Finite element method for ODEs

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1 Settings

In this report, we will solve the following ODE by the finite element method (FEM).

$$\begin{cases} -u'' + u = f, \quad x \in (0,1) \\ u(0) = 0, \quad u'(1) + u(1) = g \end{cases} \tag{1}$$

2 FEM form

The variation form can be formalized as: Find $u \in \mathbb{H}_0^1((0,1))$, for all $\phi \in \mathbb{H}_0^1((0,1))$

$$\int_0^1 (u\phi + u'\phi')\ \mathrm{d}x + u(1)\phi(1) = \int_0^1 f\phi\ \mathrm{d}x + g\phi(1) \tag{2}$$

Then for the finite element space $\mathbb{V}_n \subset \mathbb{H}_0^1((0,1))$, suppose $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$. We construct base function $\lambda_i,\ \forall 1 \le i \le n$ below.
For $j = 1, 2, \ldots, n-1$.

$$\lambda_j(x) = \begin{cases} \frac{x_j - x}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j] \\ \frac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j] \\ 0, & \text{others} \end{cases} \tag{3}$$

For $j = n$,

$$\lambda_n(x) = \begin{cases} \frac{x_n - x}{x_n - x_{n-1}}, & x \in [x_{n-1}, x_n] \\ 0, & \text{others} \end{cases} \tag{4}$$

Assume the finite element solution is $u_h = \sum_{i=1}^n u_i\lambda_i(x)$, then by setting $\phi = \lambda_i$ in Equation 2, we get

$$\sum_{i=1}^n u_i \int_0^1 \lambda_i\lambda_j + \lambda_i'\lambda_j'\ \mathrm{d}x = \int_0^1 f\lambda_j\ \mathrm{d}x, \quad \forall j = 1, 2, \ldots, n-1 \tag{5}$$

For $j = n$,

$$u_n + \sum_{i=1}^n u_i \int_0^1 \lambda_i\lambda_n + \lambda_i'\lambda_n'\ \mathrm{d}x = \int_0^1 f\lambda_n\ \mathrm{d}x + g \tag{6}$$

In the following, $h_j = x_j - x_{j-1}$. We can explicitly calculate that, for $j = 1.2. \dots, n-1$,

$$\int_0^1 \lambda_i \lambda_j \, \mathrm{d}x = \begin{cases} \frac{h_j}{6}, & i = j-1 \\ \frac{h_j + h_{j+1}}{3}, & i = j \\ \frac{h_{j+1}}{6}, & i = j+1 \end{cases} \tag{7}$$

$$\int_0^1 \lambda_i' \lambda_j' \, \mathrm{d}x = \begin{cases} -\frac{1}{h_i} & i = j-1 \\ \frac{1}{h_j} + \frac{1}{h_{j+1}}, & i = j \\ -\frac{1}{h_{j+1}} & i = j+1 \end{cases} \tag{8}$$

$$\int_0^1 f \lambda_j \, \mathrm{d}x \approx \frac{h_j f(x_{j-1}) + 2h_j f(x_j) + 2h_{j+1} f(x_i) + h_{j+1} f(x_{j+1})}{6} \tag{9}$$

For $j = n$,

$$\int_0^1 \lambda_i \lambda_n \, \mathrm{d}x = \begin{cases} \frac{h_n}{6}, & i = n-1 \\ \frac{h_n}{3}, & i = n \end{cases} \tag{10}$$

$$\int_0^1 \lambda_i' \lambda_n' \, \mathrm{d}x = \begin{cases} -\frac{1}{h_n}, & i = n-1 \\ \frac{1}{h_n}, & i = n \end{cases} \tag{11}$$

$$\int_0^1 f \lambda_n \, \mathrm{d}x \approx \frac{h_n f(x_n)}{2} \quad \text{(middle point)} \tag{12}$$

Then we can convert the Equation 5, 6 to a linear equation in the form

$$A_n u = F_n \tag{13}$$

The explicit form of A_n will be obvious in the code fem.m, hence, we do not write it down here. Besides, the uniqueness of the solution can be directly derived from the uniqueness in the space $\mathcal{H}_0^1((0,1))$, or from the fact that A_n is diagonal dominant.

3 Numerical test

We consider two different real solutions:

- Case 1: $u(x) = \sin(10\pi x)$
- Case 2: $u(x) = \exp(-10(x-0.5)^2) - \exp(-5/2)$

$f(x), g$ will be defined accordingly. For the sake of simplicity, we only consider the uniform grid. Since A_n is symmetric, and weakly diagonal dominant, we can use conjugate gradient method to find u.

