
Electrodynamics - Homework 1

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1 PROBLEM 1

1.1 PART A

We wish to prove the following statement

$$\int_v \nabla \circ = \int_s \vec{ds} \circ \quad (1.1)$$

We assume that we may write the vector \vec{A} as some $\vec{A} = \rho(r) \cdot \vec{c}$. The divergence theorem reads

$$\int_v \nabla \cdot \vec{A} = \int_s \vec{A} \cdot \vec{ds} \quad (1.2)$$

Applying the vector ansatz gets us

$$\int_v \nabla \cdot (\rho \vec{c}) dv = \int_s (\rho \vec{c}) \cdot \vec{ds} \quad (1.3)$$

$$\int_v [\nabla \rho \cdot \vec{c} + \rho \nabla \cdot \vec{c}] dv = \int_s (\rho \vec{c}) \cdot \vec{ds} \quad (1.4)$$

$$\rightarrow \vec{c} \text{ is constant} \quad (1.5)$$

$$\vec{c} \cdot \int_v \nabla \rho dv = \vec{c} \cdot \int_s \rho \vec{ds} \quad (1.6)$$

Since the last line must be true for any \vec{c} , the integral components must be equal and we have arrived at the end of the proof for the ordinary product.

To prove the cross product case, we make the vector ansatz $\vec{A} = \vec{c} \times \vec{B}(r)$. Applying the divergence theorem yields

$$\int_v \nabla \cdot (\vec{c} \times \vec{B}) dv = \int_s (\vec{c} \times \vec{B}) \cdot \vec{ds} \quad (1.7)$$

$$\int_v \left[\vec{c} \cdot (\nabla \times \vec{B}) - \vec{B} \cdot (\nabla \times \vec{c}) \right] dv = \int_s (\vec{B} \times \vec{ds}) \cdot \vec{c} \quad (1.8)$$

$$\vec{c} \cdot \int_v \nabla \times \vec{B} dv = \vec{c} \cdot \int_s \vec{B} \times \vec{ds} \quad (1.9)$$

On the second line, the cyclic property of the vector triple product was used. Since the last line must be true for any \vec{c} , the integral operators on the vector \vec{B} must perform the same action, and thus are equal, proving the second part of the general theorem.

Next we wish to prove the following for the ordinary and dot products:

$$\int_s (\vec{ds} \times \nabla) \circ = \int_c \vec{dr} \circ \quad (1.10)$$

Using stokes theorem, which reads:

$$\int_s (\nabla \times \vec{A}) \cdot \vec{ds} = \int_c \vec{A} \cdot \vec{dr} \quad (1.11)$$

We make the vector ansatz $\vec{A} = \vec{a}\phi$, where \vec{a} is an arbitrary constant vector and ϕ is an arbitrary scalar field. Substituting this into stokes theorem yields

$$\int_s (\nabla \times \vec{a}\phi) \cdot \vec{ds} = \int_c \phi \vec{a} \cdot \vec{dr} \quad (1.12)$$

$$\int_s [\phi \nabla \times \vec{a} + \nabla \phi \times \vec{a}] \cdot \vec{ds} = \int_c \phi \vec{a} \cdot \vec{dr} \quad (1.13)$$

$$\int_s \nabla \phi \times \vec{a} \cdot \vec{ds} = \int_c \phi \vec{a} \cdot \vec{dr} \quad (1.14)$$

$$- \int_s \vec{a} \times \nabla \phi \cdot \vec{ds} = \int_c \phi \vec{a} \cdot \vec{dr} \quad (1.15)$$

$$-\vec{a} \cdot \int_s \nabla \phi \times \vec{ds} = \vec{a} \cdot \int_c \phi \vec{dr} \quad (1.16)$$

$$\vec{a} \cdot \left[\int_c \phi \vec{dr} + \int_s \nabla \phi \times \vec{ds} \right] = 0 \quad (1.17)$$

$$\vec{a} \cdot \left[\int_c \phi \vec{dr} - \int_s \vec{ds} \times \nabla \phi \right] = 0 \quad (1.18)$$

$$(1.19)$$

Since \vec{a} is arbitrary the quantity in the brackets must be zero, completing the proof for the ordinary product. Finally we want the dot product version. We make the vector ansatz $\vec{A} = \vec{a} \times \vec{P}$, where once again \vec{a} is an arbitrary constant vector. Plugging this into stokes theorem gives us

$$\int_s (\nabla \times (\vec{a} \times \vec{P})) \cdot d\vec{s} = \int_c \vec{a} \times \vec{P} \cdot d\vec{r} \quad (1.20)$$

