# 《偏微分方程数值解》

# Numerical Solutions to Partial Differential Equations

numpde_lecture_1_c1.pdf

School of Mathematical Sciences
Peking University

# Numerical Methods for PDEs

- Finite Difference Methods for Elliptic PDEs
- Finite Difference Methods for Parabolic PDEs
- Finite Difference Methods for Hyperbolic PDEs
- Finite Element Methods for Elliptic PDEs

# FDMs for Elliptic PDEs

# Definition of the elliptic PDEs — 2nd order

A 2nd order linear PDE with $n$ independent variables

$$\pm L(u) \triangleq \pm \left[ \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c \right] u = f, \qquad (1.1.1)$$

is elliptic in $\Omega$, if for every $x \in \Omega$, there exists $\alpha(x) > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \alpha(x) \sum_{i=1}^{n} \xi_i^2, \ \forall \xi \in \mathbb{R}^n \backslash \{0\}. \qquad (1.1.2)$$

(elliptic condition!)

**Note**: Eq. (2) implies the matrix $A = (a_{ij}(x))$ is positive definite.

inf α(x) is not necessarily greater than 0!

算子L在点x∈Ω处是椭圆型的，如果存在α(x)>0, s.t. (2) 式成立。
算子L在Ω内是椭圆型的，如果L在Ω内的每个点处都是椭圆型的。

# Definition of the elliptic PDE — 2nd order

- $L$ — the 2nd order linear elliptic operator;
- $a_{ij}$, $b_i$, $c$ — coefficients, functions of $x = (x_1, \ldots, x_n)$;
- $f$ — RHS term, or source term, a function of $x$;

The operator $L$ and Eq. (1) are said to be uniformly elliptic in $\Omega$, if (2) holds with

$$\inf_{x\in\Omega} \alpha(x) = \alpha_0 > 0, \tag{1.1.3}$$

i.e.,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha_0 \sum_{i=1}^{n} \xi_i^2 , \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}.$$

算子L在Ω内是一致椭圆型的，如果存在不依赖x的正数$\alpha_0>0$ s.t.(3)成立。

# Definition of the elliptic PDE — 2nd order

**Example**: $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is a linear 2nd order uniformly elliptic operator, because

$$a_{ii} = 1, \ \forall i, \quad a_{ij} = 0, \ \forall i \neq j,$$

and the Poisson equation

$$-\triangle u(x) = f(x)$$

is a linear 2nd order uniformly elliptic PDE.

# Definition of the elliptic PDEs — $2m$-th order

A linear PDE of order $2m$ with $n$ independent variables

$$\pm L(u) \triangleq \pm \left[ \sum_{k=1}^{2m} \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1,\ldots,i_k} \frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} + a_0 \right] u = f, \qquad (1.1.4)$$

is elliptic in $\Omega$ if for every $x \in \Omega$, there exists $\alpha(x) > 0$ such that

$$\sum_{i_1,\ldots,i_{2m}=1}^{n} a_{i_1,\ldots,i_{2m}}(x)\xi_{i_1} \cdots \xi_{i_{2m}} \geq \alpha(x) \sum_{i=1}^{n} \xi_i^{2m},$$

$$\alpha(x) > 0, \forall\, \xi \in \mathbb{R}^n \setminus \{0\}\ . \qquad (1.1.5)$$

Note that (5) implies the $2m$ order tensor $A = (a_{i_1,\ldots,i_{2m}})$ is positive definite.

# Definition of the elliptic PDEs — $2m$-th order

- $L$ – the $2m$-th order linear elliptic operator;
- $a_{i_1,\ldots,i_k}$, $a_0$ — coefficients, functions of $x = (x_1, \ldots, x_n)$;
- $f$ — RHS term, or source term, a function of $x$;

The operator $L$ and Eq. (4) are said to be uniformly elliptic in $\Omega$, if (5) holds with

$$\inf_{x\in\Omega} \alpha(x) = \alpha_0 > 0, \qquad (1.1.3)'$$

i.e.,

$$\sum_{i_1,\ldots,i_{2m}=1}^{n} a_{i_1,\ldots,i_{2m}}(x)\xi_{i_1}\cdots\xi_{i_{2m}} \geq \alpha_0 \sum_{i=1}^{n} \xi_i^{2m}, \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}.$$

