Dispersion Relation

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Linear advection equation

$$\frac{\partial u}{\partial t} + a\,\frac{\partial u}{\partial x} = 0 \qquad (1)$$

- Consider wavelike solution $u(x,t) = Ae^{i[kx-\omega t]}$, (1a) corresponding to initial data $u(x,0) = Ae^{i[kx]}$.
- Substituting it into (1) gives the dispersion relation for PDE (1):

$$\omega = ak \qquad (1b)$$

Thus wavelike solution of (1) becomes

$$u(x,t) = Ae^{i[kx-akt]} = Ae^{ik[x-at]}.$$

- For each real wave number k, there is a corresponding real frequency ω satisfying (1b) for PDE (1) such that (1a) is a solution of (1).
- Now it is obvious that the wavelike solution (1a) propagates rightward with t at the speed $c(k) := \omega(k)/k$, which is called the phase speed.

- But the evolution of a wave packet containing several wave numbers will be more complicated. Let an initial distribution $u(x,0)$ located approximately at the origin have the Fourier transform $\hat{u}(k)$. Then at time $t \geq 0$, the solution (ignoring normalization factors) can be written

$$u(x,0) = \int_{-\infty}^{\infty} \hat{u}(k) e^{-i[\omega(k)t - kx]} dk = \int_{-\infty}^{\infty} \hat{u}(k) e^{-it[\omega(k) - kx/t]} dk \qquad \text{(1c)}$$

- Suppose x/t is held fixed as $t \to \infty$. This corresponds to moving our eyes rightward at a fixed speed x/t =const. After a long time, what will we see? The answer comes from observing that as t increases, the exponential in (1c) oscillates more and more rapidly with k, hence tends to cancel to 0 as $t \to \infty$. Such cancellation will evidently take place everywhere except for any k of stationary phase, at which

$$\frac{d}{dk}\left(\omega(k) - \frac{kx}{t}\right) = 0 \Leftrightarrow \frac{d}{dk}\omega(k) = \frac{x}{t}$$

- As $t \to \infty$, therefore, our eyes will see only any wave numbers that satisfy this equation. In other words, energy associated with wave number k moves asymptotically at the group speed $C(k) \coloneqq \frac{d}{dk}\omega(k)$.

consider a signal consisting of two superimposed sine waves with slightly different frequencies and wavelengths, i.e., a signal with the amplitude function

$$A(x,t) = \cos\left[(k-\Delta k)x - (\omega - \Delta\omega)t\right] + \cos\left[(k+\Delta k)x - (\omega + \Delta\omega)t\right]$$

$$\cos\left[(kx - \omega t) \pm (\Delta kx - \Delta\omega t)\right]$$

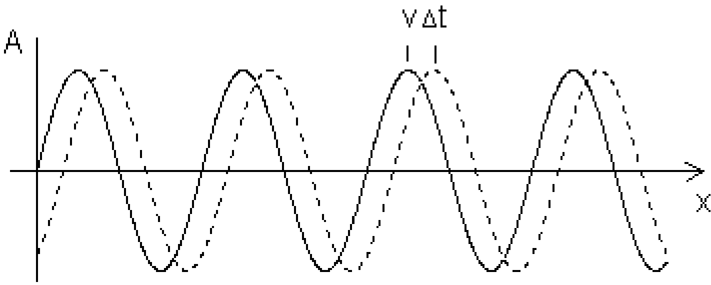
$$= \cos(kx - \omega t)\cos(\Delta kx - \Delta\omega t) \mp \sin(kx - \omega t)\sin(\Delta kx - \Delta\omega t)$$

So, adding the two terms of A(x,t) together, the products of sines cancel out, and we can express the overall signal as

