CSC 320
Foundations of Computer Science

Lecture 7

Instructor: Dr. Ulrike Stege

Territory Acknowledgement

We acknowledge and respect the lək̓ʷəŋən peoples on whose traditional territory the university stands and the Songhees, Esquimalt and W̱SÁNEĆ peoples whose historical relationships with the land continue to this day.

This meeting will be recorded

"Please be aware our sessions are being screen-recorded to allow students who are not able to attend to watch later and will be posted in Brightspace."

Deadlines; Assessment

10%
Quizzes
Quiz 1-8: 1% each
Quiz 9: 2%

25%
Assignments
Assignment 1-5: 5% each

25%
Midterms
Midterm 1: 10%
Midterm 2: 15%

40%
Final Exam

May

S	M	T	W	T	F	S
			3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	1	2	3
4	5	6	7	8	9	10

June

S	M	T	W	T	F	S
28	29	30	31	1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	1
2	3	4	5	6	7	8

July

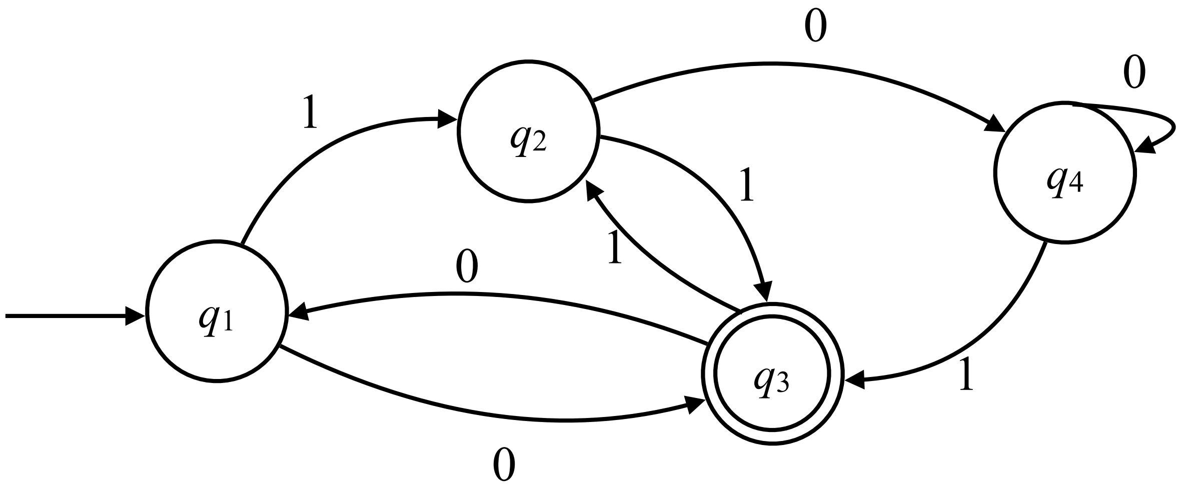
S	M	T	W	T	F	S
25	26	27	28	29	30	1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	

Timed quizzes (~30 min)
Review before starting quiz

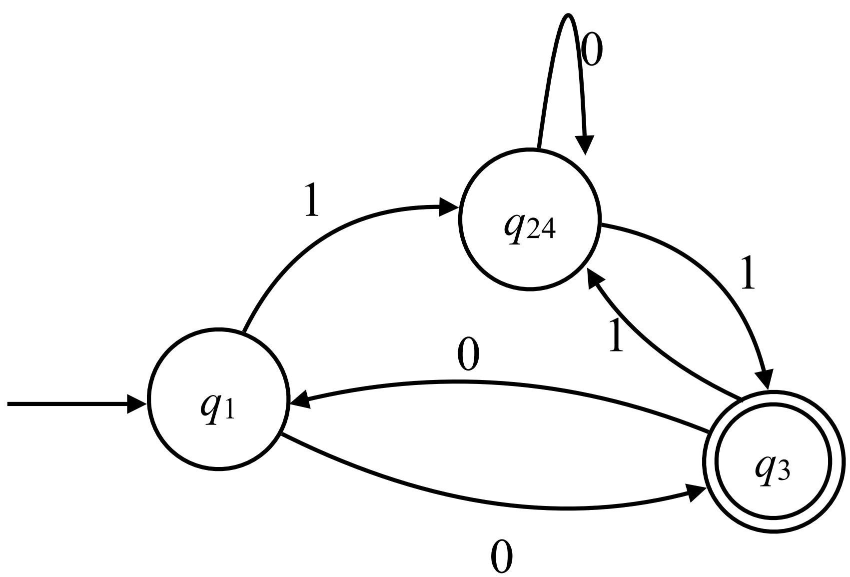
Last time ....

