Numerical Solutions to Partial Differential Equations

numpde_lecture_11_c4.pdf

School of Mathematical Sciences
Peking University

Initial-Boundary Value Problems of Evolution Equations

考虑下列发展方程的初边值问题

$$\frac{\partial u}{\partial t}(x,t) = L(u(x,t)) + f(x,t), \qquad \forall (x,t) \in \Omega \times (0, t_{\max}], \tag{4.1.1a}$$

$$g(u(x,t)) = g_0(x,t), \qquad \forall (x,t) \in \partial\Omega_1 \times (0, t_{\max}], \tag{4.1.1b}$$

$$u(x,0) = u^0(x), \qquad \forall x \in \Omega, \tag{4.1.1c}$$

where $L(\cdot)$ is a (linear) differential operator acting on u with respect to x, and is assumed not explicitly depend on the time t.

Initial-Boundary Value Problems of Evolution Equations

Definition 4.1

An initial value problem is said to be well posed with respect to the norm $\|\cdot\|$ of a Banach space $\mathbb{X}$, if it holds

1. for any given initial data $u^0 \in \mathbb{X}$, *i.e.* $\|u^0\| < \infty$, there exists a solution;

存在唯一解

2. there is a constant $C > 0$, such that, if v, w are the solutions of the problem with initial data v^0, $w^0 \in \mathbb{X}$ respectively, then

对初值连续依赖性

$$\|v(\cdot,t) - w(\cdot,t)\| \le C\|v^0(\cdot) - w^0(\cdot)\|, \quad \forall\, t \in [0, t_{\max}]. \tag{4.1.6}$$

Remark: Similarly, we can define the well-posedness for the initial-boundary value problems.

Uniformly Well Conditioned Difference Schemes

We consider the difference scheme of the following form

$$B_1 U^{m+1} = B_0 U^m + F^m. \qquad (4.1.2)$$

1. The difference operators B_1 and B_0 are independent of m.
2. 线性差分算子 $(B_\alpha U^m)_{\mathbf{j}'} = \sum_{\mathbf{j}\in J_\Omega} b^\alpha_{\mathbf{j}',\mathbf{j}} U^m_{\mathbf{j}}, \quad \forall \mathbf{j}' \in J_\Omega, \quad \alpha = 0, 1.$ (4.1.3)
3. F^m contains information on the inhomogeneous boundary conditions as well as the source term of PDE.
4. $B_1 = O(\tau^{-1})$, B_1 is invertible, and B_1^{-1} is uniformly well conditioned, *i.e.* there exists a constant $K > 0$, such that

$$\|B_1^{-1}\| \le K\tau. \qquad (4.1.4)$$

5. Under the above assumptions, we can rewrite the difference scheme as

$$U^{m+1} = B_1^{-1}\left[B_0 U^m + F^m\right]. \qquad (4.1.5)$$

Remark: In vector case of dimension p, $U^m_{\mathbf{j}} \in R^p$, and $b^\alpha_{\mathbf{j}',\mathbf{j}} \in R^{p\times p}$.

Truncation Error of Difference Schemes

Definition 4.2

Suppose that u is an exact solution to the problem, define

$$T^m := B_1 u^{m+1} - [B_0 u^m + F^m], \tag{4.2.1}$$

as the truncation error of the scheme for the problem.

Remark: By properly scaling the coefficients of the scheme (so that $F^m = f^m$), the definition is consistent with $T^m := \{[B_1(\Delta_{+t} + 1) - B_0] - [\partial_t - L]\}u^m$.

