Numerical Solutions to Partial Differential Equations

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What is a Modified Equation of a Difference Scheme?

4.5节(167页)

1. Let h, τ be the spatial and temporal step sizes.
2. Let $U^{m+1} = B_1^{-1}[B_0 U^m + F^m]$ be a difference scheme.
3. Let $\{U_j^m\}_{m\geq 0, j\in J}$, be a solution to the scheme.
4. Let $P = P_{h,\tau}$ be a parameterized differential
5. Let $\mathbb{X}_{h,\tau} = \{\tilde{U} \text{ smooth} : \tilde{U}_j^m = U_j^m, \forall m \geq 0, j \in J\}$.
6. If $P\tilde{U} = 0$, for some $\tilde{U} \in \mathbb{X}_{h,\tau}$, $\forall h, \tau$, then the differential equation $Pu = 0$ is called a modified equation of the difference scheme $U^{m+1} = B_1^{-1}[B_0 U^m + F^m]$.
7. The qth order modified equation: $P\tilde{U} = O(\tau^q + h^q)$, for some $\tilde{U} \in \mathbb{X}_{h,\tau}$, $\forall h, \tau$.

How to Derive the Modified Equation — an Example

Such P is not unique. We want $P = D + H_x$, with $Du = 0$ the original equation, H_x a higher order partial differential operator with respect to x.

1. 1D advection equation: $u_t + au_x = 0, \quad a > 0.$ (4.5.1)
2. Upwind scheme: $\frac{U_j^{m+1} - U_j^m}{\tau} + a\frac{U_j^m - U_{j-1}^m}{h} = 0.$ (4.5.2)
3. Let $\tilde{U}$ be smooth and $\tilde{U}_j^m = U_j^m$.
4. Taylor expand $\tilde{U}$ at (x_j, t_m)

$$\tilde{U}_j^{m+1} = \left[\tilde{U} + \tau\tilde{U}_t + \frac{1}{2}\tau^2\tilde{U}_{tt} + \frac{1}{6}\tau^3\tilde{U}_{ttt} + \cdots\right]_j^m, \tag{4.5.3}$$

$$\tilde{U}_{j-1}^m = \left[\tilde{U} - h\tilde{U}_x + \frac{1}{2}h^2\tilde{U}_{xx} - \frac{1}{6}h^3\tilde{U}_{xxx} + \cdots\right]_j^m, \tag{4.5.4}$$

How to Derive the Modified Equation — an Example

5. Hence,

$$0 = \frac{\tilde{U}_j^{m+1} - \tilde{U}_j^m}{\tau} + a\frac{\tilde{U}_j^m - \tilde{U}_{j-1}^m}{h}$$
$$= \left[\tilde{U}_t + a\tilde{U}_x\right]_j^m + \frac{1}{2}\left[\tau\tilde{U}_{tt} - ah\tilde{U}_{xx}\right]_j^m + \frac{1}{6}\left[\tau^2\tilde{U}_{ttt} + ah^2\tilde{U}_{xxx}\right]_j^m + O(\tau^3 + h^3). \quad (4.5.5)$$

6. $\tilde{U}_t + a\tilde{U}_x = 0$, the first order modified equation. (original one) (4.5.6)
7. $\tilde{U}_t + a\tilde{U}_x = \frac{1}{2}\left[ah\tilde{U}_{xx} - \tau\tilde{U}_{tt}\right]$: the second order. (4.5.7)
8. $\tilde{U}_t + a\tilde{U}_x = \frac{1}{2}\left[ah\tilde{U}_{xx} - \tau\tilde{U}_{tt}\right] - \frac{1}{6}\left[ah^2\tilde{U}_{xxx} + \tau^2\tilde{U}_{ttt}\right]$, the 3rd. (4.5.8)
9. But the latter two are not in the preferred form.

