Numerical Solutions to Partial Differential Equations

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Embedding Operator and Embedding Relation of Banach Spaces

嵌入定理深刻地刻划Sobolev空间之间或Sobolev空间与其它函数空间之间的关系.
在近代PDE理论研究中起着重要的作用.

1. $\mathbb{X}$, $\mathbb{Y}$: Banach spaces with norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$.

X嵌入(连续地)到Y 的定义Y

2. If $x \in \mathbb{X} \Rightarrow x \in \mathbb{Y}$, & $\exists$ const. $C > 0$ independent of x s.t. $\|x\|_{\mathbb{Y}} \leq C\|x\|_{\mathbb{X}}$, $\forall x \in \mathbb{X}$, then the identity map $I : \mathbb{X} \to \mathbb{Y}$, $I\,x = x$ is called an embedding operator, and the corresponding embedding relation is denoted by $\mathbb{X} \hookrightarrow \mathbb{Y}$.
3. The embedding operator $I : \mathbb{X} \to \mathbb{Y}$ is a bounded linear map.
4. If, in addition, I is happened to be a compact map, then, the corresponding embedding is called a compact embedding, and is denoted by $\mathbb{X} \overset{c}{\hookrightarrow} \mathbb{Y}$.

设X,Y是赋范线性空间，T是X到Y的连续算子.如果T把定义域中任何有界集映射成Y中的列紧集，则称T是紧算子或全连续算子.
紧算子是一类重要的有界算子, 它最接近于有限维空间上的线性算子.
设A是度量空间X中的无穷集，如果A中的任一无穷子集必有一个收敛的点列，就称A是X中的列紧集

The Sobolev Embedding Theorem

Theorem 5.5 Page 191

Let Ω be a bounded connected domain with a Lipschitz continuous boundary $\partial\Omega$, then

$$\mathbb{W}^{m+k,p}(\Omega) \hookrightarrow \mathbb{W}^{k,q}(\Omega), \ \forall\, 1 \leq q \leq \frac{np}{n-mp}, \ k \geq 0, \quad \text{if } \ m < n/p;$$

$$\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,q}(\Omega), \ \forall\, 1 \leq q < \frac{np}{n-mp}, \ k \geq 0, \quad \text{if } \ m < n/p;$$

$$\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,q}(\Omega), \ \forall\, 1 \leq q < \infty, \ k \geq 0, \quad \text{if } \ m = n/p;$$

$$\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^{k}(\overline{\Omega}), \quad \forall\, k \geq 0, \quad \text{if } \ m > n/p.$$

Def: 称**X嵌入(连续地)到Y**, 如果X包含于Y, X到Y具有连续内射即存在C>0, s.t. 对X中任何元素x, x的Y范数不超过x的X范数的C倍: ||x||_Y\leq C||x||_X.

Rem: 嵌入定理5.5: Sobolev空间这函数的较"低"阶范数可以被较"高"阶范数控制. 一般地讲, 反之不真.

Trace of a Function and Trace Operators

迹的概念对PDE至关重要, 这是Ω闭包上连续函数在其边界上取值的通常概念的推广.

1. Since the n dimensional Lebesgue measure of a Lipschitz continuous boundary $\partial\Omega$ is zero, a function in $\mathbb{W}^{m,p}(\Omega)$ is generally not well defined on $\partial\Omega$.
2. $\mathbb{C}^\infty(\overline{\Omega})$ is dense in $\mathbb{W}^{m,p}(\Omega)$ for $1 \le p < \infty$.
3. For $u \in \mathbb{W}^{m,p}(\Omega)$, let $\{u_k\} \subset \mathbb{C}^\infty(\overline{\Omega})$ be such that
$$\|u_k - u\|_{m,p,\Omega} \longrightarrow 0, \quad \text{as } \; k \to \infty,$$
4. If, for any such a sequence, $u_k|_{\partial\Omega} \to \nu(u)$ in $\mathbb{L}^q(\partial\Omega)$, then, we call $u|_{\partial\Omega} \triangleq \nu(u) \in \mathbb{L}^q(\partial\Omega)$ the trace of u on $\partial\Omega$, and call $\nu : \mathbb{W}^{m,p}(\Omega) \to \mathbb{L}^q(\partial\Omega)$, $\nu(u) = u|_{\partial\Omega}$ the trace operator.

