有限元简介

1.. 两点边值问题

求解如下方程

$$
\begin{cases} -u''(x) = f(x), x \in \Omega := (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1}
$$

弱导数: 对于函数$v(x)$, 若存在$v_g(\int_a^b |v_g| dx < \infty)$, 满足

$$
\int_a^b v\varphi'(x)dx = -\int_a^b v_g \varphi(x)dx, \forall \varphi(x) \in C_0^\infty(a,b)
$$

则称$v_g(x)$是$v(x)$的弱导数, 也称为广义导数, 记为$v'(x)$.

例:

$$
v(x) = \begin{cases} x, & 0 \leqslant x \leqslant 1; \\ -x, & -1 \leqslant x < 0. \end{cases} \qquad \text{弱导数: } v_g(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1; \\ -1, & -1 \leqslant x < 0. \end{cases}
$$

$$
H_0^1(\Omega) := \left\{ v : \int_\Omega v^2 dx < \infty, \int_\Omega v_g^2 dx < \infty, v(0) = v(1) = 0 \right\}
$$

定义

$$
\varphi(t) = \begin{cases} 2t, & 0 \leqslant t \leqslant \frac{1}{2}, \\ 1, & t > \frac{1}{2}. \end{cases}
$$

对$v(x) \in H_0^1(\Omega)$, 令$w(x) = v(x)\varphi(x)$, 于是, 有

$$
\begin{aligned}
w(x) &= w(0) + \int_0^x w'(t)dt \\
&= \int_0^x [v'(t)\varphi(t) + v(t)\varphi'(t)]dt \\
&\leqslant \int_0^1 |v'(t)|dt + 2\int_0^1 |v(t)|dt \\
&\leqslant C\|v\|_{H^{1,1}}(\Omega)
\end{aligned}
$$

因此,$\forall x \geqslant \frac{1}{2}$, 有

$$
v(x) = w(x) \leqslant C\|v\|_{H^{1,1}(\Omega)}
$$

同理可证明$\forall 0 \leqslant x < \frac{1}{2}$, 有

$$
v(x) \leqslant C\|v\|_{H^{1,1}(\Omega)}
$$

对$\forall v(x) \in H_0^1(\Omega), x, y \in (0,1)$, 有

$$|v(x)-v(y)| = |\int_x^y v'(t)dt| \leqslant |x-y|^{1/2}(\int_a^b |v'(t)|^2 dt)^{1/2}$$

因此$v(x)$ Holder连续.

2.. 有限元方法

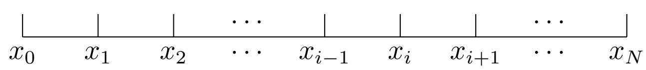
在方程(1)的两端同乘以$v(x) \in H_0^1(\Omega)$, 有

$$-\int_0^1 u''(x)v(x)dx = \int_0^1 f(x)v(x)dx$$

$$\int_0^1 u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx \tag{2}$$

问题(2)是问题(1)的变分问题. 将区间$[0,1]$进行剖分:

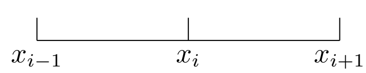
图 1



$x_i, i = 0, 1, 2, \cdots, N$称为剖分的节点, $[x_i, x_{i+1}]$称为单元.
在内节点x_i上, 令$h_i = x_i - x_{i-1}$, 定义

$$\varphi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h_i}, & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1}-x}{h_{i+1}}, & x \in [x_i, x_{i+1}] \\ 0, & \text{其他} \end{cases}$$

图 2



定义: $V_h = \left\{ v_h = \sum\limits_{i=1}^{N-1} v_i \varphi_i(x), v_i \in R \right\}$
有限元问题: 求$u_h = \sum\limits_{j=1}^{N-1} u_j \varphi_j(x)$, 使得

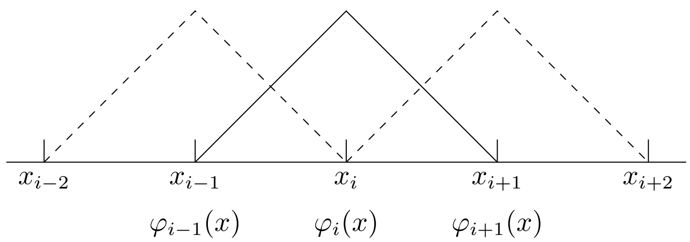
$$\int_0^1 u_h'(x)v_h'(x)dx = \int_0^1 v_h(x)f(x)dx, \forall v_h(x) \in V_h$$

等价的: $\int_0^1 u_h'(x)\varphi_j'(x)dx = \int_0^1 \varphi_j f(x)dx, j = 1, 2, \cdots, N-1$, 也即是

$$\sum_{i=1}^{N-1} u_i \int_0^1 \varphi_i'(x)\varphi_j'(x)dx = \int_0^1 f(x)\varphi_j(x)dx$$

只有$j = i-1, i, i+1$时, φ_j与φ_i有共同的支集.