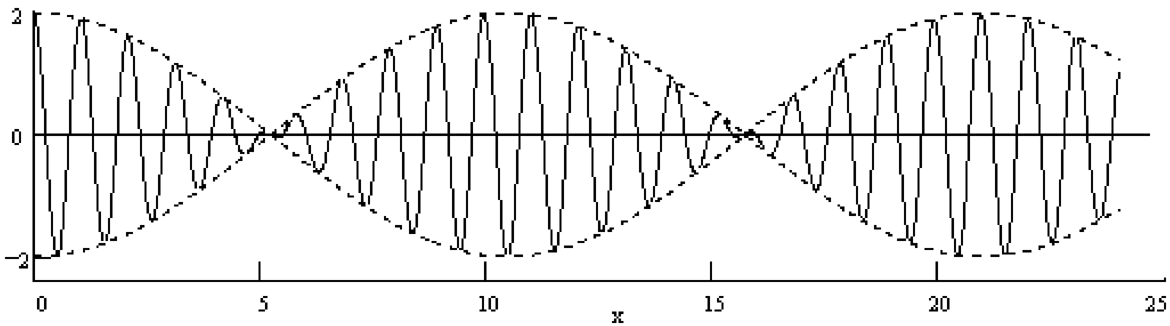
$$A(x,t) = 2\cos(kx - \omega t)\cos(\Delta kx - \Delta\omega t)$$

This can be somewhat loosely interpreted as a simple sinusoidal wave with the angular velocity ω, the wave number k, and the modulated amplitude 2cos(Δkx - Δωt). In other words, the amplitude of the wave is itself a wave, and the phase velocity of this modulation wave is v = Δω/Δk=$d\omega/dk$.

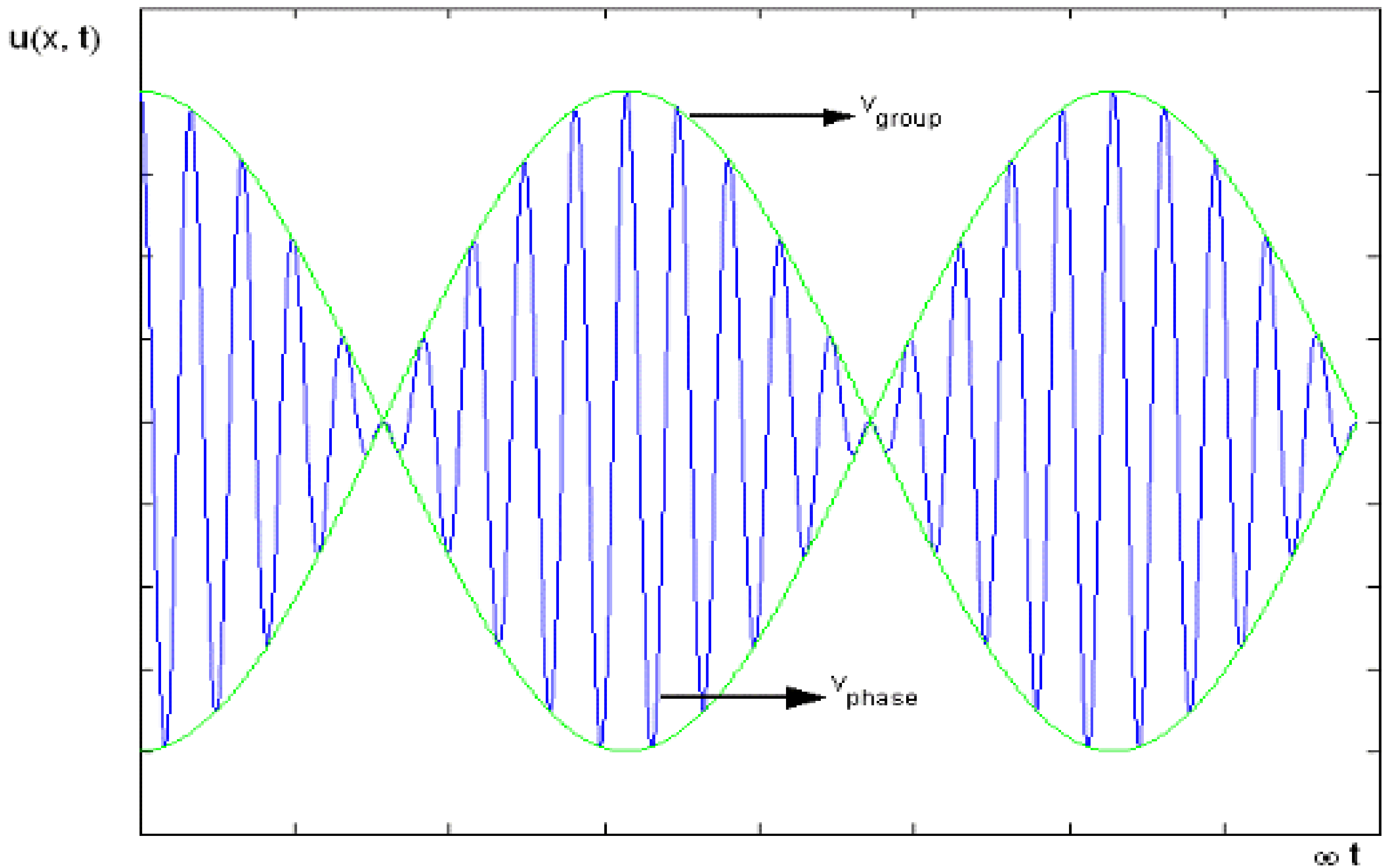
- This $\frac{d\omega}{dk}$ is the phase velocity of the amplitude envelope wave, but since each amplitude wave contains a group of internal waves, this speed is usually called the *group velocity*.



