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Numerical Solutions of Partial Differential Equations
偏微分方程数值解

——Poisson 方程的FDM

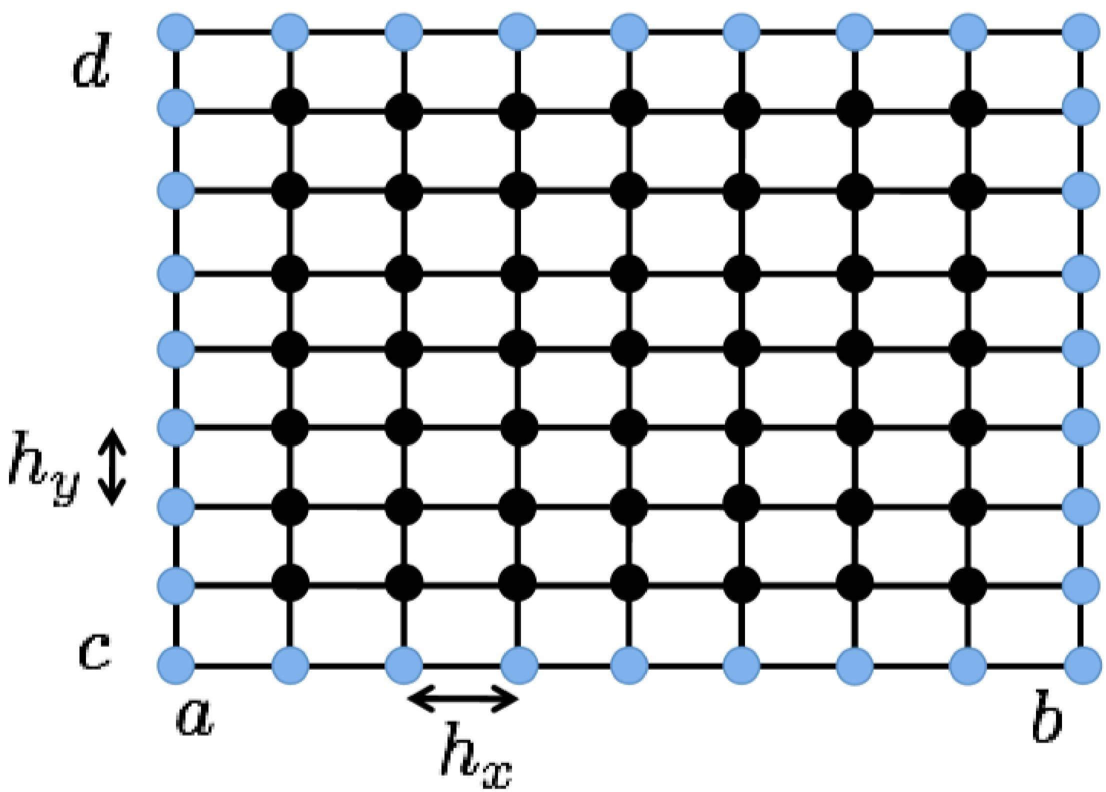
Huazhong Tang
School of Mathematical Sciences
Peking University, P.R. China

Grid/mesh for a simple 2D domain.

长方形网格

网格线 $x_i = a + ih_x, i = 0,1,2, \dots, M,$ $h_x = \dfrac{b-a}{M}$

网格线 $y_j = c + jh_y, j = 0,1,2, \dots, N,$ $h_y = \dfrac{d-c}{N}$



网格步长

FDM for Poisson eq.

$-\Delta u = f$ in Ω

$u = g$ on $\partial\Omega$

Ω=[0,1]X[0,1]