Consistency of Difference Schemes

Definition 4.3

The difference scheme is said to be consistent with the problem, if for all sufficiently smooth exact solution u of the problem, the truncation error satisfies

$$T_{\mathbf{j}}^{m} \to 0, \qquad \text{as } \tau(h) \to 0, \quad \forall m\tau \le t_{\max}, \ \forall \mathbf{j} \in J_{\Omega}. \tag{4.2.2}$$

In particular, the difference scheme is said to be consistent with the problem in the norm $\|\cdot\|$, if

$$\tau \sum_{l=0}^{m-1} \|T^{l}\| \to 0, \qquad \text{as } \tau(h) \to 0, \quad \forall m\tau \le t_{\max}. \tag{4.2.3}$$

Order of Accuracy of Difference Schemes

Definition 4.4

The difference scheme is said to have order of accuracy p in τ and q in h, if p and q are the largest integers so that

$$|T_{\mathbf{j}}^m| \le C\left[\tau^p + h^q\right], \qquad \text{as } \tau(h) \to 0, \quad \forall m\tau \le t_{\max}, \quad \forall \mathbf{j} \in J_\Omega, \tag{4.2.4}$$

for all sufficiently smooth solutions to the problem, where C is a constant independent of $\mathbf{j}$, τ and h.

Remark: Similarly, we can define the order of accuracy of a scheme with respect to a norm $\|\cdot\|$, if

$$\tau \sum_{l=0}^{m-1} \|T^l\| \le C\left[\tau^p + h^q\right], \qquad \text{as } \tau(h) \to 0, \quad \forall m\tau \le t_{\max}.$$

Convergence of Difference Schemes

Definition 4.5

The difference scheme is said to be convergent in the norm $\|\cdot\|$ with respect to the problem, if the difference solution U^m to the scheme satisfies

$$\|U^m - u^m\| \to 0, \qquad \text{as} \;\; \tau(h) \to 0, \quad m\tau \to t \in [0, t_{\max}], \tag{4.2.5}$$

for all initial data u^0 with which the problem is well posed in the norm $\|\cdot\|$.

Remark: Similarly, we can define the order of convergence of a scheme to be p in τ and q in h with respect to the norm $\|\cdot\|$, if

$$\|U^m - u^m\| \leq C\left[\tau^p + h^q\right], \qquad \text{as} \;\; \tau(h) \to 0, \quad m\tau \to t \in [0, t_{\max}].$$

Lax-Richtmyer Stability of Finite Difference Schemes

Definition 4.6

A finite difference scheme is said to be stable with respect to a norm $\|\cdot\|$ and a given refinement path $\tau(h)$, if there exists a constant $K_1 > 0$ independent of τ, h, V^0 and W^0, such that

$$\|V^m - W^m\| \le K_1 \|V^0 - W^0\|, \quad \forall\, m\tau \le t_{\max}, \tag{4.3.1}$$

as long as $V^m - W^m$ is a solution to the homogeneous difference scheme (meaning $F^m = 0$ for all m) with initial data $V^0 - W^0$.

Stability and Uniform Well-Posedness of Finite Difference Schemes

The stability of finite difference schemes are closely related to the uniform well-posedness of the corresponding discrete problems.

For linear problems, since $V^m - W^m = B_1^{-1} B_0 (V^{m-1} - W^{m-1})$, thus, a scheme is (Lax-Richtmyer) stable if and only if

$$\| \left(B_1^{-1} B_0 \right)^m \| \leq K_1, \quad \forall \, m\tau \leq t_{\max}. \tag{4.3.2}$$

Remark: In general, the uniform well-posedness of the scheme with respect to the boundary data and source term can be derived from the Lax-Richtmyer stability and the uniform invertibility of the scheme.

Lax Equivalence Theorem

We always assume below that, for any initial data u^0 with which the corresponding problem is well posed, there exists an sequence of sufficiently smooth solutions v_α, *s.t.* $\lim_{\alpha\to\infty} \|v_\alpha^0 - u^0\| \to 0$.

Theorem 4.1

For a uniformly solvable linear finite difference scheme which is consistent with a well-posed linear evolution problem, the stability is a necessary and sufficient condition for its convergence.

1. The original continuous problem must be well-posed.
2. Linear and linear, can't emphasize more! Not true if nonlinear.
3. The consistency is also a crucial condition.
4. Do not forget uniform solvability of the scheme.
5. "Stability $\Leftrightarrow$ Convergence" (not hold if miss any condition).

