

Math, Problem Set #5, Convex Analysis

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Exercise 7.1

Let V be a vector space and let $S \subseteq V$. We wts that $\text{conv}(S)$ is convex.

Let $x, y \in \text{conv}(S)$. Then, by the definition of $\text{conv}(S)$ we have that, $x = \sum \alpha_i \tilde{x}_i$ and $y = \sum \beta_i \tilde{y}_i$, where $\{\alpha_i\}$ and $\{\beta_i\}$ are sets of coefficients, and $\{\tilde{x}_i\}$ and $\{\tilde{y}_i\}$ are elements of S .

Now, let $\lambda \in (0, 1)$. Notice that:

$$\begin{aligned}\lambda x + (1 - \lambda)y &= \lambda \sum \alpha_i \tilde{x}_i + (1 - \lambda) \sum \beta_i \tilde{y}_i \\ &= \sum \lambda \alpha_i \tilde{x}_i + \sum (1 - \lambda) \beta_i \tilde{y}_i \\ &= \sum \delta_j \tilde{z}_j\end{aligned}$$

which is a convex combination of elements of S . Therefore, $\lambda x + (1 - \lambda)y \in \text{conv}(S)$. Thus $\text{conv}(S)$ is convex.

Exercise 7.2

Let V be an inner product space and let $a \in V$ and $b \in \mathbb{R}$.

First, consider the hyperplane:

$$P(a, b) = \{x \in V : \langle a, x \rangle = b\}$$

Let $x, y \in P(a, b)$ and let $\lambda \in (0, 1)$. Call $z = \lambda x + (1 - \lambda)y$. Notice that:

$$\begin{aligned}\langle a, z \rangle &= \langle a, \lambda x + (1 - \lambda)y \rangle \\ &= \lambda \langle a, x \rangle + (1 - \lambda) \langle a, y \rangle \\ &= \lambda b + (1 - \lambda)b \\ &= b\end{aligned}$$

Hence, $z \in P(a, b)$. Therefore, $P(a, b)$ is a convex set.

Consider now, wlog, the closed halfspace:

$$H(a, b) = \{x \in V : \langle a, x \rangle \leq b\}$$

Let $x, y \in H(a, b)$ and let $\lambda \in (0, 1)$. Call $z = \lambda x + (1 - \lambda)y$. Notice that:

$$\begin{aligned}\langle a, z \rangle &= \langle a, \lambda x + (1 - \lambda)y \rangle \\ &= \lambda \langle a, x \rangle + (1 - \lambda) \langle a, y \rangle \\ &\leq \max\{\langle a, x \rangle, \langle a, y \rangle\} \\ &\leq b\end{aligned}$$

Hence, $z \in H(a, b)$. Therefore, $H(a, b)$ is a convex set.

Exercise 7.4

I will first prove the suggested points.

(i)

$$\begin{aligned}\|x - y\|^2 &= \|x - p + p - y\|^2 \\ &= \langle x - p + p - y, x - p + p - y \rangle \\ &= \langle x - p, x - p \rangle + 2\langle x - p, p - y \rangle + \langle p - y, p - y \rangle \\ &= \|x - p\|^2 + \|y - p\|^2 + 2\langle x - p, p - y \rangle\end{aligned}$$

(ii)

If $y \neq p$, then $\|y - p\|^2 > 0$. Moreover, $\langle x - p, p - y \rangle \geq 0$. Therefore:

$$\begin{aligned}\|x - y\|^2 &> \|x - p\|^2 \\ \Rightarrow \|x - y\| &> \|x - p\|\end{aligned}$$

(iii)

$$\begin{aligned}\|x - z\|^2 &= \|x - \lambda y - (1 - \lambda)p\|^2 \\ &= \|x - p + \lambda(p - y)\|^2 \\ &= \langle x - p + \lambda(p - y), x - p + \lambda(p - y) \rangle \\ &= \langle x - p, x - p \rangle + 2\langle x - p, \lambda(p - y) \rangle + \langle \lambda(p - y), \lambda(p - y) \rangle \\ &= \|x - p\|^2 + \lambda^2\|p - y\|^2 + 2\lambda\langle x - p, \lambda(p - y) \rangle\end{aligned}$$

