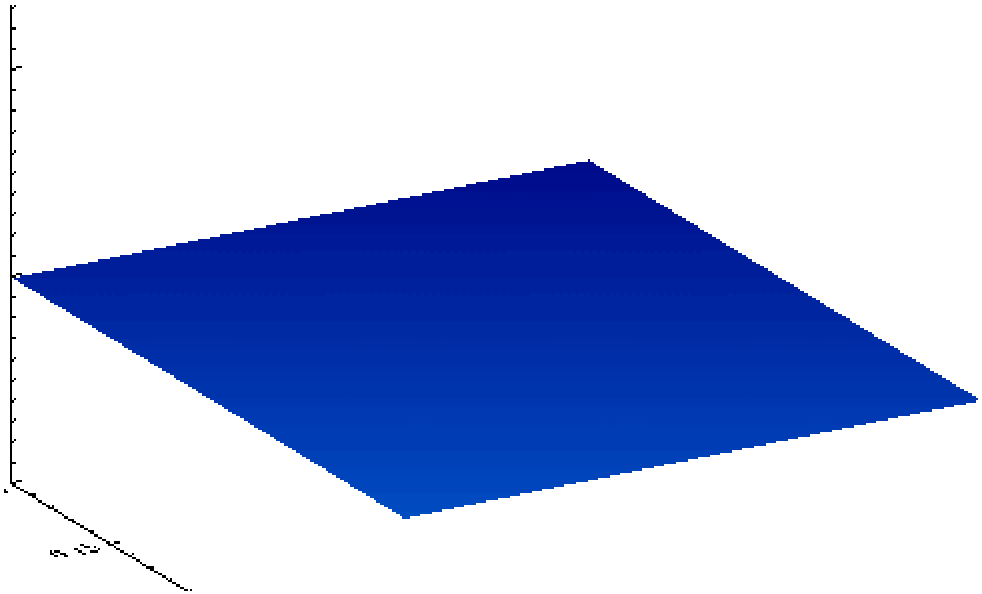
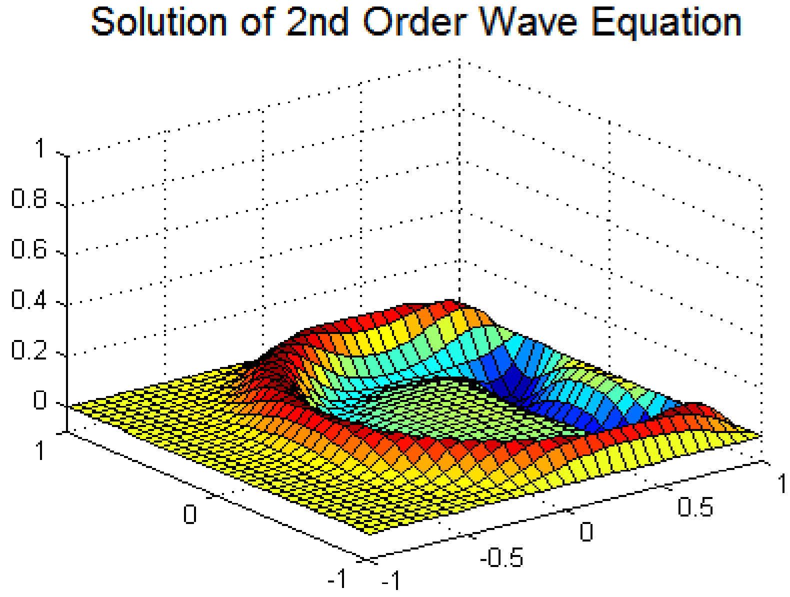
Wave equation

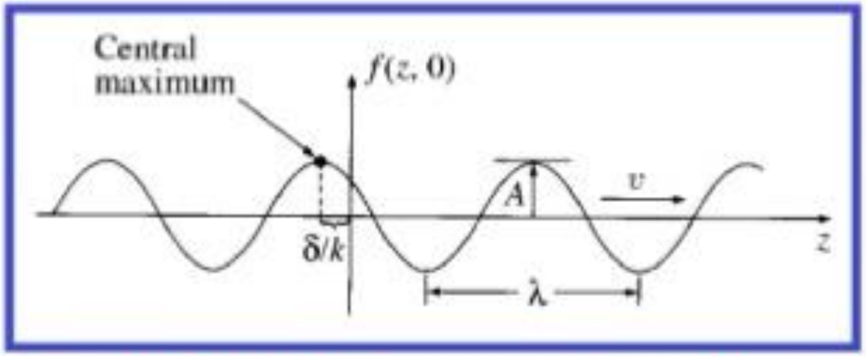




Waves in One Dimension

Sinusoidal Waves 正弦波

Terminology



$$f(z,t) = A\cos[k(z - vt) + \delta]$$

$$f(z,t) = A\cos(kz - \omega t + \delta)$$

Complex Notation. In view of **Euler's Formula**

$$e^{i\theta} = \cos\theta + i\sin\theta \longrightarrow f(z,t) = \mathrm{Re}[Ae^{i(kz-\omega t+\delta)}]$$

