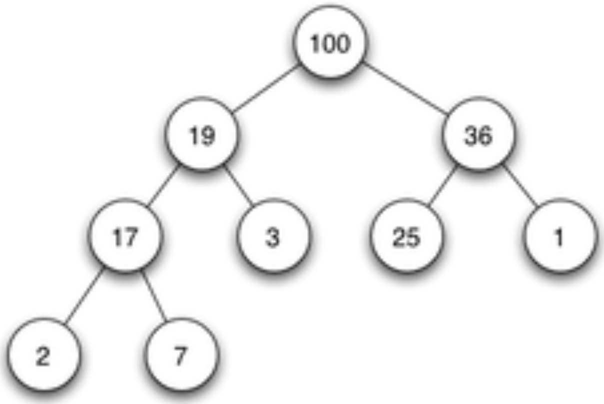
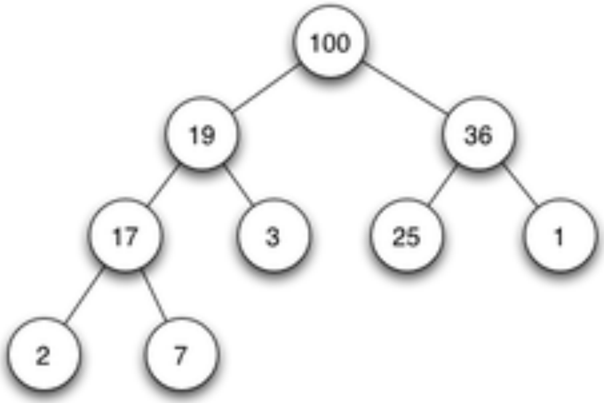


One Kind of Binary Tree ADTs

Heaps and Priority Queues

Heap

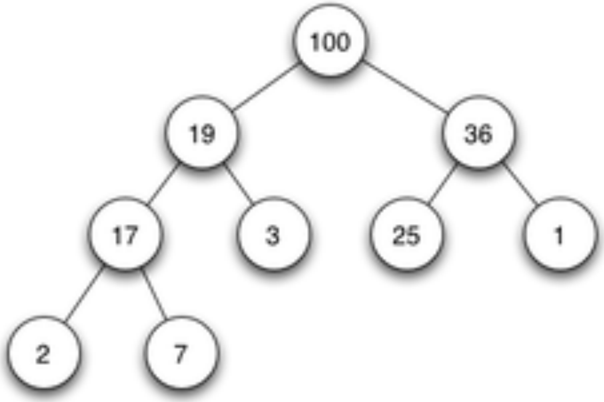
- A binary tree storing keys at its nodes



Heap

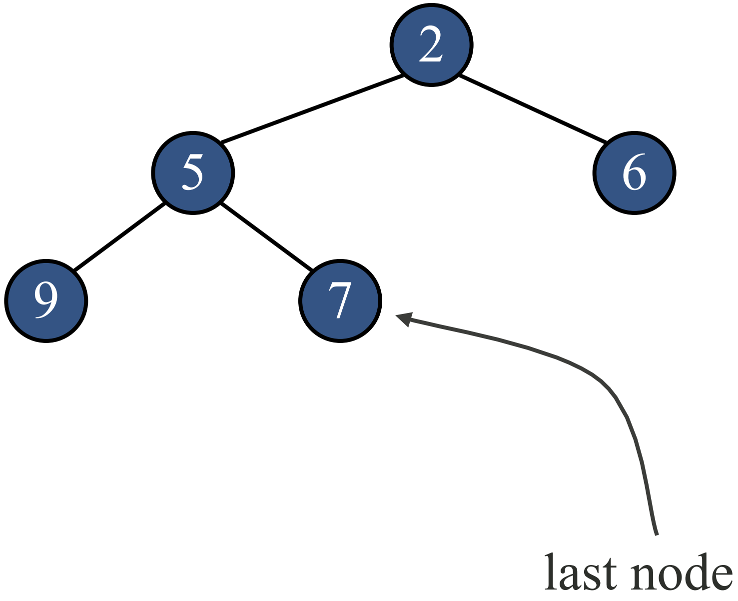
Satisfy the following properties:

- Heap-Order:
  - for every internal node v other than the root,
  - ***Maxheap: key***(***v***) <= ***key***(***parent***(***v***))
  - ***Minheap: key***(***v***) >= ***key***(***parent***(***v***))
- A Complete Binary Tree:
  - let ***h*** be the height of the heap
  - for $i = 0, \ldots, h - 1$, there are 2^i nodes of depth i
  - at depth $h - 1$, the internal nodes are to the left of the external nodes



Heap

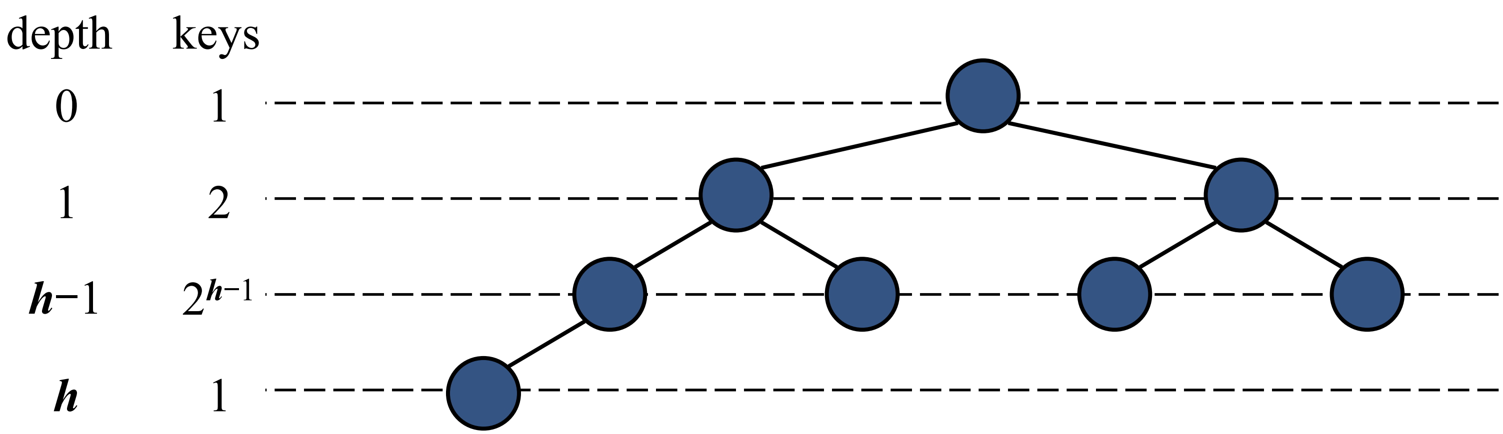
- The last node of a heap is the rightmost node with the maximal depth