The Lax equivalence theorem is the fundamental theorem in the analysis of finite difference methods for the numerical solution of partial differential equations. It states that for a consistent finite difference method for a well-posed linear initial value problem, the method is convergent if and only if it is stable.

Proof of Sufficiency of the Lax Equivalence Theorem (stability $\Rightarrow$ convergence)

(1A) If $u \in \mathbb{X}$ is a sufficiently smooth solution of the problem, then, by the definition of the truncation error, we have

$$B_1(U^{m+1} - u^{m+1}) = B_0(U^m - u^m) - T^m, \tag{4.3.3}$$

or equivalently $U^{m+1} - u^{m+1} = (B_1^{-1}B_0)(U^m - u^m) - B_1^{-1}T^m.$ (4.3.4)

Recursively, and by assuming $U^0 = u^0$, we obtain

$$U^m - u^m = -\sum_{l=0}^{m-1} (B_1^{-1}B_0)^l B_1^{-1} T^{m-l-1}. \tag{4.3.5}$$

Thus, by the uniform solvability $\|B_1^{-1}\| \le K\tau$ and the stability $\underbrace{\|(B_1^{-1}B_0)^l\| \le K_1}_{(4.3.2)}$, we have $\|U^m - u^m\| \le KK_1\tau \sum_{l=0}^{m-1} \|T^l\|$, $\forall m > 0$. (4.3.6)

Therefore, by the definition of the consistency, we have

$$\lim_{\tau(h)\to 0} \|U^m - u^m\| = 0, \quad 0 \le m\tau \le t_{\max}.$$

Proof of Sufficiency of the Lax Equivalence Theorem (Cont'd)

(1B) For a general solution u, let v_α be the smooth solution sequence satisfying $\lim_{\alpha\to\infty}\|v_\alpha^0 - u^0\| \to 0$.

1. $\forall \varepsilon > 0$, $\exists A > 0$, such that $\|v_\alpha^0 - u^0\| < \varepsilon$ for all $\alpha > A$.
2. For fixed $\beta > A$, let V_β^m be the solution of the difference scheme with $V_\beta^0 = v_\beta^0$.
3. $\forall \varepsilon > 0$, $\exists h(\varepsilon) > 0$, s.t. $\|V_\beta^m - v_\beta^m\| < \varepsilon$, for all $h < h(\varepsilon)$.

Thus, by the stability and the uniform invertibility of the scheme and the well-posedness of the problem that, if $h < h(\varepsilon)$, then

$$\|U^m - u^m\| \le \|U^m - V_\beta^m\| + \|V_\beta^m - v_\beta^m\| + \|v_\beta^m - u^m\| \le (K_1 + 1 + C)\varepsilon. \tag{4.3.7}$$

(4.3.1)稳定性

(4.1.6) 连续依赖性

Since ε is arbitrary, this implies

$$\lim_{\tau(h)\to 0}\|U^m - u^m\| = 0, \quad 0 \le m\tau \le t_{\max}. \tag{4.3.8}$$

Proof of Necessity of the Lax Equivalence Theorem (convergence $\Rightarrow$ stability)

(2) 要证(4.3.2)成立，即算子族一致有界

1. $(B_1^{-1}B_0)_h^m$: bounded linear operators in $(\mathbb{X}, \|\cdot\|)$. ----Banach空间
2. 排序规则: $(B_1^{-1}B_0)_h^m \succ (B_1^{-1}B_0)_{\hat{h}}^{\hat{m}}$, either $h < \hat{h}$, or $h = \hat{h}$ and $m > \hat{m}$.
排序: 后〉前
3. By the resonance theorem, if, for each given $u^0 \in X$,
共鸣定理

$$S_{t_{\max}}(u^0) := \sup_{\{h>0,\ m\le\tau^{-1}t_{\max}\}} \{\|(B_1^{-1}B_0)_h^m u^0\|\} < \infty,$$

then the sequence $\{(B_1^{-1}B_0)_h^m\}$ is uniformly bounded, and (4.3.2)
consequently the scheme is stable. 第(2)步证明的目的

