Northwestern University

Quantum Mechanics, Final Exam

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1 Problem 1

Consider a wavefunction given by

$$\Psi(\vec{x}) = (x + y + 3z)f(r) \tag{1.1}$$

I wish to check if Ψ is an eigenfunction of the L^2 operator in three dimensions. I know that in the spherical coordinate basis, the L^2 operator can be represented as (Shankar Equation 12.5.36):

$$L^2 = \hbar^2 \left[\frac{1}{\sin(\theta)} \frac{\partial}{\partial\theta} \sin(\theta) \frac{\partial}{\partial\theta} + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial\phi^2} \right] \tag{1.2}$$

Using the transformations from cartiesian to spherical coordinates

$$x = r\cos(\phi)\sin(\theta) \tag{1.3}$$
$$y = r\sin(\phi)\sin(\theta) \tag{1.4}$$
$$z = r\cos(\theta) \tag{1.5}$$

The radial wavefunction reads

$$\Psi = (r\cos(\phi)\sin(\theta) + r\sin(\phi)\sin(\theta) + 3r\cos(\theta))f(r) \tag{1.6}$$
$$= r(\cos(\phi)\sin(\theta) + \sin(\phi)\sin(\theta) + 3\cos(\theta))f(r) \tag{1.7}$$

I now act upon the whole wavefunction with the L^2 operator in the coordinate basis and attempt to simplify the results. Starting from the last term first, I need two derivatives of the wavefunction with respect to ϕ.

$$\frac{d}{d\phi}\Psi = rf(r)\frac{\partial}{\partial\phi}(\cos(\phi)\sin(\theta) + \sin(\phi)\sin(\theta) + \cancel{3\cos(\theta)}) \tag{1.8}$$

$$= rf(r)(-\sin(\phi)\sin(\theta) + \cos(\phi)\sin(\theta)) \tag{1.9}$$

$$\text{2nd.} \;\rightarrow\; \text{need } \frac{\partial}{\partial\phi}(-\sin(\phi)\sin(\theta) + \cos(\phi)\sin(\theta)) \tag{1.10}$$

$$= rf(r)(-\cos(\phi)\sin(\theta) - \sin(\phi)\sin(\theta)) \tag{1.11}$$

$$\text{Times } \frac{1}{\sin^2(\theta)} \rightarrow -rf(r)\left[\frac{\cos(\phi)}{\sin(\theta)} + \frac{\sin(\phi)}{\sin(\theta)}\right] \tag{1.12}$$

We need a derivative with respect to θ:

$$\frac{\partial}{\partial\theta}\Psi = \frac{\partial}{\partial\theta}rf(r)(\cos(\phi)\sin(\theta) + \sin(\phi)\sin(\theta) + 3\cos(\theta)) \tag{1.13}$$

$$= rf(r)(\cos(\phi)\cos(\theta) + \sin(\phi)\cos(\theta) - 3\sin(\theta)) \tag{1.14}$$

$$\text{Times } \sin(\theta) \rightarrow rf(r)(\cos(\phi)\cos(\theta)\sin(\theta) + \sin(\phi)\cos(\theta)\sin(\theta) - 3\sin^2(\theta)) \tag{1.15}$$

$$\rightarrow \frac{\partial}{\partial\theta}rf(r)(\cos(\phi)\cos(\theta)\sin(\theta) + \sin(\phi)\cos(\theta)\sin(\theta) - 3\sin^2(\theta)) \tag{1.16}$$

$$= rf(r)\left[\cos(\phi)(\cos(2\theta)) + \sin(\phi)(\cos(2\theta)) - 3\sin(2\theta)\right] \tag{1.17}$$

$$\text{Times } \frac{1}{\sin(\theta)} \rightarrow rf(r)\frac{1}{\sin(\theta)}\left[\cos(\phi)(\cos(2\theta)) + \sin(\phi)(\cos(2\theta)) - 3\sin(2\theta)\right] \tag{1.18}$$

At this point I should be able to piece together the action of the whole operator on Ψ:

$$rf(r)\left[\frac{1}{\sin(\theta)}\left[\cos(\phi)(\cos(2\theta)) + \sin(\phi)(\cos(2\theta)) - 3\sin(2\theta)\right] - \left[\frac{\cos(\phi)}{\sin(\theta)} + \frac{\sin(\phi)}{\sin(\theta)}\right]\right] \tag{1.19}$$

