1. Write the power set of the set $S = \{1, 2\}$.

   Answer:
   $$\mathcal{P}(S) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}.$$

2. If $A = \{1, 2\}$ and $B = \{2, 1\}$, are the sets A and B equal? Explain.

   Answer: Yes, they are equal because sets are unordered; the elements are the same.

3. Are $\varnothing$ and $\{\varnothing\}$ the same set? Explain.

   Answer: No. $\varnothing$ is the empty set (having no elements) whereas $\{\varnothing\}$ is a set with one element (the empty set).

4. Is the set $(2, 3]$ equal to the statement $2 < x \leq 3$? Explain.

   Answer: Yes. Both notations describe the set of real numbers x such that $2 < x \leq 3$.

5. If n is an integer and $n > 3.1$, what stronger inequality must n satisfy?

   Answer: Since n is an integer, $n > 3.1$ implies that $n \geq 4$.

6. Determine the complement of the set $S = (2, 5)$.

   Answer: The complement of S (with respect to $\mathbb{R}$) is
   $$(-\infty, 2] \cup [5, \infty).$$

7. Consider the function $f : [0, 2] \to [0, 10]$ defined by $f(x) = x^2$. What is the range of f? Explain.

   Answer: The function $f(x) = x^2$ is continuous and increasing on the interval $[0, 2]$ because its derivative, $f'(x) = 2x$, is nonnegative for $x \geq 0$. This means the smallest value is at $x = 0$ and the largest at $x = 2$. Evaluating these endpoints gives:
   $$f(0) = 0^2 = 0 \quad \text{and} \quad f(2) = 2^2 = 4.$$
   Thus, the range of f is all values between 0 and 4
   $$[0, 4]$$

8. Is the function $f(x) = x^2$ injective? Explain.

   Answer: No, not on all of $\mathbb{R}$ because distinct numbers (x and $-x$) can have the same square. (Note: f is injective on $[0, \infty)$.)

9. Show that
   $$f : \{0, 1\}^2 \to \{0, 1\}^2; \quad f(a, b) = (a,\, a \oplus b)$$
   is bijective. Also show that the functions
   $$g : \{0, 1\}^2 \to \{0, 1\}^2; \quad f(a, b) = (a,\, a \wedge b)$$

and

$$h : \{0,1\}^2 \to \{0,1\}^2; \quad f(a,b) = (a,\, a \vee b)$$

are not bijective. Explain how this relates to the array storage question from Homework 1.

Answer:

$$f(a,b) = (a,\, a \oplus b),$$
$$g(a,b) = (a,\, a \wedge b),$$
$$h(a,b) = (a,\, a \vee b).$$

For f:
The first component of $f(a,b)$ is a, and the second is $a \oplus b$. Given an output $(a,\, a \oplus b)$, we can recover b uniquely by computing

$$b = a \oplus (a \oplus b).$$

Thus, f is one-to-one and onto (bijective).

For g:
When $a = 0$, regardless of whether $b = 0$ or $b = 1$,

$$0 \wedge b = 0,$$

so both $(0,0)$ and $(0,1)$ map to $(0,0)$. Hence, g is not injective (and so not bijective).

For h:
When $a = 1$, regardless of the value of b,

$$1 \vee b = 1,$$

so both $(1,0)$ and $(1,1)$ map to $(1,1)$. Thus, h is not injective.

Relation to Array Storage Question from Homework 1:
An addressing function in array storage must be injective to prevent collisions, and while f is bijective (thus collision-free), g and h are not, leading to potential data loss due to mapping different inputs to the same output.

10. Find all real solutions of

$$1 \leq \lfloor 3x + 5 \rfloor < 3.$$

Justify all your steps.

Answer: Since the floor function $\lfloor 3x + 5 \rfloor$ must equal either 1 or 2 (the only integers in $[1,3)$), consider both cases:

- If $\lfloor 3x+5 \rfloor = 1$:

$$1 \le 3x+5 < 2 \quad \Rightarrow \quad -4 \le 3x < -3 \quad \Rightarrow \quad -\frac{4}{3} \le x < -1.$$

- If $\lfloor 3x+5 \rfloor = 2$:

$$2 \le 3x+5 < 3 \quad \Rightarrow \quad -3 \le 3x < -2 \quad \Rightarrow \quad -1 \le x < -\frac{2}{3}.$$

Thus, the solution is

$$x \in \left[-\frac{4}{3}, -1\right) \cup \left[-1, -\frac{2}{3}\right) = \left[-\frac{4}{3}, -\frac{2}{3}\right).$$

11. Simplify

$$60+61+62+63+64+65+66+67$$

using a formula covered in lecture. Do not evaluate your final answer.

Answer: Using the arithmetic series sum formula,

$$S = n\left(\frac{a_1 + a_n}{2}\right),$$

with $n = 8$, $a_1 = 60$, and $a_n = 67$, the sum is

$$8\left(\frac{60+67}{2}\right).$$

12. Suppose the sequence $\{a_n\}$ is arithmetic with $a_3 = 5$ and $a_{11} = 87$. What is the common difference?

Answer: The common difference is

$$d = \frac{a_{11} - a_3}{11 - 3} = \frac{87-5}{8} = \frac{82}{8} = \frac{41}{4}.$$

13. Evaluate

$$\sum_{k=2}^{1000} \frac{3^{2k+4}}{2^{3k+5}}.$$

algebraically and simplify as much as possible. Show all steps and leave large powers unevaluated.