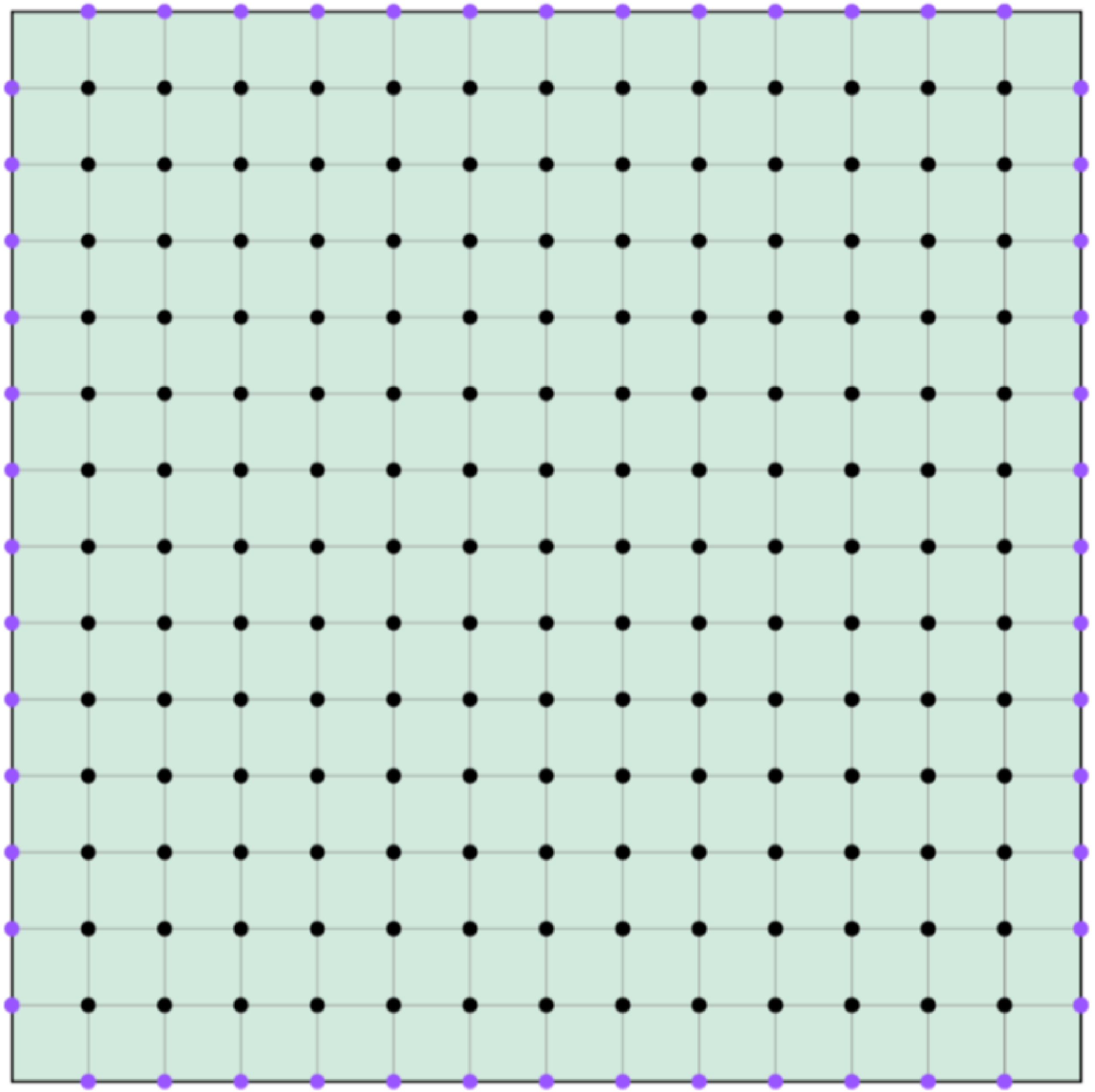
正方形网格

网格步长: h = 1/N

内部网格点: 黑色

边界网格点: 紫色

mesh

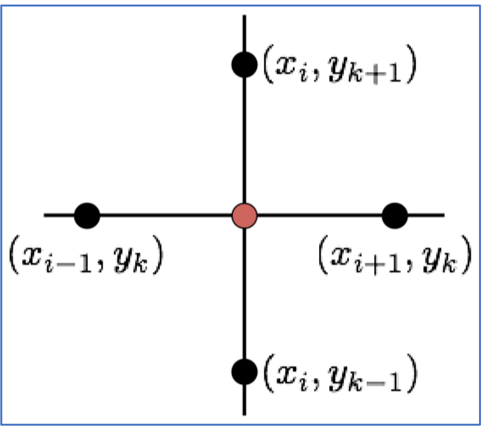


FDM for Poisson eq.