I'm not sure how exactly the trig identities make this work out, but *Mathematica* **(code included)** this simplifies to

$$h = -2(\sin(\theta)(\sin(\phi) + \cos(\phi)) + 3\cos(\theta)) \tag{1.20}$$

$$\tag{1.21}$$

Multiplying by the $-\hbar^2$ in front of the L^2 operator, we arrive at the fact that

$$L^2\Psi = 2\hbar^2\Psi \tag{1.22}$$

Meaning that Ψ is an eigenstate of L^2 with the eigenvalue $2\hbar^2$. Since we must have eigenvalues of the form $l(l+1)\hbar^2$, we know $l(l+1) = 2$, which indicates that $l = 1$ for this eigenstate. This in turn indicates that m_l can run from $-l$ to l or that m_l can take on the values $-1, 0,$ and 1. We should be able to get the probabilities of each of these states by expanding the angular part of the original state as a linear combination of spherical harmonics:

$$\begin{aligned}
\Psi(\theta,\phi) &= (\cos(\phi)\sin(\theta) + \sin(\phi)\sin(\theta) + 3\cos(\theta)) && (1.23)\\
&= \left[\left(\frac{e^{i\phi}+e^{-i\phi}}{2}\right)\sin(\theta) + \left(\frac{e^{i\phi}-e^{-i\phi}}{2i}\right)\sin(\theta) + 3\cos(\theta)\right] && (1.24)\\
&= \left[\frac{1}{2}e^{i\phi}\sin(\theta) + \frac{1}{2}e^{-i\phi}\sin(\theta) - \frac{i}{2}e^{i\phi}\sin(\theta) + \frac{i}{2}e^{-i\phi}\sin(\theta) + 3\cos(\theta)\right] && (1.25)\\
&= \left[\frac{1}{2}(1-i)e^{i\phi}\sin(\theta) + \frac{1}{2}(1+i)e^{-i\phi}\sin(\theta) + 3\cos(\theta)\right] && (1.26)
\end{aligned}$$

Where I plugged in the standard definitons of sine and cosine as complex exponentials. Now, the $l = 0$ and $l = 1$ spherical harmonics read:

$$Y_0^0 = \sqrt{\frac{3}{4\pi}}\cos(\theta) \tag{1.27}$$

$$Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{\pm i\phi} \tag{1.28}$$

I want to substitute these into the wavefunction in its current state. To do this I note that, for instance:

$$3\cos(\theta) = A\sqrt{\frac{3}{4\pi}}\cos(\theta) \tag{1.29}$$

$$\tag{1.30}$$

This is like asking "how much of that particular spherical harmonic is this term", then we can simply solve for A to get to correct coefficients.

$$\begin{aligned}
&= \left[\frac{\frac{1}{2}(1-i)}{-\sqrt{\frac{3}{8\pi}}}Y_1^1 + \frac{\frac{1}{2}(1+i)}{\sqrt{\frac{3}{8\pi}}}Y_1^{-1} + \frac{3}{\sqrt{\frac{3}{4\pi}}}Y_0^0\right] && (1.31)\\
&= \left[(i-1)\sqrt{\frac{2\pi}{3}}Y_1^1 + (1+i)\sqrt{\frac{2\pi}{3}}Y_1^{-1} + 2\sqrt{3\pi}Y_0^0\right] && (1.32)
\end{aligned}$$

The probabilities of getting each of the states are the squares of the moduli of each of the coefficients on the spherical harmonics, where Γ is the sum of the squares of the moduli of the coefficients. Each one is individual divided by total. Gamma is:

$$\Gamma = \left|(i-1)\sqrt{\frac{2\pi}{3}}\right|^2 + \left|2\sqrt{3\pi}\right|^2 + \left|(1+i)\sqrt{\frac{2\pi}{3}}\right|^2 \tag{1.33}$$

$$= \frac{44\pi}{3} \tag{1.34}$$

$$P(m_l = -1) = \frac{\left|(i-1)\sqrt{\frac{2\pi}{3}}\right|^2}{\Gamma} = \frac{1}{11} \tag{1.35}$$

$$P(m_l = 0) = \frac{\left|2\sqrt{3\pi}\right|^2}{\Gamma} = \frac{9}{11} \tag{1.36}$$

$$P(m_l = 1) = \frac{\left|(1+i)\sqrt{\frac{2\pi}{3}}\right|^2}{\Gamma} = \frac{1}{11} \tag{1.37}$$

$$\tag{1.38}$$

Note the probability of finding the system in *some state* adds up to 1.