a pure traveling sinusoidal wave



The "phase velocity" of the internal oscillations is $\frac{\omega}{k}$ =1 meter/sec, whereas the amplitude envelope wave (indicated by the dotted lines) has a phase velocity of $\frac{d\omega}{dk}$ =0.33 meter/sec. As a result, if we were riding along with the envelope, we would observe the internal oscillations moving forward from one group to the next.



Phase and group velocity are related through Rayleigh's formula:

$$v_{group} = v_{phase}\left(1 - \frac{\omega}{v_{phase}}\frac{dv_{phase}}{d\omega}\right)^{-1}$$

The **group velocity** of a wave is the velocity with which the overall shape of the waves' amplitudes—known as the *modulation* or *envelope* of the wave —propagates through space.

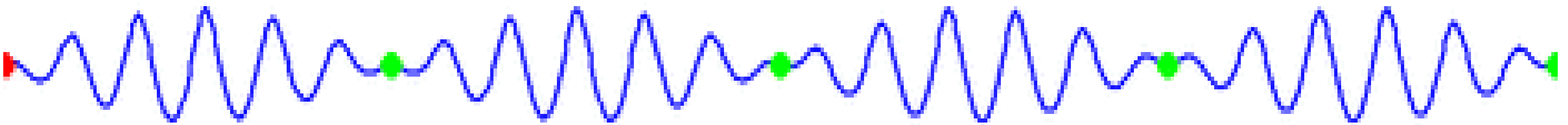


Fig.1 Frequency dispersion in groups of gravity waves on the surface of deep water. The red dot moves with the phase velocity, and the green dots propagate with the group velocity. In this deep-water case, the phase velocity is twice the group velocity. The red dot overtakes two green dots when moving from the left to the right of the figure. New waves seem to emerge at the back of a wave group, grow in amplitude until they are at the center of the group, and vanish at the wave group front. For surface gravity waves, the water particle velocities are much smaller than the phase velocity, in most cases.