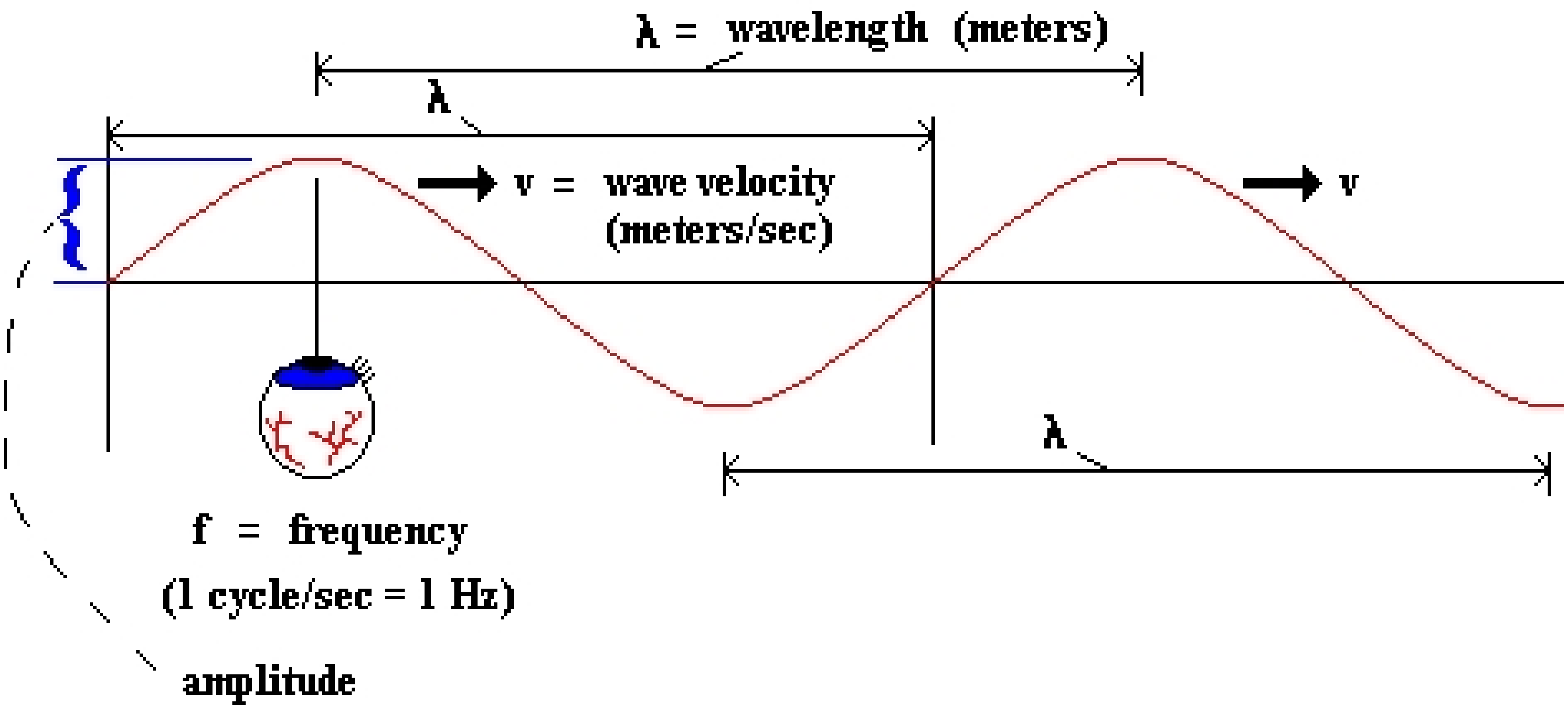
where $\mathrm{Re}(\xi)$ denotes the real part of the complex number ξ

Complex Wave Function 复的波函数

$$\boxed{\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)}}$$

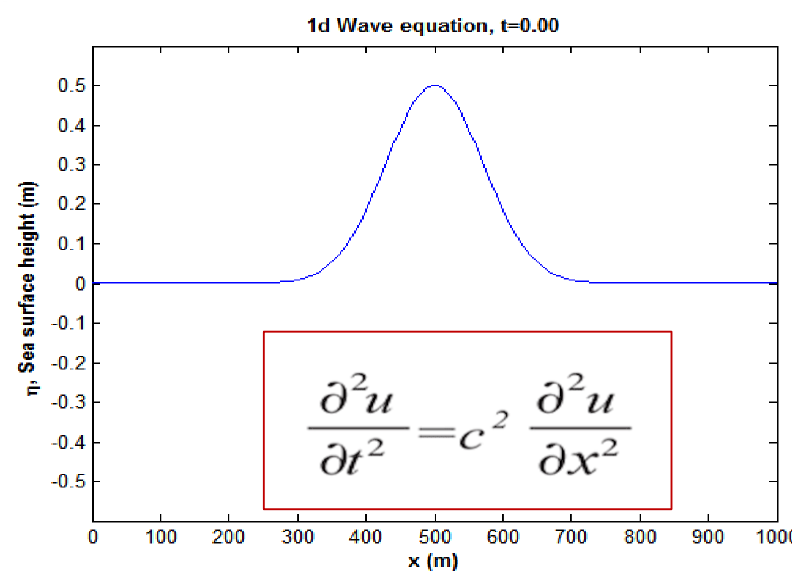
Linear Combinations of sinusoidal Waves

$$\boxed{\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k)e^{i(kz-\omega t)}\,dk.}$$