Step 1: Write

$$3^{2k+4} = 3^4 \cdot 3^{2k} = 81 \cdot 9^k, \quad 2^{3k+5} = 2^5 \cdot 2^{3k} = 32 \cdot 8^k.$$

Which simplifies to,

$$\frac{3^{2k+4}}{2^{3k+5}} = \frac{81}{32}\left(\frac{9}{8}\right)^k.$$

Step 2: Then,

$$S = \frac{81}{32} \sum_{k=2}^{1000} \left(\frac{9}{8}\right)^k.$$

Let $j = k - 2$ so that when $k = 2$, $j = 0$; then

$$\sum_{k=2}^{1000} \left(\frac{9}{8}\right)^k = \left(\frac{9}{8}\right)^2 \sum_{j=0}^{998} \left(\frac{9}{8}\right)^j.$$

Step 3: Using the geometric series formula,

$$\sum_{j=0}^{998} \left(\frac{9}{8}\right)^j = \frac{1 - \left(\frac{9}{8}\right)^{999}}{1 - \frac{9}{8}} = \frac{1 - \left(\frac{9}{8}\right)^{999}}{-\frac{1}{8}} = -8\left[1 - \left(\frac{9}{8}\right)^{999}\right].$$

Step 4: Therefore,

$$\sum_{k=2}^{1000} \left(\frac{9}{8}\right)^k = \left(\frac{9}{8}\right)^2 \cdot \left[-8\left(1 - \left(\frac{9}{8}\right)^{999}\right)\right] = -\frac{81}{64} \cdot 8\left(1 - \left(\frac{9}{8}\right)^{999}\right) = -\frac{81}{8}\left(1 - \left(\frac{9}{8}\right)^{999}\right).$$

Step 5: Substituting back,

$$S = \frac{81}{32} \cdot \left[-\frac{81}{8}\left(1 - \left(\frac{9}{8}\right)^{999}\right)\right] = -\frac{6561}{256}\left(1 - \left(\frac{9}{8}\right)^{999}\right).$$

Rewriting,

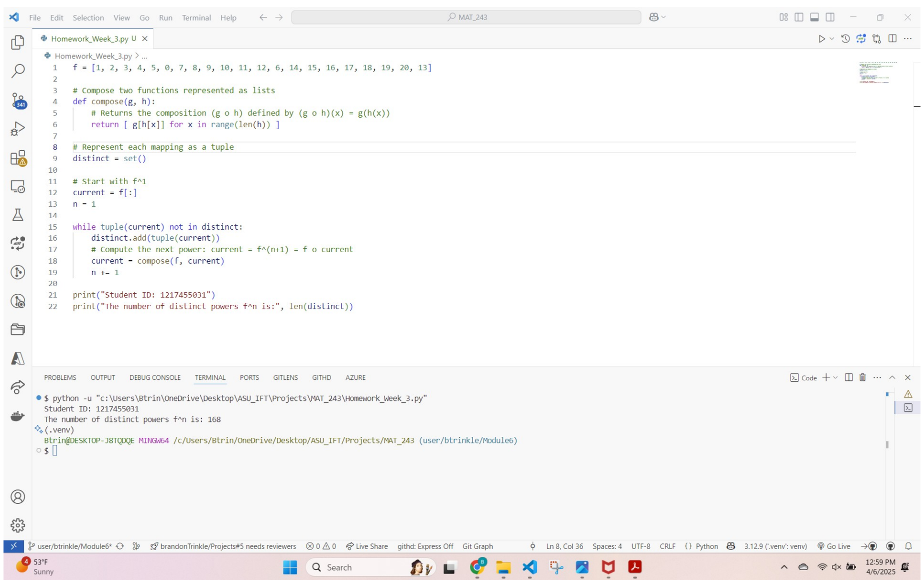
$$S = \frac{6561}{256}\left[\left(\frac{9}{8}\right)^{999} - 1\right].$$

$$\boxed{S = \frac{6561}{256}\left[\left(\frac{9}{8}\right)^{999} - 1\right]}.$$

14. (Extra Credit) Let $S = \{0, 1, \ldots, 20\}$. The function $f : S \to S$ is given by the table:

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	1	2	3	4	5	0	7	8	9	10	11

x	11	12	13	14	15	16	17	18	19	20
$f(x)$	12	6	14	15	16	17	18	19	20	13


Homework_Week_3.py
f = [1, 2, 3, 4, 5, 0, 7, 8, 9, 10, 11, 12, 6, 14, 15, 16, 17, 18, 19, 20, 13]
# Compose two functions represented as lists
def compose(g, h):
    # Returns the composition (g o h) defined by (g o h)(x) = g(h(x))
    return [g[h[x]] for x in range(len(h))]
# Represent each mapping as a tuple
distinct = set()
# Start with f^1
current = f[:]
n = 1
while tuple(current) not in distinct:
    distinct.add(tuple(current))
    # Compute the next power: current = f^(n+1) = f o current
    current = compose(f, current)
    n += 1
print("Student ID: 1217455031")
print("The number of distinct powers f^n is:", len(distinct))
$ python -u "c:\Users\Btrin\OneDrive\Desktop\ASU_IFT\Projects\MAT_243\Homework_Week_3.py"
Student ID: 1217455031
The number of distinct powers f^n is: 168
(.venv)
Btrin@DESKTOP-J8TQDQE MINGW64 /c/Users/Btrin/OneDrive/Desktop/ASU_IFT/Projects/MAT_243 (user/btrinkle/Module6)
$