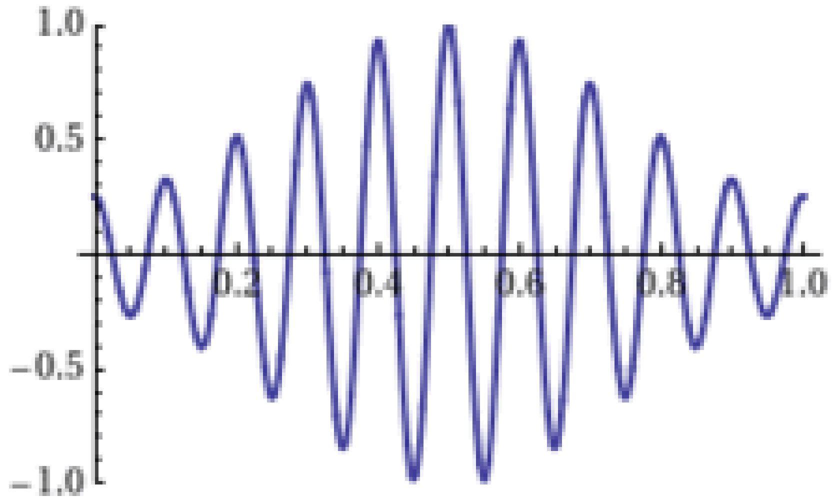
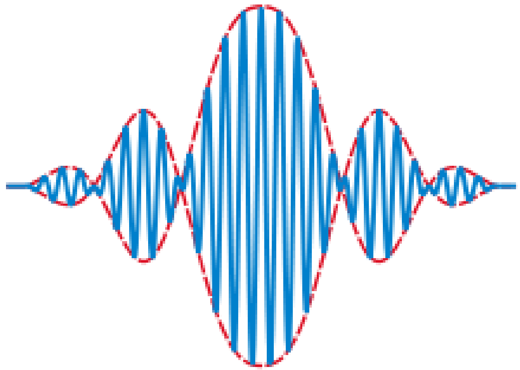


Fig.2 This shows a wave with the group velocity and phase velocity going in different directions. The group velocity is positive (i.e. the envelope of the wave moves rightward), while the phase velocity is negative (i.e. the peaks and troughs move leftward).

Fig.3 Solid line: A wave packet. Dashed line: The *envelope* of the wave packet. The envelope moves at the group velocity.



