Re-expressing RBF

Author

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Proof. Hypothesis with movable centers:

$$f(x) = \sum_k c_k e^{-\|x-w_k\|^2}$$

let $\tilde{x}_k = \|x - w_k\| \geq 0$. Thus, notice:

$$e^{-\tilde{x}_k^2} \approx \tilde{t}(\tilde{x}_k) = max(0, 1 - \tilde{x}_k)$$

Hence:

$$f(x) = \sum_k c_k e^{-\tilde{x}_k^2} \approx \sum_k c_k \tilde{t}(\tilde{x})$$

Notice the following two relations:

$$\tilde{x}_k^2 = \|x - w_k\|^2 = 2(1 - \langle x, w_k \rangle)$$

and:

$$\tilde{t}(x) = |\tilde{x}_k - 1|_+ - |\tilde{x}_k|_+ + 1$$

where $|z|_+ = max(0, z)$. Hence:

$$f(x) \approx \sum_k c_k \tilde{t}(\tilde{x}) = \sum_k c_k (|\tilde{x}_k - 1|_+ - |\tilde{x}_k|_+ + 1)$$

Notice $\tilde{x} \geq 0 \implies |\tilde{x}|_+ = \tilde{x}$.

$$f(x) \approx \sum_k c_k \tilde{t}(\tilde{x}) = \sum_k c_k |\tilde{x}_k - 1|_+ - \sum_k c_k \tilde{x}_k + \sum_k c_k$$

Lets analyze

$$\sum_k c_k |\tilde{x}_k - 1|_+ \tag{1}$$

in more detail:
Notice $|\tilde{x}_k - 1|_+ = |\sqrt{2}\sqrt{1 - \langle x, w_k \rangle} - 1|_+ = \sqrt{2}|\sqrt{1 - \langle x, w_k \rangle} - \frac{\sqrt{2}}{2}|_+$

Let $\langle x, w_k \rangle = cos(\theta_k)$ where $\theta_k = \angle(x, w_k)$ is the angle between x and w_k (normalized) vectors.

Hence we have:

$$|\tilde{x}_k - 1|_+ = \sqrt{2}|\sqrt{1 - cos(\theta_k)} - \frac{\sqrt{2}}{2}|_+$$

Notice that the above function $|\sqrt{1 - cos(\theta_k)} - \frac{\sqrt{2}}{2}|_+$ is periodic. So we can get its Fourier approximation:

$$|\sqrt{1 - cos(\theta_k)} - \frac{\sqrt{2}}{2}|_+ \approx a_0 + \sum_n a_n cos(n\theta_k) \tag{2}$$

Now lets try to express the above in terms of only inner products $\langle x, w_k \rangle$ (or equivalently, as a function of $cos(\theta_k)$). It can be shown that:

$$cos(n\theta_k) = \frac{1}{2}\left(\left(cos(\theta_k) + \sqrt{cos^2(\theta_k) - 1}\right)^n + \left(cos(\theta_k) - \sqrt{cos^2(\theta_k) - 1}\right)^n\right)$$

Hence, makes Fourier approximation (2) become:

$$|\sqrt{1 - cos(\theta_k)} - \frac{\sqrt{2}}{2}|_+ \approx \sum_n a_n \frac{1}{2}\left(\left(\langle x, w_k \rangle + \sqrt{\langle x, w_k \rangle^2 - 1}\right)^n + \left(\langle x, w_k \rangle - \sqrt{\langle x, w_k \rangle^2 - 1}\right)^n\right)$$

this makes the final equation (1) look as follows:

$$f(x) \approx \sum_k c_k |\tilde{x}_k - 1|_+ \approx \sum_{n,k} c_k a_n \frac{1}{2}\left(\left(\langle x, w_k \rangle + \sqrt{\langle x, w_k \rangle^2 - 1}\right)^n + \left(\langle x, w_k \rangle - \sqrt{\langle x, w_k \rangle^2 - 1}\right)^n\right)$$

$\square$