- What is the concept of reduction?
- DFA state minimization
- Non-regular languages
- Pumping Lemma for regular languages

Recap
DFA: State Minimization



$q_2 \sim q_4$



Today

- Nonregular languages
- Pumping Lemma for regular languages
- Using the Pumping Lemma for regular languages to show that a language is nonregular
- Proof of Pumping Lemma for regular languages

Yes or no?

- Is every finite language regular?
- Why/why not?

Pumping Lemma for regular languages

- If L is a regular language, then there is a natural number p (the *pumping length*) where:

    - if s is any string in L of length at least p (ie, $s \in L$, $|s| \geq p$), then s can be divided into $s = xyz$ satisfying the following

        1. for each $i \geq 0$: $xy^iz \in L$
        2. $|y| > 0$ (ie, $y \neq \varepsilon$)
        3. $|xy| \leq p$

Notes:
(1) y^i means the concatenation of i copies of string y
(2) Conditions 1–3 hold **for all** strings in L that are of length at least p

Pumping Lemma for regular languages

If L is a regular language, then there is a $p \in \mathbb{N}$ where:

if $s \in L$, $|s| \geq p$, then s can be divided into $s = xyz$ satisfying:

1. for each $i \geq 0$: $xy^iz \in L$
2. $|y| > 0$ (ie, $y \neq \varepsilon$)
3. $|xy| \leq p$

What if L is finite?

Example: $L = \{\epsilon, a, b, aa, ab, ba, bb\}$

Using the Pumping Lemma: Example

Prove: $L = \{0^n 1^n | n \geq 0\}$ is non-regular

Assuming L is regular, all properties of PL must hold for L:

In particular, PL gives us some p.

If $s \in L$ and $|s| \geq p$ then there is a way to rewrite s as $s = xyz$ with

1. for each $i \geq 0$: $xy^iz \in L$
2. $y \neq \epsilon$
3. $|xy| \leq p$

Using the Pumping Lemma: Example (3)

Prove: $L = \{0^n 1^n | n \geq 0\}$ is non-regular

- Let p be pumping length given by pumping lemma

- Choose $\boldsymbol{s = 0^p 1^p}$

  - $s \in L$ and $|s| \geq p$ (since $|s| = 2p$)

PL guarantees:

- s can be rewritten as $s = xyz$ with

  1. $xy^iz \in L$, for all $i \geq 0$

  2. $y \neq \epsilon$

  3. $|xy| \leq p$

- We don't know the value of p
- Actually: there is no way to know the value of p in the case that L is not a regular language
- Must work with general p

- s is string that we hope is a counterexample
- must reach a contradiction to properties of PL

- For this, length of s must depend on p since $|s| \geq p$ is required

Using the Pumping Lemma: Example (4)

Prove: $L = \{0^n 1^n | n \geq 0\}$ is non-regular

- Does there really exist some rewriting for s, $s = 0^p 1^p = xyz$ s.t. PL properties hold?
  - Recall: Trying to show that L is not regular
  - Show that $s = 0^p 1^p$ is counterexample for PL properties
    - Prove: There is ***no single rewriting*** of $s = 0^p 1^p$ into $s = 0^p 1^p = xyz$ s.t. three properties hold
  - Note: Just one rewriting of s that satisfies PL properties would show that string s **is not** a counterexample
  - Therefore we must consider all possibilities to rewrite s:
    - Because of property 2 (ie, $y \neq \epsilon$), y cannot be the empty string and therefore
      - **Case 1:** String y consists of 0s only
      - **Case 2:** String y consists only of 1s
      - **Case 3:** String y consists of both 0s and 1s

$$x\, y\, z$$

$$\underbrace{0 \ldots . 0}_{p \text{ many}}\,\underbrace{1 \ldots 1}_{p \text{ many}}$$

Using the Pumping Lemma: Example (5)

Prove: $L = \{0^n1^n | n \geq 0\}$ is non-regular

- **Case 1: String y consists of 0s only**
  String $xyyz$ has more 0s than 1s. Therefore $xyyz \notin L$, violating property 1 of PL. Contradiction.

- **Case 2: String y consists only of 1s**
  Just like above: String $xyyz$ has more 1s than 0s. Therefore $xyyz \notin L$, violating property 1 of PL. Contradiction.

- **Case 3: String y consists of 0s followed by 1s**
  String $xyyz$ may have the same number of 0s and 1s, but they are out of order (some 1s occur before 0s). Hence $xyyz \notin L$, violating property 1 of PL. Contradiction.