图 3



故有

$$u_{i-1}\int_0^1 \varphi_{i-1}'(x)\varphi_i'(x)dx + u_i\int_0^1 \varphi_i'(x)\varphi_i'(x)dx + u_{i+1}\int_0^1 \varphi_{i+1}'(x)\varphi_i'(x)dx$$
$$= \int_0^1 f\varphi_i(x)dx, \text{令} f_i = \int_0^1 f\varphi'(x)dx$$

有:

$$\int_0^1 \varphi_{i-1}'(x)\varphi_i'(x)dx$$
$$= \int_{x_{i-1}}^{x_i} \varphi_{i-1}'(x)\varphi_i'(x)dx = \int_{x_{i-1}}^{x_i} \left(-\frac{1}{h_i}\right)\left(\frac{1}{h_i}\right)dx = -\frac{1}{h_i}$$

与

$$\int_0^1 \varphi_{i+1}'(x)\varphi_i'(x)dx = \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h_{i+1}}\right)\left(\frac{1}{h_{i+1}}\right)dx$$
$$= -\frac{1}{h_{i+1}}$$

$$\int_0^1 \varphi_i'(x)\varphi_i'(x)dx = \int_{x_{i-1}}^{x_i} \varphi_i'(x)\varphi_i'(x)dx + \int_{x_i}^{x_{i+1}} \varphi_i'(x)\varphi_i'(x)dx$$
$$= \frac{1}{h_i} + \frac{1}{h_{i+1}}$$

即

$$(4) \quad -\frac{u_{i-1}}{h_i} + (\frac{1}{h_i} + \frac{1}{h_{i+1}})u_i - \frac{u_{i+1}}{h_{i+1}} = f_i, \quad i = 1, 2, \cdots, N-1$$

令 $a_i = \frac{1}{h_i} + \frac{1}{h_{i+1}}, i = 1, 2, \cdots N-2$, 并令

$$A_h = \begin{pmatrix} a_1 & -\frac{1}{h_2} & & & & \\ -\frac{1}{h_2} & a_2 & -\frac{1}{h_3} & & & \\ & -\frac{1}{h_3} & a_3 & -\frac{1}{h_4} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\frac{1}{h_{N-2}} & a_{N-2} & -\frac{1}{h_{N-1}} \\ & & & & -\frac{1}{h_{N-1}} & a_{N-1} \end{pmatrix}$$

与:

$$U_h = (u_1, u_2, \cdots, u_{N-1})^T, \quad F_h = (f_1, f_2, \cdots, f_{N-1})^T$$

(4)的矩阵形式为:

$$(5) \qquad A_h U_h = F_h$$

$$f_i = \int_{x_{i-1}}^{x_{i+1}} \phi_i(x) f(x) dx \approx \frac{h_i f(x_{i-1})}{6} + \frac{h_i f(x_i)}{3} + \frac{h_{i+1} f(x_i)}{3} + \frac{h_{i+1} f(x_{i+1})}{6}$$

3.. 误差估计

有如下关系式:

$$\int_0^1 u'v'dx = \int_0^1 vfdx$$
$$\int_0^1 u'v_h'(x)dx = \int_0^1 v_h fdx \quad \forall v_n \in V_n$$
$$\int_0^1 u_h'(x)v_h'(x)dx = \int_0^1 v_h(x)fdx \quad \forall v_h \in V_h$$

可得正交性:

$$\int_0^1 (u' - u_h')v_h'dx = 0$$

故$\forall w_h \in V_h$, 有

$$\begin{aligned}
&\int_0^1 (u' - u_h')^2 dx \\
&= \int_0^1 (u' - u_h')(u' - w_h')dx + \int_0^1 (u' - u_h')(w_h' - u_h')dx (= 0, 令 v_h = w_h - u_h 即可得) \\
&= \int_0^1 (u' - u_h')(u' - w_h')dx \\
&\leqslant \left(\int_0^1 (u' - u_h')^2 dx\right)^{1/2} \left(\int_0^1 (u' - w_h')^2 dx\right)^{1/2}
\end{aligned}$$

即有: $\int_0^1 (u'-u_h')^2dx \leqslant \int_0^1 (u'-w_h')^2dx$, 任对$w_h\in V_h$成立, 也即是

$$\int_0^1 (u'-u_h')^2dx \leqslant \inf_{w_h\in V_h}\int_0^1 (u'-w_h')^2dx$$

定义插值函数$I_hu\in V_h$,

$$(I_hu)(x_i)=u(x_i), i=1,2,\cdots,N$$

变点展开技术: 在单元$[x_i,x_{i+1}]$上, 有:

$$u(x)-I_hu(x)=u(x)-u(x_i)\varphi_i(x)-u(x_{i+1})\varphi_{i+1}(x)$$

展开

$$\begin{aligned} u(x_{i+1})&=u(x)+u'(x)(x_{i+1}-x)+\int_x^{x_{i+1}} u''(t)(x_{i+1}-t)dt\\ u(x_i)&=u(x)+u'(x)(x_i-x)+\int_x^{x_i} u''(t)(x_i-t)dt \end{aligned}$$

