Examples for the energy method for stability of numerical methods

Huazhong Tang

School of Mathematical Sciences, Peking University, P.R. China

October 24, 2019

Abstract

This draft gives several examples on application of the energy method for the stability of numerical methods.

Keywords: TBA

1 Schemes for heat equation

Consider 1D heat equation

$$u_t = au_{xx},\ x \in \mathbb{R},\ t > 0, \tag{1.1}$$

with the initial data

$$u(x,0) = u_0(x),$$

where $a > 0$. Assuming

$$||u_0||_2 < \infty.$$

Remark: We can consider the initial-boundary value problem similarly.

1.1 Explicit scheme

Under a uniform mesh $\{t_n = n\tau, x_j = jh : n \in \mathbb{Z}^+ \cup \{0\}, j \in \mathbb{Z}\}$, the explicit scheme is

$$\frac{u_j^{n+1} - u_j^n}{\tau} = a\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2},\ j \in \mathbb{Z}, \tag{1.2}$$

or

$$u_j^{n+1} = u_j^n - \nu\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right), \ j \in \mathbb{Z}. \tag{1.3}$$

Multiplying (1.2) by $\frac{1}{2}(u_j^{n+1} + u_j^n)$ gives

$$\frac{1}{2\tau}(u_j^{n+1})^2 - \frac{1}{2\tau}(u_j^n)^2 = \frac{a}{2h^2}(u_j^{n+1} + u_j^n)\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right). \tag{1.4}$$

Substituting (1.3) into the RHS of (1.4) yields

$$\begin{aligned}\frac{1}{2}(u_j^{n+1})^2 - \frac{1}{2}(u_j^n)^2 =& \frac{\nu}{2}\left[2u_j^n + \nu(u_{j+1}^n - 2u_j^n + u_{j-1}^n)\right]\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)\\ =&\nu u_j^n\left[(u_{j+1}^n - u_j^n) - (u_j^n - u_{j-1}^n)\right] + \frac{\nu^2}{2}\left[(u_{j+1}^n - u_j^n) - (u_j^n - u_{j-1}^n)\right]^2\end{aligned} \tag{1.5}$$

Multiplying it by h and summing w.r.t. j yields

$$\begin{aligned}\sum_j \frac{h}{2}(u_j^{n+1})^2 - \sum_j \frac{h}{2}(u_j^n)^2 =& \sum_j h\nu u_j^n\left[(u_{j+1}^n - u_j^n) - (u_j^n - u_{j-1}^n)\right] + \sum_j h\frac{\nu^2}{2}\left[(u_{j+1}^n - 2u_j^n + u_{j-1}^n)\right]^2\\ =& -\sum_j h\nu(u_{j+1}^n - u_j^n)^2 + \sum_j h\frac{\nu^2}{2}\left[(u_{j+1}^n - u_j^n) - (u_j^n - u_{j-1}^n)\right]^2\\ \le& -\sum_j h\nu(u_{j+1}^n - u_j^n)^2 + \sum_j h\frac{\nu^2}{2}\left[2(u_{j+1}^n - u_j^n)^2 + 2(u_j^n - u_{j-1}^n)^2\right]\\ =& -\nu(1-2\nu)\sum_j h(u_{j+1}^n - u_j^n)^2.\end{aligned} \tag{1.6}$$

So, if $0 < 2\nu \le 1$, then

$$\sum_j \frac{h}{2}(u_j^{n+1})^2 \le \sum_j \frac{h}{2}(u_j^n)^2, \quad \text{i.e. } ||u^{n+1}||_{energy} \le ||u^n||_{energy}.$$

The explicit scheme is stable in the energy norm. It is obvious that such energy norm is equivalent to the ℓ^2 norm. Thus, the explicit scheme is ℓ^2 stable. □

1.2 Implicit scheme

The implicit scheme is

$$\frac{u_j^{n+1} - u_j^n}{\tau} = a\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2}, \ j \in \mathbb{Z}, \tag{1.7}$$

or

$$u_j^{n+1} - \nu \left[(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) \right] = u_j^n. \tag{1.8}$$