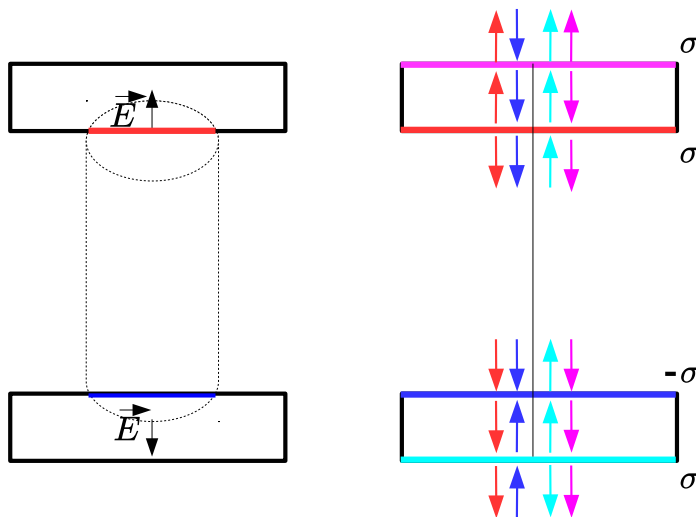
$$\int_s \left[\vec{a}(\nabla \cdot \vec{P}) - \vec{P}(\nabla \cdot \vec{a}) + (\vec{P} \cdot \nabla) \vec{a} - \vec{P}(\vec{a} \cdot \nabla) \right] \cdot d\vec{s} = \int_c \vec{P} \times d\vec{r} \cdot \vec{a} \quad (1.21)$$

$$\vec{a} \cdot \int_v \left[d\vec{s}(\nabla \cdot \vec{P}) - \nabla(\vec{P} \cdot d\vec{s}) \right] = \vec{a} \cdot \int_c \vec{P} \times d\vec{r} \quad (1.22)$$

$$(1.23)$$

Again, \vec{a} is arbitrary allowing us to equate the integral parts, completing the proof.

2 PROBLEM 2



2.1 PART A

The field must be zero inside both conductors. Furthermore, infinite sheets project field lines in a uniform fashion from their origin out to infinity. Consider the gaussian cylinder in figure A. Nominally the electric field \vec{E} would point normally out of the endcaps of the cylinder and would be proportional to ρ_{enc} . However the restriction that $\vec{E} = 0$ inside the conductors leads us to the conclusion that $\rho_{enc} = 0$. Unless the plates are uncharged this indicates that the charges must be equal and opposite.

2.2 PART B

The visual argument follows figure B. Given from part A that the surface charge densities on the inner surfaces are equal and opposite. We define the bottom surface of the upper plate $\sigma_{1,b}$ as positive and thus the top surface of the bottom plate $\sigma_{2,t} = -\sigma_{1,b}$. Pictured here is the case where the surface charge densities on the outer surfaces $\sigma_{1,t}$ and $\sigma_{2,b}$ carry positive charge, although the argument is exactly the same for the negative case. The arrows in the figure correspond to the direction of the electric field in a region due to the surface charge of the corresponding color. Arrows for negative charges always point towards the source surface. Arrows for positive charges always point away from the source surface. We see that inside the conductors, the red and blue fields from the inside surfaces already balance. This indicates that the outer, cyan and magenta surfaces must contribute in such a manner as to be opposite in the interior regions of the plates. This can only happen if they are equal in magnitude and sign.

