



## The design of equity-indexed annuities

Phelim Boyle<sup>a</sup>, Weidong Tian<sup>b,\*</sup>

<sup>a</sup> Wilfrid Laurier University, Ontario, Canada N2L 3C5

<sup>b</sup> University of Waterloo, Ontario, Canada N2L 3G1

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### ABSTRACT

There is a rich variety of tailored investment products available to the retail investor in every developed economy. These contracts combine upside participation in bull markets with downside protection in bear markets. Examples include equity-linked contracts and other types of structured products. This paper analyzes these contracts from the investor's perspective rather than the issuer's using concepts and tools from financial economics. We analyze and critique their current design and examine their valuation from the investor's perspective. We propose a generalization of the conventional design that has some interesting features. The generalized contract specifications are obtained by assuming that the investor wishes to maximize end of period expected utility of wealth subject to certain constraints. The first constraint is a guaranteed minimum rate of return which is a common feature of conventional contracts. The second constraint is new. It provides the investor with the opportunity to outperform a benchmark portfolio with some probability. We present the explicit form of the optimal contract assuming both constraints apply and we illustrate the nature of the solution using specific examples. The paper focusses on equity-indexed annuities as a representative type of such contracts but our approach is applicable to other types of equity-linked contracts and structured products.

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### 1. Introduction

This paper discusses and critiques equity-linked contracts. These products are important since they constitute a popular class of investment contracts for retail investors. We analyze them from the investor's perspective. This viewpoint contrasts with much of the actuarial literature which is based on the issuer's perspective and focusses on the pricing, hedging and risk management of these contracts. We discuss the optimality of the conventional design of these contracts from the consumer's viewpoint and note that in general the design is inefficient. We propose a new type of product which we call the *generalized contract*. Under a generalized contract an investor has the opportunity to beat a benchmark index with some probability and also has a guaranteed minimum rate of return, irrespective of market performance. We give the optimal contract design of this generalized contract and explore its properties using specific examples.

At this stage, a brief description of equity-indexed annuities may be helpful. In a single premium contract, the customer(investor) pays an initial amount (the premium) to the

insurer. Suppose that the contract matures in say five years. At maturity the payoff is based on the performance of some reference index which could be, for example, a stock market index. The contract participates in the gains (if any) in the reference portfolio during this period. The detailed arrangements of how this participation is calculated vary, but usually the participation has some call option features. In addition these contracts provide a floor of protection if the market does poorly. For example the guaranteed floor may be a return of the initial investment which means that the investor has an embedded put option. In some contracts there is a maximum or cap on the investor's return. The cap can limit the return over the life of the contract or it may operate period by period. In some circumstances the investor has an early redemption option to cancel or surrender the contract. In this paper we use the equity-indexed annuity (EIA) as a representative example of this class of contracts in our analysis. These represent a very popular<sup>1</sup> class of structured products sold by insurance companies in USA with a rich variety of design features.

At first sight the EIA appears to deliver the best of both worlds: an investment that goes up when the stock market goes up but provides a guaranteed floor if the market collapses.

\* Corresponding address: University of Waterloo, Department of Statistics and Actuarial Science, 200 University Avenue West, N2L 3G1 Ontario, Canada. Tel.: +1 519 8854567.

E-mail addresses: [pboyle@wlu.ca](mailto:pboyle@wlu.ca) (P. Boyle), [wdtian@uwaterloo.ca](mailto:wdtian@uwaterloo.ca) (W. Tian).

<sup>1</sup> According to the industry group LIMRA, sales of EIA were around 24 billion US dollars in 2004, up from \$ 14.4 billion in 2003, and the sales volume is even bigger in 2005.

However, these contracts have some features that may reduce their attractiveness to retail investor. First, many contracts have high initial commissions payable to the agent who sells the contract to the investor. Second, the contract design is often complicated making it difficult for the consumer to understand the product. Third the design may not really suit the consumer's needs. One of the main aims of this paper is to address this last feature. Specifically, we discuss how to design a contract that is optimal for the customers under certain assumptions. Our proposed contract is generally more complicated than existing products. However this optimal contract provides a useful benchmark.

There is one aspect of our analysis that deserves some discussion. This arises from our use of a framework where financial institutions such as insurance companies exist and the extent to which we can simultaneously use the assumptions of no-arbitrage and complete markets in the same framework. Strictly speaking, in a world of perfect information, frictionless markets and a complete set of marketed securities there is no role for financial intermediaries. In this world consumers can use the existing securities to maximize their expected utility. However, in practice individual consumers face significant transaction costs and informational costs if they want to trade directly to form their optimal portfolio. Merton<sup>2</sup> has discussed the role of financial intermediaries in this setting and noted that individuals use financial institutions which face lower transaction costs to perform these functions. Indeed as Merton and Bodie (2005) note

... in a modern well developed financial system the lowest cost transactor may have marginal trading costs close to zero and can trade almost continuously. Our approach uses a similar perspective to Merton's. Financial institutions sell contracts to their customers. The institutions are able to replicate payoffs with essentially no transaction costs. However an individual customer may be quite willing to pay a higher premium for the contract than its strict no-arbitrage value because he(she) faces significant transaction costs.

There is an extensive actuarial literature on the pricing, hedging and risk management of these contracts. See for example Biffis and Millossovich (2006), Nielsen (2006), Hardy (2003) and Tiong (2000). This approach is termed the *fair premium* approach and it can be traced back to Black and Scholes (1973), Merton (1973), Brennan and Schwartz (1976), Boyle and Schwartz (1977). Since the present value of the claim is often calculated by employing the no-arbitrage principle, we argue that the (no-arbitrage) present value of the payoff is just the first step in investigating this kind of contract from the consumer's perspective. This follows from the discussion in the previous paragraph. The issuer will be able to charge the customer more than the no-arbitrage value of the contract.

The aim of this paper is to critique the conventional structure and design of EIAs and suggest a possible generalization. In the spirit of the papers of Arrow (1974) and Raviv (1979), we investigate the optimal design of an EIA under the expected utility framework.<sup>3</sup> The expected utility framework is natural for this analysis because the contract is an investment vehicle and it is not reasonable to assume that investors are risk-neutral. Moore and

Young (2005) also use the expected utility approach in the context of the design of a perpetual equity-indexed annuity.

In a conventional EIA, both the guaranteed return and the equity-indexed return are incorporated into one guaranteed payoff. Therefore, the optimal design problem for an EIA becomes a standard optimal portfolio selection problem (See Cox and Huang (1989), Merton (1971) and Pliska (1986)) subject to a minimal guarantee on the terminal wealth. The optimal payoff under the EIA is the optimal terminal wealth in this optimal portfolio selection problem, and it is derived explicitly in Theorem 3.1 of Section 3.4. We show that the current payoff structure of the conventional EIA is not optimal for most investors.<sup>4</sup> We are also able to investigate the impact of different investor risk preferences on the return to the investor.

In the second part of the paper we introduce a new type of equity-linked product which we call a generalized equity index annuity. Boyle and Tian (in press) have derived the explicit solution for the optimal contract in this case under fairly general conditions. Here we develop the analysis using the equity-indexed annuity as the prototype but our results hold for other types of structured products with appropriate modifications. We assume that the preferred contract will maximize the investor's expected utility of terminal wealth subject to two constraints. The first constraint is that there is a minimum guaranteed rate of return on the contract. This constraint is a very pervasive feature in these types of tailored retail products. The second constraint gives the investor an opportunity to outperform some benchmark with a certain probability  $\alpha$ . We provide justification for this type of constraint later but, for now, we note that in the special case when  $\alpha = 1$ , this aspect of the contract corresponds to the equity participation feature of existing contracts. Since we do not account for the mortality and surrender features, the contracts considered in this paper are very similar to structured products<sup>5</sup> which are sold by banks.

Under some plausible assumptions we are able to solve for the optimal contract design of the generalized equity-indexed annuity. The explicit solution depends on the investor's utility function, the chosen benchmark and the confidence level as well as certain capital market parameters. The optimal payoff at maturity displays an interesting dependence on the level of the underlying index. In general there will be discontinuities and payoff need not be an increasing function of the index. The discontinuity arises from the existence of the probabilistic constraint and Basak and Shapiro (2001) document the same type of behavior in their study of VaR.

The rest of the paper is organized as follows. In Section 2, we discuss in some detail a popular type of EIA known as a point-to-point EIA. We examine the valuation and design of this contract from the investor's perspective. We extend the analysis to other types of EIAs in Section 3. Section 4 describes our proposals for a generalized EIA. We also exhibit and discuss the solution to the optimal design of the generalized contract. Section 5 describes the explicit construction of the optimal design for a specific type of generalized contract and gives several numerical examples to help describe its main features. Section 6 concludes the paper. Some technical formulae are given in the Appendices A and B.

<sup>2</sup> Merton and Bodie (2005) state: "But in the presence of substantial informational and transactional costs it is not realistic to posit that the only process for individuals to establish their optimal portfolios is to trade each separate security for themselves directly in the markets. Instead individuals are likely to turn to financial organizations such as mutual funds and pension funds that can provide pooled portfolio management services at a much lower cost than individuals can provide for themselves".

<sup>3</sup> There is also some research on the optimal design without using the utility framework. See Doherty and Eeckhoudt (1995), Doherty and Schlesinger (1983), Gollier and Schlesinger (1996) and Schlesinger (1997).

<sup>4</sup> This is not surprising that these insurance contracts are not optimal or Pareto-efficient. For instance, Brennan (1993) developed a nice analysis of the non-optimality of some insurance contracts. Since we explore the investor's perspective by including risk preferences, our analysis is different from Brennan's (1993).

<sup>5</sup> Generally speaking, a structured product is often based on a complicated underlying index, while an EIA may have a complicated formula for computing the investor's return. In USA structured products are registered under the Securities Act as securities and most EIAs are registered as insurance products. See Francis et al. (2000) for a discussion of structured products.

## 2. Analysis of point-to-point EIAs

In this section, we consider a simple example of a popular EIA contract known as a point-to-point EIA. We will often refer to this contract throughout the paper.