这样

$$\begin{aligned} u(x)-I_hu(x)=u(x)&-\left(u(x)+u'(x)(x_i-x)\right)\varphi_i(x)\\ &-\left(u(x)+u'(x)(x_{i+1}-x)\right)\varphi_{i+1}(x)\\ &-\int_x^{x_i} u''(t)(x_i-t)dt\ \varphi_i(x)\\ &-\int_x^{x_{i+1}} u''(t)(x_{i+1}-t)dt\ \varphi_{i+1}(x)\\ =&-u'(x)\left((x_i-x)\varphi_i(x)+(x_{i+1}-x)\ \varphi_{i+1}(x)\right)\\ &-\int_x^{x_i} u''(t)(x_i-t)dt\ \varphi_i(x)\\ &-\int_x^{x_{i+1}} u''(t)(x_{i+1}-t)dt\ \varphi_{i+1}(x)\\ =&-\int_x^{x_i} u''(t)(x_i-t)dt\ \varphi_i(x)\\ &-\int_x^{x_{i+1}} u''(t)(x_{i+1}-t)dt\ \varphi_{i+1}(x) \end{aligned}$$

于是:

$$\begin{aligned} |u(x)-I_hu(x)|\leqslant&\left(\int_{x_i}^{x_{i+1}}(u''(t))^2dt\right)^{1/2}\left(\int_{x_i}^{x_{i+1}}(x_i-t)^2dt\right)^{1/2}\varphi_i(x)\\ &+\left(\int_{x_i}^{x_{i+1}}(u''(t))^2dt\right)^{1/2}\left(\int_{x_i}^{x_{i+1}}(x_{i+1}-t)^2dt\right)^{1/2}\varphi_{i+1}(x)\\ =&\frac{1}{\sqrt{3}}\left(\int_{x_i}^{x_{i+1}}(u''(t))^2dt\right)^{1/2}h_{i+1}^{3/2} \end{aligned}$$

于是

$$\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}(u(x)-I_hu(x))^2dx \leqslant \frac{1}{3}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}(u''(x))^2dx\ h_{i+1}^4$$
$$=\frac{1}{3}\sum_{i=0}^{N-1}h_{i+1}^4\int_{x_i}^{x_{i+1}}(u''(x))^2dx$$

令一方面,

$$\begin{aligned}(u(x)-I_hu(x))' &= u'(x)+\frac{u\left(x_i\right)}{h_{i+1}}-\frac{u(x_{i+1})}{h_{i+1}}\\ &= u'(x)+\frac{u(x)+u'(x)\left(x_i-x\right)}{h_{i+1}}-\frac{u(x)+u'(x)\left(x_{i+1}-x\right)}{h_{i+1}}\\ &\quad+\frac{1}{h_{i+1}}\int_x^{x_i}u''(t)\left(x_i-t\right)dt-\frac{1}{h_{i+1}}\int_x^{x_{i+1}}u''(t)\left(x_{i+1}-t\right)dt\\ &= \frac{1}{h_{i+1}}\left(\int_x^{x_i}u''(t)\left(x_i-t\right)dt-\int_x^{x_{i+1}}u''(t)\left(x_{i+1}-t\right)dt\right)\end{aligned}$$

于是

$$|(u(x)-I_hu(x))'| \leqslant \frac{2}{\sqrt{3}h_{i+1}}\left(\int_{x_i}^{x_{i+1}}(u''(x))^2dx\right)^{1/2}h_{i+1}^{3/2}$$

这样:

$$\int_0^1\left((u(x)-I_hu(x))'\right)^2dx \leq \frac{4}{3}\sum_{i=0}^{N-1}h_{i+1}^2\int_{x_i}^{x_{i+1}}\left(u''(x)\right)^2dx$$

4.. 二维泊松问题及有限元方法

$$\begin{cases}-\Delta u=f(x,y), & x\in\Omega\\ u|_{\partial\Omega}=0\end{cases} \qquad (6)$$

定义:

$$H_0^1(\Omega)=\left\{v:\int_\Omega v^2dxdy+\int_\Omega|\nabla v|^2dxdy<\infty\right\}$$

在(6)两边同时乘以$v(x,y)$, 分步积分有:

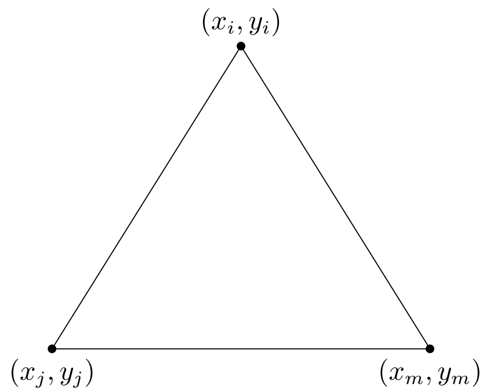
$$\int_\Omega -\Delta uvdxdy=\int_\Omega \nabla u\nabla vdxdy$$