2.3 PART C

The fields themselves can be found by imposing the following conditions from the problem description:

$$\sigma_{1,T} + \sigma_{1,B} = q_1 \quad (2.1)$$

$$\sigma_{2,T} + \sigma_{2,B} = q_2 \quad (2.2)$$

$$(2.3)$$

Plugging in the fact that $\sigma_{2,T} = -\sigma_{1,B}$ and that $\sigma_{2,B} = \sigma_{1,T}$ gives us

$$\sigma_{1,T} + \sigma_{1,B} = q_1 \quad (2.4)$$

$$\sigma_{1,T} - \sigma_{1,B} = q_2 \quad (2.5)$$

Adding the two equations gives us

$$\sigma_{1,T} = \sigma_{2,B} = \frac{q_1 + q_2}{2} \quad (2.6)$$

$$\rightarrow \vec{E}_{out} = \frac{q_1 + q_2}{2\epsilon_0} \quad (2.7)$$

Whereas subtracting the two equations gives us

$$\sigma_{1,B} = -\sigma_{2,T} = \frac{q_1 - q_2}{2} \quad (2.8)$$

$$\rightarrow \vec{E}_{in} = \frac{q_1 - q_2}{2\epsilon_0} \quad (2.9)$$

If $q_1 = -q_2 = Q$, then $E_{in} = \frac{2Q}{\epsilon_0}$ and $E_{out} = 0$, which makes sense as this corresponds to a standard parallel plate capacitor: No field outside and double the field from two parallel plates inside.

3 PROBLEM 3

Consider figure A. This is the normal situation in which both plates of the capacitor are copper, and gets us the normal geometric capacitance $C_g = \frac{Q_g}{V_g}$. In figure B, we replace one of the plates with graphene, whose structure allows for the density of charge carriers within the material to vary in the presence of appropriate forces driving such a change. Let the change in the energy due to such a process be defined by the total differential:

$$\Delta E = \frac{\partial E}{\partial N} \Delta N \quad (3.1)$$

$$(3.2)$$

Given ΔN new charge carriers within the graphene, the total change in charge would be $\Delta Q = q_e \Delta N$. Solving for ΔN gives us:

$$\Delta N = \frac{\Delta Q}{q_e} \quad (3.3)$$

$$\Delta E = \frac{\partial E}{\partial N} \frac{\Delta Q}{q_e} \quad (3.4)$$

$$\rightarrow \text{Divide by } q_e \quad (3.5)$$

$$\frac{\Delta E}{q_e} = \frac{\partial E}{\partial N} \frac{\Delta Q}{q_e^2} \quad (3.6)$$

But the left hand side is just the definition of ΔV as potential energy per unit charge. This allows us to directly relate ΔV and ΔQ :

$$\Delta V = \frac{\partial E}{\partial N} \frac{\Delta Q}{q_e^2} \quad (3.7)$$

$$\frac{\Delta Q}{\Delta V} = C_q = q_e^2 \frac{\partial N}{\partial E} \quad (3.8)$$

The derivative on the right is the density of states - thus if we know the geometric capacitance, and we know the *measured* capacitance, we can directly infer the density of states as long as we know the total charge of a charge carrier.

4 PROBLEM 4

If the separation distance between the two conductors is large relative to their size, we can consider them to be point charges. The voltages on each of the conductors is the superposition of two terms - one due to the charge on the conductor itself and its self capacitance

$$V_1 = \frac{q_1}{c_1} + \frac{1}{4\pi\epsilon_0} \frac{q_2}{r} \quad (4.1)$$

$$V_2 = \frac{q_2}{c_2} + \frac{1}{4\pi\epsilon_0} \frac{q_1}{r} \quad (4.2)$$

If we let $c_0 = 4\pi\epsilon_0 r$, then we have

$$V_1 = \frac{q_1}{c_1} + \frac{q_2}{c_0} \quad (4.3)$$

$$V_2 = \frac{q_2}{c_2} + \frac{q_1}{c_0} \quad (4.4)$$

Solving both equations for q_1 gives us

$$V_1 - \frac{q_2}{c_0} = \frac{q_1}{c_1} \rightarrow q_1 = V_1 c_1 - \frac{q_2 c_1}{c_0} \quad (4.5)$$

$$V_2 - \frac{q_2}{c_2} = \frac{q_1}{c_0} \rightarrow q_1 = V_2 c_0 - \frac{q_2 c_0}{c_2} \quad (4.6)$$