反证法: Suppose for some $u^0 \in \mathbb{X}$, $\exists \tau_k \to 0$, $m_k\tau_k \to t \in [0, t_{\max}]$, s.t.

$$\lim_{k\to\infty} \|(B_1^{-1}B_0)_{h_k}^{m_k} u^0\| = \infty. \qquad \textbf{(4.3.9)}$$

共鸣定理(一致有界性原理或 Banach-Steinhaus定理): In its basic form, it asserts that for a family of continuous linear operators (and thus bounded operators) whose domain is a Banach space, pointwise boundedness is equivalent to uniform boundedness in operator norm.

Proof of Necessity of the Lax Equivalence Theorem (Cont'd)

4. u, U with initial data u^0, w and W with initial data $w^0 = 0$.

$$(B_1^{-1}B_0)_h^m u^0 = U_h^m - W_h^m = (U_h^m - u^m) + (u^m - w^m) + (w^m - W_h^m). \quad \textbf{(4.3.10)}$$

5. Convergence $\Rightarrow$
已知条件

$$\lim_{k\to\infty}\left(\|U_{h_k}^{m_k} - u^{m_k}\| + \|w^{m_k} - W_{h_k}^{m_k}\|\right) = 0. \quad \textbf{(4.3.11)}$$

6. Well-posedness $\Rightarrow$ $\|u^m - w^m\| \le C\|u^0 - w^0\| = C\|u^0\|$. **(4.3.12)**
已知条件

7. Therefore $\lim_{k\to\infty}\|(B_1^{-1}B_0)_{h_k}^{m_k}u^0\| \le C\|u^0\|$. (A contradiction). **(4.3.13)**

■

How to Choose a Norm for Stability Analysis

1. Maximum principle $\Rightarrow \|\cdot\|_\infty$ a good candidate.
2. Discrete maximum principle $\Rightarrow \|\cdot\|_\infty$ stability.
3. Constant-coefficient equation, periodic boundary conditions

   $\Rightarrow \|\cdot\|_2$ a good choice.
4. Fourier analysis method $\Rightarrow \|\cdot\|_2$ stability conditions.
5. Local Fourier analysis $\Rightarrow$ necessary $\|\cdot\|_2$ stability conditions.

- More general cases $\Rightarrow \|\cdot\|_1$, $\|\cdot\|_p$, energy norm, etc.

Small Disturbance on a Scheme Does Not Change Its Stability

Theorem 4.2

Suppose $U^{m+1} = B_1^{-1} B_0 U^m$ *is stable in* $\|\cdot\|$. *Suppose* $E(\tau)$ *is uniformly bounded in* $\|\cdot\|$. *Then, the scheme*

$$U^{m+1} = \left[B_1^{-1} B_0 + \tau E(\tau)\right] U^m \tag{4.4.1}$$

is also stable in the norm $\|\cdot\|$.

Small Disturbance on a Scheme Does Not Change Its Stability

Proof: P162

- $(A+B)^2 = A^2 + AB + BA + B^2$, in general $(A+B)^m$ has 2^m terms, there are C_j^m terms having j of B and $(m-j)$ of A multiplied together in various order. 第j项中有j个因子B, (m-j)个因子A

已知条件
- $\|(B_1^{-1}B_0)^k\| \leq K_1$, for all k, and $\|E(\tau)\| \leq K_2$.

Therefore, if $m\tau \leq t_{\max}$, we have

$$\left\| \left[B_1^{-1}B_0 + \tau E(\tau)\right]^m \right\| \leq \sum_{j=0}^{m} C_j^m K_1^{j+1} (\tau K_2)^j$$

$$= K_1(1 + K_1K_2\tau)^m \leq K_1 e^{t_{\max} K_1 K_2}.$$

■

Fourier Analysis in Constant-Coefficient and Periodic Case ($\Omega = (-1,1)^n$)

1. Coefficients of a scheme: $b^{\alpha}_{\mathbf{j}',\mathbf{j}} = b^{\alpha}_{(\mathbf{j}-\mathbf{j}')}$ (see (4.1.3)).