变分问题: 求$u(x,y)\in H_0^1(\Omega)$, 使得

$$\int_\Omega \nabla u\nabla vdxdy=\int_\Omega f(x,y)v(x,y)dxdy, \forall v\in H_0^1(\Omega)$$

重心坐标:

图 4



求线性函数$u = ax + by + c$, 使得

$$\begin{cases} ax_i + by_i + c = u_i \\ ax_j + by_j + c = u_j \\ ax_m + by_m + c = u_m \end{cases}$$

这样有:

$$a = \frac{\begin{vmatrix} u_i & y_i & 1 \\ u_j & y_j & 1 \\ u_m & y_m & 1 \end{vmatrix}}{\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_m & y_m & 1 \end{vmatrix}}, \quad b = \frac{\begin{vmatrix} x_i & u_i & 1 \\ x_j & u_j & 1 \\ x_m & u_m & 1 \end{vmatrix}}{\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_m & y_m & 1 \end{vmatrix}}, \quad c = \frac{\begin{vmatrix} x_i & y_i & u_i \\ x_j & y_j & u_j \\ x_m & y_m & u_m \end{vmatrix}}{\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_m & y_m & 1 \end{vmatrix}}$$

也即是

$$a = \frac{1}{2\Delta_e}\left[\begin{vmatrix} y_j & 1 \\ y_m & 1 \end{vmatrix} u_i + \begin{vmatrix} y_m & 1 \\ y_i & 1 \end{vmatrix} u_j + \begin{vmatrix} y_i & 1 \\ y_j & 1 \end{vmatrix} u_m\right]$$

$$b = \frac{1}{2\Delta_e}\left[-\begin{vmatrix} x_j & 1 \\ x_m & 1 \end{vmatrix} u_i - \begin{vmatrix} x_m & 1 \\ x_i & 1 \end{vmatrix} u_j - \begin{vmatrix} x_i & 1 \\ x_j & 1 \end{vmatrix} u_m\right]$$

$$c = \frac{1}{2\Delta_e}\left[\begin{vmatrix} x_j & y_j \\ x_m & y_m \end{vmatrix} u_i + \begin{vmatrix} x_m & y_m \\ x_i & y_i \end{vmatrix} u_j + \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} u_m\right]$$

其中: $2\Delta_e = \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_m & y_m & 1 \end{vmatrix}$.

定义:

$$\lambda_i=\frac{1}{2\Delta_e}\left[\begin{vmatrix}y_j & 1\\ y_m & 1\end{vmatrix}x-\begin{vmatrix}x_j & 1\\ x_m & 1\end{vmatrix}y+\begin{vmatrix}x_j & y_j\\ x_m & y_m\end{vmatrix}\right]$$
$$\lambda_j=\frac{1}{2\Delta_e}\left[\begin{vmatrix}y_m & 1\\ y_i & 1\end{vmatrix}x-\begin{vmatrix}x_m & 1\\ x_i & 1\end{vmatrix}y+\begin{vmatrix}x_m & y_m\\ x_i & y_i\end{vmatrix}\right]$$
$$\lambda_m=\frac{1}{2\Delta_e}\left[\begin{vmatrix}y_i & 1\\ y_j & 1\end{vmatrix}x-\begin{vmatrix}\Delta x_i & 1\\ x_j & 1\end{vmatrix}y+\begin{vmatrix}x_i & y_i\\ x_j & y_j\end{vmatrix}\right]$$

则有: $u=u_i\lambda_i+u_j\lambda_j+u_m\lambda_m$. 同时也即是

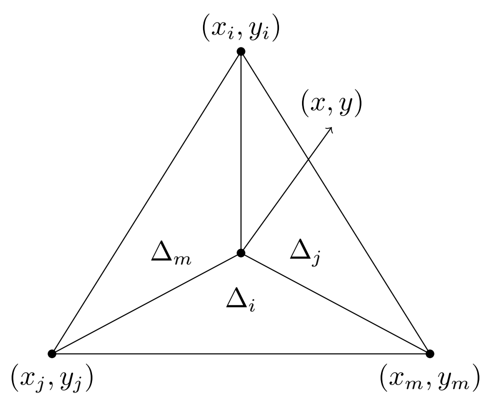
$$\lambda_i=\frac{2\Delta_i}{2\Delta_e},\lambda_j=\frac{2\Delta_j}{2\Delta_e},\lambda_m=\frac{2\Delta_m}{2\Delta_e}$$

其中

$$2\Delta_i=\begin{vmatrix}x & y & 1\\ x_j & y_j & 1\\ x_m & y_m & 1\end{vmatrix},\quad 2\Delta_j=\begin{vmatrix}x & y & 1\\ x_m & y_m & 1\\ x_i & y_i & 1\end{vmatrix},\quad 2\Delta_m=\begin{vmatrix}x & y & 1\\ x_i & y_i & 1\\ x_j & y_j & 1\end{vmatrix}$$