Height of a Heap

- Theorem:

  A heap storing ***n*** keys has height ***O*** (log ***n***)



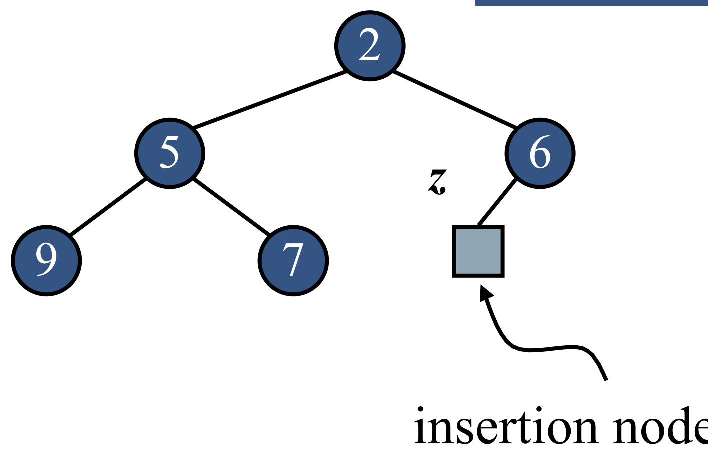
Height of a Heap

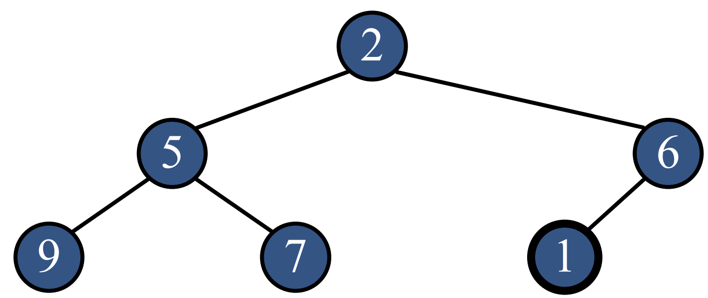
Proof: (we apply the complete binary tree property)

- Let $\boldsymbol{h}$ be the height of a heap storing $\boldsymbol{n}$ keys
- Since there are 2^i keys at depth $\boldsymbol{i} = 0, \ldots, \boldsymbol{h} - 1$ and at least one key at depth $\boldsymbol{h}$, we have $\boldsymbol{n} \geq 1 + 2 + 4 + \ldots + 2^{h-1} + 1$
- Thus, $\boldsymbol{n} \geq 2^h$, i.e., $\boldsymbol{h} \leq \log \boldsymbol{n}$

Insertion

- Insert a key ***k*** to the heap
  - a complete binary tree
  - heap order
- The algorithm consists of three steps
  - Find the insertion node ***z*** (the new last node)
  - Store ***k*** at ***z***
  - Restore the heap-order property (discussed next)

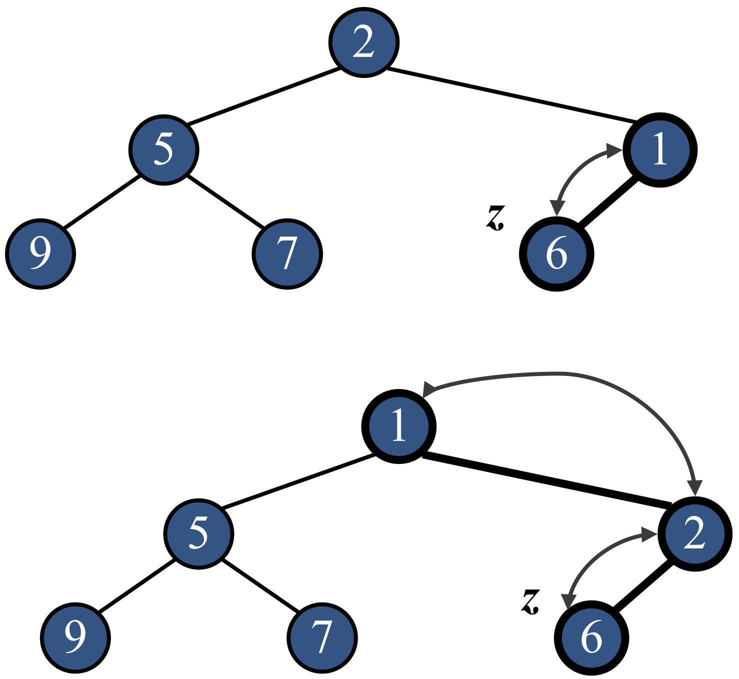




Upheap

- After the insertion of a new key ***k***, the heap-order property may be violated

- Algorithm upheap restores the heap-order property by swapping ***k*** along an upward path from the insertion node



Upheap

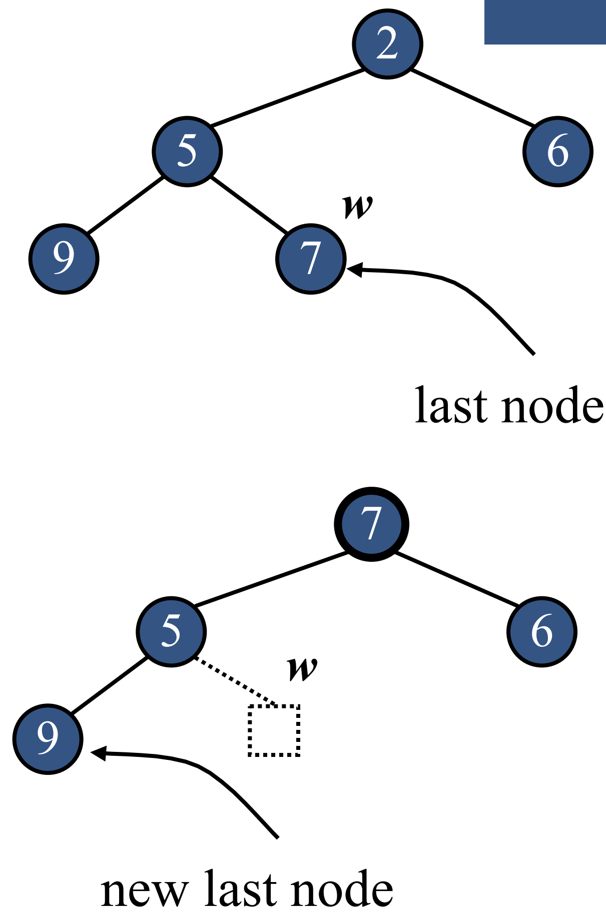
- Upheap terminates when the key $\boldsymbol{k}$ reaches the root or a node whose parent has a key smaller than or equal to $\boldsymbol{k}$