2. Fourier mode for a one-step scalar scheme: $U^m_{\mathbf{j}} = \lambda^m_{\mathbf{k}} e^{\mathrm{i}\pi\mathbf{k}\cdot\mathbf{j}h}$,

   amplification factor: $\lambda_{\mathbf{k}} = \dfrac{\sum_j b^0_{(\mathbf{j}-\mathbf{j}')} e^{\mathrm{i}\pi\mathbf{k}\cdot(\mathbf{j}-\mathbf{j}')h}}{\sum_j b^1_{(\mathbf{j}-\mathbf{j}')} e^{\mathrm{i}\pi\mathbf{k}\cdot(\mathbf{j}-\mathbf{j}')h}} = \dfrac{\hat{B}_0(\mathbf{k})}{\hat{B}_1(\mathbf{k})}$.

3. In vector cases, apply the discrete Fourier transform:

$$\hat{U}(\mathbf{k}) = \frac{1}{(\sqrt{2}N)^n} \sum_{j_1=-N+1}^{N} \cdots \sum_{j_n=-N+1}^{N} U_{\mathbf{j}} e^{-\mathrm{i}\pi\mathbf{k}\cdot\mathbf{j}\frac{1}{N}}, \qquad \textbf{(4.4.2)}$$

   discrete inverse Fourier transform (*i.e.* Fourier expansion):

$$U_{\mathbf{j}} = \frac{1}{(\sqrt{2})^n} \sum_{k_1=-N+1}^{N} \cdots \sum_{k_n=-N+1}^{N} \hat{U}(\mathbf{k}) e^{\mathrm{i}\pi\mathbf{k}\cdot\mathbf{j}\frac{1}{N}}. \qquad \textbf{(4.4.3)}$$

Fourier Analysis in Vector Case (Constant-Coefficient and Periodic)

4. To obtain the characteristic equations, substitute the Fourier mode $U_{\mathbf{j}}^m(\mathbf{k}) = \hat{U}^m(\mathbf{k})e^{\mathrm{i}\pi\mathbf{k}\cdot\mathbf{j}\frac{1}{N}}$ into the **homogeneous** scheme $\sum_{\mathbf{j}} b^1_{\mathbf{j}',\mathbf{j}}\hat{U}^{m+1}(\mathbf{k})e^{\mathrm{i}\pi\mathbf{k}\cdot\mathbf{j}\frac{1}{N}} = \sum_{\mathbf{j}} b^0_{\mathbf{j}',\mathbf{j}}\hat{U}^m(\mathbf{k})e^{\mathrm{i}\pi\mathbf{k}\cdot\mathbf{j}\frac{1}{N}}$. **(4.1.2)**

(4.1.3)

5. Denote $\hat{B}_\alpha = \sum_{\mathbf{j}} b^\alpha_{\mathbf{j}',\mathbf{j}}e^{\mathrm{i}\pi\mathbf{k}\cdot(\mathbf{j}-\mathbf{j}')\frac{1}{N}} = \sum_{\mathbf{j}} b^\alpha_{(\mathbf{j}-\mathbf{j}')}e^{\mathrm{i}\pi\mathbf{k}\cdot(\mathbf{j}-\mathbf{j}')\frac{1}{N}}$.

