CSC 320 - Tutorial 1

1. Set Theory
2. Terminology
3. Proof Strategies

Set Theory

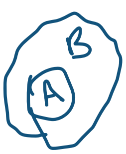
- A **set** is group of unique objects
- Objects in a set are called **elements** or **members**

  $x \in S$ x is an element of set S

  $x \notin S$ x is not an element of S

- The **empty set** ($\varnothing$) is a set containing zero elements
- Sets can be **finite** or **infinite**
- Sets can be **countable** or **uncountable.**

$S_1 = \{1, 13, 49\}$	finite set	countable
$S_3 = \{x \mid x \text{ is a real number}\}$	infinite set	uncountable



- For sets A and B, we say A is a **subset** of B ($A \subseteq B$) if every member of A is also a member of B
- For sets A and B, we say A is a **proper subset** of B ($A \subsetneq B$) if every member of A is also a member of B and $A \neq B$
- The **Power Set** (P(A)) the set containing all possible subsets of A

  $S = \{a, b, c\} \rightarrow P(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

- **Union** ($A \cup B$)

  $S = \{x \mid x \in A \text{ or } x \in B\}$

- **Intersection** ($A \cap B$)

  $S = \{x \mid x \in A \text{ and } x \in B\}$

- **Complement** ($\bar{A}$)

  $S = \{x \mid x \notin A\}$

- **Set difference** (A - B)

  $S = \{x \mid x \in A \text{ and } x \notin B\}$

- **Cartesian/Cross product** (A x B)

  $S = \{ (\underline{x}, \underline{y}) \mid x \in \underline{A} \text{ and } y \in \underline{B}\}$

  tuple

Terminology

- An **alphabet** (Σ) is a finite set of symbols

  $\Sigma_1 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ 275

  $\Sigma_2 = \{a, b, c, \ldots, x, y, z\}$

  $\Sigma_3 = \{0, 1\}$

- A **string** is a made up of zero or more symbols
- The **empty string** denoted as ε (epsilon) contains zero symbols ε
- The length of a string w is denoted as $|w|$

  $|\text{hello}| = 5$ $|1010| = 4$ $|\varepsilon| = 0$

- Strings can be **concatenated**

  Given $w_1 = 1001$ $w_2 = 1100$ then $w = w_1w_2 = 10011100$

- A **superscript** denotes the number of occurrences of the string that precedes

  $0^4 = 0000$ $\#^2 = \#\#$ $(101)^2 = 101101$

- Σ^* is the set of all possible strings over the alphabet Σ

  Σ^* is **infinite** and **countable** (you can try the proof using $\Sigma = \{0, 1\}$)

- The set of all languages is the set containing all subsets of Σ^* a.ka. $P(\Sigma^*)$

  $P(\Sigma^*)$ is **infinite** and **uncountable**

  $\{0, 1\}$ Σ symbols

  $\{\varepsilon, 0, 1, 111, 00, 10101101, \ldots\}$ Σ^* strings

- A **language** is a set of strings over some alphabet

  The set of binary numbers is a language over {0, 1}

- Languages can be **concatenated**

  $L = L_1L_2 = \{ w_1w_2 \mid w_1 \in L_1 \text{ and } w_2 \in L_2 \}$

- Given language L, **L*** (Kleene star) is the set of all strings obtained by concatenating zero or more strings in L
- Given language L, $\mathbf{L}^+$ (closure) = LL^*

Language Practice

1. What are some elements in Σ* if Σ = {a, b}

$$\Sigma^* = \{a, b, \varepsilon, aa, ba, b\ldots b, abab, \ldots\}$$

   a. Is Σ* finite or infinite?

   b. Is Σ* countable or uncountable?

2. Given L = {0, 1, 00, 11}

   a. What is the Kleene star of L, **L*** ?

$w_1 = 101$ $w_2 = \varepsilon$

zero or more

$w_1 w_2 = w_1$

$$L^* = \{01, 000, \varepsilon, 011100, 1111111, \ldots\}$$

   b. What is the Kleene plus of L, **L⁺** ?

