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Single Detector Log Likelihood

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0.1 Vectorization

Equation 24 from Arxiv 15502.05370*v*1.*pdf* for the network log likelihood reads

$$\begin{aligned}\ln \mathcal{L} = &\frac{D_{ref}}{D} Re \sum_{k} \sum_{(l,m)} [F_k Y_{lm}]^* Q_{k,lm} \\ &- \left[\frac{D_{ref}}{2D}\right]^2 \sum_{k} \sum_{(l,m),(l',m')} \left[|F_k|^2 Y^*_{l,m} Y_{l',m'} U_{k,(l,m),(l'm')}\right] \\ &- \left[\frac{D_{ref}}{2D}\right]^2 \sum_{k} \sum_{(l,m),(l',m')} Re \left[F_k^2 Y_{l,m} Y_{l'm'} V_{k,(l,m),(l'm')}\right]\end{aligned} \tag{0.1}$$

Consider a single detector, thus dropping the sum over k. The first term is of the form $\vec{A} \cdot \vec{B} = \sum_{i=0}^{d} A_i B_i$, so if $Q_{k,(l,m)}$ were a simple vector, we could write it as

$$-\left[\frac{D_{ref}}{2D}\right]^2 \sum_{k} \sum_{(l,m),(l',m')} \left[|F_k|^2 Y^*_{l,m} Y_{l',m'} U_{k,(l,m),(l'm')}\right] = -\left[\frac{D_{ref}}{2D}\right]^2 F * \vec{Y} \cdot \vec{Q} \tag{0.2}$$

However the $Q_{k,(l,m)}$ are actually harmonic mode time series and not single values. We desire a vector whose values are the likelihoods at each point in the time series.

Consider the case where we have only the $(2,-2),(2,0)$ and $(2,2)$ modes. If we write all the mode time series $Q^0, Q^1, Q^2 ...$ as the columns of a matrix , then the desired result is obtained with

$$F * \begin{bmatrix} Q^0_{2,-2} & Q^0_{2,+0} & Q^0_{2,+2} \\ Q^1_{2,-2} & Q^1_{2,+0} & Q^1_{2,+2} \\ Q^2_{2,-2} & Q^2_{2,+0} & Q^2_{2,+2} \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} (Y_{2,-2}) \\ (Y_{2,+0}) \\ (Y_{2,+2}) \end{bmatrix} = \begin{bmatrix} Q^0_{2,-2}Y_{2,-2} + Q^0_{2,-2}Y_{2,+0} + Q^0_{2,+2}Y_{2,+2} \\ Q^1_{2,-2}Y_{2,-2} + Q^1_{2,-2}Y_{2,+0} + Q^1_{2,+2}Y_{2,+2} \\ Q^2_{2,-2}Y_{2,-2} + Q^2_{2,-2}Y_{2,+0} + Q^2_{2,+2}Y_{2,+2} \\ \vdots \end{bmatrix}$$

(0.3)

With $\vec{Y}$ and $\mathbf{Q}$ defined as the matrix and vector above respectively, we have for the first term

$$\frac{D_{ref}}{D} Re \left[\mathbf{Q} \left(F\vec{Y} \right)^* \right] \tag{0.4}$$

The second term is a sum once over all the possible combinations of $(l,m),(l',m')$ pairs using the $U_{(l,m),(l',m')}$ cross terms. Its result is a scalar quantity made up of terms like

$$Y^*_{2,-2}Y_{2,-2}U_{(2,-2),(2,-2)} + Y^*_{2,-2}Y_{2,+0}U_{(2,-2),(2,+0)} + Y^*_{2,-2}Y_{2,+2}U_{(2,-2),(2,+2)} \tag{0.5}$$
$$+ Y^*_{2,+0}Y_{2,-2}U_{(2,+0),(2,-2)} + Y^*_{2,+0}Y_{2,+0}U_{(2,+0),(2,+0)} + Y^*_{2,+0}Y_{2,+2}U_{(2,+0),(2,+2)} \tag{0.6}$$
$$+ Y^*_{2,+2}Y_{2,-2}U_{(2,+2),(2,-2)} + Y^*_{2,+2}Y_{2,+0}U_{(2,+2),(2,+0)} + Y^*_{2,+2}Y_{2,+2}U_{(2,+2),(2,+2)} \tag{0.7}$$