If we know that Ψ is an energy eigenfunction with eigenvalue E, we can find the potential using the schrodinger equation. To do this we would rewrite the radial schrodinger equation as

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{L^2}{2mr^2}\right]u = Eu \tag{1.39}$$

Since we have the eigenvalues of L^2, all we would do is take a bunch of derivatives of the r-dependent part, and then solve algebraically for $V(r)$. We have a wavefunction $\Psi = rf(r)g(\theta,\phi) = u(r)g(\theta,\phi)$ using the standard convention for the definiton of $u(r)$. Substituting in $-2\hbar^2$ for L^2 and expanding, we get

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r)u + \frac{\hbar^2 u}{mr^2} = Eu \tag{1.40}$$

$$Eu + \frac{\hbar^2}{m}\left[\frac{1}{2}\frac{d^2u}{dr^2} - \frac{u^2}{r^2}\right] = V(r) \tag{1.41}$$

Completing the problem.

2 Problem 2

We are asked to investigate a two spin-$\frac{1}{2}$, mass m particle system interacting through the potential

$$V(r) = \frac{g}{r}\sigma_1 \cdot \sigma_2 \tag{2.1}$$

Where g is a constant greater then zero and σ_i are the pauli spin matrices for the $i'th$ particle.
I am interpreting σ_1 and σ_2 as some kind of vector of pauli matrices including the x, y, and z components. The total spin operator is given by

$$S = \frac{\hbar}{2}(\sigma_1 + \sigma_2) \tag{2.2}$$

So that $S \cdot S$ is given by

$$\frac{\hbar^2}{4}(\sigma_1^2 + \sigma_2^2 + 2\sigma_1 \cdot \sigma_2) \tag{2.3}$$

But we know that $\sigma_i^2 = \sigma_i \cdot \sigma_i = \sigma_i^x \sigma_i^x + \sigma_i^y \sigma_i^y + \sigma_i^z \sigma_i^z$. Now, any component pauli matrix squared equals the identity. So we have that $(\sigma_i^j)^2 = I$ for all of the components. This gets us to the fact that $\sigma_i^2 = 3I$.

Now we can write $S \cdot S$ as:

$$S \cdot S = \frac{\hbar^2}{4}(3I + 3I + 2\sigma_1 \cdot \sigma_2) \tag{2.4}$$

$$= \frac{\hbar^2}{4}(6I + 2\sigma_1 \cdot \sigma_2) \tag{2.5}$$

Rewriting the left hand side using the known eigenvalues for the $S \cdot S$ operator we have

$$\cancel{\hbar^2} S(S+1) = \frac{\cancel{\hbar^2}}{4}(6I + 2\sigma_1 \cdot \sigma_2) \tag{2.6}$$

From here we can solve the equation for the term $\sigma_1 \cdot \sigma_2$:

$$4S(S+1) = 6I + 2\sigma_1 \cdot \sigma_2 \tag{2.7}$$

$$4S(S+1) - 6I = 2\sigma_1 \cdot \sigma_2 \tag{2.8}$$

$$2S(S+1) - 3I = \sigma_1 \cdot \sigma_2 \tag{2.9}$$

Substituting this expression into the potential we have that

$$V(r) = \frac{g}{r}\left[2S(S+1) - 3I\right] \tag{2.10}$$

We can plug in a few values of S to examine bound states. Evidently, if the total spin $S = 1$, then $V(r) = \frac{g}{r}[4 - 3I]$ which is greater then zero and gives rise to repulsive potentials. The $S = 0$ state has potential $V(r) = -\frac{3g}{r}$ which is attractive and gives rise to bound states.

Since this is a quantum mechanical two body problem with an attractive potential, the energy eigenvalues for this system must be something like the same ones for a hydrogen atom. Those are

$$E_n = \frac{m}{2\hbar^2 n^2}\left[\frac{e^2}{4\pi\epsilon_0}\right]^2 \tag{2.11}$$

Where the potential was $V(r) = -\frac{1}{4\pi\epsilon_0 r}$. Therefore it is not unreasonable to expect energy eigenvalues something like:

$$E_n = \frac{m}{2\hbar^2 n^2}[3g]^2 = \frac{9mg}{2\hbar^2 n^2} \tag{2.12}$$

For the spin dependent potential.

