1.\ \lvert 0_{\hat{4}}\rangle)$
0.00257874	$\{\{0_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}, 1_{\hat{4}}\}\}$	$\lvert 0_{\hat{1}}\rangle \otimes \lvert 0_{\hat{2}}\rangle \otimes \lvert 1_{\hat{3}}\rangle \otimes (-1.\ \lvert 1_{\hat{4}}\rangle)$
0.00257874	$\{\{0_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}, 0_{\hat{4}}\}\}$	$\lvert 0_{\hat{1}}\rangle \otimes \lvert 1_{\hat{2}}\rangle \otimes \lvert 0_{\hat{3}}\rangle \otimes (-1.\ \lvert 0_{\hat{4}}\rangle)$
0.00257874	$\{\{0_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}, 1_{\hat{4}}\}\}$	$\lvert 0_{\hat{1}}\rangle \otimes \lvert 1_{\hat{2}}\rangle \otimes \lvert 0_{\hat{3}}\rangle \otimes (-1.\ \lvert 1_{\hat{4}}\rangle)$
0.00257874	$\{\{0_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}, 0_{\hat{4}}\}\}$	$\lvert 0_{\hat{1}}\rangle \otimes \lvert 1_{\hat{2}}\rangle \otimes \lvert 1_{\hat{3}}\rangle \otimes (-1.\ \lvert 0_{\hat{4}}\rangle)$
0.00257874	$\{\{0_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}, 1_{\hat{4}}\}\}$	$\lvert 0_{\hat{1}}\rangle \otimes \lvert 1_{\hat{2}}\rangle \otimes \lvert 1_{\hat{3}}\rangle \otimes (-1.\ \lvert 1_{\hat{4}}\rangle)$
0.00257874	$\{\{1_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}, 0_{\hat{4}}\}\}$	$\lvert 1_{\hat{1}}\rangle \otimes \lvert 0_{\hat{2}}\rangle \otimes \lvert 0_{\hat{3}}\rangle \otimes (-1.\ \lvert 0_{\hat{4}}\rangle)$
0.961319	$\{\{1_{\hat{1}}, 0_{\hat{2}}, 0_{\hat{3}}, 1_{\hat{4}}\}\}$	$\lvert 1_{\hat{1}}\rangle \otimes \lvert 0_{\hat{2}}\rangle \otimes \lvert 0_{\hat{3}}\rangle \otimes \lvert 1_{\hat{4}}\rangle$
0.00257874	$\{\{1_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}, 0_{\hat{4}}\}\}$	$\lvert 1_{\hat{1}}\rangle \otimes \lvert 0_{\hat{2}}\rangle \otimes \lvert 1_{\hat{3}}\rangle \otimes (-1.\ \lvert 0_{\hat{4}}\rangle)$
0.00257874	$\{\{1_{\hat{1}}, 0_{\hat{2}}, 1_{\hat{3}}, 1_{\hat{4}}\}\}$	$\lvert 1_{\hat{1}}\rangle \otimes \lvert 0_{\hat{2}}\rangle \otimes \lvert 1_{\hat{3}}\rangle \otimes (-1.\ \lvert 1_{\hat{4}}\rangle)$
0.00257874	$\{\{1_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}, 0_{\hat{4}}\}\}$	$\lvert 1_{\hat{1}}\rangle \otimes \lvert 1_{\hat{2}}\rangle \otimes \lvert 0_{\hat{3}}\rangle \otimes (-1.\ \lvert 0_{\hat{4}}\rangle)$
0.00257874	$\{\{1_{\hat{1}}, 1_{\hat{2}}, 0_{\hat{3}}, 1_{\hat{4}}\}\}$	$\lvert 1_{\hat{1}}\rangle \otimes \lvert 1_{\hat{2}}\rangle \otimes \lvert 0_{\hat{3}}\rangle \otimes (-1.\ \lvert 1_{\hat{4}}\rangle)$
0.00257874	$\{\{1_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}, 0_{\hat{4}}\}\}$	$\lvert 1_{\hat{1}}\rangle \otimes \lvert 1_{\hat{2}}\rangle \otimes \lvert 1_{\hat{3}}\rangle \otimes (-1.\ \lvert 0_{\hat{4}}\rangle)$
0.00257874	$\{\{1_{\hat{1}}, 1_{\hat{2}}, 1_{\hat{3}}, 1_{\hat{4}}\}\}$	$\lvert 1_{\hat{1}}\rangle \otimes \lvert 1_{\hat{2}}\rangle \otimes \lvert 1_{\hat{3}}\rangle \otimes (-1.\ \lvert 1_{\hat{4}}\rangle)$
Probability	Measurement	State