Equating the two expressions gives

$$V_1 c_1 - \frac{q_2 c_1}{c_0} = V_2 c_0 - \frac{q_2 c_0}{c_2} \quad (4.7)$$

$$V_1 c_1 - V_2 c_0 = \frac{q_2 c_1}{c_0} - \frac{q_2 c_0}{c_2} \quad (4.8)$$

$$= q_2 \left[\frac{c_1}{c_0} - \frac{c_0}{c_2} \right] \quad (4.9)$$

$$= q_2 \left[\frac{c_2 c_1 - c_0^2}{c_0 c_2} \right] \quad (4.10)$$

$$q_2 = (V_1 c_1 - V_2 c_0) \left[\frac{c_0 c_2}{c_2 c_1 - c_0^2} \right] \quad (4.11)$$

Letting β denote the unitless factor in brackets in the above equation we have

$$q_2 = \beta (V_1 c_1 - V_2 c_0) \quad (4.12)$$

Repeating the process solving the equations for q_2 now yields:

$$V_1 - \frac{q_1}{c_1} = \frac{q_2}{c_0} \rightarrow q_2 = V_1 c_0 - \frac{q_1 c_0}{c_1} \quad (4.13)$$

$$V_2 - \frac{q_1}{c_0} = \frac{q_2}{c_2} \rightarrow q_2 = V_2 c_2 - \frac{q_1 c_2}{c_0} \quad (4.14)$$

Equating the two expressions gives

$$V_1 c_0 - \frac{q_1 c_0}{c_1} = V_2 c_2 - \frac{q_1 c_2}{c_0} \quad (4.15)$$

$$V_1 c_0 - V_2 c_2 = \frac{q_1 c_0}{c_1} - \frac{q_1 c_2}{c_0} \quad (4.16)$$

$$= \left[\frac{c_0}{c_1} - \frac{c_2}{c_0} \right] q_1 \quad (4.17)$$

$$= \left[\frac{c_0^2 - c_1 c_2}{c_0 c_1} \right] q_1 \quad (4.18)$$

$$q_1 = (V_1 c_0 - V_2 c_2) \left[\frac{c_0 c_1}{c_0^2 - c_1 c_2} \right] \quad (4.19)$$

Letting γ denote the factor in brackets in the above equation we have

$$q_1 = \gamma (V_1 c_0 - V_2 c_2) \quad (4.20)$$

The force between the two conductors is given by Coulomb's Law:

$$F = \frac{K q_1 q_2}{r^2} \quad (4.21)$$

$$= \frac{K}{r^2} \beta \gamma (V_1 c_0 - V_2 c_2) (V_1 c_1 - V_2 c_0) \quad (4.22)$$

$$= \frac{K}{r^2} \left[\frac{c_0 c_1}{c_0^2 - c_1 c_2} \right] \left[\frac{c_0 c_2}{c_2 c_1 - c_0^2} \right] (V_1 c_0 - V_2 c_2) (V_1 c_1 - V_2 c_0) \quad (4.23)$$

$$= -\frac{K}{r^2} \left[\frac{c_0^2 c_1 c_2}{(c_0^2 - c_1 c_2)^2} \right] (V_1 c_0 - V_2 c_2) (V_1 c_1 - V_2 c_0) \quad (4.24)$$

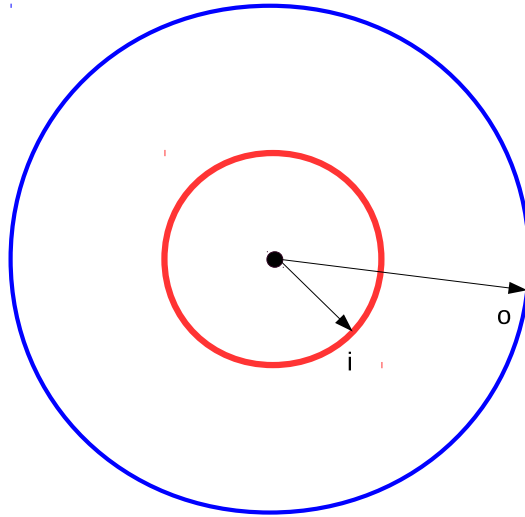
$$= \frac{K}{r^2} \left[\frac{c_0^2 c_1 c_2}{(c_0^2 - c_1 c_2)^2} \right] (V_1 c_0 - V_2 c_2) (V_2 c_0 - V_1 c_1) \quad (4.25)$$