Trace of a Function and Trace Operators

5. If ν is continuous, we say $\mathbb{W}^{m,p}(\Omega)$ embeds into $\mathbb{L}^q(\partial\Omega)$, and denote the embedding relation as $\mathbb{W}^{m,p}(\Omega) \hookrightarrow \mathbb{L}^q(\partial\Omega)$.

6. Trace operators, as well as corresponding embedding and compact embedding, into other Banach spaces defined on the whole or a part of $\partial\Omega$ can be defined similarly.

Obviously, under the conditions of the embedding theorem, $\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^k(\partial\Omega)$, if $m > n/p$. In general, we have the following trace theorem.

The Trace Theorem

Theorem 5.6

If the boundary $\partial\Omega$ of a bounded connected open domain Ω is an order $m \geq 1$ continuously differentiable surface, then, we have

$$\mathbb{W}^{m,p}(\Omega) \hookrightarrow \mathbb{L}^q(\partial\Omega), \quad \text{for } 1 \leq q \leq \frac{(n-1)p}{n-mp}, \quad \text{if} \quad m < n/p;$$

$$\mathbb{W}^{m,p}(\Omega) \hookrightarrow \mathbb{L}^q(\partial\Omega), \quad \text{for } 1 \leq q < \infty, \quad \text{if} \quad m = n/p.$$

Trace 定理表示：(1)W^{m,p}(Ω)中函数的迹属于L^q(∂Ω)；(2) 迹算子是连续的.

In addition, if $m = 1$ and $p = q = 2$, and if the boundary $\partial\Omega$ is a Lipschitz continuous surface, then, we have in particular

$$\underbrace{\mathbb{H}^1(\Omega)}_{m=1,p=2} \hookrightarrow \mathbb{L}^2(\partial\Omega).$$

注意，Th5.6中的嵌入记号不等同于前面的. 有的书上并不用该嵌入记号。

Remarks on $\mathbb{H}_0^1(\Omega)$ and $\mathbb{H}_0^2(\Omega)$

For a bounded connected open domain Ω with Lipschitz continuous boundary $\partial\Omega$,

p=2

1. by definition the Hilbert space $\mathbb{H}_0^m(\Omega)$ is the closure of $\mathbb{C}_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_m = \|\cdot\|_{m,2} := \|\cdot\|_{m,2,\Omega}$;

   m=1,p=2

2. in particular, $\mathbb{H}_0^1(\Omega) = \{u \in \mathbb{H}^1(\Omega) : u|_{\partial\Omega} = 0\}$;

   给出了H^1_0的清晰刻划：H^1中迹为0的函数构成的Hilbert空间

m=2,p=2

3. $\mathbb{H}_0^2(\Omega) = \{u \in \mathbb{H}^2(\Omega) : u|_{\partial\Omega} = 0,\ \partial_\nu u|_{\partial\Omega} = 0\}$, where $\partial_\nu u|_{\partial\Omega}$ is the outer normal derivative of u in the sense of trace.

   给出了H^2_0的清晰刻划：　H^2中函数及其法向导数的迹为0的函数构成的Hilbert空间

Derivation of a Variational Form

1. The Dirichlet boundary value problem of the Poisson equation

$$-\triangle u = f, \quad \forall x \in \Omega, \qquad u = \bar{u}_0, \quad \forall x \in \partial\Omega. \tag{5.2.4}$$

2. Assume the problem admits a classical solution $u \in \mathbb{C}^2(\overline{\Omega})$.

3. For any test function $v \in \mathbb{C}_0^\infty(\Omega)$, by the Green's formula,

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} v \, \partial_\nu u \, dx = \int_\Omega fv \, dx. \tag{5.2.5}$$

Derivation of a Variational Form

4. Let $a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$; $(\cdot, \cdot)$ the inner product of $\mathbb{L}^2(\Omega)$.
5. By the denseness of $\mathbb{C}_0^\infty(\Omega)$ in $\mathbb{H}_0^1(\Omega)$, we are lead to

$$\boxed{a(u, v) = (f, v), \quad \forall v \in \mathbb{H}_0^1(\Omega).} \qquad (5.2.6)$$

6. u does not have to be in $\mathbb{C}^2(\overline{\Omega})$ to satisfy such a variational equation, $u \in \mathbb{H}^1(\Omega)$ makes sense.
   满足(5.2.6)的u不必是二次连续可微，只要是H^1的.