TraditionalForm gives the results in a notation closer to the one used in papers and books:

```
TraditionalForm[N[qm]]
```

Probability	Measurement	State
0.00257874	$(0_1\ 0_2\ 0_3\ 0_4)$	$\lvert 0\rangle \otimes \lvert 0\rangle \otimes \lvert 0\rangle \otimes (-1.\ \lvert 0\rangle)$
0.00257874	$(0_1\ 0_2\ 0_3\ 1_4)$	$\lvert 0\rangle \otimes \lvert 0\rangle \otimes \lvert 0\rangle \otimes (-1.\ \lvert 1\rangle)$
0.00257874	$(0_1\ 0_2\ 1_3\ 0_4)$	$\lvert 0\rangle \otimes \lvert 0\rangle \otimes \lvert 1\rangle \otimes (-1.\ \lvert 0\rangle)$
0.00257874	$(0_1\ 0_2\ 1_3\ 1_4)$	$\lvert 0\rangle \otimes \lvert 0\rangle \otimes \lvert 1\rangle \otimes (-1.\ \lvert 1\rangle)$
0.00257874	$(0_1\ 1_2\ 0_3\ 0_4)$	$\lvert 0\rangle \otimes \lvert 1\rangle \otimes \lvert 0\rangle \otimes (-1.\ \lvert 0\rangle)$
0.00257874	$(0_1\ 1_2\ 0_3\ 1_4)$	$\lvert 0\rangle \otimes \lvert 1\rangle \otimes \lvert 0\rangle \otimes (-1.\ \lvert 1\rangle)$
0.00257874	$(0_1\ 1_2\ 1_3\ 0_4)$	$\lvert 0\rangle \otimes \lvert 1\rangle \otimes \lvert 1\rangle \otimes (-1.\ \lvert 0\rangle)$
0.00257874	$(0_1\ 1_2\ 1_3\ 1_4)$	$\lvert 0\rangle \otimes \lvert 1\rangle \otimes \lvert 1\rangle \otimes (-1.\ \lvert 1\rangle)$
0.00257874	$(1_1\ 0_2\ 0_3\ 0_4)$	$\lvert 1\rangle \otimes \lvert 0\rangle \otimes \lvert 0\rangle \otimes (-1.\ \lvert 0\rangle)$
0.961319	$(1_1\ 0_2\ 0_3\ 1_4)$	$\lvert 1\rangle \otimes \lvert 0\rangle \otimes \lvert 0\rangle \otimes \lvert 1\rangle$
0.00257874	$(1_1\ 0_2\ 1_3\ 0_4)$	$\lvert 1\rangle \otimes \lvert 0\rangle \otimes \lvert 1\rangle \otimes (-1.\ \lvert 0\rangle)$
0.00257874	$(1_1\ 0_2\ 1_3\ 1_4)$	$\lvert 1\rangle \otimes \lvert 0\rangle \otimes \lvert 1\rangle \otimes (-1.\ \lvert 1\rangle)$
0.00257874	$(1_1\ 1_2\ 0_3\ 0_4)$	$\lvert 1\rangle \otimes \lvert 1\rangle \otimes \lvert 0\rangle \otimes (-1.\ \lvert 0\rangle)$
0.00257874	$(1_1\ 1_2\ 0_3\ 1_4)$	$\lvert 1\rangle \otimes \lvert 1\rangle \otimes \lvert 0\rangle \otimes (-1.\ \lvert 1\rangle)$
0.00257874	$(1_1\ 1_2\ 1_3\ 0_4)$	$\lvert 1\rangle \otimes \lvert 1\rangle \otimes \lvert 1\rangle \otimes (-1.\ \lvert 0\rangle)$
0.00257874	$(1_1\ 1_2\ 1_3\ 1_4)$	$\lvert 1\rangle \otimes \lvert 1\rangle \otimes \lvert 1\rangle \otimes (-1.\ \lvert 1\rangle)$
Probability	Measurement	State

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