In a standard point-to-point EIA, the buyer pays a single premium at the beginning. The contract typically credits a return that is linked to an external reference portfolio which is often an equity index. The seller also guarantees that the contract will pay at least a minimum rate of interest on the investment. In this way the purchaser has a floor guarantee. In addition the purchaser participates in the upside growth of the reference index. More formally, let  $g$  denote the minimum guaranteed rate and  $k$  the participation rate. Assume that the initial premium paid by the customer is  $x_0$ , and the contract has a term of  $T$  years. At contract maturity the investor will receive the following payoff<sup>6</sup>:

$$\Gamma = \max \left\{ x_0 e^{gT}, x_0 \left( \frac{S_T}{S_0} \right)^k \right\}.$$

The no-arbitrage value of this payoff is given by

$$E[\xi_T \Gamma] = y_0 \quad (2.1)$$

where  $\xi_T$  is the state-price density in this market. Since the premium must be at least as large as the no-arbitrage value of the payoff  $\Gamma$ , we have

$$x_0 \geq y_0. \quad (2.2)$$

We will focus on the case that  $x_0 > y_0$  because the insurer charges a higher premium than the no-arbitrage value of the payoff.<sup>7</sup>

Before proceeding we discuss the trade-off between the participation rate  $k$  and the minimum guaranteed rate  $g$ . For this purpose, we first imagine a hypothetical break even point-to-point EIA contract for which  $x_0 = y_0$  holds. This condition will imply a relationship between the contract parameters  $k$  and  $g$ . Let us denote the members of this set by

$$\{\hat{k}, \hat{g}\}. \quad (2.3)$$

We can derive the feasible values of  $\hat{k}$  and  $\hat{g}$  by finding the parameter combinations for which the no-arbitrage value of the payoff under the EIA is  $x_0$ . In other words  $\hat{k}$  and  $\hat{g}$  satisfy

$$x_0 = E[\xi_T \hat{\Gamma}], \quad (2.4)$$

where

$$\hat{\Gamma} = \max \left\{ x_0 e^{\hat{g}T}, x_0 \left( \frac{S_T}{S_0} \right)^{\hat{k}} \right\}.$$

By using (2.4), we can adjust either the participation rate  $k$ , or the guaranteed rate  $g$  or both so that the no-arbitrage value of the payoff is less than that of the single premium  $x_0$ . In practice this means that we pick  $k < \hat{k}$  or  $g < \hat{g}$  or both together.

<sup>6</sup>Returns are quoted as continuously compounded rates in this paper. The return of EIA is  $\frac{1}{T} \log(\frac{\Gamma}{x_0})$  which is greater than or equal to the product of participation rate  $k$  times the return of the index, that is  $k * \frac{1}{T} \log(\frac{S_T}{S_0})$ , where  $\Gamma$  denotes the terminal payoff of EIA. Then  $X_T \geq x_0 (\frac{S_T}{S_0})^k$ .

<sup>7</sup>Palmer (2006) indicates that in USA, the average first year commission rates for EIAs in the last few years is about 8% of the premium. The loaded premium  $x_0$  is often charged by insurer to against non-hedgeable risks such as mortality and surrender options, and generate business profit. A full equilibrium discussion of the relationship between the no-arbitrage value  $y_0$  of the payoff  $\Gamma$  and the premium  $x_0$  is beyond the scope of this article.

We can illustrate this point in the case where  $S_T$  has a log-normal distribution. Under this assumption there is a closed form expression for  $y_0$ . We assume that

$$\frac{dS}{S} = \mu dt + \sigma dW(t) \quad (2.5)$$

where  $\mu$  is the drift,  $\sigma$  is the diffusion and  $W(t)$  is a standard Brownian motion under the real world measure  $P$ . We assume that  $\mu > r$  where  $r$  is the risk-free rate. In this case (2.1) becomes

$$y_0 = e^{(g-r)T} x_0 \Phi(\alpha) + x_0 e^{(k-1)rT + \frac{1}{2}k(k-1)\sigma^2 T} \Phi(-\alpha + k\sigma\sqrt{T}) \quad (2.6)$$

where

$$\alpha = \frac{g - k(r - \frac{1}{2}\sigma^2)}{k\sigma} \sqrt{T}$$

and  $\Phi(x)$  is the cumulative normal distribution function. The condition  $y_0 < x_0$  becomes

$$e^{gT} \Phi(\alpha) + e^{kT(r + \frac{k-1}{2}\sigma^2)} \Phi(-\alpha + k\sigma\sqrt{T}) < e^{rT}. \quad (2.7)$$

Hence for our specimen EIA, where the index has a log-normal distribution, Eq. (2.4) becomes

$$e^{\hat{g}T} \Phi(\alpha) + e^{\hat{k}T(r + \frac{\hat{k}-1}{2}\sigma^2)} \Phi(-\alpha + \hat{k}\sigma\sqrt{T}) = e^{rT}. \quad (2.8)$$

This last equation determines the relationship between  $\hat{k}$  and  $\hat{g}$ . We can solve it numerically to compute  $\hat{k}$  and  $\hat{g}$ . Panel (A) in Fig. 1 displays the trade-off between the contract parameters  $\hat{k}$  and  $\hat{g}$  for a representative set of parameters. For instance, when  $\hat{g} = 2\%$ , the highest participation rate  $\hat{k} = 60.22\%$ ; if  $\hat{g} = 3\%$ , then  $\hat{k}$  is 48.6%.

We now explore numerically how the magnitude of the difference  $(x_0 - y_0)$  is related to the reduction in the participation rate,  $(\hat{k} - k)$ . To measure the loss to the investors, we use the ratio of  $x_0 - y_0$  to  $x_0$ , which is

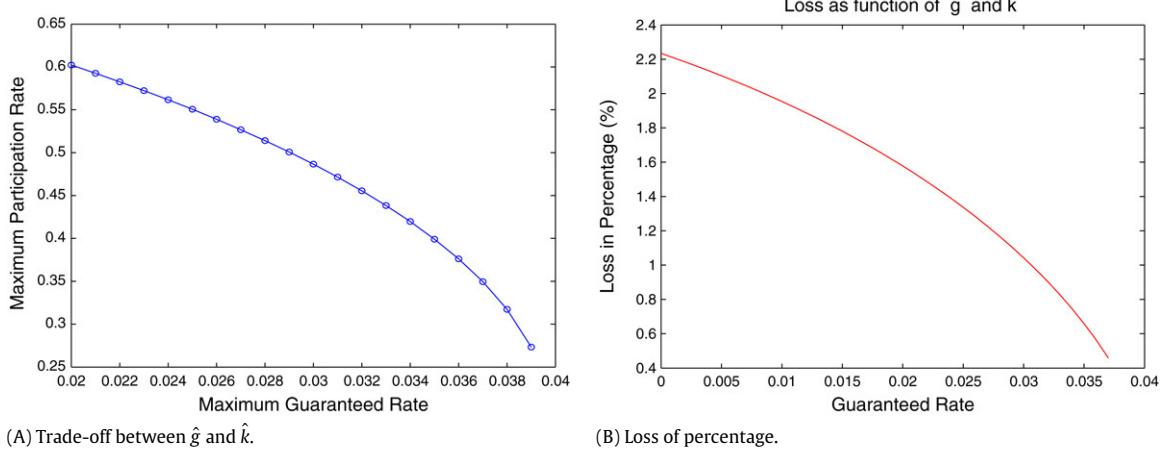
$$1 - \frac{y_0}{x_0} \quad (2.9)$$

to denote the *loss percentage*. Panel (B) in Fig. 1 displays the loss percentage as a function of  $k$  and  $g$  where the participation rate  $k$  equals  $0.90\hat{k}$ .  $\hat{k}$  is determined by Eq. (2.8) where  $g = \hat{g}$ . The loss percentage varies from around 2.2% when  $g = 0\%$  to 0.4% when  $g = 3.75\%$ .

## 3. Improving the design of existing EIAs

In this section we discuss how we can improve the design of an EIA from the investor's viewpoint. We use the expected utility framework to compare different payoffs. First we briefly review the classic Merton portfolio selection problem. Then we will show how the design of the conventional EIA can be improved from the investor's perspective. We consider an investor who wants to maximize expected utility of terminal wealth and in addition have a payoff at least as good as that of an EIA. We find the optimal contract in this setting. This contract maximizes the investor's expected utility subject to the payoff being at least as good as that of the selected EIA.

This section is in four parts. First we review the Merton model for optimal portfolio selection. Then we show how to modify the design of the traditional point-to-point EIA to better incorporate the investor's preferences. The payoff on the point-to-point EIA is taken to be the benchmark. In the third part we discuss some other examples of conventional EIAs and in the last part we discuss how to enhance the design of any conventional EIA by taking into account the investor's preferences regarding the basic EIA payoff as the guaranteed benchmark.



**Fig. 1.** Trade-off and loss of percentage. Index is log-normally distributed. Parameters: contract maturity is 5 years, risk-free rate is 4%, index volatility is 20%. Panel (A) reports the trade-off between  $\hat{k}$  and  $\hat{g}$ . In Panel (B), we choose  $k = 0.9\hat{k}$ , and  $\hat{k}$  is displayed in Panel (A) when  $g = \hat{g}$ . The loss of percentage, which is defined in (2.9), is displayed in Panel (B). Note that both  $\mu$  and  $x_0$  are not involved in both Panel (A) and Panel (B).

### 3.1. The Merton solution

We first consider the general problem of how an investor selects an optimal portfolio. This is a classical problem in financial economics and it was solved<sup>8</sup> by Robert Merton in 1971. Merton's solution and some of the later extensions will be useful in providing a framework for analyzing the design of equity-linked contracts. We will show how the solution is modified when there is a guarantee. However in this subsection we do not directly discuss equity-indexed annuities. We focus on the solution to the optimal portfolio problem and the structure of the payoff assuming that the investor makes the optimal decision.

We consider an investor with initial wealth  $x_0$  and a utility function given by  $u(\cdot)$ , where  $u$  is strictly increasing, strictly concave and twice differentiable. The available assets consist of a risky asset  $S$  and the risk-free asset. We assume no-arbitrage, no frictions and that the market is complete. Given these assumptions there exists a unique state-price process  $\{\xi_t\}$ . We assume that the investor wishes to maximize the expected utility of terminal wealth, denoted by  $X_T$ , over time horizon  $T$ . The investor's optimal terminal wealth is given by

$$X_T^* = I(\lambda^m \xi_T), \quad (3.10)$$

where the multiplier,  $\lambda^m > 0$ , solves

$$E[\xi_T I(\lambda^m \xi_T)] = x_0,$$

and  $I(\cdot)$  is the inverse of the investor's marginal utility function  $u'(\cdot)$ . We refer to the payoff in formula (3.10) as the Merton solution.

Now we modify the investor's objective function by including a guarantee at maturity. In this case the investor wishes to maximize the expected utility of terminal wealth  $X_T$  subject to the constraint

$$X_T \geq x_0 e^{gT}, \quad (3.11)$$

where  $g$  is the guaranteed rate. This is a natural constraint given how often it occurs in practice. We can also find the solution<sup>9</sup> to this problem. In this case

$$X_T^* = \max\{I(\lambda^g \xi_T), x_0 e^{gT}\}, \quad (3.12)$$

<sup>8</sup> This material is well known but it is briefly reviewed here since the approach plays a key role in our later analysis. Merton was able to derive the optimal solution under fairly general conditions. See Merton (1971), Cox and Huang (1989) and Pliska (1986).

<sup>9</sup> Both this solution and the solution of example in Section 3.2 follow from Theorem 3.1 in Section 3.4.

where

$$E[\xi_T \max\{I(\lambda^g \xi_T), x_0 e^{gT}\}] = x_0.$$

This last solution gives the optimal payoff when there is a guarantee and the new Lagrange multiplier  $\lambda^g$  will differ from the Merton one,  $\lambda^m$ . In fact  $\lambda^g > \lambda^m$ . It turns out that the investor's expected utility is reduced when we impose the constraint. This result is evident from inspection of the two payoffs. However it is noteworthy given the prevalence of such guarantees that their inclusion serves to reduce the investor's expected utility relative to the unconstrained problem.