Since none of the cases is possible, we **cannot rewrite s** satisfying the properties of PL. Therefore: L is **not** regular

Another example

- We show that $L = \{w \in \{0,1\}^* \mid w$ has an equal number of 0s and 1s$\}$ is not regular

- Assume that L is regular

- Let p be the pumping length given by pumping lemma

- $s = ?$

Crucial what string $s \in L$ of length at least p to choose from L to derive contradiction for all possible rewritings of s

- Not every long string in L might yield contradiction

- Just one string in L of length at least p that does not satisfies the PL properties sufficient to prove language not regular.

- When choosing string as counterexample, think about what properties might make language not regular

Another example (continued)

- We show that $L = \{w \in \{0,1\}^* \mid w$ has an equal number of 0s and 1s$\}$ is not regular
- Assume that L is regular
- Let p be the pumping length given by pumping lemma
- We choose $s = 1^p0^p$: $s \in L$ and $|s| \geq p$

Another example (continued)

- $L = \{w \in \{0,1\}^* \mid w$ has an equal number of 0s and 1s$\}$

- Let $s = 1^p0^p$: $s \in L$ and $|s| \geq p$

- According to PL there is a rewriting $s = xyz$ satisfying

1. $xy^iz \in L$, for all $i \geq 0$
2. $y \neq \epsilon$
3. $|xy| \leq p$

- We show in all cases, $xy^iz \in L$ for all $i \geq 0$ not satisfied (implying that s is counterexample)

- Note: $|xy| \leq p$ (Property 3) and $y \neq \epsilon$ (Property 2)
    - y must consist of 1s only (and consists of at least one 1)
    - Consider case 1. for $i = 0$: xz
        - $xz \notin L$ since it contains less 1s than 0s! Contradiction

How can we show correctness of Pumping Lemma (PL)?

If L is a regular language, then there is a $p \in \mathbb{N}$ where:

if $s \in L$, $|s| \geq p$, then s can be divided into $s = xyz$ satisfying:

1. for each $i \geq 0$: $xy^iz \in L$
2. $|y| > 0$ (ie, $y \neq \varepsilon$)
3. $|xy| \leq p$

Proof Idea

Consider properties of DFA M if $L(M) = L$ for $p = |Q|$

What does M's computation look like for a string of length at least p?

Observation

If L satisfies PL: What do strings in L look like? Each string in L is either:

1. of length shorter than p, or

2. of length at least p

Note: Pumping Lemma only talks about strings in L that are of length at least p

Proving the Pumping Lemma

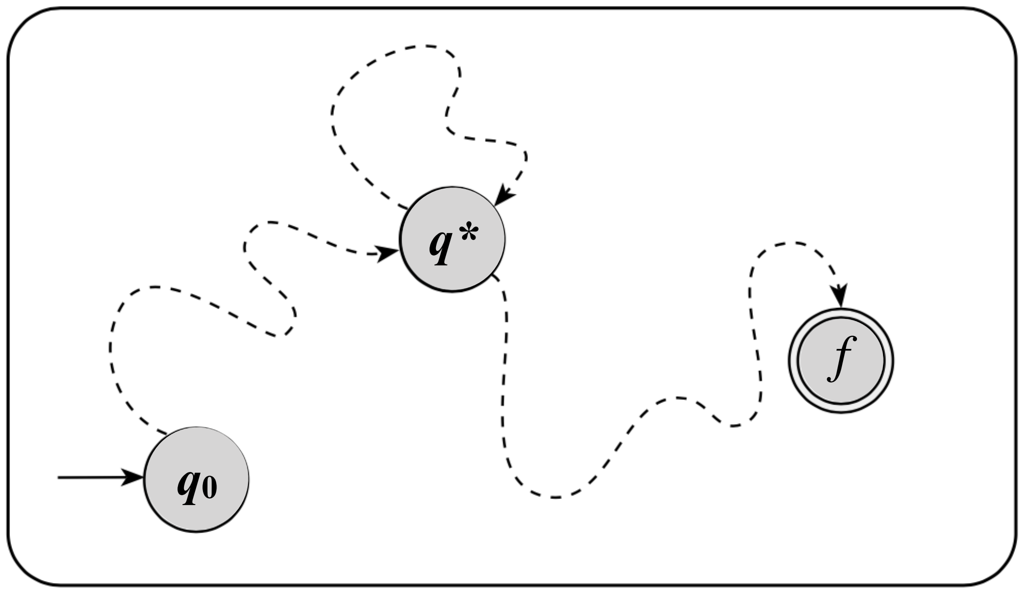
- Proof uses a famous concept: **Pigeon hole principle**
- Fancy name for fact that: **if p pigeons are placed into fewer than p holes, some hole has to have more than one pigeon in it**