3 Problem 3

We are asked to find the eigenstates for S^2 and S_z for a three spin-$\frac{1}{2}$ particle system including no orbital angular momentum. We look to express the $2x2x2 = 8$ eigenstates in terms of the individual basis states. Since $[S^2, S_z] = 0$, we know that the two operators share a common eigenbasis. So it will suffice just to find the eigenstates of S^2. Since there are 3 such particles, the maximum possible value of S is equal to $\frac{3}{2}$. This means that m can take on values of $-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}$, and $\frac{3}{2}$.

We search for linear combinations of the individual basis states:

$$1.\ |\uparrow\uparrow\uparrow\rangle \tag{3.1}$$
$$2.\ |\downarrow\uparrow\uparrow\rangle \tag{3.2}$$
$$3.\ |\uparrow\downarrow\uparrow\rangle \tag{3.3}$$
$$4.\ |\uparrow\uparrow\downarrow\rangle \tag{3.4}$$
$$5.\ |\downarrow\downarrow\uparrow\rangle \tag{3.5}$$
$$6.\ |\downarrow\uparrow\downarrow\rangle \tag{3.6}$$
$$7.\ |\uparrow\downarrow\downarrow\rangle \tag{3.7}$$
$$8.\ |\downarrow\downarrow\downarrow\rangle \tag{3.8}$$

We begin by showing that $|\uparrow\uparrow\uparrow\rangle$ is an eigenstate of S^2. We can do so by showing that it is in fact just an eigenstate of S_z. If $S_z^T = S_z^1 + S_z^2 + S_z^3$, then

$$
\begin{aligned}
S_z^T \left|\uparrow\uparrow\uparrow\right\rangle &= (S_z^1 + S_z^2 + S_z^3)\left|\uparrow\uparrow\uparrow\right\rangle && (3.9)\\
&= S_z^1\left|\uparrow\right\rangle\left|\uparrow\right\rangle\left|\uparrow\right\rangle + \left|\uparrow\right\rangle S_z^2\left|\uparrow\right\rangle + \left|\uparrow\right\rangle\left|\uparrow\right\rangle S_z^3\left|\uparrow\right\rangle && (3.10)\\
&= \frac{\hbar}{2}\left|\uparrow\right\rangle\left|\uparrow\right\rangle\left|\uparrow\right\rangle + \left|\uparrow\right\rangle\frac{\hbar}{2}\left|\uparrow\right\rangle + \left|\uparrow\right\rangle\left|\uparrow\right\rangle\frac{\hbar}{2}\left|\uparrow\right\rangle && (3.11)\\
&= \frac{3\hbar}{2}\left|\uparrow\uparrow\uparrow\right\rangle && (3.12)
\end{aligned}
$$

Which confirms that $\left|\uparrow\uparrow\uparrow\right\rangle$ is already an eigenstate both of S_z and S^2. We can now get at least three other eigenstates of S^2 by acting the lowering operator on the top state:

$$
\begin{aligned}
S_-^T \left|\uparrow\uparrow\uparrow\right\rangle &= (S_-^1 + S_-^2 + S_-^3)\left|\uparrow\uparrow\uparrow\right\rangle && (3.13)\\
&= S_-^1\left|\uparrow\right\rangle\left|\uparrow\right\rangle\left|\uparrow\right\rangle + \left|\uparrow\right\rangle S_-^2\left|\uparrow\right\rangle\left|\uparrow\right\rangle + \left|\uparrow\right\rangle\left|\uparrow\right\rangle S_-^3\left|\uparrow\right\rangle && (3.14)\\
&= \hbar\left|\downarrow\uparrow\uparrow\right\rangle + \hbar\left|\uparrow\downarrow\uparrow\right\rangle + \hbar\left|\uparrow\uparrow\downarrow\right\rangle && (3.15)\\
\text{Normalize.} \rightarrow &\frac{\hbar}{\sqrt{3}}\left(\ \left|\downarrow\uparrow\uparrow\right\rangle + \left|\uparrow\downarrow\uparrow\right\rangle + \left|\uparrow\uparrow\downarrow\right\rangle\ \right) && (3.16)
\end{aligned}
$$

