The planetary N–body problem

[*]Essay 4 on the course "Numerical Analysis".

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I. Introduction

A numerical model of the Solar System is a special case of n-body problem, where one of the masses is much larger than all the others. In the following, we refer to this problem as the planetary problem. A prototypical example of a planetary problem is the Sun-Jupiter-Saturn system, where the mass of the Sun is about 100 times larger than the masses of Jupiter or Saturn.

An approximate solution to the problem is to decompose it into $n-1$ pairs of star-planet Kepler problems, treating interactions among the planets as perturbations. Hence, the core effort to long-term modeling will be analyzing the stability under such perturbations.

II. The N-body Problem

We consider that we have $n+1$ particles (n planets and the Sun) interacting between each other due to their mutual gravitational attraction.

Assume $u_i$are the the position of the $n+1$ bodies with respect to the centre of mass, for $i = 0, 1, \cdots, n$. By gravitational attraction, we have

$$\ddot{u}_i = \sum_{0\le j\le n, j\neq i} G m_j \frac{u_j - u_i}{|u_j - u_i|^3} \tag{II.1}$$

Besides, II.1 can be seemed as the Hamiltonian equations associated to the Hamiltonian function

$$H = \sum_i \frac{|p_i|^2}{2m_i} - \sum_{i<j} \frac{G m_i m_j}{|u_i - u_j|} \tag{II.2}$$

where $p_i = m_i u_i$ is the conjugated momenta.

In an appropriate set of coordinates:

$$H = H_A(p, q) + \varepsilon H_B(q) \tag{II.3}$$

where H_A corresponds to the Keplerian motion and H_B to the Planetary interactions.

A. Jacobi Coordinates

The Jacobi corordinates were introduced in [Wisdom and Holman, 1991]. We consider the position of each planet (P_i) w.r.t. the central mass of the previous planets ($P_0, \cdots, P_{i-1}$), which means

$$u_0' \leftarrow \frac{\sum_{j=0}^n m_j u_j}{\mu_n}, \quad u_i' \leftarrow u_i - \frac{\sum_{j=0}^i m_j u_j}{\mu_i} \tag{II.4}$$

where $\mu_i = \sum_{j=0}^i m_j$.

Also, the mass is scaled. $m_0' = \mu_n$ and $m_i' = m_i \mu_{i-1}/\mu_i$. By a direct computation, we have

$$\begin{aligned} H = &\sum_{i=1}^n \left(\frac{|p_i'|^2}{2m_i'} - \frac{G m_i m_0}{|u_i'|} \right) \\ &+ \sum_{i=1}^n \left(\frac{G m_i m_0}{|u_i'|} - \frac{G m_i m_0}{|u_i - u_0|} \right) - \sum_{0<i<j} \frac{G m_i m_j}{|u_i - u_j|} \end{aligned} \tag{II.5}$$

The second sum, which we may call the indirect perturbation, contains differences of nearly equal quantities and is actually of the same order as the direct interaction terms. The Hamiltonian now separates into a sum of n nonin-

teracting Kepler Hamiltonians, and a smaller interaction Hamiltonian, as desired:

$$H = H_K + H_I \tag{II.6a}$$

$$H_K = \sum_{i=1}^{n} \left(\frac{|p_i'|^2}{2m_i'} - \frac{Gm_i m_0}{|u_i'|} \right) \tag{II.6b}$$

$$H_I = \sum_{i=1}^{n} \left(\frac{Gm_i m_0}{|u_i'|} - \frac{Gm_i m_0}{|u_i - u_0|} \right) - \sum_{0<i<j} \frac{Gm_i m_j}{|u_i - u_j|} \tag{II.6c}$$

B. Heliocentric Coordinates

Heliocentric coordinates [Kinoshita et al., 1991] considers relative position of each planet w.r.t. the Sun (P_0), i.e.