- Since a heap has height $\boldsymbol{O}(\log \boldsymbol{n})$, upheap runs in $\boldsymbol{O}(\log \boldsymbol{n})$ time

- Insertion of a heap runs in $\boldsymbol{O}(\log \boldsymbol{n})$ time

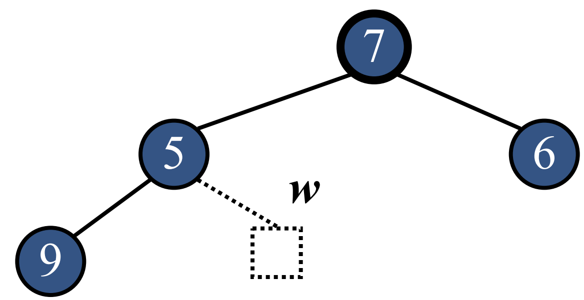
RemoveMin

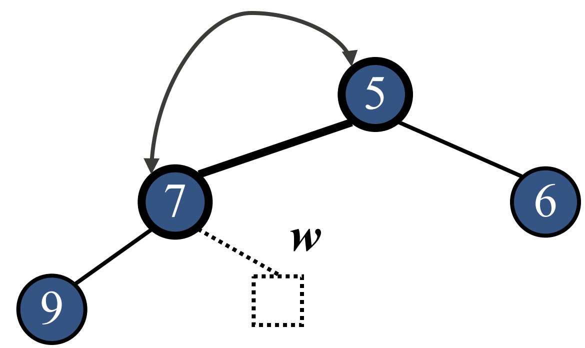
- Removal of **the root** key from the heap
- The removal algorithm consists of three steps
  - Replace the root key with the key of the last node ***w***
  - Remove ***w***
  - Restore the heap-order property (discussed next)



Downheap

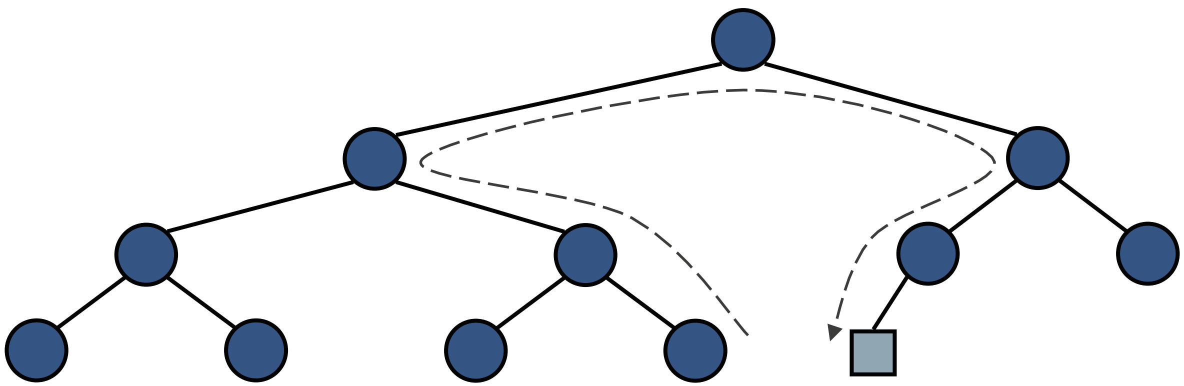
- After replacing the root key with the key ***k*** of the last node, the heap-order property may be violated
- Algorithm downheap restores the heap-order property by swapping key ***k*** along a downward path from the root
  - Find the minimal child c
  - Swap k and c if c<k





Updating the Last Node

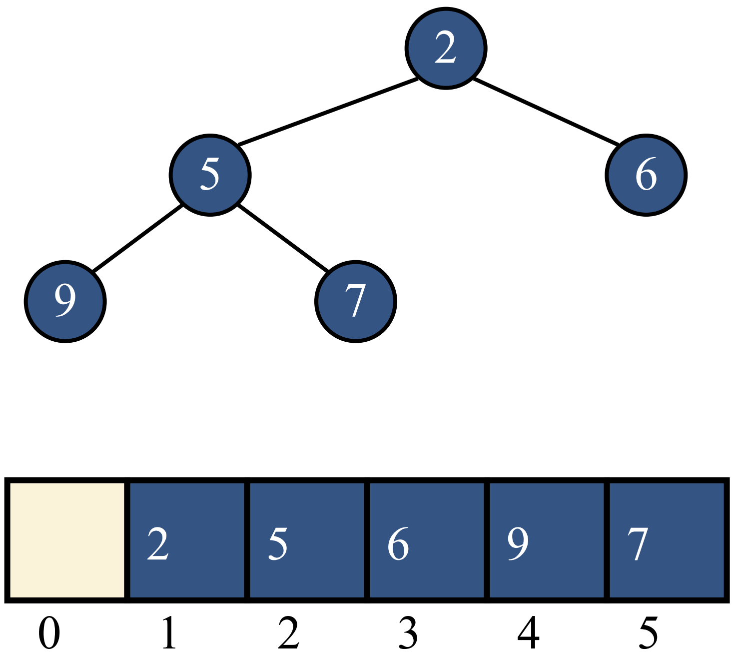
- The insertion node can be found by traversing a path of $O(\log n)$ nodes
  - Go up until a left child or the root is reached
  - If a left child is reached, go to the right child
  - Go down left until a leaf is reached



- Similar algorithm (swap left/right) for updating the last node after a removal

Array-based Implementation

- We can represent a heap with n keys by means of an array of length $n + 1$
- The cell of at rank 0 is not used
- For the node at rank i
  - the left child is at rank $2i$
  - the right child is at rank $2i + 1$
- Insert at rank $n + 1$
- Remove at rank n
- Use a ***growthable array***