6. We have $\hat{B}_1\hat{U}^{m+1}(\mathbf{k}) = \hat{B}_0\hat{U}^m(\mathbf{k})$. **(4.4.4)**

7. Characteristic equations in the space of frequencies:

$$\hat{U}^{m+1}(\mathbf{k}) = \hat{B}_1(\mathbf{k})^{-1}\hat{B}_0(\mathbf{k})\hat{U}^m(\mathbf{k}), \qquad \textbf{(4.4.5)}$$

Fourier Analysis in Vector Case (Constant-Coefficient and Periodic)

9. Amplification matrix: $G(\mathbf{k}) = \hat{B}_1(\mathbf{k})^{-1}\hat{B}_0(\mathbf{k}) \in \mathbb{C}^{p\times p}$, **(4.4.6)**

10. Fourier mode: $U_{\mathbf{j}}^m = \lambda_{\mathbf{k}}^m e^{\mathrm{i}\pi\mathbf{k}\cdot\mathbf{j}h}\hat{U}(\mathbf{k})$ with $\lambda_{\mathbf{k}}\hat{U}(\mathbf{k}) = G(\mathbf{k})\hat{U}(\mathbf{k})$. **(4.4.7)**
    代数特征值问题

11. If $G(\mathbf{k})$ is a normal matrix, *i.e.* $\bar{G}^T G = G\bar{G}^T$, then G has p orthogonal eigenvectors. Therefore $\|U^m\|_2 = |\lambda_{\mathbf{k}}|^m \|U^0\|_2$. **(4.4.8)**

12. In the general case, $\hat{U}^m(\mathbf{k}) = [G(\mathbf{k})]^m\,\hat{U}^0(\mathbf{k})$, therefore **(4.4.9)**
    $\mathbb{L}^2$ stability $\Leftrightarrow \|[G(\mathbf{k})]^m\|_2 \le K_1, \quad \forall\,\mathbf{k}, \forall\, m\tau \le t_{\max}$. **(4.4.11)**

von Neumann Condition of $\mathbb{L}^2$ Stability

Theorem 4.3

A necessary condition for the difference scheme to be $\mathbb{L}^2$ stable along a refinement path is that there exists a constant $K \geq 0$ such that every eigenvalue $\lambda_{\mathbf{k}}$ of the amplification matrix $G(\mathbf{k})$ satisfies

$$|\lambda_{\mathbf{k}}| \leq 1 + K\tau, \quad \forall\, \tau \leq t_{\max}, \quad \forall\, \mathbf{k}. \tag{4.4.12}$$

The von Neumann Condition of $\mathbb{L}^2$ Stability

Proof:

1. $\mathbb{L}^2$ stable $\Rightarrow$ for $|\lambda_{\mathbf{k}}|^m \leq K_1$ for $m = [t_{\max}/\tau]$.
2. Let $f(x) = x^s$, $(s < 1)$, since $f''(x) = s(s-1)x^{s-2} < 0$, $f(x)$ is a concave function of x, hence $f(x) \leq f(1) + f'(1)(x-1)$.

- If $K_1 \leq 1 \Rightarrow |\lambda_{\mathbf{k}}| \leq 1$. ok!

3. If $K_1 > 1 \Rightarrow |\lambda_{\mathbf{k}}| \leq K_1^{\frac{1}{m}} \leq 1 + \frac{1}{m}(K_1 - 1)$.
4. If $2\tau \leq t_{\max}$, $m \geq \dfrac{t_{\max}}{\tau} - 1 \geq \dfrac{t_{\max}}{2\tau}$.
5. The theorem holds for $K = 2(K_1 - 1)/t_{\max}$.

(4.4.12)

■

In General, the von Neumann Condition Is Not Sufficient for Stability

1. If $G(\mathbf{k})$ is a normal matrix, then the von Neumann condition is also a sufficient condition for a scheme to be $\mathbb{L}^2$ stable.
2. The conclusion is however not valid if $G(\mathbf{k})$ is not normal.
3. In general, we can not expect to obtain a sufficient condition by the local Fourier analysis (see Exercise 4.3).
4. In practical computations, the constant K can be a problem. If K is too big, then the von Neumann condition can fail to provide useful information on the grid ratio.