为下图所示区域面积

图 5



且满足$\lambda_i((x_k,y_k))=\delta_{ik}=\begin{cases}1, i=k\\ 0, i\neq k\end{cases}$, 同时还有:

$$\begin{aligned}&\frac{\partial\lambda_i}{\partial x}=\frac{1}{2\Delta_e}\left(y_j-y_m\right),\quad \frac{\partial\lambda_i}{\partial y}=\frac{x_m-x_j}{2\Delta_e}\\ &\frac{\partial\lambda_j}{\partial x}=\frac{1}{2\Delta e}\left(y_m-y_i\right),\quad \frac{\partial\lambda_j}{\partial y}=\frac{x_i-x_m}{2\Delta e}\\ &\frac{\partial\lambda_m}{\partial x}=\frac{y_i-y_j}{2\Delta e},\quad \frac{\partial\lambda_m}{\partial y}=\frac{x_j-x_i}{2\Delta_i}\end{aligned}$$

与:

$$\begin{aligned}1 &= \lambda_i + \lambda_j + \lambda_m \\ x &= x_i\lambda_i + x_j\lambda_j + x_m\lambda_m \\ y &= y_i\lambda_i + y_j\lambda_j + y_m\lambda m\end{aligned}$$

这样就有:

$$\begin{aligned}x &= (x_i - x_m)\,\lambda_i + (x_j - x_m)\,\lambda_j + x_m \\ y &= (y_i - x_m)\,\lambda_i + (y_j - y_m)\,\lambda_j + y_m\end{aligned}$$

也即有

$$\left|\frac{\partial(x,y)}{\partial\left(\lambda_i,\lambda_j\right)}\right| = \left|\begin{array}{cc} x_i - x_m & x_j - x_m \\ y_i - y_m & y_j - y_m \end{array}\right| = 2\Delta_e$$

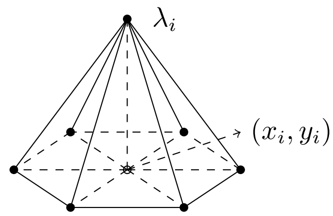
令:

$$V_h := \left\{v \in C^0(\Omega) : v|_T \in P_1(T), \forall T, v|_{\partial\Omega} = 0\right\}$$

有限元问题: 求$u_h \in V_h$, 使得

$$\int_\Omega \nabla u_h \cdot \nabla v_h dxdy = \int_\Omega f v_h dxdy, \forall v_h \in V_h$$

图 6



$u_h = \sum\limits_{j\in\mathcal{N}_0} u_j\lambda_j$, 其中$\mathcal{N}_0$表示内节点的集合.

$$\int_\Omega \nabla u_h \cdot \nabla v_h dxdy = \sum_e \int_e \nabla u_h \cdot \nabla v_h dxdy$$

$$u_h|_e = u_i\lambda_i + u_j\lambda_j + u_m\lambda_m = [\lambda_i\ \lambda_j\ \lambda_m]\begin{bmatrix}u_i\\u_j\\u_m\end{bmatrix}$$

$$v_h|_e = v_i\lambda_i + u_j\lambda_j + u_m\lambda_m = [v_i\ v_j\ v_m]\begin{bmatrix}\lambda_i\\\lambda_j\\\lambda_m\end{bmatrix}$$

于是

$$\int_e \nabla u_h \nabla v_h dxdy = [v_i\ v_j\ v_m] \int_e \begin{bmatrix} \nabla\lambda_i \\ \nabla\lambda_j \\ \nabla\lambda_m \end{bmatrix} \cdot [\nabla\lambda_i\ \nabla\lambda_j\ \nabla\lambda_m] dxdy \begin{bmatrix} u_i \\ u_j \\ u_m \end{bmatrix}$$
$$= [v_i\ v_j\ v_m] \int_e \begin{pmatrix} \nabla\lambda_i \cdot \nabla\lambda_i & \nabla\lambda_i \cdot \nabla\lambda_j & \nabla\lambda_i \cdot \nabla\lambda_m \\ \nabla\lambda_j \cdot \nabla\lambda_i & \nabla\lambda_j \cdot \nabla\lambda_j & \nabla\lambda_j \cdot \nabla\lambda_m \\ \nabla\lambda_m \cdot \nabla\lambda_i & \nabla\lambda_m \cdot \nabla\lambda_j & \nabla\lambda_m \cdot \nabla\lambda_m \end{pmatrix} dxdy \begin{bmatrix} u_i \\ u_j \\ u_m \end{bmatrix}$$

令

$$\mathrm{K_e} := \int_e \begin{pmatrix} \nabla\lambda_i \cdot \nabla\lambda_i & \nabla\lambda_i \cdot \nabla\lambda_j & \nabla\lambda_i \cdot \nabla\lambda_m \\ \nabla\lambda_j \cdot \nabla\lambda_i & \nabla\lambda_j \cdot \nabla\lambda_j & \nabla\lambda_j \cdot \nabla\lambda_m \\ \nabla\lambda_m \cdot \nabla\lambda_i & \nabla\lambda_m \cdot \nabla\lambda_j & \nabla\lambda_m \cdot \nabla\lambda_m \end{pmatrix} dxdy$$

这样有:

$$\int_\Omega \nabla u_h \cdot \nabla v_h dxdy = \sum_e [v_i\ v_j\ v_m]\, \mathrm{K_e} \begin{bmatrix} u_i \\ u_j \\ u_m \end{bmatrix}$$

其中$\mathrm{K_e}$称为单元刚度矩阵.(注: $u_i = v_i = 0$, 若i为边界节点).