How to Derive the Modified Equation — an Example (continue)

10. By

$$\left[\tilde{U}_t + a\tilde{U}_x\right] + \tfrac{1}{2}\left[\tau\tilde{U}_{tt} - ah\tilde{U}_{xx}\right] + \tfrac{1}{6}\left[\tau^2\tilde{U}_{ttt} + ah^2\tilde{U}_{xxx}\right] = O(\tau^3 + h^3) \qquad (4.5.5)$$

$$\Rightarrow \quad \tilde{U}_{xt} = -a\tilde{U}_{xx} + \frac{1}{2}\left[ah\tilde{U}_{xxx} - \tau\tilde{U}_{xtt}\right] + O(\tau^2 + h^2), \qquad (4.5.9)$$

$$\begin{aligned} \Rightarrow \quad \tilde{U}_{tt} &= -a\tilde{U}_{xt} + \frac{1}{2}\left[ah\tilde{U}_{xxt} - \tau\tilde{U}_{ttt}\right] + O(\tau^2 + h^2) \\ &= a^2\tilde{U}_{xx} - \frac{1}{2}\left[a^2h\tilde{U}_{xxx} - ah\tilde{U}_{xxt} - a\tau\tilde{U}_{xtt} + \tau\tilde{U}_{ttt}\right] + O(\tau^2 + h^2). \end{aligned} \qquad (4.5.10)$$

$$\Rightarrow \quad \tilde{U}_t + a\tilde{U}_x = \frac{1}{2}ah(1-\nu)\tilde{U}_{xx} + O(\tau^2 + h^2). \qquad (4.5.11)$$

How to Derive the Modified Equation — an Example (continue)

11. Hence, $\tilde{U}_t + a\tilde{U}_x = \frac{1}{2}ah(1-\nu)\tilde{U}_{xx}$ is the 2nd order modified equation. (4.5.11)

Similarly, we have the 3rd order modified equation:

$$\tilde{U}_t + a\tilde{U}_x = \frac{1}{2}ah(1-\nu)\tilde{U}_{xx} - \frac{1}{6}ah^2(1-\nu)(1-2\nu)\tilde{U}_{xxx}. \tag{4.5.12}$$

Derive the Modified Equation by Difference Operator Calculus

1. Express a difference operator by a series of differential operators (Taylor expansion). For example, $\triangle_{+t} = e^{\tau \partial_t} - 1$. (4.5.13)

2. Formally inverting the expression, a differential operator can then be expressed by a power series of a difference operator.

3. For example, $\partial_t = \tau^{-1} \ln(1 + \tau \mathcal{D}_{+t})$, where $\mathcal{D}_{+t} := \tau^{-1} \triangle_{+t}$. (4.5.14)
   This yields $\partial_t = \mathcal{D}_{+t} - \frac{\tau}{2}\mathcal{D}_{+t}^2 + \frac{\tau^2}{3}\mathcal{D}_{+t}^3 - \frac{\tau^3}{4}\mathcal{D}_{+t}^4 + \cdots$. (4.5.15)

4. For a difference scheme $\mathcal{D}_{+t} U_j^m = \mathcal{A}_x U_j^m = (\sum_{k=0}^{\infty} \alpha_k \partial_x^k) U_j^m$, substitute $\mathcal{D}_{+t}$ by $\sum_{k=0}^{\infty} \alpha_k \partial_x^k$ in the series expression of ∂_t, and collect the terms with the same powers of ∂_x, we are led to the modified equation
$$\left[\partial_t - \sum_{k=0}^{\infty} \beta_k \partial_x^k\right] \tilde{U} = 0.$$

Derive Modified Equation by Difference Operator Calculus — an Example

1. Advection-diffusion equation: $u_t + au_x = cu_{xx}$, $x \in \mathbb{R}$, $t > 0$. (4.5.16)
2. Explicit scheme: $\frac{U_j^{m+1} - U_j^m}{\tau} + a\frac{U_{j+1}^m - U_{j-1}^m}{2h} = c\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{h^2}$. (4.5.17)
3. By Taylor series expansions of $\triangle_{0x} U_j^m$ and $\delta_x^2 U_j^m$, we have

$$\mathcal{D}_{+t}\tilde{U} = \left\{ -a\left[\partial_x + \frac{1}{6}h^2\partial_x^3 + \cdots\right] + c\left[\partial_x^2 + \frac{1}{12}h^2\partial_x^4 + \cdots\right]\right\}\tilde{U}. \tag{4.5.18}$$