Multiplying (1.8) by $\frac{1}{2}(u_j^{n+1} + u_j^n)$ gives

$$\begin{aligned}
\frac{1}{2}(u_j^{n+1})^2 - \frac{1}{2}(u_j^n)^2 =& \frac{\nu}{2}(u_j^{n+1} + u_j^n)(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) \\
=& \frac{\nu}{2}[2u_j^{n+1} - \nu(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})](u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) \\
=& \nu u_j^{n+1}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) - \frac{\nu^2}{2}\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right)^2 \\
\leq& \nu u_j^{n+1}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}).
\end{aligned}$$

Summing it w.r.t. j and shifting it gives

$$\begin{aligned}
\sum_j h\frac{1}{2}(u_j^{n+1})^2 - \sum_j h\frac{1}{2}(u_j^n)^2 \leq& \nu \sum_j h\Big[u_j^{n+1}(u_{j+1}^{n+1} - u_j^{n+1}) - u_j^{n+1}(u_j^{n+1} - u_{j-1}^{n+1})\Big] \\
=& -\nu \sum_j h(u_{j+1}^{n+1} - u_j^{n+1})^2 \leq 0,
\end{aligned}$$

due to $\nu > 0$.

1.3 Crank-Nicolson scheme

The CN scheme is

$$\frac{u_j^{n+1} - u_j^n}{\tau} = a\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2h^2} + a\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{2h^2}, \ j \in \mathbb{Z}, \tag{1.9}$$

or

$$u_j^{n+1} - u_j^n = \frac{\nu}{2}\left[(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + (u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) \right]. \tag{1.10}$$

Multiplying (2.6) by $\frac{1}{2}(u_j^{n+1} + u_j^n) =: u_j^*$ gives

$$\frac{1}{2}(u_j^{n+1})^2 - \frac{1}{2}(u_j^n)^2 = \nu u_j^*[(u_{j+1}^* - u_j^*) - (u_j^* - u_{j-1}^*)].$$

Noting $\nu > 0$ and summing it w.r.t. j yields

$$\begin{aligned}
\sum_j h\frac{1}{2}(u_j^{n+1})^2 - \sum_j h\frac{1}{2}(u_j^n)^2 =& \sum_j h\nu u_j^*[(u_{j+1}^* - u_j^*) - (u_j^* - u_{j-1}^*)] \\
=& -\nu \sum_j h(u_{j+1}^* - u_j^*)^2 \leq 0.
\end{aligned}$$

□

1.4 Du fort-Frankel scheme

$$\frac{u_j^{n+1}-u_j^{n-1}}{2\tau}=a\frac{u_{j+1}^n-(u_j^{n+1}+u_j^{n-1})+u_{j-1}^n}{2h^2},\ j\in\mathbb{Z}. \tag{1.11}$$

The readers are referred to [1, 3].

1.5 Semi-discrete scheme

$$\frac{du_j(t)}{dt}=a\frac{u_{j+1}(t)-2u_j^n(t)+u_{j-1}(t)}{h^2},\ j\in\mathbb{Z}, \tag{1.12}$$

Multiplying (1.2) by $u_j(t)$ gives

$$\frac{d\frac{1}{2}(u_j(t))^2}{dt}=au_j(t)\frac{u_{j+1}(t)-2u_j(t)+u_{j-1}(t)}{h^2} \tag{1.13}$$

Multiplying it by h and summing w.r.t. j yields

$$\begin{aligned}\frac{d}{dt}\left(\sum_j h\frac{1}{2}\big(u_j(t)\big)^2\right)=&a\sum_j\frac{1}{h}u_j(t)(u_{j+1}(t)-u_j(t))-a\sum_j\frac{1}{h}u_j(u_j-u_{j-1})\\=&-a\sum_j\frac{1}{h}(u_{j+1}(t)-u_j(t))^2\leq 0.\end{aligned} \tag{1.14}$$

