Enhanced Cheat Sheet for Logic

Below is an enhanced cheat sheet that includes both the key concepts and the meanings of the symbols used. You can use this as a quick reference while studying.

# Symbols and Their Meanings

p, q, r, a: Propositional variables representing statements.

→: “Implies” or “if … then.”

∨: Logical “or” (disjunction).

∧: Logical “and” (conjunction).

¬: Logical “not” (negation).

∀: “For all” (universal quantifier).

∃: “There exists” (existential quantifier).

# 1. Propositions

## Definition:

A proposition is a declarative statement that can be assigned a truth value (true or false).

## Example:

“Frederic Gauss is the greatest mathematician of all time.” Although subjective, it is a proposition because it makes an assertion that is either true or false.

# 2. Important Logical Equivalences

## Conditional and Disjunction

### Conditional as a Disjunction:

p → q ≡ ¬p ∨ q

Meaning: “If p then q” is equivalent to “either not p or q”.

Symbols Used:

p → q: “if p then q”

¬p: “not p”

∨: “or”

## Contrapositive

Contrapositive of a Conditional:

p → q ≡ ¬q → ¬p

Meaning: If “p implies q” is true, then “if q is false, p must be false” is also true.

## Negation of a Conditional

Negation of a Conditional Statement:

¬(p → q) ≡ p ∧ ¬q

Meaning: Saying “it is not true that if p then q” means p is true and q is false.

Symbols Used:

¬(p → q): Negation of the conditional

∧: “and”

## De Morgan’s Identities

For Conjunction:

¬(p ∧ q) ≡ ¬p ∨ ¬q

For Disjunction:

¬(p ∨ q) ≡ ¬p ∧ ¬q

Meaning: These rules distribute negation over “and” and “or”.

# 3. Proof Example: (p ∧ q) → r ≡ (p → r) ∨ (q → r)

## Step-by-Step:

1. Start with: (p ∧ q) → r

2. Convert to Disjunction:

(p ∧ q) → r ≡ ¬(p ∧ q) ∨ r

Explanation: Use the equivalence p → q ≡ ¬p ∨ q.

3. Apply De Morgan’s Identity:

¬(p ∧ q) ≡ ¬p ∨ ¬q

So, the expression becomes: ¬p ∨ ¬q ∨ r

4. Express p → r and q → r:

p → r ≡ ¬p ∨ r and q → r ≡ ¬q ∨ r

5. Combine the Disjunctions:

(p → r) ∨ (q → r) ≡ (¬p ∨ r) ∨ (¬q ∨ r)

By associativity and commutativity, this simplifies to: ¬p ∨ ¬q ∨ r

6. Conclusion: Both expressions simplify to the same form, so the equivalence holds.

# 4. Negating Statements with Quantifiers

## a. Negate: ∀x ∃y (xy ≤ 0 → (−1 ≤ y < 0))

Negate the Statement:

¬[∀x ∃y (xy ≤ 0 → (−1 ≤ y < 0))]

Switch Quantifiers:

≡ ∃x ∀y ¬(xy ≤ 0 → (−1 ≤ y < 0))

Negate the Conditional:

¬(A → B) ≡ A ∧ ¬B

Here, ¬(xy ≤ 0 → (−1 ≤ y < 0)) ≡ (xy ≤ 0) ∧ ¬(−1 ≤ y < 0)

Final Negated Form:

∃x ∀y [(xy ≤ 0) ∧ ¬(−1 ≤ y < 0)]

## b. Negate “Every planet in this solar system revolves around the Sun.”

Logical Form:

∀x (P(x) → R(x))

P(x): “x is a planet in this solar system”

R(x): “x revolves around the Sun”

Negate the Statement:

¬∀x (P(x) → R(x)) ≡ ∃x ¬(P(x) → R(x))

Apply Negation of Conditional:

¬(P(x) → R(x)) ≡ P(x) ∧ ¬R(x)

Final Form:

∃x [P(x) ∧ ¬R(x)]

(There exists at least one planet that does not revolve around the Sun.)

