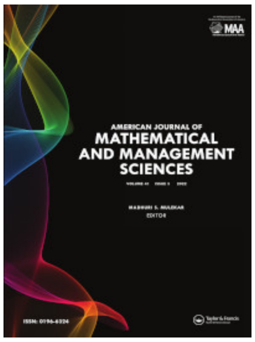

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Emergency Service Location Problem with Ring Roads

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ABSTRACT

A key purpose of a service network design is to determine the best possible location(s) for each service center. The facility services must provide a quick and easy response to callers within a reasonable distance, especially in urgent cases. It's well known that this problem is NP-hard, non-convex and non-differentiable optimization problem. However, if we make some simplifying assumptions, the problem could be solved within a polynomial time. This article introduces an algorithm for solving the problem of determining the best possible locations of the emergency service centers, concerning the case in which these centers are located on simple closed curves (e.g., ring roads). The proposed model can be applied in designing the emergency centers on ring roads in new cities, which have become one of the most important designs in solving traffic congestion problems. The problem is mathematically formulated, an algorithm for solving the problem under simplifying assumptions is proposed, the mathematics behind the algorithm is given, and the algorithm is illustrated by a numerical example.

KEYWORDS AND PHRASES
Analytical solution; computational algorithm; emergency location; ring roads; simple closed curve

1. Introduction

Accessing emergency service sites to aid citizens in the shortest possible time is one of the top priorities to be considered in real life. Facility service centers such as ambulances, medical centers, fire stations, and other ones must be easily accessible with no delay, especially when it relates to the lives and properties of the people. Therefore, it is important to find the optimum location(s) for these centers that minimize the maximum distance (or time, cost, etc.) from each demand point (caller) to its nearest service center. Moreover, to guarantee a good enough level of the service system, the service should be delivered within a specified limit (reasonable distance) especially in the worst expected circumstances. A special version from the class of emergency service location problem is investigated by Zimmermann (1991) who stated that a finite number of callers are to be served from a finite number of facility services in a given network, such that every caller in the service system is delivered service at the minimum possible cost with a focus on the most distant one. This form had considered a modulated version of the n center location problem, studied by Cuninghame-Green (1984). Also, it's a well-known problem in location theory for its use in modeling the location of emergency facility services.

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The idea of the problem is based on the concept of determining the p-center of a graph such that the maximum distance to a vertex from its closest center p is minimized (Hakimi, 1964). This concept is discussed and developed by many authors like Minieka (1970, 1977), Halfin (1974), and Hakimi et al. (1978). However, the general form of this problem is classified as NP-hard type that can't be solved in a polynomial time (Hudec, 1992). Later on, Zimmermann (1992) used the max-separable optimization criteria introduced in Jajou and Zimmermann (1989) to figure out an efficient solution to such a version of the location problem. And, under additional constraints, he solved it algorithmically in a polynomial time (Zimmermann, 1992). A generalization of this algorithm, including the emptiness of the feasible solution set and other extensive details are introduced by Tharwat and Zimmermann (2000). In those algorithms, they considered two available routes from each center to each caller. The problem is handled for the penalized and unpenalized case with three available routes situation (Tharwat & Hassan, 2007; Tharwat et al., 2009). Numerous researchers have addressed this special form of the location problem using various forms and approaches. For example, Tharwat and Zimmermann (2010) used a max-separable objective function and min-separable inequality constraints to solve one class of separable optimization problems. To be more realistic, Hassan (2018) solved the problem, assuming different weights to the callers reflecting their different priorities. The emergency location problems have several applications in many fields such as the medical field (Ahmadi-Javid et al., 2017; Benabdouallah & Bojji, 2018; Jánošíková et al., 2015), the firefighting (Hogg, 1968) and Aktaş et al., 2013, and the police system (De Leo, 2017; Swersey, 1994). A review on successive location models applied to emergency services can also be found in Eiselt (1992), Li et al. (2011), Marianov (2017), Pirabán et al. (2019), Başar et al. (2012), and Zhu et al. (2019).

Most of the above treatments assumed that the locations of the service centers are on traditional roads (line segments). In this research, we considered the ring roads (closed curves) as an alternative study. As in most of the modern urban design, it became very essential to avoid the traffic congestion problem and to offer fast routes between the different points of crowded cities. We introduce an investigation to the emergency service location problem when the service centers are located on closed curves, assuming that each caller is connected to its nearest service center by two available routes.

The remainder of this paper is organized as follows. In Section 2, the problem is thoroughly explained and formulated along with the assumptions for determining the problem's feasible regions. A well-structured and efficient algorithm for solving the proposed problem is given in Section 3. Section 4 provides the mathematical basis behind the algorithm as well as the analysis of all possible cases of the presumed service center locations. Section 5 gives a detailed numerical example to illustrate the algorithm step by step along with the expected different scenarios for the optimum locations of the service centers. Finally, conclusions and future direction points are given in Section 6.