Proof of Pumping Lemma

Given DFA $M = (Q, \Sigma, \delta, q_0, F)$ with $L(M) = L$ and $p = |Q|$

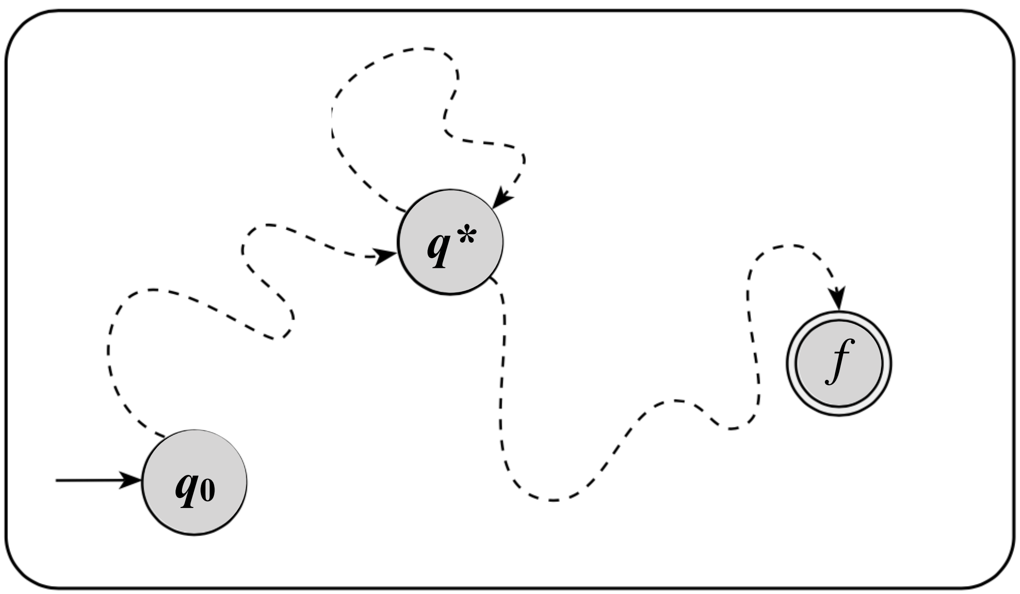
Let $s \in L$ and $|s| = n \geq p;$ consider sequence of states of M for input s

- Reading first symbol: starts from start state q_0
- Then: sequence of states until it reaches end of s in state f
    - Since $s \in L$: $f \in F$
- Since $|s| = n$: sequence of states of M computing s has length $n+1$
    - $q_n = f$
    - Since $n = |s| \geq p$: $n + 1 > p = |Q|$
- Therefore: sequence of states computing s contains (at least one) repeated state (**pigeonhole principle**)
- Assume: q^* is **first repeated** state in sequence of states accepting s


q*
q0
f

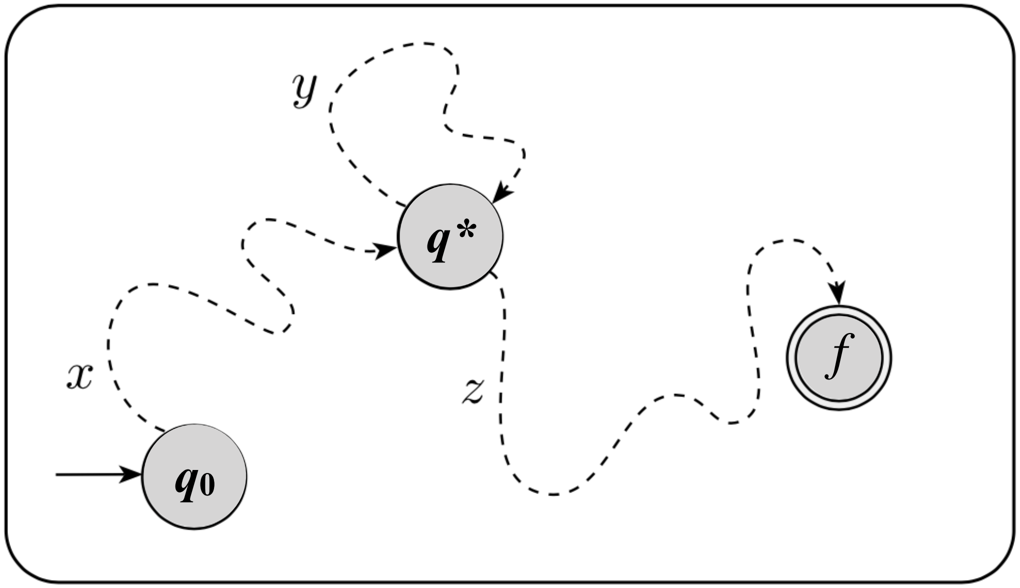
Proof of pumping lemma (2)