# Definition of the elliptic PDEs

**Example**: the 2$m$-th order harmonic equation

$$(-\triangle)^m u = f$$

is a linear 2$m$-th order uniformly elliptic PDE, and $\triangle^m$ is a linear 2$m$-th order uniformly elliptic operator, because

$$a_{i_1,\dots,i_{2m}}(x) = 1, \quad \text{if the indexes appear in pairs;}$$

$$a_{i_1,\dots,i_{2m}}(x) = 0, \quad \text{otherwise.}$$

In particular, the biharmonic equation $\triangle^2 u = f$ is a linear 4th order uniformly elliptic PDE, and $\triangle^2$ is a linear 4-th order uniformly elliptic operator.

n个自变量p个应变量的方程组(1.1.6)是在点x处是椭圆型的，如果存在α(x)>0, s.t. (1.1.7) 式成立

# Steady state convection-diffusion equation

1. $x \in \Omega \subset \mathbb{R}^n$;
2. $\mathbf{v}(x)$: the fluid velocity at $x$;
3. $u(x)$: the fluid density at $x$;
4. $a(x) > 0$: the diffusive coefficient;
5. $f(x)$: the density of the source or sink of the substance.
6. $J$: the diffusion flux (amount of substance per unit area per unit time)

- Fick's law: $J = -a(x)\nabla u(x)$.

  菲克定律是指在不依靠宏观的混合作用发生的传质现象时，描述分子扩散过程中传质通量与浓度梯度之间关系的定律.

# Steady state convection-diffusion equation

For an arbitrary open subset $\omega \subset \Omega$ with piecewise smooth boundary $\partial\omega$, Fick's law says the substance brought into $\omega$ by diffusion per unit time is given by

Fick's law

$$\int_{\partial\omega} J \cdot (-\nu(x))\, ds = \int_{\partial\omega} a(x)\nabla u(x) \cdot \nu(x)\, ds,$$

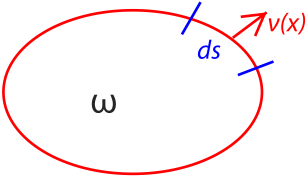
ν(x)是边界的单位外法向量

while the substance brought into $\omega$ by the flow per unit time is

$$\int_{\partial\omega} u(x)\mathbf{v}(x) \cdot (-\nu(x))\, ds$$

通过边界流入的

and the substance produced in $\omega$ by the source per unit time is




$$\int_{\omega} f(x)\, dx.$$

源或汇引起的变化

# Steady state convection-diffusion equation

Therefore, the net change of the substance in $\omega$ per unit time is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega} u(x)\,dx = \int_{\partial\omega} a(x)\nabla u(x)\cdot\nu(x)\,ds - \int_{\partial\omega} u(x)\mathbf{v}(x)\cdot\nu(x)\,ds + \int_{\omega} f(x)\,dx.$$

By the steady state assumption, $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega} u(x)\,dx = 0$, for arbitrary $\omega$, and by the divergence theorem (or Green's formula or Stokes formula), this leads to the steady state convection-diffusion equation in the integral form

$$\int_{\omega} \{\nabla\cdot(a\nabla u - u\,\mathbf{v}) + f\}\,dx = 0, \quad \forall\omega \tag{1.1.10}$$

# Steady state convection-diffusion equation

The term $-[a(x)\nabla u(x) - u(x)\mathbf{v}(x)]$ is named as the substance flux, since it represents the speed that the substance flows.

Assume that $\nabla \cdot (a\nabla u - u\,\mathbf{v}) + f$ is smooth, then, we obtain the steady state convection-diffusion equation in the differential form

$$-\nabla \cdot (a(x)\nabla u(x) - u\,\mathbf{v}) = f(x), \quad \forall x \in \Omega. \qquad (1.1.11)$$

In particular, if $\mathbf{v} = 0$ and $a = 1$, we have the steady state diffusion equation $-\triangle u = f$. ---Poisson 方程

# Boundary conditions for the elliptic equations

For a complete steady state convection-diffusion problem, or problems of elliptic equations in general, we also need to impose proper boundary conditions.

Three types of most commonly used boundary conditions:

First type $$u = u_D, \quad \forall x \in \partial\Omega; \tag{1.1.12}$$

Second type $$\frac{\partial u}{\partial \nu} = g, \quad \forall x \in \partial\Omega; \tag{1.1.13}$$

ν是边界的单位外法向量

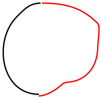
Third type $$\frac{\partial u}{\partial \nu} + \alpha u = g, \quad \forall x \in \partial\Omega; \tag{1.1.14}$$

where $\alpha \geq 0$, and $\alpha > 0$ at least on some part of the boundary (*physical meaning: higher density produces bigger outward diffusion flux*).

密度越高，向外的扩散通量越大。

# Boundary conditions for the steady state convection-diffusion equation

- 1st type boundary condition — Dirichlet boundary condition;
- 2nd type boundary condition — Neumann boundary condition;
- 3rd type boundary condition — Robin boundary condition;
- Mixed-type boundary conditions — different types of boundary conditions imposed on different parts of the boundary.