basic wave equation

$$\boxed{v = \lambda f}$$



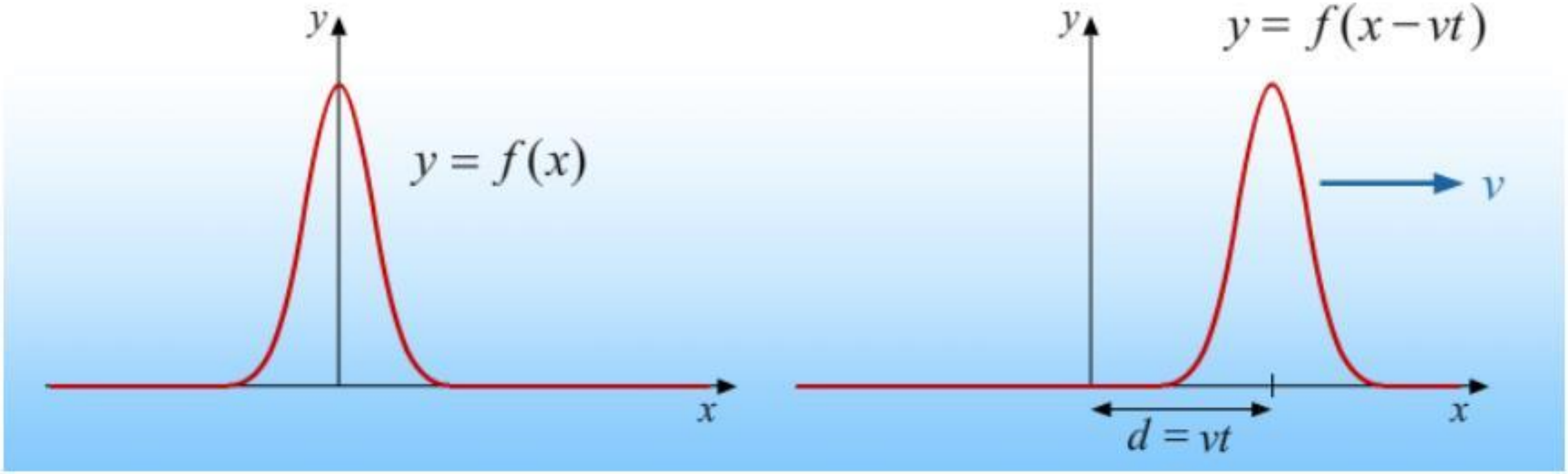
The Wave Equation

Wave Equation

$$\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2}$$

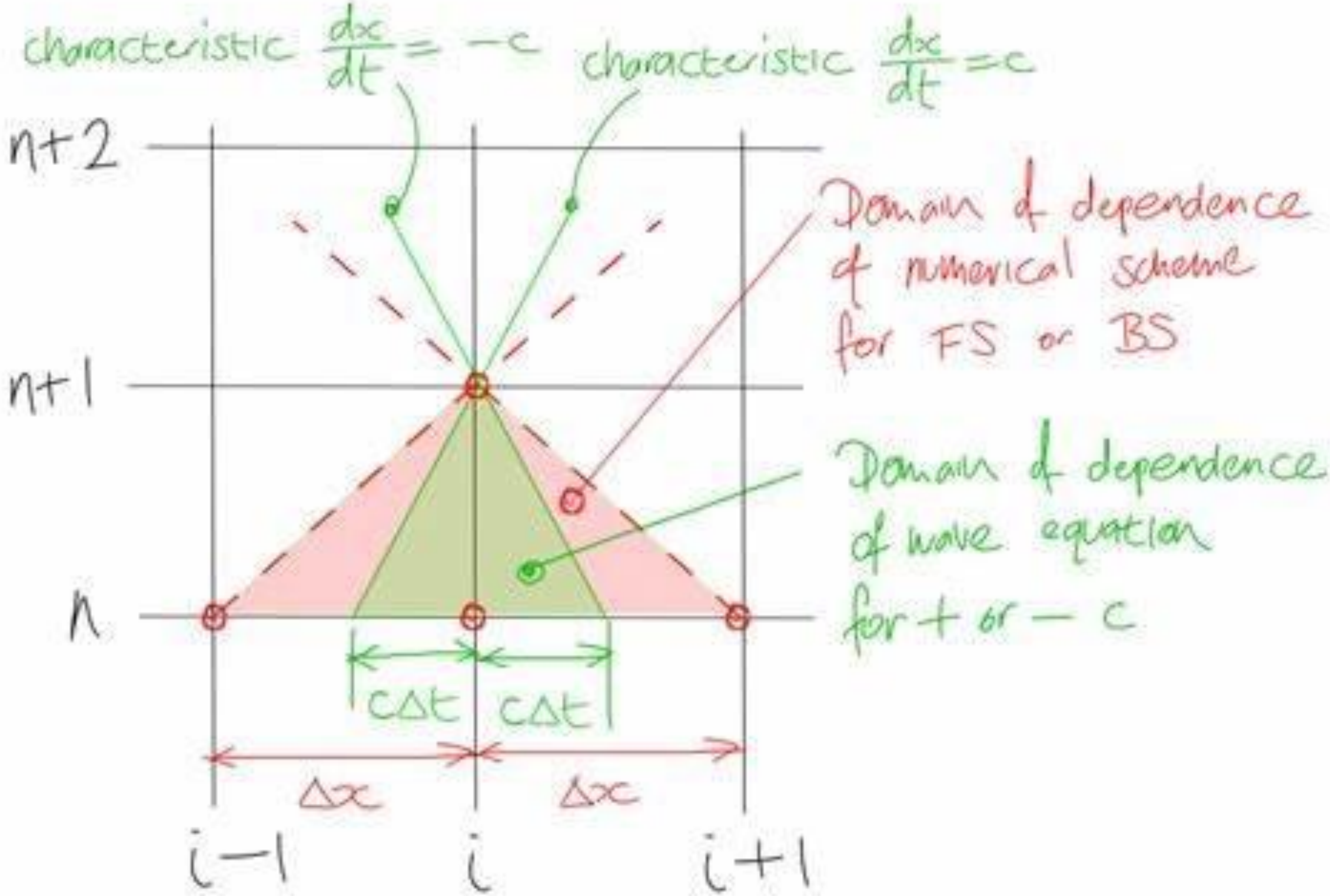
Solutions to the Wave Equation

$$y(x,t) = f(x - vt)$$



Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

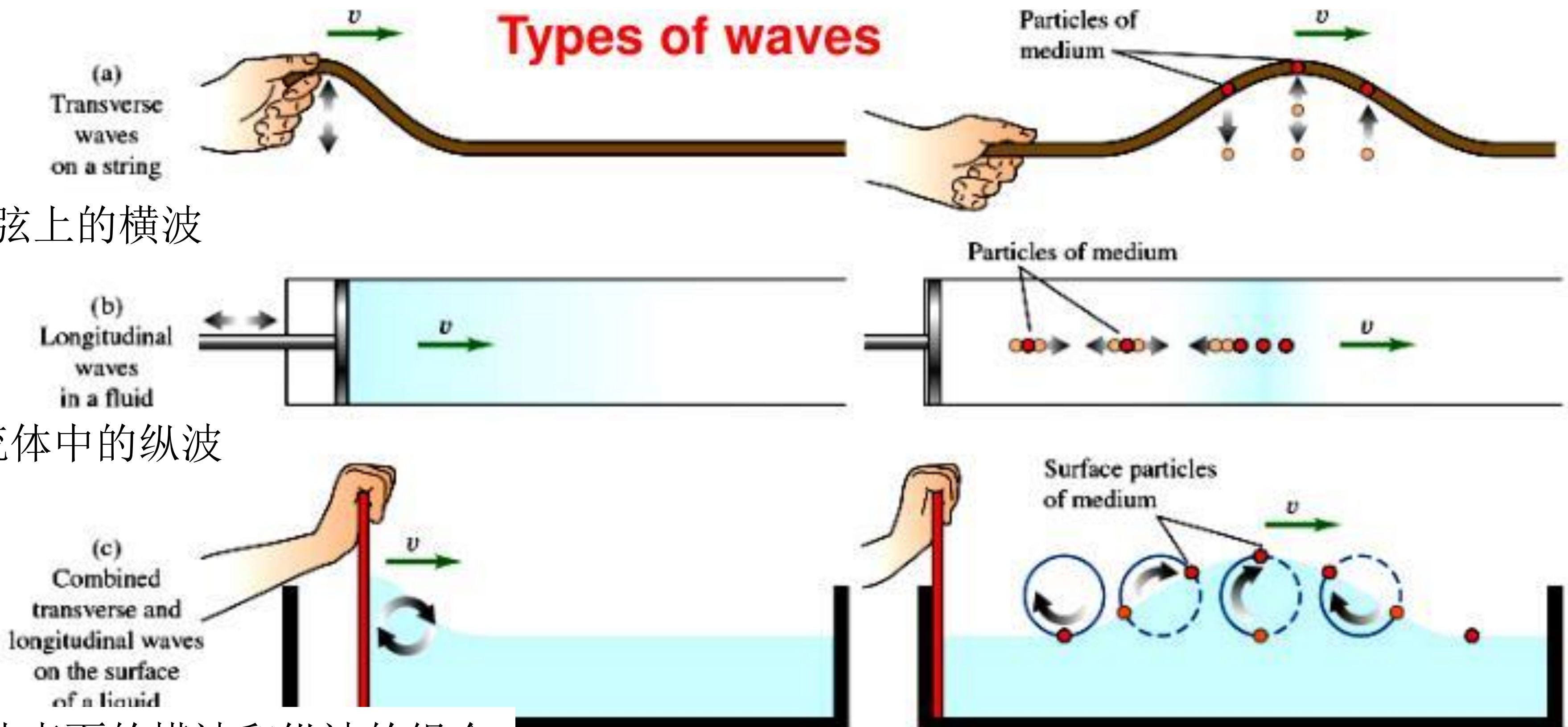


Mechanical Waves and Wave Equation

波是在介质或真空中传播的非局部扰动.

波在没有物质的大量流动(bulk flow)的情况下把能量从一个地方带到另一个地方.