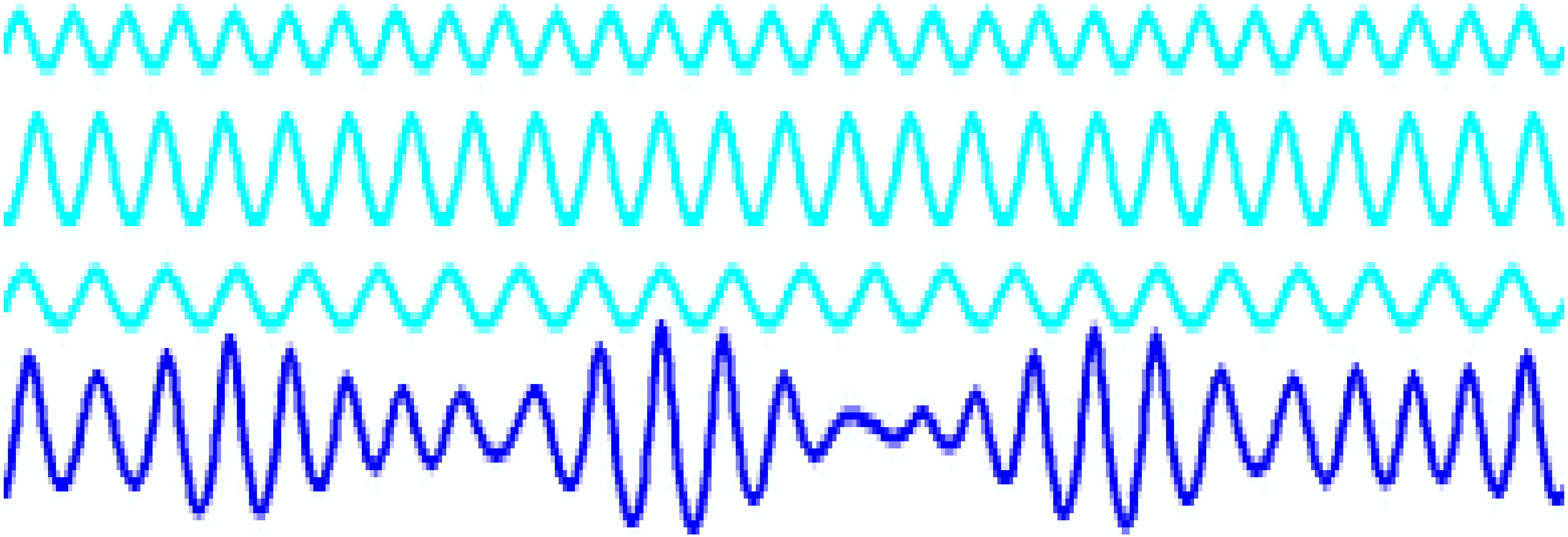


Fig.4 Distortion of wave groups by higher-order dispersion effects, for surface gravity waves on deep water (with $v_g = \frac{1}{2} v_p$). The superposition of three wave components – with respectively 22, 25 and 29 meter wavelengths, fitting in a periodic horizontal domain of 2 km length – is shown. The wave amplitudes of the components are respectively 1, 2 and 1 meter.

$$u_t = -u_x.$$

This equation is nondispersive: its dispersion relation is the linear equation $\omega(k) = k$.

$$[D_0(jh)U^n]_\nu = \frac{1}{2jh}[U^n_{\nu+j} - U^n_{\nu-j}].$$

mesh ratio $\lambda = k/h$.

Familar difference schemes that we will consider are *leap frog*,

(1.8a) LF: $U^{n+1} - U^{n-1} = -2kD_0(h)U^n$,

Crank-Nicolson (an implicit scheme),

(1.8b) CN: $U^{n+1} - U^n = -\frac{k}{2}D_0(h)(U^n + U^{n+1})$,

and *fourth-order leap frog* (fourth order in space, second order in time),

(1.8c) LF4: $U^{n+1} - U^{n-1} = -2k[\frac{4}{3}D_0(h) - \frac{1}{3}D_0(2h)]U^n$.

dispersion relations turn out to be

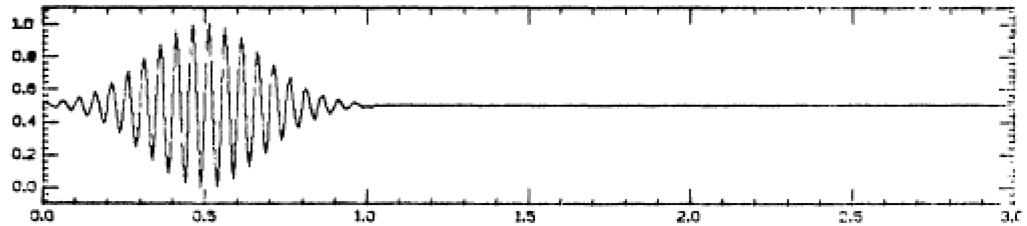
(1.9a) LF: $\sin \omega k = \lambda \sin \xi h$,

(1.9b) CN: $2 \tan \frac{\omega k}{2} = \lambda \sin \xi h$,

(1.9c) LF4: $\sin \omega k = \frac{4\lambda}{3} \sin \xi h - \frac{\lambda}{6} \sin 2\xi h$.