The above corresponds to the $\left|\frac{3}{2}, \frac{1}{2}\right\rangle$ state. By the same logic we can show that the bottom state, corresponding to $\left|\downarrow\downarrow\downarrow\right\rangle$ or $\left|\frac{3}{2}, -\frac{3}{2}\right\rangle$ is also an eigenstate of S^2 and therefore S_z:

$$
\begin{aligned}
S_z^T \left|\downarrow\downarrow\downarrow\right\rangle &= (S_z^1 + S_z^2 + S_z^3)\left|\downarrow\downarrow\downarrow\right\rangle && (3.17)\\
&= -\frac{3\hbar}{2}\left|\downarrow\downarrow\downarrow\right\rangle && (3.18)
\end{aligned}
$$

Applying now a *raising* operator to this eigenstate gives

$$
\begin{aligned}
S_+^T \left|\downarrow\downarrow\downarrow\right\rangle &= (S_+^1 + S_+^2 + S_+^3)\left|\downarrow\downarrow\downarrow\right\rangle && (3.19)\\
&= S_+^1\left|\downarrow\right\rangle\left|\downarrow\right\rangle\left|\downarrow\right\rangle + \left|\downarrow\right\rangle S_+^2\left|\downarrow\right\rangle\left|\downarrow\right\rangle + \left|\downarrow\right\rangle\left|\downarrow\right\rangle S_+^3\left|\downarrow\right\rangle && (3.20)\\
&= \hbar\left|\uparrow\downarrow\downarrow\right\rangle + \hbar\left|\downarrow\uparrow\downarrow\right\rangle + \hbar\left|\downarrow\downarrow\uparrow\right\rangle && (3.21)\\
\text{Normalize.} \rightarrow &\frac{\hbar}{3}\left(\ \left|\uparrow\downarrow\downarrow\right\rangle + \left|\downarrow\uparrow\downarrow\right\rangle + \left|\downarrow\downarrow\uparrow\right\rangle\ \right) && (3.22)
\end{aligned}
$$

I can get at least two more states by just adding a spin-up or spin-down particle to a singlet for two particles:

$$
\frac{\hbar}{\sqrt{2}}\left(\ \left|\uparrow\downarrow\uparrow\right\rangle + \left|\downarrow\uparrow\uparrow\right\rangle\ \right) \qquad (3.23)
$$

$$
\frac{\hbar}{\sqrt{2}}\left(\ \left|\uparrow\downarrow\downarrow\right\rangle + \left|\downarrow\uparrow\downarrow\right\rangle\ \right) \qquad (3.24)
$$

I get the last two by treating two spin-$\frac{1}{2}$ particles as a single particle of spin 1 and using the $1x\frac{1}{2}$ entry in the Clebsch-Gordan table:

$$\frac{\hbar}{\sqrt{6}}(\ 2\left|\uparrow\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\uparrow\right\rangle + \left|\uparrow\downarrow\uparrow\right\rangle\ ) \tag{3.25}$$

$$\frac{\hbar}{\sqrt{6}}(\ 2\left|\downarrow\downarrow\uparrow\right\rangle + \left|\uparrow\downarrow\downarrow\right\rangle \left|\downarrow\uparrow\downarrow\right\rangle\ ) \tag{3.26}$$

Or, really, just finding one and then flipping all of the spins over.

4 Problem 4

We are asked to estimate the energy levels for a quartic potential in one dimension of the form λx^4. The Schrodinger equation for this situation reads:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \lambda x^4\Psi = E\Psi \tag{4.1}$$

I will use the WKB approximation. Following *Griffiths* equation 8.51, we have that the wavefunctions must match up in the region between the turning points x_1 and x_2.

$$\int_{x_1}^{x_2} p(x)dx = \left(n - \frac{1}{2}\right)\pi\hbar \tag{4.2}$$

At a turning point, we must have that the potential energy is equal to the energy:

$$E = \lambda x^4 \tag{4.3}$$

$$\frac{E}{\lambda} = x^4 \tag{4.4}$$

$$\left(\frac{E}{\lambda}\right)^{\frac{1}{4}} = x_{1,2} \tag{4.5}$$

The potential is an even function allowing us to double the integral for $p(x)$ and plug in the value we just found for the upper limit:

$$2\int_0^{\left(\frac{E}{\lambda}\right)^{\frac{1}{4}}} \sqrt{2m(E - \lambda x^4)}dx \tag{4.6}$$