2. Problem Formulation and Modelling

This section introduces a mathematical formulation to the problem of determining the location of n emergency service centers, each of these centers has to be located on a ring road (simple closed curve). We assume that each caller for service has a fixed

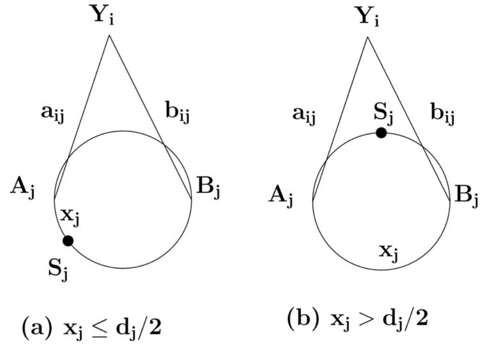


Figure 1. The distance between the service center and the exit points.

position and is connected to each service center by two available routes. The object is to determine the optimum locations for these centers such that the maximum distance between each caller and its nearest service center is minimized. The description of the problem is as follows.

Assume that there are m callers, each one is located at a fixed point Y_i, $i \in M = \{1, ..., m\}$, these points are to be served by n-service centers S_j, $j \in N = \{1, ..., n\}$. Each service center S_j has to be placed on a given simple closed curve D_j, $j \in N$. Let d_j be the length of D_j, as the shape of each curve D_j does not influence the formulation of the problem, without loss of generality, each D_j can be represented using a circle with a circumference d_j. Each of these circles D_j is connected to each caller through two exits A_j and B_j which lie on two endpoints of the diagonal of this circle, such that the length of the arc A_jB_j equals half of the circumference d_j, that's $|A_jB_j| = d_j/2$ (Figure 1). For each $i \in M$ and $j \in N$, let a_{ij} be the distance between Y_i and A_j. Similarly, let b_{ij} be the distance between Y_i and B_j. We assume that these distances are known. For simplicity, the distances a_{ij} and b_{ij} are represented as linear segments in Figure 1.

To quantitatively identify the position S_j for each service center on each curve D_j, we define the distance x_j as the length of the directed arc A_jS_j anticlockwise direction, so that x_j is unique, and lies in the interval $[0, d_j)$, Figure 1. (If S_j is at the point B_j, then $x_j = d_j/2$). If a caller Y_i is assigned to the center S_j, then the response is delivered through one of the two exits A_j or B_j, and as the curve D_j is closed, the server will use the shorter of the two arcs joining S_j with each of the two exits. Therefore, we have four possible routes between S_j and Y_i. Two of these routes are obtained from Figure 1(a), where $x_j \leq d_j/2$, and the others are obtained from Figure 1(b), where $x_j > d_j/2$. Accordingly,

- If $x_j \leq d_j/2$, then the available routes from S_j to Y_i are $S_jA_jY_i$ with length $x_j + a_j$ and $S_jB_jY_i$ with length $d_j/2 - x_j + b_{ij}$, and
- If $x_j > d_j/2$, then the available routes from S_j to Y_i are $S_jA_jY_i$ with length $d_j - x_j + a_{ij}$, and $S_jB_jY_i$ with length $x_j - d_j/2 + b_{ij}$.

Therefore, we have the following four viable linear functions $r_k(x_j), k = 1, ..., 4$. From Figure 1(a), we have

$$r_1(x_j) = x_j + a_{ij}, \quad x_j \in [0, d_j/2), \tag{1}$$

$$r_2(x_j) = d_j/2 - x_j + b_{ij}, \quad x_j \in [0, d_j/2), \tag{2}$$

and from Figure 1(b), we have

$$r_3(x_j) = x_j - d_j/2 + b_{ij}, \quad x_j \in [d_j/2, d_j), \tag{3}$$

$$r_4(x_j) = d_j - x_j + a_{ij}, \quad x_j \in [d_j/2, d_j). \tag{4}$$

Then, the service will be delivered from S_j to Y_i through the shortest route of these four routes with distance $r_{ij}(x_j)$, which is represented by

$$r_{ij}(x_j) = \min_{k=1,\dots,4} \{r_k(x_j)\}, \quad i \in M, \ j \in N. \tag{5}$$

Equation (5) represents the minimum distance which must be covered if Y_i is served from the center S_j.

Now, we assume that each caller will be served by its nearest service center, say $S_t, t \in N$, with the smallest available distance from S_j to Y_i over all $j \in N$. Accordingly, we have

$$\min_{j \in N} r_{ij}(x_j) = r_{it}(x_t) \quad \text{for some } t \in N. \tag{6}$$

Finally, the performance quality of the emergency service management system in the above description is obtained by (minimizing) the following function over all $i \in M$

$$f(x_1, \dots, x_n) = \max_{i \in M} \min_{j \in N} r_{ij}(x_j), \quad \text{for each } (x_1, \dots, x_n) : 0 \le x_j \le d_j, \ j \in N. \tag{7}$$

In addition, we also need to guarantee a good enough level of the performance quality of this service system. Consequently, it's reasonable to impose a specified threshold α so that the maximum distance, which must be covered from S_j to each caller in the system, does not exceed α, i.e., $f(x_1, \dots, x_n) \le \alpha$.

