Optimal Asset Allocation in a Single Stock Portfolio

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Optimal Allocation without Transaction Costs

- At time t, an investor holds a portfolio composed of shares of stock worth a value of $Y(t)$ and cash worth a value of $X(t)$. The total value of the portfolio is $Z(t) = X(t) + Y(t)$.

- Specifically, we assume that the stock evolves by geometric Brownian Motion with constant expected return μ and volatility σ, while the cash position has a constant return rate r. In other words,

$$dY(t) = \mu Y(t)dt + \sigma Y(t)dB(t)$$

$$dX(t) = rX(t)dt$$

Optimal Allocation without Transaction Costs

- The investor has to choose an adapted investment strategy, $Y(t)$, that maximizes the expected utility at a final time, T:

$$f(z,t) = \sup E_{z,t}[U(Z(T))],$$

where $Z(t) = X(t) + Y(t) = z$, for $t \leq T$.

- Applying Itô to $Z(t)$, we obtain

$$dZ(t) = [(\mu - r)Y(t) + rZ(t)]dt + \sigma Y(t)dB(t)$$

Optimal Allocation without Transaction Costs

- Our goal is to find the optimal trading strategy and to evaluate the associated "value function" $f(z,t)$ – the optimal utility of final time wealth, if the system starts in state z at time t.
- The value function will solve a non-linear Partial Differential Equation (the so called "Hamilton Jacobi Bellman" equation).
- Underlying the derivation of the HJB equation is the **Dynamic Programming Principle**: an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

No TC: Deriving the HJB Equation (Heuristically)

- First, notice that since trading is free, our investor will choose a trading strategy that is based solely on the total worth of his portfolio. This means that $Y(t) = y(Z(t), t)$, for some trading strategy function y.

- The Dynamic Programming Principle implies that

$$f(z,t) = \sup E_{z,t}[f(z', t')], \tag{1}$$

where $t' = t + dt$ and $z' = Z(t')$. Assuming that $f(z, t)$ is sufficiently smooth (as we will see it is often not the case), we can apply standard rules from Itô Calculus to find an expression for the differential of f.

No TC: Deriving the HJB Equation (Heuristically)

- Applying Itô to the value function and using our expression for $dZ(t)$, we obtain

$$df(z,t) = \{f_t + [(\mu - r)Y(t) + rZ(t)]f_z + \frac{1}{2}\sigma^2 Y(t)^2 f_{zz}\}dt + f_z \sigma Y(t) dB(t)$$

Thus, since the expectation of the stochastic integral equals zero

$$E_{z,t} f(z', t') = f(z,t) + \int_t^{t+dt} \{f_t + [(\mu - r)Y(s) + rZ(s)]f_z$$

$$+ \frac{1}{2}\sigma^2 Y(s)^2 f_{zz}\} ds$$

$$= f(z,t) + \{f_t + [(\mu - r)Y(t) + rZ(t)]f_z + \frac{1}{2}\sigma^2 Y(t)^2 f_{zz}\}dt + o(dt)$$

No TC: Deriving the HJB Equation (Heuristically)

- Now we plug the expression for $E_{z,t} f(z', t')$ into (1). After cancelling the $f(z, t)$ term, dividing by dt, and taking the limit, we finally obtain

$$0 = \sup_{y} \{ f_t + [(\mu - r)y + rZ] f_z + \frac{1}{2} \sigma^2 y^2 f_{zz} \}, \tag{2}$$

  which is the HJB equation for the problem with no transaction costs.

- By taking the derivative with respect to y of the right hand side of (2), we get the formula for the optimal trading strategy (the so called Merton value):

$$m(Z, t) = -\frac{(\mu - r) f_z}{\sigma^2 f_{zz}}$$

  (Note that m is positive since $f_z > 0$ and $f_{zz} < 0$, which can be established from the properties of utility functions).

Optimal Allocation with Transaction Costs

- The Merton strategy (the $m(Z,t)$ above) implies that the investor will trade to keep the same fixed proportion of stock and cash at each instant in time.
- To the idealized model, where trading is free, we can add proportional transaction costs, where the investor loses a fraction proportional to a small parameter, ϵ, of the value of each trade she makes.
- In this case, "trading at all times" is no longer an optimal solution, as it would cause the investor to incur in infinite losses. Here, the optimal strategy allows the portfolio to move freely within the interior of a **No Transaction Region**, in which no trade occurs.
- This region stretches a distance γ (proportional to the transaction cost parameter $\epsilon^{1/3}$), and is centered around the ideal Merton strategy. (See for example Goodman, Ostrov [1], Shreve, Soner [2]).

