1. **Claim ($P(n)$):**
$$3\sum_{i=1}^{n-1} 4^i \;=\; 4^n - 4, \quad n \geq 2.$$

   Proof by induction:

   Base case ($n = 2$):
$$3\sum_{i=1}^{1} 4^i = 3 \cdot 4 = 12, \quad 4^2 - 4 = 16 - 4 = 12.$$

   Thus $P(2)$ holds.

   Inductive step: Assume $P(k)$ holds for some $k \geq 2$:
$$3\sum_{i=1}^{k-1} 4^i = 4^k - 4.$$

   We show $P(k+1)$:
$$3\sum_{i=1}^{k} 4^i = 3\Big(\sum_{i=1}^{k-1} 4^i\Big) + 3 \cdot 4^k \;=\; (4^k - 4) + 3 \cdot 4^k \;=\; 4^{k+1} - 4.$$

   $\boxed{\text{Hence } P(k+1) \text{ holds. By induction}}$
$$3\sum_{i=1}^{n-1} 4^i = 4^n - 4 \quad \forall\, n \geq 2.$$

2. **Claim:** Let $a_1 = 1$ and $a_{n+1} = 2a_n + 2^n$. Then
$$a_n = n\,2^{\,n-1}.$$

   Proof by induction:

   Base case ($n = 1$): $a_1 = 1$ and $1 \cdot 2^0 = 1$.

   Inductive step: Assume $a_k = k\,2^{\,k-1}$. Then
$$a_{k+1} = 2a_k + 2^k = 2\big(k\,2^{\,k-1}\big) + 2^k = k\,2^k + 2^k = (k+1)\,2^k.$$

   Thus $a_{k+1} = (k+1)2^k$. By induction,
$$\boxed{a_n = n\,2^{\,n-1}}.$$

3. **Claim:** For

$$A(m,n) = \begin{cases} 2n, & m = 0, \\ 0, & m \geq 1,\ n = 0, \\ 2, & m \geq 1,\ n = 1, \\ A\big(m-1,\ A(m,n-1)\big), & m \geq 1,\ n \geq 2, \end{cases}$$

Given $A(1,n) = 2^n$.

Proof by induction on n:

Base case ($n = 1$): $A(1,1) = 2$ and $2^1 = 2$.

Inductive step: Assume $A(1,k) = 2^k$. Then for $n = k+1 \geq 2$,

$$A(1,k+1) = A\big(0,\ A(1,k)\big) = 2 \cdot A(1,k) = 2 \cdot 2^k = 2^{k+1}.$$

By induction,

$$\boxed{A(1,n) = 2^n}.$$

4. **Recursive definitions:**

(a) Positive integers *not* divisible by 4:

$$S = \{1,2,3\}, \quad \text{and if } n \in S, \text{ then } n+4 \in S.$$

(b) Positive integers that are powers of 2:

$$T = \{1\}, \quad \text{and if } n \in T, \text{ then } 2n \in T.$$

$$\boxed{\textbf{Answer: } S = \{1,2,3\} \cup \{n+4 : n \in S\}, \qquad T = \{1\} \cup \{2n : n \in T\}.}$$

5. **Structural induction on**

$$S:\ 7 \in S,\ x \in S \implies 2x+3 \in S,\ x \in S \implies x^2+8 \in S.$$

Claim: Every member of S is a positive integer ending in digit 7.

Definition: n ends in 7 iff $n \equiv 7 \pmod{10}$.

Base: $7 \equiv 7 \pmod{10}$.

Inductive step: Assume $x \equiv 7 \pmod{10}$. Then

$$2x+3 \equiv 2\cdot 7+3 = 17 \equiv 7 \pmod{10}, \quad x^2+8 \equiv 7^2+8 = 57 \equiv 7 \pmod{10}.$$

Both end in 7. Hence by structural induction the claim holds.

$$\boxed{\textbf{Answer: } \text{All members of } S \text{ are positive integers ending in the digit 7.}}$$

6. **Counterexample:** Find $n \equiv 7 \pmod{10}$ not in S.

   Take $n = 27$. If $27 \in S$ it must come from

   $$27 = 7, \quad 27 = 2x + 3, \quad 27 = x^2 + 8.$$

   But $2x + 3 = 27 \implies x = 12 \not\equiv 7 \pmod{10}$, and $x^2 + 8 = 27$ has no integer solution. Hence $27 \notin S$.

   $\boxed{\textbf{Answer: } n = 27.}$

7. **Bit-strings B with**

   $$001 \in B, \quad b \in B \implies 11b,\, 10b,\, 0b \in B.$$

   Let $a_n = \#\{b \in B : |b| = n\}$ for $n \geq 2$.

   Compute: $a_2 = 0$, $a_3 = 1$ (only "001"). For $n \geq 4$, each length-n string arises by prefixing

   $$0b \text{ (b of length $n-1$)}, \quad \text{or} \quad 11b \text{ or } 10b \text{ (b of length $n-2$)}.$$

   Thus

   $$a_n = a_{n-1} + 2\,a_{n-2}, \quad n \geq 4, \quad a_2 = 0,\ a_3 = 1.$$

   $\boxed{\textbf{Answer: } a_2 = 0,\ a_3 = 1,\ a_n = a_{n-1} + 2a_{n-2} \text{ for } n \geq 4.}$

8. **Closed form for**

   $$a_n - a_{n-1} - 2a_{n-2} = 0, \quad a_2 = 0,\ a_3 = 1.$$

   $$\begin{aligned} \text{Characteristic equation:} \quad r^2 - r - 2 = 0 &\implies r = 2,\ -1, \\ a_n &= \alpha\, 2^n + \beta\,(-1)^n, \\ a_2 = 0: \quad 4\alpha + \beta &= 0, \\ a_3 = 1: \quad 8\alpha - \beta &= 1, \implies \alpha = \tfrac{1}{12}, \beta = -\tfrac{1}{3}. \end{aligned}$$

   Hence

   $$a_n = \tfrac{1}{12}\,2^n \;-\; \tfrac{1}{3}\,(-1)^n.$$

   $\boxed{\textbf{Answer: } a_n = \tfrac{1}{12}2^n - \tfrac{1}{3}(-1)^n.}$

9. **Extra Credit**