A Variational Form of Dirichlet BVP of the Poisson Equation

Definition 5.5

If $u \in \mathbb{V}(\bar{u}_0; \Omega) = \{u \in \mathbb{H}^1(\Omega) : u|_{\partial\Omega} = \bar{u}_0\}$ satisfies the variational equation

$$a(u, v) = (f, v), \quad \forall v \in \mathbb{H}_0^1(\Omega), \tag{5.2.6}$$

then, u is called a weak solution of the Dirichlet boundary value problem of the Poisson equation; the corresponding variational problem is called a variational form, or weak form, of the Dirichlet boundary value problem of the Poisson equation; and the function spaces $\mathbb{V}(\bar{u}_0; \Omega)$ and $\mathbb{H}_0^1(\Omega)$ are called respectively the trial and test function spaces of the variational problem.

trial function--试探函数; test function--检验函数

- Obviously, the classical solution, if exists, is a weak solution.
- Let $\tilde{u} \in \mathbb{H}^1(\Omega)$ and $\tilde{u}|_{\partial\Omega} = \bar{u}_0$, then $\mathbb{V}(\bar{u}_0; \Omega) = \tilde{u} + \mathbb{H}_0^1(\Omega)$.

The Relationship Between Weak and Classical Solutions

Theorem 5.7

(1) *Let $f \in \mathbb{C}(\overline{\Omega})$ and $\bar{u}_0 \in \mathbb{C}(\partial\Omega)$. If $u \in \mathbb{C}^2(\overline{\Omega})$ is a classical solution of the Dirichlet boundary value problem of the Poisson equation, then, it must also be a weak solution of the problem.* (2) *On the other hand, if u is a weak solution of the Dirichlet boundary value problem of the Poisson equation, and in addition $u \in \mathbb{C}^2(\overline{\Omega})$, then it must also be a classical solution of the problem.*

- The classical solution, if exists, is a weak solution, follows directly from the derivation of the variational form of the problem.
- We only need to show the second part of the theorem.

Proof of Weak Solution + $u \in \mathbb{C}^2(\overline{\Omega}) \Rightarrow$ Classical Solution

1. Let u be a weak solution, and $u \in \mathbb{C}^2(\overline{\Omega})$.
2. Since u is a weak solution and $u \in \mathbb{C}^2(\overline{\Omega})$, by the Green's formula:
$$\int_\Omega (\triangle u + f)\, v\, dx = 0, \quad \forall v \in \mathbb{C}_0^\infty(\Omega).$$
Variational lemma
3. $\triangle u + f$ is continuous $\Rightarrow -\triangle u = f$, $\forall x \in \Omega$.
4. By the definition of trace, $u|_{\partial\Omega} = \bar{u}_0$ also holds in the classical sense.
5. u is a classical solution of the Dirichlet BVP of the Poisson equation. ■

Another Variational Form of Dirichlet BVP of the Poisson Equation

- The quadratic functional $J(v) = \frac{1}{2}\, a(v,\ v) - (f,\ v)$ on $\mathbb{H}^1(\Omega)$.
- Its Fréchet differential $J'(u)v = a(u,\ v) - (f,\ v)$. (5.2.7)
- The weak form above is simply $J'(u)v = 0,\ \forall v \in \mathbb{H}_0^1(\Omega)$.

Definition 5.6

If $u \in \mathbb{V}(\bar{u}_0; \Omega)$ is a minima of the functional $J(\cdot)$ in $\mathbb{V}(\bar{u}_0,\ \Omega)$, meaning

$$\boxed{J(u) = \min_{v \in \mathbb{V}(\bar{u}_0;\Omega)} J(v),} \qquad (5.2.8)$$

then, u is called a weak solution of the Dirichlet BVP of the Poisson equation. The corresponding functional minimization problem is called a variational form (or weak form) of the Dirichlet BVP of the Poisson Equation.

Theorem 5.8

The weak solutions of the two variational problems are equivalent.
in Def 5.5 & 5.6

Proof: (1) Let $u \in \mathbb{V}(\bar{u}_0; \Omega)$ be a minima of J in $\mathbb{V}(\bar{u}_0; \Omega)$, then

$$J'(u)v = 0, \quad \forall v \in \mathbb{H}_0^1(\Omega); \;\; \Rightarrow \;\; a(u,\, v) = (f,\, v), \;\; \forall v \in \mathbb{H}_0^1(\Omega).$$

(2) Let $u \in \mathbb{V}(\bar{u}_0; \Omega)$ satisfy the above equation. Then, by the symmetry of the bilinear form $a(u,\, v)$, we have

$$J(v) - J(u) = \underbrace{a(u,\, v-u) - (f,\, v-u)}_{=0,\text{due to } (5.2.6)} + \frac{1}{2}\, a(v-u,\, v-u).$$