怎样组装总刚度矩阵?

$$\mathrm{K_e} \Rightarrow \begin{pmatrix} k_{ii} & \cdots & k_{ij} & \cdots & k_{im} \\ \vdots & & \vdots & & \vdots \\ k_{ji} & \cdots & k_{jj} & \cdots & k_{jm} \\ \vdots & & \vdots & & \vdots \\ k_{mi} & \cdots & k_{mj} & \cdots & k_{mm} \end{pmatrix}, \qquad \mathrm{K}_{N\times N}$$

右端项

$$\int_e f v_h dxdy = [v_i\ v_j\ v_m] \int_e \begin{pmatrix} f\lambda_i \\ f\lambda_j \\ f\lambda_m \end{pmatrix} dxdy$$

令:

$$F_e = \int_e \begin{pmatrix} f\lambda_i \\ f\lambda_j \\ f\lambda_m \end{pmatrix} dxdy$$

$$F_e \Rightarrow \begin{pmatrix} f_i \\ \vdots \\ f_j \\ \vdots \\ f_m \end{pmatrix} \qquad F_N$$

$$A_{N\times N}U_N = F_N$$

5.. 误差估计

类似一维问题, 有

$$\int_\Omega (\nabla u - \nabla u_h)\cdot(\nabla u - \nabla u_h)\,dxdy = \inf_{v_h\in V_h}\int_\Omega |\nabla u - \nabla u_h|^2\,dxdy$$

定义插值函数$I_h u$, 使得

$$I_h u = \sum_{i\in\mathcal{N}} u\left(x_i, y_i\right)\lambda_i$$

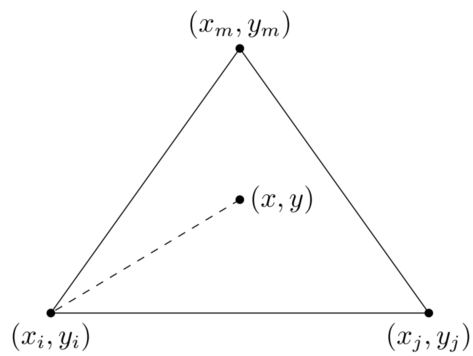
在Δ_e上(顶点为$(x_i, y_i), (x_j, y_j), (x_m, y_m)$), 有

$$\begin{aligned} u(x,y) - I_h u(x,y) = u(x,y) &- u(x_i,y_i)\lambda_i - u(x_j,y_j)\lambda_j \\ &- u(x_m,y_m)\lambda_m \end{aligned}$$

令$A_i = (x_i, y_i), A = (x, y)$, 作变点展开

$$\begin{aligned} u\left(x_i, y_i\right) = u(x,y) &+ \nabla u(x,y)\cdot\left(x_i - x, y_i - y\right)^T \\ &+ \frac{\left(x_i - x\right)^2}{2}\int_0^1 s\partial_{xx}u\left(A_i + s\left(A - A_i\right)\right)ds \\ &+ \frac{\left(y_i - y\right)^2}{2}\int_0^1 s\partial_{yy}u\left(A_i + s\left(A - A_i\right)\right)ds \\ &+ \left(x_i - x\right)\left(y_i - y\right)\int_0^1 s\partial_{xy}u\left(A_i + s\left(A - A_i\right)\right)ds \end{aligned}$$

图 7



作极坐标变化, 得

$$
\begin{aligned}
u\left(x_i, y_i\right) = {} & u(x, y) + \nabla u(x, y) \cdot \left(x_i - x, y_i - y\right)^T \\
& + \frac{(\cos\theta_i)^2}{2} \int_0^{l_{i(x,y)}} \partial_{xx} u\left((x_i, y_i) - t(\cos\theta_i, \sin\theta_i)\right) t dt \\
& + \frac{(\sin\theta_i)^2}{2} \int_0^{l_{i(x,y)}} \partial_{yy} u\left((x_i, y_i) - t(\cos\theta_i, \sin\theta_i)\right) t dt \\
& + \cos\theta_i \sin\theta_i \int_0^{l_{i(x,y)}} \partial_{xy} u\left((x_i, y_i) - t(\cos\theta_i, \sin\theta_i)\right) t dt.
\end{aligned}
$$