As it stands this integral is completely impossible. Make the substitution:

$$z = x^4 \tag{4.7}$$
$$x = z^{\frac{1}{4}} \tag{4.8}$$
$$dz = 4x^3 dx \tag{4.9}$$
$$= 4(z^{\frac{1}{4}})^3 dx \tag{4.10}$$
$$= 4z^{\frac{3}{4}} dx \tag{4.11}$$
$$\rightarrow = dx = \frac{1}{4} z^{-\frac{3}{4}} dz \tag{4.12}$$

We also need to transform the upper limit of integration:

$$(\frac{E}{\lambda}^{\frac{1}{4}})^4 = \frac{E}{\lambda} \tag{4.13}$$

So we have

$$2 \int_0^{\frac{E}{\lambda}} \sqrt{2m(E - \lambda z)} \left[\frac{1}{4} z^{-\frac{3}{4}} \right] dz \tag{4.14}$$

Integrating with Mathematica (code attached) I get

$$\frac{E^{3/4} \sqrt{2\pi m} \Gamma\left(\frac{5}{4}\right)}{\sqrt[4]{\lambda} \Gamma\left(\frac{7}{4}\right)} = \left(n - \frac{1}{2}\right) \pi \hbar \tag{4.15}$$

Solving this expression for E gives me

$$E = \left[\left(n - \frac{1}{2}\right) \pi \hbar \frac{\Gamma(\frac{7}{4})(4\lambda^{\frac{1}{4}})}{\Gamma(\frac{1}{4})\sqrt{2\pi m}} \right]^{\frac{4}{3}} \tag{4.16}$$

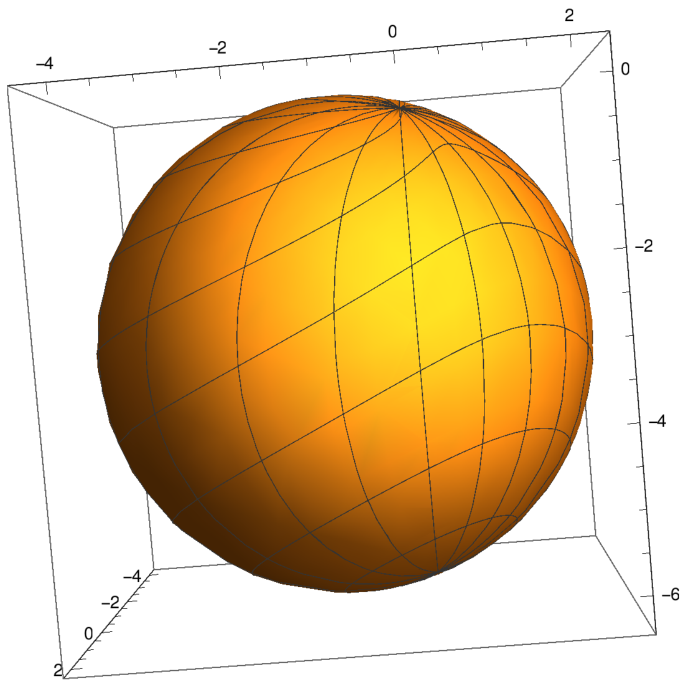
These should be the energy levels for a quartic potential.

In[20]:= FullSimplify[(1 / Sin[θ]) (Cos[ϕ] Cos[2 θ] + Sin[ϕ] Cos[2 θ] - 3 Sin[2 θ]) - (Cos[ϕ] / Sin[θ] + Sin[ϕ] / Sin[θ])]

Out[20]= -2 (3 Cos[θ] + Sin[θ] (Cos[ϕ] + Sin[ϕ]))

In[21]:= SphericalPlot3D[-2 (3 Cos[θ] + Sin[θ] (Cos[ϕ] + Sin[ϕ])), {θ, 0, π}, {ϕ, 0, 2 π}]

Out[21]=



In[31]:= ans = Integrate[2 Sqrt[2 m (p - λ z)] ((1 / 4) z^(-3 / 4)), {z, 0, (p / λ)}]

Out[31]= ConditionalExpression$\left[\dfrac{\sqrt{m\,p}\,\sqrt{2\,\pi}\,\left(\frac{p}{\lambda}\right)^{1/4}\,\text{Gamma}\left[\frac{5}{4}\right]}{\text{Gamma}\left[\frac{7}{4}\right]},\ m\,p > 0\ \&\&\ p\,\lambda > 0\right]$

In[33]:= Solve[ans == (n - (1 / 2)) π h, p]