Chapter 5 Appendix

泛函分析综合运用函数论, 几何学, 现代数学的观点来研究无限维向量空间上的算子和极限理论, **可看作无限维的分析学**.

泛函分析是研究拓扑线性空间到拓扑线性空间之间满足各种拓扑和代数条件的映射的分支学科.

泛函的研究对象：

- 空间—赋予某些数学结构的非空集合。

包括
- 距离空间
- 赋范空间

- 算子—两个集合间的映射。

包括
- 有界线性算子
- 泛函

•有限维线性赋范空间—线性代数研究对象

•无限维线性赋范空间—泛函分析研究对象

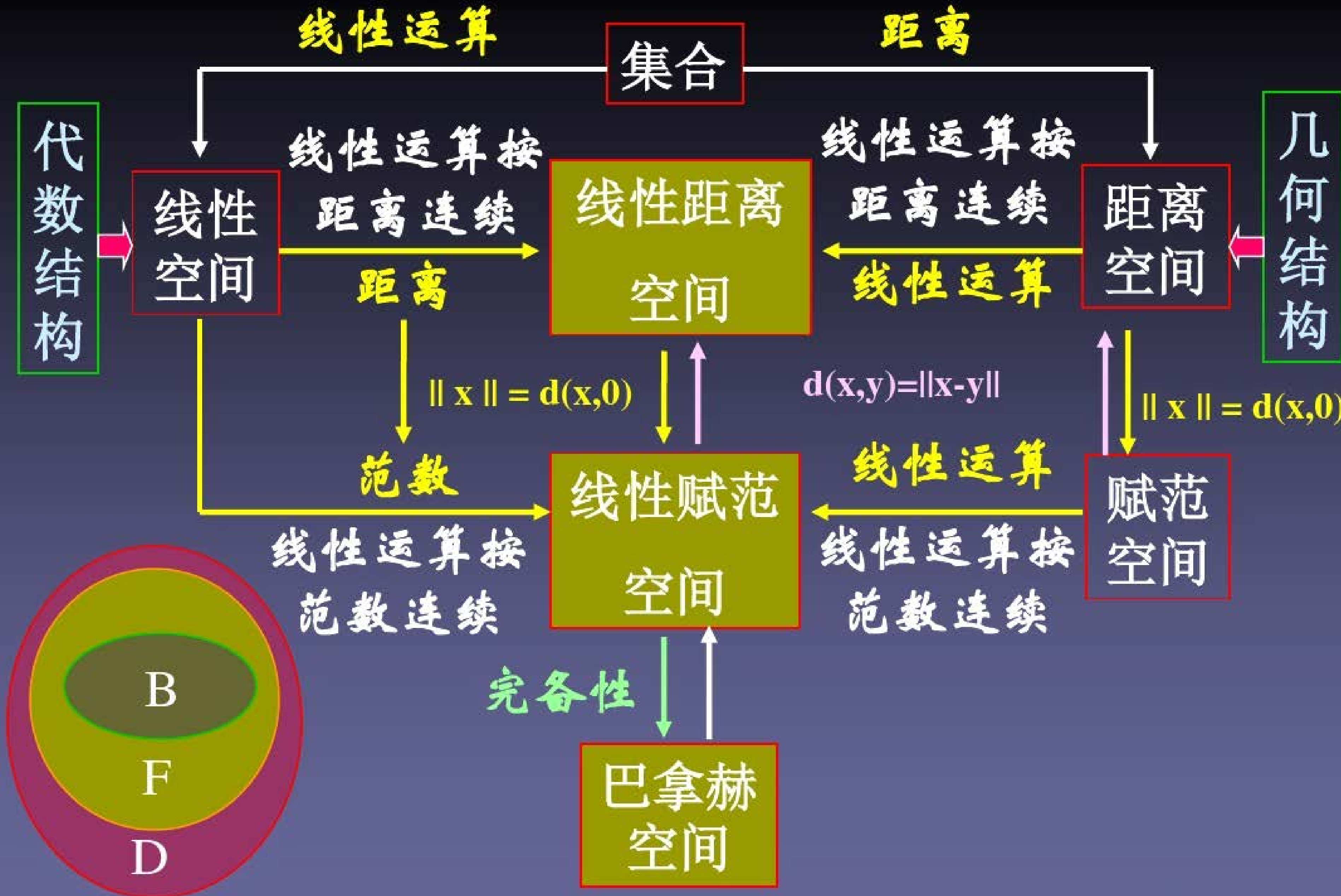
线性空间+范数⇒线性赋范空间

线性赋范空间+完备性⇒巴拿赫空间