Initial wave packet

$$u(x,0) = e^{-16(x-1/2)^2} \sin \xi x,$$



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Upwind scheme for (1) with assuming $a > 0$

$$u_j^{n+1} = u_j^n - \nu(u_j^n - u_{j-1}^n),\ \nu = a\tau/h. \qquad (2)$$

Consider wavelike or exponential behavior

$$u_j^n = e^{i[kx_j - \omega t_n]}.$$

The amplification factor of upwind scheme

$$A(k) = u_j^{n+1}/u_j^n = e^{-i\omega\tau} = 1 - \nu(1 - e^{-ikh}), \qquad (2b)$$

which is a equation satisfied by the dispersion relation of (2).

The amplification factor of PDE (1) is from the previous solution $A_{exact}(k) = u(x_j, t_{n+1})/\, u(x_j, t_n) = e^{-ika\tau} = e^{-ik\nu h}$.

Amplitude /phase errors

- If the exact amplification factor during a stepsize h for $\varphi = e^{ikx}$ is $A_{exact}(k)$, i.e. if $L\,[\varphi] = 0, -\infty < x < \infty$, φ(x,0)= e^{ikx}, then φ(x,t)= $A_{exact}(k)e^{ikx}$,
- then if $a(k) \coloneqq A(k)/A_{exact}(k)$ is the relative amplification rate, $|a(k) - 1|$ is the amplitude error.

  $|a(k)| < 1 \Rightarrow$ damping; $|a(k)| > 1 \Rightarrow$ amplifying; $|a(k)| = 1$

$\Rightarrow$ neutral.

- and if $R \coloneqq \arg(A(k))/\arg(A_{exact}(k))$ is the relative phase change: $R < 1 \Rightarrow$ decelerating; $R > 1 \Rightarrow$ accelerating.
- For upwind scheme (2) for advection eqn.,

$$A(k) = 1 - \nu(1 - e^{-ikh}),$$

$$A_{exact}(k) = u(x_j, t_{n+1})/\,u\left(x_j, t_n\right) = e^{-ika\tau} = e^{-ik\nu h}.$$

arg(z)=argument of a complex number z

Cont'd

Upwind scheme for (1) with assuming $a > 0$

$u_j^{n+1} = u_j^n - \nu(u_j^n - u_{j-1}^n)$, $\nu = a\tau/h$. (2)

If $\omega = \omega_r + \mathrm{i}\omega_i$ is the solution of (2b), then

Im $\omega_i < 0 \Rightarrow$ damping;

Im $\omega_i > 0 \Rightarrow$ amplifying;

Im $\omega_i = 0 \Rightarrow$ neutral.

$\omega_r/\omega_{exact} > 1 \Rightarrow$ accelerating;

$\omega_r/\omega_{exact} < 1 \Rightarrow$ decelerating.

Modified Eqn

Consider 1D advection eqn

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0. \qquad (c>0) \qquad (5)$$

and its upwind scheme or Donor-Cell Approximation on a uniform mesh

$$\frac{u_i^{n+1}-u_i^{n}}{\Delta t}+c\frac{u_i^{n}-u_{i-1}^{n}}{\Delta x}=0 \qquad (6)$$

Expanding the grid function to a smooth function and using Taylor expansion gives

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=\boxed{-\frac{\Delta t}{2}u_{tt}+\frac{c\Delta x}{2}u_{xx}-\frac{(\Delta t)^2}{6}u_{ttt}-\frac{c(\Delta x)^2}{6}u_{xxx}+O(\Delta x^3+\Delta t^3)} \qquad (7)$$

truncation error

Leading term

Accuracy in the sense of truncation error;
Consistency

Modified Eqn

Dispersion Error: occurs when the leading terms in modified eqn have odd-order derivatives. They are characterized by oscillations or small wiggles in the solution, mostly in the form of moving waves.

