CSC 320
Foundations of Computer Science

Lecture 5

Instructor: Dr. Ulrike Stege

Territory Acknowledgement

We acknowledge and respect the lək̓ʷəŋən peoples on whose traditional territory the university stands and the Songhees, Esquimalt and W̱SÁNEĆ peoples whose historical relationships with the land continue to this day.

This meeting will be recorded

"Please be aware our sessions are being screen-recorded to allow students who are not able to attend to watch later and will be posted in Brightspace."

Deadlines; Assessment

10%	25%	25%	40%
Quizzes	Assignments	Midterms	Final Exam
Quiz 1-8: 1% each Quiz 9: 2%	Assignment 1-5: 5% each	Midterm 1: 10% Midterm 2: 15%	

May

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			3	4	5	[illegible]
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	1	2	3
4	5	6	7	8	9	10

June

S	M	T	W	T	F	S
28	29	30	31	1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	1
2	3	4	5	6	7	8

July

S	M	T	W	T	F	S
25	26	27	28	29	30	1
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9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	

Timed quizzes (~30 min)
Review before starting quiz

Quizzes

- To help you review lecture materials
- Feedback on understanding; use of terminology
- Quiz 3: Will include a bit more on regular expressions in addition to new material

Assignments

- Checking your understanding
- Problem solving
- Ability to write up solutions/proofs clearly

Highly recommended: practice!

Last time ....

- The languages of regular expressions are exactly the regular languages

Equivalence of Regular Languages and Languages of Regular Expressions

- Admin
- Lectures
- Tutorials
- Additional Materials
  - Article on P vs. NP problem (MIT Technology
  - Example: DFA to RegEx

Today …

- A note on reductions
- DFA state minimization
- Non-regular languages

Reductions

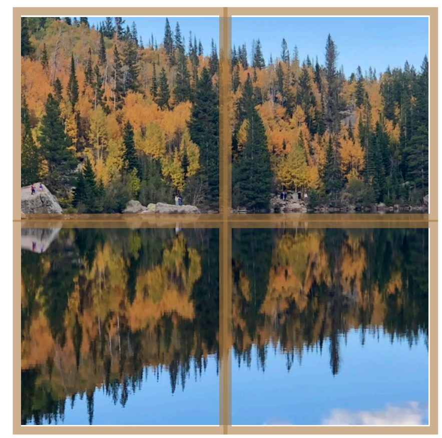
- Important concept that we will use later in the course
- What are reductions?
    - A reduction is an algorithm that transforms one problem/language (Problem A) into another (Problem B)
    - Reductions can be used to show, eg, Problem B is *at least as difficult* as Problem A

Solving x' of B will also solve x of A

Metaphor for Reduction

- Problem: Increasing brightness in a room with dirty windows

Reduce problem to window cleaning



Clean window solves problem

If the window can be cleaned, then the room brightness is increased

Back to Regular Languages ….

DFA State Minimization

- Given a DFA
- Goal: reduce number of states without changing language recognized
  1. Remove unreachable states
  2. Identify/collapse states that yield same result
     - Maintain determinism

Your turn

- You are to design, for some language L over Σ with $L \neq \Sigma^*$ and $L \neq \emptyset$, a DFA that has as few states as possible

- What's the smallest number of states such a DFA can have at best? E.g. $|Q| \geq ?; |F| \geq ?$

What kind of states can/ cannot be collapsed?

- Don't collapse accept and non-accept states 

  - If there exists a string w and states p, q such that
    - when w is processed starting at state p M yields acceptance and
    - when w is processed starting at state q M yields non-acceptance
    - Then do not collapse p and q

State equivalence

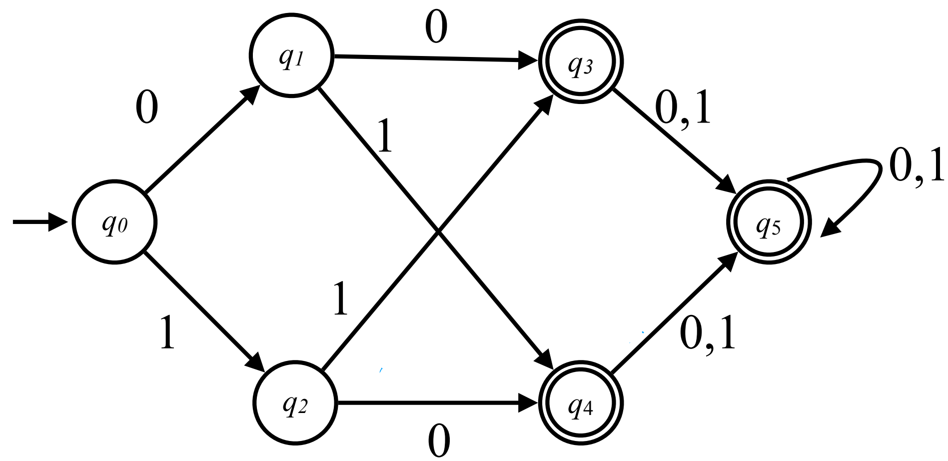
- Two states p, q of DFA M are **equivalent** ($\boldsymbol{p \sim q}$) if and only if *for all* $w \in \Sigma^*$
  - computation of M for w starting at state p ends in accept state if and only if computation of M for w starting at state q ends in accept state

State minimization algorithm

Identifies all pairs of states that are equivalent

- Let M be a DFA with no inaccessible state
- Write down all pairs $\{p,q\}$ of states in M (initially unmarked)
- For each unordered pair $\{p,q\}$: mark $\{p,q\}$ if $p \in F$ and $q \notin F$ (or vice versa)
- Repeat until no changes occur:
  - If there exists an unmarked pair $\{p,q\}$ such that:
    - $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$
    - then mark $\{p,q\}$

Note: The algorithm produces state equivalence:
$\{p,q\}$ unmarked if and only if $p \sim q$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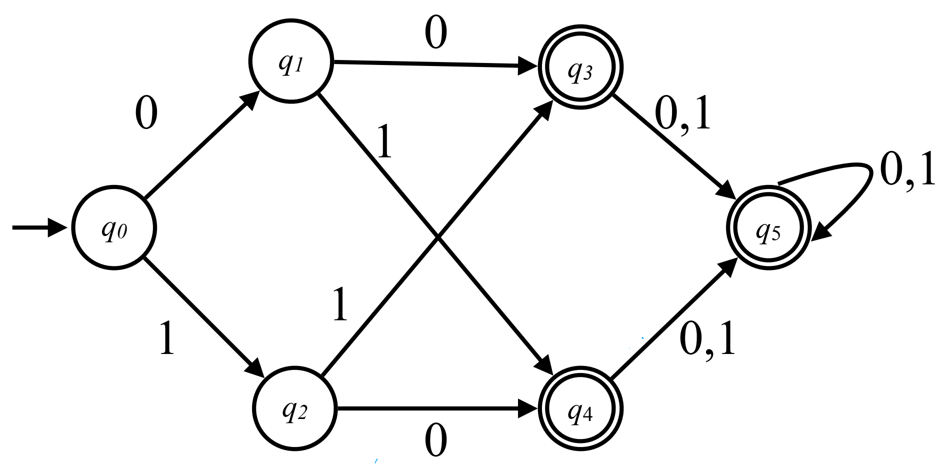

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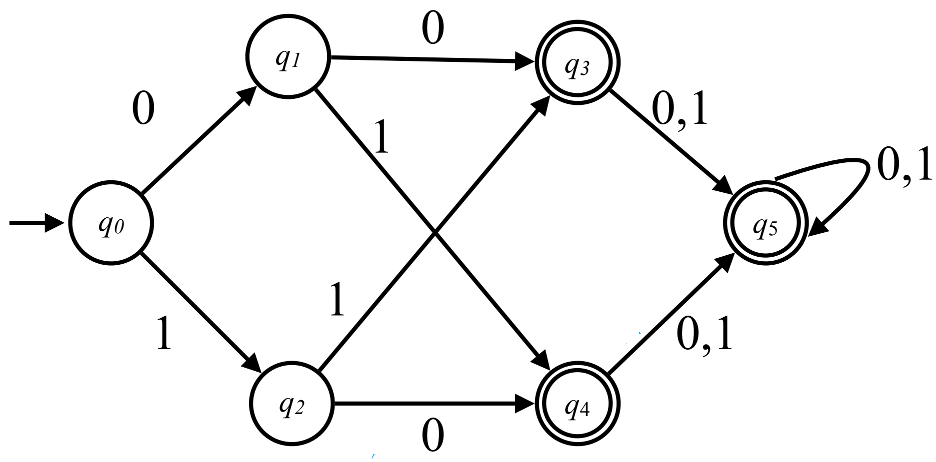

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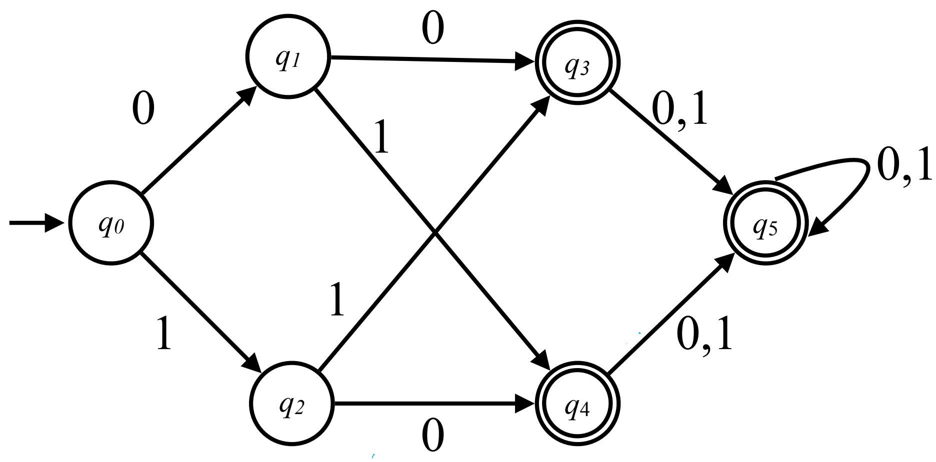

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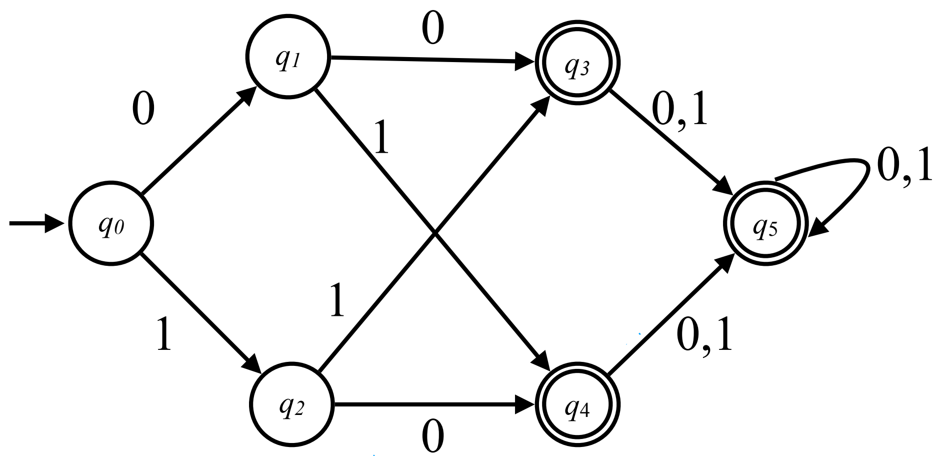

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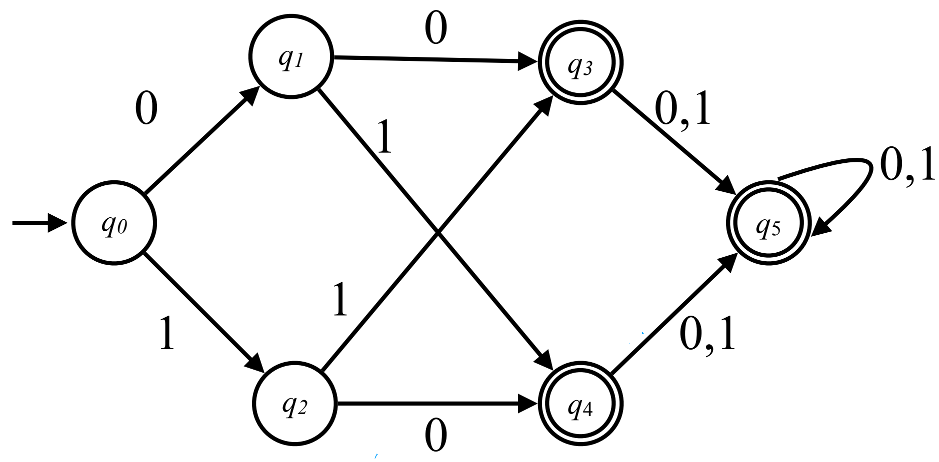

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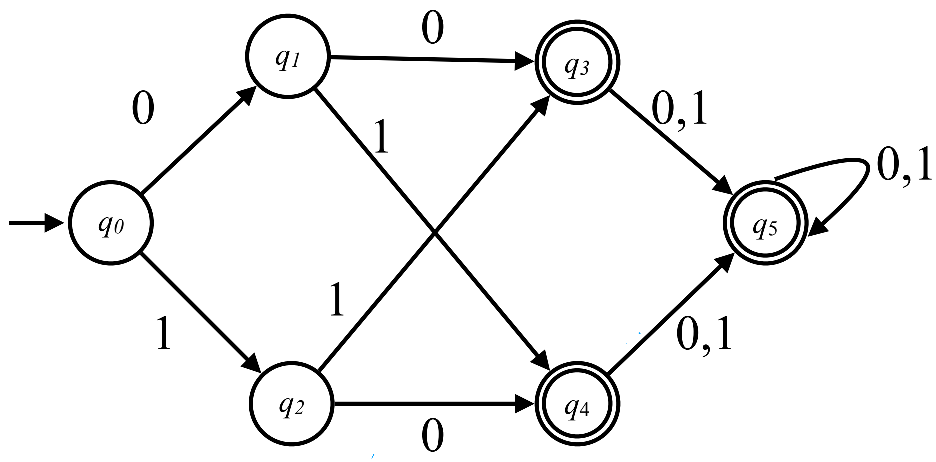

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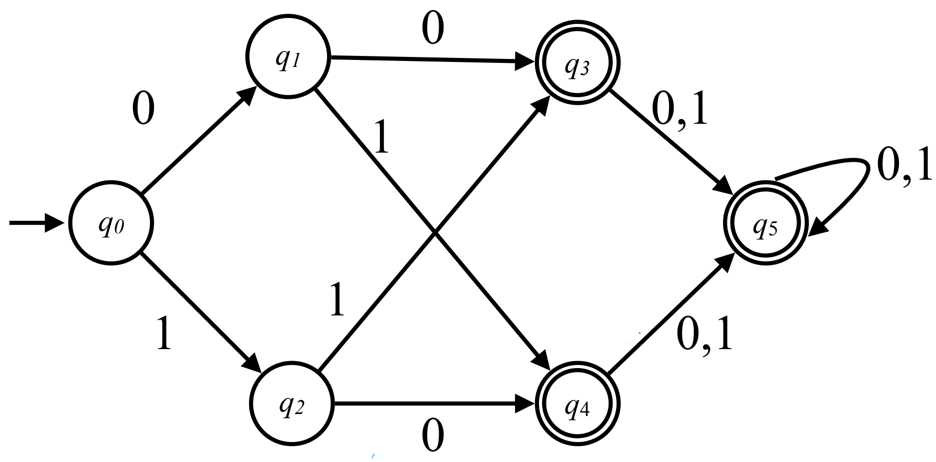

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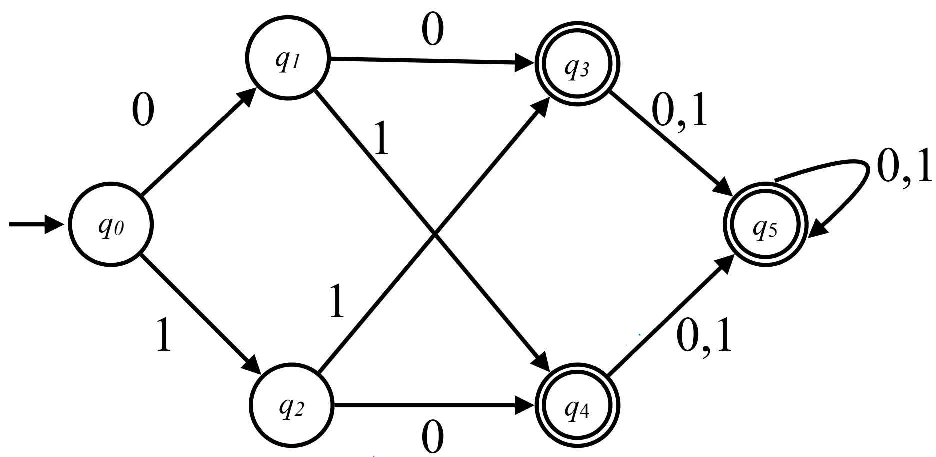