机械波是介质中粒子位置的波扰动.



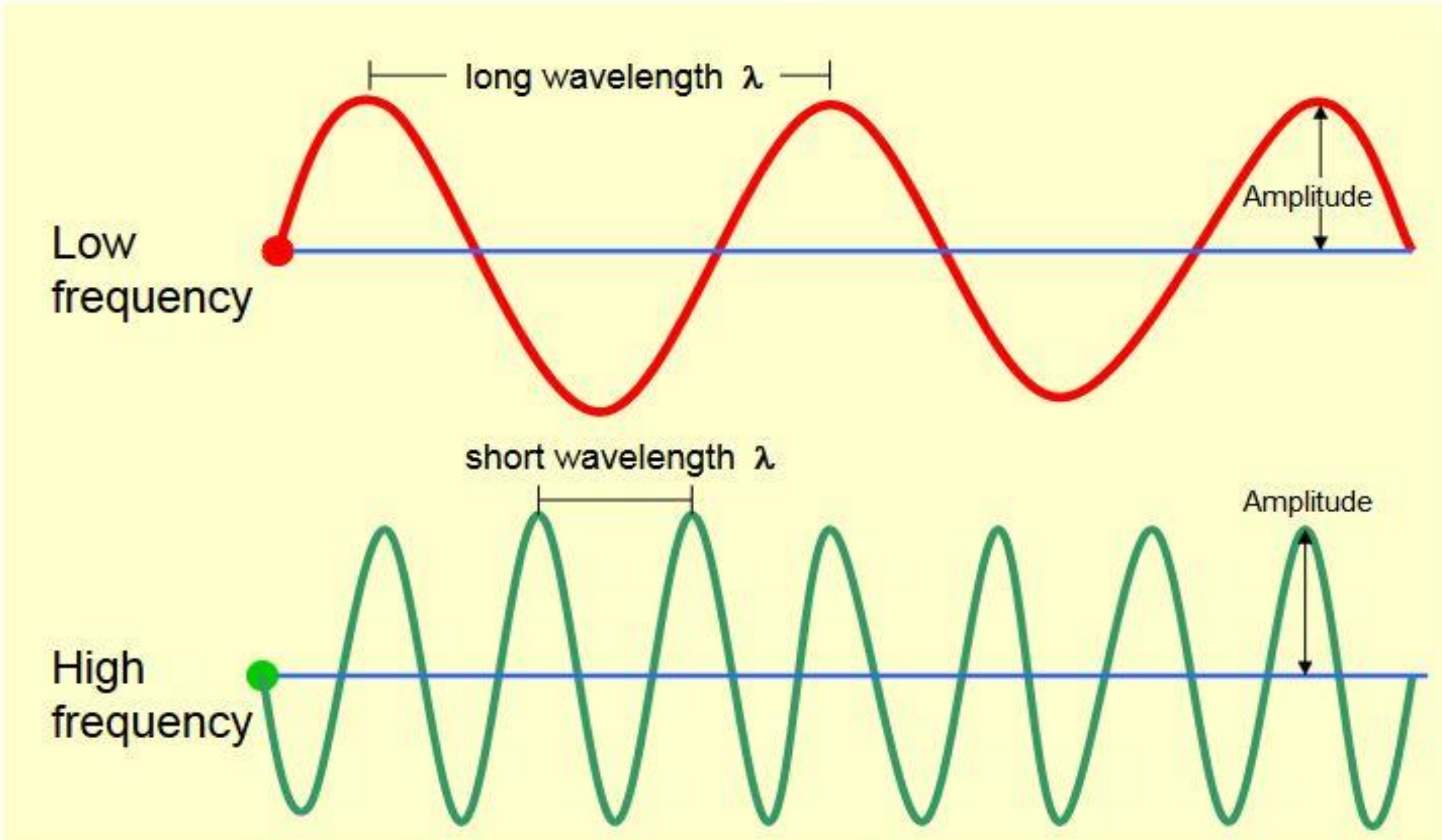
弦上的横波

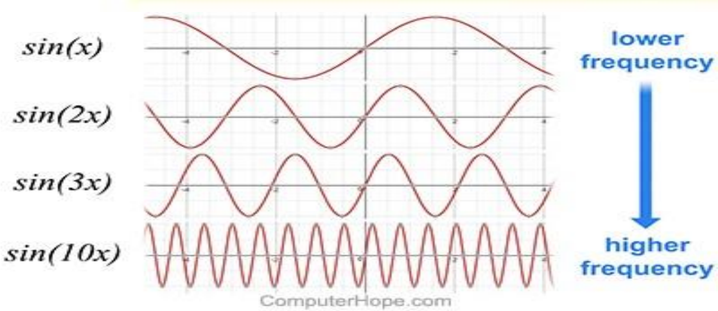
流体中的纵波

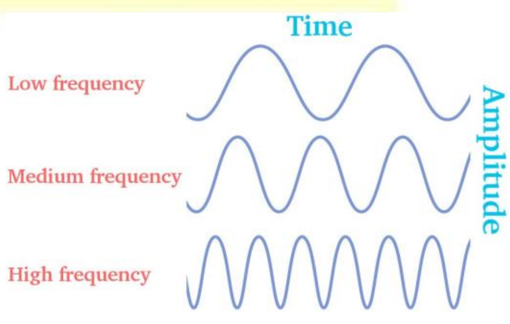
液体表面的横波和纵波的组合

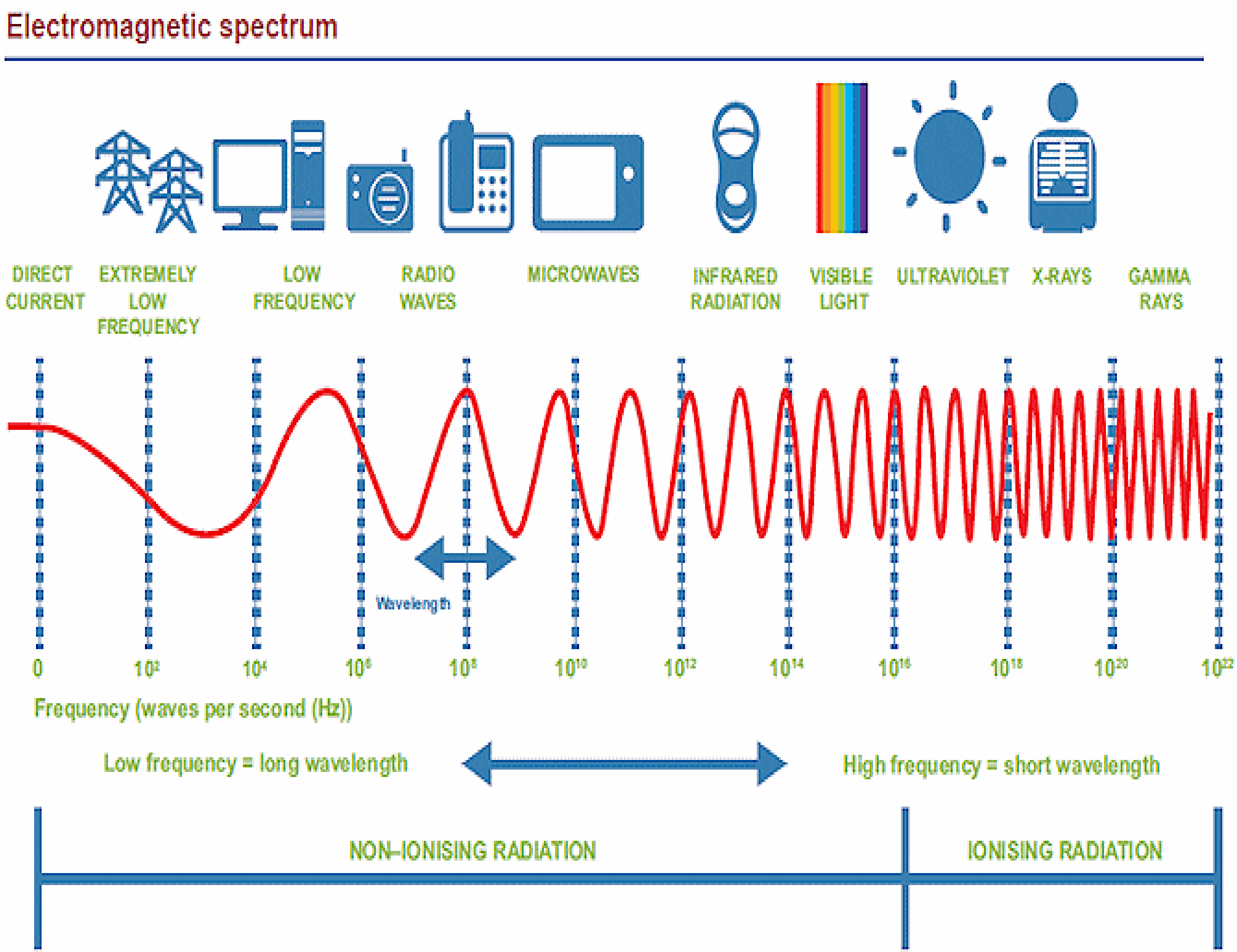
Electromagnetic waves (light), plasma waves, gravitational waves, …

Waves


long wavelength λ
Amplitude
Low frequency
short wavelength λ
Amplitude
High frequency

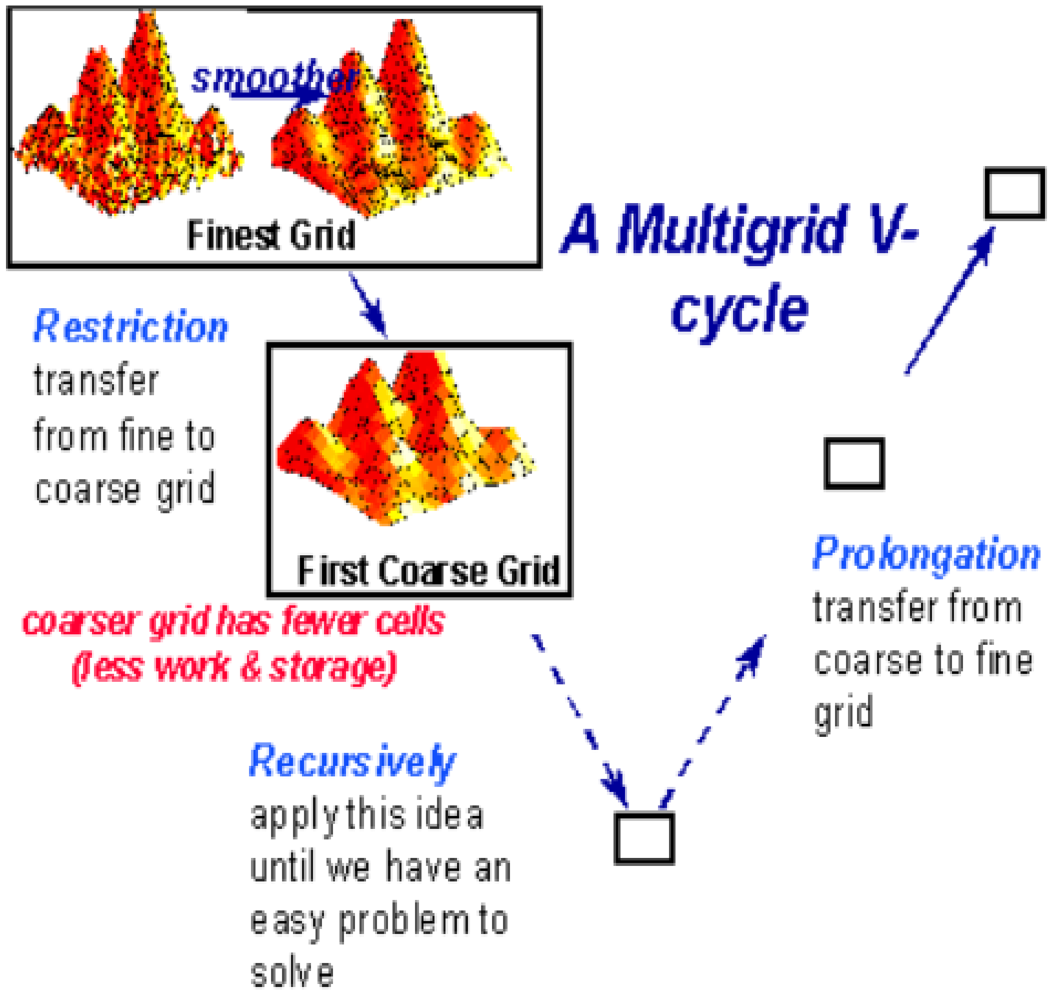

sin(x)
sin(2x)
sin(3x)
sin(10x)
ComputerHope.com
lower frequency
higher frequency


Time
Low frequency
Medium frequency
High frequency
Amplitude


Electromagnetic spectrum
DIRECT CURRENT
EXTREMELY LOW FREQUENCY
LOW FREQUENCY
RADIO WAVES
MICROWAVES
INFRARED RADIATION
VISIBLE LIGHT
ULTRAVIOLET
X-RAYS
GAMMA RAYS
Wavelength
0
10²
10⁴
10⁶
10⁸
10¹⁰
10¹²
10¹⁴
10¹⁶
10¹⁸
10²⁰
10²²
Frequency (waves per second (Hz))
Low frequency = long wavelength
High frequency = short wavelength
NON–IONISING RADIATION
IONISING RADIATION