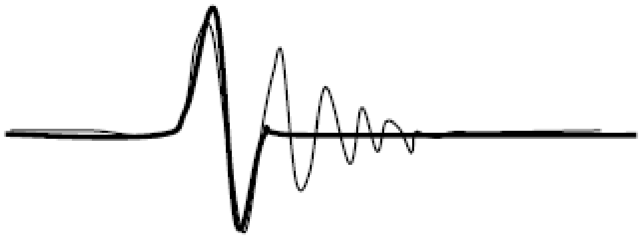

Figure: the thick line is the true solution and thin line is the numerical solution.

It's called dispersion error because waves of different wavelengths propagate at different speed (i.e., wave speed =a function of *k*) in the numerical solution due to numerical approximations – causing dispersion of waves.

For the PDE $\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0$, all Fourier components should move at the same speed, *c*.

Modified Eqn

Dissipation Error – occurs when the leading terms have even-order derivatives. They are characterized by a loss of wave amplitude. The effect is also called artificial viscosity, which is implicit in the numerical solution.

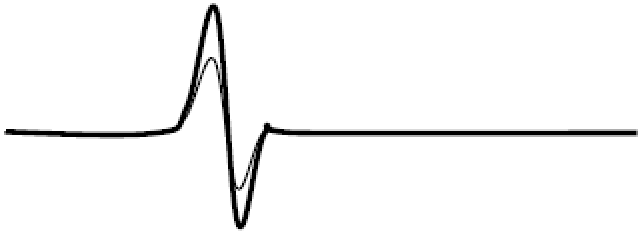

Figure: the thick line is the true solution and thin line is the numerical solution.

The combined effect of dissipation and dispersion is often called diffusion.

Modified Eqn

To replace u_{tt} in right hand side of (7), we take (Eq.(7))$_t$ **[note here we use the FDE not the PDE]** →

$$u_{tt} + cu_{xt} = -\frac{\Delta t}{2}u_{ttt} + \frac{c\Delta x}{2}u_{xxt} - \frac{(\Delta t)^2}{6}u_{tttt} - \frac{c(\Delta x)^2}{6}u_{xxxt} + \ldots \qquad (8)$$

and $-c$ (Eq(7))$_x$

$$-cu_{tx} - c^2u_{xx} = \frac{c\Delta t}{2}u_{ttx} - \frac{c^2\Delta x}{2}u_{xxx} + \frac{c(\Delta t)^2}{6}u_{tttx} + \frac{c^2(\Delta x)^2}{6}u_{xxxx} + \ldots \qquad (9)$$

then add (8) and (9) →

$$u_{tt} = c^2u_{xx} + \Delta t\left(\frac{-u_{ttt}}{2} + \frac{c}{2}u_{ttx} + O(\Delta t)\right) + \Delta x\left(\frac{c}{2}u_{xxt} - \frac{c^2}{2}u_{xxx} + O(\Delta x)\right). \qquad (10)$$

Similarly, we can obtain other time derivatives, u_{ttt} found in (7) and (10) and u_{ttx} and u_{xxt} found in (10).

$$\begin{aligned} u_{ttt} &= -c^3u_{xxx} + O(\Delta x + \Delta t), \\ u_{ttx} &= c^2u_{xxx} + O(\Delta x + \Delta t), \\ u_{xxt} &= -cu_{xxx} + O(\Delta x + \Delta t). \end{aligned} \qquad (11)$$

Combining (7), (10) and (11) →

$$u_t + cu_x = \frac{c\Delta t}{2}(1-\mu)u_{xx} - \frac{c(\Delta x)^2}{6}\left(2\mu^2 - 3\mu + 1\right)u_{xxx} + O(\Delta x^3, \Delta x^2\Delta t, \Delta t^2\Delta x, \Delta t^3) \qquad (12)$$

where $\mu = \frac{c\Delta t}{\Delta x}$.

