THE EFFECT OF SMALL TRANSACTION COSTS ON SINGLE AND MULTIPLE STOCK PORTFOLIOS

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Abstract.

1. Introduction.

2. Single stock portfolio.

2.1. **Merton problem.** We summarize the analysis for the optimal investment problem when transaction costs are zero (the well-known Merton problem). In this setting, no cost is incurred for selling and buying the stock, and our portfolio position can thus be freely rebalanced at any instant in time. More precisely, at time t we hold a portfolio composed of shares of a stock worth Y(t) dollars and a cash amount of X(t) dollars. The total worth of the portfolio is denoted by $Z(t) = X(t) + Y(t)$. We further assume that the stock's dollar value evolves according to a geometric Brownian motion with constant expected return μ and volatility σ, while the cash amount $X(t)$ can be invested at a constant return rate r. In other words,

$$dX(t) = rX(t)dt, \tag{2.1}$$

$$dY(t) = \mu Y(t)dt + \sigma Y(t)dB(t), \tag{2.2}$$

where $B(t)$ is a standard Brownian motion. The differential for $Z(t)$, the total worth of our portfolio, is given by

$$dZ(t) = [(\mu - r)Y(t) + rZ(t)]dt + \sigma Y(t)dB(t). \tag{2.3}$$

If our portfolio is worth z dollars at time t, we want to find an adapted trading strategy $\{Y(s)\}_{t\leq s\leq T}$ which maximizes the expected utility of $Z(T)$, the total dollar value of the portfolio at a subsequent time T. We can denote the maximum expected utility by f, that is,

$$f(t,z) = \sup\left\{E_{t,z}[U(Z(T))]\right\}, \tag{2.4}$$

where the supremum is taken over all admissible trading strategies. Because trading is free, we can rebalance our portfolio at any instant in time by purchasing or selling the stock as it is best. The decomposition of our portfolio into cash and stock is therefore irrelevant, and the optimal trading strategy will be a function of $Z(t)$ only; that is $Y(t) = y(t, Z)$, for some function y. The dynamic programming principle implies that

$$f(t, z) = \sup \left\{ E_{t,z}[f(t', z')] \right\}, \tag{2.5}$$

where $t' = t + dt$, $z' = Z(t')$ and the supremum is taken over all admissible trading strategies $\{Y(s)\}_{t \le s \le t'}$. We can now use equation (2.5) to informally derive the Hamilton-Jacobi-Bellman equation satisfied by f. If we assume that the value function f is sufficiently smooth, we can express its differential by a straightforward application of Itô's lemma. We find that

$$df = \{f_t + [(\mu - r)Y + rZ]f_z + \frac{1}{2}\sigma^2 Y^2 f_{zz}\}dt + \sigma f_z Y dB, \tag{2.6}$$

where we used the standard Itô calculus fact $\langle B \rangle_t = t$ (for a generic stochastic process $X(t)$, we use $\langle X \rangle_t$ to denote its quadratic variation). Equations (2.5) and (2.6), together with $E_{z,t}[\int_t^{t'} (...) dB(s)] = 0$ (for sufficiently integrable integrand, the Itô integral is a martingale), imply the following equalities

$$\begin{aligned} f(t, z) &= \sup\{E_{t,z}[f(t', z')]\} \\ &= \sup\{[f(t, z) + f_t + [(\mu - r)y + rz]f_z + \frac{1}{2}\sigma^2 y^2 f_{zz}]dt + o(dt)\}, \end{aligned}$$

where the supremum is understood to be taken over y (over trading strategies on the infinitesimal time-interval $[t, t + dt]$). We can now pull $f(t, z)$ out of the supremum (so that it cancels with the left side), divide by dt and take the limit as $dt \to 0$. We conclude that the Hamilton-Jacobi-Bellman equation for the value function is

$$0 = \sup_y \{f_t + [(\mu - r)y + rz]f_z + \frac{1}{2}\sigma^2 y^2 f_{zz}\}, \tag{2.7}$$

subject to terminal condition $f(T, z) = U(z)$. By differentiating the right hand side with respect to y and setting the derivative to zero, we find an expression for the optimal trading strategy, the so called Merton strategy:

$$m(t, Z) = -\frac{(\mu - r) f_z(t, Z)}{\sigma f_{zz}(t, Z)}. \tag{2.8}$$

This expression implies that the optimal strategy is to incessantly trade in order to keep a fixed fraction of total wealth $Z(t)$ invested in the stock. Note that m is always positive since $f_z > 0$ and $f_{zz} < 0$ (these two inequalities follow from the fact that our utility function $U(.)$ is strictly increasing and concave).