We can examine the difference between these two solutions in the case where the risky asset has a log-normal distribution since this case is analytically very tractable. In this case  $\xi_T$  also has a log-normal distribution, and is given by

$$\xi_T = a \left( \frac{S_T}{S_0} \right)^{-b}, \quad (3.13)$$

where

$$a = \exp \left\{ \frac{\theta}{\sigma} \left( \mu - \frac{1}{2} \sigma^2 \right) T - \left( r + \frac{1}{2} \theta^2 \right) T \right\}, \quad b = \frac{\theta}{\sigma},$$

$$\theta = \frac{\mu - r}{\sigma}.$$

We assume that the investor has a log utility function and initial wealth  $x_0$ . In this case the optimal wealth under the Merton solution is

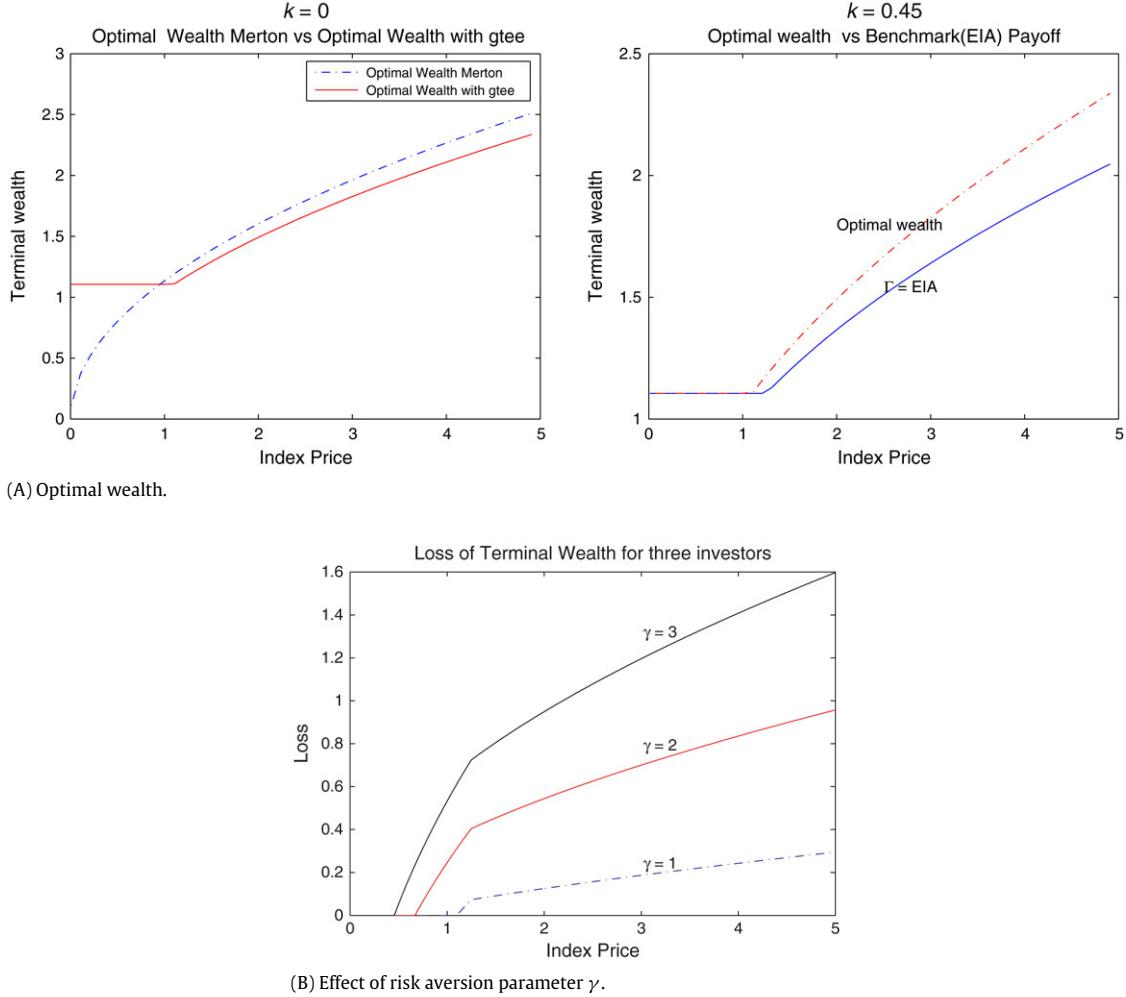
$$X_T^* = I(\lambda^m \xi_T) = I \left( \frac{\xi_T}{x_0} \right) = \frac{x_0}{a} \left( \frac{S_T}{S_0} \right)^b. \quad (3.14)$$

From Eq. (3.12), the corresponding optimal terminal wealth when there is a guarantee is

$$X_T^* = \max\{I(\lambda^g \xi_T), x_0 e^{gT}\} = \max \left\{ \frac{1}{\lambda^g a} \left( \frac{S_T}{S_0} \right)^b, x_0 e^{gT} \right\}. \quad (3.15)$$

We use a simple numerical example to compare these two solutions. This comparison is displayed in the left figure of Panel (A) in Fig. 2. Assume that  $x_0 = 1$ ,  $S_0 = 1$ ,  $T = 5$ ,  $\mu = 6\%$ ,  $\sigma = 20\%$ ,  $g = 2\%$ ,  $r = 4\%$ . In this case  $b = .5$  and  $a = \frac{1}{1.1331}$  so that the optimal terminal wealth in the Merton solution becomes

$$X_T^* = 1.1331 \sqrt{S_T}. \quad (3.16)$$



**Fig. 2.** Optimal wealth and risk aversion. In Panel (A) the investor has log utility and the index is log-normally distributed. Parameters:  $x_0 = 1, S_0 = 1, T = 5, \mu = 6\%, \sigma = 20\%, g = 2\%, r = 4\%$ . The left figure in Panel (A) displays Merton's optimal wealth, which is determined in Eq. (3.10), and the optimal wealth with a guaranteed 2% return, determined by Eq. (3.15). In the right figure in Panel (A), the participation rate  $k = 0.45$ . The optimal wealth in this case is displayed as well as the payoff  $\Gamma$ . Panel (B) reports the loss of terminal wealth for different levels of risk aversion. We assume that the investor has CRRA utility with risk aversion parameter  $\gamma$ , where  $\gamma = 1, 2$  and  $\gamma = 3$ .

When there is a guarantee rate  $g = .02$ , we find that  $\lambda^g = 1.0746$ , and the correspond optimal wealth in Eq. (3.12) is given by

$$X_T^* = \max\{1.1052, 1.0544\sqrt{S_T}\}. \quad (3.17)$$

Comparing expressions (3.16) and (3.17) we see that for large values of  $S_T$  the Merton unconstrained solution, (3.16), will give higher terminal wealth than (3.17). For low values of  $S_T$  the wealth under the constrained problem is greater than the wealth under the Merton solution because of the guarantee.

### 3.2. Optimal design of point-to-point EIA

This discussion of the Merton optimal solution sets us up nicely to consider a point-to-point EIA in the same framework. To be specific consider a point-to-point EIA with a fixed guarantee of  $x_0 e^{gT}$  at maturity and participation rate  $k$ . We now use the expected utility framework to analyze how to design the optimal contract for an investor who desires a payoff that is similar to a point-to-point EIA. More precisely we assume that the investor wishes to have a payoff that at least matches the payoff under a given point-to-point EIA. We assume that the investor wishes to maximize expected utility of terminal wealth with the constraint that the payoff has to be at least as good as that under the point-to-point EIA. This analysis is similar to that of the last subsection except that the

investor's optimal payoff will now contain the payoff of the point-to-point EIA as the guaranteed payoff.

We assume that the investor has a utility function  $u(\cdot)$  and that the terminal wealth under the EIA is  $\Gamma$  where

$$\Gamma = \max \left\{ x_0 e^{gT}, x_0 \left( \frac{S_T}{S_0} \right)^k \right\}.$$

Hence the optimal terminal wealth of this investor is equivalent to the solution of the following maximization problem

$$\max E[u(X_T)]$$

subject to the constraint that the investor's terminal wealth  $X_T$  satisfies

$$X_T \geq \max \left\{ x_0 e^{gT}, x_0 \left( \frac{S_T}{S_0} \right)^k \right\}, \quad (3.18)$$

where  $x_0$  is the initial wealth.

The optimal form of  $X_T$  can be derived using the general theory of portfolio selection in the Merton framework. It is similar to the optimal design we considered in the last subsection. The optimal payoff at time  $T$  is given by

$$X_T^* = \max \left\{ I(\lambda^g \xi_T), x_0 e^{gT}, x_0 \left( \frac{S_T}{S_0} \right)^k \right\}, \quad (3.19)$$

where  $\lambda^g$  is a positive number satisfying

$$E \left[ \xi_T \max \left\{ I(\lambda^g \xi_T), x_0 e^{gT}, x_0 \left( \frac{S_T}{S_0} \right)^k \right\} \right] = x_0 \quad (3.20)$$

and  $I(\cdot)$  is the inverse function of  $u'(\cdot)$ . The claim

$$\max \left\{ I(\lambda^g \xi_T), x_0 e^{gT}, x_0 \left( \frac{S_T}{S_0} \right)^k \right\}$$

is the optimal payoff for an investor with utility function  $u(\cdot)$ . Eq. (3.20) means that the no-arbitrage value of the optimal payoff is equal to  $x_0$ . Of course this is the ideal contract from the consumer's viewpoint and there is no allowance for transaction costs. In practice the premium  $x_0$  will exceed the no-arbitrage value of the contract.

We now consider a numerical example when  $\Gamma$  is the payoff under a specific point-to-point EIA. We assume the same numerical parameters as in the last subsection, but in this case the participation rate  $k = 45\%$ . For this example  $g = .02 = \hat{g}$  and  $k = 45\%$  is less than  $\hat{k} = 60.22\%$  so  $x_0 > y_0$ . In fact,  $y_0 = .9617$ . The guaranteed contract payoff,  $\Gamma$ , provides the greater of a guaranteed return 2%, and participation in the market with  $k = 45\%$ . The optimal terminal wealth for this case is

$$\begin{aligned} X_T^* &= \max \left\{ x_0 e^{gT}, x_0 \left( \frac{S_T}{S_0} \right)^k, \frac{1}{\lambda^g a} \left( \frac{S_T}{S_0} \right)^b \right\} \\ &= \max\{1.1052, S_T^{0.45}, 1.0541\sqrt{S_T}\}, \end{aligned} \quad (3.21)$$

and

$$\Gamma = \max \left\{ x_0 e^{gT}, x_0 \left( \frac{S_T}{S_0} \right)^k \right\} = \max\{1.1052, S_T^{0.45}\}.$$

The right figure of Panel (A) in Fig. 2 displays the optimal wealth  $X_T^*$  and the guaranteed payoff  $\Gamma$ . As one can see from Fig. 2, the payoff under this optimal contract is greater than  $\Gamma$  for a wider range of values of the market index  $S$ . It is clear that the contract payoff  $\Gamma$  is not the optimal terminal wealth.<sup>10</sup>

If  $k = \hat{k}$ , the optimal wealth in (3.19) is

$$X_T^* = \max \left\{ x_0 e^{gT}, x_0 \left( \frac{S_T}{S_0} \right)^{\hat{k}} \right\}. \quad (3.22)$$

This fact follows easily from Theorem 3.1 in Section 3.4.