$$u_0' = u_0, \quad u_i' = u_i - u_0 \tag{II.7}$$

and

$$p_0' = \sum_{i=0}^{n} p_i, \quad p_i' = p_i \tag{II.8}$$

Similarly, the Hamiltonian is naturally split into

$$H = H_K + H_I \tag{II.9a}$$

$$H_K = \sum_{i=1}^{n} \left(\frac{1}{2}|p_i'|^2 \frac{m_0 + m_i}{m_0 m_i} - \frac{Gm_i m_0}{|u_i'|} \right) \tag{II.9b}$$

$$H_I = \sum_{0<i<j} \left(\frac{p_i' \cdot p_j'}{m_0} - G\frac{m_i m_j}{|u_i - u_j|} \right) \tag{II.9c}$$

C. Jacobi vs Heliocentric coordinates

In both cases we have $H = H_K + H_I$. But:

$$H_{\mathrm{J}} = H_A(p,q) + \varepsilon H_B(q) \tag{II.10a}$$

$$H_{\mathrm{H}} = H_A(p,q) + \varepsilon(H_B(q) + H_C(p)) \tag{II.10b}$$

Note that the size of the perturbation in Jacobi coordinates is smaller that the size of the perturbation in Heliocentric coordinates, giving a better approximation of the real dynamics. However, the expressions in Heliocentric coordinates are easier to handle, and do not require a specific order on the planets.

III. Symplectic Splitting Methods for Hamiltonian Systems

A. Lie Formalism

Let $H(q,p)$ be a Hamiltonian, where $z = (q,p)$ are a set of canonical coordinates. Then we have

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \{H, z\} := L_H z \tag{III.1}$$

where $\{F,G\} = F_q G_p - F_p G_q$ is the Poisson Bracket.

The formal solution of III.1 at time $t = \tau$ that start at time $t = \tau_0$ is given by

$$z(\tau) = \exp(\tau L_H) z(\tau_0) \tag{III.2}$$

The main idea is to build approximations for $\exp(\tau L_H)$ that preserve the symplectic character.

We focus on the special case $H = H_A + \varepsilon H_B$, where H_A and H_B are integrable on its own. Thus,

$$\exp(\tau L_H) = \exp(\tau(L_{H_A} + \varepsilon L_{H_B}))$$

Since We recall that H_A and H_B are integrable, hence we can compute $\exp(\tau(L_{H_A})$ and $\exp(\tau(L_{H_B})$ explicitly. We want to build an approximation to $\exp(\tau(A + \varepsilon B))$.

We want to construct symplectic integrators $S_n(r)$ that approximate $\exp(\tau(A + \varepsilon B))$

$$S_n(r) = \prod_{i=1}^{n} \exp(a_i \tau A) \exp(b_i \tau \varepsilon B) \tag{III.3}$$

Such integrators are very easy to implement on a computer, as they consist in the sequence of operators.

By the Baker-Campbell-Hausdorff (BHC) formula for the product of two exponential of non-commuting operators X and Y:

$$\exp X \exp Y = \exp Z$$

where

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] - [Y,[Y,X]]) + \cdots \tag{III.4}$$

Let $S_n(r) = \exp(\tau K)$, the BCH theorem ensures us that K lies in the Lie algebra generated by A and B, and it can be expanded by

$$\begin{aligned}\tau K = &\tau p_{1,0} A + \varepsilon\tau p_{1,1} B + \varepsilon\tau^2 p_{2,1}[A,B] \\ &+ \varepsilon\tau^3 p_{3,1}[A,[A,B]] + +\varepsilon^2\tau^3 p_{3,2}[B,[B,A]] \\ &+ \varepsilon\tau^4 p_{4,1}[A,[A,[A,B]]] + \varepsilon^2\tau^4 p_{4,2}[A,[B,[B,A]]] \\ &+ \varepsilon^3\tau^4 p_{4,3}[B,[B,[B,A]]] + \cdots\end{aligned} \tag{III.5}$$

where $p_{i,j}$ are polynomials of a_i and b_i. Clearly, we must have

$$p_{1,0} = p_{1,1} = 1, \quad p_{i,j} = 1, \text{ for } i = 2, \cdots, p \tag{III.6}$$