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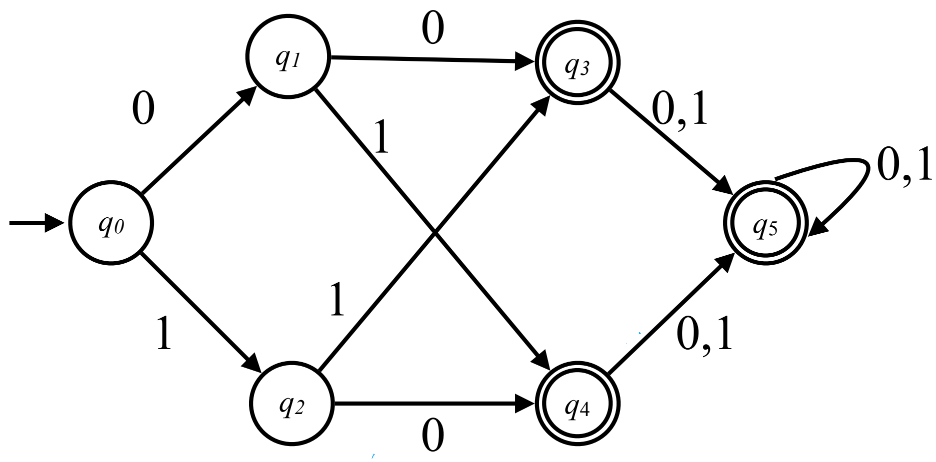

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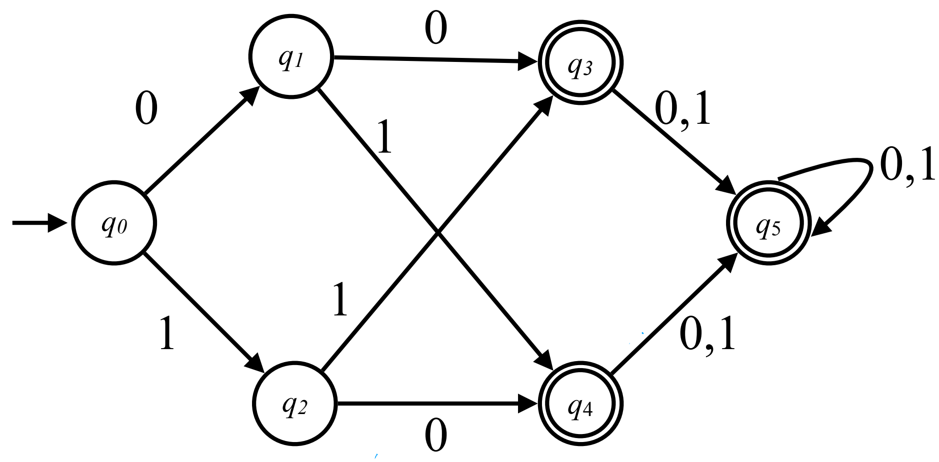

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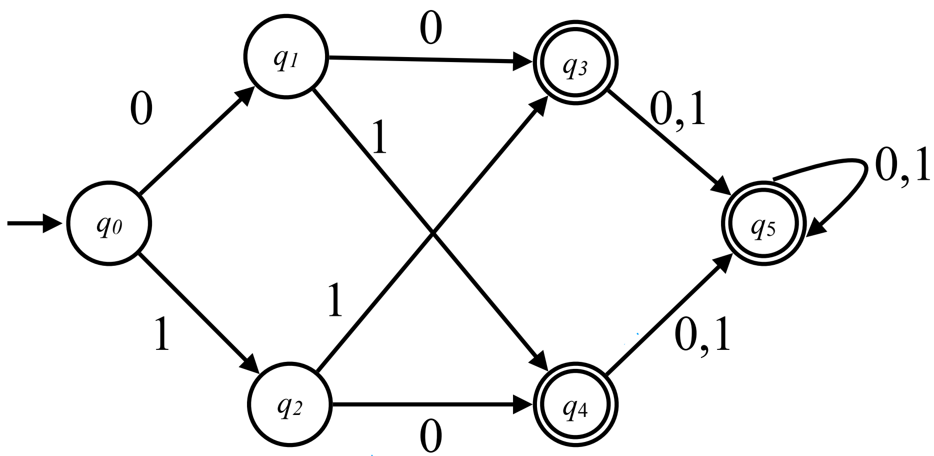

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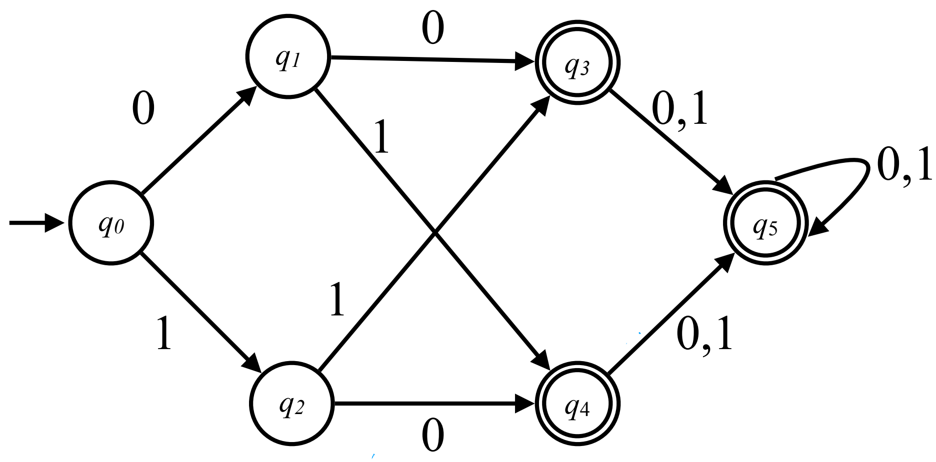

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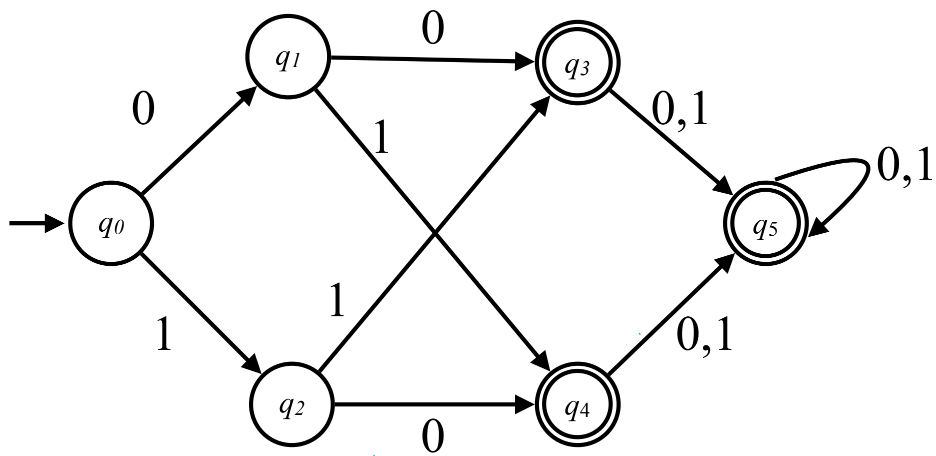

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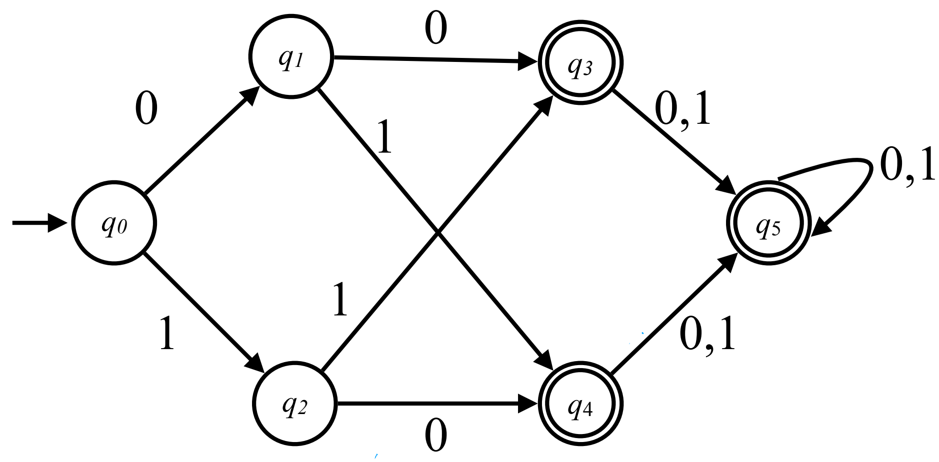

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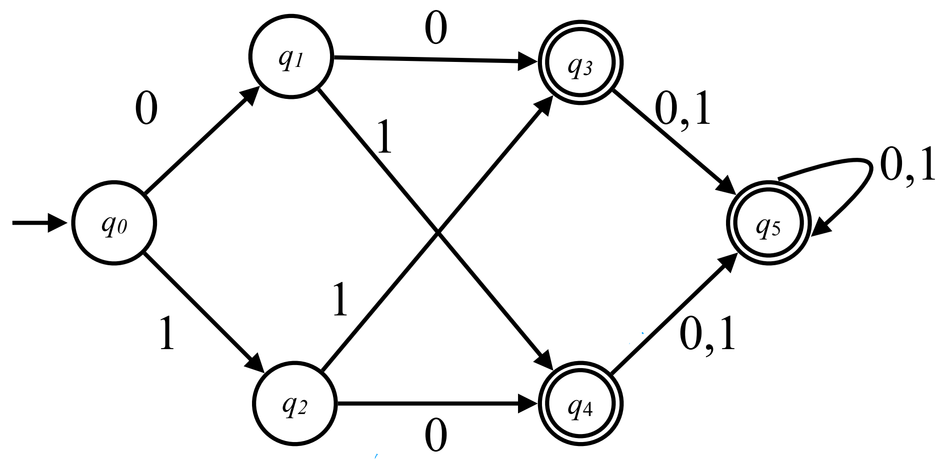

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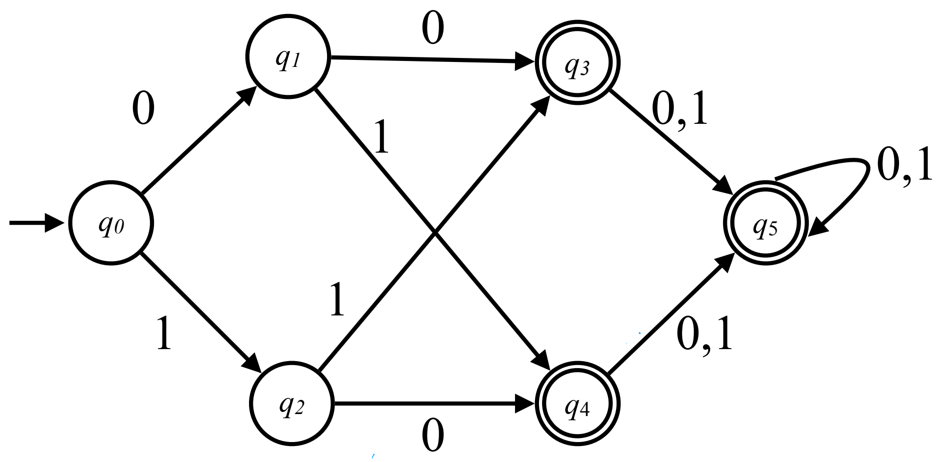

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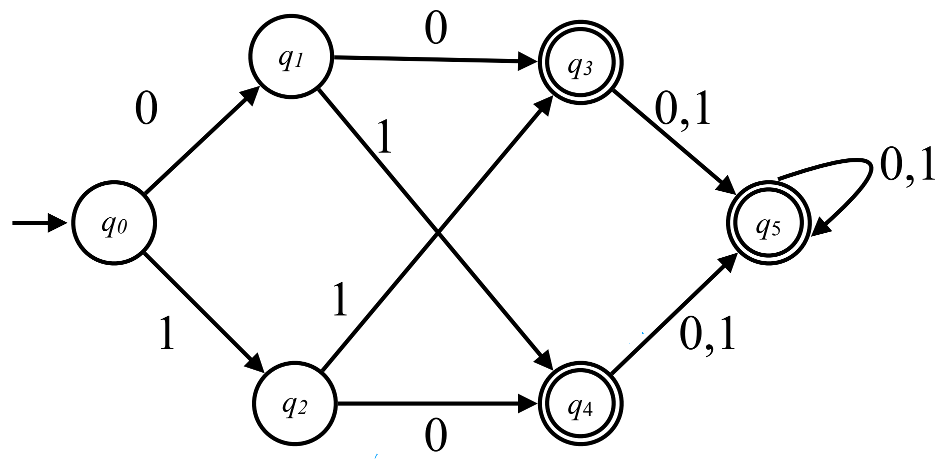

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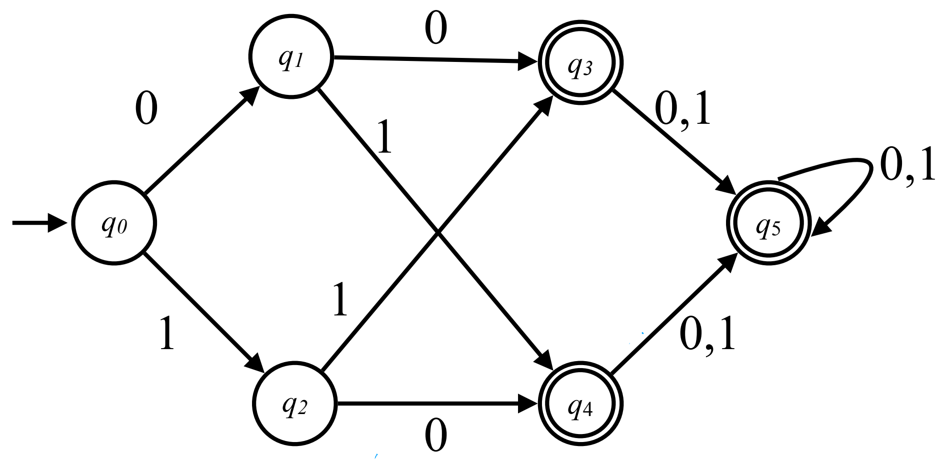

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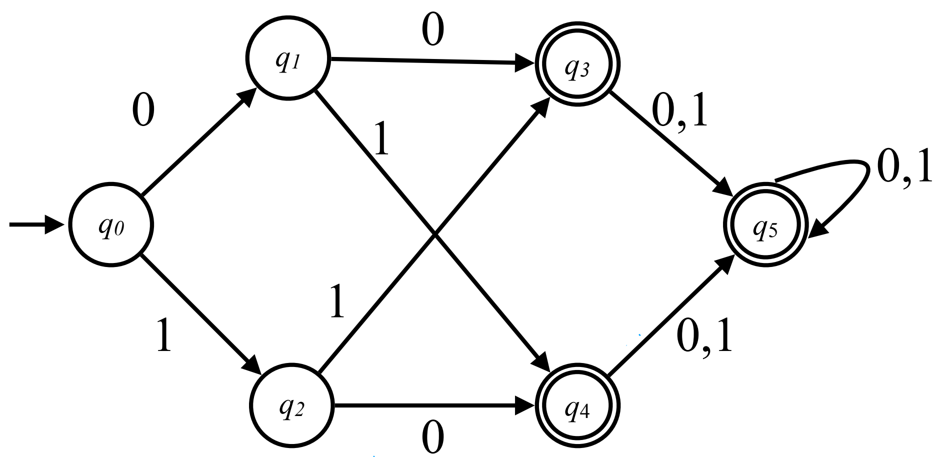

