46-921, Fall 2022: Homework #2

Due 3:30 PM EDT, Wednesday, September 14

Note: When you are asked to approximate the distribution of an estimator, it is **not** sufficient to just say "It is approximately normal."

1. Suppose that $X_1, X_2, \ldots, X_n$ are iid with the Gamma(α, β) distribution. Determine the method of moments estimators for α and β.

   Comment: This is an important case because maximum likelihood does not admit a closed form for the estimators for α and β.

2. During lecture we compared three different estimators for λ when working with an iid sample from the Exponential(λ) distribution. We concluded that, based on MSE, the "adjusted" method of moments estimator is the best choice.

   Now, conduct a simulation experiment to address the following question: In the case where $n = 20$ and $\lambda = 10$, what proportion of the time does the adjusted method of moments estimator come closer to the true value of λ than does the "worst" of the three estimators (the method of moments estimator based on the second moment)? Be sure to submit your Python code and results.

3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. from the *Pareto distribution*:

$$f_X(x) = \begin{cases} \dfrac{\alpha\lambda^\alpha}{x^{\alpha+1}} & x > \lambda,\ \alpha > 0,\ \lambda > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

   For this exercise we will assume that λ is a known constant.

   (a) Find the MLE for α, call it $\widehat{\alpha}$.

   (b) Approximate the distribution of $\widehat{\alpha}$.

   (c) A sample of size 100 is taken from a population that we are willing to assume has the Pareto distribution with $\lambda = 2$. It holds that $\sum_i \log(x_i) = 107.39$. Make a statement regarding our best estimate of α and attach a standard error to the esimator.

4. Suppose that $X_1, X_2, \ldots, X_n$ are iid from the Poisson(θ) distribution.

   (a) What is the MLE for θ in this case?

   (b) What is the MLE for $P(X_i = 0)$? (Note that the probability is the same for all i.)

   (c) What is the Fisher Information $I(\theta)$?

   (d) Use part (c) to construct a $100(1-\alpha)\%$ confidence interval for θ. Your interval should not depend on the unknown θ.