We will say that a method $S_n(\tau)$ has order p if $K = A + \varepsilon B + o(\tau^p)$. It is easy to check that

$$\begin{aligned} p_{0,1} &= a_1 + a_2 + \cdots + a_n = 1 \\ p_{1,1} &= b_1 + b_2 + \cdots + b_n = 1 \end{aligned} \tag{III.7}$$

We will say that the method $S_n(\tau)$ has n stages if it requires n evaluations of $\exp(\tau A)$ and $\exp(\tau B)$ per step-size.

In general $\varepsilon \ll \tau$, so we are more interested in killing the error terms with small powers of ε. We will

find the coefficients such that:

and has order $(s_1, s_2, \cdots)$ if it satisfies

$$|\tau K - \tau(A + \varepsilon B)| = \mathcal{O}(\sum_{i=1}^{m} \varepsilon^i \tau^{s_i+1}) \tag{III.8}$$

B. $(2n, 2)$ *methods*

A symmetric schemes can be constructed to only kill the terms of order $k = 1, 2, \cdots, 2n$. Four types of symmetric schemes, i.e. $S_n(\tau)^{-1} = S_n(-\tau)$, are constructed and analyzed in [Laskar and Robutel, 2001]. The main advantage is that

1) We only need n stages to have a method of order $(2n, 2)$.
2) We can guarantee that for all n the coefficients a_i, b_i will always be positive.

C. Killing Higher-order Terms

The above scheme can be written as

$$K = (A+\varepsilon B)+\varepsilon\tau^{2n}(\text{some term})+\varepsilon^2\tau^2 p_{3,2}[B,[B,A]]+ \tag{III.9}$$

and there are in the literature several options to kill the terms of order $\tau^2\varepsilon^2[B,[B,A]]$.

a) Composition method: [Kinoshita et al., 1991] showed that if a symplectic methods of order $2k$ w.r.t. τ, then it is possible to find a new method of order $2k + 2$ by taking

$$S^m(\tau)S(c\tau)S^m(\tau)$$

where $c = -(2m)^{1/(2k+1)}$. With this simple composition methods we can transform any of the $(2n, 2)$ methods described above to $(2n, 4)$ method.

b) The corrector term: This option was proposed by [Laskar and Robutel, 2001]. If A is quadratic in p and B depends only of q then $[B,[B,A]]$, depending only on the q,is integrable. Hence, this procedure only works in Jacobi coordinates.

Define $C = \{\{A, B\}, B\}$, an additional step

$$SC_n(\tau) = \exp(-\frac{1}{2}\tau^3\varepsilon^2 bL_c)S_n(\tau)\exp(-\frac{1}{2}\tau^3\varepsilon^2 bL_c)$$

c) Adding an extra stage: [Mclachlan, 1995] proposed to add an additional stage to reduce the $\varepsilon^2\tau^2$ term. The result of [Suzuki, 1991] tells us that it is not possible to simultaneously get rid of the two terms $\{\{A, B\}, B\}$ and $\{A, \{A, B\}\}$ with integrators having only positive values for the a_i, b_i constants.

Since $\sum a_i = \sum d_i = 1$, having only positive constants ensures that the values of the constants become smaller as the order of the integrator increases. This prevents explosion of the coefficients of the remainders which are polynomial in the a_i, b_i.

D. Compensated Summation

Let y_0 and $\{\delta_n\}_{n\geq 0}$ be given and assume that we want to compute the terms $y_{n+1} = y_n + \delta_n$.

We start with $e = 0$, and for $n = 0, 1, 2, \cdots$ do,

$$a \leftarrow y_n, e \leftarrow e+\delta_n, y_{n+1} \leftarrow a+e, e \leftarrow e+(a-y_{n+1})$$

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