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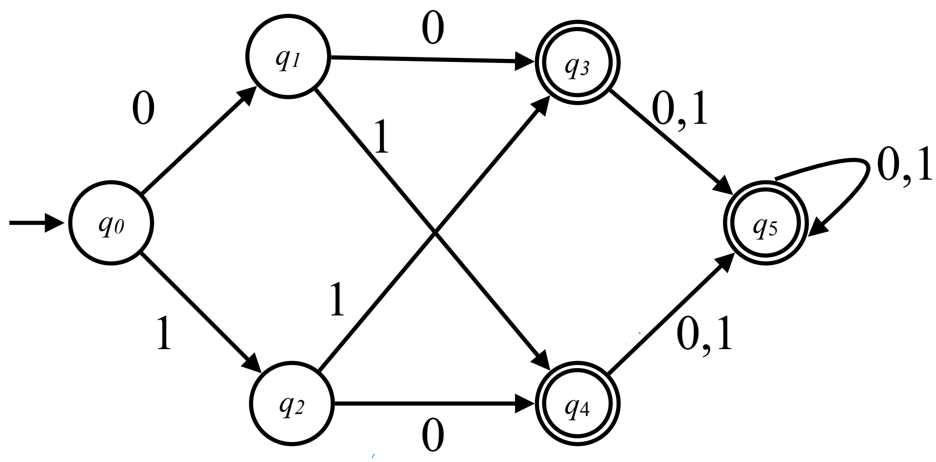

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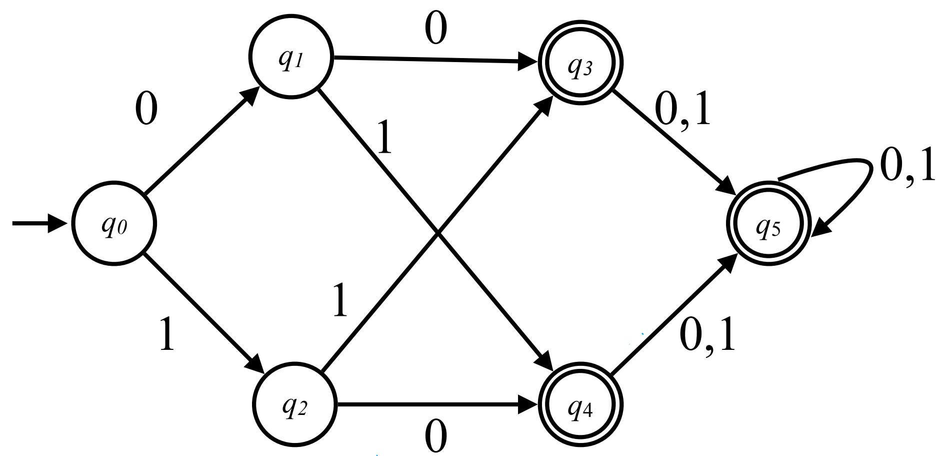

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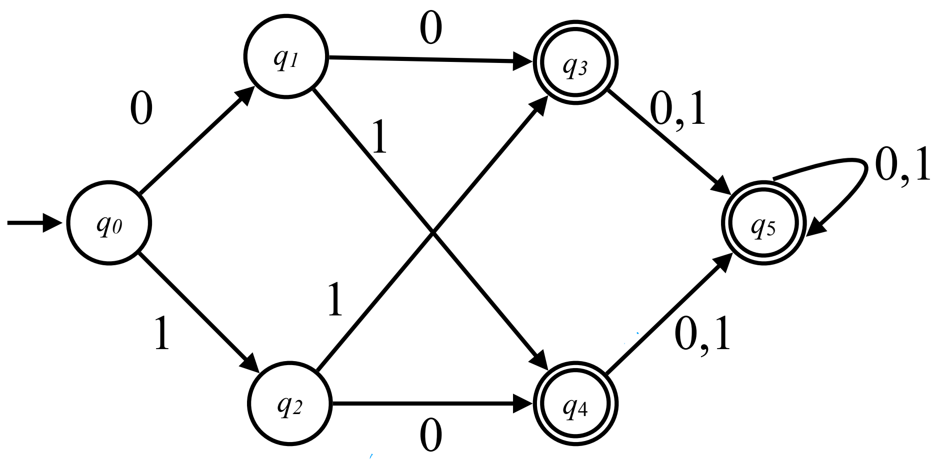
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boxed{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

For each pair $\{p,q\}$: mark $\{p,q\}$ if $p \in F$ and $q \notin F$



$\{q_0, q_1\}$ $\{q_1, q_2\}$ $\boldsymbol{\{q_2, q_3\}}$ $\{q_3, q_4\}$ $\{q_4, q_5\}$

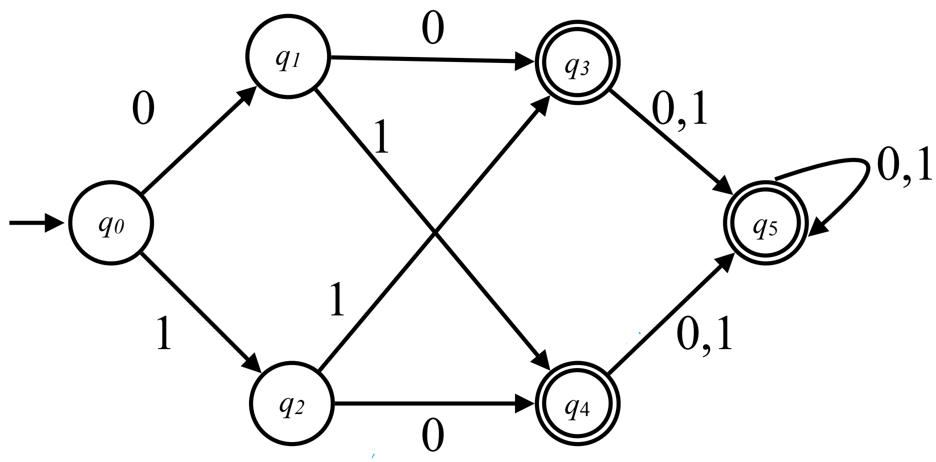
$\{q_0, q_2\}$ $\boldsymbol{\{q_1, q_3\}}$ $\boldsymbol{\{q_2, q_4\}}$ $\{q_3, q_5\}$

$\boldsymbol{\{q_0, q_3\}}$ $\boldsymbol{\{q_1, q_4\}}$ $\boldsymbol{\{q_2, q_5\}}$

$\boldsymbol{\{q_0, q_4\}}$ $\boldsymbol{\{q_1, q_5\}}$

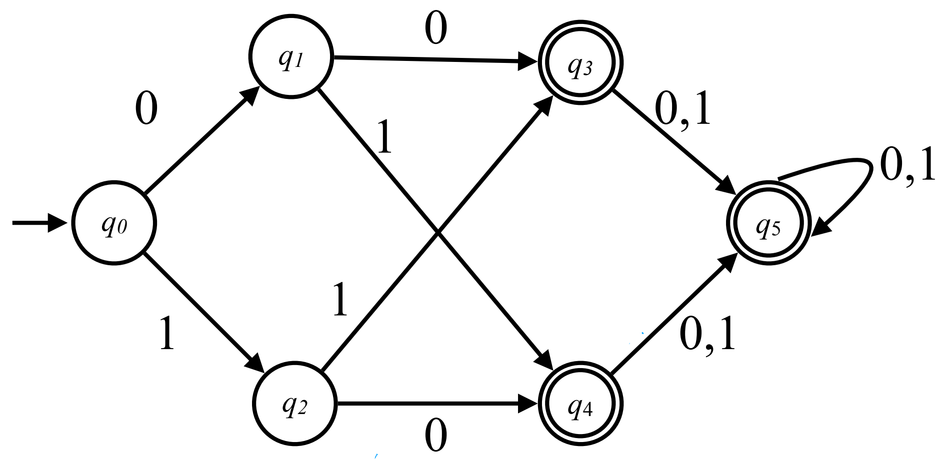
$\boldsymbol{\{q_0, q_5\}}$

For each pair $\{p,q\}$: mark $\{p,q\}$ if $p \in F$ and $q \notin F$



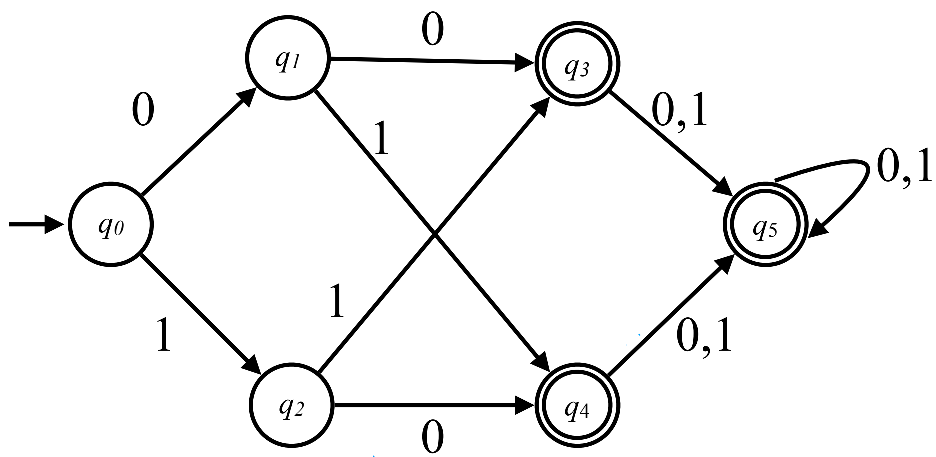
$\{q_0, q_1\}$	$\{q_1, q_2\}$	$\mathbf{\{q_2, q_3\}}$	$\boxed{\{q_3, q_4\}}$	$\{q_4, q_5\}$
$\{q_0, q_2\}$	$\mathbf{\{q_1, q_3\}}$	$\mathbf{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\mathbf{\{q_0, q_3\}}$	$\mathbf{\{q_1, q_4\}}$	$\mathbf{\{q_2, q_5\}}$		
$\mathbf{\{q_0, q_4\}}$	$\mathbf{\{q_1, q_5\}}$			
$\mathbf{\{q_0, q_5\}}$				

For each pair $\{p,q\}$: mark $\{p,q\}$ if $p \in F$ and $q \notin F$



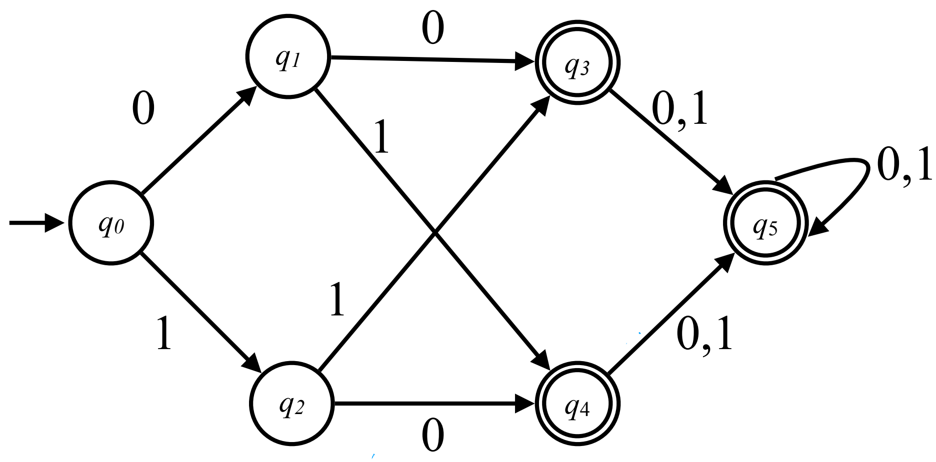
$\{q_0, q_1\}$	$\{q_1, q_2\}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\{q_0, q_2\}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\boxed{\{q_3, q_5\}}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

For each pair $\{p,q\}$: mark $\{p,q\}$ if $p \in F$ and $q \notin F$



$\{q_0, q_1\}$	$\{q_1, q_2\}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\boxed{\{q_4, q_5\}}$
$\{q_0, q_2\}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

For each pair $\{p,q\}$: mark $\{p,q\}$ if $p \in F$ and $q \notin F$



$\{q_0, q_1\}$ $\{q_1, q_2\}$ $\boldsymbol{\{q_2, q_3\}}$ $\{q_3, q_4\}$ $\{q_4, q_5\}$

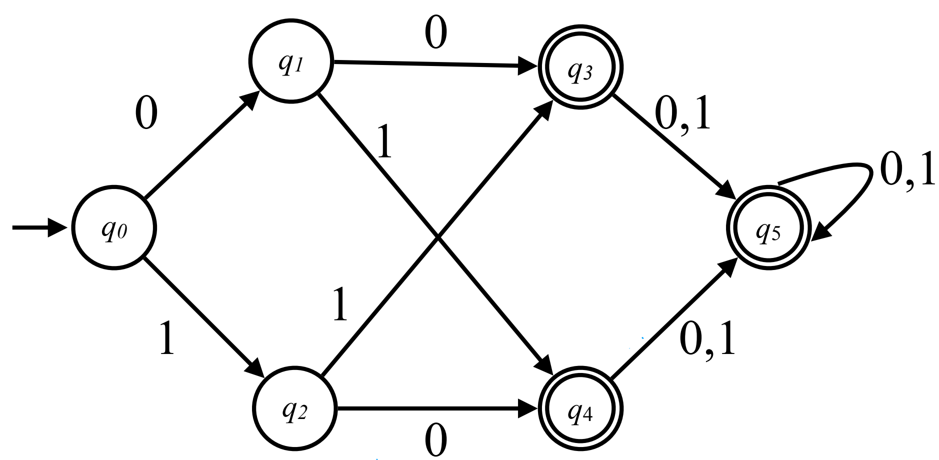
$\{q_0, q_2\}$ $\boldsymbol{\{q_1, q_3\}}$ $\boldsymbol{\{q_2, q_4\}}$ $\{q_3, q_5\}$

$\boldsymbol{\{q_0, q_3\}}$ $\boldsymbol{\{q_1, q_4\}}$ $\boldsymbol{\{q_2, q_5\}}$

$\boldsymbol{\{q_0, q_4\}}$ $\boldsymbol{\{q_1, q_5\}}$

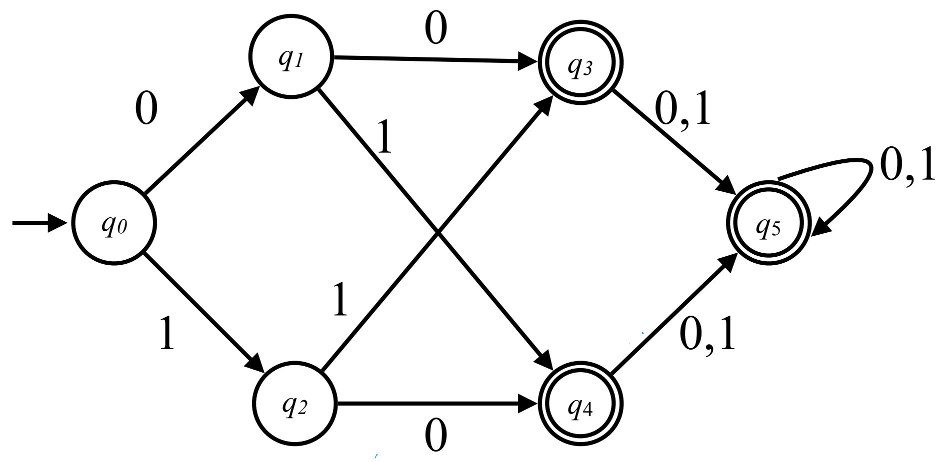
$\boldsymbol{\{q_0, q_5\}}$

Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{q_0, q_1\}$	$\{q_1, q_2\}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\{q_0, q_2\}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

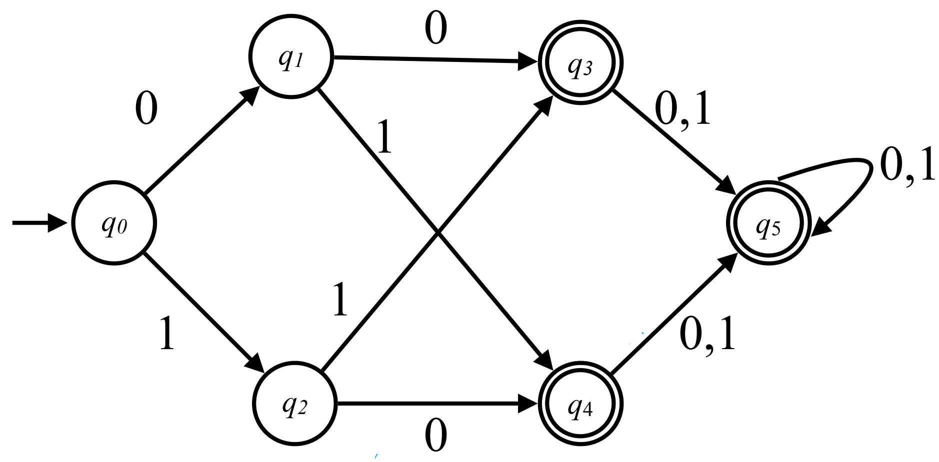
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{\delta(q_0,0), \delta(q_1,0)\}$