Formally, let  $X_T^*$  denote the optimal wealth in this case when the guaranteed payment is  $\Gamma$ . It is well known that (either directly from its construction, or Theorem 3.1), the constraint  $E[\xi_T X_T^*] = x_0$  is binding. As we have noted the current no-arbitrage value of  $\Gamma$  is  $y_0 < x_0$ . Hence the loss to the investor in terms of current value is  $(x_0 - y_0)$ . This does not depend on the investor's utility function. At time  $T$  the loss to the investor is  $(X_T^* - \Gamma)$  which will depend on the investor's utility function because  $X_T^*$  depends on the investor's utility function.

We now illustrate how the investor's loss at maturity depends on her utility function. We compare the optimal terminal wealth with the benchmark  $\Gamma$  for three different levels of investor risk aversion. Panel (B) in Fig. 2 displays the loss of terminal wealth

corresponding to the CRRA utility function  $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ ,  $\gamma > 0$  for  $\gamma = 1, 2, 3$ . Note that the log-utility function corresponds to the case  $\gamma = 1$ . When the risk aversion parameter  $\gamma = 1$ , the loss percentage is fairly small. When the risk aversion parameter  $\gamma$  increases, the loss increases. If  $\gamma = 2$ , the loss could be as high as 20% of the initial investment amount when the index  $S$  moves from 1\$ to 2\$ after five years. The loss could reach 45% when  $\gamma = 3$ . This implies that the more risk averse the investor, the higher the loss. It is interesting to note that the prevailing wisdom seems to be that an EIA might be more appropriate for conservative investors. In fact, as Wachter (2003) showed that the best strategy for the most conservative investor is to buy and hold the risk-free bond. However our findings is intuitive. For a conservative investor, the percentage investment in the index is small. Hence if the index moves up substantially the conservative investor will have foregone a profitable investment opportunity.

We can use the analysis of this section to discuss implications for the design of EIAs. It is clear from our discussion that investors with different levels of risk aversion would prefer different contract designs. We saw that the optimal  $X_T^*$  varies with the investor's utility function. However in practice the investor in an EIA receives a payoff  $\Gamma$  which is offered by the issuer. Even in this case there is scope for variation in contract design. If we assume a point-to-point EIA with parameters  $\{k, g\}$ , then the issuer can design a range of contracts with current no-arbitrage value,  $y_0 < x_0$ , by varying  $k$  and  $g$ . Investors will not be indifferent among these contracts. Typically there will be a preferred contract in this set. Suppose we label the set of feasible pairs  $\{k, g\}$  by the set  $A$ . For  $a \in A$  the corresponding EIA is  $\Gamma_a$ . Then an investor with utility function  $u(\cdot)$  will have a preferred EIA obtained as follows

$$\max_{a \in A} E[u(\Gamma_a)].$$

This suggests that insurers should offer several EIAs with different contract parameters to appeal to different investor clienteles. Anecdotal evidence suggests that this happens in practice.

### 3.3. Examples of conventional style EIAs

So far we have discussed only the plain vanilla point-to-point EIA. The point-to-point EIA is one of the simplest types of EIA designs in the market. The return on this type of contract can be modified in various ways.

In this subsection we describe other EIA contracts which can be analyzed within the framework used in the last section. Here are some examples.

**Example 1 (Interest Rate Cap and Floor).** In some EIA policies, the index-linked interest rate is capped at a specific level known as the cap rate. In the simplest case the total return is capped. In this case the payoff is

$$\Gamma = x_0 \max \left\{ e^{gT}, \min \left\{ \left( \frac{S_T}{S_0} \right)^k, e^{cT} \right\} \right\}. \quad (3.1)$$

However in some designs the interest rate is capped monthly. Assume that  $T_0 = 0 < T_1 < \dots < T_N = T$  is the term over which the rate is capped. The return of this EIA payoff,  $\frac{1}{T} \log(\frac{\Gamma}{x_0})$ , is given by

$$\max \left\{ g, k \sum_{i=1}^N \min \left\{ \frac{1}{T_i - T_{i-1}} \log \left( \frac{S(T_i)}{S(T_{i-1})} \right), c \right\} \right\}, \quad (3.2)$$

where  $c$  is the monthly cap.

<sup>10</sup> Boyle and Tian (in press) show that, in the CRRA utility function family, there is just a single value for the risk aversion parameter that could make the payoff  $\Gamma$  optimal for the investor. An additional necessary condition is that  $y_0 = x_0$  otherwise  $\Gamma$  could not be optimal. In general, it is known that any given investment plan is only optimal for a very special class of utility functions. See Black and Perold (1992), Cox and Leland (2000), He and Leland (1993) and He and Huang (1994).

**Example 2** (High Water Market). In this case, the index-linked rate is based on the increase in index value from the index level at the beginning of the term to the highest index value during the contract's term. The payoff is

$$\Gamma = x_0 \left( \frac{\max S_t}{S_0} \right)^k, \quad (3.3)$$

where  $\max S_t$  denotes the highest level along the path

$$\max S_t = \max\{S(0), S(T_1), \dots, S(T_{N-1}), S(T_N)\}.$$

**Example 3** (Ratchet EIA). This is also known as the cliquet design. In this design the contract will earn, at the end of every period, the return on the index over that period with a participation rate  $k$ , or the minimum guaranteed return  $g$ , whichever is higher. The benchmark becomes

$$\Gamma = x_0 \prod_{i=1}^n \max \left\{ e^g, \left( \frac{S_{t_i}}{S_{t_{i-1}}} \right)^k \right\}. \quad (3.4)$$

This list of EIAs, of course, is far from complete. An EIA might combine more than one feature in the list. For instance the high water mark feature could also have an interest rate cap. We refer to Streiff and DiBiase (1999) and Hardy (2003) for a more extensive description of the different designs. We can summarize the different design features in terms of the payoff as follows. The payoff on the EIA is a non-negative random variable  $\Gamma$  which is a function of the reference index and the contract parameters. Note that a surrender option is often embedded inside EIA. The surrender option gives policyholders to give up the contract and receive certain surrender value. The discussion on the optimal design of EIA with a surrender option adds technical complications and is not pursued in this paper.

#### 3.4. Improving the design of a conventional EIA

In this section we discuss how to improve the design of any type of conventional EIA. We assume that the basic EIA contract has a payoff of  $\Gamma$  at the end of the term. We use the same framework as before. The basic EIA payoff is taken to be the benchmark. In this case the optimal design can be recast as the following optimization problem

$$\max E[u(X_T^{x_0, \pi})]$$

where  $x_0$  is the initial investment,  $\pi$  is a self-financing strategy such that the terminal wealth  $X_T^{x_0, \pi}$  is subject to the constraint that

$$X_T^{x_0, \pi} \geq \Gamma. \quad (3.5)$$

To derive an explicit expression for the optimal contract we assume that the financial market is *complete*. According to Harrison and Kreps (1979) and Harrison and Pliska (1981), there exists a unique state-price density process  $\{\xi_T\}$ . For technical reasons we assume that  $\xi_T$  has finite moments, positive or negative. In addition we will also make use of the following assumption on the continuity and range of the state-price density.

**Assumption 1.** The distribution function of  $\xi_T$  is continuous and  $P(\xi_T < x) > 0$  for every  $x > 0$ ,  $T > 0$ .

The following theorem<sup>11</sup> solves the optimization problem.

**Theorem 3.1.** (1) There exists a self-financing trading strategy  $\pi$  such that  $X_T^{x_0, \pi} \geq \Gamma$  if and only if  $E[\xi_T \Gamma] \leq x_0$ .

(2) If the guaranteed benchmark,  $\Gamma$  satisfies  $E[\xi_T \Gamma] < x_0$ , then there exists a unique optimal feasible terminal wealth, namely  $X_T^{eia} \equiv \max\{I(\lambda^g \xi_T), \Gamma\}$  such that the budget constraint is binding. More precisely,

$$E[\xi_T \max\{I(\lambda^g \xi_T), \Gamma\}] = x_0.$$

Furthermore,  $\lambda^g$  is unique by virtue of Assumption 1.

(3) If  $E[\xi_T \Gamma] = x_0$ , then the only feasible terminal wealth is  $\Gamma$ , a.s.

Therefore, for any given EIA with payoff  $\Gamma$ , the optimal design or payoff for an investor with utility function  $u(\cdot)$  should be  $\max\{I(\lambda^g \xi_T), \Gamma\}$ . The loss (profit) of the terminal wealth for the investor (issuer) is  $\max\{I(\lambda^g \xi_T), \Gamma\} - \Gamma$ . By Theorem 3.1, the present value of the loss of the terminal wealth

$$E[\xi_T (\max\{I(\lambda^g \xi_T), \Gamma\} - \Gamma)] = x_0 - y_0. \quad (3.6)$$

$x_0 - y_0$  is the initial loss for any investor who buys the EIA and this is independent of the investor's risk preference. However, the loss of the terminal wealth will be different for different investors.

The contract design in Theorem 3.1 is more widely known in the finance literature as “*portfolio insurance*” since  $\Gamma$  represents the insurance level for the portfolio's terminal wealth.<sup>12</sup> The payoff  $\Gamma$  is also referred as the benchmark in the remainder of this paper. Using Theorem 3.1 we are able to design an optimal EIA type contract with a payoff which is at least equal to  $\Gamma$ .

#### 4. The generalized EIA

In this section we expand the EIA contract to include a new design feature and we refer to this new design as a *generalized EIA*. This new design extends the concept of a conventional EIA in a significant way. The contract retains some of the features of a conventional EIA. Boyle and Tian (in press) have obtained an explicit solution for the optimal design of this generalized EIA under fairly general conditions. In this paper we will quote their results without proofs. This section is organized as follows. In Section 4.1 we discuss the motivation for introducing the generalized EIA contract. We then explain how the optimal design of the generalized EIA can be viewed as a (non-convex) optimization problem. Next we give the explicit optimal design of the generalized EIA contract. Lastly we discuss several extensions of the generalized EIA.

##### 4.1. Motivation

In practice, many investors require safety of their principal but they also hanker after high returns. We have seen a trade-off between these requirements in a point-to-point EIA in Section 2. The guarantee has a cost. The higher the guarantee rate, the lower the participation rate in a point-to-point EIA. We can tweak the design of the EIA but the no-arbitrage principle limits what can be accomplished. However it is possible to redesign the contract to make it better suit investor preferences. We now describe how this can be done.