Therefore, the set of feasible solutions (locations) $x_j, \ j \in N$ of the service centers $S_j, \ j \in N$ such that the performance quality of the service delivered to each caller in the system $Y_i, \ i \in M$ is "good enough service", is defined as follows:

$$L(\alpha) = \{x = (x_1, \dots, x_n) | 0 \le x_j \le d_j, \ j \in N \ \& f(x_1, \dots, x_n) \le \alpha\}. \tag{8}$$

Accordingly, the emergency location problem, say P1, can be formulated as follows:

$$\begin{aligned} &\text{P1 : Objective function} && \mathit{min}\ f(x_1, \dots, x_n) \\ &\text{subject to} \quad x \in L(\alpha), && \\ & && 0 \le x_j \le d_j, \ j \in \{1, 2, \dots, n\}. \end{aligned} \tag{9}$$

The general form of the problem P1 represented in (9) is classified as NP-hard, that is, it cannot be solved in a polynomial time (Hudec, 1992). It's also a non-convex and non-differentiable optimization problem (Zimmermann, 1991).

The following are the two assumptions that had been investigated by Zimmermann (1991, 1992), for which P1 is no longer an NP-hard problem as it can be solved in a polynomial time. We will recall these assumptions in our case to suggest an efficient algorithm for our problem.

Assumption 1. For each service center S_j, assume that the set of points $\{Y_i,\ i \in M\}$ can be rearranged in ascending or descending order so that the distance of both exits A_j and B_j from $Y_{i_1}(j)$ is smaller than or equal the distance of these points from $Y_{i_2}(j)$, and so on up to $Y_{i_m}(j)$. That is, for each fixed $j,\ j \in N$, say j^*, there exists a permutation $\pi_{j^*} = (i_1(j^*), i_2(j^*), \dots, i_m(j^*)) \approxeq i_1, i_2, \dots, i_m$ of indices $(1, 2, \dots, m)$ such that the following inequalities hold.

$$(a_{i_1 j^*}, b_{i_1 j^*}) \leq (a_{i_2 j^*}, b_{i_2 j^*}) \leq \dots \leq (a_{i_{m-1} j^*}, b_{i_{m-1} j^*}) \leq (a_{i_m j^*}, b_{i_m j^*}), \tag{10}$$

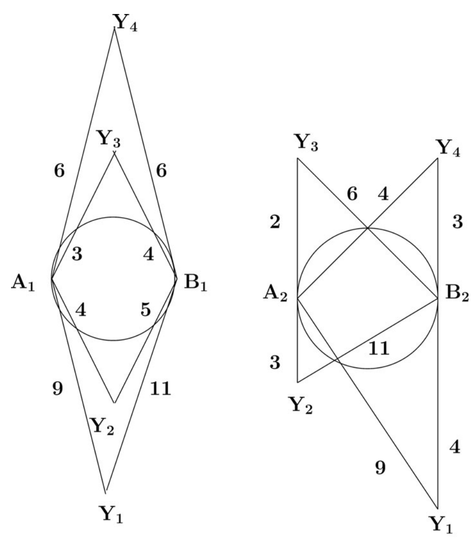
where $(a_{i_1 j^*}, b_{i_1 j^*}) \leq (a_{i_2 j^*}, b_{i_2 j^*})$ means that $a_{i_1 j^*} \leq a_{i_2 j^*}$ and $b_{i_1 j^*} \leq b_{i_2 j^*}$. If the above inequalities are satisfied, the pairs of distances are said to be nested.

In Figure 2(a), we can see that Assumption 1 is satisfied. That is, the callers in the example, given $m = 4$, will be served in ascending order as $Y_{3j}, Y_{2j}, Y_{4j}, Y_{1j}$, since we have $(a_{3j}, b_{3j}) = (3, 4) \leq (a_{2j}, b_{2j}) = (4, 5) \leq (a_{4j}, b_{4j}) = (6, 6) \leq (a_{1j}, b_{1j}) = (9, 11)$. In Figure 2(b), Assumption 1 is violated because the points $Y_{3j}, Y_{2j}, Y_{4j}, Y_{1j}$ cannot be rearranged in ascending nor descending order because we have $(a_{3j}, b_{3j}) = (2, 6) \not\leq (a_{2j}, b_{2j}) = (3, 11) \not\leq (a_{4j}, b_{4j}) = (4, 3) \not\leq (a_{1j}, b_{1j}) = (9, 4)$. Thus, for each j only those pairs (a_{ij}, b_{ij}) can be ordered for which the set $\{x_j | r_{ij}(\alpha) \leq \alpha\}$ is non-empty.