Recall: Priority Queue ADT

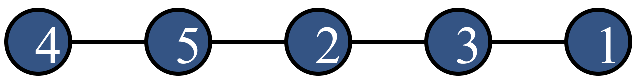
- A priority queue dequeues entries in order according to their keys
- Each entry is a pair (key, value)
- Main methods of the Priority Queue ADT
  - insert(k, x)
    inserts an entry with key k and value x
  - removeMin()
    removes and returns the entry with smallest key
  - min()
    returns, but does not remove, an entry with smallest key
  - size(), isEmpty()

A sequence of priority queue methods:

Method	Return Value	Priority Queue Contents
insert(5,A)		{ (5,A) }
insert(9,C)		{ (5,A), (9,C) }
insert(3,B)		{ (3,B), (5,A), (9,C) }
min( )	(3,B)	{ (3,B), (5,A), (9,C) }
removeMin( )	(3,B)	{ (5,A), (9,C) }
insert(7,D)		{ (5,A), (7,D), (9,C) }
removeMin( )	(5,A)	{ (7,D), (9,C) }
removeMin( )	(7,D)	{ (9,C) }
removeMin( )	(9,C)	{ }
removeMin( )	null	{ }
isEmpty( )	true	{ }

Sequence-based Priority Queue

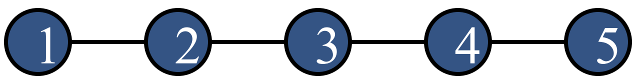
- Implementation with an unsorted list



- Performance:
  - insert takes $O(1)$ time since we can insert the item at the beginning or end of the sequence
  - removeMin and min take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

Sequence-based Priority Queue

- Implementation with a sorted list



- Performance:
  - insert takes $O(n)$ time since we have to find the place where to insert the item
  - removeMin and min take $O(1)$ time, since the smallest key is at the beginning

Priority Queue Sort

- We can use a priority queue to sort a set of comparable elements
  1. Insert the elements one by one with a series of insert operations
  2. Remove the elements in sorted order with a series of removeMin operations
- The running time of this sorting method depends on the priority queue implementation

Algorithm ***PQ-Sort***(***S***, ***C***)

Input sequence ***S***, comparator ***C*** for the elements of ***S***

Output sequence ***S*** sorted in increasing order according to ***C***

$P \leftarrow$ priority queue with comparator C

while $\neg S.isEmpty()$

$e \leftarrow S.removeFirst()$

$P.insert(e, \emptyset)$

while $\neg P.isEmpty()$

$e \leftarrow P.removeMin().getKey()$

$S.addLast(e)$

Selection-Sort

- Selection-sort is the variation of PQ-sort where the priority queue is implemented with an unsorted sequence

- Running time of Selection-sort:
  1. Inserting the elements into the priority queue with n insert operations takes $O(n)$ time
  2. Removing the elements in sorted order from the priority queue with n removeMin operations takes time proportional to

$$1 + 2 + \ldots + n$$

- Selection-sort runs in $O(n^2)$ time

Selection-Sort Example

	Sequence S	Priority Queue P
Input:	(7,4,8,2,5,3,9)	()
Phase 1		
(a)	(4,8,2,5,3,9)	(7)
(b)	(8,2,5,3,9)	(7,4)
..	..	..
(g)	()	(7,4,8,2,5,3,9)
Phase 2		
(a)	(2)	(7,4,8,5,3,9)
(b)	(2,3)	(7,4,8,5,9)
(c)	(2,3,4)	(7,8,5,9)
(d)	(2,3,4,5)	(7,8,9)
(e)	(2,3,4,5,7)	(8,9)
(f)	(2,3,4,5,7,8)	(9)
(g)	(2,3,4,5,7,8,9)	()

Insertion-Sort

- Insertion-sort is the variation of PQ-sort where the priority queue is implemented with a sorted sequence

- Running time of Insertion-sort:
  1. Inserting the elements into the priority queue with n insert operations takes time proportional to

$$1 + 2 + \ldots + n$$

  2. Removing the elements in sorted order from the priority queue with a series of n removeMin operations takes $O(n)$ time

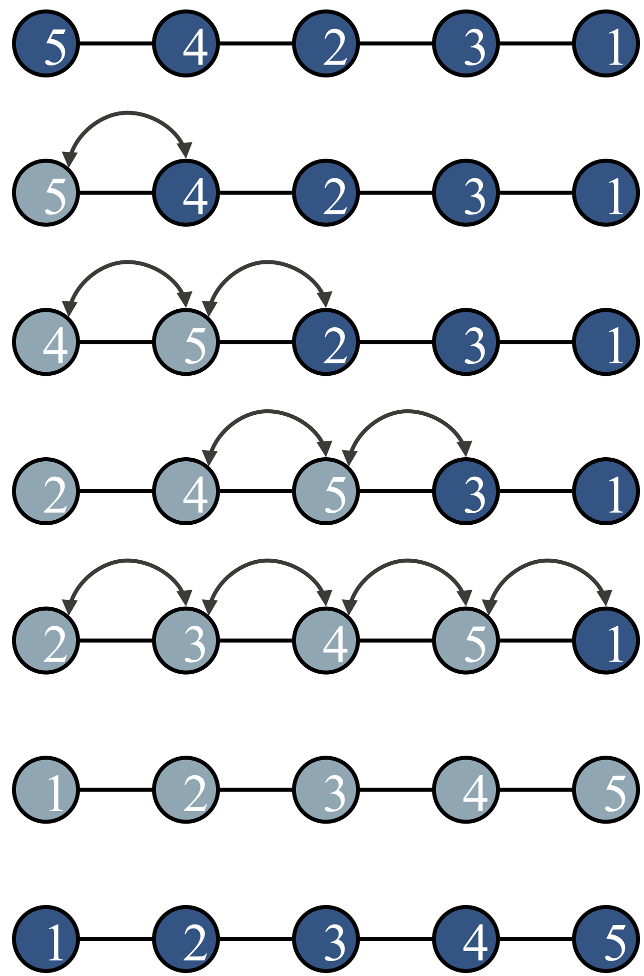
- Insertion-sort runs in $O(n^2)$ time

Insertion-Sort Example

	Sequence S	Priority queue P
Input:	(7,4,8,2,5,3,9)	()
Phase 1		
(a)	(4,8,2,5,3,9)	(7)
(b)	(8,2,5,3,9)	(4,7)
(c)	(2,5,3,9)	(4,7,8)
(d)	(5,3,9)	(2,4,7,8)
(e)	(3,9)	(2,4,5,7,8)
(f)	(9)	(2,3,4,5,7,8)
(g)	()	(2,3,4,5,7,8,9)
Phase 2		
(a)	(2)	(3,4,5,7,8,9)
(b)	(2,3)	(4,5,7,8,9)
..	..	..
(g)	(2,3,4,5,7,8,9)	()