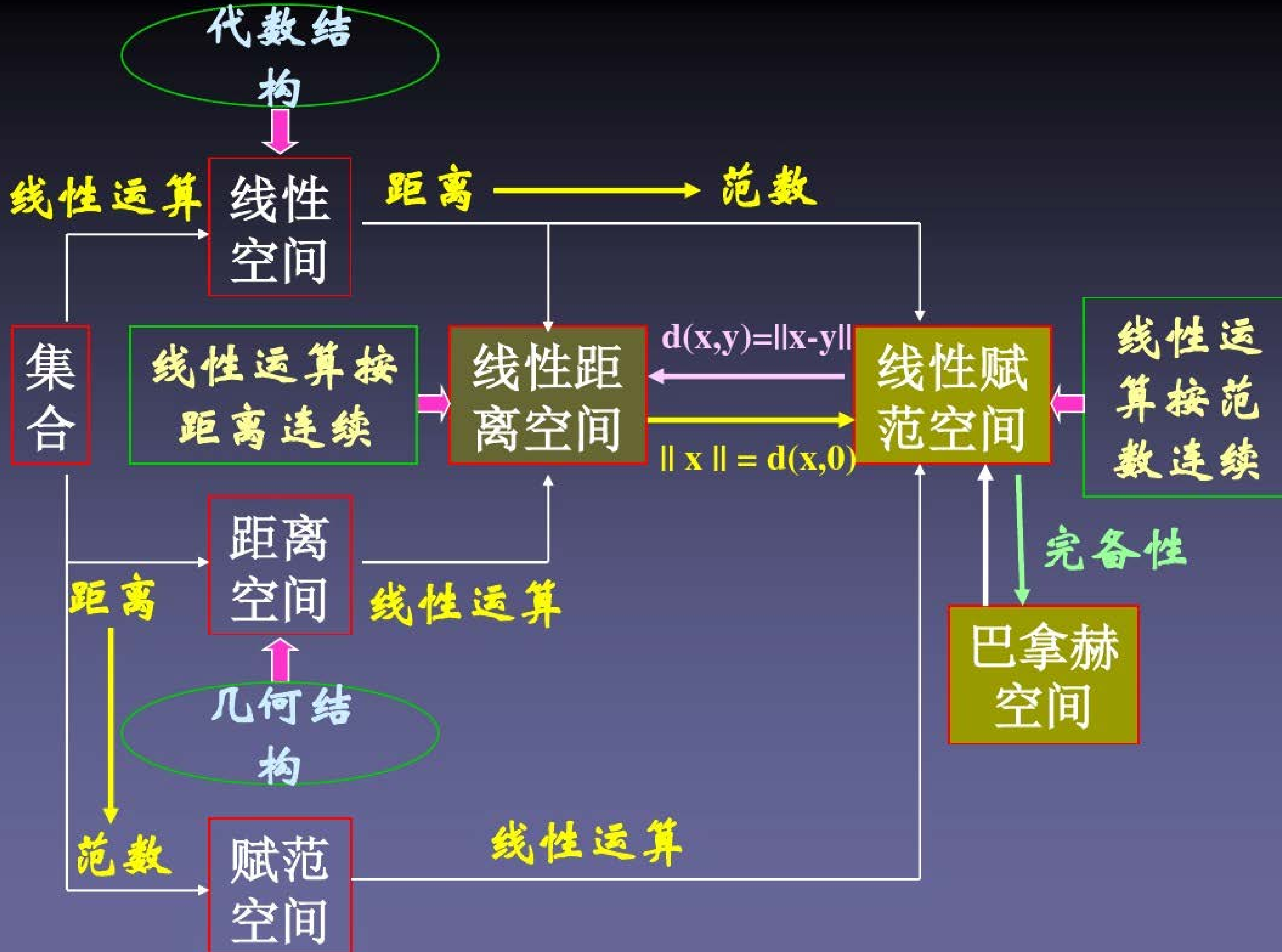
线性空间+内积⇒内积空间

内积空间+完备性⇒希尔伯特空间

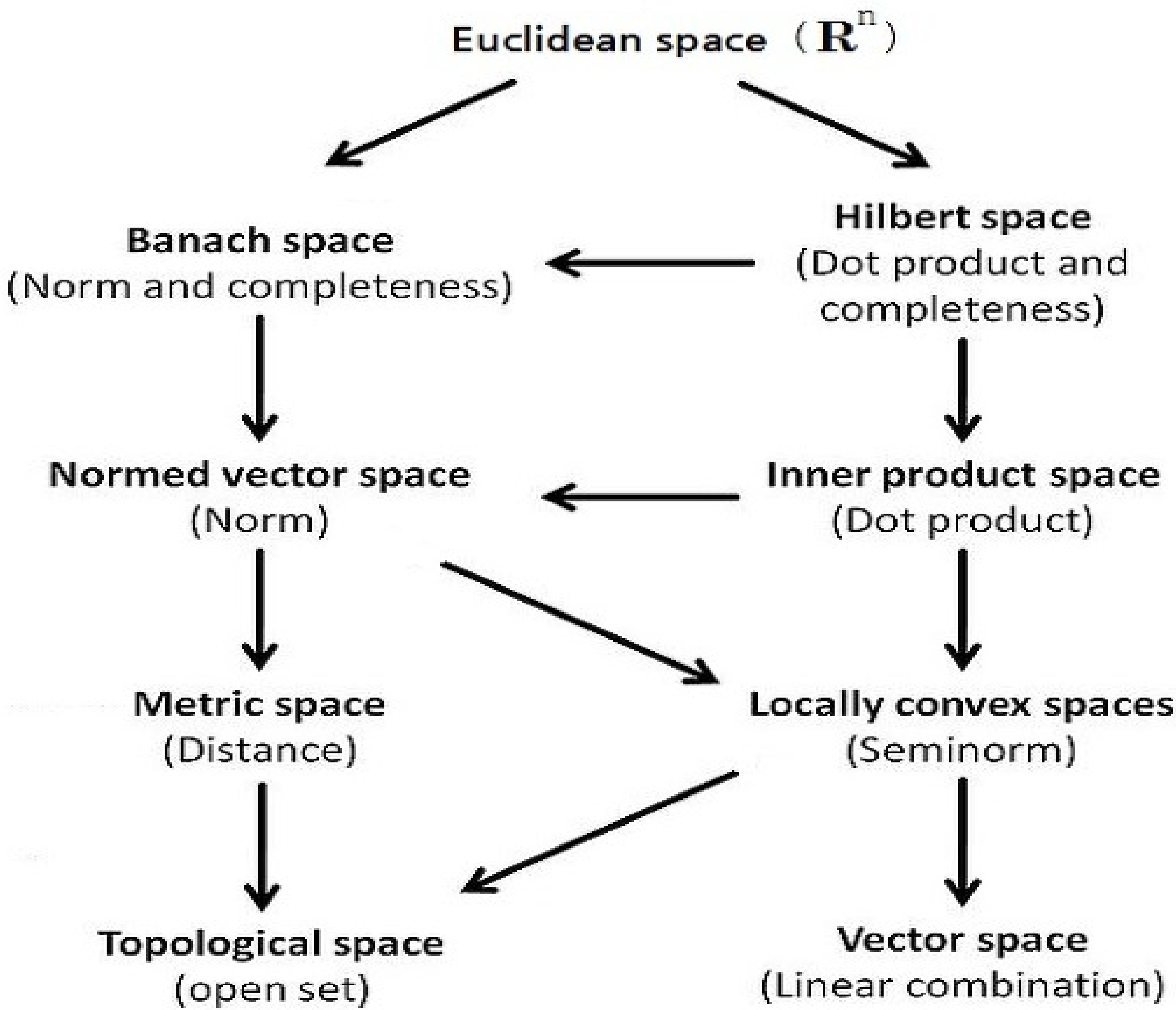
泛函分析正是把空间的代数结构和几何结构进行结合的研究 才得到了许多有实用价值的结果.