5-point difference operator Δ_h defined by

$$\Delta_h u(ih,jh) := \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}$$

$$= \frac{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}}{h^2}$$



Taylor 展开

$$\Delta_h u(ih,jh) - \Delta u(ih,jh) = \frac{h^2}{12}\left[\frac{\partial^4 u}{\partial x^4}(\xi, jh) + \frac{\partial^4 u}{\partial y^4}(ih, \eta)\right] \qquad (*0)$$

Theorem 2.1. If $u \in C^4(\overline{\Omega})$, then $\lim_{h\to 0} ||\Delta_h u - \Delta u||_{L^\infty(\Omega_h)} = 0.$

Lemma 3.4. *If $u \in C^4(\Omega)$, then*

$$\|\Delta_h u_I - (\Delta u)_I\|_{\infty,\Omega_h\setminus\Gamma_h} \le \frac{h^2}{6}\max\left\{\left\|\frac{\partial^4 u}{\partial x^4}\right\|_{\infty,\Omega}, \left\|\frac{\partial^4 u}{\partial y^4}\right\|_{\infty}\right\}$$

https://www.math.uci.edu/~chenlong/226/FDM.pdf

FDM for Poisson eq.

$$-\Delta_h u_h = f \text{ on } \Omega_h, \quad u_h = g \text{ on } \Gamma_h.$$

N = 4

$$-\begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{pmatrix} = \begin{pmatrix} h^2 f_{1,1} - u_{1,0} - u_{0,1} \\ h^2 f_{2,1} - u_{2,0} \\ h^2 f_{3,1} - u_{3,0} - u_{4,1} \\ h^2 f_{1,2} - u_{0,2} \\ h^2 f_{2,2} \\ h^2 f_{3,2} - u_{4,2} \\ h^2 f_{1,3} - u_{0,3} - u_{1,4} \\ h^2 f_{2,3} - u_{2,4} \\ h^2 f_{3,3} - u_{4,3} - u_{3,4} \end{pmatrix}$$

自然顺序

$$-\begin{pmatrix} A & I & O \\ I & A & I \\ O & I & A \end{pmatrix}$$

3×3 blocks

FDM for Poisson eq.

$u_{1,1}, u_{2,1}, \ldots, u_{N-1,1}, u_{1,2}, \ldots, u_{N-1,N-1},$

自然顺序

$$-\begin{pmatrix} A & I & O & \cdots & O & O \\ I & A & I & \cdots & O & O \\ O & I & A & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & I & A \end{pmatrix}$$

(N − 1) × (N − 1) blocks

Notice that the matrix has many special properties:

- it is sparse with at most 5 elements per row nonzero
- it is block tridiagonal, with tridiagonal and diagonal blocks
- it is symmetric
- it is diagonally dominant
- its diagonal elements are positive, all others nonpositive
- it is positive definite

注意：这些是针对上页的系数矩阵，左侧有个负号！

FDM for Poisson eq.

$$-\nabla^2 u = f$$

$$u(x,y)|_\Gamma = g(x,y)$$

Top BC

Bottom BC

Left BC

$$\begin{bmatrix} \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} & \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} & \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} & \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} & \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ & \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} & \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \end{bmatrix} \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \\ U_{1,2} \\ U_{2,2} \\ U_{3,2} \\ U_{1,3} \\ U_{2,3} \\ U_{3,3} \end{bmatrix} = h^2 \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{1,2} \\ f_{2,2} \\ f_{3,2} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3} \end{bmatrix} + \begin{bmatrix} U_{1,0} + U_{0,1} \\ U_{2,0} \\ U_{3,0} + U_{4,1} \\ U_{0,2} \\ 0 \\ U_{4,2} \\ U_{1,4} + U_{0,3} \\ U_{2,4} \\ U_{3,4} + U_{4,3} \end{bmatrix}$$

Right BC

FDM for Poisson eq.

Q1: 离散问题是否存在唯一解?
A1: 检查系数矩阵是否可逆.
A2: 利用离散的最大/小值原理

Q2: 误差估计?
A1: 利用离散的最大/小值原理

THEOREM (Discrete Maximum Principle). *Let v be a function on $\bar{\Omega}_h$ satisfying*

$$\Delta_h v \geq 0 \ \text{on} \ \Omega_h.$$

Then $\max_{\Omega_h} v \leq \max_{\Gamma_h} v$. *Equality holds if and only if v is constant.*

Remark 1. The analogous discrete minimum principle, obtained by reversing the inequalities and replacing max by min, holds.
Remark 2. This is a discrete analogue of the maximum principle for the Laplace operator.

FDM for Poisson eq.

反证法

PROOF. Suppose $\max_{\Omega_h} v \geq \max_{\Gamma_h} v$. Take $x_0 \in \Omega_h$ where the maximum is achieved. Let x_1, x_2, x_3, and x_4 be the nearest neighbors. Then

$$4v(x_0) = \sum_{i=1}^{4} v(x_i) - h^2 \Delta_h v(x_0) \leq \sum_{i=1}^{4} v(x_i) \leq 4v(x_0),$$

since $v(x_i) \leq v(x_0)$. Thus equality holds throughout and v achieves its maximum at all the nearest neighbors of x_0 as well. Applying the same argument to the neighbors in the interior, and then to their neighbors, etc., we conclude that v is constant. □

FDM for Poisson eq.