Exercise 7.6

Call $LC = \{x \in \mathbb{R}^n : f(x) \leq c\} \subseteq \mathbb{R}^n$, the lower contour set at point c of f . Let $x, y \in LC$, let $\lambda \in (0, 1)$ and call $z = \lambda x + (1 - \lambda)y$. Notice that:

$$\begin{aligned}f(z) &= f(\lambda x + (1 - \lambda)y) \\ &\leq \lambda f(x) + (1 - \lambda)f(y) \quad (\text{by convexity of } f) \\ &\leq \max\{f(x), f(y)\} \\ &\leq c\end{aligned}$$

Therefore, $z \in LC$ and LC is convex. In other words, the lower contour set of a convex function is convex.

Exercise 7.7

Let $\theta \in (0, 1)$ and $x, y \in C$.

$$\begin{aligned} f(\theta x + (1 - \theta)y) &= \sum \lambda_i f_i(\theta x + (1 - \theta)y) \\ &= \sum \lambda_i (\theta f_i(x) + (1 - \theta)f_i(y)) \quad \text{by convexity of each } f_i \\ &= \theta \sum \lambda_i f_i(x) + (1 - \theta) \sum \lambda_i f_i(y) \\ &= \theta f(x) + (1 - \theta)f(y) \end{aligned}$$

Therefore, f is convex. One direct consequence of the result is that the sum of convex functions is a convex function.

Exercise 7.13

Let f be convex and bounded above. We wts that f is constant.

Suppose not. Then, wlog, take x, y such that $f(x) < f(y)$. Take $\lambda \in (0, 1)$ and let $z = \lambda x + (1 - \lambda)y$. By convexity of f we have that:

$$\begin{aligned} f(z) &\leq \lambda f(x) + (1 - \lambda)f(y) \\ \implies f(x) &\geq \frac{f(z)}{\lambda} + \frac{(1 - \lambda)}{\lambda} f(y) \end{aligned}$$

Notice that $\frac{(1 - \lambda)}{\lambda} > 0, \forall \lambda \in (0, 1)$. Now, take the limit as $\lambda \rightarrow 0^+$:

$$f(x) \rightarrow +\infty$$

so that f is unbounded above, a contradiction. Thus, f must be constant.

Exercise 7.20

Let f be convex and $-f$ be convex. By the definition of convexity it follows that $\forall \lambda \in (0, 1)$ and $\forall x, y \in \text{dom}(f)$:

$$f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y)$$

To show that f is affine, we want to show that $\forall \lambda \in \mathbb{R}$ and $\forall x, y \in \text{dom}(f)$:

$$f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y)$$

In other words, we wts that f preserves affine combinations.

We have already studied the case in which $\lambda \in (0, 1)$. Let's now focus on the case in which $\lambda > 0$. Let $z = \lambda x + (1 - \lambda)y$. Notice that:

$$\begin{aligned} x &= \frac{z}{\lambda} - \frac{(1 - \lambda)}{\lambda}y \\ &= tz + (1 - t)y \end{aligned}$$

where $t = 1/\lambda$. Therefore:

$$f(x) = tf(z) + (1 - t)f(y)$$

which implies that:

$$\begin{aligned} f(z) &= \frac{f(x)}{t} - \frac{(1 - t)}{t}f(y) \\ &= \lambda f(x) + (1 - \lambda)f(y) \end{aligned}$$

Let's now focus on the case $\lambda \in (-1, 0)$. Again, let $z = \lambda x + (1 - \lambda)y$ and notice that:

$$\begin{aligned} y &= \frac{z}{(1 - \lambda)} - \frac{\lambda}{(1 - \lambda)}x \\ &= tz + (1 - t)x \end{aligned}$$

where $t = \frac{1}{(1 - \lambda)} \in (0, 1)$. Therefore we have that:

$$f(y) = tf(z) + (1 - t)f(x)$$

which implies that:

$$\begin{aligned} f(z) &= \frac{f(y)}{t} - \frac{(1 - t)}{t}f(x) \\ &= \lambda f(x) + (1 - \lambda)f(y) \end{aligned}$$

The final case in which $\lambda < -1$ follows directly from the previous two. Therefore, f is affine.