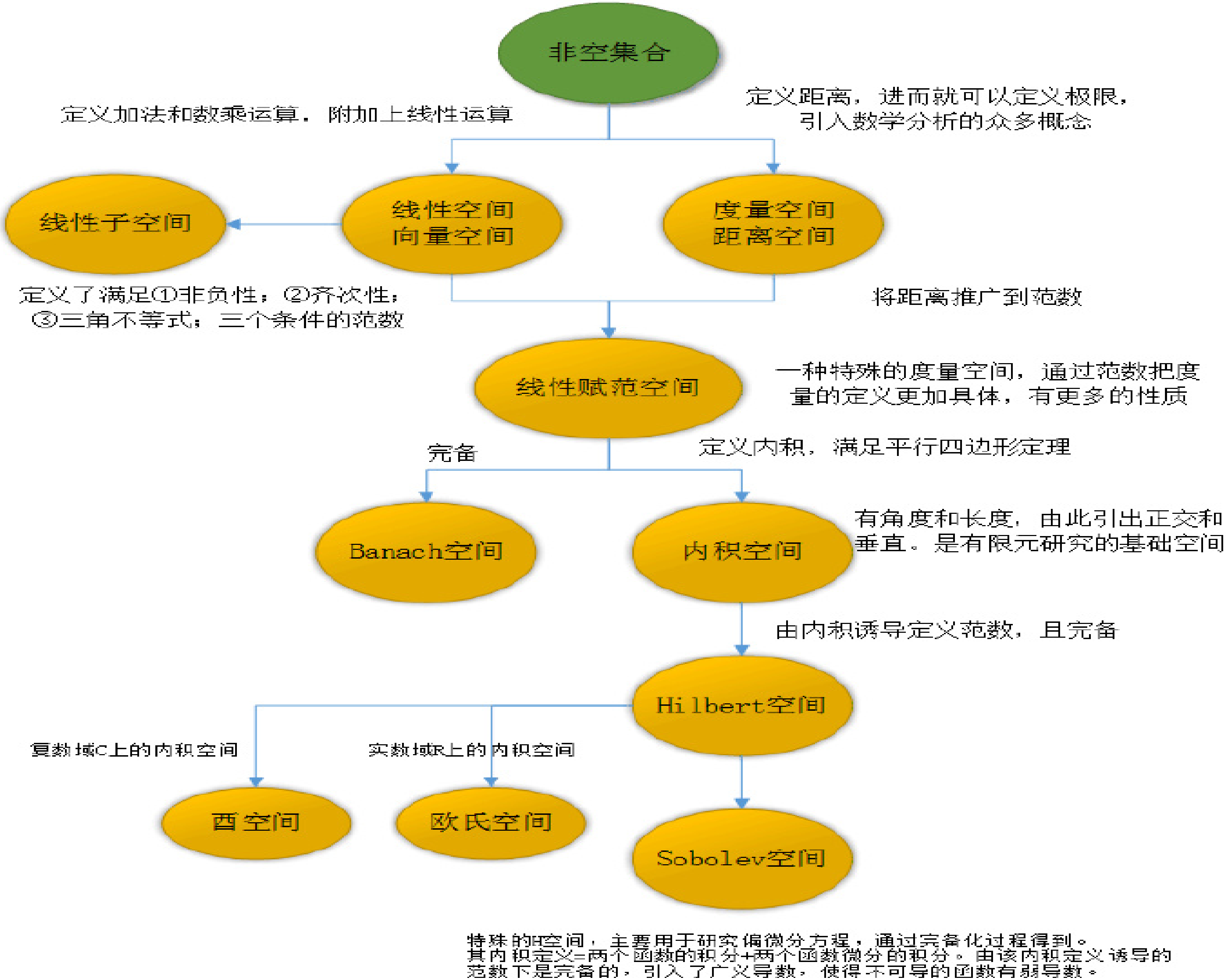


最常用距离空间R^n, m, $C[a,b]$, l^p, $L^p[a,b]$ 又都是线性空间



代数结构
线性运算
线性空间
距离
范数
集合
线性运算按距离连续
线性距离空间
d(x,y)=||x-y||
|| x || = d(x,0)
线性赋范空间
线性运算按范数连续
完备性
巴拿赫空间
距离
距离空间
线性运算
几何结构
范数
赋范空间
线性运算


Euclidean space ($\mathbf{R}^n$)
Banach space
(Norm and completeness)
Hilbert space
(Dot product and completeness)
Normed vector space
(Norm)
Inner product space
(Dot product)
Metric space
(Distance)
Locally convex spaces
(Seminorm)
Topological space
(open set)
Vector space
(Linear combination)


非空集合
定义加法和数乘运算，附加上线性运算
定义距离，进而就可以定义极限，
引入数学分析的众多概念
线性子空间
线性空间
向量空间
度量空间
距离空间
定义了满足①非负性；②齐次性；
③三角不等式；三个条件的范数
将距离推广到范数
线性赋范空间
一种特殊的度量空间，通过范数把度
量的定义更加具体，有更多的性质
完备
定义内积，满足平行四边形定理
Banach空间
内积空间
有角度和长度，由此引出正交和
垂直。是有限元研究的基础空间
由内积诱导定义范数，且完备
Hilbert空间
复数域C上的内积空间
实数域R上的内积空间
酉空间
欧氏空间
Sobolev空间
特殊的H空间，主要用于研究偏微分方程，通过完备化过程得到。
其内积定义=两个函数的积分+两个函数微分的积分。由该内积定义诱导的
范数下是完备的，引入了广义导数，使得不可导的函数有弱导数。

Spaces

Distance:

1. $d(x,y)=0$ if and only if $x=y$,
2. $d(x,y) = d(y,x)$,
3. $d(x,y) \leq d(x,z) + d(z,y)$.

metric space

Norm

1. $||x||=0$ if and only if $x=0$.
2. $||a\,x|| = |a|\,||x||$, for all scalars a.
3. $||x+y|| \leq ||x|| + ||y||$

normed linear space

Banach space

Inner product

1. $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$.
2. $\langle \alpha v, w\rangle = \alpha \langle v, w\rangle$.
3. $\langle v, w\rangle = \langle w, v\rangle$.
4. $\langle v, v\rangle \geq 0$ and equal if and only if $v = 0$.