4. The modified equation obtained:

$$\begin{aligned}\tilde{U}_t + a\tilde{U}_x &= \frac{1}{2}\left[2c - a^2\tau\right]\tilde{U}_{xx} - \frac{1}{6}\left[ah^2 - 6ac\,\tau + 2a^3\tau^2\right]\tilde{U}_{xxx} \\ &\quad + \frac{1}{12}\left[ch^2 - 2a^2\tau h^2 - 6c^2\tau + 12a^2c\tau^2 - 3a^4\tau^3\right]\tilde{U}_{xxxx} + \cdots .\end{aligned} \tag{4.5.20}$$

What is the use of a Modified Equation

1. Difference solutions approximate higher order modified equation with higher order of accuracy.
2. Well-posedness of the modified equations provides useful information on the stability of the scheme.
3. Amplitude and phase errors of the modified equations on the Fourier mode solutions provide the corresponding information for the scheme.
4. Convergence rate of the solution of the modified equation to the solution of the original equation also provides the corresponding information for the scheme.
5. In particular, the dissipation and dispersion of the solutions of the modified equations can be very useful.

Dissipation and Dispersion Terms of the Modified Equation

1. Fourier mode $e^{\mathrm{i}(kx+\omega t)} \Rightarrow$ modified eqn. $\tilde{U}_t = \sum_{m=0}^{\infty} a_m \partial_x^m \tilde{U}$. (4.5.21)
2. Notice $\partial_x^m e^{\mathrm{i}(kx+\omega t)} = (\mathrm{i}k)^m e^{\mathrm{i}(kx+\omega t)} \Rightarrow$ dispersion relation:
$$\omega(k) = \sum_{m=1}^{\infty} (-1)^{m-1} a_{2m-1} k^{2m-1} - \mathrm{i} \sum_{m=0}^{\infty} (-1)^m a_{2m} k^{2m}. \tag{4.5.22}$$
3. Denote $\omega(k) = \omega_0(k) + \mathrm{i}\omega_1(k)$, where
$$\omega_0(k) := \sum_{m=1}^{\infty} (-1)^{m-1} a_{2m-1} k^{2m-1}, \quad \omega_1(k) := -\sum_{m=0}^{\infty} (-1)^m a_{2m} k^{2m}. \tag{4.5.23}$$
4. The Fourier mode solution $e^{\mathrm{i}(kx+\omega(k)t)} = e^{-\omega_1(k)t} e^{\mathrm{i}(kx+\omega_0(k)t)}$.
5. Even order spatial derivative terms change the amplitude.
6. Odd order spatial derivative terms change the phase speed.
7. Even and odd order terms are called dissipation and dispersion terms of the modified equations respectively.

Dissipation and Dispersion of Modified Equation — an Example

1. Consider third order modified equation of the upwind scheme for the advection equation with $a > 0$ as an example:
$$\tilde{U}_t + a\tilde{U}_x = \frac{1}{2}ah(1-\nu)\tilde{U}_{xx} - \frac{1}{6}ah^2(1-\nu)(1-2\nu)\tilde{U}_{xxx}.$$
2. We have here $a_0 = 0$, $a_1 = -a$, $a_2 = \frac{1}{2}ah(1-\nu)$, $a_3 = -\frac{1}{6}ah^2(1-\nu)(1-2\nu)$, $a_m = 0$, $m \geq 4$.
3. Thus, we have
$$\omega_0(k) = -ak + \frac{1}{6}a(1-\nu)(1-2\nu)k^3h^2, \quad -\omega_1(k) = -\frac{1}{2}a(1-\nu)k^2h.$$
4. If CFL condition is not satisfied $\Rightarrow -\omega_1(k) > 0 \Rightarrow$ unstable.
5. For $kh \ll 1 \Rightarrow$ relative phase error $O(k^2h^2)$.