$\boxed{\{q_0, q_1\}}$	$\{q_1, q_2\}$	**$\{q_2, q_3\}$**	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\{q_0, q_2\}$	**$\{q_1, q_3\}$**	**$\{q_2, q_4\}$**	$\{q_3, q_5\}$	
$\{q_0, q_3\}$	**$\{q_1, q_4\}$**	**$\{q_2, q_5\}$**		
$\{q_0, q_4\}$	**$\{q_1, q_5\}$**			
$\{q_0, q_5\}$				

Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{q_1, q_3\}$

$\boxed{\{q_0, q_1\}}$ $\{q_1, q_2\}$ $\boldsymbol{\{q_2, q_3\}}$ $\{q_3, q_4\}$ $\{q_4, q_5\}$

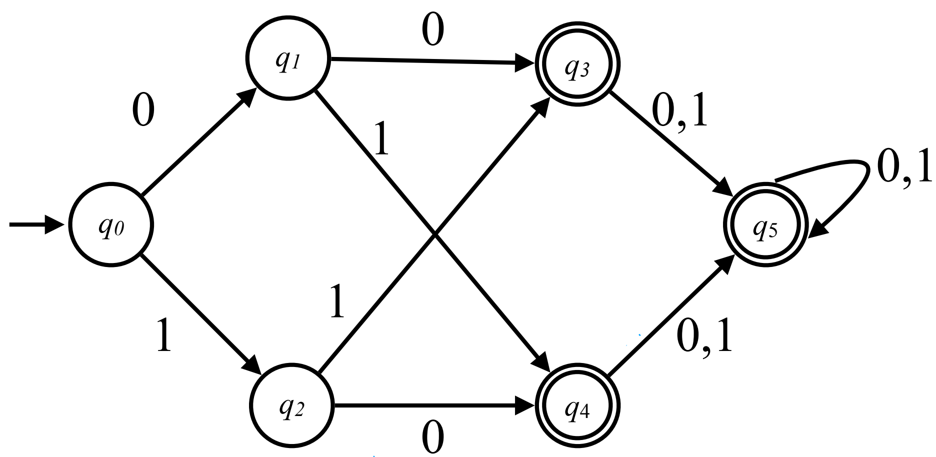
$\{q_0, q_2\}$ $\boldsymbol{\{q_1, q_3\}}$ $\boldsymbol{\{q_2, q_4\}}$ $\{q_3, q_5\}$

$\boldsymbol{\{q_0, q_3\}}$ $\boldsymbol{\{q_1, q_4\}}$ $\boldsymbol{\{q_2, q_5\}}$

$\boldsymbol{\{q_0, q_4\}}$ $\boldsymbol{\{q_1, q_5\}}$

$\boldsymbol{\{q_0, q_5\}}$

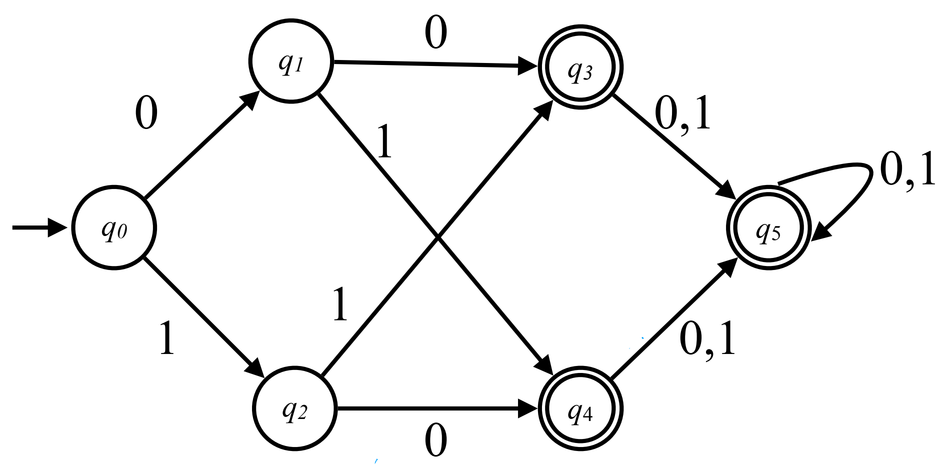
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{q_1, q_3\}$

$\boldsymbol{\{q_0, q_1\}}$	$\{q_1, q_2\}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\{q_0, q_2\}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

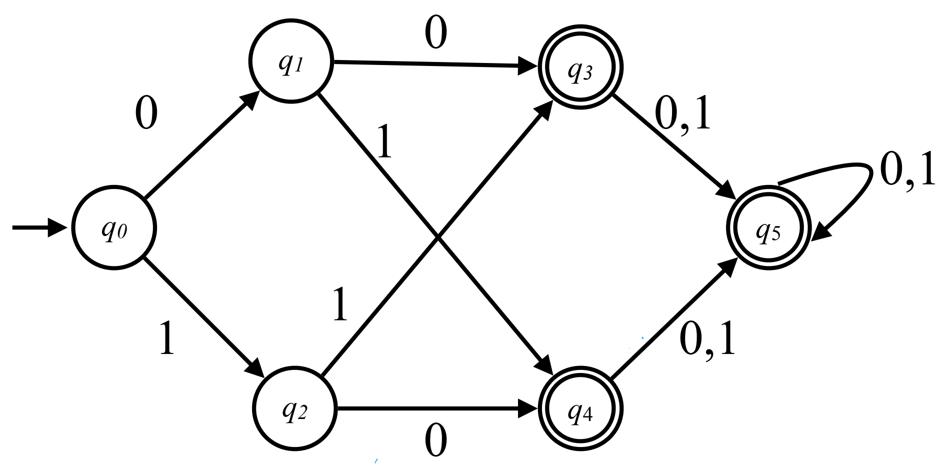
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a),\ \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{\delta(q_0,0),\ \delta(q_2,0)\}$

$\boldsymbol{\{q_0, q_1\}}$	$\{q_1, q_2\}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\boxed{\{q_0, q_2\}}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

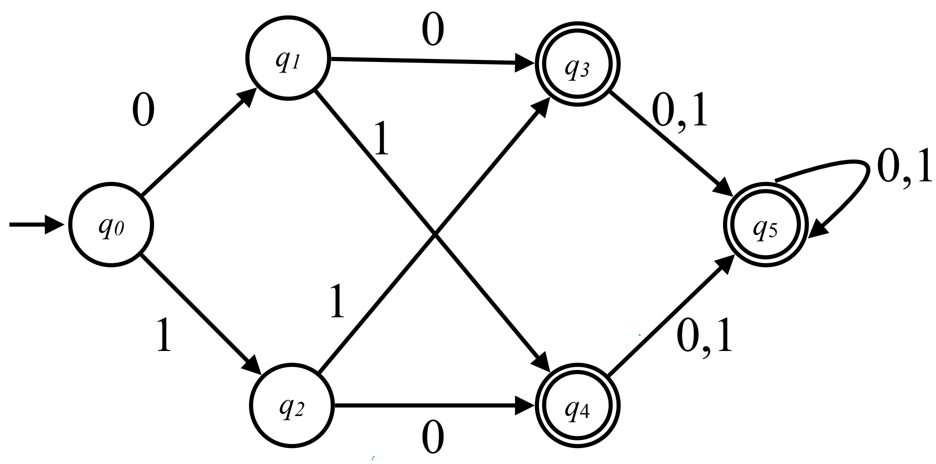
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{q_1, q_4\}$

$\boldsymbol{\{q_0, q_1\}}$	$\{q_1, q_2\}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\boxed{\{q_0, q_2\}}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

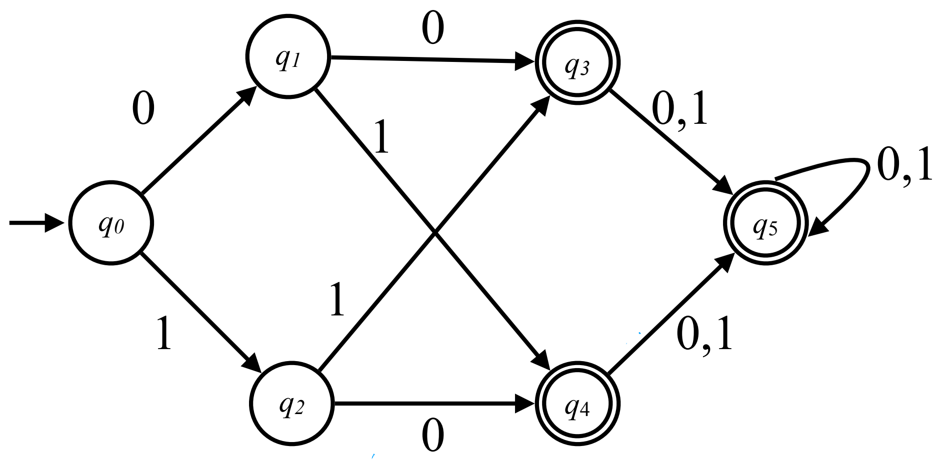
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{q_1, q_4\}$

$\boldsymbol{\{q_0, q_1\}}$	$\{q_1, q_2\}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\boldsymbol{\{q_0, q_2\}}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

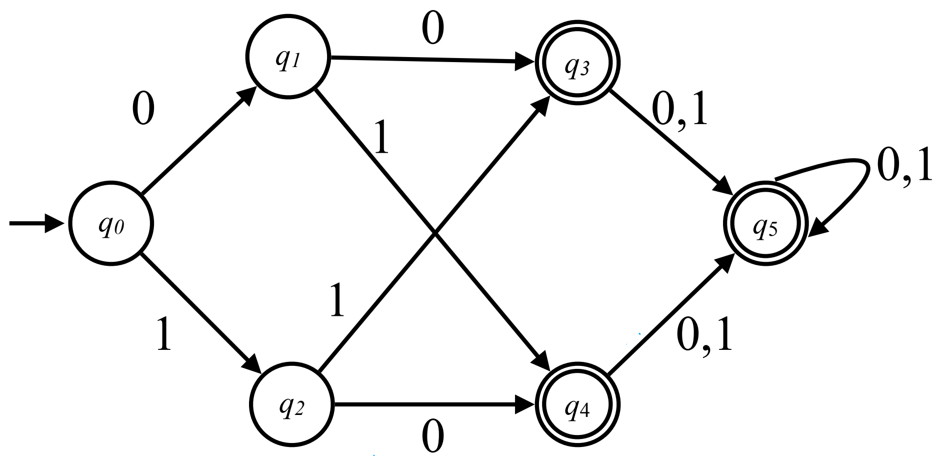
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a),\ \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{\delta(q_1,0),\ \delta(q_2,0)\}$

$\boldsymbol{\{q_0, q_1\}}$	$\boxed{\{q_1, q_2\}}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\boldsymbol{\{q_0, q_2\}}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

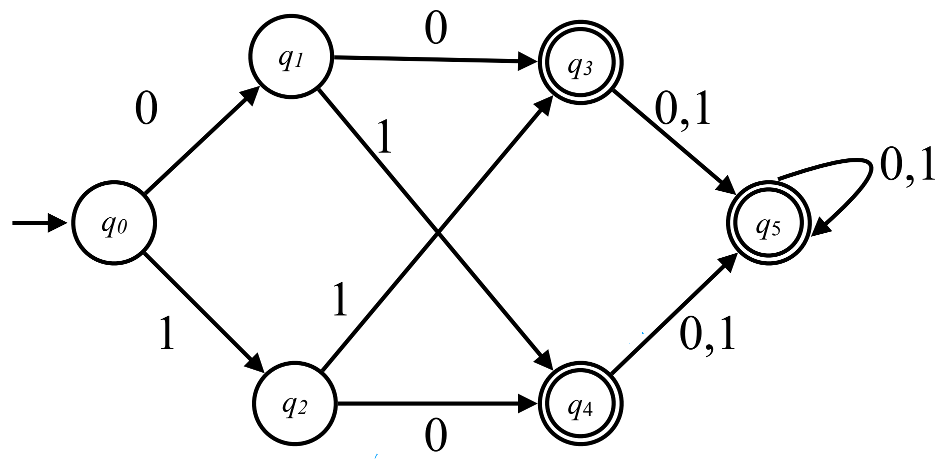
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{q_3, q_4\}$

$\boldsymbol{\{q_0, q_1\}}$	$\boxed{\{q_1, q_2\}}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\boldsymbol{\{q_0, q_2\}}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

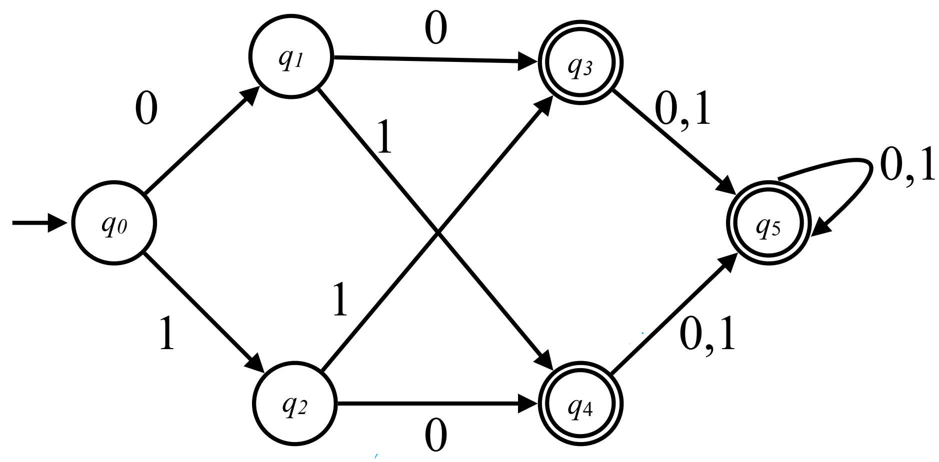
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{\delta(q_1,1), \delta(q_2,1)\}$

$\boldsymbol{\{q_0, q_1\}}$	$\boxed{\{q_1, q_2\}}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\boldsymbol{\{q_0, q_2\}}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

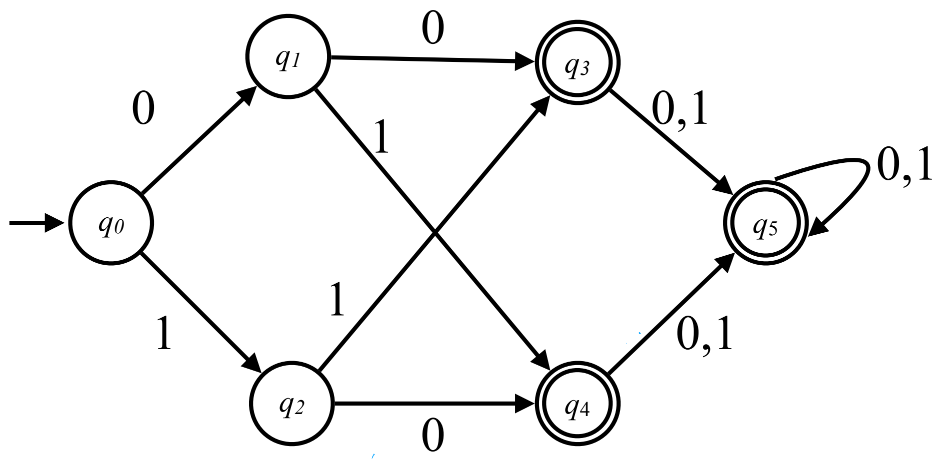
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a\in\Sigma$ then mark $\{p,q\}$



$\{q_4, q_3\}$

$\boldsymbol{\{q_0, q_1\}}$	$\boxed{\{q_1, q_2\}}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\boldsymbol{\{q_0, q_2\}}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

