MAT 243 – Discrete Math Structures: Exam 2 Comprehensive Cheat Sheet

# I. Proof Techniques

## 1. Direct Proof

• Assume the hypothesis holds for an arbitrary element (state the domain).

• Use definitions, algebraic manipulation, and known theorems to deduce the conclusion.

• Example: To show “if x is even, then x + 3 is odd”, let x = 2k and write:

x + 3 = 2k + 3 = 2(k + 1) + 1 (odd number).

## 2. Proof by Contraposition

• To prove 'if P then Q', prove instead 'if not Q then not P.'

• Assume ¬Q and show that ¬P follows.

• Useful when the direct proof is not straightforward.

## 3. Proof by Contradiction

• Assume the negation of the statement to be proved.

• Deduce a logical contradiction (e.g., violation of a known property).

• Conclude the original statement must be true.

## 4. Proofs with Quantifiers

• Universal Statements (∀x P(x)):

– Prove by letting x be an arbitrary element and showing P(x) holds.

– Disprove by finding a counterexample: ∃x such that ¬P(x).

• Existential Statements (∃x P(x)):

– Prove by providing a specific example x₀ with P(x₀) true.

– Disprove by showing ∀x ¬P(x).

• Order matters: ∀x∃y P(x,y) vs. ∃y∀x P(x,y).

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# II. Sets and Set Operations

## 1. Basic Definitions

• A set is an unordered collection of distinct objects (e.g., A = {1, 2, 3}).

• The empty set, ∅, contains no elements, while {∅} is a singleton (one element).

## 2. Power Set

• P(S) is the set of all subsets of S.

– Example: If S = {1, 2}, then P(S) = {∅, {1}, {2}, {1,2}}.

## 3. Subsets

• A ⊆ B means every element of A is in B.

• A ⊂ B (proper subset) means A ⊆ B but A ≠ B (no set is a proper subset of itself).

## 4. Set Operations

• Union (A ∪ B): Elements in A or B (or both). Example: {1,2,3} ∪ {3,4,5} = {1,2,3,4,5}.

• Intersection (A ∩ B): Common elements. Example: {1,2,3} ∩ {3,4,5} = {3}.

• Difference (A − B): Elements in A that are not in B. Example: {1,2,3} − {2,3} = {1}.

• Complement: Relative to a universal set U. Example: If S = (2,5) ⊆ ℝ, then Sᶜ = (−∞,2] ∪ [5,∞).

• Symmetric Difference (A △ B): Elements in exactly one of A or B. Example: {1,2,3} △ {3,4,5} = {1,2,4,5}.

• Cartesian Product (A × B): Set of all ordered pairs (a, b) with a ∈ A and b ∈ B.

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# III. Functions

## 1. Definitions

• A function f: A → B assigns every element x ∈ A exactly one element f(x) ∈ B.

• Domain: The set A of inputs.

• Codomain: The set B of possible outputs.

• Range: {f(x) | x ∈ A} (the actual outputs).

## 2. Properties

• Injective (one-to-one): f(x₁) = f(x₂) implies x₁ = x₂.

• Surjective (onto): For every y ∈ B, there exists x ∈ A such that f(x) = y.

• Bijective: Both injective and surjective; f has an inverse.

• Well-Defined: Every input maps to an element within the codomain.

## 3. Images and Preimages

• For X ⊆ A, f(X) = {f(x) | x ∈ X}.

• For Y ⊆ B, f⁻¹(Y) = {x ∈ A such that f(x) ∈ Y}.

## 4. Review Notes

• A function is surjective if its range equals its codomain.

• In finite sets of equal size, injectivity implies surjectivity.

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# IV. Floor and Ceiling Functions

## 1. Definitions

• Floor function ⌊x⌋: The greatest integer ≤ x.

• Ceiling function ⌈x⌉: The smallest integer ≥ x.

• Inequalities: For any real x, x − 1 < ⌊x⌋ ≤ x and x ≤ ⌈x⌉ < x + 1.

## 2. Example Problem

• Solve: 2 ≤ ⌈2x + 5⌉ < 5. (Rewrite in terms of 2x + 5 and solve for x.)

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# V. Sequences, Summation, and Sigma Notation

## 1. Sequences

• A sequence is an ordered list denoted {aₙ}, where n is an index.

• Indexing: Zero-based (a₀ is the first term) or one-based (a₁ is the first term).

## 2. Types of Sequences

• Arithmetic: aₙ = a + n·d, where d is the constant difference.

• Geometric: aₙ = a · qⁿ, where q is the constant ratio.

## 3. Recursive Definitions

• A sequence can be defined recursively (e.g., aₙ = aₙ₋₁ + 2 with a₀ = 0).

## 4. Summation and Sigma Notation

• Express sums using Σ notation.

• Practice index shifts and writing closed forms.

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# VI. Additional Review Exam Concepts

## 1. Function Properties (Review Q1–Q2)

• Understand domain, codomain, and range.

• For f: A → B with |A| = |B|, injectivity implies surjectivity.

• If f is not surjective, then its range is a proper subset of the codomain.

## 2. Proof by Contradiction (Review Q3)

• For “if x is rational and y is irrational then x+y is irrational”, assume:

- x is rational, y is irrational, and x+y is rational, then derive a contradiction.

## 3. Set Operations (Review Q6–Q7)

• Find power sets and Cartesian products (e.g., power set of A = {1, {a, t}}, compute A × A).

• Understand membership vs. subset:

- {1,2} ⊆ {1,2,3,A,c} is true.

- {1} ∈ {1,2,3,A,c} is true.

- ∅ ⊆ S is always true; ∅ ∈ S only if ∅ is an element of S.

## 4. Quantified Statements (Review Q8)

• Practice proving:

a. ∃x ∀y (5y − xy = y) for all real numbers.

b. ∀x ∃y (y is odd and x − 1 < y ≤ x + 1) for positive integers.

c. ∀x ∃y ∃z (y ≠ x + z) for all integers.

## 5. Floor/Ceiling & Function Images (Review Q9–Q10)

• Evaluate f([−1,5]) when f is the ceiling function.

• Disprove: For all real x, y, ⌊x + y⌋ = ⌊x⌋ + ⌊y⌋.

## 6. Function Analysis (Review Q11)

• For f: (–2,2) → (–4,0] defined by f(x) = –x², check if f is increasing/decreasing, injective, surjective, etc.

## 7. Summation Problems (Review Q12–Q15)

• Practice index shifts in summation.

• Write sums in sigma notation and find closed forms.

• Example: Express and evaluate the sum 26 + 30 + 34 + … + 554.

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# VII. Final Reminders

• Always state your assumptions and domain clearly in proofs.

• Justify every logical step.

• Use proper notation: distinguish between ∈ and ⊆, and ordered pairs (a, b) vs. sets {a, b}.

• Practice writing proofs with quantifiers and review common pitfalls.

• Use visual aids (Venn diagrams, number lines) to understand sets and intervals.

• Revisit homework (Week 3) and the Review Exam 2 document to cover all topics.

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Good luck on Exam 2 – Study hard and review these concepts thoroughly!

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