$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$

# General framework of Finite Difference Methods

1. Discretize the domain $\Omega$ by introducing a grid;
2. Discretize the function space by introducing grid functions;
3. Discretize the differential operators by properly defined difference operators;
4. Solve the discretized problem to get a finite difference solution;
5. Analyze the approximate properties of the finite difference solution.

# Dirichlet boundary value problem of the Poisson equation

第1.2小节

$$\begin{cases} -\triangle u(x) = f(x), & \forall x \in \Omega, \\ u(x) = u_D(x), & \forall x \in \partial\Omega, \end{cases} \tag{1.2.1}$$

where $\Omega = (0,\ 1) \times (0,\ 1)$ is a rectangular region.

# Discretize $\Omega$ by introducing a grid

1. Space (spatial) step sizes: $\triangle x = \triangle y = h = 1/N$;
2. Index set of the grid nodes: $J = \{(i,j) : (x_i, y_j) \in \overline{\Omega}\}$;
3. Index set of grid nodes on the Dirichlet boundary: $J_D = \{(i,j) : (x_i, y_j) \in \partial\Omega\}$;
4. Index set of interior nodes: $J_\Omega = J \setminus J_D$.

For simplicity, both $(i,j)$ and $(x_i, y_j)$ are called grid nodes.

# Discretize the function space by introducing grid functions

- $u_{i,j} = u(x_i, y_j)$, exact solution restricted on the grid;
- $f_{i,j} = f(x_i, y_j)$, source term restricted on the grid;
- $U_{i,j}$, numerical solution on the grid;
- $V_{i,j}$, a grid function.

# Discretize differential operators by difference operators

- $\dfrac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\triangle x^2} \approx \partial_x^2 u;$
- $\dfrac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\triangle y^2} \approx \partial_y^2 u;$

The Poisson equation $-\triangle u(x) = f(x)$ is discretized as the 5 point difference scheme

$$-L_h U_{i,j} \triangleq \frac{4U_{i,j} - U_{i-1,j} - U_{i,j-1} - U_{i+1,j} - U_{i,j+1}}{h^2} = f_{i,j}, \quad \forall (i,j) \in J_\Omega. \tag{1.2.2}$$

The Dirichlet boundary condition is discretized as

$$U_{i,j} = u_D(x_i, y_j), \quad \forall (i,j) \in J_D. \tag{1.2.3}$$

# Solution of the discretized problem

The discrete system

$$-L_h U_{i,j} \triangleq \frac{4U_{i,j} - U_{i-1,j} - U_{i,j-1} - U_{i+1,j} - U_{i,j+1}}{h^2} = f_{i,j}, \quad \forall (i,j) \in J_\Omega, \tag{1.2.2}$$

$$U_{i,j} = u_D(x_i, y_j), \quad \forall (i,j) \in J_D, \tag{1.2.3}$$

is a system of linear algebraic equations, whose matrix is symmetric positive definite. Consequently, there is a unique solution.

***Proof: see Page 6***

# Analyze the Approximate Property of the Discrete Solution

1. Approximation error: $e_{i,j} = U_{i,j} - u_{i,j}$;

2. The error equation:

$$-L_h e_{i,j} \triangleq \frac{4e_{i,j} - e_{i-1,j} - e_{i,j-1} - e_{i+1,j} - e_{i,j+1}}{h^2} = T_{i,j}, \ \forall (i,j) \in J_\Omega; \quad (1.2.4)$$

3. The local truncation error

$$T_{i,j} := [(L_h - L)u]_{i,j} = L_h u_{i,j} - (Lu)_{i,j} = L_h u_{i,j} + f_{i,j}, \quad \forall (i,j) \in J_\Omega.$$

4. $\|e_h\| = \|(-L_h)^{-1} T_h\| \le \|(-L_h)^{-1}\| \|T_h\|$.