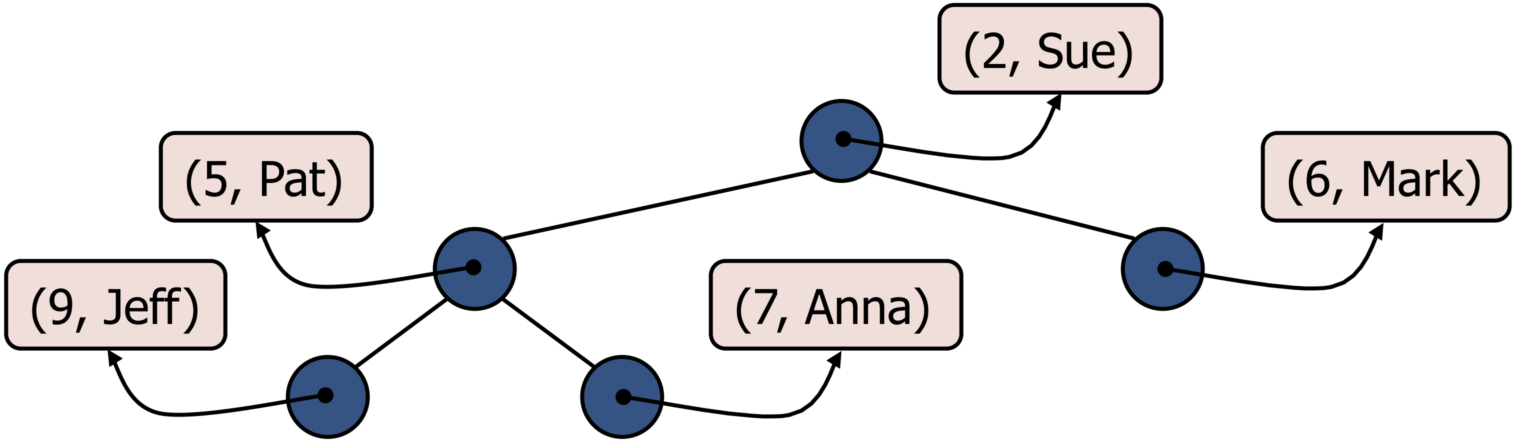
In-place Insertion-Sort (Bubble Sort)

- Instead of using an external data structure, we can implement selection-sort and insertion-sort in-place
- A portion of the input sequence itself serves as the priority queue
- For in-place insertion-sort
  - We keep sorted the initial portion of the sequence
  - We can use swaps instead of modifying the sequence



Heaps and Priority Queues

- We can use a heap to implement a priority queue
- We store a (key, element) item at each internal node
- We keep track of the position of the last node



Heap-Sort

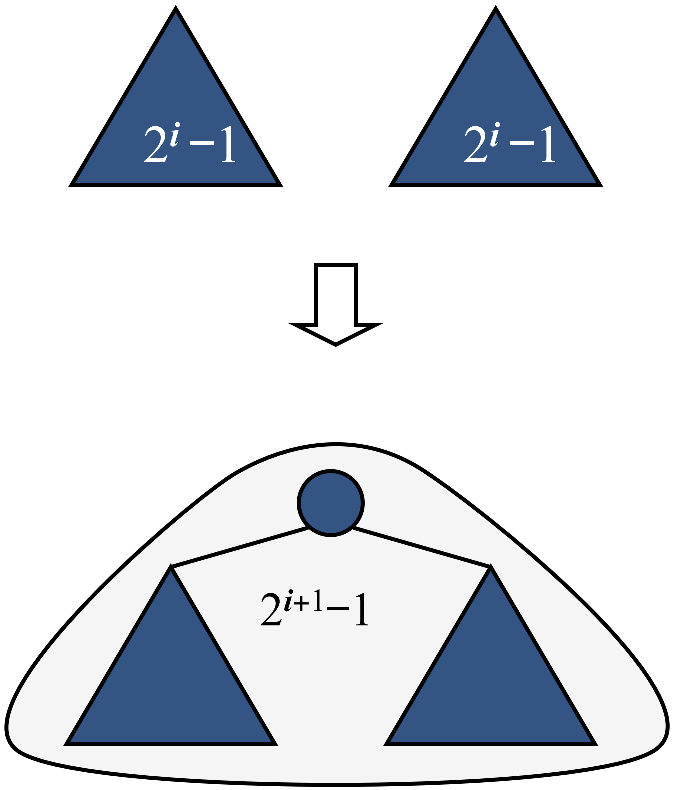
- Consider a priority queue with n items implemented by means of a heap
  - the space used is $O(n)$
  - methods insert and removeMin take $O(\log n)$ time
  - methods size, isEmpty, and min take time $O(1)$ time
- Using a heap-based priority queue, we can sort a sequence of n elements in $O(n \log n)$ time
- The resulting algorithm is called heap-sort
- Heap-sort is much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort

A Faster Heap-Sort

- Insert n keys one by one taking O(n log n) times
- If we know all keys in advance, we can save the construction to O(n) times by bottom up construction

