Formalization of:
Existence and Uniqueness of Homotopy Lifts
&
Universal Covering Spaces

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Thm 1 (Existence and Uniqueness of Homotopy Lifts) Let $X, Y, \tilde{X}$ be topological spaces. Let maps be defined as follows

$F : Y \times [0,1] \xrightarrow{cont.} X$
$p : \tilde{X} \xrightarrow{cont.} X$
hp : p is a covering map
$\tilde{F}_0 : Y \times \{0\} \xrightarrow{cont.} X$
$h : F|_{Y \times \{0\}} = p \circ \tilde{F}_0$

$$\vdash \exists! \tilde{F} : Y \times [0,1] \xrightarrow{cont.} \tilde{X}, (p \circ \tilde{F} = F) \wedge (\tilde{F}|_{Y \times \{0\}}) = \tilde{F}_0)$$

$$\begin{tikzcd}
 & \tilde{X} \arrow[d, "p"] \\
Y \times [0,1] \arrow[ur, "\tilde{F}_0"] \arrow[ur, dashed, "\exists! \tilde{F}"'] \arrow[r, Rrightarrow, "F"'] & X
\end{tikzcd}$$

Thm 2 (Uniqueness of Homotopy Lifts, General Case) Let $X, Y, \tilde{X}$ be topological spaces. Let maps be defined as follows

$F : Y \times [0,1] \xrightarrow{cont.} X$

$p : \tilde{X} \xrightarrow{cont.} X$

hp : p is a covering map

$H_1, H_2 : Z \xrightarrow{cont.} \tilde{X}$

$hH : F \circ H_1 = F \circ H_2$

$hconn : \forall y \in Y, \exists x \in \mathrm{ConnComp}(y), H_1(x) = H_2(x)$

$\vdash H_1 = H_2$

$$\begin{tikzcd}
 & \tilde{X} \arrow[d, "p"] \\
Z \arrow[ur, bend left=40, "H_1"] \arrow[ur, bend right=30, "H_2"'] \arrow[r, "F"'] & X
\end{tikzcd}$$

Thm 3 (Existence of the Universal Covering)

Let X be a topological space satisfying the conditions

- Path-connected
- Locally path-connected
- Semilocally simply connected

Then there exists a covering space

$$\pi_\infty : \widetilde{X_\infty} \to X$$

which is simply connected (i.e. $\pi_1(\widetilde{X}_\infty) = 0$) and satisfies a universal property

$$\begin{tikzcd}
& \widetilde{X_\infty} \arrow[dl, dotted, "\exists!"'] \arrow[d, "\pi_\infty"] \\
\widetilde{X} \arrow[r, "p"] & X
\end{tikzcd}$$

Let X be a topological space satisfying the conditions

- Path-connected
- Locally path-connected
- Semilocally simply connected

Then there exists a basis of X whose elements are both path-connected and semilocally simply connected.

Lifting this basis constructs a basis for the topology of $\widetilde{X}_\infty$