Euclidean space

Hilbert space

https://www.math.colostate.edu/~pauld/M645/BasicHS.pdf

(2.1.1) Definition. *A* **bilinear form,** $b(\cdot,\cdot)$, *on a linear space* V *is a mapping* $b : V \times V \longrightarrow \mathbb{R}$ *such that each of the maps* $v \mapsto b(v,w)$ *and* $w \mapsto b(v,w)$ *is a linear form on* V*. It is* **symmetric** *if* $b(v,w) = b(w,v)$ *for all* $v,w \in V$*. A (real)* **inner product**, *denoted by* $(\cdot,\cdot)$, *is a symmetric bilinear form on a linear space* V *that satisfies*

(a) $(v,v) \geq 0 \; \forall v \in V$ *and*

(b) $(v,v) = 0 \Longleftrightarrow v = 0.$

S.C. Brenner & L.R. Scott, The Mathematical Theory of Finite Element Methods, 3rd Edition, Springer, 2008.

(2.1.2) Definition. *A linear space* V *together with an inner product defined on it is called an* **inner-product space** *and is denoted by* $(V,(\cdot,\cdot))$.

(2.1.3) Examples. The following are examples of inner-product spaces.

(i) $V = \mathbb{R}^n, (x,y) := \sum_{i=1}^{n} x_i y_i$

(ii) $V = L^2(\Omega), \Omega \subseteq \mathbb{R}^n, (u,v)_{L^2(\Omega)} := \int_\Omega u(x)v(x)dx$

(iii) $V = W_2^k(\Omega), \Omega \subseteq \mathbb{R}^n, (u,v)_k := \sum_{|\alpha|\leq k}(D^\alpha u, D^\alpha v)_{L^2(\Omega)}$

Notation. The inner-product space (iii) is often denoted by $H^k(\Omega)$. Thus, $H^k(\Omega) = W_2^k(\Omega)$.

Hilbert space

- The term Euclidean space refers to a finite dimensional linear space with an inner product.
- A Euclidean space is always complete by virtue of the fact that it is finite dimensional (and we are taking the scalars here to be the real numbers which have been constructed to be complete).
- An infinite dimensional inner product space which is complete for the norm induced by the inner product is called a **Hilbert space**.
- A Hilbert space is in many ways like a Euclidean space (which is why finite dimensional intuition often works in the infinite dimensional Hilbert space setting) but there are some ways in which the infinite dimensionality leads to subtle differences we need to be aware of.

https://www.math.colostate.edu/~pauld/M645/BasicHS.pdf

Hilbert space

- The subspace M is said to be **closed** if it contains all its limit points; i.e., every sequence of elements of M that is Cauchy for the H-norm, converges to an element of M.
- In a Euclidean space every subspace is closed but in a Hilbert space this is not the case.
- Every finite dimensional subspace of a Hilbert space H is closed.

Distance:

1. d(x,y)=0 if and only if x=y,
2. d(x,y) = d(y,x),
3. d(x,y) $\leq$ d(x,z) + d(z,y).

Norm

1. ||x||=0 if and only if x=0.
2. ||a x|| =|a| ||x||, for all scalars a.
3. ||x+y|| $\leq$ ||x|| + ||y||

inner product satisfies

1. $\langle u+v,\ w\rangle = \langle u,\ w\rangle + \langle v,\ w\rangle$.
2. $\langle \alpha\, v,\ w\rangle = \alpha\, \langle v,\ w\rangle$.
3. $\langle v,\ w\rangle = \langle w,\ v\rangle$.
4. $\langle v,\ v\rangle \geq 0$ and equal if and only if $v = 0$.

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Linear Functional

- A real valued function defined on H, is said to be a ***functional*** on H. The functional, L, is said to be:

(a) **Linear** if, for all x and y in H, $L(C_1x + C_2y) = C_1Lx + C_2Ly$, for all scalars C_1, C_2.

(b) **Bounded** if there exists a constant C such that $|Lx| \leq C\|x\|_H$ for all x in H

(c) **Continuous** if $\|x_n - x\|_H \to 0$ implies that $|Lx_n - Lx| \to 0$

It is not difficult to show that the only example of a linear functional on a Euclidean space E is *Lx :=(x, z)_E* for some z in E, fixed.

Riesz Representation Theorem For every continuous linear functional f on Hilbert space H there exists a unique element z_f in H such that $f(x) = (x, z_f)_H$ for all x in H.

$$\|z_f\|_H = \|f\|_{H^*}$$

Riesz定理建立了Hilbert空间H和它的对偶空间H*之间的一个等距同构(isometric isomorphism), 即存在一个保持距离的一一映射.

https://www.math.colostate.edu/~pauld/M645/BasicHS.pdf

Bilinear Forms

- A real valued function *a(x, y)* defined on $H \times H$ is said to be:

(a) **Bilinear** if, for all $x_1, x_2, y_1, y_2 \in H$ and all scalars C_1, C_2

$$a(C_1 x_1 + C_2 x_2, y_1) = C_1 a(x_1, y_1) + C_2 a(x_2, y_1)$$

$$a(x_1, C_1 y_1 + C_2 y_2) = C_1 a(x_1, y_1) + C_2 a(x_2, y_1)$$

(b) **Bounded** if there exists a constant $b > 0$ such that,

$$|a(x,y)| \leq b\|x\|_H \|y\|_H \quad \text{for all x,y in H}$$

(c) **Continuous** if $x_n \to x$, and $y_n \to y$ in H, implies $a(x_n, y_n) \to a(x,y)$ in R

(d) **Symmetric** if $a(x,y) = a(y,x)$ for all x,y in H

(e) **Positive or coercive** if there exists a $C > 0$ such that

$$a(x,x) \geq C\|x\|_H^2 \quad \text{for all x in H}$$

https://www.math.colostate.edu/~pauld/M645/BasicHS.pdf

Bilinear Forms

- It is not hard to show that for both linear functionals and bilinear forms, boundedness is equivalent to continuity.
- If a(x,y) is a bilinear form on $H \times H$, and F(x) is a linear functional on H, then $\Phi(x)=a(x, x)/2+F(x)+Const$ is called a **quadratic functional on H**.
- In a Euclidean space a quadratic functional has a unique extreme point located at the point where the gradient of the functional vanishes. This result generalizes to the infinite dimensional situation.

Lemma 3 Suppose $a(x,y)$ is a positive, bounded and symmetric bilinear form on Hilbert space H, and F(x) is a bounded linear functional on H. Consider the following problems

(a) minimize $\Phi(x) = a(x,x)/2 - F(x) + Const$ over H

(b) find x in H satisfying $a(x,y) = F(y)$ for all y in H.

Then

i) x in H solves (a) if and only if x solves (b)

ii) there is at most on x in H solving (a) and (b)

iii) there is at least one x in H solving (a) and (b)

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Bilinear Forms

- Lemma 3 requires that the bilinear form *ax, y* be symmetric. For application to existence theorems for partial differential equations, this is an unacceptable restriction. Fortunately, the most important part of the result remains true even when the form is not symmetric.

Lax-Milgram Lemma- Suppose a(u,v) is a positive and bounded bilinear form on Hilbert space H; i.e.,

$$|a(u,v)| \leq \alpha \|u\|_H \|v\|_H \quad \forall u,v \in H$$

and

$$a(u,u) \geq \beta \|u\|_H^2 \quad \forall u \in H.$$

Suppose also that $F(v)$ is a bounded linear functional on H. Then there exists a unique U in H such that

$$a(U,v) = F(v) \quad \forall v \in H.$$

https://www.math.colostate.edu/~pauld/M645/BasicHS.pdf

Proof of Lax-Milgram Lemma

Proof- For each fixed u in H, the mapping $v \rightsquigarrow a(u,v)$ is a bounded linear functional on H. It follows that there exists a unique $z_u \in H$ such that

$$a(u,v) = (z_u, v)_H \quad \forall v \in H.$$

Let $Au = z_u$; i.e., $a(u,v) = (Au, v)_H \ \forall u \in H$. Clearly A is a linear mapping of H into H, and since

$$\|Au\|_H^2 = |(Au, Au)_H| = |a(u, Au)| \le \alpha \|u\|_H \|Au\|_H$$

it is evident that A is also bounded. Note further, that

$$\beta \|u\|_H^2 \le a(u,u) = (Au, u)_H \le \|Au\|_H \|u\|_H$$

i.e., $\beta \|u\|_H \le \|Au\|_H \quad \forall u \in H.$

This estimate implies that A is one-to one and that R_A, the range of A, is closed in H. Finally, we will show that $R_A = H$. Since the range is closed, we can use the projection theorem to write, $H = R_A \oplus R_A^\perp$. If $u \in R_A^\perp$, then

$$0 = (Au, u)_H = a(u,u) \ge \beta \|u\|_H^2; \quad \text{i.e.,} \quad R_A^\perp = \{0\}.$$

Since $F(v)$ is a bounded linear functional on H, it follows from the Riesz theorem that there is a unique $z_F \in H$ such that $F(v) = (z_F, v)_H$ for all v in H. Then the equation $a(u,v) = F(v)$ can be expressed as

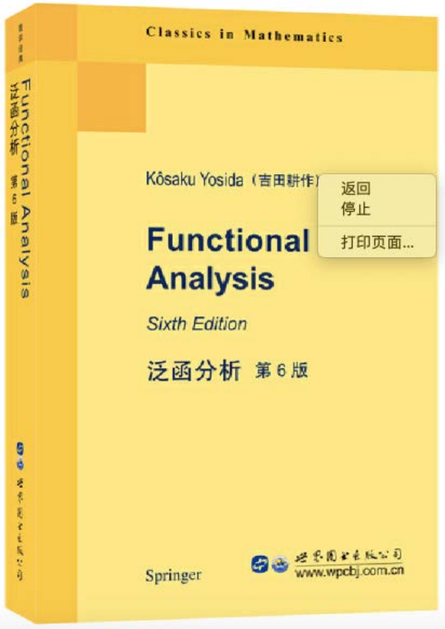
$$(Au, v)_H = (z_F, v)_H \quad \forall v \in H; \quad \text{i.e.,} \quad Au = z_F.$$