Dissipation and Dispersion of Modified Equation — another Example

1. Consider the Lax-Wendroff scheme of the advection equation:

$$\frac{U_j^{m+1} - U_j^m}{\tau} + a\frac{U_{j+1}^m - U_{j-1}^m}{2h} = \frac{1}{2}a^2\tau\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{h^2}. \qquad (4.5.24)$$

2. The modified equation (compare with (4.5.16) and (4) on p.8 of this slides):

$$\tilde{U}_t + a\tilde{U}_x = -\frac{1}{6}ah^2(1-\nu^2)\tilde{U}_{xxx} - \frac{1}{8}ah^3\nu(1-\nu^2)\tilde{U}_{xxxx} + \cdots. \qquad (4.5.25)$$

3. For $kh \ll 1$, dispersion and dissipation components of $\omega(k)$:

$$\omega_0(k) \approx a_1k - a_3k^3 = -ak\left(1 - \frac{1}{6}(1-\nu^2)k^2h^2\right),$$

$$-\omega_1(k) \approx a_0 - a_2k^2 + a_4k^4 = -\frac{1}{8}a\nu(1-\nu^2)k^4h^3.$$

4. $\nu^2 > 1 \Rightarrow -\omega_1(k) > 0 \Rightarrow$ unstable.
5. For $kh \ll 1 \Rightarrow$ phase lag, relative phase error $O(k^2h^2)$.

Necessary Stability Conditions Given by the Modified Equation

1. $-\omega_1 = \sum_{m=0}^{\infty}(-1)^m a_{2m} k^{2m} > 0 \Rightarrow$ the scheme is unstable.

2. In the case of $a_0 = 0$, a finite difference scheme is generally unstable if $a_2 < 0$, or $a_2 = 0$ but $a_4 > 0$.

3. The case when $a_0 = 0$, $a_2 > 0$, $a_4 > 0$ is more complicated. For $kh \ll 1$, Fourier mode solutions are stable, for kh big, say $kh = \pi$, they can be unstable, in particular, high frequency modes are unstable when $a_{2m} = 0$, $\forall m > 2$.

Remark: In fact, for high frequency modes, $-\omega_1 = \sum_{m=0}^{\infty}(-1)^m a_{2m} k^{2m}$ does not necessarily make sense, since it may not converge in general.

Necessary Stability Conditions Given by the Modified Equation

4. In general, the modified equation can only provide necessary conditions for the stability of a difference scheme.

5. For most schemes, the instability appears most easily in the lowest or highest end of Fourier mode solutions.

6. It makes sense to derive the modified equation for the highest end (or oscillatory component) of Fourier mode solutions.

Derivation of Modified Equation for Oscillatory Component

1. For the highest frequencies, $kh = \pi - k'h$, where $k'h \ll 1$.
2. The instability of the highest frequency Fourier mode also shows simultaneously in the form of the time step oscillation, *i.e.* $\arg(\lambda_k) \approx \pi$ for $kh \approx \pi$. Denote $\hat{\lambda}_{k'} = |\lambda_k| e^{\mathrm{i}(\arg(\lambda_k)-\pi)}$, then $\lambda_k = |\lambda_k| e^{\mathrm{i}\arg(\lambda_k)} = -\hat{\lambda}_{k'}$, and $\lambda_k^m = (-1)^m \hat{\lambda}_{k'}^m$.
3. It makes sense to write the oscillatory Fourier modes as $(-1)^{m+j}(U^o)_j^m = \lambda_k^m e^{\mathrm{i}kjh} = (-1)^{m+j} \hat{\lambda}_{k'}^m e^{-\mathrm{i}k'jh}$.