The distinguishing feature of a generalized EIA contract is that it gives the investor an opportunity to beat (or match) the performance of some selected benchmark index at some confidence level. For example, under a generalized EIA an investor might earn 125% of the index return with an 80% confidence level.

<sup>11</sup> This theorem is well known in the finance literature and occurs in different places in different versions. We refer to Dybvig et al. (1999) and Boyle and Tian (2007) for its proof in the current context.

<sup>12</sup> See Black and Perold (1992), Brennan and Schwartz (1988), Brennan and Solanki (1988), Basak (1995), Benninga and Blume (1985), Grossman and Zhou (1996), Leland (1980), and Tepila (2001).

This generalized EIA is similar to an *event-driven* EIA in some sense but the design idea is very different. To illustrate this point we consider two (physical) events as follows:

$$A = \{S_T \geq S_0 e^{aT}\}, \quad B = \{\max_{0 \leq t \leq T} S_t \geq L\} \quad (4.1)$$

where  $a > 0, L > 0$  are determined by

$$P(A) = P(B) = 0.8. \quad (4.2)$$

The corresponding event-driven EIAs have payoffs

$$\begin{aligned} \Gamma_1 &= x_0 \max \left\{ e^{gT}, \left( \frac{S_T}{S_0} \right)^{1.25} 1_A \right\}, \\ \Gamma_2 &= x_0 \max \left\{ e^{gT}, \left( \frac{S_T}{S_0} \right)^{1.25} 1_B \right\}, \end{aligned} \quad (4.3)$$

respectively. The optimal designs of the EIAs  $\Gamma_1$  or  $\Gamma_2$  could be determined by [Theorem 3.1](#). The buyer of either one of these EIAs has at least an 80% chance of obtaining 125% of the index return but the set of events which coincide with the higher return is different in each case. For the EIA with payoff  $\Gamma_1$ , the event is triggered if the final index price  $S_T$  is high enough, while for the EIA with payoff  $\Gamma_2$ , the event is depends on the path of the index price process  $\{S_t : 0 \leq t \leq T\}$ .

Assuming now the investor is either not sure or not interested in a specific event. We now show how the concept of generalized EIA can be introduced using these examples to build a new type of contract. In this new contract, the risk for the investor is determined solely by the probability of achieving the desired return. The benchmark return is based on the realized index return. There are several justifications to support this type of probability constraints. First, from the no-arbitrage principle, if an investor wants to beat the risk-free rate she must bear some risk. Second, many investors do seem to care about beating a benchmark. Third, there is often uncertainty about the model and its parameters. Whenever there is model risk and estimation risk, probability constraints provide a way to cope with model misspecification error.

We illustrate the design of the generalized EIA using the point-to-point structure as an example. Assume that  $g$  is the guaranteed rate, the benchmark is  $\Gamma$  and that the confidence level is  $\alpha$ . The payoff,  $X_T$ , of the generalized EIA, satisfies two constraints:

$$X_T \geq x_0 e^{gT}, \quad P\left(X_T \geq x_0 \left(\frac{S_T}{S_0}\right)^k\right) \geq \alpha. \quad (4.4)$$

There is an important difference between a conventional EIA and a generalized EIA, in terms of the payoff structure. The issuer and the investor have very different perspectives. The issuer would like to provide the desired payoff as economically as possible. In addition the issuer may have other objectives as well. For example, one possible objective for the issuer is to minimize the expected return of the payoff:

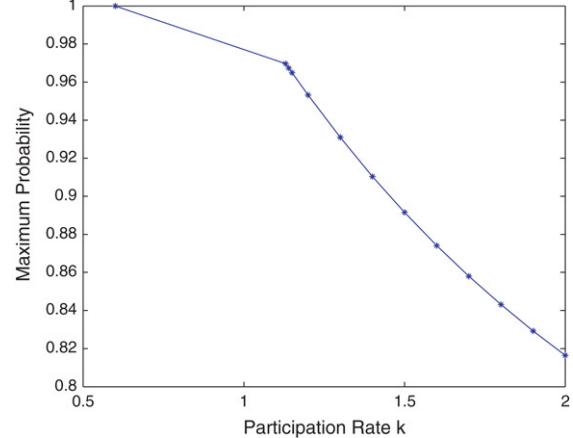
$$\min E[V(X_T^{x_0, \pi})]$$

where  $V(\cdot)$  denotes the utility function of the issuer, the terminal wealth  $X_T^{x_0, \pi}$  is subject to (4.4) and  $\pi$  is a self-financing strategy.<sup>13</sup>

In contrast, the investor wishes to maximize the expected utility of terminal wealth subject to the constraints (4.4). We focus on the investor's perspective in this paper. The optimal design for the investor can be recast as the following optimization problem

$$\max E[u(X_T^{x_0, \pi})]$$

where  $\pi$  is a self-financing strategy such that the terminal wealth  $X_T^{x_0, \pi}$  is subject to the constraints in (4.4).



**Fig. 3.** Trade-off between  $\hat{\alpha}$  and the participation rate  $k$ . Investor has log utility and the index is log-normally distributed. Parameters:  $T = 5, \mu = 6\%, r = 4\%, \sigma = 20\%$ . The guaranteed rate  $g = 2\%$  is given. The maximum probability  $\hat{\alpha}$  is defined in (4.5), as a function of the participation rate  $k$ . We calculate  $\hat{\alpha}$  by using the analysis in Spivak and Cvitanic (1999), Boyle and Tian (in press). The point on the curve where  $\hat{\alpha} = 1$  corresponds to  $\hat{k} = 0.6022$ , which is determined by Eq. (2.8) where  $\hat{g} = g = 2\%$ . The maximum probability  $\hat{\alpha}$  is decreasing with respect to the participation rate  $k$ .

#### 4.2. Optimal design of the generalized EIA contract

In this subsection we present the optimal design of the generalized EIA. We assume that  $g$  is the guaranteed rate and  $f = e^{gT}$ . We assume that  $\Gamma$  is a positive (a.s.) random variable and that the payoff  $X_T$  satisfies

$$P(X_T \geq \Gamma) \geq \alpha, \quad X_T \geq f x_0$$

where  $\alpha \in (0, 1)$  is the specified physical probability.

We have noted that there is precise trade-off between  $\hat{g}$  and  $\hat{k}$  in Section 2 for a point-to-point EIA. For a generalized point-to-point EIA, the participation rate  $k$  could be increased substantially with the same guaranteed rate  $g$ . To illustrate the impact of the confidence level  $\alpha$ , we define

$$\hat{\alpha} := \max_{\pi} P\left(X_T^{x_0, \pi} \geq x_0 \left(\frac{S_T}{S_0}\right)^k\right) \quad (4.5)$$

where  $\pi$  denotes a self-financing strategy, and  $X_T^{x_0, \pi}$  is the terminal wealth by following the strategy  $\pi$ , starting from the initial amount  $x_0$ , and the terminal wealth satisfies

$$X_T^{x_0, \pi} \geq f x_0.$$

Given a participation rate  $k$ ,  $\hat{\alpha}$  is the maximum possible probability that the terminal wealth beats  $x_0 \left(\frac{S_T}{S_0}\right)^k$ . Conversely, given  $\hat{\alpha}$ , we can determine the maximum possible participation rate  $k$  such that the terminal wealth beats  $x_0 \left(\frac{S_T}{S_0}\right)^k$  with confidence level  $\hat{\alpha}$ . In particular, when  $\hat{\alpha} = 1$ , the maximum possible participation rate  $k$  is  $\hat{k}$  in Eq. (2.4).<sup>14</sup> Fig. 3 displays the trade-off between the participation rate  $k$  and the maximum probability  $\hat{\alpha}$ , when the guaranteed rate is  $g$ .

We now state the general result for the optimal generalized EIA contract. For the complete technical details and proofs we refer to Boyle and Tian (in press). To state the result we need certain definitions and technical details. We do not discuss these definitions or attempt to motivate them in this section since our

<sup>13</sup> This optimal design problem for the issuer is studied in [Bernard et al. \(submitted for publication\)](#).

<sup>14</sup> For a complete discussion of  $\hat{\alpha}$  we refer to Spivak and Cvitanic (1999) and Boyle and Tian (in press).

focus here is on applications rather than mathematical proofs. We assume that  $\Gamma$  is a general benchmark index. It is convenient to define the following functions for any  $\lambda > 0, x \geq 0$ ,

$$G(\lambda, x) = P(\max\{I(\lambda\xi_T), fx_0\} < \Gamma, h(\lambda, \xi_T) \geq x), \quad (4.6)$$

and

$$H(\lambda) = P(\max\{I(\lambda\xi_T), fx_0\} < \Gamma), \lambda > 0 \quad (4.7)$$

and let

$$\lambda_\alpha := \text{Sup}\{\lambda : H(\lambda) < 1 - \alpha\}, \quad (4.8)$$

where

$$h(\lambda, \xi_T) = u(\max\{I(\lambda\xi_T), fx_0\}) - \lambda\xi_T \max\{I(\lambda\xi_T), fx_0\} + \lambda\xi_T \Gamma - u(\Gamma). \quad (4.9)$$

Under **Assumption 1**, there exists a unique positive  $\lambda^g$  such that

$$E[\xi_T \max\{I(\lambda^g \xi_T), fx_0, \Gamma\}] = x_0. \quad (4.10)$$

We always assume that  $\lambda_\alpha < \lambda^g$ . Otherwise, this generalized EIA contract reduces to the classical (conventional) EIA, and **Theorem 3.1** presents the optimal design already. Define

$$G^{-1}(\lambda, 1 - \alpha) = \{x > 0 : G(\lambda, x) := 1 - \alpha\}, \forall \lambda > 0. \quad (4.11)$$

We impose one more technical assumption that ensures the continuity of certain functions.

**Assumption 2.**  $G(\lambda, x)$  is jointly continuous with respect to both  $\lambda$  and  $x$ . Moreover, for any  $\lambda \geq \lambda_\alpha$ , there exists at most one member in  $G^{-1}(\lambda, 1 - \alpha)$ .