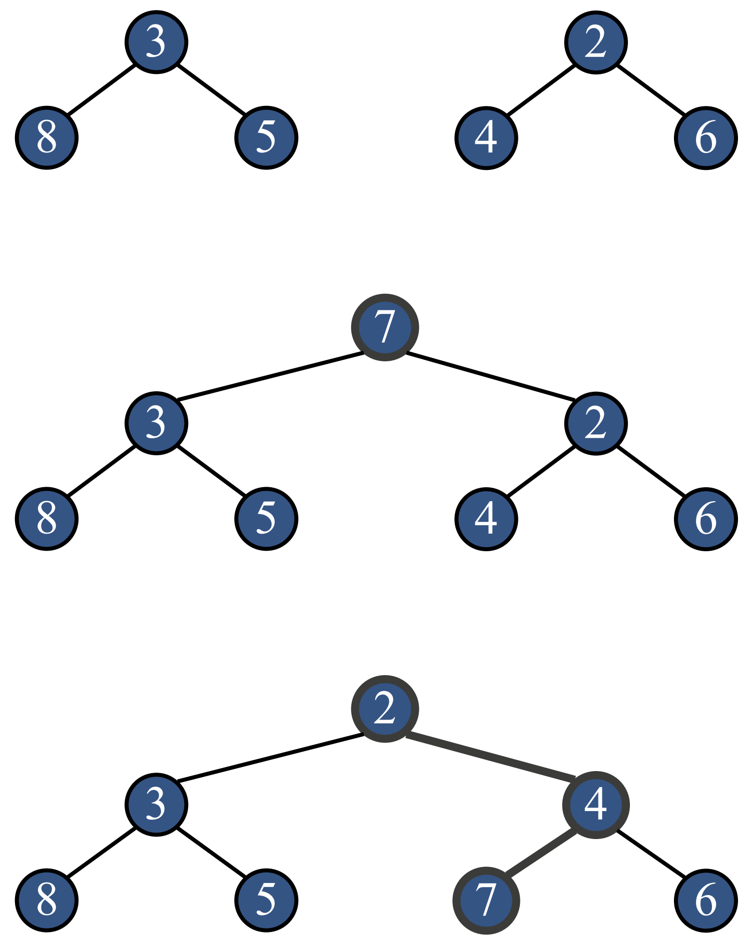
Bottom-up Heap Construction

- We can construct a heap storing n given keys in using a bottom-up construction with log n phases
- In phase i, pairs of heaps with $2^i - 1$ keys are merged into heaps with $2^{i+1}-1$ keys

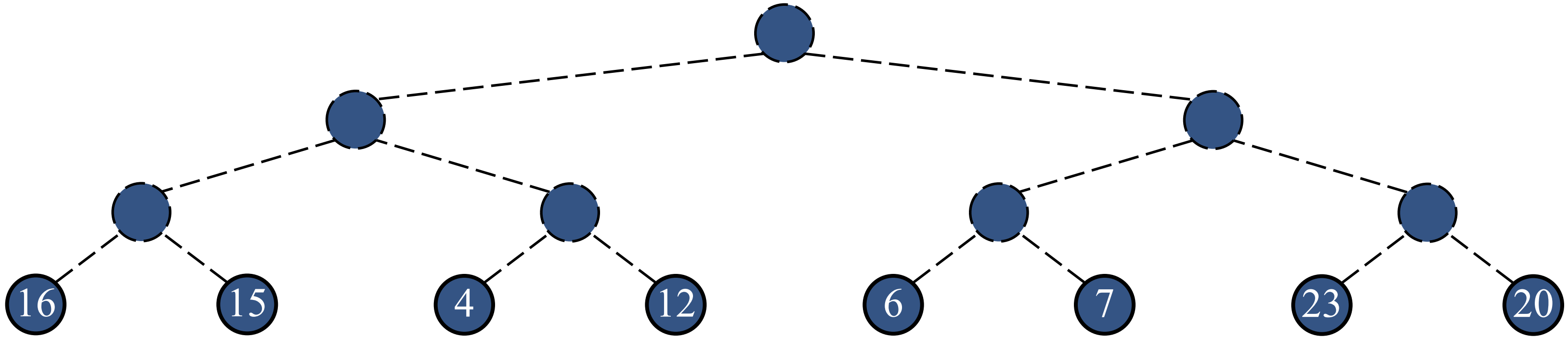


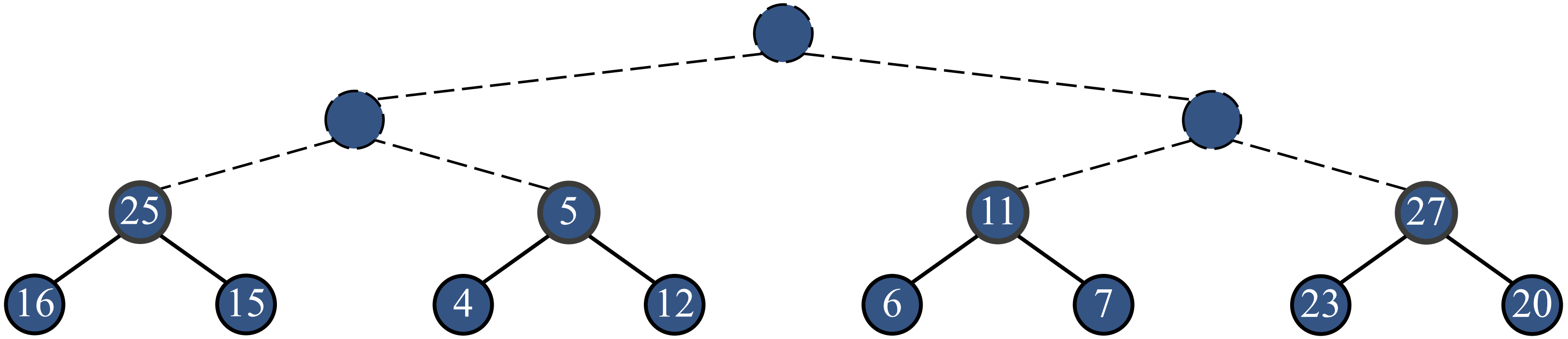
Merging Two Heaps

- Given two heaps and a key ***k***, we create a new heap with the root node storing ***k*** and with the two heaps as subtrees
- We perform downheap to restore the heap-order property