Noting that $K c_0 = r$, and rearranging, we have

$$F = 4\pi\epsilon_0 c_1 c_2 \left[\frac{(4\pi\epsilon_0 V_1 - V_2 c_2) (4\pi\epsilon_0 V_2 - V_1 c_2)}{\left((4\pi\epsilon_0 r)^2 - c_1 c_2 \right)^2} \right] \quad (4.26)$$

Completing the proof.

5 PROBLEM 5



The coefficients in the capacitance matrix are the result of the geometry of the system and not the configuration of the potentials or charges on the system - in fact the voltages and charges in any physical scenario will adjust themselves such that they obey the following definition for the c_{ij} :

$$c_{ij} = \frac{\partial q_i}{\partial V_j} \quad (5.1)$$

We can thus charge the system in any manner we see fit and the c_{ij} will always be the same, meaning we can invent whatever scenario is most convenient to produce an

expression for $q_i(V_j)$ whose derivative we can take. For element c_{11} we need to relate V_1 and q_1 , we designate these V_i and q_i . Let the outer shell be grounded at potential $V_o = 0$ and the inner shell be at some potential v_i . Both shells will carry charges q_o and q_i respectively. The potential v_i can be found starting from the definition and using the field from the inner spherical shell with charge q_i :

$$V_b - V_a = - \int_a^b \vec{E} \cdot \vec{dl} \quad (5.2)$$

$$V_i - V_o = - \int_{r_o}^{r_i} \frac{kq_i}{r^2} \hat{r} \cdot \hat{r} dr \quad (5.3)$$

$$V_i = -kq_i \left[-\frac{1}{r} \right]_{r_o}^{r_i} \quad (5.4)$$

$$= -kq_i \left[\frac{1}{r_o} - \frac{1}{r_i} \right] \quad (5.5)$$

$$= -kq_i \left[\frac{r_i - r_o}{r_o r_i} \right] \quad (5.6)$$

$$V_i = kq_i \left[\frac{r_o - r_i}{r_o r_i} \right] \quad (5.7)$$

Solving for q_i and taking a derivative gets us

$$q_i = \frac{V_i}{k} \left[\frac{r_o r_i}{r_o - r_i} \right] \quad (5.8)$$

$$\frac{\partial q_i}{\partial V_i} = \left[\frac{r_o r_i}{k(r_o - r_i)} \right] = c_{ii} = c_{11} \quad (5.9)$$

This quantity is positive and has units of inverse k times length, which is capacitance. In a similar manner we solve for $c_{12} = c_{21} = c_{io}$. This time we choose to ground the *inner* shell at $V_i = 0$ and let the outer shell exist at potential V_o , because we want to relate q_i with V_o . The potential would then be given by the following, since the electric field is exactly the same:

$$V_o - V_i = - \int_{r_i}^{r_o} \frac{kq_i}{r^2} \hat{r} \cdot \hat{r} dr \quad (5.10)$$

$$V_o = -kq_i \left[-\frac{1}{r} \right]_{r_i}^{r_o} \quad (5.11)$$

$$= kq_i \left[\frac{1}{r_o} - \frac{1}{r_i} \right] \quad (5.12)$$

$$V_o = kq_i \left[\frac{r_i - r_o}{r_i r_o} \right] \quad (5.13)$$

$$(5.14)$$

Solving for q_i and taking a derivative gets us

$$q_i = \frac{V_o}{k} \left[\frac{r_i r_o}{r_i - r_o} \right] \quad (5.15)$$

$$\frac{\partial q_i}{\partial V_o} = \frac{r_i r_o}{k (r_i - r_o)} = c_{io} = c_{12} \quad (5.16)$$