Boyle and Tian (in press) show that **Assumption 2** implies the existence of a unique positive number  $d(\lambda, \alpha)$  such that

$$P(\max\{I(\lambda\xi_T), fx_0\} < \Gamma, h(\lambda, \xi_T) \geq d(\lambda, \alpha)) = 1 - \alpha. \quad (4.12)$$

Define  $X_{\lambda, \alpha}(T)$  as follows:

$$X_{\lambda, \alpha}(T) = \begin{cases} I(\lambda\xi_T), & \text{if } I(\lambda\xi_T) \geq \Gamma > fx_0 \\ \max\{I(\lambda\xi_T), fx_0\}, & \text{if } \max\{I(\lambda\xi_T), fx_0\} < \Gamma, \\ & h(\lambda, \xi_T) \geq d(\lambda, \alpha) \\ \Gamma, & \text{if } \max\{I(\lambda\xi_T), fx_0\} < \Gamma, \\ & h(\lambda, \xi_T) < d(\lambda, \alpha) \\ \max\{I(\lambda\xi_T), fx_0\}, & \text{if } \Gamma \leq fx_0. \end{cases}$$

We are now in a position to present the general result which is proved in Boyle and Tian (in press).<sup>15</sup>

**Theorem 4.1.** Under **Assumptions 1** and **2**, and assuming that

$$\lim_{\lambda \rightarrow \infty} \{E[\xi_T fx_0 1_{\{\Gamma > fx_0, h(\lambda, \xi_T) \geq d(\lambda, \alpha)\}}] + E[\xi_T \Gamma 1_{\{\Gamma > fx_0, h(\lambda, \xi_T) < d(\lambda, \alpha)\}}] + E[\xi_T fx_0 1_{\{\Gamma \leq fx_0\}}]\} < x_0 \quad (4.13)$$

then there exists a self-financing process  $\pi_t^*$  with terminal wealth  $X^{geqp}(T)$  such that

$$X^{geqp}(T) \geq fx_0, \quad P(X^{geqp}(T) \geq \Gamma) \geq \alpha.$$

And  $E[u(X^{geqp}(T))] \geq E[u(X^{x_0, \pi}(T))]$  for any self-financing process  $\pi_t$  whose terminal wealth is subject to the constraint conditions:

$$X^{x_0, \pi}(T) \geq fx_0, \quad P(X^{x_0, \pi}(T) \geq \Gamma) \geq \alpha.$$

Moreover, we can choose

$$X^{geqp}(T) = X_{\lambda^*, \alpha}(T)$$

for some positive real number  $\lambda^* > \lambda_\alpha$ .

<sup>15</sup> There is a major obstacle in tackling this optimization problem under this non-convex constraint. The solution given here solves the problem for the complete market framework. The corresponding optimal design problem in an incomplete market is not solved yet.

In the next section we will illustrate how to make use of **Theorem 4.1** to explicitly construct the optimal payoff (terminal wealth) of the generalized EIA. The optimal payoff of EIA in **Theorem 4.1** is designed from investor's perspective, but it is helpful for both investor and issuer. If the issuer were to dynamically replicate this optimal payoff, hedging raises important practical<sup>16</sup> issues that we do not explore in this paper.

### 4.3. Extensions

In the generalized EIA contract, only a guaranteed rate is imposed. Actually it is possible to embed a conventional EIA into a generalized EIA. We explain the main points in this subsection.

Given an EIA contract with a payoff structure  $\Gamma_0$  at the end of the term. We assume that  $\Gamma$  is a benchmark and  $\alpha$  is a specific probability. In the generalized EIA, the payoff  $X_T$  satisfies

$$X_T \geq \Gamma_0, \quad P(X_T \geq \Gamma) \geq \alpha.$$

The optimal design of the generalized EIA is reduced to a maximum expected utility problem subject to the last set of constraints on the terminal wealth  $X_T$ . **Theorem 4.1** can be extended easily to this case where  $fx_0$  is replaced by  $\Gamma_0$ .

We assumed a constant risk-free interest rate in this paper for convenience. We can readily extend the results to a deterministic term structure of interest rates. Moreover, it is also possible to use a stochastic interest rate model in the analysis.

## 5. Example of generalized EIA contract

In this section we illustrate the details of our optimal solution for the generalized EIA by using a specific example. The example is instructive since it reveals a number of interesting properties of the optimal contract. We assume that the benchmark is  $\Gamma = x_0(\frac{\xi_T}{\xi_0})^k$  and that the guaranteed is  $g$ . We also assume that  $u(x) = \log(x)$  to simplify some technical issues.

Let us briefly recall the steps involved in deriving the optimal solution. We base the construction on the general solution given in **Theorem 4.1**. We consider a class of feasible terminal wealths  $X_{\lambda, \alpha}$ , and the optimal  $X^{geqp}(T)$  will belong to this class. The optimal terminal wealth corresponds to the payoff for the optimal generalized EIA. The construction of  $X_{\lambda, \alpha}$ , by definition, depends on the properties of some auxiliary functions  $h(\lambda, \xi_T)$ ,  $G(\lambda, x)$  and  $H(\lambda)$ . For instance, using  $H(\lambda)$  we can determine a special  $\lambda$  which we call  $\lambda_\alpha$ . From  $G(\lambda, x)$  we determine  $d(\lambda, \alpha)$ . Finally, we can find the optimal  $\lambda^* \geq \lambda_\alpha$  from the binding constraint that the no-arbitrage value of the claim  $X_{\lambda^*, \alpha}$  is  $x_0$ .

To determine the exact functional form of  $h(\lambda, \xi_T)$ , we consider two regions. Over the region  $\{I(\lambda\xi_T) \geq fx_0\}$ , we have  $h(\lambda, \xi_T) = h_1(\lambda, \xi_T)$ ; over the region  $\{I(\lambda\xi_T) < fx_0\}$ ,  $h(\lambda, \xi_T) = h_2(\lambda, \xi_T)$ , where

$$h_1(\lambda, \xi_T) = -\log(\lambda x_0 a^{\frac{k}{b}}) - 1 + \left(\frac{k}{b} - 1\right) \log(\xi_T) + \lambda x_0 a^{\frac{k}{b}} \xi_T^{1-\frac{k}{b}}, \quad (5.1)$$

and

$$h_2(\lambda, \xi_T) = \log(fa^{-\frac{k}{b}}) - \lambda f x_0 \xi_T + \frac{k}{b} \log(\xi_T) + \lambda x_0 a^{\frac{k}{b}} \xi_T^{1-\frac{k}{b}}. \quad (5.2)$$

<sup>16</sup> There are important technical issues in hedging such as trading restrictions, transaction costs, and the discontinuity of the payoff. We do not address the hedging problem in this paper but refer to Boyle and Vorst (1992), Edirisinghe et al. (1993), Naik and Uppal (1994), and Leland (1985).

We can determine  $d(\lambda, \alpha)$  explicitly from the two functions  $h(\lambda, \xi_T)$  and  $H(\lambda)$ . The explicit construction of  $X_{\lambda,\alpha}(T)$  will be given shortly. At this stage we mention that it is easy to check the validity of both **Assumptions 1** and **2** when  $k \neq b$ . Since the assumptions in **Theorem 4.1** are verified, the optimal terminal wealth is determined by **Theorem 4.1**.

We now give the explicit expression of  $X_{\lambda,\alpha}(T)$ . The form of this expression depends on the relative magnitudes of  $k$  and  $b$ . The cases  $k < b$ ,  $k = b$  and  $k > b$  are all different and we use different symbols for the optimal terminal wealth in these three cases. In each case we have a family of the (optimal) terminal wealth, where each family is indexed by a positive number  $\lambda \geq \lambda_\alpha$ . We use the following notation for these families in the three different cases.

1. When  $k > b$ , there are two subcases. The terminal wealth is either  $Z_{\lambda,\alpha}(T)$  or  $V_{\lambda,\alpha}(T)$ , corresponding to different values of  $\lambda$ .
2. When  $k < b$ , the terminal wealth is  $Y_{\lambda,\alpha}(T)$ .
3. When  $k = b$ , the terminal wealth is  $W_{\lambda,\alpha}(T)$ .

The explicit expressions for  $Z_{\lambda,\alpha}(T)$ ,  $V_{\lambda,\alpha}(T)$ ,  $Y_{\lambda,\alpha}(T)$ ,  $W_{\lambda,\alpha}(T)$  are quite complicated and are given in [Appendix A](#).

In the rest of this section, we discuss this point-to-point generalized EIA contract for the three cases  $k > b$ ,  $k < b$  or  $k = b$  respectively.

### 5.1. Case one: $k > b$

As we explained earlier, given a confidence level  $\alpha$ , there exists a maximum participation rate  $k$ . This trade-off is captured by the equation:

$$\begin{aligned} E[\xi_T f x_0 1_{\{\xi_T < \bar{\xi}_{1-\alpha}\}}] + E[x_0 a^{\frac{k}{b}} \xi_T^{1-\frac{k}{b}} 1_{\{\bar{\xi}_{1-\alpha} \leq \xi_T < c\}}] \\ + E[\xi_T f x_0 1_{\{\xi_T \geq c\}}] \leq x_0 \end{aligned} \quad (5.3)$$

where both parameter  $c$  and  $\bar{\xi}_{1-\alpha}$  are defined in formula [\(A.1\)](#), [\(A.2\)](#) of [Appendix A](#), respectively. Then there exists a feasible solution which satisfies the two constraints in the generalized EIA contract; namely achieving at least the guaranteed return and the probability of beating the benchmark  $\Gamma$ . Furthermore the solution for the optimal contract (optimal wealth) exists in this case. Note that

$$\lambda_\alpha = \frac{1 - \frac{k-b}{b}}{\xi_{1-\alpha}} a^{-\frac{k}{b}}, \quad (5.4)$$

and we assume that  $\lambda_\alpha < \lambda^g$ .

The implementation procedure is as follows:

- Step 1.** Check whether the trade-off condition [\(5.3\)](#) holds or not. If this condition is satisfied, go to the next step. Otherwise, stop.
- Step 2.** Check whether  $E[\xi_T Z_{\lambda_\beta,\alpha}(T)] \leq x_0$ . If so, there exists a  $\lambda^* \in [\lambda_\alpha, \lambda_\beta]$  such that  $E[\xi_T Z_{\lambda^*,\alpha}(T)] = x_0$ . We solve for the optimal  $\lambda^*$  which is the solution of the equation  $E[\xi_T Z_{\lambda^*,\alpha}(T)] = x_0$ . If not, go to the next step.
- Step 3.** There exists a  $\lambda^* > \lambda_\beta$  such that  $E[\xi_T V_{\lambda^*,\alpha}(T)] = x_0$ . Solve for the optimal  $\lambda^*$  in the equation  $E[\xi_T V_{\lambda^*,\alpha}(T)] = x_0$ .

The optimal terminal wealth  $Z_{\lambda,\alpha}(T)$  and  $V_{\lambda,\alpha}(T)$  are displayed when  $k > b$  in [Fig. 4](#). Assuming first  $k = 0.65$ . In this case,  $\lambda_\alpha = 1.0258$ ,  $\lambda_\beta = 1.1073$  and  $\lambda^g = 1.0804$ . Since  $E[\xi_T Z_{\lambda_\beta,\alpha}(T)] = 0.9971 \leq x_0$ , there exists one optimal wealth with the form  $Z_{\lambda,\alpha}(T)$  for some  $\lambda^* \in (\lambda_\alpha, \lambda_\beta]$ . Panel (A) of [Fig. 4](#) displays the optimal wealth  $Z_{\lambda^*,\alpha}(T)$  for  $\lambda^* = 1.0906 \in (\lambda_\alpha, \lambda_\beta)$ .<sup>17</sup> When  $k = 0.75$  and other contract parameters are kept the same, since

$E[\xi_T Z_{\lambda_\beta,\alpha}(T)] = 1.0132 > x_0$ , then there exists one optimal wealth with the form  $V_{\lambda,\alpha}(T)$  for some  $\lambda > \lambda_\beta$ . Panel (B) in [Fig. 4](#) displays the optimal wealth  $V_{\lambda^*,\alpha}(T)$  for  $\lambda^* = 1.2133 > \lambda_\beta$  in the case of  $k = 0.75$ .

