46-921, Fall 2022: Homework #3

Due 3:30 PM, September 21

1. Consider the following probability density function:

$$f_X(x) = e^{-\left[(x-\mu)+e^{-(x-\mu)}\right]}, \quad -\infty < x < \infty$$

This is a *Gumbel Distribution.*

Write a function that, when passed a vector of data values assumed to be iid from the above distribution, returns both the MLE for μ and the approximate standard error for the estimator.

Demonstrate via simulations that your function works. You can generate draws from this distribution by using the following fact: If U is Uniform$(0,1)$, then

$$\mu - \log(-\log(U))$$

has the Gumbel distribution. Your simulations should also demonstrate that your stated SE is (approximately) correct.

Include your code in a Python notebook.

Comment: You do not need to go overboard on the simulations. Choose two reasonable values for n, along with two choices for μ, and see how it performs.

2. Suppose that $X_1, X_2, \ldots, X_n$ are iid random variables. Each X_i takes three possible values (call these 1, 2, and 3, but the names are unimportant). The random variables are such that

$$P(X_i = 1) = p_1$$

and

$$P(X_i = 2) = p_2,$$

and, of course,

$$P(X_i = 3) = 1 - p_1 - p_2.$$

The restrictions

$$p_1 + p_2 < 1, \quad p_1 > 0, \quad p_2 > 0$$

are placed on the parameters. (This is a special case of the *multinomial distribution* which is a generalization of the binomial distribution.)

It is not difficult to show (and not too surprising) that the MLE for p_j equals the sample proportion of the X_i which equal j.

(a) What is the asymptotic distribution of the MLE $(\widehat{p}_1, \widehat{p}_2)$? (Don't just say it's normal. Derive the covariance matrix.)

Hint: Start by defining n_j to be the number of the X_i which are equal to j for $j = 1, 2, 3$. Of course, $n = n_1 + n_2 + n_3$. The likelihood function is then

$$L(\theta) = p_1^{n_1} p_2^{n_2} (1 - p_1 - p_2)^{n_3}.$$

(b) Derive a $100(1-\alpha)\%$ confidence interval for $\log(p_1/(1-p_1))$.

3. Consider the example in lecture where we estimated the pair (α, σ^2) from the geometric Brownian motion. Suppose that instead I wanted to estimate (α, σ), i.e., I want to determine the asymptotic distribution of the MLE for this parameter vector.

   What is the asymptotic distribution of the MLE $(\widehat{\alpha}, \widehat{\sigma})$? (Don't just say it's normal. Derive the covariance matrix.)

   Hint: Don't make this more work than it needs to be. Take the result we derived in lecture, and use it as a starting point.