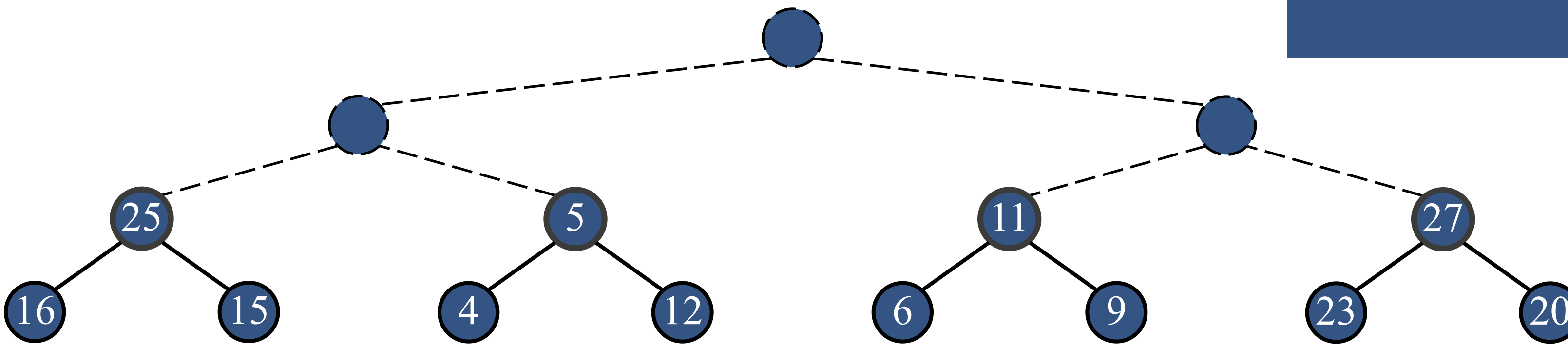


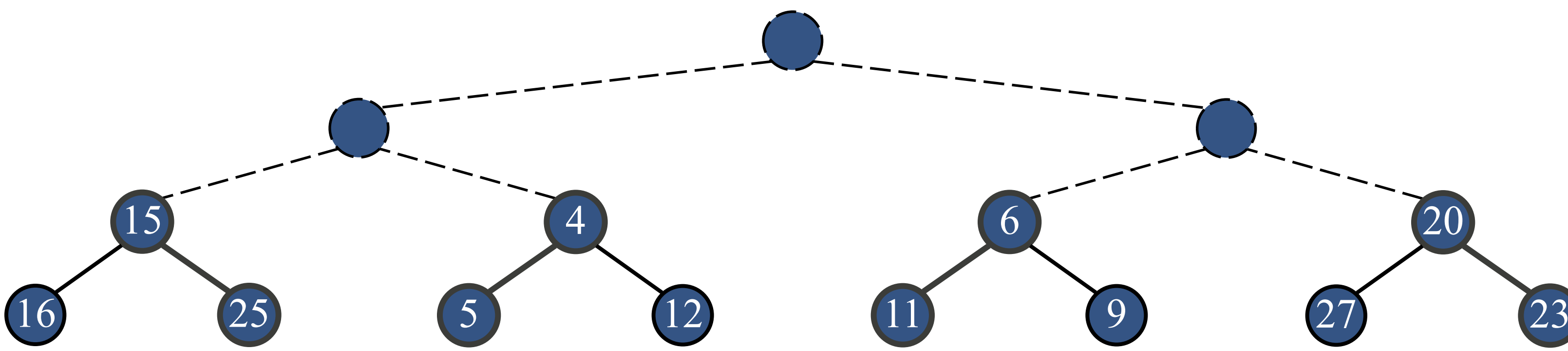
An Example of Bottom-up Construction

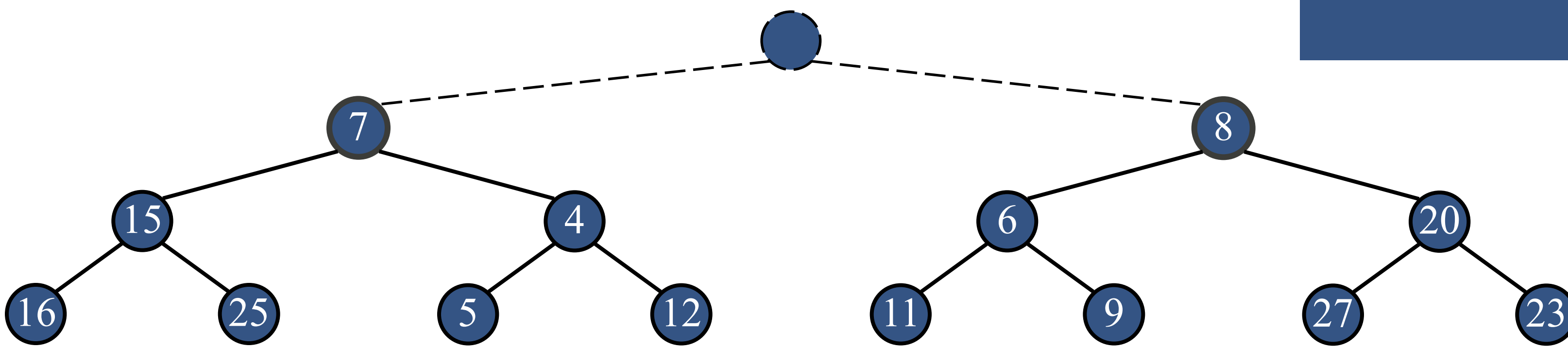




Restore the order for each one



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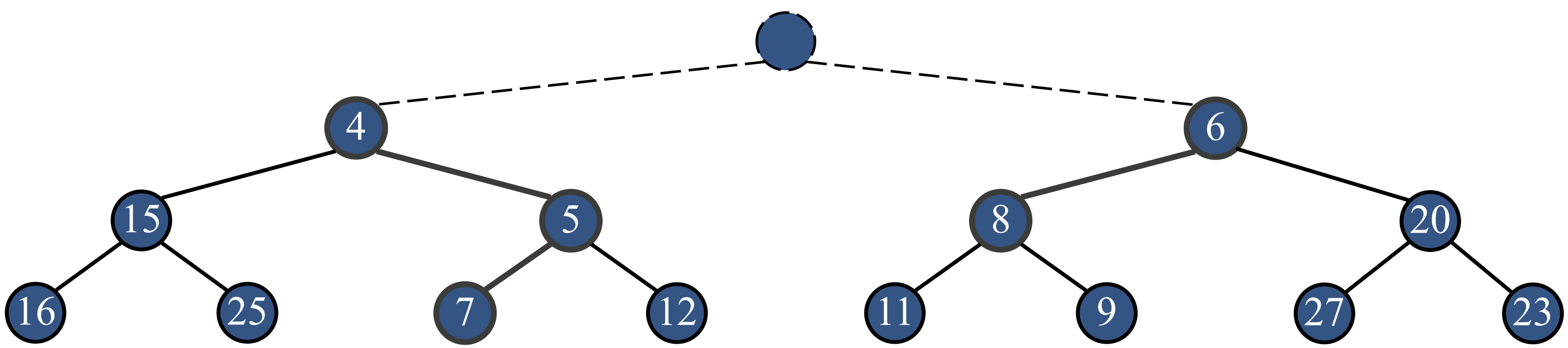
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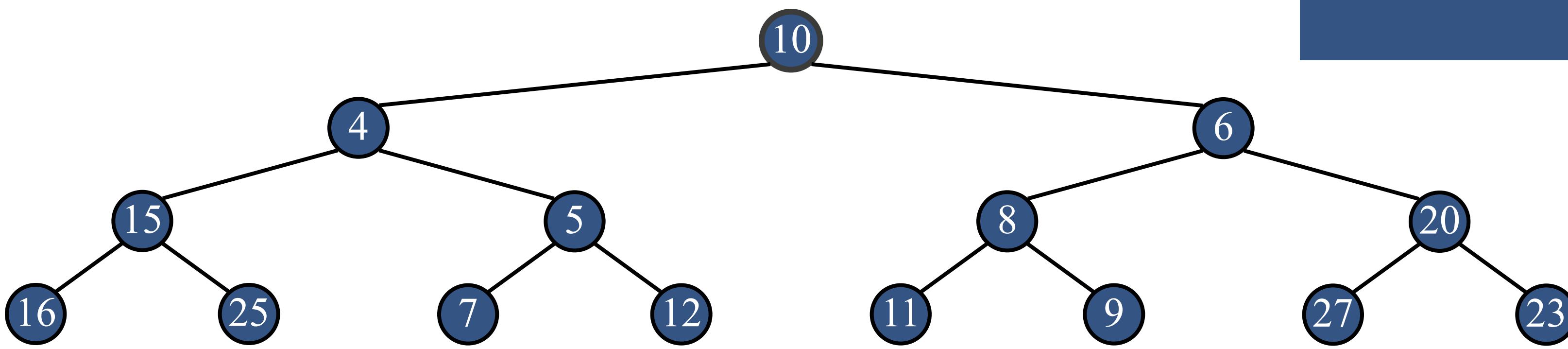
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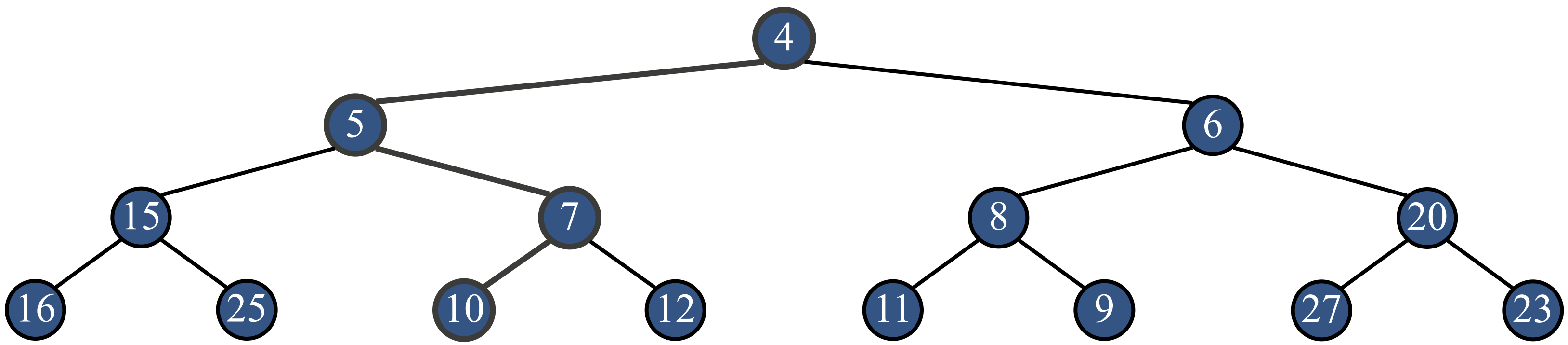