It is instructive to compare this generalized EIA with a conventional EIA. From Eq. [\(2.8\)](#) we know that the maximum participation rate  $\hat{k} = .6022$ . In the above generalized EIAs the participation rate  $k$  is either 0.65 or 0.75 both of which are greater than  $\hat{k}$ . The return of the generalized EIA can be higher than the return of EIA in some scenarios. The risk for the generalized EIA, however, is that the higher participation rate is realized only with a probability of 85%. The optimal wealth of the EIA with participation rate 0.60 is displayed in both Panel (A) and Panel (B) of [Fig. 4](#).

In Panel (A) we see that for some index movements the optimal payoff under the generalized EIA beats the optimal payoff under the standard EIA. On the other hand, the optimal payoff of the standard EIA beats the generalized EIA when the index price moves up significantly. This feature seems surprising because the generalized EIA with higher participation rate seems better than the EIA with smaller participation rate. However it is intuitive from the motivation of the generalized EIA. The generalized EIA has the potential to have a higher return with probability  $\alpha$ . Therefore, the risk of having a lower return has a probability of  $1 - \alpha$ . Thus the optimal payoff under the generalized EIA has a quite different profile from the optimal payoff under a conventional EIA. The major point is that except for a small probability (smaller than  $1 - \alpha = 15\%$ ), the optimal payoff under the generalized EIA beats the optimal payoff under the conventional EIA. We can observe a similar pattern from Panel (B) for  $k = 0.75$ .

### 5.2. Case two: $k < b$

This solution here is more complicated than in the previous case. The trade-off between the participation rate  $k$  and the probability  $\alpha$  is captured by the following condition:

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} E[\xi_T \Gamma 1_{\{a_\lambda \leq \xi_T \leq y_\lambda\}} \cup \{\bar{y}_\lambda \leq \xi_T < c\}] \\ + E[f x_0 \xi_T 1_{\{y_\lambda \leq \xi_T \leq \bar{y}_\lambda\}} \cup \{\xi_T \geq c\}}] < x_0. \end{aligned} \quad (5.5)$$

The rest of the procedure is similar to the first case  $k > b$  with some technical differences. In the current case, we need to solve for three variables  $\{y_\lambda, \bar{y}_\lambda, \lambda\}$  simultaneously. They are determined by the following three equations:

$$P(y_\lambda < \xi_T < \bar{y}_\lambda) = 1 - \alpha, \quad (5.6)$$

and

$$h_1(\lambda, y_\lambda) = h_2(\lambda, \bar{y}_\lambda), \quad (5.7)$$

and

$$E[\xi_T Y_{\lambda,\alpha}(T)] = x_0. \quad (5.8)$$

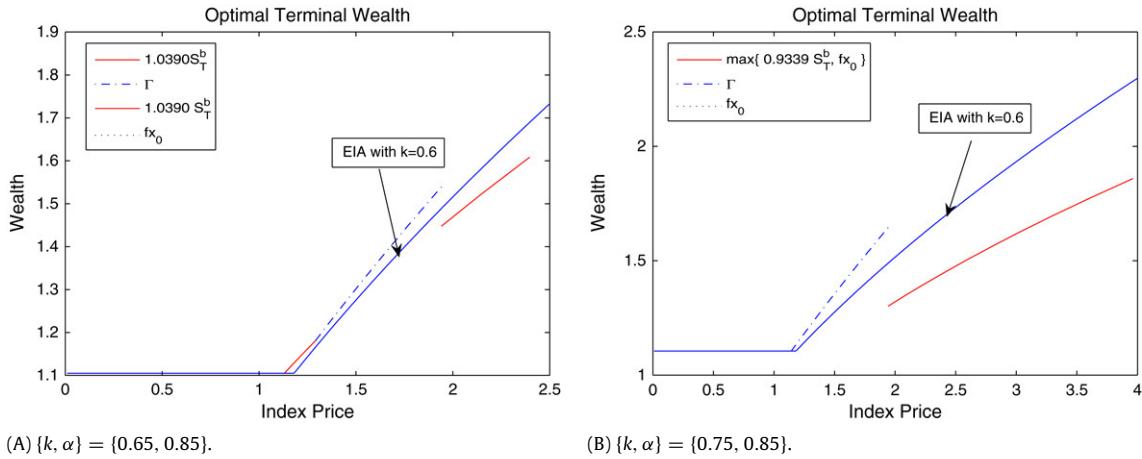
The procedure is then similar to the first case and we omit the details.

### 5.3. Case three: $k = b$

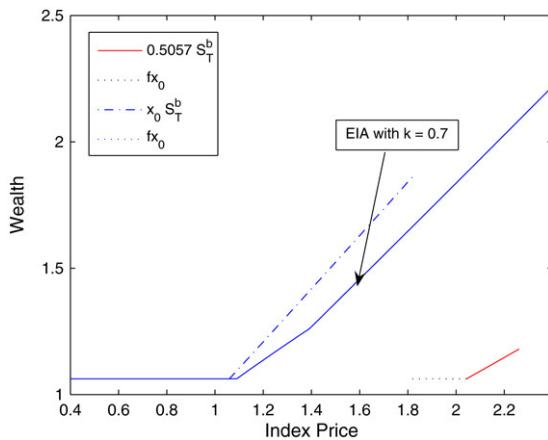
The last case is a special but it is the most challenging one from a theoretical perspective. The reason is as follows. Because  $k = b$ , [Assumption 2](#) does not hold and [Theorem 4.1](#) cannot be used directly. The trade-off between  $k$  and  $\alpha$  is captured by the following conditions:

$$\begin{aligned} f x_0 \bar{\xi}_{1-\alpha} E[1_{\{\xi_T \leq \bar{\xi}_{1-\alpha}\}}] + a x_0 E[1_{\{\bar{\xi}_{1-\alpha} < \xi_T < \frac{a}{f}\}}] \\ + f x_0 E[\xi_T 1_{\{\xi_T \geq \frac{a}{f}\}}] > x_0, \end{aligned} \quad (5.9)$$

<sup>17</sup> The calculation of  $\lambda^*$  in this paper is based on formulae in [Appendix B](#).



**Fig. 4.** Optimal Wealth of a Generalized EIA Contract with  $k > b$ . Investor has log utility and the index is log-normal distributed. Parameters:  $T = 5$ ,  $\mu = 6\%$ ,  $r = 4\%$ ,  $\sigma = 20\%$ ,  $g = 2\%$ ,  $x_0 = 1$ ,  $S_0 = 1$ . Note that  $b = 0.5$  in this case. Panel (A) displays the optimal payoff under the generalized EIA with  $k = 0.65$  and  $\alpha = 0.85$ . In Panel (B), the participation rate  $k = 0.75$  with the same confidence level  $\alpha = 0.85$ . The optimal payoff under a conventional EIA with participation rate 0.6 is also provided for comparison purposes.



**Fig. 5.** Optimal wealth of a generalized EIA contract with  $k = b$ . Investor has log utility and the index is log-normally distributed. Parameters:  $T = 5$ ,  $\mu = 7\%$ ,  $r = 4\%$ ,  $\sigma = 17\%$ ,  $g = 1.2\%$ ,  $\alpha = 0.8$ ,  $k = 1.0381$ . Note that  $b = k = 1.0381$  in this case. The optimal wealth is displayed in this Figure. Under these contract parameters, the maximum participation rate  $\hat{k} = 0.7135$  by solving Eq. (2.8). The optimal payoff under the conventional EIA with participation rate 0.7 is displayed as well for comparison purpose.

and

$$\begin{aligned} &fx_0E[\xi_T 1_{\{\xi_T \leq \bar{\xi}_{1-\alpha}\}}] + ax_0E[1_{\{\bar{\xi}_{1-\alpha} < \xi_T < \frac{g}{f}\}}] \\ &+ fx_0E[\xi_T 1_{\{\xi_T \geq \frac{g}{f}\}}] < x_0. \end{aligned} \quad (5.10)$$

If so, there exists a positive number  $\lambda^* > \frac{1}{fx_0\bar{\xi}_{1-\alpha}}$  such that  $W_{\lambda^*, \alpha}(T)$  represents the optimal payoff of the generalized EIA.

Fig. 5 displays the optimal payoff under the generalized EIA when  $k = b$ . In this Figure  $T = 5$ ,  $\mu = 7\%$ ,  $r = 4\%$ ,  $\sigma = 17\%$ ,  $g = 1.2\%$ ,  $\alpha = 0.8$ . Then  $b = 1.0381$  and  $\bar{\xi}_{1-\alpha} = 0.5434$ . The optimal  $\lambda^* = 1.9567 > \frac{1}{fx_0\bar{\xi}_{1-\alpha}} = 1.7332$ . In this case  $W_{\lambda^*, \alpha}(T)$  is the optimal payoff when  $\lambda^* = 1.9567$ . Fig. 5 also shows the terminal wealth of a conventional point-to-point EIA with participation rate 0.7. Similar to Fig. 4, we observe that the optimal payoff for the generalized EIA is higher than the optimal wealth of EIA when the index moves down or moves up moderately. However, if the market moves up significantly, with a probability  $1 - 80\% = 20\%$ , the conventional EIA performs better.

## 6. Conclusions

In this paper we analyzed the properties of EIAs from the investors' perspective. We showed that in general they are not efficient contracts for the consumer. We proposed a new type of EIA, the generalized EIA contract. This contract combines a guaranteed return together with the opportunity to beat the index. We provided the explicit solution for the optimal contract design in this case and presented several examples to illustrate the key features of these new contracts. The payoffs of these contracts display some interesting features including discontinuities and non-monotonicity with respect to the underlying index.

There is no doubt that these new contracts have some features that might make it difficult for insurance companies to start issuing them. The optimal payoff depends on the investor's preferences. One way to deal with this would be to issue a range of contracts that cover a broad range of risk aversion levels. The optimal payoffs tend to be complicated but they can be expressed in terms of the underlying index. However, the discontinuities make the contracts difficult to hedge. We do not explore this point here but note that investment banks now routinely sell complicated derivatives such as barrier options. The fact that the payoff may decrease when the index rises may not be appealing. It turns out that there are structured products<sup>18</sup> known as Index-Linked Securities available on the American Stock Exchange which have discontinuous payoffs that are not always increasing functions of the Index but we feel that in general most investors might not be attracted to such contracts. In summary there are several practical implementation problems associated with our new proposed design. We hope that our paper will help stimulate research on the problem of optimal contract design for equity-indexed annuities.