## c. Negate “Some people in Arizona don’t like to drive on highways.”

Logical Form:

∃x [A(x) ∧ ¬D(x)]

A(x): “x is a person in Arizona”

D(x): “x likes to drive on highways”

Negate the Statement:

¬∃x [A(x) ∧ ¬D(x)]

Switch Quantifier:

≡ ∀x ¬[A(x) ∧ ¬D(x)]

Apply De Morgan’s:

¬[A(x) ∧ ¬D(x)] ≡ ¬A(x) ∨ D(x)

Final Form:

∀x [¬A(x) ∨ D(x)]

(Every person either is not in Arizona or likes to drive on highways.)

# 5. Translating Statements with “Necessary”, “Sufficient”, “Only If”, and “Unless”

Original Statement:

“If there are no clouds, then I can see the stars.”

Symbolically:

Let C = “there are clouds” and S = “I can see the stars.”

Then: ¬C → S

Rephrasings:

Only If: “I can see the stars only if there are no clouds.” (If S then ¬C.)

Necessary: “For me to see the stars, it is necessary that there are no clouds.”

Sufficient: “The absence of clouds is sufficient for me to see the stars.”

Unless: “I can see the stars unless there are clouds.” (Interpreted as: if there are clouds, then I cannot see the stars, i.e., C → ¬S.)

# 6. Expressing Statements in “If… then” Form

## a. “I get my work done on time only if I don’t procrastinate.”

Let:

W = “Work is done on time”

P = “I procrastinate”

Translation:

W → ¬P

## b. “It is necessary to have wind to sail across the lake.”

Let:

S = “Sail across the lake”

W = “Have wind”

Translation:

S → W

## c. “To be able to vote in the upcoming election it is sufficient to have an American passport.”

Let:

V = “Vote in the upcoming election”

P = “Have an American passport”

Translation:

P → V

## d. “Unless we live more economically we are going to use up all our resources.”

Let:

E = “Live more economically”

U = “Use up all resources”

Translation:

¬E → U

# 7. Expressing Conditionals as Disjunctions

## a. Rewrite ¬p → ¬q:

Use the equivalence:

¬p → ¬q ≡ ¬(¬p) ∨ ¬q

Simplify ¬(¬p) to p:

≡ p ∨ ¬q

## b. Rewrite a → ¬p:

Use the equivalence:

a → ¬p ≡ ¬a ∨ ¬p

# 8. Expressing Disjunction as a Conditional

Express p ∨ q using a conditional:

p ∨ q ≡ ¬p → q

Meaning: “If not p then q” is equivalent to “p or q.”

# 9. Expressing Conjunction Using Conditional and Negation

## Express p ∧ ¬q:

Observation:

The negation of a conditional gives a conjunction:

¬(p → q) ≡ p ∧ ¬q

Thus:

p ∧ ¬q ≡ ¬(p → q)

# 10. Expressing ¬p ∧ ¬q Using Conditional and Negation

Start with De Morgan’s Law:

¬p ∧ ¬q ≡ ¬(p ∨ q)

Express the disjunction as a conditional:

p ∨ q ≡ ¬p → q

Combine with negation:

¬(p ∨ q) ≡ ¬(¬p → q)

Final Expression:

¬p ∧ ¬q ≡ ¬(¬p → q)

# 11. Evaluating Arguments

## Valid Argument Example:

Premises:

Every bug is an insect: ∀x (B(x) → I(x))

Elephants are not insects: ∀x (E(x) → ¬I(x))

Trombone is an elephant: E(Trombone)

Conclusion:

¬B(Trombone)

Reasoning:

Since every bug must be an insect and Trombone is not an insect (being an elephant), it cannot be a bug.

## Invalid Argument Example:

Premises:

Every bug is an insect: ∀x (B(x) → I(x))

Wizzy-Fizz is an insect: I(Wizzy-Fizz)

Conclusion:

B(Wizzy-Fizz)

Reasoning:

Just because every bug is an insect does not mean every insect is a bug. The argument incorrectly infers the converse.