Since $v - u \in \mathbb{H}_0^1(\Omega)$, ~~we are lead to~~

$$J(v) - J(u) = \frac{1}{2}\, a(v-u,\, v-u) \geq 0, \quad \forall v \in \mathbb{V}(\bar{u}_0; \Omega).$$

Therefore, $u \in \mathbb{V}(\bar{u}_0; \Omega)$ is the unique minima of J in $\mathbb{V}(\bar{u}_0; \Omega)$. ■

Existence and Uniqueness of Weak Solutions

Theorem 5.9 P195

Let Ω be a bounded connected domain with Lipschitz continuous $\partial\Omega$. Let $f \in \mathbb{L}^2(\Omega)$. Suppose $\{u_0 \in \mathbb{H}^1(\Omega) : u_0|_{\partial\Omega} = \bar{u}_0\} \neq \emptyset$. Then, the Dirichlet BVP of the Poisson equation has a unique weak solution.

考虑用Lax-Migram定理证明该定理， 为此要验证a(u,v)是一个连续的双线性形式, 且满足强制条件.
前者是显然的, 后者的验证需要用Poincare-Friedrichs不等式(5.2.3).

Proof of Existence and Uniqueness Theorem on Weak Solutions

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1. Define $F(v) = (f,\ v) - a(u_0,\ v)$ on $\mathbb{V} = \mathbb{H}_0^1(\Omega)$.
2. By the Poincaré-Friedrichs inequality (see Theorem 5.4) ~~that~~

==> a(u,v)满足强制条件

$$\exists\ const.\ \alpha(\Omega) > 0,\quad s.t.\quad a(v,\ v) \geq \alpha \|v\|^2_{1,2,\Omega},\quad \forall v \in \mathbb{V},$$

3. By the Lax-Milgram lemma (see Theorem 5.1), the variational problem
$$\begin{cases} \text{Find}\ \ u \in \mathbb{V} \text{ such that} \\ a(u, v) = F(v), \quad \forall v \in \mathbb{V}, \end{cases}$$
has a unique solution.
4. u solves the above problem $\Leftrightarrow$ $u + u_0$ is a weak solution. of (5.2.6) ■

The BC $u(0) = 0$ is called *essential* as it appears in the variational formulation explicitly, i.e., in the definition of V. This type of BC also frequently goes by the proper name "Dirichlet." The BC $u'(1) = 0$ is called *natural* because it is incorporated implicitly. This type of BC is often referred to by the name "Neumann."

Derivation of a Variational Form for Neumann BVP of the Poisson Eqn

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1. The Neumann BVP of the Poisson equation

$$-\triangle u = f, \ \ \forall x \in \Omega, \qquad \partial_\nu u = g, \ \ \forall x \in \partial\Omega. \tag{5.2.9}$$

2. Assume the problem admits a classical solution $u \in \mathbb{C}^2(\overline{\Omega})$.

3. For any test function $v \in \mathbb{C}^\infty(\overline{\Omega})$, by the Green's formula,

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} v \, \partial_\nu u \, dx = \int_\Omega fv \, dx.$$

Derivation of a Variational Form for Neumann BVP of the Poisson Eqn

4. Let $(g,\ v)_{\partial\Omega} = \int_{\partial\Omega} g\, v\, ds$, $a(u,\ v)$ and $(f,\ v)$ as before.

5. By the denseness of $\mathbb{C}^\infty(\overline{\Omega})$ in $\mathbb{H}^1(\Omega)$, ~~we are lead to~~

$$\boxed{a(u,\ v) = (f,\ v) + (g,\ v)_{\partial\Omega}, \quad \forall v \in \mathbb{H}^1(\Omega).} \qquad (5.2.10)$$

6. u does not have to be in $\mathbb{C}^2(\overline{\Omega})$ to satisfy such a variational equation, $u \in \mathbb{H}^1(\Omega)$ makes sense.
满足(5.2.6)的u不必是二次连续可微，只要是H^1的.

A Variational Form of the Neumann BVP of the Poisson Equation

Definition 5.7

$u \in \mathbb{H}^1(\Omega)$ is said to be a weak solution of the Neumann BVP of the Poisson equation, if it satisfies

$$a(u, v) = (f, v) + (g, v)_{\partial\Omega}, \quad \forall v \in \mathbb{H}^1(\Omega), \tag{5.2.10}$$

which is called the variational form (or weak form) of the Neumann BVP of the Poisson equation.