- We show we can rewrite s as $s = xyz$
  - x: substring of s appearing processed by M before reaching q^*
  - y: substring between first two appearances of q^*
  - z: suffix of s



Pr a (2)

- We show
  - x: subs ching q^*
  - y: subs
  - z: suffi



1. $xy^iz \in L$, for all $i \geq 0$
2. $y \neq \epsilon$
3. $|xy| \leq p$

- We prove: $s = xyz$ satisfies the three properties of PL

1. Computing M on xz implies $xz \in L$; computing $xyyz$ implies $xyyz \in L$; computing xy^iz for $i > 2$ implies $xy^iz \in L$. Thus $xy^iz \in L$ for $i \geq 0$

2. $|y| > 0$ since no state in M can be repeated without processing at least one symbol

3. $|xy| \leq p$: If $|xy| > p$ then computing first $p{+}1$ symbols results in a state repetition, but q^* is first repeated state when computing s

Concludes PL proof

Another example

- We show that $L = \{w \in \{0,1\}^* \mid w$ has an equal number of 0s and 1s$\}$ is not regular

- Assume that L is regular

- Let p be the pumping length given by pumping lemma

- $s = ?$

Counterexample: choice of s

It is crucial what string $s \in L$ of length at least p to choose from L to derive contradiction for all possible rewritings of s

- Not every long string in L might yield contradiction
- Just determining for one string in L of length at least p that does not satisfies the PL properties is sufficient to prove language not regular
- When choosing string as counterexample, think about what properties might make language nonregular

Another example (continued)

- We show that $L = \{w \in \{0,1\}^* \mid w$ has an equal number of 0s and 1s$\}$ is not regular
- Assume that L is regular
- Let p be the pumping length given by pumping lemma
- We choose $s = 1^p0^p$: $s \in L$ and $|s| \geq p$
- According to PL there is a rewriting $s = xyz$ satisfying all three properties
- We show in all cases, $xy^iz \in L$ for all $i \geq 0$ not satisfied (therefore s is counterexample)
- Note: $|xy| \leq p$ (Property 3) and $y \neq \epsilon$ (Property 2)
  - y must consist of 1s only (and consists of at least one 1)
  - Consider xz
    - $xz \notin L$ since it contains less 1s than 0s! Contradiction.

Your turn

1. Show that $L = \{0^i 1^j \mid i > j\}$ is nonregular

$s_1 = 0^p 1^p$

$s_2 = 0^p 1^{\frac{p}{2}}$

$s_3 = 0^{\frac{p}{2}} 1^p$

$s_4 = 0^{p+1} 1^p$

$s_5 = 0^p 1^{p+1}$

$s_6 = 0^p 1^{p-1}$

$s_7 = 0^{p-1} 1^p$

Your turn

1. Show that $L_1 = \{0^i 1^j \mid i > j\}$ is nonregular
2. Show that $L_2 = \{0^i 1^j \mid i < j\}$ is nonregular

$s_1 = 0^p 1^p$

$s_2 = 0^p 1^{\frac{p}{2}}$

$s_3 = 0^{\frac{p}{2}} 1^p$

$s_4 = 0^{p+1} 1^p$

$s_5 = 0^p 1^{p+1}$

$s_6 = 0^p 1^{p-1}$

$s_7 = 0^{p-1} 1^p$

Your turn

$s_1 = 0^p 1^p$

$s_2 = 0^p 1^{\frac{p}{2}}$

$s_3 = 0^{\frac{p}{2}} 1^p$

$s_4 = 0^{p+1} 1^p$

$s_5 = 0^p 1^{p+1}$

$s_6 = 0^p 1^{p-1}$

$s_7 = 0^{p-1} 1^p$

1. Show that $L_1 = \{0^i 1^j \mid i > j\}$ is nonregular
2. Show that $L_2 = \{0^i 1^j \mid i < j\}$ is nonregular
3. Show that $L_3 = \{0^i 1^j \mid i \leq j\}$ is nonregular

Your turn (2)

Explain why $L = \{0^i 1^j \mid i \leq j < 121\}$ is regular

Next

- Context-free languages
    - Grammars
    - Pushdown automata