K. W. Morton & D. Mayers, Numerical Solution of Partial Differential Equations, 2nd ed., CUP, 2005
http://mathcenter.hust.edu.cn/Chinese/rcpy/bsyjs.htm

Derivation of Modified Equation for Oscillatory Component

4. The finite difference solution can often be decomposed as $U_j^m = (U^s)_j^m + (-1)^{m+j}(U^o)_j^m$, *i.e.* the smooth and oscillatory components of the difference solution. (4.5.26)

5. The modified equation studied previously is for the smooth component $\tilde{U} = \tilde{U}^s$.

6. The modified equation for the oscillatory component $\tilde{U} = (-1)^{m+j}\tilde{U}^o$ can be derived in a similar way.

Derivation of Modified Equation for Oscillatory Component — an Example

1. Let the oscillatory component $\tilde{U}_j^m = (-1)^{m+j}(\tilde{U}^o)_j^m$ be a smooth function satisfying $\tilde{U}_j^m = U_j^m$.
2. Substitute it into the explicit scheme of the heat equation $u_t = cu_{xx}$: $U_j^{m+1} = (1-2\mu)U_j^m + \mu\left(U_{j-1}^m + U_{j+1}^m\right)$.
3. Taylor expanding $(\tilde{U}^o)_{j-1}^m$ and $(\tilde{U}^o)_{j+1}^m$ at $(\tilde{U}^o)_j^m$ yields

$$\begin{aligned}(\tilde{U}^o)_j^{m+1} &= (2\mu-1)(\tilde{U}^o)_j^m + \mu\left((\tilde{U}^o)_{j-1}^m + (\tilde{U}^o)_{j+1}^m\right)\\ &= \left\{(4\mu-1) + 2\mu\left[\frac{1}{2}h^2\partial_x^2 + \frac{1}{24}h^4\partial_x^4 + \cdots\right]\right\}(\tilde{U}^o)_j^m. \qquad (4.5.27)\end{aligned}$$

Derivation of Modified Equation for Oscillatory Component — an Example

4. Rewrite the scheme into the form of $\mathcal{D}_{+t}(\tilde{U}^o)_j^m = \sum_{k=0}^{\infty} \alpha_k \partial_x^k (\tilde{U}^o)_j^m$.

   Remember $\mathcal{D}_{+t} = \tau^{-1}\triangle_+$, $\mu = c\tau/h^2$, we have

$$\mathcal{D}_{+t}\tilde{U}^o = \left\{ 2\tau^{-1}(2\mu - 1) + c\left[\partial_x^2 + \frac{1}{12}h^2\partial_x^4 + \cdots\right]\right\}\tilde{U}^o. \qquad (4.5.28)$$

5. Since $\partial_t = \mathcal{D}_{+t} - \frac{\tau}{2}\mathcal{D}_{+t}^2 + \frac{\tau^2}{3}\mathcal{D}_{+t}^3 - \frac{\tau^3}{4}\mathcal{D}_{+t}^4 + \cdots$,

6. Denote $\xi = 2\tau^{-1}(2\mu - 1)$, and

$$a_0 = \xi - \frac{1}{2}\xi^2\tau + \frac{1}{3}\xi^3\tau^2 - \frac{1}{4}\xi^4\tau^3 + \cdots = \tau^{-1}\ln(1 + \xi\tau). \qquad (4.5.29)$$

Derivation of Modified Equation for Oscillatory Component — an Example

7. Therefore, the modified equation of the oscillatory component $\tilde{U}^o$ has the form

$$\partial_t \tilde{U}^o = \tau^{-1} \ln(1 + 2(2\mu - 1))\tilde{U}^o + \sum_{m=1}^{\infty} a_{2m}\partial_x^{2m}\tilde{U}^o. \tag{4.5.30}$$

8. Consequently, if $2\mu > 1$, the oscillatory component $\tilde{U}^o$ will grow exponentially, which implies that the difference scheme is unstable for the highest frequency Fourier modes.

Coercivity of the Differential Operator $L(\cdot)$ and the Energy Inequality

4.6节(P175)

Consider $\partial_t u = L(u)$, where L a partial differential operator with respect to x, and L does not explicitly depends on t.