The set of points for which the center S_j can be located on D_j to serve each caller Y_i within a distance $\alpha > 0$ is defined as follows

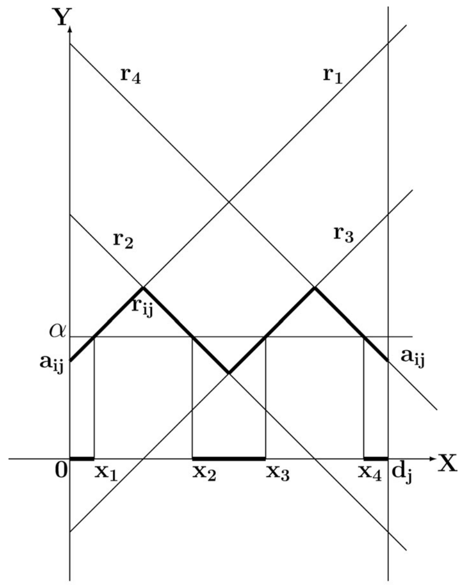
$$V_{ij}(\alpha) = \{x_j : 0 \leq x_j \leq d_j \ and \ r_{ij}(x_j) \leq \alpha\}, \forall\ i \in M,\ j \in N. \tag{11}$$

Assumption 2. For a fixed $j,\ j \in N$, say j^* assume that there exists a permutation $\pi_{j^*} = (i_1(j^*), i_2(j^*), \dots, i_m(j^*)) \approxeq i_1, i_2, \dots, i_m$ of the indices $(1, 2, \dots, m)$ such that the following nested relationships hold.



(a) Nested (b) Not Nested

Figure 2. (a) Assumption 1 is satisfied (nested) and (b) Assumption 1 is violated (not nested).



$$V_{ij}(\alpha) = [0, x_1] \cup [x_2, x_3] \cup [x_4, d_j]$$

Figure 3. Possible locations (bold horizontal lines) of a service center for one customer.

$$V_{i_1j^*}(\alpha) \supseteq V_{i_2j^*}(\alpha) \supseteq \ldots \supseteq V_{i_mj^*}(\alpha). \tag{12}$$

This means that the sets $V_{ij^*}(\alpha)$ are nested, where the intersection of its elements is the set $V_{i_mj^*}(\alpha)$ (last one in (12)). Figure 3 illustrates one of the sets $V_{ij^*}(\alpha)$, which are indicated by bold horizontal lines.

In general, the process of finding the intersection of n sets is not a polynomial time process. But, Assumption 2 allows us to determine this intersection in a reasonable time. From the previous discussion, it follows that the problem P1 in (9) can be reformulated as follows

$$\begin{array}{ll} \text{P2 : Objectivefunction} & \min_{\alpha} f(x_1, \ldots, x_n) \\ \text{subject to} \quad L(\alpha) \neq \phi. & \end{array} \tag{13}$$

We are now ready to present the proposed algorithm for finding the solution to Problem P2.

3. The Proposed Algorithm

The input and output variables for the problem are as follows:

Input:

- The number of callers m at fixed (different) positions Y_i,
- The number n of emergency service centers S_j,
- The distance a_{ij} from Y_i to the exit A_j, $\forall\ i \in M$ and $\forall\ j \in N$,
- The distance b_{ij} from Y_i to the exit B_j, $\forall\ i \in M$ and $\forall\ j \in N$,
- The length (circumference) d_j of the closed curve D_j, $\forall\ j = 1, \ldots n$, and
- The maximum threshold $\tilde{\alpha}$(optional).

Output:

- The optimal value of α, and
- The optimal solution x_j^{opt}.

For the optimization problem formulated in Section 2, we now propose the following algorithm for solving the problem P2 in (13).

1. Check the validity of Assumption 1. If it is invalid, write "Assumption 1 is invalid" and stop, otherwise, continue.
2. Calculate the values $\delta_{ij} = min\{a_{ij}, b_{ij}\}$, $\forall\ i \in M$ and $\forall\ j \in N$ as in (16).
3. Calculate the values $\tilde{\alpha}_i = min_{j\in\{1,\dots,n\}}\delta_{ij}$, $\forall\ i \in M$ as in (18), and $\tilde{\alpha} = max\{\tilde{\alpha}_i : i = 1,\dots,m\}$ as in (20).
4. If the threshold α is not given, then set $\alpha = \tilde{\alpha}$, else, go to Step 6.
5. If $\tilde{\alpha} \le \alpha$, then set $\alpha = \tilde{\alpha}$ and go to Step 6, else write "Infeasible solution" and stop.
6. $\forall\ i \in M$ and $\forall\ j \in N$, calculate the value $\rho_{ij} = max\{r_{ij}(x_j) : 0 \le x_j \le d_j\}$ as in (17).
7. Calculate the values $\mu_{ij}^{*}(\alpha), \mu_{ij}^{**}(\alpha), \nu_{ij}^{*}(\alpha)$, and $\nu_{ij}^{**}(\alpha)$ (see Lemma 2).
8. Calculate the matrix $V_{ij}(\alpha)$ by applying the cases that are investigated in Section4.
9. Find an index $j(i)$ such that: $V_{ij}(\alpha) \ne \phi, \forall\ i \in M$ (see Corollary 1). If there are more than one such index, break the tie arbitrarily. If $\forall\ j \in N, V_{ij}(\alpha) = \phi$, then write "The problem is infeasible" and stop.
10. Compute $P_j(\alpha) = \{i : i \in \{1,\dots,m\}, V_{ij}(\alpha) \ne \phi\}$, $\forall\ j \in N$ as in (14).
11. Compute $V_j(\alpha) = \cap_{i\in P_j(\alpha)} V_{ij}(\alpha)$, $\forall\ j \in N$ as in (15).
12. If $P_j(\alpha) \ne \phi, \forall\ j \in N$, then the optimal solution is $x_j^{opt} \in V_j(\alpha)$, otherwise $x_j^{opt} \in [0, d_j)$.