$$= -c_{11} \quad (5.17)$$

We note here that the off diagonal coefficients are apparently, in this case, the negative of the upper left hand element. The final element is c_{22} , solving for which requires us to relate V_o and q_o . The trick here is that we need to get q_o in our expression somehow. This is in contrast to the previous three coefficients which can be found just by investigating the field between the two spheres due to q_i . This time we ground the inner surface so that $V_i = 0$ and note that the potential $V_\infty = 0$, at $r = \infty$. We investigate the field at $r > r_o$. By gauss law, the electric field is proportional to the sum of the enclosed charges:

$$\vec{E} = \frac{k (q_i + q_o)}{r^2} \hat{r} \quad (5.18)$$

$$V_o - V_\infty = - \int_\infty^{r_o} \frac{k (q_i + q_o)}{r^2} \hat{r} \cdot \hat{r} dr \quad (5.19)$$

$$= \frac{k (q_i + q_o)}{r_o} \quad (5.20)$$

We need to somehow eliminate q_1 from the equation. We can thankfully use the fundamental linear relationship for q_1 in this system, noting that $V_i = 0$ in this particular physical scenario:

$$q_i = c_{11} V_1 + c_{12} V_2 \quad (5.21)$$

$$= c_{io} V_o \quad (5.22)$$

We know c_{12} from the previous direct calculation. Substituting it in we would get

$$V_o = \frac{kV_o \left(\frac{r_i r_o}{k(r_i - r_o)} \right) + kq_o}{r_o} \quad (5.23)$$

$$= \frac{kV_o r_i}{r_o k(r_i - r_o)} + \frac{kq_o}{r_o} \quad (5.24)$$

$$V_o = \frac{V_o r_i}{r_i - r_o} + \frac{kq_o}{r_o} \quad (5.25)$$

$$V_o \left(1 - \frac{r_i}{r_i - r_o} \right) = \frac{kq_o}{r_o} \quad (5.26)$$

$$V_o \left(\frac{r_i - r_o - r_i}{r_i - r_o} \right) = \frac{kq_o}{r_o} \quad (5.27)$$

$$V_o \left(\frac{r_o}{r_o - r_i} \right) = \frac{kq_o}{r_o} \quad (5.28)$$

$$q_o = \frac{V_o}{k} \left(\frac{r_o^2}{r_o - r_i} \right) \quad (5.29)$$

$$\frac{\partial q_o}{\partial V_o} = \frac{r_o^2}{k(r_o - r_i)} = c_{oo} = c_{22} \quad (5.30)$$

We can now form the whole matrix:

$$\begin{bmatrix} q_i \\ q_o \end{bmatrix} = \frac{r_o}{k(r_o - r_i)} \begin{bmatrix} r_i & -r_i \\ -r_i & r_o \end{bmatrix} \begin{bmatrix} V_i \\ V_o \end{bmatrix} \quad (5.31)$$

To relate this to what we would normally call mutual capacitance, we need to define $q_o = -q_i$, and go back to the definition

$$V_i = -p_{ii}Q + p_{io}Q \quad (5.32)$$

$$V_o = -p_{oi}Q + p_{oo}Q \quad (5.33)$$

The potential difference $V_o - V_i = (p_{ii} + p_{oo} - 2p_{io})Q$, and $C_{mutual} = \frac{Q}{\Delta V}$, so $C_{mutual} = (p_{ii} + p_{oo} - 2p_{io})^{-1}$. The matrix P can be found by the inversion rule for symmetric 2x2 matrices:

$$P = C^{-1} = \frac{1}{c_{ii}c_{oo} - c_{io}^2} \begin{bmatrix} c_{oo} & -c_{io} \\ -c_{io} & c_{ii} \end{bmatrix} \quad (5.34)$$

This implies that the following formula for C_{mutual} :

$$C_{mutual} = \frac{c_{ii}c_{oo} - c_{io}^2}{c_{oo} + c_{ii} - 2c_{io}} \quad (5.35)$$

$$= \frac{\left[\frac{r_o}{k(r_o - r_i)} \right]^2 (r_i r_o - r_i^2)}{\left[\frac{r_o}{k(r_o - r_i)} \right] (r_o + r_i - 2r_i)} \quad (5.36)$$

$$= \frac{r_o r_i}{k(r_o - r_i)} \checkmark \quad (5.37)$$

This completes the proof.