1. Coerciveness condition of the differential operator $L(\cdot)$:

注：这里的强制性条件不同于Lax-Milgram定理(P185)中的.

$$\int_\Omega L(u)u\,dx \le C\|u\|_2^2, \qquad \forall u \in \mathbb{X}, \tag{4.6.1}$$

2. Since $u_t(x,t) = L(u(x,t))$, we are led to $\frac{d}{dt}\|u\|_2^2 \le 2C\|u\|_2^2$. (4.6.2)

3. By the Gronwall inequality, we have

$$\|u(\cdot,t)\|_2^2 \le e^{2Ct}\|u^0(\cdot)\|_2^2, \qquad \forall t \in [0, t_{\max}]. \tag{4.6.3}$$

(4.6.2)=> (exp(-2Ct) ||u||^2_2)' \leq 0=>关于t积分(4.6.3)

4. In particular, the $\mathbb{L}^2(\Omega)$ norm of the solution decays exponentially, if $C < 0$; and it is non-increasing, if $C = 0$.

Coercivity of the Differential Operator $L(\cdot)$ and the Energy Inequality

1. The inequalities in similar forms as above with various norms are generally called energy inequalities, and the corresponding norm is called the energy norm.

2. In such cases, we hope that the difference solution satisfies corresponding discrete energy inequality.

Energy Inequality for Runge-Kutta Time-Stepping Numerical Schemes

Theorem 4.4

Suppose that a difference scheme has the form

$$U^{m+1} = \sum_{i=0}^{k} \frac{(\tau L_\triangle)^i}{i!} U^m, \tag{4.6.4}$$

Suppose that the difference operator $L_\triangle$ is coercive in the Hilbert space $(\mathbb{X}, \langle\cdot,\cdot\rangle)$, i.e. *there exist a constant $K > 0$ and an increasing function $\eta(h) : \mathbb{R}_+ \to \mathbb{R}_+$, such that*

$$\langle L_\triangle U, U\rangle \le K\|U\|^2 - \eta\|L_\triangle U\|^2, \quad \forall U. \tag{4.6.5}$$

Then, for $k = 1, 2, 3, 4, \cdots$, there exists a constant $K' \ge 0$ s.t.

$$\|U^{m+1}\| \le (1 + K'\tau)\|U^m\|, \quad \text{if } \tau \le 2\eta. \quad (4.6.6) \tag{4.6.7}$$

In particular, if $K \le 0$, we have $K' = 0$ and $\|U^{m+1}\| \le \|U^m\|$. (4.6.8)

Note: $K' > 0 \Rightarrow$ Lax-Richtmyer stable; $K' = 0 \Rightarrow$ strongly stable.

Proof of the Theorem for $k = 1$

Without loss of generality, assume $K \geq 0$.

For $k = 1$. By the definition and the coercivity of $L_\triangle$, we have

$$\begin{aligned} \|U^{m+1}\|^2 &\overset{(4.6.4)}{=} \|(I + \tau L_\triangle)U^m\|^2 \\ &= \|U^m\|^2 + 2\tau\langle L_\triangle U^m, U^m\rangle + \tau^2\|L_\triangle U^m\|^2 \\ &\overset{(4.6.5)}{\leq} (1 + 2K\tau)\|U^m\|^2 + \tau(\tau - 2\eta)\|L_\triangle U^m\|^2. \end{aligned} \tag{4.6.9}$$

Therefore, the conclusion of the theorem holds for $K' = K$.

Proof of the Theorem for 2 ($k \geq 3$ is left as an Exercise)

For $k = 2$, $I + \tau L_\triangle + \frac{1}{2}(\tau L_\triangle)^2 = \frac{1}{2}I + \frac{1}{2}(I + \tau L_\triangle)^2$. Therefore, (4.6.10)

$$\begin{aligned}
\|U^{m+1}\| &\overset{(4.6.4)}{=} \|(\frac{1}{2}I + \frac{1}{2}(I+\tau L_\triangle)^2)U^m\| \\
&\leq \frac{1}{2}\|U^m\| + \frac{1}{2}(1+K\tau)^2\|U^m\| \\
&\overset{(4.6.9)}{\leq} (1 + K(1+\eta K)\tau)\,\|U^m\|, \qquad \text{if } \tau \leq 2\eta.
\end{aligned} \tag{4.6.11}$$

So, the conclusion of the theorem holds for $K' = K(1 + \eta K)$.