4. Mathematical Background for the Solution

In this section, we provide some definitions and results which are important to construct the computational algorithm for solving the problem P2 in (13). Then we provide a solution to the problem.

The problem P2 has a solution if each caller in the service system, Y_i, is served by at least one service center S_j. This requires the set $L(\alpha)$ in (8) to be non-empty, which implies the following results, which are found in Tharwat and Zimmermann (2000).

Corollary 1. *$L(\alpha) \ne \phi \Longleftrightarrow \forall\ i \in \{1,\dots,m\}$, $\exists\ j(i) \in \{1,\dots,n\}$, such that $V_{ij(i)}(\alpha) \ne \phi$.*

Corollary 2. *If Assumption 2 is fulfilled and $V_{i_1 j^*}(\alpha) = \dots = V_{i_{l-1} j^*}(\alpha) = \phi, V_{i_l j^*}(\alpha) \ne \phi$, then it holds that: $V_j(\alpha) = V_{i_l j^*}(\alpha)$, where*

$$P_j(\alpha) = \{i : i \in \{1,\dots,m\},\ \ V_{ij}(\alpha) \ne \phi\},\ \ \forall\ j \in N, \tag{14}$$

and

$$V_j(\alpha) = \cap_{i \in P_j(\alpha)} V_{ij}(\alpha), \;\; \forall \; j \in N. \tag{15}$$

Note that the lowest and the highest values of $r_{ij}(x_j)$ determine δ_{ij} and ρ_{ij}, respectively, as follows:

$$\delta_{ij} = min\{r_{ij}(x_j) : 0 \leq x_j \leq d_j\} = min\{a_{ij}, b_{ij}\}, \;\; \forall \; i \in M, \;\; j \in N \tag{16}$$

and

$$\rho_{ij} = max\{r_{ij}(x_j) : 0 \leq x_j \leq d_j\}. \tag{17}$$

Note that δ_{ij} is the min of $r_{ij}(x_j)$, which means that $\delta_{ij} = min\{r_{ij}(0), r_{ij}(d_j/2)\} = min\{a_{ij}, b_{ij}\}$, and ρ_{ij} value is reached at one of the intersection of the basic linear functions, and this can be found in each case separately as will be discussed here.

Lemma 1. *Let*

$$\tilde{\alpha}_i = min\{\alpha, \alpha \in R^+ : \exists \text{ at least one } j \in \{1, \dots, n\} \text{ such that } V_{ij}(\alpha) \neq \phi\}, \forall i \in M \tag{18}$$

then

$$\tilde{\alpha}_i = min_{j \in \{1,\dots,n\}} \delta_{ij}, \forall \; i \in M. \tag{19}$$

Corollary 3. *The optimal solution of the optimization problem P2 is given by*

$$\tilde{\alpha} = max\{\tilde{\alpha}_i : \;\; i = 1, \dots, m\}. \tag{20}$$

Proof. If $\tilde{\alpha} = max\{\tilde{\alpha}_i : \; i = 1, \dots, m\}$, then $\alpha \geq max\{\tilde{\alpha}_i : \; i = 1, \dots, m\}$. But, $\tilde{\alpha}_i = min_{j \in \{1,\dots,n\}} \delta_{ij}$, therefore $\forall \; i \in \{1, 2, \dots m\} \exists \; j^*$ such that $\delta_{ij^*} \leq \alpha$. This means that, $V_{ij^*}(\alpha) \neq \phi$ and α is a solution of the problem P2.

On the other hand, assume that $\alpha \leq max\{\tilde{\alpha}_i : \; i = 1, \dots, m\}$, so $\exists \; i^*$ such that $\tilde{\alpha}(i^*) > \alpha$. Then, $\forall j \in \{1, 2, \dots n\}$, $\delta_{i^*j} > \alpha$ and $V_{i^*j}(\alpha) = \phi$, which means that α is not a solution for P2. Then, the minimum value of the solution set for P2 (the optimum solution) is $\alpha = max\{\tilde{\alpha}_i : \; i = 1, \dots, m\}$. □

For completing the final solution of P2 in (13), we need to determine the set $V_{ij}(\alpha)$ defined in (11), for each i, j, which represents the possible locations that can be imposed for any service center S_j to serve the caller Y_i within a given threshold α. For this purpose, we have the following lemma.

Lemma 2. *If $V_{ij}(\alpha) \neq \phi$, then at least one of the following four conditions is satisfied:*

1. $\mu^*_{ij}(\alpha) = \alpha - a_{ij} \geq 0$.
2. $\mu^{**}_{ij}(\alpha) = d_j - \alpha + a_{ij} \leq d_j$.
3. $\nu^*_{ij}(\alpha) = d_j/2 + b_{ij} - \alpha \leq d_j/2$.
4. $\nu^{**}_{ij}(\alpha) = \alpha + d_j/2 - b_{ij} \geq d_j/2$.