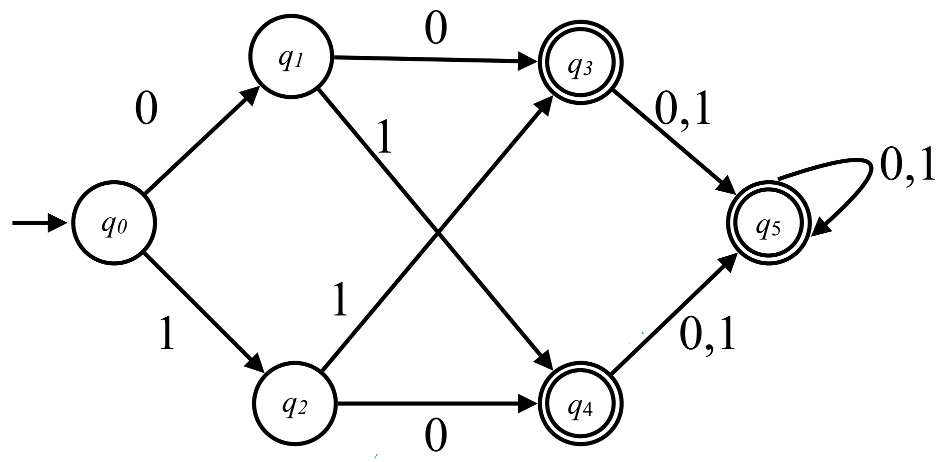
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{\delta(q_3,0), \delta(q_4,0)\}$

$\{q_0, q_1\}$	$\{q_1, q_2\}$	**$\{q_2, q_3\}$**	$\boxed{\{q_3, q_4\}}$	$\{q_4, q_5\}$
$\{q_0, q_2\}$	**$\{q_1, q_3\}$**	**$\{q_2, q_4\}$**	$\{q_3, q_5\}$	
$\{q_0, q_3\}$	**$\{q_1, q_4\}$**	**$\{q_2, q_5\}$**		
$\{q_0, q_4\}$	**$\{q_1, q_5\}$**			
$\{q_0, q_5\}$				

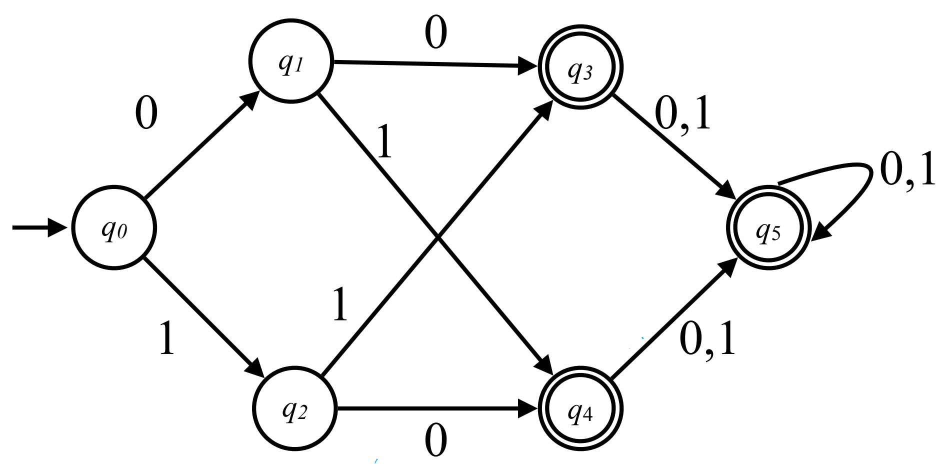
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a),\ \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{q_5\}$

$\boldsymbol{\{q_0, q_1\}}$	$\{q_1, q_2\}$	$\boldsymbol{\{q_2, q_3\}}$	$\boxed{\{q_3, q_4\}}$	$\{q_4, q_5\}$
$\boldsymbol{\{q_0, q_2\}}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{\delta(q_3,1), \delta(q_4,1)\}$

$\{q_0, q_1\}$ $\{q_1, q_2\}$ **$\{q_2, q_3\}$** $\boxed{\{q_3, q_4\}}$ $\{q_4, q_5\}$

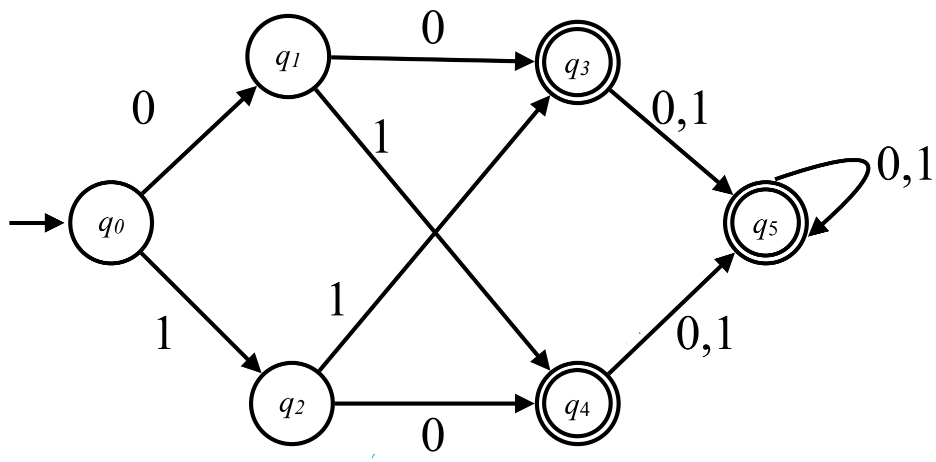
$\{q_0, q_2\}$ **$\{q_1, q_3\}$** **$\{q_2, q_4\}$** $\{q_3, q_5\}$

$\{q_0, q_3\}$ **$\{q_1, q_4\}$** **$\{q_2, q_5\}$**

$\{q_0, q_4\}$ **$\{q_1, q_5\}$**

$\{q_0, q_5\}$

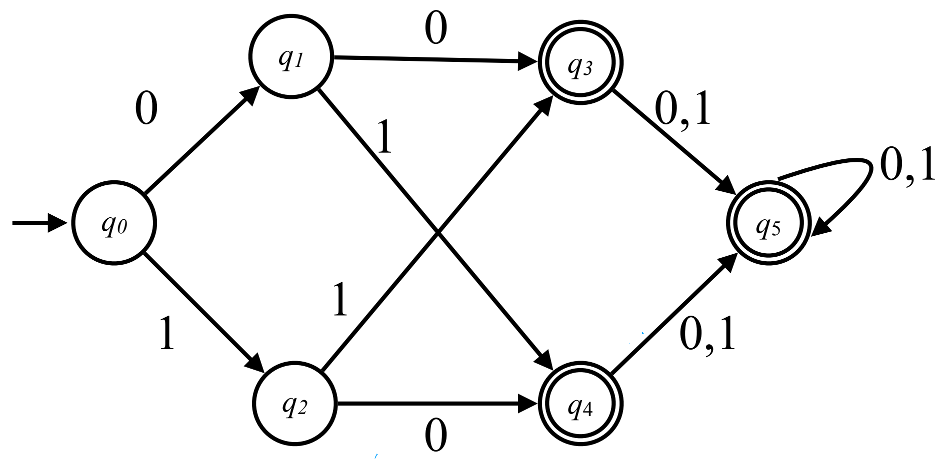
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{q_5\}$

$\mathbf{\{q_0, q_1\}}$	$\{q_1, q_2\}$	$\mathbf{\{q_2, q_3\}}$	$\boxed{\{q_3, q_4\}}$	$\{q_4, q_5\}$
$\mathbf{\{q_0, q_2\}}$	$\mathbf{\{q_1, q_3\}}$	$\mathbf{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\mathbf{\{q_0, q_3\}}$	$\mathbf{\{q_1, q_4\}}$	$\mathbf{\{q_2, q_5\}}$		
$\mathbf{\{q_0, q_4\}}$	$\mathbf{\{q_1, q_5\}}$			
$\mathbf{\{q_0, q_5\}}$				

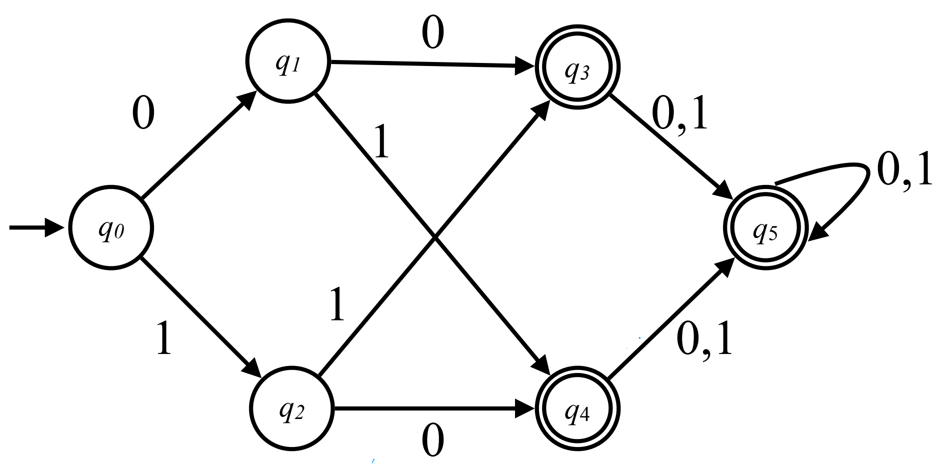
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{\delta(q_3,0), \delta(q_5,0)\}$

$\boldsymbol{\{q_0, q_1\}}$	$\{q_1, q_2\}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\boldsymbol{\{q_0, q_2\}}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\boxed{\{q_3, q_5\}}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

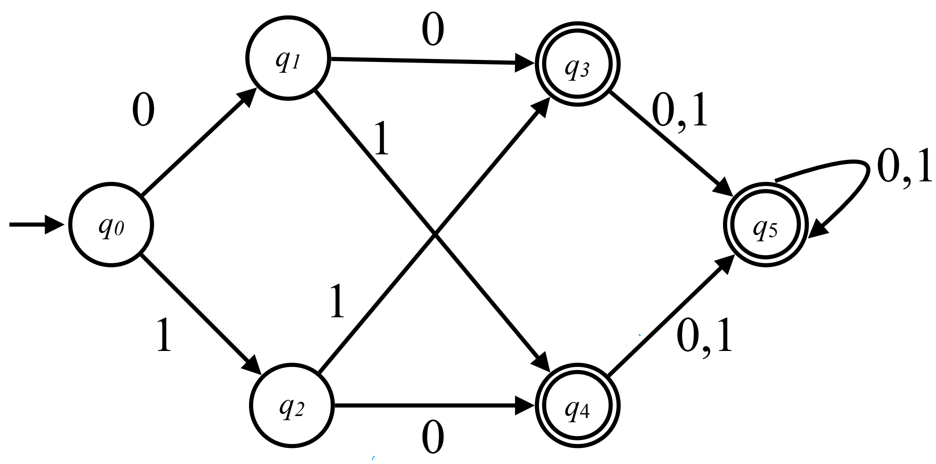
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{q_5\}$

$\mathbf{\{q_0, q_1\}}$	$\{q_1, q_2\}$	$\mathbf{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\mathbf{\{q_0, q_2\}}$	$\mathbf{\{q_1, q_3\}}$	$\mathbf{\{q_2, q_4\}}$	$\boxed{\{q_3, q_5\}}$	
$\mathbf{\{q_0, q_3\}}$	$\mathbf{\{q_1, q_4\}}$	$\mathbf{\{q_2, q_5\}}$		
$\mathbf{\{q_0, q_4\}}$	$\mathbf{\{q_1, q_5\}}$			
$\mathbf{\{q_0, q_5\}}$				

Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{\delta(q_3,1), \delta(q_5,1)\}$

$\boldsymbol{\{q_0, q_1\}}$	$\{q_1, q_2\}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\boldsymbol{\{q_0, q_2\}}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\boxed{\{q_3, q_5\}}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
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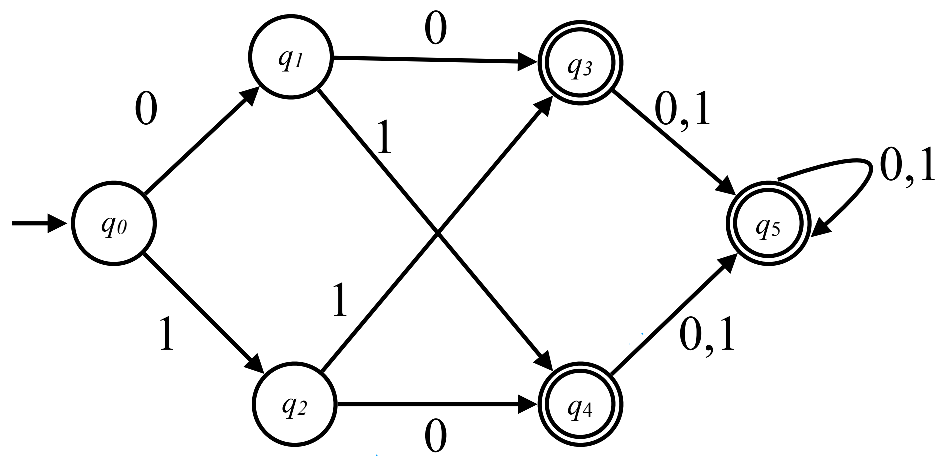

$\{q_5\}$

$\boldsymbol{\{q_0, q_1\}}$ $\{q_1, q_2\}$ $\boldsymbol{\{q_2, q_3\}}$ $\{q_3, q_4\}$ $\{q_4, q_5\}$

$\boldsymbol{\{q_0, q_2\}}$ $\boldsymbol{\{q_1, q_3\}}$ $\boldsymbol{\{q_2, q_4\}}$ $\boxed{\{q_3, q_5\}}$

$\boldsymbol{\{q_0, q_3\}}$ $\boldsymbol{\{q_1, q_4\}}$ $\boldsymbol{\{q_2, q_5\}}$

$\boldsymbol{\{q_0, q_4\}}$ $\boldsymbol{\{q_1, q_5\}}$

$\boldsymbol{\{q_0, q_5\}}$

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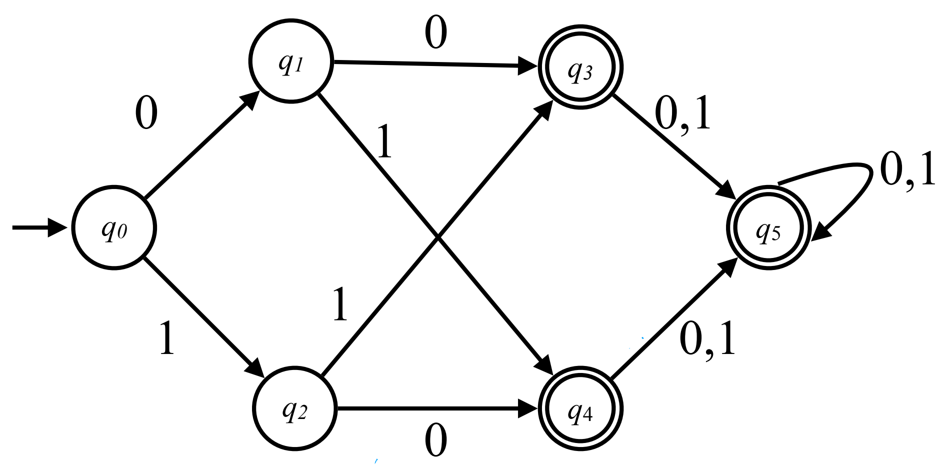
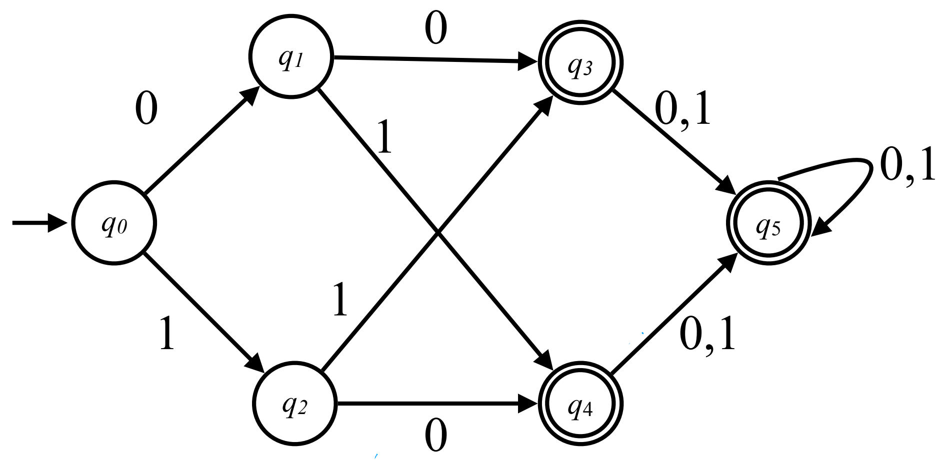