The semi-discrete scheme is stable in the energy norm. It is obvious that such energy norm is equivalent to the ℓ^2 norm. Thus, the semi-discrete scheme is ℓ^2 stable. □

2 Schemes for convection equation

Consider 1D convective/advective/transport equation

$$u_t-au_x,\ x\in\mathbb{R},\ t>0, \tag{2.1}$$

with the initial data

$$u(x,0)=u_0(x),$$

where $a>0$. Assuming

$$||u_0||_2<\infty.$$

2.1 Explicit upwind scheme

The first-order explicit upwind scheme is

$$\frac{u_j^{n+1}-u_j^n}{\tau}+a\frac{u_j^n-u_{j-1}^n}{h}=0,\ \ j\in\mathbb{Z}, \tag{2.2}$$

or

$$u_j^{n+1}=u_j^n-\nu(u_j^n-u_{j-1}^n),\ \ \nu=a\tau/h,\ \ j\in\mathbb{Z}, \tag{2.3}$$

Assuming $0\le\nu\le1$ and squaring (2.4) gives

$$\begin{aligned}(u_j^{n+1})^2=&\big[u_j^n(1-\nu)+\nu u_{j-1}^n\big]^2=(1-\nu)^2(u_j^n)^2+\nu^2(u_{j-1}^n)^2+2\nu(1-\nu)u_j^nu_{j-1}^n\\ \le&(1-\nu)^2(u_j^n)^2+\nu^2(u_{j-1}^n)^2+\nu(1-\nu)\big[(u_j^n)^2+(u_{j-1}^n)^2\big]\end{aligned} \tag{2.4}$$

Summing it w.r.t.j and shifting some terms in j as well as using $0\le\nu\le1$, gives

$$\begin{aligned}\sum_j h(u_j^{n+1})^2\le&\sum_j h\left\{(1-\nu)^2(u_j^n)^2+\nu^2(u_{j-1}^n)^2+\nu(1-\nu)\big[(u_j^n)^2+(u_{j-1}^n)^2\big]\right\}\\ =&\sum_j h(1-2\nu+\nu^2+\nu^2+\nu-\nu^2)(u_j^n)^2=\sum_j h(1-\nu+\nu^2)(u_j^n)^2\le\sum_j h(u_j^n)^2,\end{aligned}$$

i.e. the explicit 1st-order upwind scheme is ℓ^2 stable. □

We can also multiply (2.2) by $(u_j^{n+1}+u_j^n)$ and get

$$\begin{aligned}(u_j^{n+1})^2-(u_j^n)^2=&-\nu(u_j^{n+1}+u_j^n)\left(u_j^n-u_{j-1}^n\right)\\ =&-\nu\big(2u_j^n-\nu(u_j^n-u_{j-1}^n)\big)\left(u_j^n-u_{j-1}^n\right)\\ =&-2\nu u_j^n(u_j^n-u_{j-1}^n)+\nu^2(u_j^n-u_{j-1}^n)^2\\ \le&-\nu(u_j^n)^2-\nu(u_j^n-u_{j-1}^n)^2+\nu^2(u_j^n-u_{j-1}^n)^2\le0,\end{aligned}$$

if $0\le\nu\le1$. Then summing it w.r.t.j gives the ℓ^2 stability of the scheme. □

2.2 Crank-Nicolson scheme

The CN scheme is

$$\frac{u_j^{n+1}-u_j^n}{\tau}=a\frac{u_{j+1}^n-u_{j-1}^n}{4h}+a\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{4h},\ \ j\in\mathbb{Z}, \tag{2.5}$$

or

$$u_j^{n+1}-u_j^n=\frac{\nu}{4}\left[(u_{j+1}^n-u_{j-1}^n)+(u_{j+1}^{n+1}-u_{j-1}^{n+1})\right],\ \ \nu=a\tau/h>0, \tag{2.6}$$