Theorem. There is a unique solution to the discrete BVP.

PROOF. Since we are dealing with a square linear system, it suffices to show nonsingularity, i.e., that if $\Delta_h u_h = 0$ on Ω_h and $u_h = 0$ on Γ_h, then $u_h \equiv 0$. Using the discrete maximum and the discrete minimum principles, we see that in this case u_h is everywhere 0. $\square$

Using the maximum principle & comparison function gives

Theorem. The solution u_h to discrete BVP satisfies

$$\boxed{\|u_h\|_{L^\infty(\bar{\Omega}_h)} \leq \frac{1}{8}\|f\|_{L^\infty(\Omega_h)} + \|g\|_{L^\infty(\Gamma_h)}.} \qquad (2.3)$$

This is a a statement of maximum norm stability and stability result in the sense that it states that the mapping *(f,g)* → u_h is bounded uniformly with respect to *h*.

FDM for Poisson eq.

PROOF. We introduce the comparison function $\phi(x) = [(x_1 - 1/2)^2 + (x_2 - 1/2)^2]/4$, which satisfies $\Delta_h \phi = 1$ on Ω_h, and $0 \le \phi \le 1/8$ on $\bar{\Omega}_h$. Set $M = \|f\|_{L^\infty(\Omega_h)}$. Then

$$\Delta_h(u_h + M\phi) = \Delta_h u_h + M \ge 0,$$

←比较函数的选取目的

见脚注

so

maximum principle

$$\max_{\Omega_h} u_h \le \max_{\Omega_h}(u_h + M\phi) \le \max_{\Gamma_h}(u_h + M\phi) \le \max_{\Gamma_h} g + \frac{1}{8}M.$$

Thus u_h is bounded above by the right-hand side of (2.3). A similar argument applies to $-u_h$ giving the theorem. □

https://www.math.uci.edu/~chenlong/226/FDM.pdf

$|\Delta_h u_h| = |f| \le M \Rightarrow -\Delta_h u_h \le M$

FDM for Poisson eq.

By applying the stability result to the error $u - u_h$, we can bound the error in terms of the **consistency error** $\Delta_h u_h - \Delta u$

Theorem Let u be the solution of the Dirichlet problem of Poisson eq. and u_h the solution of the discrete problem. Then

$$\boxed{\|u - u_h\|_{L^\infty(\bar{\Omega}_h)} \leq \frac{1}{8}\|\Delta u - \Delta_h u\|_{L^\infty(\bar{\Omega}_h)}.} \quad (*1)$$

PROOF. Since $\Delta_h u_h = f = \Delta u$ on Ω_h, $\Delta_h(u - u_h) = \Delta_h u - \Delta u$. Also, $u - u_h = 0$ on 看作右端函数 Γ_h. Apply last theorem 即(2.3) (with u_h replaced by $u - u_h$), we obtain the theorem. □

Theorem 3.5. *Let u be the solution of the Dirichlet problem (6) and u_h the solution of the discrete problem (14). If $u \in C^4(\Omega)$, then*

$$\|u_I - u_h\|_{\infty,\Omega_h} \leq Ch^2,$$

with constant

$$C = \frac{1}{48}\max\left\{\left\|\frac{\partial^4 u}{\partial x^4}\right\|_{\infty,\Omega}, \left\|\frac{\partial^4 u}{\partial y^4}\right\|_{\infty,\Omega}\right\}.$$

convergence results on FDS

← Lem3.4 + (*1)

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FDM for Poisson eq.

Combining (*1) with Theorem 2.1, we obtain error estimates.

COROLLARY *If $u \in C^2(\bar{\Omega})$, then*

$$\lim_{h\to 0} \|u - u_h\|_{L^\infty(\bar{\Omega}_h)} = 0.$$

If $u \in C^4(\bar{\Omega})$, then

(*1)+第四页(*0)→

$$\|u - u_h\|_{L^\infty(\bar{\Omega}_h)} \le \frac{h^2}{48} M_4,$$

where $M_4 = \max(\|\partial^4 u/\partial x_1^4\|_{L^\infty(\bar{\Omega})}, \|\partial^4 u/\partial x_2^4\|_{L^\infty(\bar{\Omega})})$.

THEOREM 2.1. *If $v \in C^4(\bar{\Omega})$, then*

$$\lim_{h\to 0} \|\Delta_h v - \Delta v\|_{L^\infty(\Omega_h)} = 0.$$