$\{\delta(q_4,0), \delta(q_5,0)\}$

$\boldsymbol{\{q_0, q_1\}}$	$\{q_1, q_2\}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\boxed{\{q_4, q_5\}}$
$\boldsymbol{\{q_0, q_2\}}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a),\ \delta(q,a)\}$ is marked for some $a\in\Sigma$ then mark $\{p,q\}$



$\{q_5\}$

$\mathbf{\{q_0, q_1\}}$	$\{q_1, q_2\}$	$\mathbf{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\boxed{\{q_4, q_5\}}$
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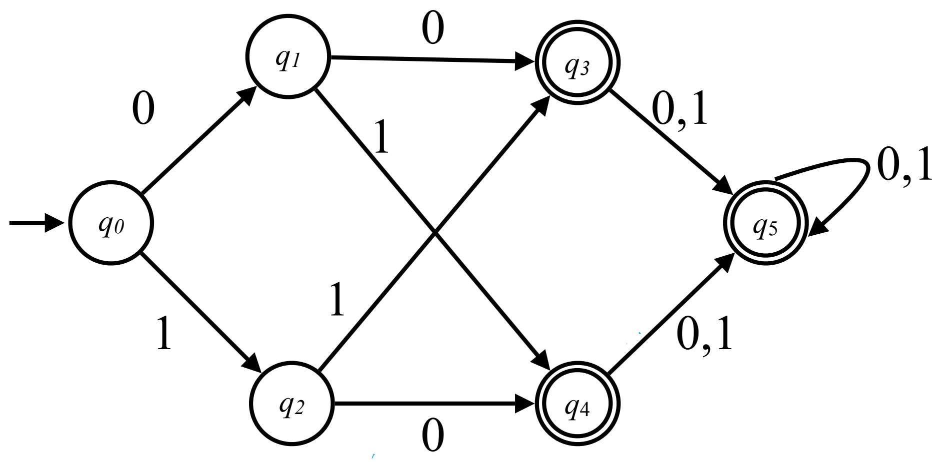

$\{\delta(q_4,1),\ \delta(q_5,1)\}$

$\boldsymbol{\{q_0, q_1\}}$	$\{q_1, q_2\}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\boxed{\{q_4, q_5\}}$
$\boldsymbol{\{q_0, q_2\}}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
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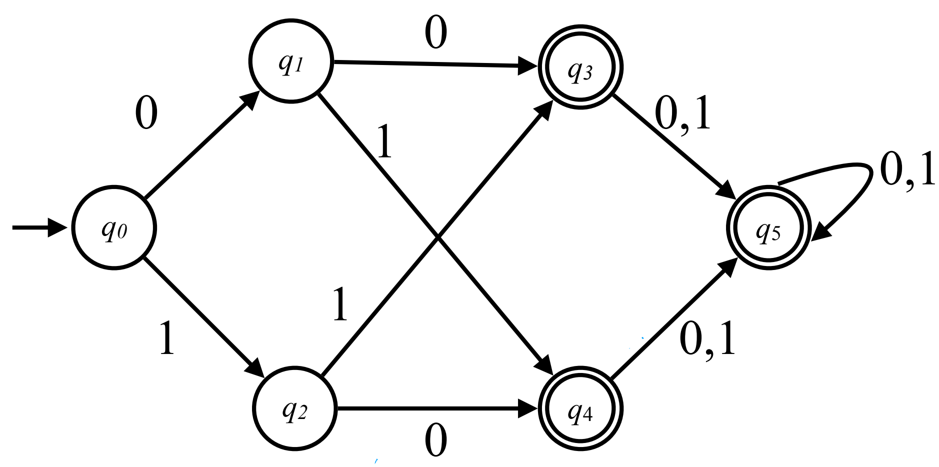

$\{q_5\}$

$\{q_0, q_1\}$	$\{q_1, q_2\}$	**$\{q_2, q_3\}$**	$\{q_3, q_4\}$	$\boxed{\{q_4, q_5\}}$
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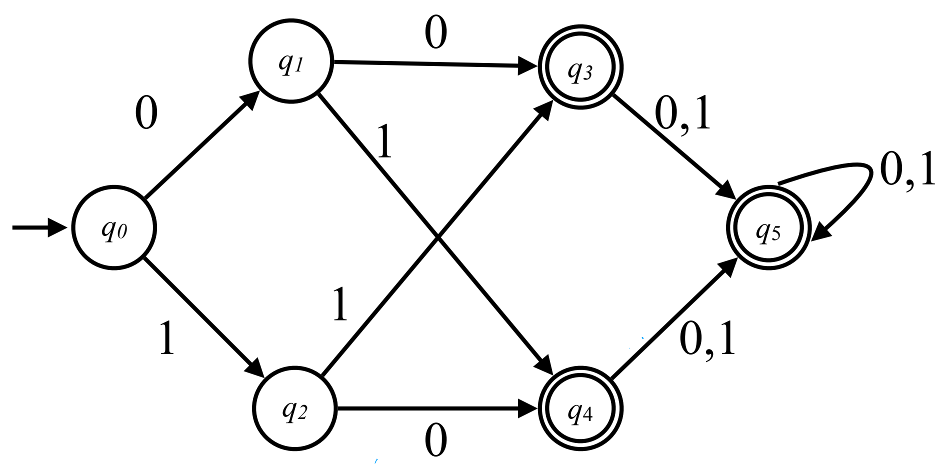
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a), \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{q_0, q_1\}$	$\{q_1, q_2\}$	**$\{q_2, q_3\}$**	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\{q_0, q_2\}$	**$\{q_1, q_3\}$**	**$\{q_2, q_4\}$**	$\{q_3, q_5\}$	
$\{q_0, q_3\}$	**$\{q_1, q_4\}$**	**$\{q_2, q_5\}$**		
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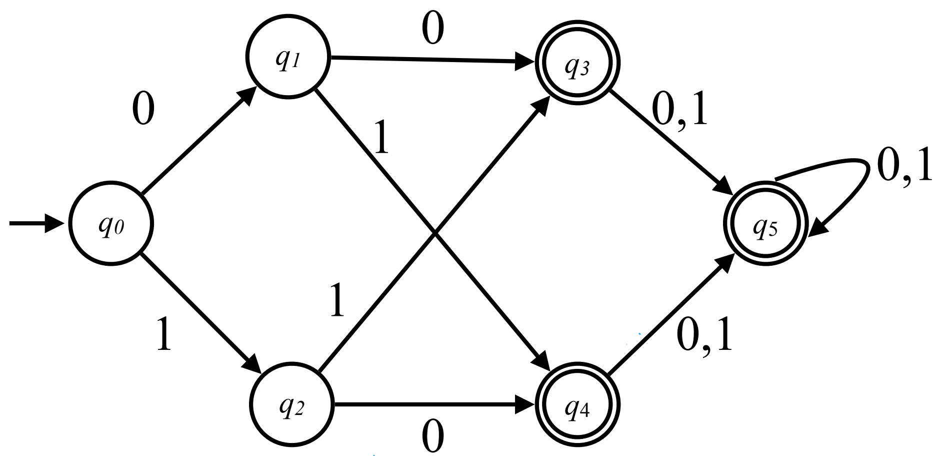

$\{q_0, q_1\}$	$\boxed{\{q_1, q_2\}}$	**$\{q_2, q_3\}$**	$\{q_3, q_4\}$	$\{q_4, q_5\}$
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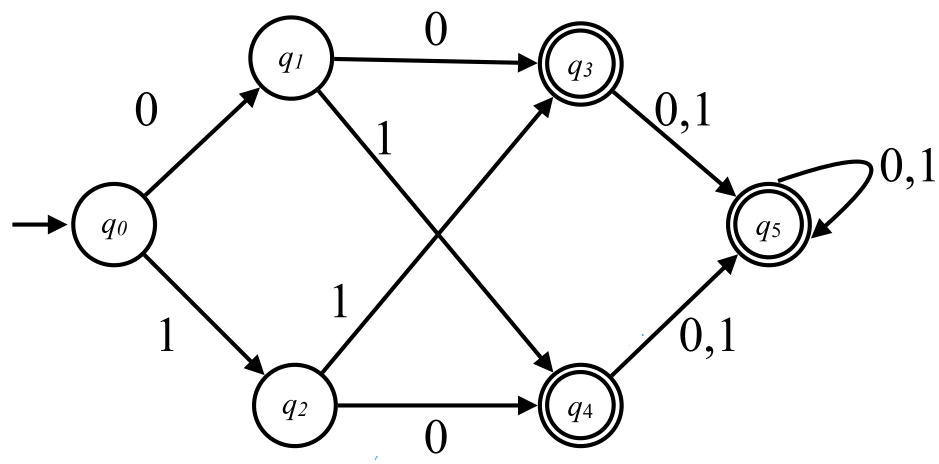

$\{q_0, q_1\}$	$\{q_1, q_2\}$	**$\{q_2, q_3\}$**	$\boxed{\{q_3, q_4\}}$	$\{q_4, q_5\}$
$\{q_0, q_2\}$	**$\{q_1, q_3\}$**	**$\{q_2, q_4\}$**	$\{q_3, q_5\}$	
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$\{q_0, q_4\}$	**$\{q_1, q_5\}$**			
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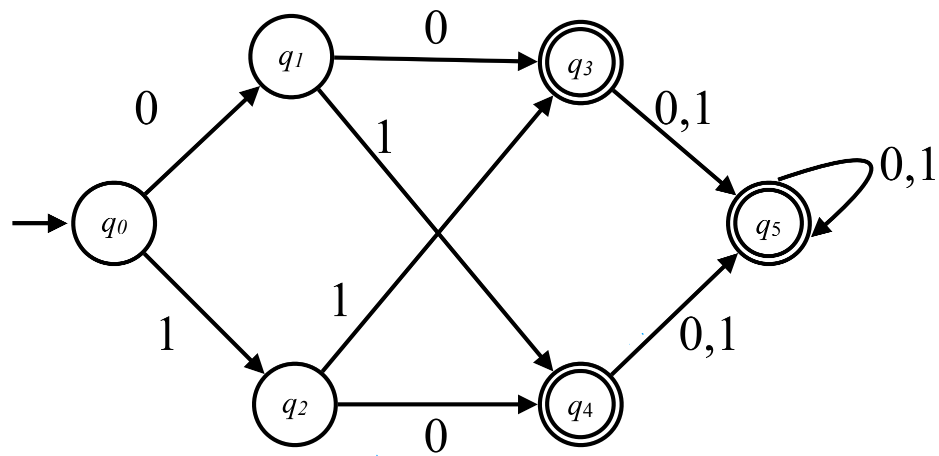
Repeat until no changes occur: If there exists an unmarked pair $\{p,q\}$ such that $\{\delta(p,a),\, \delta(q,a)\}$ is marked for some $a \in \Sigma$ then mark $\{p,q\}$



$\{q_0, q_1\}$	$\{q_1, q_2\}$	$\{q_2, q_3\}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\{q_0, q_2\}$	$\{q_1, q_3\}$	$\{q_2, q_4\}$	$\boxed{\{q_3, q_5\}}$	
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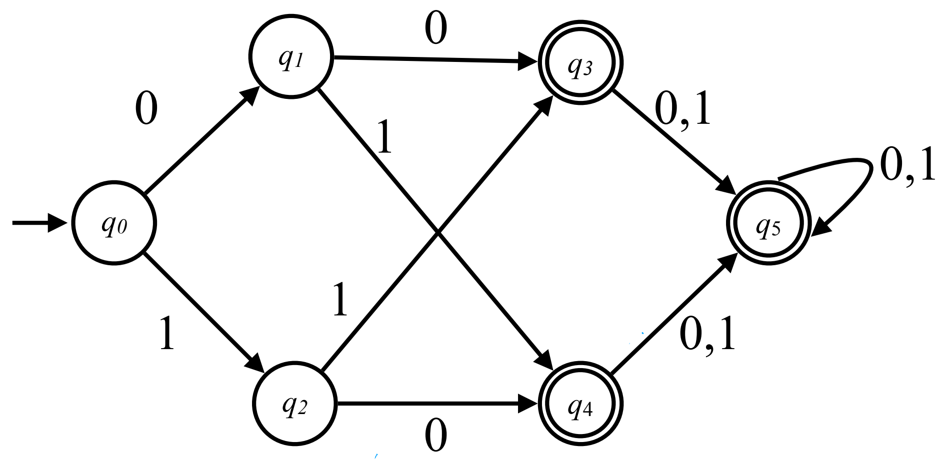
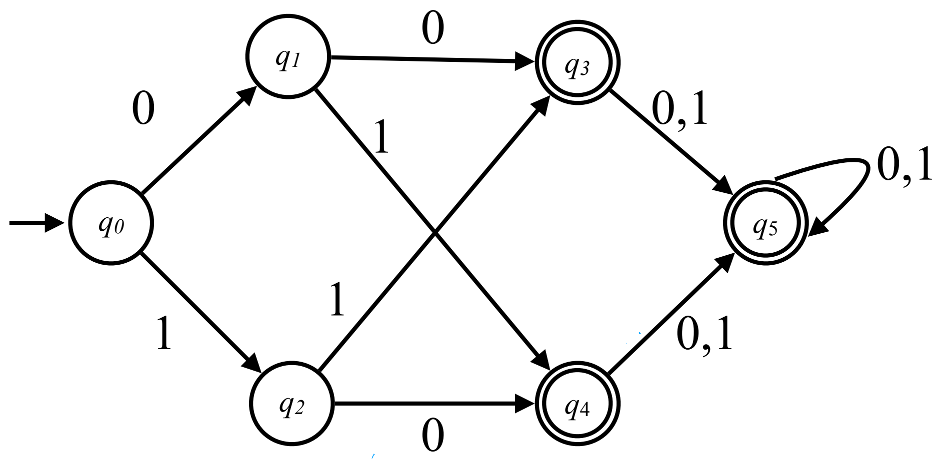


$\boldsymbol{\{q_0, q_1\}}$	$\{q_1, q_2\}$	$\boldsymbol{\{q_2, q_3\}}$	$\{q_3, q_4\}$	$\boxed{\{q_4, q_5\}}$
$\boldsymbol{\{q_0, q_2\}}$	$\boldsymbol{\{q_1, q_3\}}$	$\boldsymbol{\{q_2, q_4\}}$	$\{q_3, q_5\}$	
$\boldsymbol{\{q_0, q_3\}}$	$\boldsymbol{\{q_1, q_4\}}$	$\boldsymbol{\{q_2, q_5\}}$		
$\boldsymbol{\{q_0, q_4\}}$	$\boldsymbol{\{q_1, q_5\}}$			
$\boldsymbol{\{q_0, q_5\}}$				

The algorithm produces state equivalence: $\{p,q\}$ unmarked iff $p \sim q$



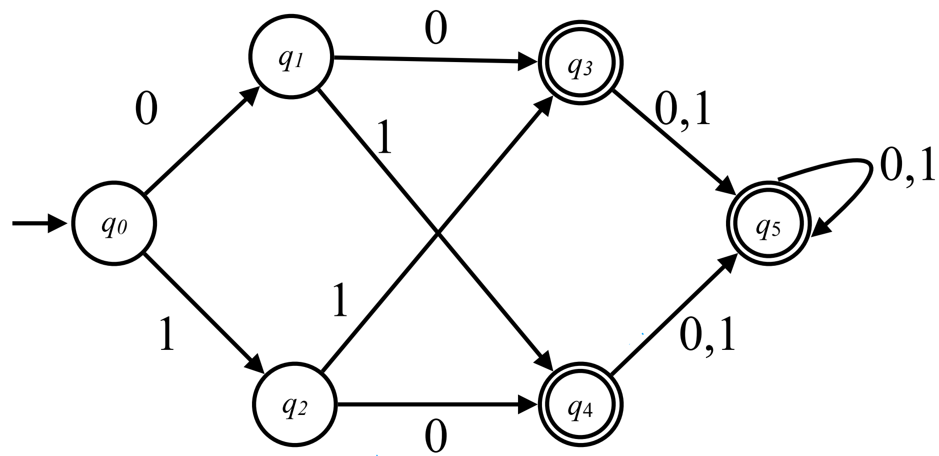
$\{q_0, q_1\}$	$\{q_1, q_2\}$	$\{q_2, q_3\}$	$\{q_3, q_4\}$	$\{q_4, q_5\}$
$\{q_0, q_2\}$	$\{q_1, q_3\}$	$\{q_2, q_4\}$	$\{q_3, q_5\}$	
$\{q_0, q_3\}$	$\{q_1, q_4\}$	$\{q_2, q_5\}$		
$\{q_0, q_4\}$	$\{q_1, q_5\}$			
$\{q_0, q_5\}$				