1. Obviously, the classical solution, if exists, is a weak solution.
2. Here, both the trial and test function spaces are $\mathbb{H}^1(\Omega)$.
   trial function--试探函数; test function--检验函数
3. If u is a solution, then, $u + const.$ is also a solution.
4. Taking $v \equiv 1$ as a test function, we obtain a necessary condition for the existence of a solution

$$\boxed{\int_\Omega f\, dx + \int_{\partial\Omega} g\, ds = 0.} \tag{5.2.11}$$

The Relationship Between Weak and Classical Solutions

Theorem 5.10

(1) *Let $f \in \mathbb{C}(\overline{\Omega})$ and $g \in \mathbb{C}(\partial\Omega)$. If $u \in \mathbb{C}^2(\overline{\Omega})$ is a classical solution of the Neumann boundary value problem of the Poisson equation, then, it must also be a weak solution of the problem.*

(2) *On the other hand, if u is a weak solution of the Neumann boundary value problem of the Poisson equation, and in addition $u \in \mathbb{C}^2(\overline{\Omega})$, then it must also be a classical solution of the problem.*

- The classical solution, if exists, is a weak solution, follows directly from the derivation of the variational form of the problem.
- We only need to show the second part of the theorem.

Proof of Weak Solution + $u \in \mathbb{C}^2(\overline{\Omega}) \Rightarrow$ Classical Solution

1. Let u be a weak solution, and $u \in \mathbb{C}^2(\overline{\Omega})$.
2. By the Green's formula,
$$\int_\Omega (\triangle u + f)\, v\, dx = 0, \quad \forall v \in \mathbb{C}_0^\infty(\Omega).$$
Variational lemma
3. $\triangle u + f$ is continuous $\Rightarrow -\triangle u = f$, $\forall x \in \Omega$.
4. By this and by the Green's formula, we have
$$\int_{\partial\Omega} (\partial_\nu u - g)\, v\, ds = 0, \quad \forall v \in \mathbb{C}^\infty(\overline{\Omega}).$$
Variational lemma
5. $(\partial_\nu u - g)$ is continuous $\Rightarrow \partial_\nu u = g$, $\forall x \in \partial\Omega$.
6. u is a classical solution of the Neumann BVP of the Poisson equation. ∎

Existence of Weak Solutions for the Neumann BVP of Poisson Eqn.

P199 **Theorem 5.12**

(1) *Let Ω be a bounded connected domain with Lipschitz continuous boundary $\partial\Omega$. Let $f \in \mathbb{L}^2(\Omega)$ and $g \in \mathbb{L}^2(\partial\Omega)$ satisfy the relation $\int_\Omega f\,dx + \int_{\partial\Omega} g\,ds = 0$. Let $\mathbb{V}_0 = \left\{u \in \mathbb{H}^1(\Omega) : \int_\Omega u\,dx = 0\right\}$, and $F : \mathbb{V}_0 \to \mathbb{R}$ be defined by $F(v) = (f,\ v) + (g,\ v)_{\partial\Omega}$. Then, the* (5.2.16)
variational problem

$$\begin{cases} \text{Find } \ u \in \mathbb{V}_0 \text{ such that} \\ a(u,v) = F(v), \quad \forall v \in \mathbb{V}_0, \end{cases} \tag{5.2.17}$$

has a unique solution, which is a weak solution of the Neumann BVP of the Poisson equation. (2) *On the other hand, if u is a weak solution of the Neumann BVP of the Poisson equation, then $\tilde{u} \triangleq u - \frac{1}{\mathrm{meas}\Omega}\int_\Omega u\,dx \in \mathbb{V}_0$ is a solution to the above variational problem.*

- The second part of the theorem is left as an exercise.

Proof of the Existence Theorem for the Neumann BVP of Poisson Eqn.

To prove the first part of the theorem, we need to show

- $a(\cdot, \cdot)$ is a continuous, $\mathbb{V}_0$-elliptic bilinear form on $\mathbb{V}_0$.
- $F(v)$ is a continuous linear form on $\mathbb{V}_0$.
- If u is a solution of the variational problem, then, it is also a weak solution of the Neumann BVP of the Poisson equation.

■

The second and third claims above can be verified by definitions, and are left as exercises.