Multigrid method

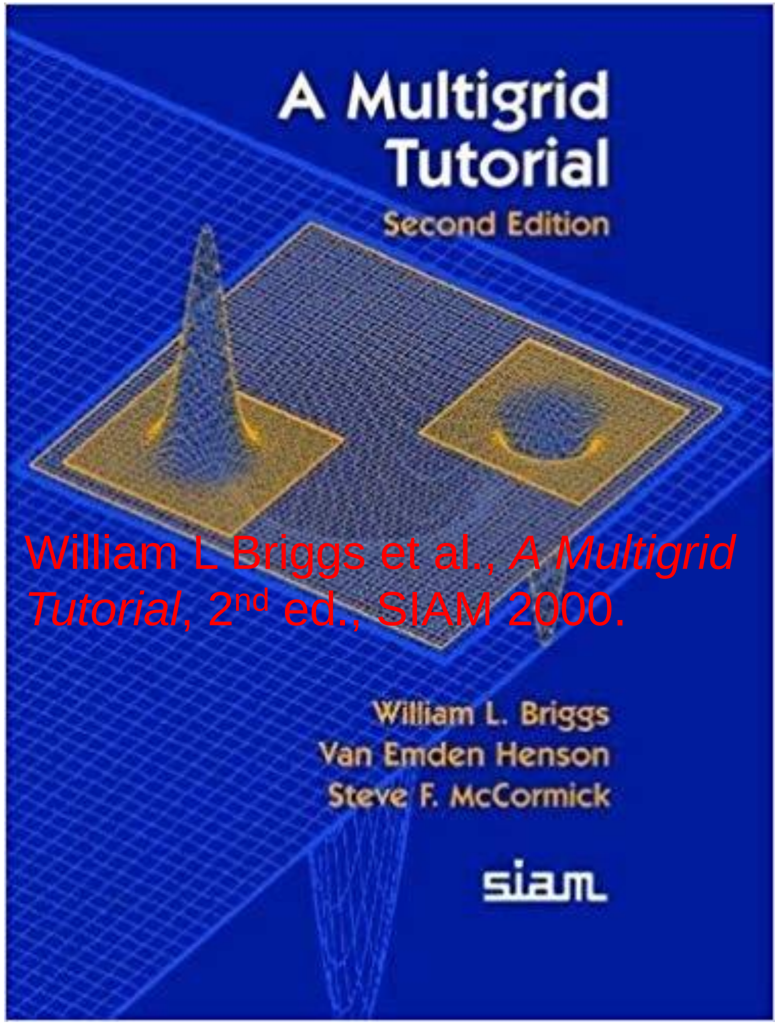
- MG uses a relaxation method to damp high-frequency error
- Low-frequency error can be accurately and efficiently solved for on coarser grid
- Recursive application to each consecutive system of leads to MG V-cycle.



Multigrid method

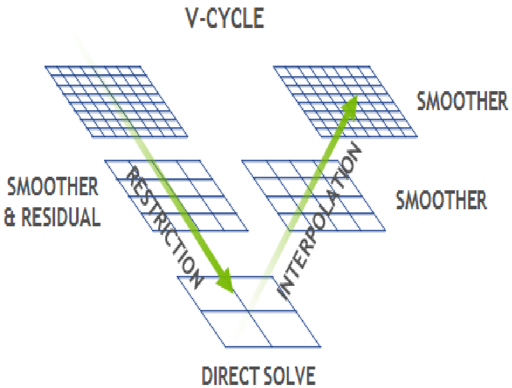
Solution Methods: Iterative Techniques (PDF)	
Iterative Methods: Multigrid Techniques (PDF)	

https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-920j-numerical-methods-for-partial-differential-equations-sma-5212-spring-2003/lecture-notes/lec7_notes.pdf



William L Briggs et al., *A Multigrid Tutorial*, 2nd ed., SIAM 2000.