Proof. If $V_{ij}(\alpha) \neq \phi$, then $\exists \; x_j \in [0, d_j)$ such that $r_{ij}(x_j) \leq \alpha \;\; \Rightarrow \;\; \delta_{ij} \leq \alpha$. But, $\delta_{ij} = a_{ij} \;\; or \;\; \delta_{ij} = b_{ij}$, then $a_{ij} \leq \alpha \;\; or \;\; b_{ij} \leq \alpha$. Clearly, we can conclude the following.

1. If $a_{ij} \leq \alpha$, then $a_{ij} - \alpha \leq 0 \Rightarrow {\mu^*}_{ij}(\alpha) \geq 0$.

2. If $a_{ij} \leq \alpha$, then $d_j + (a_{ij} - \alpha) \leq d_j \Rightarrow \mu_{ij}^{**}(\alpha) \leq d_j$.
3. If $b_{ij} \leq \alpha$, then $b_{ij} - \alpha \leq 0 \Rightarrow {\nu^*}_{ij}(\alpha) \leq d_j/2$.
4. If $b_{ij} \leq \alpha$, then $\alpha - b_{ij} \geq 0 \Rightarrow \nu_{ij}^{**}(\alpha) \geq d_j/2$. □

The different situations of the set $V_{ij}(\alpha)$ in terms of the values a_{ij}, b_{ij}, d_j *and* α are investigated here in this section. In fact, the set $V_{ij}(\alpha)$, is a subset of the interval $[0, dj)$ in the form of the finite set of points, or union of intervals, or one point or an interval, or even an empty set. It's important to know that these situations of $V_{ij}(\alpha)$ will result from the different cases of the relations among the values of the given parameters. Those possible cases are deduced as, (1) Case 1: If $a_{ij} > b_{ij}$, (2) Case 2: If $b_{ij} > a_{ij}$, and (3) Case 3: If $a_{ij} = b_{ij}$. In addition, each of these cases will be branching into sub-cases (Table 1). Because the derivations of $V_{ij}(\alpha)$ for most of the cases are roughly similar, we discuss these derivations and results for only cases 1(a) and 1(b) in Table 1.

Case 1(a): The set $V_{ij}(\alpha)$ for sub-case 1(a) in Table 1 is derived as follows:

1. $\delta_{ij} = b_{ij}$.
2. $\rho_{ij} = (d_j/2 + b_{ij} + a_{ij})/2$.
3. $[(3d_j/2 - b_{ij} + a_{ij})/2, d_j]$ is non empty.
4. The inequality $b_{ij} - d_j/2 < a_{ij} < b_{ij} + d_j/2 < d_j + a_{ij}$ holds.
5. The function $r_{ij}(x_j)$ is determined by

$$r_{ij}(x_j) = \begin{cases} r_1(x_j) & \textit{if} \quad x_j \in [0, (d_j/2 + b_{ij} - a_{ij})/2), \\ r_2(x_j) & \textit{if} \quad x_j \in [(d_j/2 + b_{ij} - a_{ij})/2, d_{ij}/2), \\ r_3(x_j) & \textit{if} \quad x_j \in [d_{ij}/2, (3d_j/2 - b_{ij} + a_{ij})/2), \\ r_4(x_j) & \textit{if} \quad x_j \in [(3d_j/2 - b_{ij} + a_{ij})/2, d_j). \end{cases}$$

Accordingly, the set $V_{ij}(\alpha)$ can be derived as follows:

i. If $\alpha < \delta_{ij}$, then $V_{ij}(\alpha) = \phi$.
ii. If $\alpha = \delta_{ij}$, then $V_{ij}(\alpha) = \{d_j/2\}$.
iii. If $\delta_{ij} < \alpha < a_{ij}$, then $V_{ij}(\alpha) = [d_j/2 - (\alpha - b_{ij}),\ d_j/2 + (\alpha - b_{ij})]$.
iv. If $a_{ij} = \alpha$, then $V_{ij}(\alpha) = \{0\} \cup [d_j/2 - (\alpha - b_{ij}),\ d_j/2 + (\alpha - b_{ij})]$.
v. If $a_{ij} < \alpha < \rho_{ij}$, then $V_{ij}(\alpha) = [0, \alpha - a_{ij}] \cup [d_j/2 - (\alpha - b_{ij}),\ d_j/2 + (\alpha - b_{ij})] \cup [d_j - (\alpha - a_{ij}),\ d_j)$.
vi. If $\alpha \geq \rho_{ij}$, then $V_{ij}(\alpha) = [0, d_j)$.

Case 1(b): For the derivation of the set $V_{ij}(\alpha)$ in sub-case 1(b) in Table 1, we have the following two possibilities:

- The inequality $b_{ij} - d_j/2 < a_{ij} < b_{ij} + d_j/2 < d_j + a_{ij}$ holds if $a_{ij} < b_{ij} + d_j/2$. In this case, the derivation of the set $V_{ij}(\alpha)$ is the same as sub-case 1(a).
- The inequality $b_{ij} - d_j/2 < b_{ij} + d_j/2 < a_{ij} < d_j + a_{ij}$ holds if $a_{ij} > b_{ij} + d_j/2$. In this case, we have:
  1. $\delta_{ij} = b_{ij}$.
  2. $\rho_{ij} = d_j/2 + b_{ij}$.
  3. $[d_j/2,\ (3d_j/2 - b_{ij} + a_{ij})/2)$ is non empty.