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<sup>18</sup> See for example the Absolute Buffer Notes issued by Lehman Brothers on the American Stock Exchange.

## Appendix A

The expression for  $\bar{\xi}_{1-\alpha}$  under our assumptions is:

$$\bar{\xi}_{1-\alpha} = \exp \left[ -\theta \sqrt{T} \Phi^{-1}(\alpha) - \left( r + \frac{\theta^2}{2} \right) T \right]. \quad (\text{A.1})$$

We also have :

$$\begin{aligned} a_\lambda &:= (\lambda x_0)^{\frac{b}{k-b}} a^{\frac{k}{k-b}}, & b_\lambda &:= \frac{1}{\lambda f x_0}, & c &:= a f^{-\frac{b}{k}}, \\ \lambda_\beta &:= \frac{1}{a x_0} f^{\frac{b}{k}-1}. \end{aligned} \quad (\text{A.2})$$

Here are the explicit expressions for  $Z_{\lambda,\alpha}(T)$ ,  $V_{\lambda,\alpha}(T)$ ,  $W_{\lambda,\alpha}(T)$  and  $Y_{\lambda,\alpha}(T)$ .

$$\begin{aligned} Z_{\lambda,\alpha}(T) &= \begin{cases} \frac{1}{a\lambda} \left( \frac{S_T}{S_0} \right)^b, & \text{if } S_T \geq S_0 \left( \frac{\bar{\xi}_{1-\alpha}}{a} \right)^{-1/b} \\ \Gamma, & \text{if } S_0 \left( \frac{a_\lambda}{a} \right)^{-1/b} < S_T < S_0 \left( \frac{\bar{\xi}_{1-\alpha}}{a} \right)^{-1/b} \\ \frac{1}{a\lambda} \left( \frac{S_T}{S_0} \right)^b, & \text{if } S_0 \left( \frac{b_\lambda}{a} \right)^{-1/b} < S_T \leq S_0 \left( \frac{a_\lambda}{a} \right)^{-1/b} \\ f x_0, & \text{if } S_T \leq S_0 \left( \frac{b_\lambda}{a} \right)^{-1/b} \end{cases} \\ V_{\lambda,\alpha}(T) &= \begin{cases} \max \left\{ \frac{1}{a\lambda} \left( \frac{S_T}{S_0} \right)^b, f x_0 \right\} & \text{if } S_T > S_0 \left( \frac{\bar{\xi}_{1-\alpha}}{a} \right)^{-1/b} \\ \Gamma, & \text{if } S_0 \left( \frac{c}{a} \right)^{-1/b} < S_T \\ & \leq S_0 \left( \frac{\bar{\xi}_{1-\alpha}}{a} \right)^{-1/b} \\ f x_0, & \text{if } S_T \leq S_0 \left( \frac{c}{a} \right)^{-1/b} \end{cases} \\ W_{\lambda,\alpha}(T) &= \begin{cases} \frac{1}{a\lambda} \left( \frac{S_T}{S_0} \right)^b, & \text{if } S_T \geq S_0 \left( \frac{b_\lambda}{a} \right)^{-1/b} \\ f x_0, & \text{if } S_0 \left( \frac{\bar{\xi}_{1-\alpha}}{a} \right)^{-1/b} \leq S_T < S_0 \left( \frac{b_\lambda}{a} \right)^{-1/b} \\ x_0 \left( \frac{S_T}{S_0} \right)^b, & \text{if } S_0 \left( \frac{1}{f} \right)^{-1/b} < S_T < S_0 \left( \frac{\bar{\xi}_{1-\alpha}}{a} \right)^{-1/b} \\ f x_0, & \text{if } S_T \leq S_0 \left( \frac{1}{f} \right)^{-1/b}. \end{cases} \end{aligned}$$

The last family is relatively complicated. It is defined as follows

$$\begin{aligned} Y_{\lambda,\alpha}(T) &= \begin{cases} \frac{1}{a\lambda} \left( \frac{S_T}{S_0} \right)^b, & \text{if } S_T \geq S_0 \left( \frac{a_\lambda}{a} \right)^{-1/b} \\ \Gamma, & \text{if } S_0 \left( \frac{y_\lambda}{a} \right)^{-1/b} \leq S_T < S_0 \left( \frac{a_\lambda}{a} \right)^{-1/b} \\ \max \left\{ \frac{1}{a\lambda} \left( \frac{S_T}{S_0} \right)^b, f x_0 \right\}, & \text{if } S_0 \left( \frac{\bar{y}_\lambda}{a} \right)^{-1/b} < S_T < S_0 \left( \frac{y_\lambda}{a} \right)^{-1/b} \\ \Gamma, & \text{if } S_0 \left( \frac{c}{a} \right)^{-1/b} < S_T \leq S_0 \left( \frac{\bar{y}_\lambda}{a} \right)^{-1/b} \\ f x_0, & \text{if } S_T \leq S_0 \left( \frac{c}{a} \right)^{-1/b} \end{cases} \end{aligned}$$

where  $y_\lambda$  and  $\bar{y}_\lambda$  are determined by the equations

$$P(y_\lambda < \bar{\xi}_T < \bar{y}_\lambda) = 1 - \alpha, \quad (\text{A.3})$$

and

$$h_1(\lambda, y_\lambda) = h_2(\lambda, \bar{y}_\lambda), \quad (\text{A.4})$$

where  $h_1(\cdot)$  and  $h_2(\cdot)$  are defined by formula (5.1) and (5.2), respectively.

## Appendix B. Formulae

Note that

$$E[\xi_T^m \mathbf{1}_{\{a \leq \xi_T \leq b\}}] = \exp \left\{ m \mu_\xi + \frac{1}{2} m^2 \sigma_\xi^2 \right\} \left\{ \Phi \left( \frac{\log(b) - \mu_\xi}{\sigma_\xi} \right) - \Phi \left( \frac{\log(a) - \mu_\xi}{\sigma_\xi} \right) \right\}$$

where  $\mu_\xi = -(r + \frac{1}{2}\theta^2)T$ ,  $\sigma_\xi = \theta\sqrt{T}$ .

Then we have

$$\begin{aligned} E[\xi_T Z_{\lambda,\alpha}(T)] &= \frac{1-\alpha}{\lambda} + \frac{1}{\lambda} \left\{ \Phi \left( \frac{\log(b_\lambda) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \right. \\ &\quad \left. - \Phi \left( \frac{\log(a_\lambda) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \right\} \\ &\quad + f x_0 e^{-rT} \left\{ 1 - \Phi \left( \frac{\log(b_\lambda) + (r - \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \right\} \\ &\quad + x_0 a^{\frac{k}{b}} \exp(A) \left\{ \Phi \left( \frac{\log(a_\lambda) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} + \left( \frac{k}{b} - 1 \right) \theta\sqrt{T} \right) \right. \\ &\quad \left. - \Phi \left( \frac{\log(\bar{\xi}_{1-\alpha}) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} + \left( \frac{k}{b} - 1 \right) \theta\sqrt{T} \right) \right\}. \end{aligned}$$

For  $\lambda \geq \frac{1}{f x_0 \bar{\xi}_{1-\alpha}}$ , we have

$$\begin{aligned} E[\xi_T V_{\lambda,\alpha}(T)] &= \frac{1}{\lambda} \Phi \left( \frac{\log(b_\lambda) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \\ &\quad + f x_0 e^{-rT} \left\{ 1 - \Phi \left( \frac{\log(c) + (r - \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \right. \\ &\quad \left. + \Phi \left( \frac{\log(\bar{\xi}_{1-\alpha}) + (r - \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \right\} \\ &\quad - \Phi \left( \frac{\log(b_\lambda) + (r - \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \\ &\quad + x_0 a^{\frac{k}{b}} \exp(A) \left\{ \Phi \left( \frac{\log(c) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} + \left( \frac{k}{b} - 1 \right) \theta\sqrt{T} \right) \right. \\ &\quad \left. - \Phi \left( \frac{\log(\bar{\xi}_{1-\alpha}) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} + \left( \frac{k}{b} - 1 \right) \theta\sqrt{T} \right) \right\}. \end{aligned}$$

If  $\lambda < \frac{1}{f x_0 \bar{\xi}_{1-\alpha}}$ , then

$$\begin{aligned}
E[\xi_T V_{\lambda,\alpha}(T)] &= \frac{1-\alpha}{\lambda} \\
&+ f x_0 e^{-rT} \left\{ 1 - \Phi \left( \frac{\log(c) + (r - \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \right\} \\
&+ x_0 a^{\frac{k}{b}} \exp(A) \left\{ \Phi \left( \frac{\log(c) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} + \left( \frac{k}{b} - 1 \right) \theta\sqrt{T} \right) \right. \\
&- \left. \Phi \left( \frac{\log(\bar{\xi}_{1-\alpha}) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} + \left( \frac{k}{b} - 1 \right) \theta\sqrt{T} \right) \right\} \\
E[\xi_T Y_{\lambda,\alpha}(T)] &= \frac{1}{\lambda} \Phi \left( \frac{\log(a_\lambda) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \\
&+ a^{\frac{k}{b}-1} x_0 \exp(A) \left\{ \Phi \left( \frac{\log(y_\lambda) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right. \right. \\
&+ \left( \frac{k}{b} - 1 \right) \theta\sqrt{T} \left. \right) - \Phi \left( \frac{\log(a_\lambda) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right. \\
&+ \left. \left. \left( \frac{k}{b} - 1 \right) \theta\sqrt{T} \right) \right\} + E[\xi_T \max\{I(\lambda\xi_T), f x_0\} \mathbf{1}_{\{y_\lambda < \xi_T < \bar{y}_\lambda\}}] \\
&+ a^{\frac{k}{b}-1} x_0 \exp(A) \left\{ \Phi \left( \frac{\log(c) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} + \left( \frac{k}{b} - 1 \right) \theta\sqrt{T} \right) \right. \\
&- \left. \Phi \left( \frac{\log(\bar{y}_\lambda) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} + \left( \frac{k}{b} - 1 \right) \theta\sqrt{T} \right) \right\} \\
&+ f x_0 e^{-rT} \left\{ 1 - \Phi \left( \frac{\log(c) + (r - \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \right\} \\
E[\xi_T W_{\lambda,\alpha}(T)] &= \frac{1}{\lambda} \Phi \left( \frac{\log(b) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \\
&+ a x_0 \left\{ \Phi \left( \frac{\log(\frac{a}{f}) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \right. \\
&- \left. \Phi \left( \frac{\log(\bar{\xi}_{1-\alpha}) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \right\} \\
&+ f x_0 e^{-rT} \left\{ \Phi \left( \frac{\log(\bar{\xi}_{1-\alpha}) + (r - \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \right. \\
&- \left. \Phi \left( \frac{\log(b_\lambda) + (r - \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right) \right\} \\
&+ 1 - \Phi \left( \frac{\log(\frac{a}{f}) + (r - \frac{1}{2}\theta^2)T}{\theta\sqrt{T}} \right).
\end{aligned}$$

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