CSC 320 - Tutorial 1

1. Set Theory
2. Terminology
3. Proof Strategies

Set Theory

- A **set** is group of unique objects
- Objects in a set are called **elements** or **members**

$x \in S$	x is an element of set S
$x \notin S$	x is not an element of S

- The **empty set** ($\varnothing$) is a set containing zero elements
- Sets can be **finite** or **infinite**
- Sets can be **countable** or **uncountable.**

$S_1 = \{1, 13, 49\}$	finite set	countable
$S_3 = \{x \mid x \text{ is a real number}\}$	infinite set	uncountable

- For sets A and B, we say A is a **subset** of B ($A \subseteq B$) if every member of A is also a member of B
- For sets A and B, we say A is a **proper subset** of B ($A \subsetneq B$) if every member of A is also a member of B and $A \neq B$
- The **Power Set** (P(A)) the set containing all possible subsets of A

  $S = \{a, b, c\} \rightarrow P(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
- **Union** ($A \cup B$)

  $S = \{x \mid x \in A \text{ or } x \in B\}$
- **Intersection** ($A \cap B$)

  $S = \{x \mid x \in A \text{ and } x \in B\}$
- **Complement** ($\bar{A}$)

  $S = \{x \mid x \notin A\}$
- **Set difference** (A - B)

  $S = \{x \mid x \in A \text{ and } x \notin B\}$
- **Cartesian/Cross product** (A x B)

  $S = \{ (x, y) \mid x \in A \text{ and } y \in B\}$

Terminology

- An **alphabet** (Σ) is a finite set of symbols

  Σ_1 = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

  Σ_2 = {a, b, c, …, x, y, z}

  Σ_3 = {0, 1}

- A **string** is a made up of zero or more symbols
- The **empty string** denoted as ε (epsilon) contains zero symbols
- The length of a string w is denoted as |w|

  |hello| = 5 |1010| = 4 |ε| = 0

- Strings can be **concatenated**

  Given w_1 = 1001 w_2 = 1100 then w = w_1w_2 = 10011100

- A **superscript** denotes the number of occurrences of the string that precedes

  0^4 = 0000 $\#^2$ = ## $(101)^2$ = 101101

- Σ^* is the set of all possible strings over the alphabet Σ

  Σ^* is **infinite** and **countable** (you can try the proof using Σ = {0, 1})

- The set of all languages is the set containing all subsets of Σ^* a.ka. $P(\Sigma^*)$

  $P(\Sigma^*)$ is **infinite** and **uncountable**

- A **language** is a set of strings over some alphabet

  The set of binary numbers is a language over {0, 1}

- Languages can be **concatenated**

  $L = L_1L_2 = \{ w_1w_2 \mid w_1 \in L_1 \text{ and } w_2 \in L_2 \}$

- Given language L, **L*** (Kleene star) is the set of all strings obtained by concatenating zero or more strings in L
- Given language L, **L^+** (closure) = LL*

Language Practice

1. What are some elements in Σ^* if $\Sigma = \{a, b\}$

    a. Is Σ^* finite or infinite?

    b. Is Σ^* countable or uncountable?

2. Given $L = \{0, 1, 00, 11\}$

    a. What is the Kleene star of L, $\mathbf{L^*}$?

    b. What is the Kleene plus of L, $\mathbf{L^+}$?

3. Describe the following sets:
   a. $S_1 = \{x \mid x = 2m \text{ where } m > 5\}$

   b. $S_2 = \{0^i1^j \mid i = j \text{ and } i, j \in N\}$

Proof Strategies

<u>Proof by Contradiction</u>

1. Assume the the opposite of what you want to prove
2. Proceed with proof with your assumption from step 1
3. Find a contradiction

<u>Proof by Construction</u>

1. Create / construct the object to prove it exists

Proof by induction

1. Prove that some property holds for a base case
2. Assume the property holds for some case x
3. Show that if the property holds for case x then it must hold for case x+1
4. Show that the property holds for all cases $i \geq x$

Contrapositive proof

Given $p \rightarrow q$, instead prove $\sim q \rightarrow \sim p$ *this proof might be easier

Proof Examples

1. Prove that $\sqrt{2}$ is an irrational number by contradiction.

2. Prove that there exists a program that can be used to calculate the sum of two numbers (proof by construction).

3. Assume that numbers are coloured either red or blue. Assume that 1 is colored blue. Also assume that if x is blue then x+1 is blue.
    a. What can we deduce about the set of natural numbers? What color are they?

    b. What proof strategy did we employ to reach this conclusion?

4. [Bonus] Prove that for all natural numbers n, $1+ 2 + 3+ .. + n = (n (n+1)) / 2$ (proof by induction)