Equivalent:

$q_1 \sim q_2$

$q_3 \sim q_4 \sim q_5$

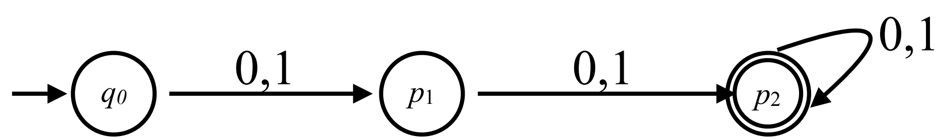
State diagram for minimized automaton



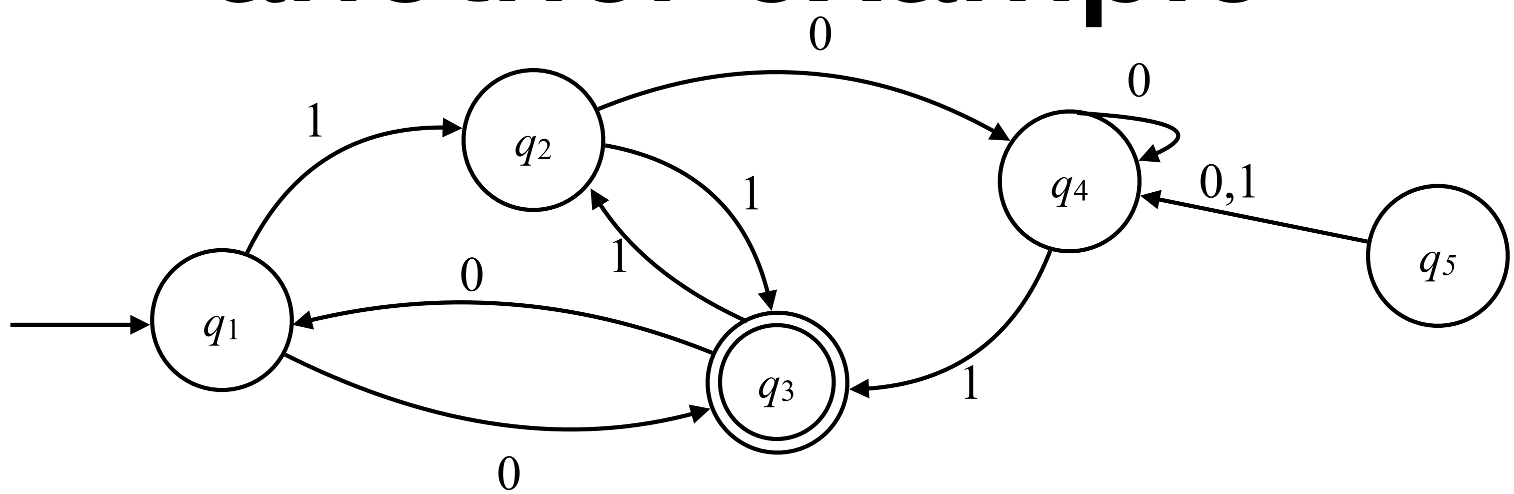
Equivalent:

p_1: $q_1 \sim q_2$

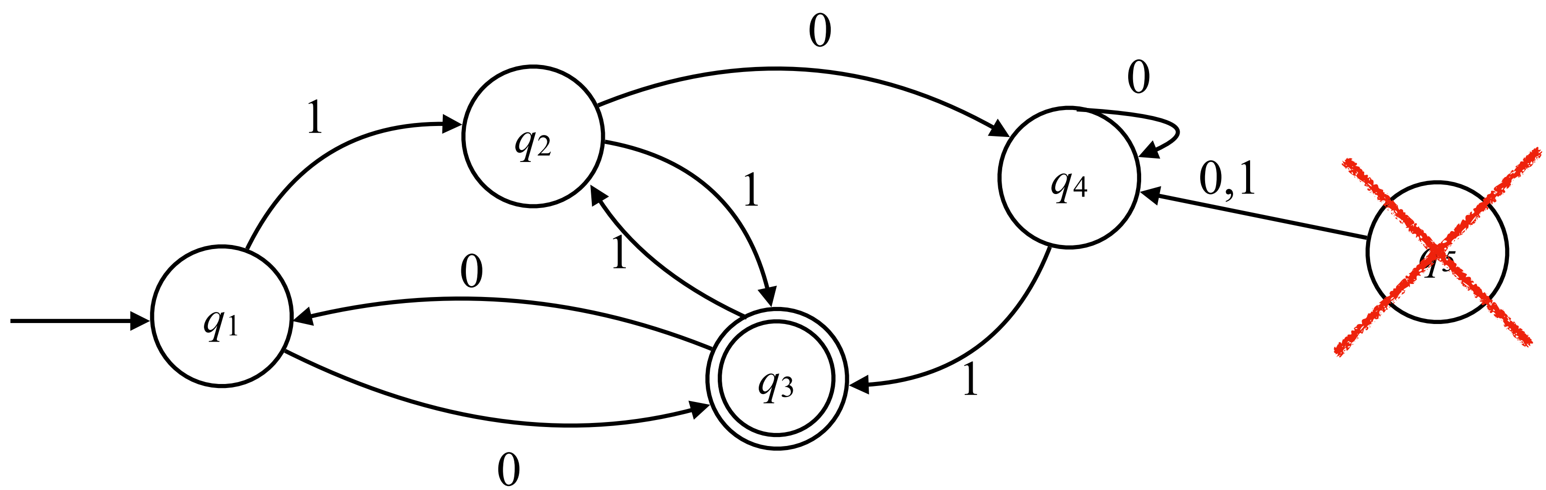
p_2: $q_3 \sim q_4 \sim q_5$



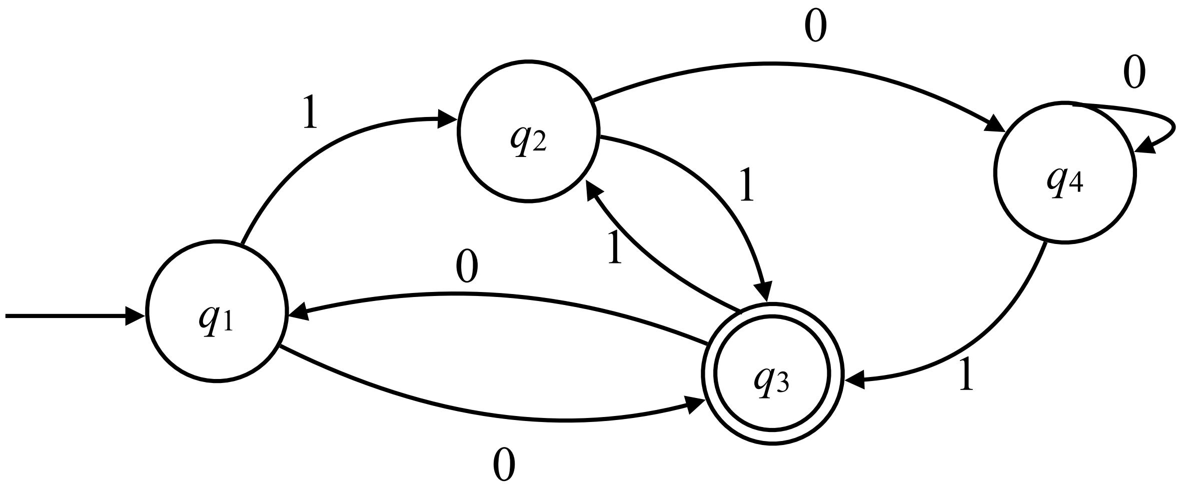
DFA State Minimization: another example



Remove unreachable state



List pairs of states

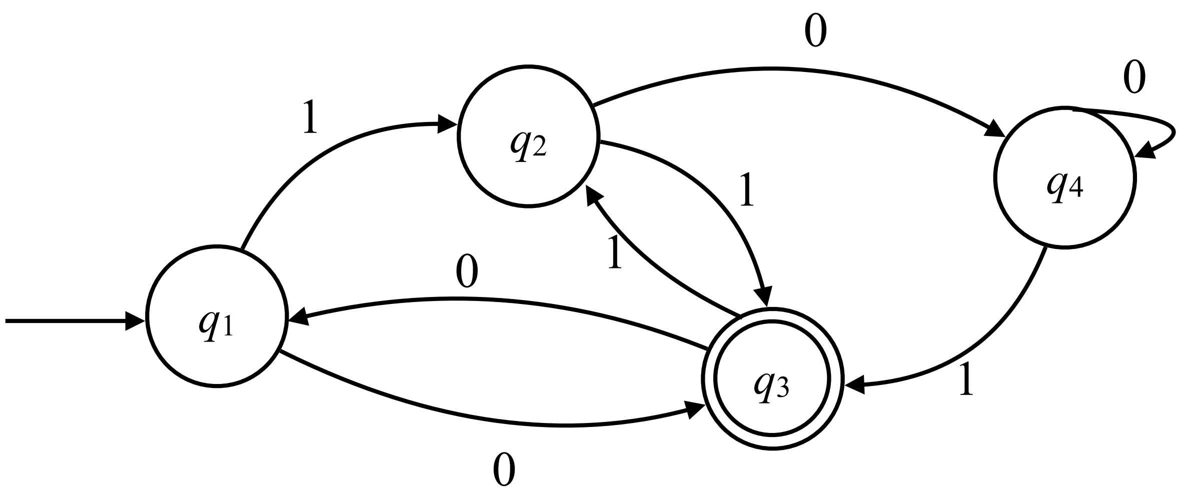


$\{q_1, q_2\}$ $\{q_2, q_3\}$

$\{q_1, q_3\}$ $\{q_2, q_4\}$

$\{q_1, q_4\}$ $\{q_3, q_4\}$

Mark pairs of states where one state is an accept state and the other one is not



$\{q_1, q_2\}$

$\cancel{\{q_1, q_3\}}$

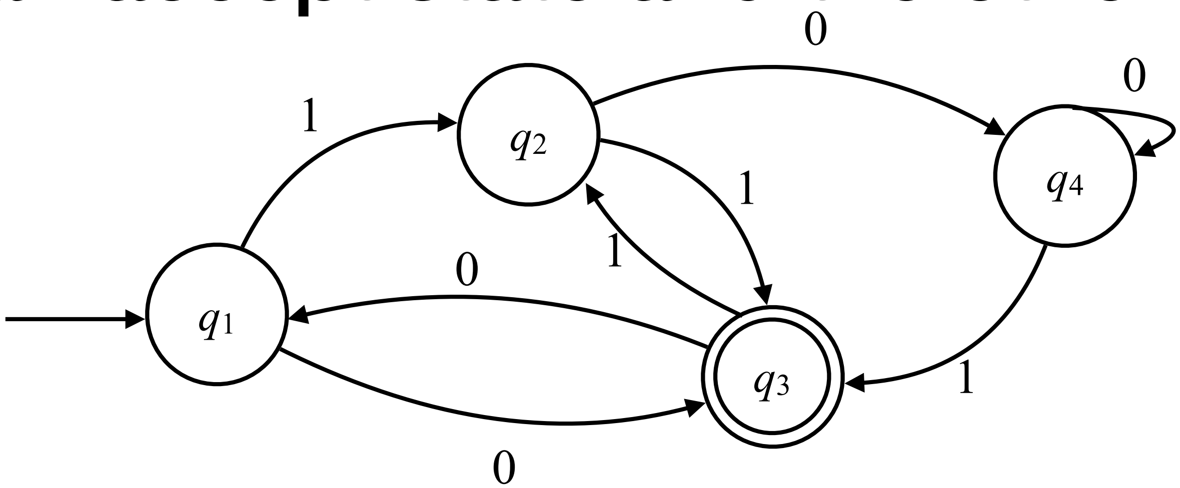
$\{q_1, q_4\}$

$\cancel{\{q_2, q_3\}}$

$\{q_2, q_4\}$

$\cancel{\{q_3, q_4\}}$

Mark pairs of states $\{q_i, q_j\}$ where for some a in $\{\delta(q_i,a), \delta(q_j,a)\}$ one state is an accept state and the other one is not



$\boxed{\cancel{\{q_1, q_2\}}}$

$\cancel{\{q_1, q_3\}}$

$\{q_1, q_4\}$

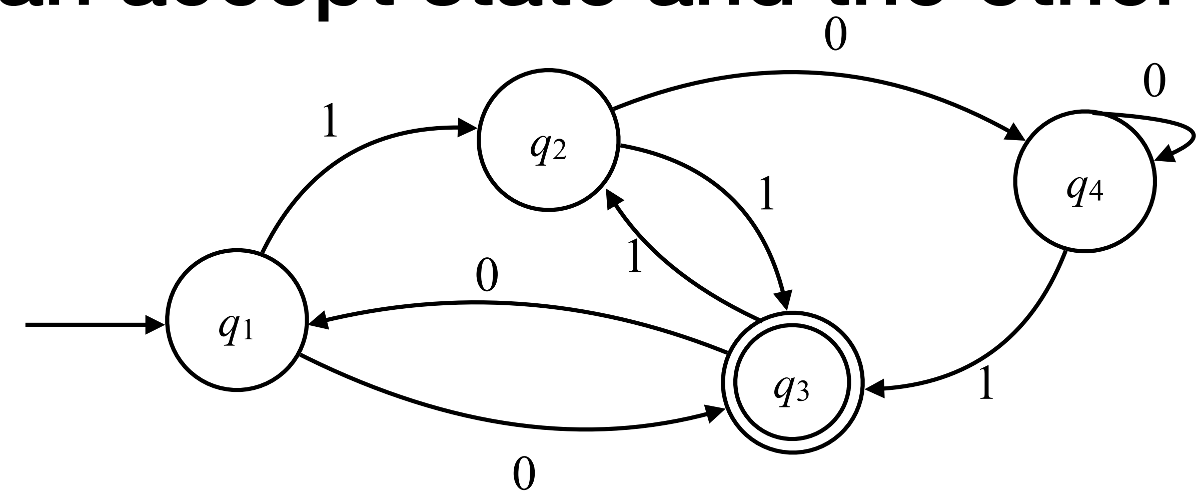
$\cancel{\{q_2, q_3\}}$

$\{q_2, q_4\}$

$\cancel{\{q_3, q_4\}}$

$\boxed{\delta(q_1,0) = q_3 \text{ , } \delta(q_2, 0) = q_4}$

Mark pairs of states $\{q_i, q_j\}$ where for some a in $\{\delta(q_i,a), \delta(q_j,a)\}$ one state is an accept state and the other one is not

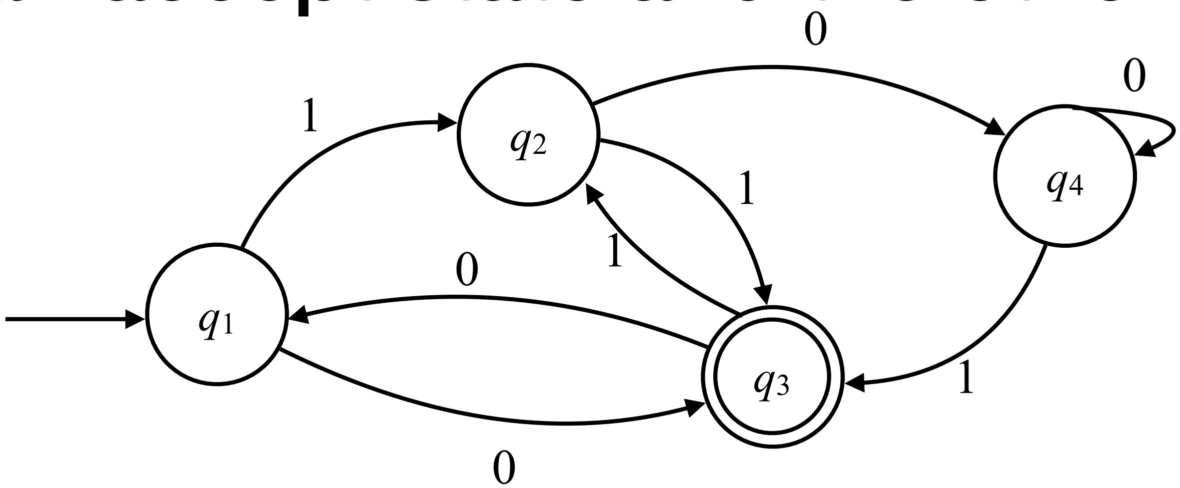


$\cancel{\{q_1, q_2\}}$ $\cancel{\{q_2, q_3\}}$

$\cancel{\{q_1, q_3\}}$ $\{q_2, q_4\}$ $\boxed{\delta(q_1,0) = q_3\ ,\ \delta(q_4, 0) = q_4}$