2.2. **Trading with transaction costs.** To the idealized Merton model, where trading is free, one can add proportional transaction costs, where we lose a fraction proportional to a small constant ϵ of each trade we make. Let $M(t)$ be the dollar amount spent on buying the stock up to time t, and $L(t)$ be the dollar worth of all stock sold up to time t. We update the differentials in (2.1) and (2.2) by

$$dX(t) = rX(t)dt + (1-\epsilon)dL(t) - (1+\epsilon)dM(t), \tag{2.9}$$

$$dY(t) = \mu Y(t)dt + \sigma Y(t)dB(t) + dM(t) - dL(t). \tag{2.10}$$

In this setting, the controls are $dL(t)$ and $dM(t)$, that is to say how much stock to sell or buy (in dollar terms) at instant t. Note that a unit increase in M removes $(1+\epsilon)$ units form X but adds only a unit of Y. Similarly, a unit increase in L removes a unit from Y but adds only $(1-\epsilon)$ units of X. Intuitively, the Merton strategy (2.8) applied to this setting can no longer be an optimal solution; because the Brownian path has an infinite variation, the costs associated with "trading at all times" would quickly outweight the benefits of precisely following the optimal rebalancing prescribed by $m(t, Z)$. We therefore expect transaction costs to induce a narrow region, built around the Merton strategy, in which it is optimal not to trade. We call this region the *hold region*, and identify it by $\mathcal{H}$. The resulting optimal trading strategy is a singular control that keeps our portfolio inside $\mathcal{H}$, and we expect trading to occur only at boundary $\partial\mathcal{H}$ (Shreve and Soner [2] showed that this is the case for the single-stock portfolio problem with power law utility). In presence of transaction costs, the value function f depends on both x and y:

$$f(t,x,y) = \sup\left\{E_{t,x,y}[U(Z(T))]\right\}, \tag{2.11}$$

where this time $Z(T) = X(T) + (1-\epsilon)Y(T)$ (due to the cost of liquidating the stock position at a final time T) and the supremum is taken over the set of increasing sales and purchase processes $\{L(s)\}_{t\le s\le T}$ and $\{M(s)\}_{t\le s\le T}$. For f sufficiently smooth, (2.9), (2.10) and Itô's lemma tell us that

$$\begin{aligned} df &= \left[f_t + rXf_x + \mu Y f_y + \frac{\sigma^2}{2}Y^2 f_{yy}\right]dt \\ &\quad + dM\left[f_y - f_x(1+\epsilon)\right] + dL\left[-f_y + f_x(1-\epsilon)\right] + \sigma Y f_y dB, \end{aligned} \tag{2.12}$$

where we used $\langle B\rangle_t = t$, and $\langle B, M\rangle_t = \langle B, L\rangle_t = 0$ (for any two stochastic processes $X^1(t)$, $X^2(t)$, we use $\langle X^1, X^2\rangle_t$ to denote their covariation). By the martingale optimality principle of stochastic control, the value function f must be (i): a supermartingale for any choice of admissible trading strategies $M(t)$ and $L(t)$, and (ii): a martingale for optimal strategies. Since $M(t)$ and $L(t)$ are increasing processes (that is, $dM \ge 0$ and $dL \ge 0$), it follows that for any choice of strategies we have

$$f_y - f_x(1+\epsilon) \le 0, \tag{2.13}$$
$$-f_y + f_x(1-\epsilon) \le 0, \tag{2.14}$$

and to ensure the non-positivity of the drift term we also need

$$f_t + rxf_x + \mu y f_y + \frac{\sigma^2}{2} y^2 f_{yy} \le 0. \tag{2.15}$$

Given optimal trading strategies, when the portfolio is in the interior of the hold region $\mathcal{H}$ (that is, when $dM = dL = 0$), we must have equality in (2.15). As soon the portfolio reaches the *buy boundary* of $\mathcal{H}$ (where $dM > 0$) we must have equality in (2.13), and when it reaches the *sell boundary* of $\mathcal{H}$ (where $dL > 0$) we must have equality in (2.14). Hence, we have derived the Hamilton-Jacobi-Bellman equation satisfied by f:

$$\begin{aligned} 0 = \max\Big\{ & f_t + rxf_x + \mu y f_y + \frac{\sigma^2}{2} y^2 f_{yy}, \\ & f_y - f_x(1+\epsilon),\ -f_y + f_x(1-\epsilon) \Big\}, \end{aligned} \tag{2.16}$$

with terminal condition $f(T, x, y) = U(x + (1-\epsilon)y)$. (We note that this calculation is of course informal, as the value function for this singular control problem need not be everywhere $C^{1,2}$. The rigorous analysis uses viscosity-solution methods; for an application of such methods see Shreve and Soner [2]). We are not aware of any explicit solution to equation (2.16), even for the simplest models. As we saw, adding transaction costs leads to an increase in the number of state variables by one and introduces a free-boundary – the hold region $\mathcal{H}$ is unknown and has to be determined as part of the solution. To simplify the following analysis, it is convenient to introduce a change of variables $(X, Y) \to (\xi, Z)$, where

$$\xi(t) = Y(t) - m(Z(t), t), \tag{2.17}$$
$$Z(t) = X(t) + Y(t). \tag{2.18}$$

In particular, ξ represents the deviation of the stock position from the idealized Merton strategy (which was defined in (2.8)), while Z is the usual total dollar worth of our portfolio. As discussed above, we expect the optimal trading strategy to be a singular control which works to keep the portfolio inside the hold region $\mathcal{H}$. In terms of the new variables, this amounts to keeping the deviation from the Merton strategy, ξ, sufficiently small. Hence, guided by previous papers, we suppose that there exist small numbers $a(z,t) < 0$ and $b(z,t) > 0$ such that $dM > 0$ only when $\xi(t) = a(z,t)$, and $dL > 0$ only when $\xi(t) = b(z,t)$; in this case, the hold region $\mathcal{H}$ is fully described by $\xi(t) \in \big[a(z,t), b(z,t)\big]$, and trading merely takes place at its boundaries. We can now easily state the free-boundary problem for the value function in terms of the new variables, ξ and Z.

We have

$$f(t,\xi,z) = \sup\big\{E_{t,\xi,z}[U(Z(T))]\big\}, \tag{2.19}$$

subject to the differentials of the changed variables in (2.17) and (2.18). In the interior of the hold region $\xi(t) \in \big(a(z,t), b(z,t)\big)$, the inequality in (2.15) must be an equality; in new variables, this means that

$$\begin{aligned}0 = f_t \ &+ \left[\mu(\xi+m) - m_t - m_z(rz + (\mu-r)(\xi+m) - \frac{1}{2}\sigma^2 m_{zz}(\xi+m)^2\right] f_\xi \\ &+ \left[rz + (\mu-r)(\xi+m)\right] f_z + \left[\frac{1}{2}\sigma^2(\xi+m)^2(1-m_z)^2\right] f_{\xi\xi} + \left[\frac{1}{2}\sigma^2(\xi+m)^2\right] f_{zz} \\ &+ \left[\sigma^2(\xi+m)^2(1-m_z)\right] f_{\xi z}.\end{aligned} \tag{2.20}$$

At the buy boundary inequality (2.13) holds with equality, while at the sell boundary inequality (2.14) holds with equality; this respectively leads to boundary conditions

$$0 = -\epsilon f_z + \epsilon m_z f_\xi + f_\xi, \qquad \text{for } \ \xi(t) = a(z,t), \tag{2.21}$$

$$0 = -\epsilon f_z + \epsilon m_z f_\xi - f_\xi, \qquad \text{for } \ \xi(t) = b(z,t). \tag{2.22}$$

Lastly, we have a terminal condition given by $f(T,\xi,z) = U(z)$. Problem (2.19) is equivalent to maximizing the value function f subject to constraints (2.20), (2.21), (2.22), plus the terminal condition. Here, maximizing f means that we optimize over the free-boundary described by the functions $a(z,t)$ and $b(z,t)$, for all (z,t). We note that conditions (2.21), (2.22) are not sufficient to identify the optimal free-boundary, as these can be matched for any trading boundaries $a(z,t), b(z,t)$. For this reason, in order to fully specify the solution to the optimization problem, we need second derivative conditions at the boundary; these can be found by a standard perturbation argument from the calculus of variations. Suppose $a(z,t)$, $b(z,t)$ are the optimal trading boundaries, and define

$$\mathcal{H}(t) = \big\{\big(\xi,z\big) \ \mid \ \xi \in \big[a(z,t), b(z,t)\big]\big\}. \tag{2.23}$$