这样

$$
\begin{aligned}
\nabla u - \nabla I_h u = {} & \nabla u - \sum_{i=1}^{3} u(x_i, y_i) \nabla \lambda_i \\
= {} & -\sum_{i=1}^{3} \nabla \lambda_i \Bigg(\frac{(\cos\theta_i)^2}{2} \int_0^{l_{i(x,y)}} \partial_{xx} u\left((x_i, y_i) - t(\cos\theta_i, \sin\theta_i)\right) t dt \\
& + \frac{(\sin\theta_i)^2}{2} \int_0^{l_{i(x,y)}} \partial_{yy} u\left((x_i, y_i) - t(\cos\theta_i, \sin\theta_i)\right) t dt \\
& + \cos\theta_i \sin\theta_i \int_0^{l_{i(x,y)}} \partial_{xy} u\left((x_i, y_i) - t(\cos\theta_i, \sin\theta_i)\right) t dt \Bigg).
\end{aligned}
$$

这里用到

$$
\sum_{i=1}^{3} u(x, y) \nabla \lambda_i = 0, \quad \sum_{i=1}^{3} \nabla u \cdot (x, y)^T \nabla \lambda_i = 0,
$$

$$
\sum_{i=1}^{3} \nabla u \cdot (x_i, y_i)^T \nabla \lambda_i = \nabla u.
$$

于是

$$
\begin{aligned}
&\int_e |\nabla u - \nabla I_h u|^2 dxdy \\
&= 3\sum_{i=1}^{3} |\nabla\lambda_i|^2 \int_{\alpha_i}^{\beta_i}\int_0^{l_{\theta_i}} \left(\frac{(\cos\theta_i)^2}{2}\int_0^{l_{i(x,y)}} \partial_{xx}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right)tdt \right.\\
&\qquad + \frac{(\sin\theta_i)^2}{2}\int_0^{l_{i(x,y)}} \partial_{yy}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right)tdt \\
&\qquad \left. + \cos\theta_i\sin\theta_i\int_0^{l_{i(x,y)}} \partial_{xy}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right)tdt \right)^2 l_i dl_i d\theta_i \quad (l_i = l_{i(x,y)}).
\end{aligned}
$$

下面估计上式中的每一项:

$$
\begin{aligned}
&\int_0^{l_{i(x,y)}} \partial_{xx}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right)tdt \\
&\le \Big(\int_0^{l_{i(x,y)}} (\partial_{xx}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right))^2 tdt\Big)^{1/2}\Big(\int\int_0^{l_{i(x,y)}} tdt\Big)^{1/2} \\
&\le \Big(\int_0^{l_{\theta_i}} (\partial_{xx}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right))^2 tdt\Big)^{1/2}\Big(\int\int_0^{l_{i(x,y)}} tdt\Big)^{1/2}.
\end{aligned}
$$

因此

$$
\begin{aligned}
&\int_{\alpha_i}^{\beta_i}\int_0^{l_{\theta_i}} \left(\frac{(\cos\theta_i)^2}{2}\int_0^{l_{i(x,y)}} \partial_{xx}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right)tdt\right)^2 l_i dl_i d\theta_i \\
&\le \int_{\alpha_i}^{\beta_i} \frac{(\cos\theta_i)^4}{4}\int_0^{l_{\theta_i}} (\partial_{xx}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right))^2 tdt \left(\int_0^{l_{\theta_i}}\int_0^{l_{i(x,y)}} tdt l_i dl_i\right) d\theta_i \\
&= \frac{1}{32}\int_{\alpha_i}^{\beta_i} l_{\theta_i}^4 (\cos\theta_i)^4 \int_0^{l_{\theta_i}} (\partial_{xx}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right))^2 tdtd\theta_i \\
&\le \frac{h^4}{32}\|\partial_{xx}u\|_{0,e}^2.
\end{aligned}
$$

其它项可以类似处理, 这样

$$
\int_e |\nabla u - \nabla I_h u|^2 dxdy \le C\|\nabla^2 u\|_{0,e}^2 \sum_{i=1}^{3} |\nabla\lambda_i|^2 h^4.
$$

下面是基于泛函分析中等价模定理的误差分析

令

$$\begin{aligned}\hat{u}(\lambda_1,\lambda_2) &= u(x,y)\\ &= u\left(x_1\lambda_1+x_2\lambda_2+x_3(1-\lambda_1-\lambda_2), y_1\lambda_1+y_2\lambda_2+y_3(1-\lambda_1-\lambda_2)\right)\end{aligned}$$

引理:

$$|u|_{s,e} \leqslant \frac{h^{1-s}}{\sin^s\theta_1}|\hat{u}|_{s,\hat{e}}$$
$$|\hat{u}|_{s,\hat{e}} \leqslant \frac{h^{s-1}}{\sin\theta_1}|u|_{s,e}$$

证明: 因为$\left|\frac{\partial(x,y)}{\partial(\lambda_1,\lambda_2)}\right| = 2\Delta_e \leqslant h^2$, 其中$h$为最长边.