If we pack the $U_{(l,m),(l',m')}$ into the matrix $\mathbf{U}$ as defined below then the following set of matrix operations produces the same sum

$$\begin{bmatrix} Y^*_{2,-2} & Y^*_{2,+0} & Y^*_{2,+2} \end{bmatrix} \begin{bmatrix} U_{(2,-2),(2,-2)} & U_{(2,-2),(2,+0)} & U_{(2,-2),(2,+2)} \\ U_{(2,+0),(2,-2)} & U_{(2,+0),(2,+0)} & U_{(2,+0),(2,+2)} \\ U_{(2,+2),(2,-2)} & U_{(2,+2),(2,+0)} & U_{(2,+2),(2,+2)} \end{bmatrix} \begin{bmatrix} Y_{2,-2} \\ Y_{2,+0} \\ Y_{2,+2} \end{bmatrix}$$

because when you multiply $\mathbf{U}$ into $\vec{Y}$ this simplifies to

$$\begin{bmatrix} Y^*_{2,-2} & Y^*_{2,+0} & Y^*_{2,+2} \end{bmatrix} \begin{bmatrix} U_{(2,-2),(2,-2)}Y_{2,-2} + U_{(2,-2),(2,-2)}Y_{2,-2} + U_{(2,-2),(2,-2)}Y_{2,-2} \\ U_{(2,-2),(2,-2)}Y_{2,-2} + U_{(2,-2),(2,-2)}Y_{2,-2} + U_{(2,-2),(2,-2)}Y_{2,-2} \\ U_{(2,-2),(2,-2)}Y_{2,-2} + U_{(2,-2),(2,-2)}Y_{2,-2} + U_{(2,-2),(2,-2)}Y_{2,-2} \end{bmatrix}$$

Which becomes the desired scalar. This allows us to write the second term as

$$-\left[\frac{D_{ref}}{2D}\right]^2 \sum_k \sum_{(l,m),(l',m')} \left[|F_k|^2 Y^*_{l,m} Y_{l',m'} U_{k,(l,m),(l'm')} \right] = -\left[\frac{D_{ref}}{2D}\right]^2 |F^2| \vec{Y}^* \mathbf{U} \vec{Y} \tag{0.8}$$

We must always set up the spherical harmonic vectors based on the value of m and the cross terms in row-major form based first on m_2 and then on m_1. If we organize the matrix $\mathbf{V}$ in the same way then the same set of steps will lead us to conclude that

$$-\left[\frac{D_{ref}}{2D}\right]^2 \sum_k \sum_{(l,m),(l',m')} Re\left[F_k^2 Y_{l,m} Y_{l'm'} V_{k,(l,m),(l'm')}\right] = -\left[\frac{D_{ref}}{2D}\right]^2 Re\left[F^2 \vec{Y} \mathbf{V} \vec{Y}\right] \tag{0.9}$$

Combining the results the single detector log likelihood is

$$\ln \mathcal{L} = \frac{D_{ref}}{D} \Re\left[\mathbf{Q}\left(F\vec{Y}\right)^*\right] - \left[\frac{D_{ref}}{2D}\right]^2 \left[|F|^2 \vec{Y}^* \mathbf{U} \vec{Y} - \Re\left(F^2 \vec{Y} \mathbf{V} \vec{Y}\right)\right] \tag{0.10}$$

0.2 Implementation

As it stands the data is organized into dictionaries keyed by tuples corresponding to modes or pairs of modes. I wrote a very basic function that takes one of these dictionaries as argument and produces matrices organized as described in the previous section. I do not think it requires a huge amount of attention since it just needs to run once at startup.

I have written only the most basic implementation for this that I can think of:

```
def sdll_matrix(rholm_vals, crossTermsU, crossTermsV, Ylms, F, dist):
 #Compute single detector log likelihood with vectorized operations
 invDistMpc = distRef/dist
 Fstar = np.conj(F)

 Ylms_conj = np.conj(Ylms)
 term1 = Fstar*np.dot(rholm_vals, Ylms_conj)
 term2 = np.dot(Fstar*F*Ylms_conj, np.dot(crossTermsU, Ylms))
 term3 = np.real(np.dot(F*F*Ylms, np.dot(crossTermsV, Ylms)))
 return invDistMpc*np.real(term1) - 0.25*invDistMpc**2 * np.real((term2 + term3))
```