Multiplying (2.6) by $\frac{1}{2}(u_j^{n+1}+u_j^n) =: u_j^*$ gives

$$\frac{1}{2}(u_j^{n+1})^2 - \frac{1}{2}(u_j^n)^2 = \frac{\nu}{2}u_j^*[(u_{j+1}^* - u_{j-1}^*)].$$

Noting $\nu > 0$ and summing it w.r.t. j yields

$$\sum_j h\frac{1}{2}(u_j^{n+1})^2 - \sum_j h\frac{1}{2}(u_j^n)^2 = \frac{\nu}{2}\sum_j h[u_j^* u_{j+1}^* - u_j^* u_{j-1}^*] = 0.$$

$\square$

3 Grönwall's inequality

Theorem 3.1. *Let I denote an interval of the real line of the form $[a,+\infty)$ or $[a,b]$ or $[a,b)$ with $a<b$. Let β and u be real-valued continuous functions defined on I. If u is differentiable in the interior I° of I (the interval I without the end points a and possibly b) and satisfies the differential inequality*

$$u'(t) \le \beta(t)\, u(t), \qquad t\in I^\circ,$$

then u is bounded by the solution of the corresponding differential equation $v'(t)=\beta(t)v(t)$:

$$u(t) \le u(a)\exp\left(\int_a^t \beta(s)\,\mathrm{d}s\right)$$

for all $t \in I$.

Remark: There are no assumptions on the signs of the functions β and u.

Proof: Define the function

$$v(t) = \exp\left(\int_a^t \beta(s)\,\mathrm{d}s\right), \qquad t\in I.$$

Note that v satisfies

$$v'(t) = \beta(t)\, v(t), \qquad t\in I^\circ,$$

with $v(a)=1$ and $v(t)>0$ for all $t\in I$. By the quotient rule

$$\frac{d}{dt}\frac{u(t)}{v(t)} = \frac{u'(t)\,v(t) - v'(t)\,u(t)}{v^2(t)} = \frac{u'(t)\,v(t) - \beta(t)\,v(t)\,u(t)}{v^2(t)} \le 0, \qquad t\in I^\circ,$$

Thus the derivative of the function $u(t)/v(t)$ is non-positive and the function is bounded above by its value at the initial point a of the interval I:

$$\frac{u(t)}{v(t)} \le \frac{u(a)}{v(a)} = u(a), \qquad t\in I,$$

which is Grönwall's inequality. $\square$

Theorem 3.2. *Let I denote an interval of the real line of the form $[a,\infty)$ or $[a,b]$ or $[a,b)$ with $a<b$. Let α,β and u be real-valued functions defined on I. Assume that β and u are continuous and that the negative part of α is integrable on every closed and bounded subinterval of I.*

(a) If β is non-negative and if u satisfies the integral inequality

$$u(t) \le \alpha(t) + \int_a^t \beta(s)u(s)\,\mathrm{d}s, \qquad \forall t \in I,$$

then

$$u(t) \le \alpha(t) + \int_a^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(r)\,\mathrm{d}r\right)\mathrm{d}s, \qquad t \in I.$$

(b) If, in addition, the function α is non-decreasing, then

$$u(t) \le \alpha(t)\exp\left(\int_a^t \beta(s)\,\mathrm{d}s\right), \qquad t \in I.$$

Remarks: There are no assumptions on the signs of the functions α and u. Compared to the differential form, differentiability of u is not needed for the integral form. There is also a version of Grönwall's inequality which doesn't need continuity of β and u.

References

[1] D. Gottlieb and B. Gustafsson, Generalized Du Fort-Frankel methods for parabolic initial-boundary value problems, *SIAM J. Numer. Anal.* 13(1), (1976), 129-144.

[2] M. Lees, Alternating direction and semi-explicit difference methods for parabolic partial differential equations, *Numer. Math.*, 3(1961), 398-412.

[3] 戴伟忠, 变系数DuFort-Frankel型差分格式的收敛性和稳定性, 厦门大学学报自然科学自版, 第28卷第6期, 1989, 563-567.