The key to the first claim is to show the $\mathbb{V}_0$-ellipticity of $a(\cdot, \cdot)$ on $\mathbb{V}_0 := \left\{u \in \mathbb{H}^1(\Omega) : \int_\Omega u\, dx = 0\right\}$, *i.e.* $|u|_{1,2,\Omega} \geq \gamma_0 \, \|u\|_{1,2,\Omega}$, for some constant $\gamma_0 > 0$. In fact, we have the following stronger result.

Poincaré-Friedrichs Inequality on $\mathbb{H}^1(\Omega)$

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Theorem 5.11

Let Ω be a bounded connected domain with Lipschitz continuous boundary $\partial\Omega$. Then, there exist constants $\gamma_1 \geq \gamma_0 > 0$ such that

m=1, p=2
$$\gamma_0 \|u\|_{1,2,\Omega} \leq \underline{\left| \int_\Omega u \, dx \right| + |u|_{1,2,\Omega}} \leq \gamma_1 \|u\|_{1,2,\Omega}, \quad \forall u \in \mathbb{H}^1(\Omega). \tag{5.2.12}$$

The inequality is also named as the Poincaré-Friedrichs inequality.

Poincaré-Friedrichs Inequality on $\mathbb{H}^1(\Omega)$

Remarks:

1. The Poincaré-Friedrichs inequality given in Theorem 5.4 is on $\mathbb{W}_0^{m,p}(\Omega)$. P190
2. Another form of the ~~Poincaré~~-Friedrichs inequality, in which $\left|\int_\Omega u\,dx\right|$ is replaced by $\|u\|_{0,2,\partial\Omega_0}$, is given in Exercise 5.6. P204
3. The Poincaré-Friedrichs inequality in a more general form on $\mathbb{W}^{m,p}(\Omega)$ can also be given.

Proof of the Poincaré-Friedrichs Inequality on $\mathbb{H}^1(\Omega)$

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(1) 1. The Schwarz inequality $\Rightarrow$ the second inequality.

(2)反证第一不等式 2. Assume the first doesn't hold, *i.e.* $\exists\ \{u_k\} \subset \mathbb{H}^1(\Omega)$, $\|u_k\|_{1,2,\Omega} \equiv 1$, s.t. $\left|\int_\Omega u_k\, dx\right| + |u_k|_{1,2,\Omega} \to 0$ as $k \to 0$. **(5.2.13)**

3. A bounded set in the Hilbert space $\mathbb{H}^1(\Omega)$ is sequentially weakly precompact, and $\mathbb{H}^1(\Omega)$ compactly embeds into $\mathbb{L}^2(\Omega)$. 准弱列紧

4. $\exists$ a subsequence $\{u_k\}$, $u \in \mathbb{H}^1(\Omega)$ and $v \in \mathbb{L}^2(\Omega)$, such that

$$u_k \rightharpoonup u, \quad \text{in } \mathbb{H}^1(\Omega); \qquad u_k \to v, \quad \text{in } \mathbb{L}^2(\Omega).$$ **(5.2.14)** **(5.2.15)**

5. $|u_k|_1 \xrightarrow{(5.2.13)} 0$ and $\|u_k - v\|_0 \xrightarrow{(5.2.15)} 0 \Rightarrow \{u_k\}$ is a Cauchy sequence in $\mathbb{H}^1(\Omega)$, therefore, $\|u_k - u\|_1 \to 0 \xRightarrow{(5.2.13)} \nabla u = 0 \Rightarrow u \equiv C$. 完备性

6. $\|u_k\|_1 \equiv 1$, $|u_k|_1 \to 0$, $\|u_k - u\|_0 \to 0 \Rightarrow \|u\|_0 = 1 \Rightarrow C \neq 0$.

7. $\left|\int_\Omega u_k\, dx\right| \to 0$ and $\|u_k - u\|_0 \to 0 \Rightarrow \int_\Omega u\, dx = 0 \Rightarrow$ $C\,\mathrm{meas}(\Omega) = \int_\Omega u\, dx = 0 \Rightarrow C = 0$, a contradiction. ■

Rellich theorem: H1中的任何有界集在L2中是准紧或预紧的(precompact).

** 在拓扑空间中，一个集合为准紧的，如果该集合的闭包为紧集.

** 在完备的度量空间中，一个集合是准紧的当且仅当任何一个点列都有Cauchy子列.

Remarks on the Derivation of Variational Forms of a PDE Problem

强制或基本BC

1. Coercive (or essential) boundary conditions: those appear in the admissible function space of the variational problem.