Standard Multigrid V-cycle



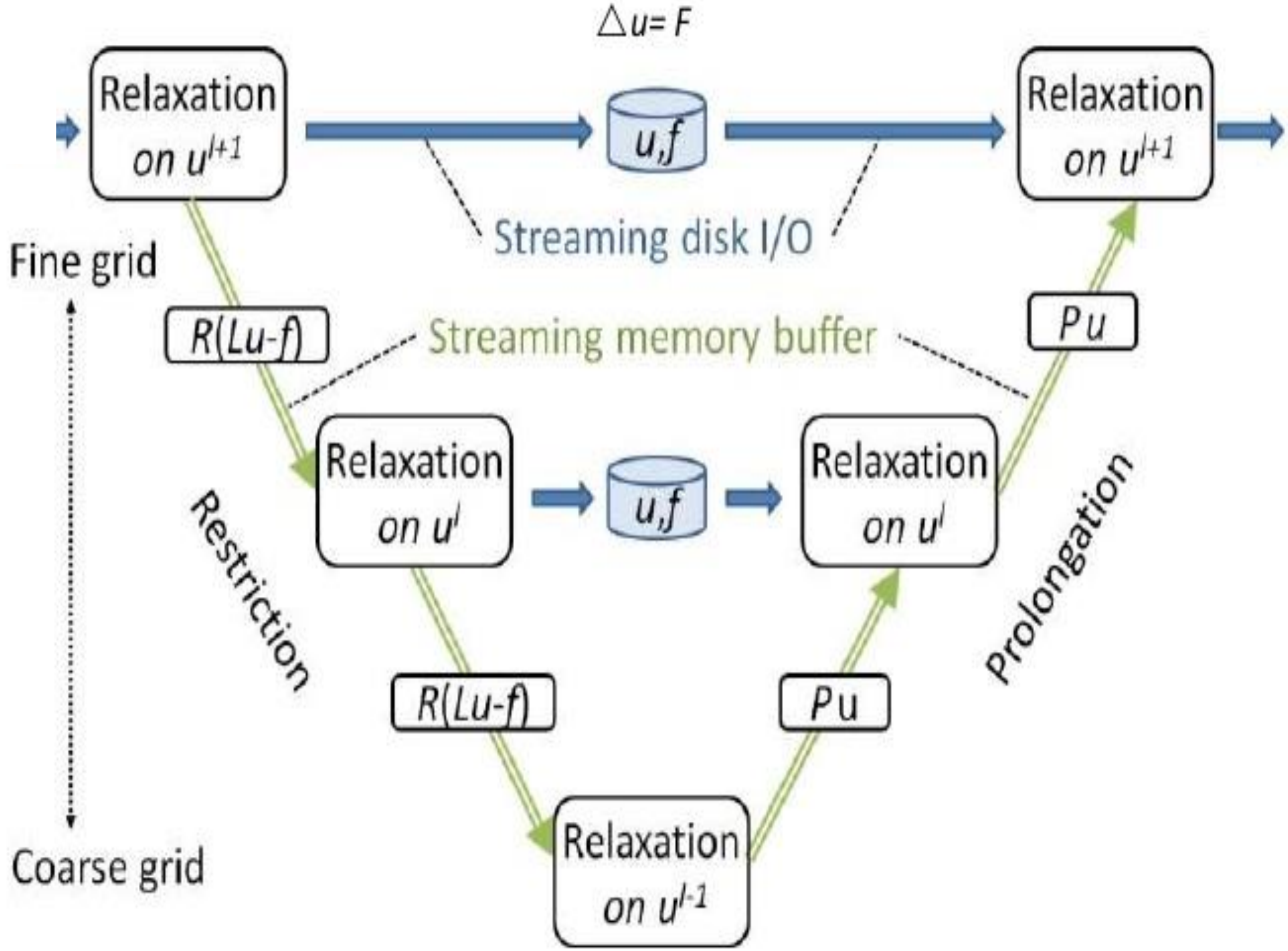
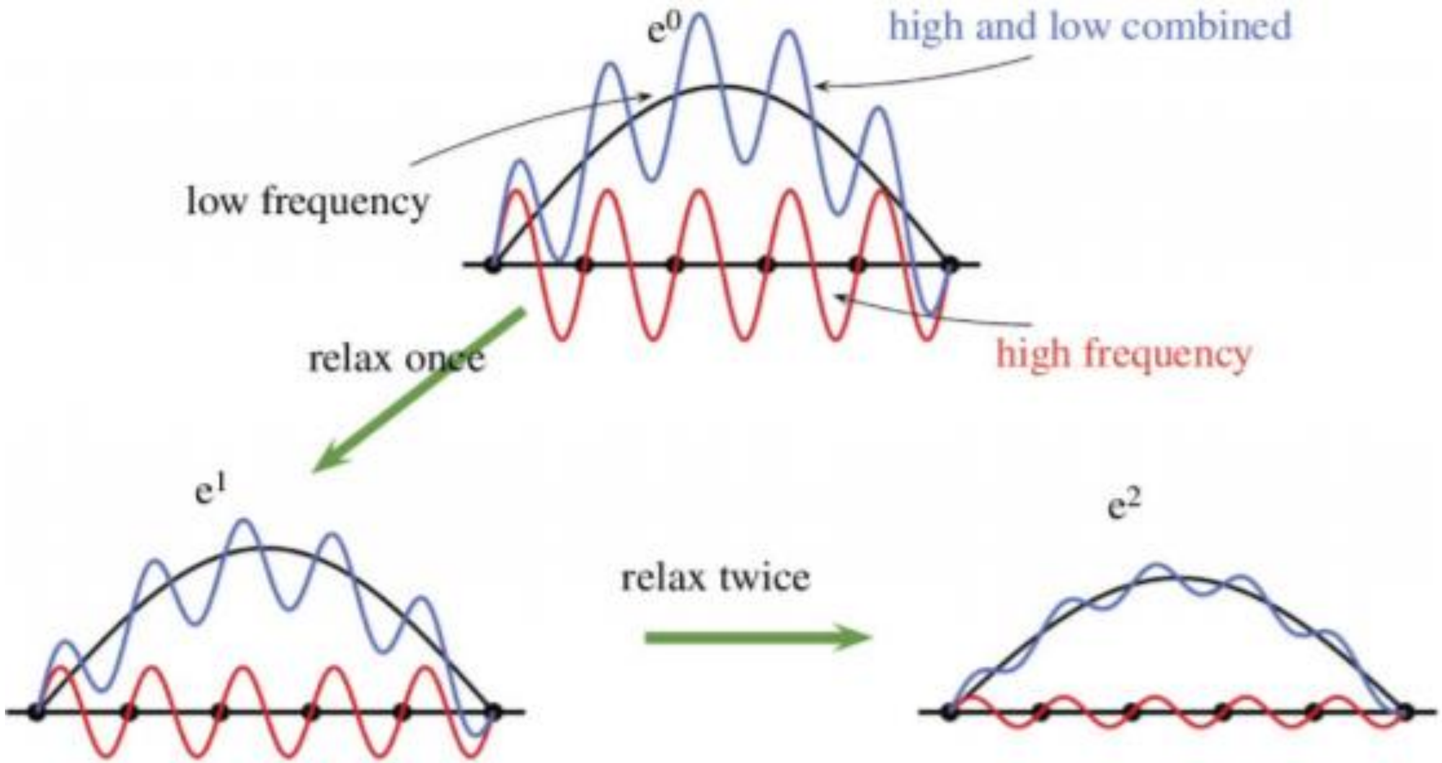


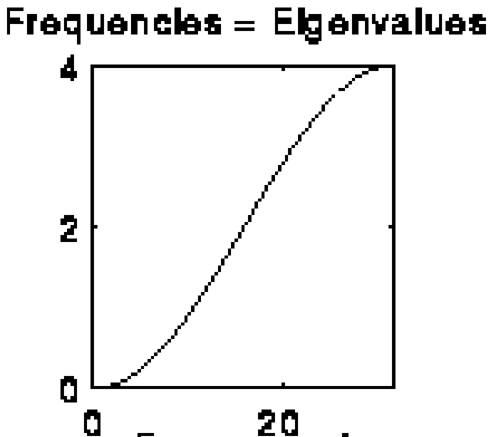
Figure 1 of "Streaming Multigrid for Gradient-Domain Operations on Large Images", SIGGRAPH 08