$L^+ = LL^*$

$L = \emptyset$

$L = \{\varepsilon\}$

$$L^+ = \{001, 0000, 0, 0011100, \ldots, 101, 1000, 1, 1011100, \ldots\}$$

$\underline{\varepsilon \in L^+}$ iff $\underline{\varepsilon \in L}$

$L = \{\varepsilon, 1, 0\}$

$L^* = \{\varepsilon, 1, 0, 101, 100, \ldots$

$L^+ = \{\varepsilon, 1, 0, \ldots$

$L^+ = L^*$

3. Describe the following sets:
   a. $S_1 = \{x \mid x = 2m \text{ where } m > 5\}$

All even numbers greater than 10

   b. $S_2 = \{0^i 1^j \mid i = j \text{ and } i, j \in N\}$

$S_2 = \{01, 0011, 000111 \ldots\}$

Equal occurrences of 0 and 1 where all 0s appear before ANY 1s.

Proof Strategies

<u>Proof by Contradiction</u>

1. Assume the the opposite of what you want to prove
2. Proceed with proof with your assumption from step 1
3. Find a contradiction

<u>Proof by Construction</u>

1. Create / construct the object to prove it exists

Proof by induction

1. Prove that some property holds for a base case
2. Assume the property holds for some case x
3. Show that if the property holds for case x then it must hold for case x+1
4. Show that the property holds for all cases i ≥ x

Contrapositive proof

Given $p \rightarrow q$, instead prove $\sim q \rightarrow \sim p$ *this proof might be easier

Proof Examples

1. Prove that √2 is an irrational number by contradiction.

Assume $\sqrt{2}$ is rational.

the exists $a, b \in \mathbb{Z}$ s.t. $\frac{a}{b} = \sqrt{2}$

let $\frac{a}{b} = \frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $\frac{m}{n}$ is fully reduced

$\frac{10}{5} = \frac{2}{1}$ either m or n or both are odd

$\sqrt{2} = \frac{m}{n} \Rightarrow m = \sqrt{2}n \Rightarrow m^2 = 2n^2 \Leftarrow$

$\therefore$ m^2 is even

if m^2 is even then m is even $\therefore$ m is even

if m is even $m = 2k$ for som $k \in \mathbb{Z}$

$(2k)^2 = 2n^2 \Rightarrow 4k^2 = 2n^2 \Rightarrow 2k^2 = n^2$

$\therefore$ n^2 is even $\therefore$ n is even

We have a contradiction $\therefore$ our original assumption was incorrect

$\therefore$ $\sqrt{2}$ is irrational.

2. Prove that there exists a program that can be used to calculate the sum of two numbers (proof by construction).

```
int main(){
    int num1, num2;
    std::cout << "Enter 2 numbers/n" ;
    std::cin >> num1 >> num2;

    std::cout << "sum" << num1 + num2;
    return 0;
}
```

3. Assume that numbers are coloured either red or blue. Assume that 1 is colored blue. Also assume that if x is blue then x+1 is blue.
    a. What can we deduce about the set of natural numbers? What color are they? 1 . . . ∞

All blue

    b. What proof strategy did we employ to reach this conclusion?

Induction

4. [Bonus] Prove that for all natural numbers n, 1+ 2 + 3+ .. + n = (n (n+1)) / 2 (proof by induction)

$1+2+\ldots+n \qquad \frac{n(n+1)}{2}$

Base Case:

$n=1 \qquad 1 \qquad \frac{1(2)}{2} = 1$ ✓

$n=2 \qquad 1+2=3 \qquad \frac{2(3)}{2} = 3$ ✓

Induction Hypothesis

Assume for some int ℓ (2) $1 \leq \ell < n$

$$\underbrace{1+2+3+\ldots+\ell} = \frac{\ell(\ell+1)}{2}$$

Induction Step

what happens for $(\ell+1)$

$$\underbrace{1+2+3+\ldots+\ell} + (\ell+1) = \frac{\ell(\ell+1)}{2} + (\ell+1)\frac{2}{2}$$

$$= \frac{\ell(\ell+1) + 2(\ell+1)}{2}$$

$$= \frac{(\ell+1)(\ell+2)}{2}$$

this is what we were expecting!

Conclusion:

$1+2+\ldots+n = \frac{n(n+1)}{2}$ holds for $n \in \mathbb{N}$

by Induction.