前面讨论中的Dirichlet BC

自然BC

2. Natural boundary conditions: those appear in the variational equation (or functional) of the variational problem.

前面讨论中的Neumann BC

基本函数空间

3. The underlying function space: determined by the highest order derivatives of the trial function u in $a(\cdot,\cdot)$.

试探函数

4. The trial function space: all functions in the underlying function space satisfying the coercive boundary condition.

检验函数

5. The test function space: u in the underlying function space, with $u = 0$ on the coercive boundary.

Remarks on the Derivation of Variational Forms of a PDE Problem

变分方程

6. The variational equation: obtained by using smooth test functions on the PDE, applying the Green's formula, and coupling the natural boundary condition.

7. Recall the BVPs of the Poisson equation.

8. $-\triangle u = f \quad \Rightarrow \quad \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} v \, \partial_\nu u \, dx = \int_\Omega fv \, dx.$

9. The underlying function space is $\mathbb{H}^1(\Omega)$.

10. $\mathbb{H}^1(\Omega) \hookrightarrow \mathbb{L}^2(\partial\Omega_0)$, $u|_{\partial\Omega_0}$ is well defined in $\mathbb{L}^2(\partial\Omega_0)$, however $u|_{\partial\Omega_0}$ does not appear in the boundary integral, therefore, the Dirichlet boundary condition on $\partial\Omega_0$ is coercive.

    Dirichlet BC是强制BC或基本BC

Remarks on the Derivation of Variational Forms of a PDE Problem

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11. $\partial_\nu u|_{\partial\Omega_1}$, which appears in the boundary integral, is not well defined in $\mathbb{L}^2(\partial\Omega_1)$ in general, therefore the 2nd and 3rd type boundary conditions appear as natural boundaries.
第2,3类BC是自然BC
12. The trial function space $\mathbb{V}(\bar{u}_0; \partial\Omega_0)$; the test one $\mathbb{V}(0; \partial\Omega_0)$.
13. The variational equation ($\partial_\nu u = g - \beta u$ on $\partial\Omega_1$):
$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega_1} \beta uv \, dx = \underbrace{\int_\Omega fv \, dx + \int_{\partial\Omega_1} gv \, dx}_{F(v)}.$$
14. The variational form of the problem:
$$\begin{cases} \text{Find} \quad u \in \mathbb{V}(\bar{u}_0; \partial\Omega_0) \text{ such that} \\ a(u, v) = F(v), \quad \forall v \in \mathbb{V}(0; \partial\Omega_0), \end{cases}$$

A Mixed Variational Form of the Dirichlet BVP of the Poisson Equation

5.3节(P 200)

1. The Poisson equation $-\triangle u = f$ can be transformed into an equivalent system of 1st order PDEs:
$$\begin{cases} p_i = \partial_i u, \quad i = 1, \ldots, n, \\ -\sum_{i=1}^{n} \partial_i p_i = f, \end{cases} \qquad\qquad x \in \Omega.$$
2. Take test functions $\mathbf{q} = (q_1, \ldots, q_n)$, $q_i \in \mathbb{C}^\infty(\overline{\Omega})$, $i = 1, \ldots, n$, and $v \in \mathbb{C}^\infty(\overline{\Omega})$.
3. By the Green's formula (applying to the integral of $\nabla u \cdot \mathbf{q}$) and the boundary condition, we see that the underlying function spaces for $\mathbf{p}$ and u are $(\mathbb{H}^1(\Omega))^n$ and $\mathbb{L}^2(\Omega)$ respectively.

*一个PDE定解问题可以有不同形式的变分形式，原BC在不同变分形式中扮演的角色会不同.

A Mixed Variational Form of the Dirichlet BVP of the Poisson Equation

4. let ν be the outward unit normal vector, and

$$a(\mathbf{p}, \mathbf{q}) = \int_\Omega \mathbf{p} \cdot \mathbf{q} \, dx = \int_\Omega \sum_{i=1}^n p_i \, q_i \, dx,$$

$$b(\mathbf{q}, u) = \int_\Omega u \, \mathrm{div}\,(\mathbf{q}) \, dx = \int_\Omega u \sum_{i=1}^n \partial_i q_i \, dx,$$

$$G(\mathbf{q}) = \int_{\partial\Omega} \bar{u}_0 \, \mathbf{q} \cdot \nu \, ds = \int_{\partial\Omega} \bar{u}_0 \sum_{i=1}^n q_i \, \nu_i \, ds,$$