Heat eqn

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} \qquad (3)$$

- Consider wavelike solution $u(x,t) = Ae^{i[kx-\omega t]}$, corresponding to initial data $u(x,0) = Ae^{i[kx]}$.
- Substituting it into (2) gives the dispersion relation for PDE (2):

$$\omega = -i\mu k^2$$

Thus wavelike solution of (2) becomes

$$u(x,t) = [Ae^{-i\mu k^2}]e^{ikx}.$$

Modified Wave Number Analysis

- Arbitrary periodic functions can be decomposed into their Fourier components, which are in the form e^{ikx}, where k is the wavenumber. For a general k, $u(x) = c_k e^{ikx}$, its exact derivative $u'(x) = ikc_k e^{ikx} = iku(x)$.
- How will a finite-difference operator δ_x approximate the derivative of $u_j = e^{ikx_j}$, $x_j = jh$.
- By definition: (ik^* is defined to be modified wave number) $\delta_x u_j = ik^* c_k e^{ikx} = ik^* u_j$. The particular form of ik^* depends on the choice of δ_x.
- Example: Central difference $\delta_x u_j = (u_{j+1} - u_{j-1})/2h$.

Using $u_j = e^{i\kappa j\Delta x}$ we have

$$\delta_x^c u_j = \frac{e^{i\kappa(j+1)\Delta x} - e^{i\kappa(j-1)\Delta x}}{2\Delta x} = \frac{e^{i\kappa\Delta x} - e^{-i\kappa\Delta x}}{2\Delta x} e^{i\kappa j\Delta x}$$

$$= \frac{e^{i\kappa\Delta x} - e^{-i\kappa\Delta x}}{2\Delta x} u_j = i\kappa_c^* u_j$$

Modified Wave Number Analysis

- Using the definition of the complex exponential $e^{i\kappa\Delta x} = cos(\kappa\Delta x) + isin(\kappa\Delta x)$ we have

$$i\kappa_c^* = i\frac{sin(\kappa\Delta x)}{\Delta x}$$

- Modified wave number $i\kappa^*$ is an approximation to $i\kappa$.

- For δ_x^c, using the infinite series expansion of $sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$$i\,\frac{sin(\kappa\Delta x)}{\Delta x} = \frac{i}{\Delta x}\left[(\kappa\Delta x) - \frac{(\kappa\Delta x)^3}{6} + O(\Delta x^5)\right] =$$

$$i\kappa\left[1 - \frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4)\right]$$

- Therefore $i\kappa_c^* = i\kappa - i\kappa\frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4) = i\kappa + O(\Delta x^2)$, a second order approximation.

Modified Wave Number Analysis

- Backward Difference:

$$\delta_x^b\, u_j = \frac{u_j - u_{j-1}}{\Delta x}$$

- Using $u_j = e^{i\kappa j \Delta x}$ we have

$$\begin{aligned}\delta_x^b u_j = \frac{e^{i\kappa j\Delta x} - e^{i\kappa(j-1)\Delta x}}{\Delta x} &= \frac{1 - e^{-i\kappa\Delta x}}{\Delta x} e^{i\kappa j\Delta x} = \\ \frac{1 - e^{-i\kappa\Delta x}}{\Delta x} u_j &= i\kappa_b^* u_j\end{aligned}$$

- Expanding in sin and cos

$$i\kappa_b^* = \frac{1 - cos(\kappa\Delta x) + isin(\kappa\Delta x)}{\Delta x}$$

- For δ_x^b, using the infinite series expansion of $sin(x)$ and $cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

$$\begin{aligned} i\kappa_b^* &= \frac{1}{\Delta x}\left[\frac{(\kappa\Delta x)^2}{2} + O(\Delta x^4)\right] \\ &+ \frac{1}{\Delta x} i \left[(\kappa\Delta x) - \frac{(\kappa\Delta x)^3}{6} + O(\Delta x^5)\right] \\ &= \frac{\kappa^2\Delta x}{2} + O(\Delta x^3) + i\kappa\left[1 - \frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4)\right] \end{aligned}$$

- Therefore $i\kappa_b^* = i\kappa + O(\Delta x)$ a first order approximation.

References

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