Math, Problem Set #3, Spectral Theory

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Due Monday, July 10th

Exercise 2

From the previous homework], we know that we can represent the derivative operator as matrix

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the roots (eigenvalues) of the characteristic polynomial of D are given by

$$\det(D - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{bmatrix} = 0$$

that is $-\lambda^3 = 0$. Hence, the unique eigenvalue of D is $\lambda = 0$, with algebraic multiplicity three and geometric multiplicity given by $Dim(Null(D))$. Since the matrix representation of the derivative operator maps vectors of the form $[a \quad b \quad c]$ (where $a, b, c \in R$) to vectors of the form $[b \quad 2c \quad 0]$, $p = [a \quad b \quad c] \in Null(D) \iff b = c = 0$. Thus, $Null(D) = Span(e_1)$ and $Dim(Null(D)) = 1$.

Exercise 4

(i)

Let A be an Hermitian matrix, i.e.,

$$\begin{bmatrix} a & x + iy \\ x - iy & b \end{bmatrix}.$$

By (4.3) we obtain that the characteristic polynomial of A is given by

$$p(\lambda) = \lambda^2 - (a + b)\lambda + ab - x^2 - y^2.$$

Therefore,

$$\Delta(p(\lambda)) = (a+b)^2 - 4(ab - x^2 - y^2) = a^2 + b^2 + 2ab - 4ab + 4x^2 + 4y^2 = (a-b)^2 + 4x^2 + 4y^2 \geq 0,$$

i.e., the discriminant is positive and the roots of the characteristic polynomial are all real.

(ii)

Let A be an Hermitian matrix, i.e.,

$$\begin{bmatrix} ix & a+ib \\ -a+ib & iy \end{bmatrix}.$$

By (4.3) we obtain that the characteristic polynomial of A is given by

$$p(\lambda) = \lambda^2 - (ix+iy)\lambda - xy + a^2 + b^2$$

Therefore,

$$\Delta(p(\lambda)) = -x^2 - y^2 + 2xy - 4a^2 - 4b^2 = -(x-y)^2 - 4(a^2+b^2) < 0$$

i.e., the discriminant is negative and the roots of the characteristic polynomial are all imaginary.

Exercise 6

Let A be an upper triangular matrix, i.e.,

$$A = \begin{bmatrix} \lambda_1 & * & * & ... & * \\ 0 & \lambda_2 & * & ... & * \\ 0 & 0 & \lambda_3 & ... & * \\ 0 & 0 & 0 & ... & * \\ 0 & 0 & ... & 0 & \lambda_n \end{bmatrix}$$

Then, λ is an eigenvalue of $A \iff (A - \lambda I)$ is singular. Thus,

$$A - \lambda I = \begin{bmatrix} \lambda_1 - \lambda & * & * & ... & * \\ 0 & \lambda_2 - \lambda & * & ... & * \\ 0 & 0 & \lambda_3 - \lambda & ... & * \\ 0 & 0 & 0 & ... & * \\ 0 & 0 & ... & 0 & \lambda_n - \lambda \end{bmatrix}$$

This matrix is again upper triangular. An upper triangular matrices is singular if and only if the product of its diagonal entries equals zero. Hence, λ is an eigenvalue of A if and only if it satisfies

$$(\lambda_1 - \lambda)(\lambda_2 - \lambda)...(\lambda_n - \lambda) = 0.$$

Each root of this equation is an eigenvalue of A. This implies that $\lambda = \lambda_i$ for all $i = 1, 2, 3, ..., n$.

Exercise 8

(i)

We know that every orthonormal set is also linearly independent. We showed in Exercise (3.8) that the set S is orthonormal, and hence linearly independent. Since $V = Span(S)$, we have that S is a basis for V.

(ii)

The matrix representation D of the derivative operator is completely determined by the map it operates on the basis S. In particular, it has to map the vector of coefficients $[a, \quad b, \quad c, \quad d]$ into the vector $[-b, \quad a, \quad -2d, \quad 2c]$. Thus,

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

(iii)

Clearly, we can pick $Span\{sin(x), cos(x)\}$ and $Span\{sin(2x), cos(2x)\}$.

Exercise 13

We need to find an eigenbasis of A. Thus,

$$\det(A - \lambda I) = (0.8 - \lambda)(0.6 - \lambda) - 0.08 = \lambda^2 - 1.4\lambda + 0.4 = 0$$

The eigenvalues of A are $\{1, 0.4\}$, and with some calculations we also find that the corresponding eigenvectors are $[1, \quad 2]^T$ and $[1, -1]^T$. Thus, let $P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$. Then,

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$$

where $P^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$.

Exercise15

Since A is semisimple it is also diagonalizable. Hence $A = PDP^{-1}$, where $Disdiagonal$. Then,

$$f(A) = P^{-1}(a_0 + a_1 D + ... + a_n D)P = P^{-1}D'P$$

where D' is still diagonal with entries $f(\lambda_i)$, i.e., $f(\lambda_i)$ are the eigenvalues of $f(A)$.

Exercise 16

(i), (ii)

Using proposition (4.3.10), we know that

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} PD^nP^{-1} = \lim_{n\to\infty} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 0.4^n \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

The answer is not dependent on the type of norm that we use.

(iii)

Using Theorem (4.3.12) we know that the eigenvalues of $f(A) = 3I + 5A + A^3$ are $f(1) = 9$ and $f(0.4) = 1.2 + 2 + 0.064 = 3.264$.

Exercise 18

We know that λ is an eigenvalue of $A \iff$ there exist a non-zero column vector x such that $Ax = \lambda x$, or (since A and A^T have the same characteristic polynomial) $A^T x = \lambda x$. Taking the transpose of both sides in the last equation, we get $x^T A = \lambda x^T$. Thus, λ is an eigenvalue of $A \iff$ there exist a non-zero row vector x such that $x^T A = \lambda x^T$.

Exercise 20

By assumption, we have that A is Hermitian, i.e., $A^H = A$ and that A is orthonormally similar to B, i.e., there exists an orthonormal matrix Q such that $A = Q^H BQ \iff B = QAQ^H$. Then,

$$B^H = [(QA)Q^H]^H = Q(QA)^H = QA^H Q^H = QAQ^H = B$$

Therefore, B is also Hermitian.

Exercise 24

If A is an Hermitian matrix, then by the First Spectral Theorem (and Corollary) we know that A is orthonormally diagonalizable and has all real eigenvalues, i.e., $D = Q^H AQ$, where D is a real diagonal matrix of eigenvalues of A. Let the vector $x = Qc$, where Q is an eigenbasis for C^n. Then,

$$\rho(x) = \frac{\langle x, Ax\rangle}{\|x\|^2} = \frac{c^H Q^H AQc}{c^H c} = \frac{c^H Dc}{c^H c} = \frac{\sum \lambda_i c_i^2}{\sum c_i^2}$$

which is real, since all the eigenvalues of A Hermitian matrix are real. Given that all eigenvalues of a Skew Hermitian matrix are all imaginary, we can go over the same proof for A Skew Hermitian to show that in this case the Rayleigh Quotient takes only imaginary values.

Exercise 25

Exercise 27

Exercise 28

Exercise 31

Exercise 32

Exercisen 33

Exercise 36

Exercise 38