Table 1. Sub-cases of all possible cases.

Sub-case	Case 1	Case 2	Case 3
(a)	$b_{ij} < a_{ij} < d_j/2$	$a_{ij} < b_{ij} < d_j/2$	$b_{ij} = a_{ij} < \frac{d_j}{2}$
(b)	$b_{ij} < d_j/2 < a_{ij}$	$a_{ij} < d_j/2 < b_{ij}$	$b_{ij} = a_{ij} > d_j/2$
(c)	$d_j/2 < b_{ij} < a_{ij}$	$d_j/2 < a_{ij} < b_{ij}$	$b_{ij} = a_{ij} = d_j/2$
(d)	$a_{ij} = d_j/2 > b_{ij}$	$a_{ij} = d_j/2 < b_{ij}$	
(e)	$b_{ij} = d_j/2 < a_{ij}$	$b_{ij} = d_j/2 > a_{ij}$	

The function $r_{ij}(x_j)$ can be determined by

$$r_{ij}(x_j) = \begin{cases} r_2(x_j) & \textit{if} \quad x_j \in \left[(d_j/2 + b_{ij} - a_{ij})/2,\ d_j/2\right), \\ r_3(x_j) & \textit{if} \quad x_j \in \left[d_j/2,\ (3d_j/2 - b_{ij} + a_{ij})/2\right). \end{cases}$$

Accordingly, the set $V_{ij}(\alpha)$ can be derived as follows:

i. If $\alpha < \delta_{ij}$, then $V_{ij}(\alpha) = \phi$.
ii. If $\alpha = \delta_{ij}$, then $V_{ij}(\alpha) = \{d_j/2\}$.
iii. If $\delta_{ij} < \alpha < \rho_{ij}$, then $V_{ij}(\alpha) = [d_j/2 - (\alpha - b_{ij}),\ d_j/2 + (\alpha - b_{ij})]$.
iv. If $\alpha \geq \rho_{ij}$, then $V_{ij}(\alpha) = [0, d_j)$.

Note that the derivation of $V_{ij}(\alpha)$ is a straightforward step after knowing the parameters a_{ij}, b_{ij}, d_j, and α.

5. Numerical Example

A new city consisting of five neighborhoods ($m = 5$) will be served by four ambulance centers ($n = 4$). Every center S_j will be located at an interior point on the *jth* ring road, where d_j is the length (in *km*) of the *jth* ring roads. Here we have $d_1 = d_2 = d_3 = 22$, $d_4 = 16$, respectively. Assume that every ring road has two exits dividing it into two equal parts. Assume also that the matrix of distances (a_{ij}, b_{ij}) from the *ith* neighborhood to each of the two exit points A_j and B_j, respectively, of the curve D_j is given by

$$[(a_{ij}, b_{ij})] = \begin{bmatrix} (3,3) & (9,10) & (6,6) & (3,5) \\ (4,6) & (8,9) & (7,8) & (5,6) \\ (6,7) & (6,7) & (9,10) & (7,8) \\ (7,7) & (5,6) & (3,2) & (9,10) \\ (8,8) & (3,3) & (5,4) & (8,9) \end{bmatrix}.$$

As shown in Figure 4, we illustrate the proposed algorithm using each of the following three scenarios.

- **Scenario 1:** If no maximum distance is given by the management, i.e., no threshold α is imposed.
- **Scenario 2:** If the management decided that the maximum distance between a caller and its nearest service center is 8, i.e., $\alpha = 8$.
- **Scenario 3:** If the management decided that the maximum distance between a caller and its nearest service center is 5, i.e., $\alpha = 5$.

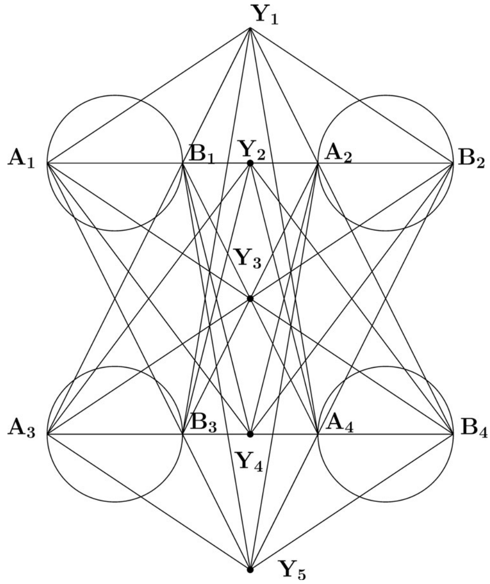


Figure 4. Numerical example-graphical illustration.