$\boxed{\cancel{\{q_1, q_4\}}}$ $\cancel{\{q_3, q_4\}}$

Mark pairs of states $\{q_i, q_j\}$ where for some a in $\{\delta(q_i,a), \delta(q_j,a)\}$ one state is an accept state and the other one is not



$\cancel{\{q_1, q_2\}}$ $\cancel{\{q_2, q_3\}}$

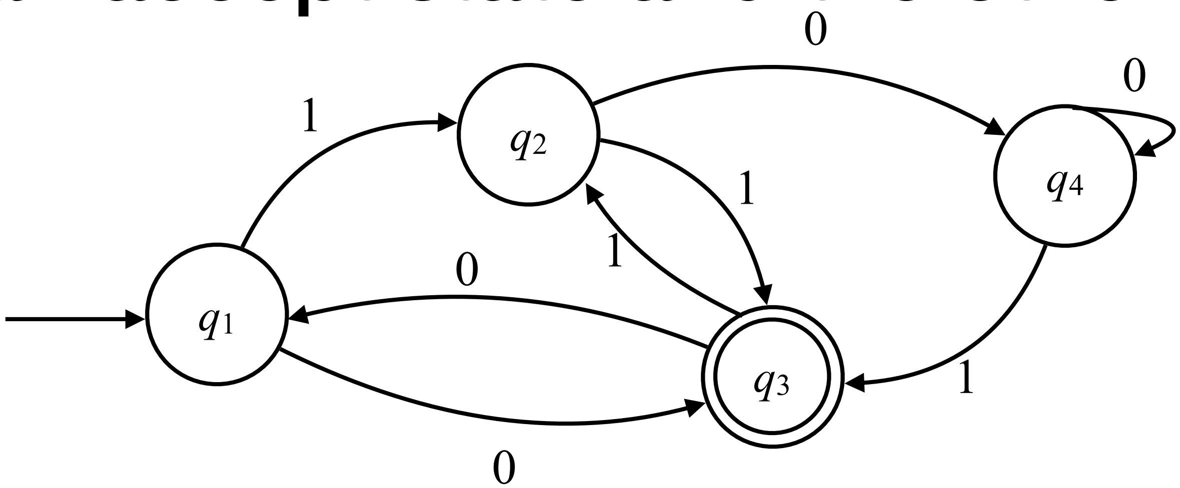
$\cancel{\{q_1, q_3\}}$ $\boxed{\{q_2, q_4\}}$

$\cancel{\{q_1, q_4\}}$ $\cancel{\{q_3, q_4\}}$

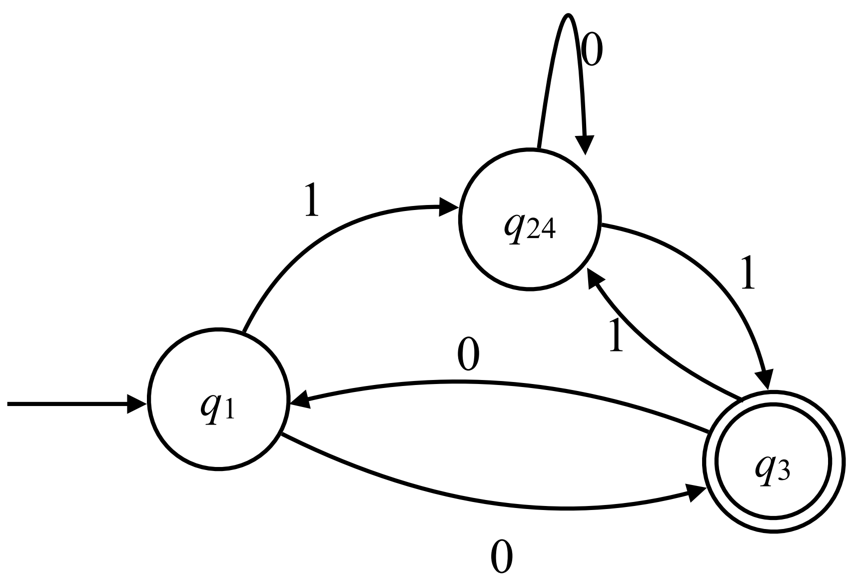
$$\boxed{\begin{array}{l}\delta(q_2,0) = q_4 \,,\, \delta(q_4, 0) = q_4 \\ \delta(q_2,1) = q_3 \,,\, \delta(q_4, 1) = q_3\end{array}}$$

$$q_2 \sim q_4$$

Mark pairs of states $\{q_i, q_j\}$ where for some a in $\{\delta(q_i,a), \delta(q_j,a)\}$ one state is an accept state and the other one is not



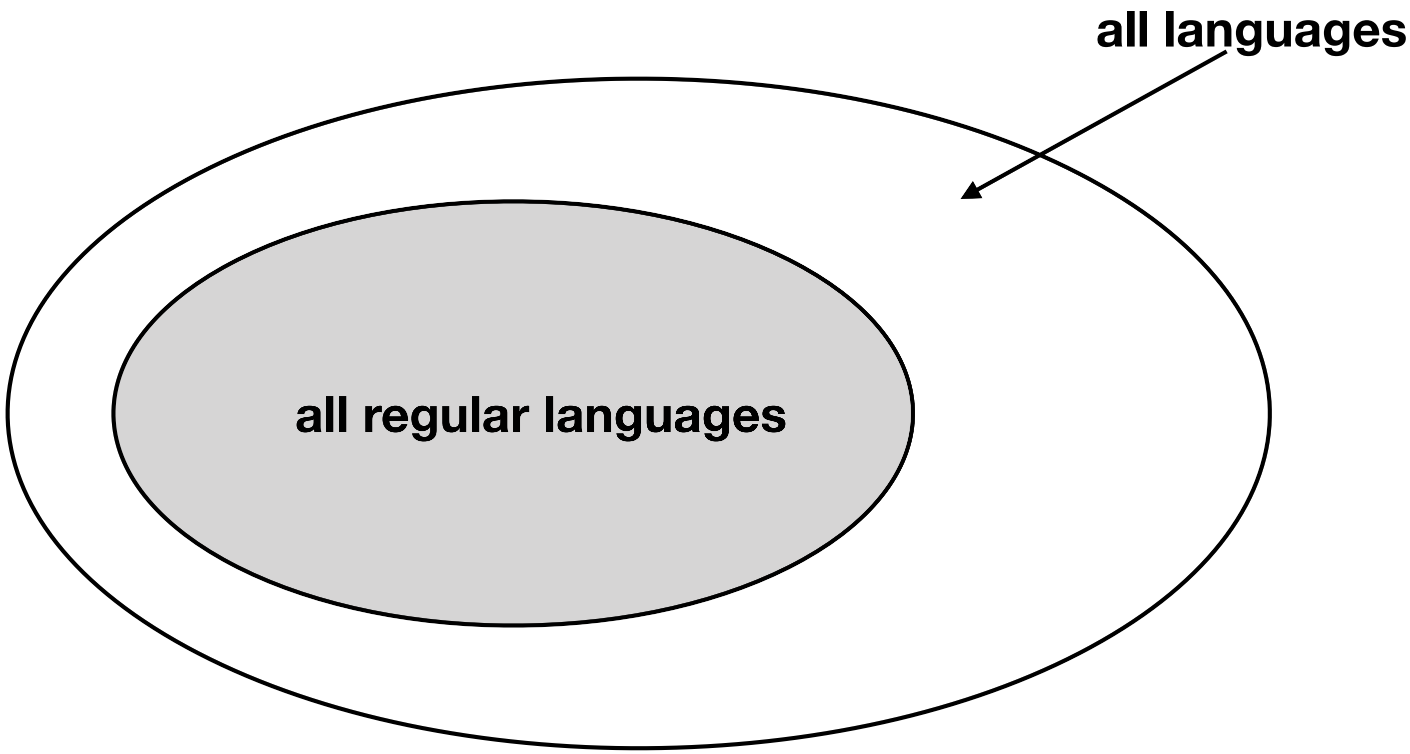
$q_2 \sim q_4$



Myhill-Nerode Theorem

- Equivalence classes for languages
- Connection between minimized DFA and regular languages (and their equivalence classes)
- **Definition**. Given $L \subseteq \Sigma^*$. For $x, y \in \Sigma^*$
  - $z \in \Sigma^*$ is a **distinguishing extension** of x and y if exactly one of xz and yz is in L
  - If such a z exists, x and y are **distinguishable** by L; **otherwise** x and y are **indistinguishable** by L
  - Given $L \subseteq \Sigma^*$. Equivalence relation $\sim_L$ on Σ^*: $x \sim_L y$ iff x and y are indistinguishable by L.
- **Theorem.** Given $L \subseteq \Sigma^*$**.** L is regular iff the number of equivalence classes of $\sim_L$ is finite. Furthermore: if L is regular, then the number of equivalence classes of $\sim_L$ equals the number of states in the minimal DFA.

Next: nonregular languages



Regular Languages

- Recall: The set of regular languages is
  - the set of all languages recognized by deterministic finite automata, and also it is the same as
  - the set of all languages recognized by nondeterministic finite automata, as well as it is the same as
  - the set of languages of all regular expressions

Consider the following language

- $L = \{w \in \{0,1\}^* \mid w$ has an equal number of occurrences of 01 and 10 as substrings$\}$
    - Is L regular?

Consider the following language

- Is L regular?
  - $L = \{w \in \{0,1\}^* \mid w$ has an equal number of occurrences of 01 and 10 as substrings$\}$
    - $\epsilon \in L$?
    - $0 \in L$? $1 \in L$?
    - $10 \in L$? $01 \in L$?
    - $101 \in L$? $010 \in L$?
    - $x \in \{0,1\}^*$: $1x1 \in L$? $0x0 \in L$?

Yes! $L = L(R)$ with $R = (\epsilon \cup 0 \cup 1) \cup 0(0 \cup 1)^*0 \cup 1(0 \cup 1)^*1$

Consider the following languages

- Are L_1, L_2, or L_3 regular?
  - $L_1 = \{w \in \{0,1\}^* \mid w$ has an equal number of 0s and 1s$\}$ **?**
  - $L_2 = \{0^n 1^n \mid n \geq 0\}$ **?**
  - $L_3 = \{0^3 1^n \mid n \geq 0\}$ ← **Regular:** $L(0001^*) = L_3$

Nonregular languages

- Recall: Given language L, if there exists a finite automaton M with $L(M) = L$ then L is regular

- Therefore: If a language is **nonregular** then **no** finite automaton exists that recognizes it

- A technique for proving that languages are nonregular:

    - Pumping lemma

Pumping Lemma

- If L is a regular language, then there is a natural number p (the *pumping length*) where:

  - if s is any string in L of length at least p (ie, $s \in L$, $|s| \geq p$), then s can be divided into $s = xyz$ satisfying the following

    1. for each $i \geq 0$: $xy^iz \in L$
    2. $|y| > 0$ (ie, $y \neq \varepsilon$)
    3. $|xy| \leq p$

Notes:
(1) y^i means the concatenation of i copies of string y
(2) Conditions 1–3 hold **for all** strings in L that are of length at least p

Next

- Investigate what the pumping lemma is and what it is good for

- Examples to show that certain languages are not regular, using the pumping lemma

- Prove why the pumping lemma is correct

- More examples to show that certain languages are not regular, using the pumping lemma

Using the Pumping Lemma (PL): Example

- Let $L = \{0^n 1^n | n \geq 0\}$. We prove that L is non-regular
  - Use PL
- **Proof by contradiction**
  - Assume that L is regular
  - Then all properties of PL must hold for L:

Using the Pumping Lemma: Example (2)

Prove: $L = \{0^n 1^n | n \geq 0\}$ is non-regular

Assuming L is regular, all properties of PL must hold for L:

In particular, PL gives us some p.

If $s \in L$ and $|s| \geq p$ then there is a way to rewrite s as $s = xyz$ with

1. for each $i \geq 0$: $xy^iz \in L$
2. $y \neq \epsilon$
3. $|xy| \leq p$

Using the Pumping Lemma: Example (3)

Prove: $L = \{0^n 1^n | n \geq 0\}$ is non-regular

- Let p be pumping length given by pumping lemma
- Choose $\boldsymbol{s = 0^p 1^p}$
  - $s \in L$ and $|s| \geq p$ (since $|s| = 2p$)

PL guarantees:

- s can be rewritten as $s = xyz$ with

1. $xy^iz \in L$, for all $i \geq 0$
2. $y \neq \epsilon$
3. $|xy| \leq p$

- We don't know the value of p
- Actually: there is no way to know the value of p in the case that L is not a regular language
- Must work with general p

- s is string that we hope is a counterexample
- must reach a contradiction to properties of PL

- For this, length of s must depend on p since $|s| \geq p$ is required

Using the Pumping Lemma: Example (4)

Prove: $L = \{0^n 1^n | n \geq 0\}$ is non-regular

- Does there really exist some rewriting for s, $s = 0^p 1^p = xyz$ s.t. PL properties hold?
  - Recall: Trying to show that L is not regular
  - Show that $s = 0^p 1^p$ is counterexample for PL properties
    - Prove: There is ***no single rewriting*** of $s = 0^p 1^p$ into $s = 0^p 1^p = xyz$ s.t. three properties hold
  - Note: Just one rewriting of s that satisfies PL properties would show that string s **is not** a counterexample
  - Therefore we must consider all possibilities to rewrite s:
    - Because of property 2 (ie, $y \neq \epsilon$), y cannot be the empty string and therefore
      - **Case 1:** String y consists of 0s only
      - **Case 2:** String y consists only of 1s
      - **Case 3:** String y consists of both 0s and 1s

$$\boldsymbol{x\ y\ z}$$

$$\underbrace{\mathbf{0 \ldots. 0}}_{p \text{ many}}\ \underbrace{\mathbf{1 \ldots 1}}_{p \text{ many}}$$

Using the Pumping Lemma: Example (5)

Prove: $L = \{0^n 1^n | n \geq 0\}$ is non-regular

- **Case 1: String y consists of 0s only**
  String $xyyz$ has more 0s than 1s. Therefore $xyyz \notin L$, violating property 1 of PL. Contradiction.

- **Case 2: String y consists only of 1s**
  Just like above: String $xyyz$ has more 1s than 0s. Therefore $xyyz \notin L$, violating property 1 of PL. Contradiction.

- **Case 3: String y consists of 0s followed by 1s**
  String $xyyz$ may have the same number of 0s and 1s, but they are out of order (some 1s occur before 0s). Hence $xyyz \notin L$, violating property 1 of PL. Contradiction.

Since none of the cases is possible, we **cannot rewrite s** satisfying the properties of PL. Therefore: L is **not** regular

Another example

- We show that $L = \{w \in \{0,1\}^* \mid w$ has an equal number of 0s and 1s$\}$ is not regular
- Assume that L is regular
- Let p be the pumping length given by pumping lemma
- $s = ?$

Crucial what string $s \in L$ of length at least p to choose from L to derive contradiction for all possible rewritings of s

- Not every long string in L might yield contradiction

- Just one string in L of length at least p that does not satisfies the PL properties sufficient to prove language not regular.

- When choosing string as counterexample, think about what properties might make language not regular

Another example (continued)

- We show that $L = \{w \in \{0,1\}^* \mid w$ has an equal number of 0s and 1s$\}$ is not regular
- Assume that L is regular
- Let p be the pumping length given by pumping lemma
- We choose $s = 1^p 0^p$: $s \in L$ and $|s| \geq p$
- According to PL there is a rewriting $s = xyz$ satisfying all three properties
- We show in all cases, $xy^i z \in L$ for all $i \geq 0$ not satisfied
  (therefore s is counterexample)
- Note: $|xy| \leq p$ (Property 3) and $y \neq \epsilon$ (Property 2)
  - y must consist of 1s only (and consists of at least one 1)
  - Consider xz
    - $xz \notin L$ since it contains less 1s than 0s! Contradiction.

Your turn

1. Show that $L = \{0^i 1^j \mid i > j\}$ is non-regular

2. Show that $L = \{0^i 1^j \mid i < j\}$ is non-regular

3. Show that $L = \{0^i 1^j \mid i \leq j\}$ is non-regular

4. Explain why $L = \{0^i 1^j \mid i \leq j < 121\}$ is regular