Optimal Allocation with Transaction Costs

- Let $M(t)$ be the dollar amount of cash spent buying stock up to time t, and $L(t)$ be the dollar worth of all stock sold up to time t. Let ϵ be the transaction cost parameter.
- When the porfolio is in the "buy region", then

$$Y(t) \longrightarrow Y(t) + dM(t)$$

$$X(t) \longrightarrow X(t) - (1+\epsilon)dM(t)$$

- When the portfolio is in the "sell region", then

$$Y(t) \longrightarrow Y(t) - dL(t)$$

$$X(t) \longrightarrow X(t) + (1-\epsilon)dL(t)$$

Optimal Allocation with Transaction Costs

- In this setting, the controls are $dM(t)$ and $dL(t)$, that is to say how much stock to sell/buy at any instant in time. Therefore, the differentials of Y and X are defined as

$$dY(t) = \mu Y(t)dt + \sigma Y(t)dB(t) + dM(t) - dL(t)$$

$$dX(t) = rX(t)dt - (1+\epsilon)dM(t) + (1-\epsilon)dL(t)$$

  Notice that a unit increase in M removes $(1+\epsilon)$ units of X but adds only a unit to Y; that is, there is a loss of ϵ dollars for every dollar spent buying stock. The same ϵ loss occurs for every dollar of stock sold.

TC case: Deriving the HJB Equation (Heuristically)

- In the transaction costs case, the investor needs to know where his portfolio is w.r.t. the idealized Merton Strategy in the (X, Y) - plane.
  - When the value of the porftolio at time t (described by the tuple $\{X(t), Y(t)\}$) deviates "too much" from the Merton Strategy m, the investor will do a trade to bring his portfolio back to the boundary of the No Transaction Region.
- The transaction costs case value function corresponds to

$$f(x, y, t) = \sup E_{x,y,t}[U(X(T) + Y(T))]$$

where $x = X(t)$, $y = Y(t)$, for $t \leq T$.

TC case: Deriving the HJB Equation (Heuristically)

- We are now ready to derive the corresponding HJB equation. As before, we apply the Dynamic Programming Principle to get

$$f(x, y, t) = \sup E_{x,y,t}[f(x', y', t')], \tag{3}$$

where$t' = t + dt$, $x' = X(t')$, and $y' = Y(t')$.

- Assuming that f is sufficiently smooth (indeed, it is not, more details below), we apply Itô to it and get

$$df(x, y, t) = \left(f_t + \mu Y(t) f_y + rX(t) f_x + \frac{\sigma^2}{2} Y(t)^2 f_{yy} \right) dt + \tag{4}$$

$$+dM(t)\left(f_y - f_x(1+\epsilon) \right) + dL(t)\left(-f_y + f_x(1-\epsilon) \right) + \sigma Y(t) f_y dB(t)$$

TC case: Deriving the HJB Equation (Heuristically)

- The procedure is now the same as before. Take expectations in (4) to get an expression for $E_{x,y,t}[f(x', y', t')]$. Then plug this into (3), divide by dt and take the limit to obtain

$$0 = \max_{dL,dY} \Bigg\{ f_t + \mu y f_y + r x f_x + \frac{\sigma^2}{2} y^2 f_{yy} + \tag{5}$$

$$+dM\Bigg(f_y - f_x(1+\epsilon)\Bigg) + dL\Bigg(- f_y + f_x(1-\epsilon)\Bigg)\Bigg\}$$

- Note that since $dM, dL \geq 0$, if either $f_y - f_x(1+\epsilon) > 0$ or $-f_y + f_x(1-\epsilon) > 0$, then the optimal action would be to set dM or dL to infinity. Therefore, we must have both $f_y - f_x(1+\epsilon) \leq 0$ and $-f_y + f_x(1-\epsilon) \leq 0$.

TC case: Deriving the HJB Equation (Heuristically)

- Let $L_0 f = f_t + \mu y f_y + rx f_x + \frac{\sigma^2}{2} y^2 f_{yy}$. Then, equation (5) can be summarized with the following HJB variational inequality:

$$0 = \max \left\{ L_0 f, f_y - f_x(1+\epsilon), -f_y + f_x(1-\epsilon) \right\} \tag{6}$$

- The set $\{(x, y) : L_0 f = 0\}$ corresponds to the "No Transaction Region".
- The set $\{(x, y) : f_y - f_x(1+\epsilon) = 0\}$ corresponds to the "Buy Region".
- The set $\{(x, y) : -f_y + f_x(1-\epsilon) = 0\}$ corresponds to the "Sell Region".