Similarly, the conclusion of the theorem for $k \geq 3$ can be proved by induction. (see Exercise 4.7)

Remark: $\tau = \kappa\eta(h)$ with $\kappa \in (0, 2]$ provides a stable refinement path.

$\mathbb{L}^2$ Stability of Upwind Scheme for Variable-Coefficient Advection Equation

例4.2 (P177)

1. The initial-boundary value problem of the advection equation

$$\begin{cases} u_t(x,t) + a(x)u_x(x,t) = 0, & 0 < x \le 1, \quad t > 0, \\ u(x,0) = u^0(x), & 0 \le x \le 1, \\ u(0,t) = 0, & t > 0, \end{cases} \tag{4.6.14}$$

2. $$U_j^{m+1} = U_j^m - \frac{a_j \tau}{h}\left(U_j^m - U_{j-1}^m\right), \quad j = 0, 1, \cdots, N. \tag{4.6.15}$$

$L_\triangle = -a(x)h^{-1}\triangle_{-x}$, $k = 1$.

3. $$0 \le a(x) \le A, \ |a(x) - a(x')| \le C|x - x'|, \ A, \ C > 0 \text{ const..} \tag{4.6.16}$$

$\mathbb{L}^2$ Stability of Upwind Scheme for Variable-Coefficient Advection Equation

4. We need to check, $\exists$ constant $K > 0$ and $\eta > 0$ s.t.

$$\langle L_\triangle U, U\rangle \le K\|U\|^2 - \eta\|L_\triangle U\|^2, \quad \forall U.$$

$$\langle L_\triangle U, U\rangle = -\sum_{j=1}^{N} a_j(U_j - U_{j-1})U_j = -\sum_{j=1}^{N} a_j(U_j)^2 + \sum_{j=1}^{N} a_j U_j U_{j-1}, \tag{4.6.17}$$

$$h\,\|L_\triangle U\|_2^2 = \sum_{j=1}^{N} a_j^2(U_j - U_{j-1})^2 \le A\sum_{j=1}^{N}\left[a_j(U_j)^2 - 2a_jU_jU_{j-1} + a_j(U_{j-1})^2\right]. \tag{4.6.18}$$

$$\begin{aligned} 2\langle L_\triangle U, U\rangle + A^{-1}h\,\|L_\triangle U\|_2^2 &\le -\sum_{j=1}^{N} a_j\left[(U_j)^2 - (U_{j-1})^2\right] \\ &= \sum_{j=1}^{N-1}(a_{j+1} - a_j)(U_j)^2 \le C\|U\|_2^2. \end{aligned} \tag{4.6.19}$$

$\mathbb{L}^2$ Stability of Upwind Scheme for Variable-Coefficient Advection Equation

5. Coercivity condition is satisfied for $K = C/2$ and $\eta = A^{-1}h/2$.

6. The conclusion of the theorem holds for $K' = K = C/2$.

7. The stability condition $\tau \leq 2\eta \Leftrightarrow A\tau \leq h$. (4.6.20)

$\mathbb{L}^2$ Stability of Upwind Scheme for Variable-Coefficient Advection Equation

Remark:

1. The stability condition $A\tau \leq h$: a natural extension of $a\tau \leq h$.
2. Constant-coefficient case: $\mathbb{L}^2$ strongly stable.
3. Variable-coefficient case: $\mathbb{L}^2$ stable in the sense of von Neumann or Lax-Richtmyer stability.
4. Variable-coefficient can cause additional error growth, and the approximation error can grow exponentially.
5. The result is typical. (Variable-coefficient, nonlinearity)

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Thank You!