$$F(v) = -\int_\Omega f \, v \, dx.$$

A Mixed Variational Form of the Dirichlet BVP of the Poisson Equation

5. Since $u \in \mathbb{L}^2(\Omega)$, $u|_{\partial\Omega}$ doesn't make sense in general, the term $\int_{\partial\Omega} \bar{u}_0\, \mathbf{q} \cdot \nu\, ds$ should be kept in the variational equation, .i.e. the Dirichlet boundary condition appears as a natural boundary condition in this case.
6. Thus, we obtain the following variational problem:

$$\begin{cases} \text{Find} \quad \mathbf{p} \in (\mathbb{H}^1(\Omega))^n, \ u \in \mathbb{L}^2(\Omega) \text{ such that} \\ a(\mathbf{p}, \mathbf{q}) + b(\mathbf{q}, u) = G(\mathbf{q}), \quad \forall \mathbf{q} \in (\mathbb{H}^1(\Omega))^n, \\ \qquad\qquad\quad b(\mathbf{p}, v) = F(v), \quad \forall\, v \in \mathbb{L}^2(\Omega). \end{cases} \tag{5.3.6}$$

Remark: Neumann boundary condition will appear as a coercive boundary condition. (Robin boundary condition does not apply here. Why?)

=>Hellinger-Reissner泛函(5.3.7)的驻点问题(5.3.8).

P202 Th5.14(Hellinger-Reissner原理)给出原问题(5.3.3-4)和变分问题(5.3.6 or 8)之间的关系.

Another Mixed Variational Form of the Dirichlet BVP of the Poisson Equation

1. If we apply the Green's formula to transform $-\int_\Omega \mathrm{div}\mathbf{p}\, v\, dx$ into the form $-\int_{\partial\Omega} v\,\mathbf{p}\cdot\nu\, ds + \int_\Omega \mathbf{p}\cdot\nabla v\, dx$ instead, then, the underlying function spaces for $\mathbf{p}$ and u are $(\mathbb{L}^2(\Omega))^n$ and $\mathbb{H}^1(\Omega)$ respectively.

2. Since $u \in \mathbb{H}^1(\Omega)$, $u|_\Omega$ makes sense, while $\mathbf{p} \in (\mathbb{L}^2(\Omega))^n$, the term $-\int_{\partial\Omega} v\,\mathbf{p}\cdot\nu\, ds$ doesn't make sense. Therefore, the Dirichlet boundary condition appears as a coercive boundary condition, while the Neumann and Robin boundary conditions appear as natural boundary conditions here in this case.

Another Mixed Variational Form of the Dirichlet BVP of the Poisson Equation

3. We have the variational problem

$$\begin{cases} \text{Find } \mathbf{p} \in (\mathbb{L}^2(\Omega))^n, \ u \in \mathbb{H}^1(\Omega), \ u|_{\partial\Omega} = \bar{u}_0 \text{ such that} \\ a(\mathbf{p}, \mathbf{q}) + b^*(\mathbf{q}, u) = 0, \qquad \forall \mathbf{q} \in (\mathbb{L}^2(\Omega))^n, \\ \qquad b^*(\mathbf{p}, v) = F(v), \qquad \forall\, v \in \mathbb{H}^1_0(\Omega), \end{cases} \tag{5.3.9}$$

where

$$b^*(\mathbf{q}, u) = -\int_\Omega \mathbf{q} \cdot \nabla u \, dx = -\int_\Omega \sum_{i=1}^n q_i \partial_i u \, dx.$$

Remarks on the Mixed Variational Forms of BVP of the Poisson Eqn.

1. The classical solution is also a solution to the mixed variational problem (named again as the weak solution).
2. Weak solution + $u \in \mathbb{C}^2(\bar{\Omega})$, $\mathbf{p} \in (\mathbb{C}^1(\bar{\Omega}))^n \Rightarrow$ u is a classical solution.
3. The weak mixed forms have their corresponding functional extremum problems.

Th5.15(Brezzi定理)中条件(2), P203,229,261

4. Under the so called B-B conditions, the weak mixed variational problems can be shown to have a unique stable solution.

B-B条件是指Babuska-Brezzi条件.
The conditions for the well posedness for variational form in mixed form of Stokes eq. is known as inf-sup condition or Ladyzhenskaya-Babuska-Breezi (LBB) condition, see [https://www.math.uci.edu/~chenlong/226/infsup.pdf]

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Thank You!