# Truncation Error of the 5 Point Difference Scheme

Suppose that the function $u$ is sufficiently smooth, then, by Taylor series expansion of $u$ on the grid node $(x_i, y_j)$, we have

$$u_{i\pm1,j} = \left[u \pm h\partial_x u + \frac{h^2}{2}\partial_x^2 u \pm \frac{h^3}{6}\partial_x^3 u + \frac{h^4}{24}\partial_x^4 u \pm \frac{h^5}{120}\partial_x^5 u + \cdots\right]_{i,j}$$

$$u_{i,j\pm1} = \left[u \pm h\partial_y u + \frac{h^2}{2}\partial_y^2 u \pm \frac{h^3}{6}\partial_y^3 u + \frac{h^4}{24}\partial_y^4 u \pm \frac{h^5}{120}\partial_y^5 u + \cdots\right]_{i,j}$$

Since $T_{i,j} = L_h u_{i,j} + f_{i,j}$ and $f_{i,j} = -\triangle u_{i,j}$, we obtain

$$T_{i,j} := \frac{1}{12}h^2(\partial_x^4 u+\partial_y^4 u)_{i,j}+\frac{1}{360}h^4(\partial_x^6 u+\partial_y^6 u)_{i,j}+O(h^6), \quad \forall (i,j) \in J_\Omega. \quad (1.2.7)$$

注：这里*u*为*PDE*的解

# Consistency and Order of Accuracy of $L_h$

1. Consistent condition of the scheme (or $L_h$ to $L$) in $l^\infty$-norm:

$$\lim_{h\to 0} T_h = \lim_{h\to 0} \max_{(i,j)\in J_\Omega} |T_{i,j}| = 0, \qquad (1.2.6)$$

2. The order of the approximation accuracy of the scheme (or $L_h$ to $L$): 2nd order approximation accuracy, since $T_h = O(h^2)$

# Stability of the Scheme

Remember that

$$\|e_h\|_\infty = \|(-L_h)^{-1}T_h\|_\infty \le \|(-L_h)^{-1}\|_\infty \|T_h\|_\infty$$

$$\lim_{h\to 0} T_h = \lim_{h\to 0} \max_{(i,j)\in J_\Omega} |T_{i,j}| = 0, \qquad \textbf{(1.2.6)}$$

therefore $\lim_{h\to 0}\|e_h\|_\infty = 0$, if $\|(-L_h)^{-1}\|_\infty$ is uniformly bounded, *i.e.* there exists a constant $C$ independent of $h$ such that

由格式或算子*Lh*的定义

$$\max_{(i,j)\in J} |U_{i,j}| \le C\left(\max_{(i,j)\in J_\Omega} |f_{i,j}| + \max_{(i,j)\in J_D} |(u_D)_{i,j}|\right). \qquad \textbf{(1.2.8)}$$

$\|(-L_h)^{-1}\|_\infty \le C$ is the stability of the scheme in $l^\infty$-norm.

# Convergence and the Accuracy of the Scheme

Remember that

$$-L_h U_{i,j} = \frac{4U_{i,j} - U_{i-1,j} - U_{i,j-1} - U_{i+1,j} - U_{i,j+1}}{h^2} = f_{i,j}, \quad \forall (i,j) \in J_\Omega.$$

**(1.2.2)**

$$-L_h e_{i,j} \triangleq \frac{4e_{i,j} - e_{i-1,j} - e_{i,j-1} - e_{i+1,j} - e_{i,j+1}}{h^2} = T_{i,j}, \quad \forall (i,j) \in J_\Omega.$$

**(1.2.4)**

therefore, since $\max_{(i,j)\in J_D} |e_{i,j}| = 0$,

$$\max_{(i,j)\in J} |U_{i,j}| \leq C \left( \max_{(i,j)\in J_\Omega} |f_{i,j}| + \max_{(i,j)\in J_D} |(u_D)_{i,j}| \right). \qquad \textbf{(1.2.8)}$$

implies also

$$\max_{(i,j)\in J} |e_{i,j}| \leq C \max_{(i,j)\in J_\Omega} |T_{i,j}| \leq C\, T_h \leq C\, h^2 \max_{(x,y)\in\overline{\Omega}} \left(M_{xxxx} + M_{yyyy}\right),$$

**(1.2.9)**

where $M_{xxxx} = \max_{(x,y)\in\overline{\Omega}} |\partial_x^4 u|$, $M_{yyyy} = \max_{(x,y)\in\overline{\Omega}} |\partial_y^4 u|$.

# The Maximum Principle and Comparison Theorem

- **Maximum principle of $L_h$**: for any grid function $\Psi$, $L_h\Psi \geq 0$, *i.e* $4\Psi_{i,j} \leq \Psi_{i-1,j} + \Psi_{i+1,j} + \Psi_{i,j-1} + \Psi_{i,j+1}$, implies that $\Psi$ can not assume nonnegative maximum in the set of interior nodes $J_\Omega$, unless $\Psi$ is a constant.