Viscosity Solutions to the HJB Equation

- In the derivation of the previous PDE's, we assumed that the value function f has the necessary number of derivatives for the argument to work. Nonetheless, it turns out that solutions to the resulting HJB equations are not everywhere differentiable.

- We need to rely on the concept of **Viscosity Solutions**, introduced by Pierre-Louis Lions and Michael G. Crandall as a generalization of the classical concept of what is meant by a 'solution' to a partial differential equation.

- If Φ is a viscosity solution to (6), Φ need not be everywhere differentiable. There may be points where $D\Phi, D^2\Phi$ do not exists, and yet Φ satisfies the equation in an appropriate generalized sense (Φ will also uniquely identify the value function of our problem).

Viscosity Solutions to the HJB Equation

- The scalar equation $F(x, u, Du, D^2u) = 0$ in a domain Ω is defined to be **Degenerate Ellipitc** if for any two symmetric matrices X, Y such that $X - Y$ is positive definite, and any values $x \in \Omega, u \in \mathbb{R}$, and $p \in \mathbb{R}^n$, we have the inequality $F(x, u, p, X) \geq F(x, u, p, Y)$.
- An upper semicontinuous function u in Ω is defined to be a **subsolution** of a degenerate elliptic equation in the viscosity sense if for any point $x_0 \in \Omega$ and any C^2 function ϕ such that $\phi(x_0) = u(x_0)$, and $\phi \geq u$ in some neighborhood of x_0, we have $F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0$.
- A lower semicontinuous function u in Ω is defined to be a **supersolution** of a degenerate elliptic equation in the viscosity sense if for any point $x_0 \in \Omega$ and any C^2 function ϕ such that $\phi(x_0) = u(x_0)$, and $\phi \leq u$ in some neighborhood of x_0, we have $F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0$.

Viscosity Solutions to the HJB Equation

- A continuous function u is a **Viscosity Solution** of the PDE if it is both a supersolution and a subsolution.
- NOTE: If u is a C^2 function that satisfies $F(x, u, Du, D^2u) = 0$ at all points in the domain, then u is also a solution in the viscosity sense. Viceversa, if u is a viscosity solution, then $F(x, u, Du, D^2u) = 0$ must hold at every point in the domain where u is C^2.

Back to the HJB Equation for the TC case

- It turns out that the value function f for the problem with transaction costs is the **unique constrained viscosity solution** of (6), which we will rewrite here for the reader

$$0 = \max \left\{ L_0 f, f_y - f_x(1+\epsilon), -f_y + f_x(1-\epsilon) \right\},$$

where $L_0 f = f_t + \mu y f_y + r x f_x + \frac{\sigma^2}{2} y^2 f_{yy}$ (see Shreve, Soner [2] for more details).

Numerical Schemes for the HJB Equation

- Consider the parabolic PDE for $(t, x) \in (0, T] \times \mathbb{R}^n$

$$u_t + F(t, x, u, Du, D^2u) = 0 \tag{7}$$

$$u(0, x) = u_0(x)$$

where F is degenerate elliptic. A numerical scheme is an equation of the form

$$S(h, t, x, u_h(t, x), [u_h]_{t,x}) = 0 \ \text{ for } (t, x) \in G_h - \{t = 0\}$$

$$u_h(0, x) = u_{0,h}(x) \ \text{ in } G_h \cap \{t = 0\}$$

where $h = (\Delta t, \Delta x)$ and $G_h = \Delta t\{0, 1, ..., n_T\} \times \Delta x Z^N$. u_h stand for the approximation of u and $[u_h]_{t,x}$ represent the value of u_h at points other than (t, x).

Numerical Schemes for the HJB Equation

- Barles and Souganidis[3] showed that if the numerical scheme defined above is **Stable**, **Consistent** and **Monotone**, then its solution u_h converges locally uniformly to the unique viscosity solution of (7).
- The scheme is **Monotone** if for $u \leq v$,

$$S(h, t, x, r, u) \geq S(h, t, x, r, v)$$

- The scheme is **Consistent** if for every smooth function $\Phi(t, x)$,

$$S(h, t, x, \Phi_h(t, x), [\Phi_h]_{t,x}) \longrightarrow_{h \to 0} \Phi_t + F(t, x, \Phi, D\Phi, D^2\Phi) = 0$$

- The scheme is **Stable** if for every $h > 0$ the scheme has a solution u_h which is uniformly bounded indipendently of h.

