

# Formalization of: Existence and Uniqueness of Homotopy Lifts & Universal Covering Spaces

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— *Topology Study Group*

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**Thm 1 (Existence and Uniqueness of Homotopy Lifts)** Let  $X, Y, \tilde{X}$  be topological spaces. Let maps be defined as follows

$$F : Y \times [0, 1] \xrightarrow{\text{cont.}} X$$

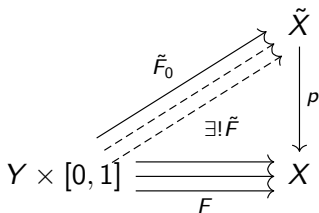
$$p : \tilde{X} \xrightarrow{\text{cont.}} X$$

$hp : p$  is a covering map

$$\tilde{F}_0 : Y \times \{0\} \xrightarrow{\text{cont.}} \tilde{X}$$

$$h : F|_{Y \times \{0\}} = p \circ \tilde{F}_0$$

$$\vdash \exists! \tilde{F} : Y \times [0, 1] \xrightarrow{\text{cont.}} \tilde{X}, (p \circ \tilde{F} = F) \wedge (\tilde{F}|_{Y \times \{0\}} = \tilde{F}_0)$$



**Thm 2 (Uniqueness of Homotopy Lifts, General Case)** Let  $X, Z, \tilde{X}$  be topological spaces. Let maps be defined as follows

$$F : Z \xrightarrow{\text{cont.}} X$$

$$p : \tilde{X} \xrightarrow{\text{cont.}} X$$

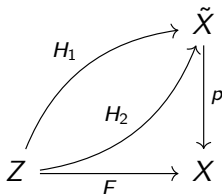
$hp : p$  is a covering map

$$H_1, H_2 : Z \xrightarrow{\text{cont.}} \tilde{X}$$

$$hH : p \circ H_1 = p \circ H_2 = F$$

$$hconn : \forall z \in Z, \exists x \in \text{ConnComp}(z), H_1(x) = H_2(x)$$

$$\vdash H_1 = H_2$$



### Thm 3 (Existence of the Universal Covering)

Let  $X$  be a topological space satisfying the conditions

- Path-connected
- Locally path-connected
- Semilocally simply connected

Then there exists a covering space

$$\pi_\infty : \widetilde{X}_\infty \rightarrow X$$

which is simply connected (i.e.  $\pi_1(\widetilde{X}_\infty) = 0$ ) and satisfies a universal property

$$\begin{array}{ccc} & \widetilde{X}_\infty & \\ \exists! \nearrow & \downarrow \pi_\infty & \\ \widetilde{X} & \xrightarrow{p} & X \end{array}$$

Let  $X$  be a topological space satisfying the conditions

- Path-connected
- Locally path-connected
- Semilocally simply connected

Then there exists a basis of  $X$  whose elements are both path-connected and semilocally simply connected.

Lifting this basis constructs a basis for the topology of  $\tilde{X}_\infty$