We set $n = 8000, 8200, 8400, \dots, 10000$ for case 1, and $4000, 4200, \dots, 6000$ for case 2. The error figure is in the folder "derivative" and "value", with file names transparent to understand.

To reproduce the results, please execute

```
convergence(choice, grid, err_type, derivative)
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where choice = 1 or 2 representing case 1 or case 2, grid = 'uniform', err_type = 'inf' or '2' represent l^∞ norm or l^2 norm, derivative = 0 or 1 representing calculating the error of the function value or the derivative.
Notice that we directly use

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norm(Err, inf)
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to calculate the l^∞ norm, but we use

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err = sqrt(sum(Err.^2.*h));
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to approximate the l^2 norm

$$(\int_0^1 (u_h - u)^2 \ \mathrm{d}x)^{1/2} \quad , or(\int_0^1 (u_h' - u')^2 \ \mathrm{d}x)^{1/2} \tag{14}$$

Table 1: Uniform grid, value

n	8000	8200	8400	8600	8800	9000	9200	9400	9600	9800	10000
log l^∞, case 1	-9.8570	-9.9063	-9.9546	-10.0016	-10.0476	-10.0925	-10.1365	-10.1795	-10.2216	-10.2628	-10.3032
log l^2 case 1	-10.4650	-10.5144	-10.5626	-10.6096	-10.6556	-10.7005	-10.7445	-10.7875	-10.8296	-10.8709	-10.9113

Table 2: Uniform grid, value

n	4000	4200	4400	4600	4800	5000	5200	5400	5600	5800	6000
log l^∞, case 2	-15.3014	-15.4006	-15.4922	-15.5796	-15.6660	-15.7490	-15.8222	-15.8997	-15.9729	-16.0512	-16.1158
log l^2 case 2	-16.0421	-16.1419	-16.2331	-16.3199	-16.4068	-16.4903	-16.5615	-16.6397	-16.7131	-16.7944	-16.8580

Table 3: Uniform grid, derivative

n	8000	8200	8400	8600	8800	9000	9200	9400	9600	9800	10000
log l^∞, case 1	-2.7847	-2.8094	-2.8336	-2.8571	-2.8801	-2.9026	-2.9246	-2.9461	-2.9672	-2.9878	-3.0081
log l^2 case 1	-3.1323	-3.1570	-3.1811	-3.2046	-3.2276	-3.2501	-3.2721	-3.2936	-3.3146	-3.3352	-3.3555

Table 4: Uniform grid, derivative

n	4000	4200	4400	4600	4800	5000	5200	5400	5600	5800	6000
log l^∞, case 2	-5.9914	-6.0402	-6.0867	-6.1312	-6.1737	-6.2145	-6.2538	-6.2915	-6.3279	-6.3630	-6.3969
log l^2 case 2	-6.6250	-6.6738	-6.7203	-6.7648	-6.8074	-6.8482	-6.8874	-6.9251	-6.9615	-6.9966	-7.0305

The l^∞ error and l^2 error of the function value is $\mathcal{O}(n^{-2})$, and the l^∞ error and l^2 error of the derivative is $\mathcal{O}(n^{-1})$, which is aligned with the theoretical results. We list the theoretical results below and postpone proofs.

$$|u(x) - I_h u(x)| \leq \frac{1}{\sqrt{3}}(\int_{x_{i-1}}^{x_i} (u''(t))^2 \text{ d}t)^{1/2} h_i^{3/2} \sim \mathcal{O}(n^{-2}) \tag{15}$$

Hence

$$\int_0^1 (u(x) - I_h u(x))^2 \text{ d}x \leq \frac{1}{3} \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (u''(t))^2 h_i^4 \text{ d}t \sim \mathcal{O}(n^{-4}) \tag{16}$$

Besides,

$$|(u(x) - I_h u(x))'| \leq \frac{2}{\sqrt{3}}(\int_{x_{i-1}}^{x_i} (u''(t))^2 \text{ d}t)^{1/2} h_i^{1/2} \sim \mathcal{O}(n^{-1}) \tag{17}$$

Hence,

$$\int_0^1 ((u(x) - I_h u(x))')^2 \text{ d}x \leq \frac{1}{3} \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (u''(t))^2 h_i^2 \text{ d}t \sim \mathcal{O}(n^{-2}) \tag{18}$$