More Practical Conditions Than the von Neumann Condition Is Necessary

例4.1

1. Initial value problem of $u_t + au_x = cu_{xx}$, $(c > 0)$. **(4.4.13)**

2. Explicit central difference scheme:

$$\frac{U_j^{m+1} - U_j^m}{\tau} + a\frac{U_{j+1}^m - U_{j-1}^m}{2h} = c\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{h^2}. \quad \textbf{(4.4.14)}$$

3. $|\lambda_k|^2 = [1 - 2\mu(1 - \cos kh)]^2 + [\nu \sin kh]^2.$ **(4.4.15)**

4. $kh = \pi \Rightarrow \mu \leq \frac{1}{2}$ a necessary condition for the $\mathbb{L}^2$ stability.

5. $\mu \leq \frac{1}{2} \quad \Rightarrow [1 - 2\mu(1 - \cos kh)]^2 \leq 1,\ \nu^2 = \frac{a^2}{c}\mu\tau \leq \frac{a^2}{2c}\tau$ **(4.4.16)**
   $\Rightarrow |\lambda_k|^2 \leq 1 + \frac{a^2}{2c}\tau \Rightarrow$ the von Neumann condition is satisfied. **(4.4.17)**

More Practical Conditions Than the von Neumann Condition Is Necessary

6. In fact, $\mu \leq \frac{1}{2} \Leftrightarrow \mathbb{L}^2$ stability.
7. Recall $|\lambda_k|^2 = [1 - 2\mu(1 - \cos kh)]^2 + [\nu \sin kh]^2$.
8. Take $\mu = \frac{1}{4}$, $\nu \geq 1$ (Péclet number $\frac{ah}{c} = \frac{\nu}{\mu} \geq 4$),
9. then, for $kh = \pi/2$, the amplification factor satisfies $|\lambda_k| \geq \frac{5}{4}$, hence, the corresponding Fourier mode grows dramatically.
10. Yet, all Fourier mode solutions to the original problem decays. (dissipation speed: e^{-ck^2})

A Practical Stability Condition for Difference Schemes

Definition 4.7

Let $\hat{u}(\mathbf{k}, t)$ be the Fourier mode solutions to the differential equation, denote

$$\alpha = \inf\{\beta \geq 0 : |\hat{u}(\mathbf{k}, t+\tau)| \leq e^{\beta\tau}|\hat{u}(\mathbf{k}, t)|, \ \ \forall\, \mathbf{k}\}. \tag{4.4.18}$$

If all of the amplification factors of the discrete Fourier mode solutions to the difference scheme satisfy

$$\boxed{|\lambda_{\mathbf{k}}| \leq e^{\alpha\tau}, \quad \forall\, \mathbf{k},} \tag{4.4.19}$$

then, the difference scheme is said to be practically stable, or to have practical stability. In particular, if $\alpha = 0$, the corresponding difference scheme is said to be strongly stable, or to have strong stability.

A Practical Stability Condition for Difference Schemes

If $\alpha > 0$,

- the discrete Fourier mode solutions can grow exponentially fast;
- however, the growth speed $\leq$ the fastest growth speed of real Fourier mode solution;
- the discrete Fourier modes are "relatively" not in growth.

A Practical Stability Condition for Difference Schemes

If $\alpha = 0$, the strong stability requires $|\lambda_{\mathbf{k}}| \leq 1$. In the above example, $|\lambda_k|^2 = [1 - 2\mu(1 - \cos kh)]^2 + [\nu \sin kh]^2$.

- $|\lambda_k| \leq 1$, $\forall k \Leftrightarrow \mu \leq \frac{1}{2}$ and $\tau \leq \frac{2c}{a^2}$ (see (3.4.10)).
- Since $\nu^2 = \left(\frac{a\tau}{h}\right)^2 = \frac{a^2}{c}\mu\tau$, $\tau \leq \frac{2c}{a^2} \Leftrightarrow \nu^2 \leq 2\mu$.
- Take $\mu = 1/4$, $\nu \leq 1/\sqrt{2} \Rightarrow |\lambda_k|^2 \leq \cos^2 \frac{1}{2}kh \leq 1$, $\forall k$.

习题 4：1，2，4　　Page 178

Thank You!