But A has been seen to be one-to-one and onto and it follows that there exists a unique $U \in H$ such that $AU = z_F$. ■

https://www.math.colostate.edu/~pauld/M645/BasicHS.pdf

压缩映射原理

压缩映射原理: X为完备的度量空间, T为X到X自身的压缩映像(即存在$0<\alpha<1$, s.t. $d(Tx,Ty)\le \alpha d(x,y)$, $\forall x, y \in X$), 则T在X中有唯一的不动点, 即X中存在唯一x, s.t. Tx=x; 且在X中任取x_0, 作$x_{m+1} = Tx_m$的迭代序列必收敛.



Banach space

Banach space is a normed vector space in which every Cauchy sequence has a limit.

Let X, Y be Banach spaces (normed vector spaces in which every Cauchy sequence has a limit).

Let A be linear $X \to Y$. Then A is continuous iff there exists $c > 0$ such that

$$\|Ax\|_Y \leq c\|x\|_X, \quad \text{for all } x \in X. \tag{1.1}$$

The space $L(X, Y)$ of linear mappings $X \to Y$, is a Banach space when endowed with the norm

$$\|A\|_{L(X,Y)} := \sup\{\|Ax\|_Y; \ \ \|x\|_X \leq 1\}. \tag{1.2}$$

Unique extension of linear mappings Let X, Y be Banach spaces, and E be a dense subspace of X.

Let $A : E \to Y$ be linear and such that for some $c > 0$:

$$\|Ae\|_Y \leq c\|e\|_X, \quad \text{for all } e \in E. \tag{1.3}$$

Then A has a unique extension to $L(X, Y)$, i.e. there exists a unique $\mathcal{A} \in L(X, Y)$, such that $\mathcal{A}e = Ae$, for all $e \in E$, and for $x \in X$ (the limit below exists):

$$\mathcal{A}x := \lim_k \{Ae_k, \quad e_k \in E, \ e_k \to x\}. \tag{1.4}$$

http://www.cmap.polytechnique.fr/~bonnans/notes/co-edp/ch1.pdf

Banach space

Topological dual Let X be a Banach space. A **linear form** over X is a linear mapping $X \to \mathbb{R}$.

We call **topological dual** (or, in short, dual) and denote by X^*, the set of **continuous** linear forms over X.

The action (duality product) of $x^* \in X^*$ over $x \in X$ is denoted by $\langle x^*, x\rangle_X$. The dual X^* is a Banach space, endowed with the **dual norm**

$$\|x^*\|_{X^*} := \sup\{|\langle x^*, x\rangle_X|; \quad \|x\|_X \le 1\}. \tag{1.7}$$

Bidual; reflexive spaces The bidual (dual of the dual) of X is denoted by X^{**}.

With $x \in X$ associate the linear form over X^*:

$$\ell_x(x^*) := \langle x^*, x\rangle_X. \tag{1.8}$$

The mapping $x \to X^{**}$, $x \mapsto \ell_x$ is **isometric**: $\|\ell_x\|_{X^{**}} = \|x\|_X$.
So, we can identify X with the image of ℓ, which is a closed subspace of X^{**}.

We say that X is **reflexive** if ℓ is onto: in that case we can identify X and X^{**}, i.e., X is the dual of X^*.

Examples of Banach spaces

- **Hilbert space** X: scalar product denoted by $(\cdot,\cdot)_X$.
  Associated norm $\|x\| := (x,x)_X^{1/2}$. Such spaces are reflexive.
  Riesz theorem: if $x^* \in X^*$, there exists $y \in X$ such that
  $$\langle x^*, x\rangle_X = (y, x)_X, \text{ for all } x \in X.$$
- Ω measurable subset of $\mathbb{R}^n$, $L^p(\Omega)$ reflexive if $p \in (1,\infty)$ and

$$L^p(\Omega)^* = L^q(\Omega), \quad 1/p + 1/q = 1, \quad p \in [1,\infty), \tag{1.12}$$

- Ω as above: $L^1(\Omega)$ and $L^\infty(\Omega)$ not reflexive.

http://www.cmap.polytechnique.fr/~bonnans/notes/co-edp/ch1.pdf

Compactness

- The key that a local property (such as continuity) can be a global (uniform) property on a set K depends on a particular property of K. It is called the compactness.
- Given a set K in (X, d), a collection of open sets $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is called an open cover of K if all U_α are open and

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha \supset K.$$

Definition

A set K in a metric space (X, d) is said to be compact if any open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of K has a finite sub-cover.

http://ocw.nctu.edu.tw/course/ana021/AAchap6.pdf

Compactness

Theorem (Heine-Borel)

In $\mathbb{R}^n$, a set K is compact if and only if it is closed and bounded.