Numerical Schemes for the HJB Equation

- Let $U(x_i, y_j, t_n) = U^n_{i,j}$, where $x_i = i\Delta x$, $y_j = j\Delta y$, $t_n = n\Delta t$. If we construct an explicit method of the form

$$U^{n+1}_{i,j} = G(h, U^n_{i,j}, \{U^n_{p,q}\}_{p,q\neq i,j}),$$

the monotonicity requirement amounts to have

$$\frac{\partial G}{\partial U^n_{i,j}} > 0, \quad \frac{\partial G}{\partial U^n_{p,q}} > 0 \text{ for all } (p,q) \neq (i,j)$$

that is, all the coefficients must be positive.

Numerical Scheme for the HJB Variational Inequality (6)

- We had the following:

$$0 = \max \left\{ L_0 f, L_1 f, L_2 f \right\},$$

where $L_1 f = f_y - f_x(1+\epsilon)$, $L_2 f = -f_y + f_x(1-\epsilon)$, and $L_0 f = f_t + \mu y f_y + r x f_x + \frac{\sigma^2}{2} y^2 f_{yy}$.

- Following Barles and Souganidis[3], we want to construct a numerical method that is stable, consistent and monotone. To do so, we will approximate the first order operators L_1 and L_2 by upwind finite difference schemes. As for the second order operator L_0, we will take an explicit time step and discretize the rest of the terms also "upwind".

Numerical Scheme for the HJB Variational Inequality (6)

- We will restrict to a rectangular domain $D \subset \mathbb{R}^2 \times \mathbb{R}$

$$D = [0, (M-1)\Delta x] \times [0, (L-1)\Delta y] \times [0, (N-1)\Delta t]$$

  where M, L are the number of grid points on the x and y axis, and $\Delta x, \Delta y > 0$ are the mesh sizes. Δt is the time step size and N is the number of steps.

- We now define the first order differences

$$D_x^+ V_{i,j}^n = \frac{V_{i+1,j}^n - V_{i,j}^n}{\Delta x}, \; D_y^+ V_{i,j}^n = \frac{V_{i,j+1}^n - V_{i,j}^n}{\Delta y}$$

$$D_x^- V_{i,j}^n = \frac{V_{i,j}^n - V_{i-1,j}^n}{\Delta x}, \; D_y^- V_{i,j}^n = \frac{V_{i,j}^n - V_{i,j-1}^n}{\Delta y}$$

Numerical Scheme for the HJB Variational Inequality (6)

- Inside the domain, the first order operators L_1 and L_2 are approximated by

$$S_1(D_x^- V_{i,j}^n, D_y^+ V_{i,j}^n) = D_y^+ V_{i,j}^n - (1+\epsilon) D_x^- V_{i,j}^n$$

$$S_2(D_x^+ V_{i,j}^n, D_y^- V_{i,j}^n) = -D_y^- V_{i,j}^n + (1-\epsilon) D_x^+ V_{i,j}^n$$

- I am still working on the upwind discretization of the operator L_0. For the sake of the exposition of the computational algorithm, let's assume that L_0 is approximated by

$$V_{i,j}^{n+1} = S_0(V_{i,j}^n, D_x V_{i,j}^n, D_y V_{i,j}^n, D_y^2 V_{i,j}^n)$$

where $D_x V_{i,j}^n, D_y V_{i,j}^n, D_y^2 V_{i,j}^n$ are finite differences for the x and y derivatives.

Numerical Scheme for the HJB Variational Inequality (6)

Algorithm:

- 1) Start from some initial condition for $\{V^0_{i,j}\}$. Take a time step with S_0 to get $\{V^1_{i,j}\}$.
- 2) Interpolate the values of V along the "trading vectors" given by S_1 and S_2. This is the same as "upwind" discretization for S_1 and S_2.
- 3) At $V^1_{i,j}$, see if the two interpolated values are greater than $V^1_{i,j}$. If yes, substitute $V^1_{i,j}$ with the largest of the two. If not, leave $V^1_{i,j}$.
- 4) Take another time step with S_0 and repeat.

References

1 J. Goodman, D. N. Ostrov, "Balancing Small Transaction Costs with Loss of Optimal Allocation in Single and Multiple Stock Portfolios"

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3 Barles, G., Daher, C. and Romano, M., Convergence of Numerical Schemes for Parabolic Equations Arising in Finance Theory

4 A. Tourin, T. Zariphopoulou, "Numerical Schemes for Investment Models with Singular Transactions"