Before applying any of these scenarios the algorithm must first check if Assumption 1 in (10) is satisfied to go to next step. In our example, Assumption 1 is satisfied, hence the algorithm proceeds to Step 2 and obtains

$$\delta_{ij} = min(a_{ij}, b_{ij}) = \begin{bmatrix} 3 & 9 & 6 & 3 \\ 4 & 8 & 7 & 5 \\ 6 & 6 & 9 & 7 \\ 7 & 5 & 2 & 9 \\ 8 & 3 & 4 & 8 \end{bmatrix}.$$

Applying Step 3 we calculate: $\tilde{\alpha}_i = (3, 4, 6, 2, 3)$ and $\tilde{\alpha} = 6$.
If the first Scenario is applied, then by Step 4, we set $\alpha = \tilde{\alpha} = 6$. By Steps 6 and 7, we have

$$V_{ij}(\alpha) = V_{ij}(6) =$$
$$\begin{bmatrix} [0,3] \cup [8,14] \cup [19,22) & \phi & \{0,11\} & [0,3] \cup [7,9] \cup [13,16) \\ [0,2] \cup \{11\} \cup [20,22) & \phi & \phi & [0,1] \cup \{8\} \cup [15,16) \\ \{0\} & \{0\} & \phi & \phi \\ \phi & [0,1] \cup \{11\} \cup [21,22] & [0,3] \cup [7,15] \cup [19,22) & \phi \\ \phi & [0,3] \cup [8,14] \cup [19,22) & [0,1] \cup [9,13] \cup [21,22) & \phi \end{bmatrix}.$$

It is easy to see that for each j, there is an i such that $V_{ij}(6)$ is nonempty, and the feasible region for every service center could be found by Steps 8–11 as:

1. $x_1 \in \{0\}$, so S_1 could be placed at the exit point A_1 to serve the customers Y_1, Y_2, and Y_3 (the callers with non-empty sets).

2. $x_2 \in \{0\}$, so S_2 should be placed at the exit point A_2 to serve the customers Y_3, Y_4, and Y_5.
3. $x_3 \in \{0, 11\}$, so S_3 could be placed either at the exit point A_3 or B_3 to serve the customers Y_1, Y_4, and Y_5.
4. $x_4 \in [0, 1] \cup \{8\} \cup [15, 16)$, so S_4 could be placed either at the point B_4 or at any point that is not far from A_4 by more than $1km$ to serve the customers Y_1 and Y_2.

The first scenario implies that the management cannot decide to offer service for all given customers in the system by only one service center, while there will be various choices to cover the given system. By Step 12, these choices are obtained by one of the following decisions.

- Constructing two service centers at $\{x_1, x_2\}$, $\{x_1, x_3\}$, or $\{x_2, x_4\}$.
- Constructing three service centers at $\{x_1, x_2, x_3\}$, $\{x_1, x_3, x_4\}$, or $\{x_2, x_3, x_4\}$.
- Constructing four service centers at $\{x_1, x_2, x_3, x_4\}$.

Usually, the decision of the management depends on various factors like the available budget, the expected number of calls received from each caller point, and the capacity of each service center.

If the second Scenario is applied, we have $\alpha = 8$ and $\alpha > \tilde{\alpha} = 6$. Therefore, by Step 5, the algorithm sets $\alpha = \tilde{\alpha} = 6$, which yields the same results as in the first scenario.

If the third Scenario is applied, we have $\alpha = 5$ and $\tilde{\alpha} = 6$, therefore the algorithm finds by Step 5 that $\alpha = 5$. Accordingly,

$$V_{ij}(\alpha) = V_{ij}(5) =$$
$$\begin{bmatrix} [0,2] \cup [9,13] \cup [20,22) & \phi & \phi & [0,2] \cup \{8\} \cup [14,16) \\ [0,1] \cup [21,22) & \phi & \phi & \{0\} \\ \phi & \phi & \phi & \phi \\ \phi & \{0\} & [0,2] \cup [8,14] \cup [20,22) & \phi \\ \phi & [0,2] \cup [9,13] \cup [20,22) & \{0\} \cup [10,12] & \phi \end{bmatrix}.$$

In the third scenario, the problem has no solution because the third customer Y_3 in the given system cannot be served by a service center within a distance less than or equal to 5. Note that, the third row in $V_{ij}(5)$ contains empty sets.

6. Conclusions and Future Directions

This work introduces a numerical procedure for solving a problem from the class of n-emergency service location problems using a new formulation. The proposed problem consists of locating n-service centers (S_j, $j \in N = \{1, \ldots, n\}$), each on an arc A_jB_j of a given simple closed curve (e.g., ring roads) D_j, $j = 1, \ldots, n$ with a length d_j, $d_j > 0$, where $|A_jB_j| = d_j/2$ and the given points $Y_1, Y_2, \ldots, Y_m$ are assumed to be served from one of the service points $S_1, S_2, \ldots, S_n$. The algorithm is illustrated by a simple numerical example but it can be used in real-life applications for locating service centers on ring-roads when planning new cities or residential areas, taking into consideration the proposed configuration.

In *future work* the problem can be investigated in the following situations.

- The solution can be derived in uncertain parameters case, like stochastic or fuzzy.
- The presented model suggests two routes between each service center and each customer, this can be extended to n routes.
- Many simulation methods can be used to derive an approximate solutions which might offer more realizable solution.

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