I shall not prove this theorem here. You can find the proof in many textbooks. For instance, [Rudin's Principle of Mathematical Analysis, pp. 40], Instead, I shall prove the following theorem which motivates the definition of Heine-Borel property..

Theorem

Any continuous function on a compact set K is uniformly continuous on K.

Rem: The set [0,\infty) is not compact. Thus, a continuous function f : [0,\infty)→R is in general not uniformly continuous.

http://ocw.nctu.edu.tw/course/ana021/AAchap6.pdf

Sequential compactness

Sequential compactness

The second notion is the sequential compactness, which is arisen from finding extremal point of a continuous function. One often faces to check whether a sequence of approximate solutions converges or not. This give the motivation of the following definition.

Definition

A set K is sequentially compact if every sequence in K has a convergent subsequence with limit in K. It is called pre-compact if its closure is sequentially compact.

http://ocw.nctu.edu.tw/course/ana021/AAchap6.pdf

Sequential compactness

Theorem (Bolzano-Weistrass)

A set in $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.

Theorem

In metric space, a subset K is compact if and only if it is sequentially compact.

Proof: see Hunter's book, pp. 24–27.

Remarks

- In the theory of point set topology, the compactness implies the sequential compactness, but not vice versa. The sequential compactness is equivalent to so-called countable compactness. Its definition is: any countable open cover has finite cover.

http://ocw.nctu.edu.tw/course/ana021/AAchap6.pdf

compact operator

In functional analysis, a **compact operator** is a linear operator *L* from a Banach space *X* to another Banach space *Y*, s.t. the image under *L* of any bounded subset of *X* is a relatively compact subset (has compact closure) of *Y*. Such an operator is necessarily a bounded operator, and so continuous.

Equivalent formulations

Let X and Y be normed spaces and $T : X \to Y$ a linear operator. Then the following statements are equivalent:

- the image of the unit ball of X under T is relatively compact in Y;
- the image of any bounded subset of X under T is relatively compact in Y;
- there exists a neighbourhood U of 0 in X and a compact subset $V \subseteq Y$ such that $T(U) \subseteq V$;
- for any bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, the sequence $(Tx_n)_{n\in\mathbb{N}}$ contains a converging subsequence.

If in addition Y is Banach, these statements are also equivalent to:

- the image of any bounded subset of X under T is totally bounded in Y.

If a linear operator is compact, then it is easy to see that it is bounded, and hence continuous.

距离空间,线性空间,赋范线性空间,Banach空间,内积空间,Hilbert空间的内在关系

如果在实数域或复数域上距离空间是完备的，该空间被称为完备距离空间. 实数域或复数域上的完备线性赋范空间被称为巴拿赫空间. 内积空间是特殊的线性赋范空间，而完备的内积空间被称为希尔伯特空间, 其上的范数由一个内积导出. 在线性空间中赋以“范数”，然后在范数的基础上导出距离，即线性赋范空间，完备的线性赋范空间称为巴拿赫空间。范数可以看出长度，线性赋范空间相当于定义了长度的空间，所有的线性赋范空间都是距离空间. 以有限维空间来说，向量的范数相当于向量的模的长度. 但是在有限维欧式空间中还有一个很重要的概念—向量的夹角，特别是两个向量的正交. 内积空间是特殊的线性赋范空间，在这类空间中可以引入正交的概念以及投影的概念，从而在内积空间中建立起相应的几何学. 用内积导出的范数来定义距离，Banach空间就成为了希尔伯特空间。

距离空间,线性空间,赋范线性空间,Banach空间,内积空间,Hilbert空间的区别

在距离空间中通过距离的概念引入了点列的极限，但是只有距离结构、没有代数结构的空间，在应用过程中受到限制。线性赋范空间和内积空间就是距离结构与代数结构相结合的产物，较距离空间有很大的优越性。线性赋范空间就是在线性空间中，给向量赋予范数，即规定了向量的长度，而没有给出向量的夹角。在内积空间中，向量不仅有长度，两个向量之间还有夹角。特别是定义了正交的概念，有无正交性概念是赋范线性空间与内积空间的本质区别。

任何内积空间都是线性赋范空间，但线性赋范空间未必是内积空间。线性赋范空间X成为内积空间的充要条件是：范数‖.‖对于一切属于X的x，y，满足平行四边形公式或中线公式。

http://ocw.nctu.edu.tw/course/ana021/AAchap6.pdf