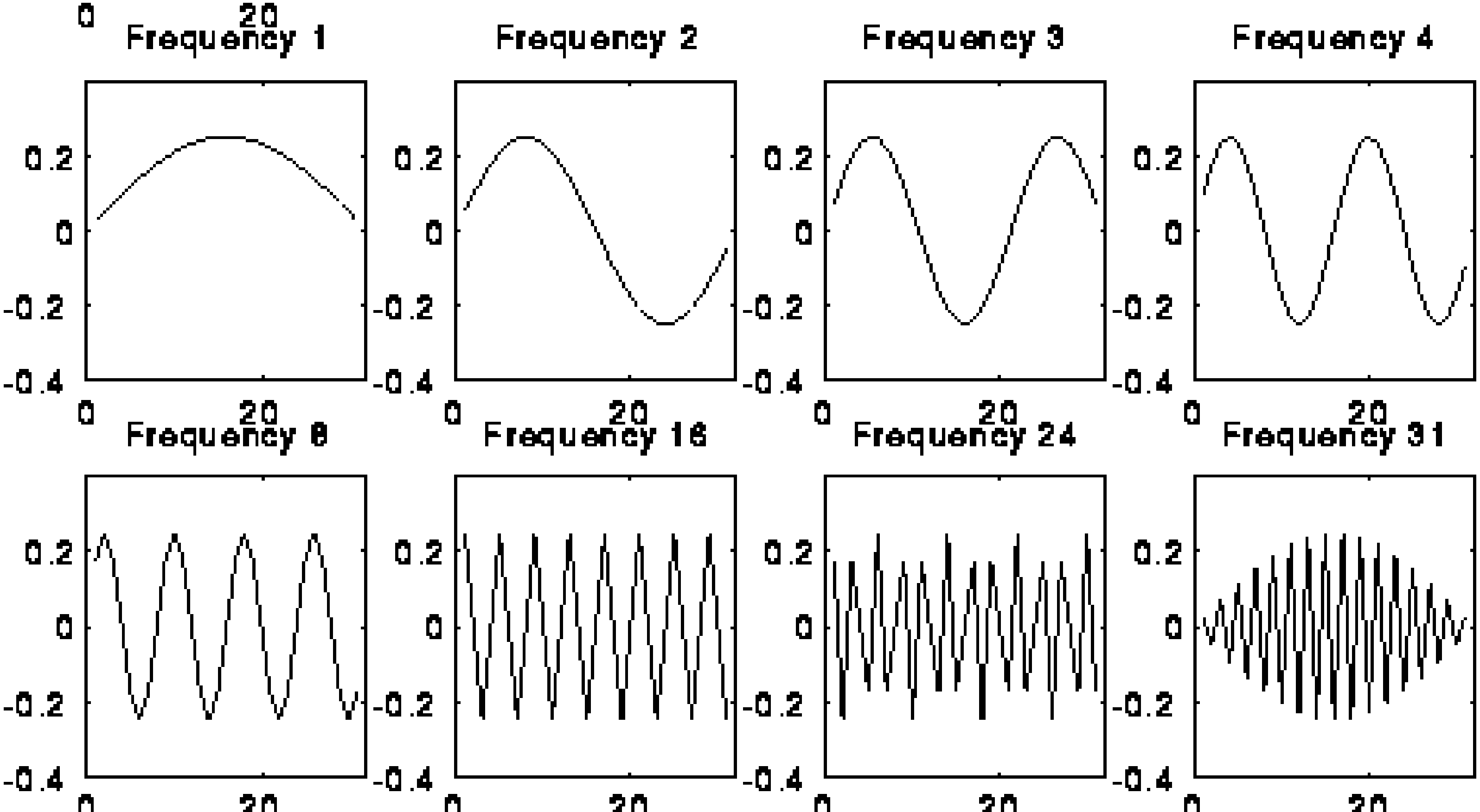
The smoothing property.

Schedule of grids for (a) V-cycle, (b) W-cycle and (c) FMV-cycle(full multigrid V cycle).

https://people.eecs.berkeley.edu/~demmel/cs267/lecture25/lecture25.html



The top plot is a plot of the eigenvalues (frequencies) of 4^5*T(5), from lowest to highest. The subsequent plots are the eigenvectors (sine-curves), starting with the lowest 4 frequencies, and then a few more frequencies up to the highest (number 31)



Semi-Lagrangian Method

Semi-Lagrangian Method

$$\boxed{\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = 0}$$

$$c_i^{n+1} = c^n_{i-u\frac{\Delta t}{\Delta x}} = \underline{c^n_{i-k-\alpha}} \qquad u\frac{\Delta t}{\Delta x} = \underline{k+\alpha} \quad k = \left[u\frac{\Delta t}{\Delta x}\right]. \tag{2.10}$$

k and α are often called the integral and the fractional Courant numbers, respectively. The expression $c^n_{i-k-\alpha}$ is to be interpreted as the value obtained from the approximate c^n values at the point $i\Delta x - u\Delta t$ by some interpolation procedure. Two interesting facts can be noticed immediately if simple linear interpolation is applied, so that

$$c_i^{n+1} = \underline{\alpha c}^n_{i-k-1} + \underline{(1-\alpha)}c^n_{i-k}. \tag{2.11}$$

First of all, if $C = u\Delta t/\Delta x < 1$, then one has in (2.11) $k = 0, C = \alpha$ and the resulting method is easily seen to be identical to the upwind method (2.3). Furthermore, it is clear that (2.11) holds for any value of the Courant number and that, since the values of the solution at the new time level $n+1$ are obtained by a linear interpolation of the values at time level n with nonnegative coefficients, the discrete maximum principle holds, i.e.

$$\min_i c_i^0 \le \min_i c_i^n \le \max_i c_i^n \le \max_i c_i^0 \tag{2.12}$$

Luca Bonaventura, An introduction to semi - Lagrangian methods for geophysical scale flows, http://www1.mate.polimi.it/~bonavent/Homepage_Luca_file/lezlag000.pdf

Semi-Lagrangian Method

for any n. This also implies stability in the maximum norm for an arbitrary Courant number. Thus, at least in a simple case the semi-Lagrangian method appear to have a great advantage over the previously reviewed Eulerian methods, since no stability condition restricts the choice of the timestep.

The semi-Lagrangian method can be easily generalized to the multidimensional case. If a constant velocity field $\mathbf{u} \in \mathbf{R}^d$ and and initial datum $c_0(x)$, $\mathbf{x} \in \mathbf{R}^d$ are considered, the multidimensional linear advection equation is given by

$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = 0. \tag{2.13}$$

As in the one dimensional case, the analytic solution is given by

$$c(\mathbf{x}, t) = c_0(\mathbf{x} - \mathbf{u}t), \tag{2.14}$$

and the semi-Lagrangian approach can be derived as in the one dimensional case, by replacing one dimensional interpolation with multidimensional interpolation techniques.

In the more general case of a space and time dependent velocity field $\mathbf{u}(\mathbf{x}, t) \in \mathbf{R}^d$, one has

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + \mathbf{u}(\mathbf{x}, t) \cdot \nabla c = 0 \tag{2.15}$$

where the usual notation dc/dt for the Lagrangian derivative has been introduced.

http://www1.mate.polimi.it/~bonavent/Homepage_Luca_file/lezlag000.pdf

Semi-Lagrangian Method

(2.17) the semi-Lagrangian methods reduces the approximation of the advection ion (2.15) to the following key steps:

- at a given time level n, compute for each mesh point $\mathbf{x}$ an approximate solution of (2.16) to determine an estimate $\mathbf{X}^*(t^n; t^{n+1}, \mathbf{x})$
- compute an approximation of (2.17) by interpolating the mesh point values at time level n at the points $\mathbf{X}^*(t^n; t^{n+1}, \mathbf{x})$.

This implies that solution of the PDE (2.15) is reduced to solution of a large set of mutually independent ODEs and to performing a multidimensional interpolation. For each of these steps, a number of classical and well studied methods is available.

$$\frac{d}{dt}\mathbf{X}(t; s, \mathbf{x}) = \mathbf{u}(\mathbf{X}(t; s, \mathbf{x}), t) \tag{2.16}$$

with initial datum at time s given by $\mathbf{X}(t; s, \mathbf{x}) = \mathbf{x}$. For smooth initial data, by the chain rule it is then possible to prove that

$$c(\mathbf{x}, t) = c_0(\mathbf{X}(0; t, \mathbf{x})). \tag{2.17}$$

Luca Bonaventura, An introduction to semi - Lagrangian methods for geophysical scale flows, http://www1.mate.polimi.it/~bonavent/Homepage_Luca_file/lezlag000.pdf