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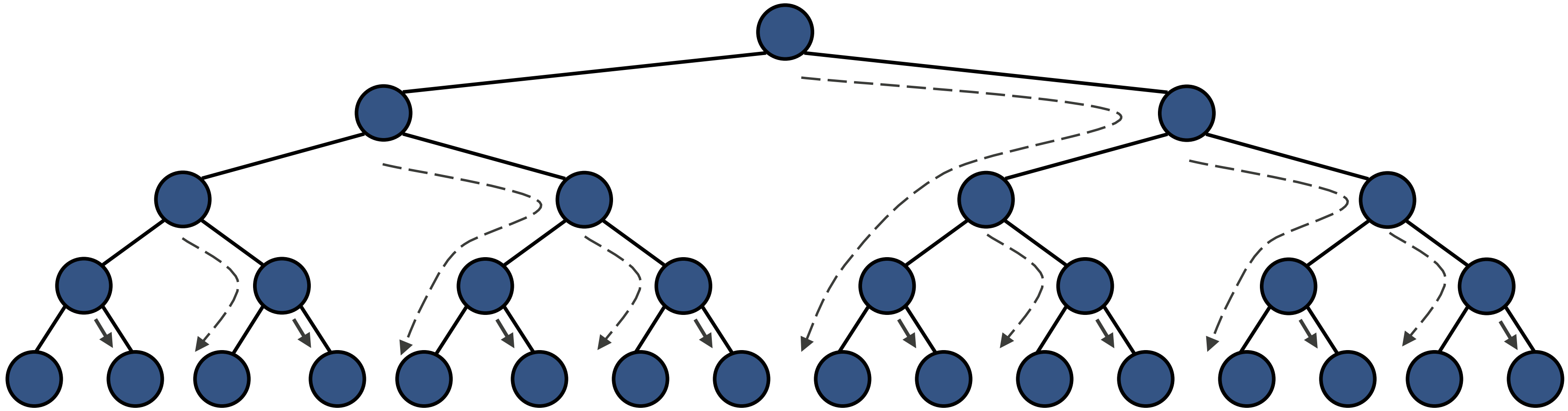
Analysis

- We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path)



Analysis

- Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is $O(n)$
- Thus, bottom-up heap construction runs in $O(n)$ time
- Bottom-up heap construction is faster than n successive insertions and speeds up the first phase of heap-sort from $O(n \log n)$ to $O(n)$