注意到不等式: $h \leqslant h_2+h_1 \leqslant 2h_2$

有

$$\max\left(\left|\frac{\partial\lambda_i}{\partial x}\right|, \left|\frac{\partial\lambda_i}{\partial y}\right|\right) \leqslant \frac{h}{2\Delta_e} \quad \leqslant \frac{h}{h_2h_3\sin\theta_1} \leqslant \frac{2}{h\sin\theta_1}$$

故:

$$\begin{aligned}\|u\|_{0,e}^2 &= \iint_e u^2dxdy = \iint_e \hat{u}\left|\frac{\partial(x,y)}{\partial(\lambda_1,\lambda_2)}\right|d\lambda_1d\lambda_2\\ &\leqslant h^2\|\hat{u}\|_{0,\hat{e}}^2\\ |u|_{1,e}^2 &= \iint_e(u_x^2+u_y^2)dxdy\\ &= \iint_{\hat{e}}\left[\left(\sum_{i=1}^2\frac{\partial\hat{u}}{\partial\lambda_i}\frac{\partial\lambda_i}{\partial x}\right)^2+\left(\sum_{i=1}^2\frac{\partial\hat{u}}{\partial\lambda_i}\frac{\partial\lambda_i}{\partial y}\right)^2\right]\left|\frac{\partial(x,y)}{\partial(\lambda_1,\lambda_2)}\right|d\lambda_1d\lambda_2\\ &\leqslant \iint_{\hat{e}}\left[\sum_{i=1}^2\left(\frac{\partial\lambda_i}{\partial x}\right)^2\sum_{i=1}^2\left(\frac{\partial\hat{u}}{\partial\lambda_i}\right)^2+\sum_{i=1}^2\left(\frac{\partial\lambda_i}{\partial y}\right)^2\sum_{i=1}^2\left(\frac{\partial\hat{u}}{\partial\lambda_i}\right)^2\right]\left|\frac{\partial(x,y)}{\partial(\lambda_1,\lambda_2)}\right|d\lambda_1d\lambda_2\\ &\leqslant \frac{C}{h^2\sin^2\theta_1}h^2\iint_{\hat{e}}\sum_{i=1}^2\left(\frac{\partial\hat{u}}{\partial\lambda_i}\right)d\lambda_1d\lambda_2\\ &= \frac{C}{\sin^2\theta_1}|\hat{u}|_{1,\hat{e}}\end{aligned}$$

类似可证$s=2$的情况和另一组不等式.

□

令: $\hat{\Pi}\hat{u} = \sum_{i=1}^3\hat{u}(\hat{A}_i)\lambda_i$

引理: $\hat{\Pi}\hat{u} = \widehat{\Pi u}$

证明: 因为$\Pi u = \sum_{i=1}^3 u(A_i)\lambda_i(x,y)$, 注意

$$\widehat{\lambda_i(x,y)} = \lambda_i, A_i \leftrightarrow \hat{A}_i, u(A_i) = \hat{u}(\hat{A}_i), i = 1,2,3$$

故有:

$$\widehat{\Pi u}=\sum_{i=1}^{3} u\left(A_i\right) \widehat{\lambda}_i(x, y)=\sum_{i=1}^{3} \hat{u}\left(\hat{A}_i\right) \lambda_i=\hat{\Pi} \hat{u}$$

$\square$

引理: 设最小内角$\theta_i \geqslant \theta_0>0$, 则有

$$|u-\Pi u|_{s, \Omega} \leqslant \frac{C}{\sin ^{s+1} \theta_0} h^{2-s}|u|_{2, \Omega}$$

证明: 对任意单元e, 证明此不等式.注意

$$\frac{1}{\sin \theta_1} \leqslant \frac{1}{\sin \theta_0}$$

故有

$$\begin{aligned}
|u-\Pi u|_{s, e} & \leq \frac{C}{\sin ^s \theta_0} h^{1-s}|\widehat{u-\Pi u}|_{s, \hat{e}} \\
& =\frac{C}{\sin ^s \theta_0} h^{1-s}\left|\hat{u}-\widehat{\Pi u}\right|_{1, \hat{e}} \\
& \leq \frac{C}{\sin ^s \theta_0} h^{1-s}\left\|\hat{u}-\widehat{\Pi u}\right\|_{2, \hat{e}} \\
& \leq \frac{C}{\sin ^s \theta_0} h^{1-s}|\hat{u}|_{2, \hat{e}} \\
& \leq \frac{C}{\sin ^s \theta_0} h^{1-s} \frac{1}{\sin \theta_0} h|u|_{2, e} \\
& =\frac{C}{\sin ^{s+1} \theta_0} h^{2-s}|u|_{2, e}
\end{aligned}$$

$Soblolev$空间等价模定理, 对一切$\hat{u} \in H_2(\hat{e})$, 有

$$\|\hat{u}\|_{2, \hat{e}} \leqslant\left(|\hat{u}|_{2, \hat{e}}+\sum_{i=1}^{3}\left|l_i(\hat{u})\right|\right)$$

且$l_i, i=1,2,3$是$H_2(\hat{e})$上的有界线性泛函, 且若有一次多项式求 p_1, 有$l_i(\hat{p})=0, i=1,2,3$, 则有$\hat{p}=0$.