In other words, $\mathcal{H}(t) \subset \mathbb{R}^2$ is the set of points in the optimized hold region at a fixed time t. Also let $\mathcal{H} = \cup_{t\le T}(t,\mathcal{H}(t))$. We can now consider the perturbed hold region

$$\mathcal{H}^\delta(t) = \big\{\big(\xi,z\big) + \delta\ h(t,\xi,z) \ \mid \ \big(\xi,z\big) \in \mathcal{H}(t)\big\}, \tag{2.24}$$

where $\delta > 0$ is a small parameter and $h : \mathcal{H} \to \mathbb{R}^2$ is a fixed arbitrary vector field with $h(T,.) = 0$; that is, we move each point $\big(\xi,z\big)$ in $\mathcal{H}(t)$ by an amount $\delta\ h(t,\xi,z)$, and the perturbed hold region coincides with the optimized one at final time T.

Correspondingly, we define $\mathcal{H}^\delta = \cup_{t\leq T}(t, \mathcal{H}^\delta(t))$. For δ sufficiently small, there exists a suboptimal value function $f(t,\xi,z,\delta)$ which satisfies PDE (2.20) in $\mathcal{H}^\delta$, boundary conditions (2.21), (2.22) on $\partial\mathcal{H}^\delta(t)$, and the terminal condition. Note that the parabolic PDE (2.20) is subject to the maximum principle, hence the first variation of the perturbed value function must vanish at $\delta = 0$; that is,

$$\dot{f}(t,\xi,z) = \frac{d}{d\delta} f(t,\xi,z,\delta)\Big|_{\delta=0} = 0 \tag{2.25}$$

The goal is to differentiate boundary conditions (2.21),(2.22) with respect to δ and use (2.25) to find the second derivative conditions. For brevity of notation, use $a_\delta = \big(\xi,z\big) + \delta\ h(t,\xi,z)$ to denote points on the perturbed *buy* boundary, and $b_\delta = \big(\xi,z\big) + \delta\ h(t,\xi,z)$ to denote points on the perturbed *sell* boundary. (2.21) and (2.22) can be rewritten in vector notation as

$$\begin{aligned} 0 &= \big(\epsilon m_z(t,a_\delta) + 1, -\epsilon\big) \cdot \nabla f(t,a_\delta,\delta), \\ 0 &= \big(\epsilon m_z(t,b_\delta) - 1, -\epsilon\big) \cdot \nabla f(t,b_\delta,\delta). \end{aligned}$$

We now differentiate the last two equations with respect to δ and take the limit as $\delta \to 0$. By the chain and product rules, together with (2.25), we obtain

$$\begin{aligned} 0 &= \nabla\big[\big(\epsilon m_z(t,a_0) + 1, -\epsilon\big) \cdot \nabla f(t,a_0)\big] \cdot h(t,a_0), \\ 0 &= \nabla\big[\big(\epsilon m_z(t,b_0) + 1, -\epsilon\big) \cdot \nabla f(t,b_0)\big] \cdot h(t,b_0), \end{aligned}$$

where we used $\frac{d}{d\delta}a_\delta = h(t,a_0)$, $\frac{d}{d\delta}b_\delta = h(t,b_0)$. Since the vector field h was arbitrary, we conclude that for all points on the optimized boundary $\partial\mathcal{H}(t)$ (and for all t), the value function must satisfy

$$0 = \nabla\big[\big(\epsilon m_z(t,z) + 1, -\epsilon\big) \cdot \nabla f(t,\xi,z)\big], \qquad \text{for } \ \xi(t) = a(z,t), \tag{2.26}$$

$$0 = \nabla\big[\big(\epsilon m_z(t,z) + 1, -\epsilon\big) \cdot \nabla f(t,\xi,z)\big], \qquad \text{for } \ \xi(t) = b(z,t). \tag{2.27}$$

These second derivative conditions are often referred as *smooth pasting conditions.* In practice, they can be derived by differentiating the first order conditions (2.21), (2.22) in any direction that is transverse to the boundary. For example, we can differentiate in the ξ direction to obtain

$$0 = -\epsilon f_{\xi z} + \epsilon m_z f_{\xi\xi} + f_{\xi\xi}, \qquad \text{for } \ \xi(t) = a(z,t) \tag{2.28}$$

$$0 = -\epsilon f_{\xi z} + \epsilon m_z f_{\xi\xi} - f_{\xi\xi}, \qquad \text{for } \ \xi(t) = b(z,t). \tag{2.29}$$

References

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