- **Comparison Theorem**: Let $F = \max_{(i,j)\in J_\Omega} |f_{i,j}|$ and $\Phi(x,y) = (x-1/2)^2 + (y-1/2)^2$, take a comparison function

$$\Psi^{\pm}_{i,j} = \pm U_{i,j} + \frac{1}{4} F\Phi_{i,j}, \quad \forall\,(i,j) \in J.$$

***(1.2.10)***
***(1.2.12)***

It is easily verified that $L_h\Psi^{\pm} \geq 0$. Thus, noticing that $\Phi \geq 0$ and by the maximum principle, we obtain

$$\pm U_{i,j} \leq \pm U_{i,j} + \frac{1}{4} F\Phi_{i,j} \leq \max_{(i,j)\in J_D} |(u_0)_{i,j}| + \frac{1}{8} F, \quad \forall\,(i,j) \in J_\Omega.$$

***(1.2.11)***
***(1.2.13)***

Consequently, $\|U\|_\infty \leq \frac{1}{8} \max_{(i,j)\in J_\Omega} |f_{i,j}| + \max_{(i,j)\in J_D} |(u_0)_{i,j}|$,

***(1.2.8)***

# The Maximum Principle and Comparison Theorem

Apply the maximum principle and comparison theorem to the error equation

$$-L_h e_{i,j} \triangleq \frac{4e_{i,j} - e_{i-1,j} - e_{i,j-1} - e_{i+1,j} - e_{i,j+1}}{h^2} = T_{i,j}, \quad \forall (i,j) \in J_\Omega.$$

we obtain

$$\|e\|_\infty \le \max_{(i,j)\in J_D} |e_{i,j}| + \frac{1}{8} T_h, \tag{1.2.14}$$

where $T_h = \max_{(i,j)\in J_\Omega} |T_{i,j}|$ is the $l^\infty$-norm of the truncation error.

# Grid and multi-index of grid

第1.3节 一般问题的差分逼近

1. Discretize $\Omega \subset \mathbb{R}^n$: introduce a grid, say by taking the step sizes $h_i = \triangle x_i$, $i = 1, \ldots, n$, for the corresponding coordinate components;
2. The set of multi-index: $J = \{\mathbf{j} = (j_1, \cdots, j_n) : x = x_{\mathbf{j}} \triangleq (j_1 h_1, \cdots, j_n h_n) \in \bar{\Omega}\}$;
3. The index set of Dirichlet boundary nodes: $J_D = \{\mathbf{j} \in J : x = (j_1 h_1, \cdots, j_n h_n) \in \partial\Omega_D\}$;
4. The index set of interior nodes: $J_\Omega = J \setminus J_D$.

For simplicity, both $(i, j)$ and $(x_i, y_j)$ are called grid nodes.

# Regular and irregular interior nodes with respect to $L_h$

1. Adjacent nodes: $\mathbf{j}, \mathbf{j}' \in J$ are adjacent, if $\sum_{k=1}^{n} |j_k - j'_k| = 1$;
2. $D_{L_h}(\mathbf{j})$: the set of nodes other than $\mathbf{j}$ used in calculating $L_h U_{\mathbf{j}}$
3. Regular interior nodes (with respect to $L_h$): $\mathbf{j} \in J_\Omega$ such that $D_{L_h}(\mathbf{j}) \subset \bar{\Omega}$;
4. Regular interior set $\mathring{J}_\Omega$: the set of all regular interior nodes;
5. Irregular interior set: $\tilde{J}_\Omega = J_\Omega \setminus \mathring{J}_\Omega$;
6. Irregular interior nodes (with respect to $L_h$): $\mathbf{j} \in \tilde{J}_\Omega$.

# The control volume, grid functions and norms

1. Control volume of the node $\mathbf{j} \in J$:
$$\omega_{\mathbf{j}} = \{x \in \Omega : \ (j_i - \frac{1}{2})h_i \le x_i < (j_i + \frac{1}{2})h_i, \ 1 \le i \le n\}, \tag{1.3.1}$$
and denote $V_{\mathbf{j}} = \text{meas}(\omega_{\mathbf{j}})$;
2. Grid function $U(x)$: extend $U_{\mathbf{j}}$ to a piecewise constant function defined on $\Omega$
$$U(x) = U_{\mathbf{j}}, \qquad \forall\, x \in \omega_{\mathbf{j}}. \tag{1.3.2}$$
3. $\mathbb{L}^p(\Omega)$ $(1 \le p \le \infty)$ norms of $U(x)$:
$$\|U\|_p = \Big\{ \sum_{j \in J} V_{\mathbf{j}} |U_{\mathbf{j}}|^p \Big\}^{1/p}, \qquad \|U\|_\infty = \max_{\mathbf{j} \in J} |U_{\mathbf{j}}|.$$

# Basic Difference Operators

1. $1^{st}$-order forward: $\triangle_{+x} v(x, x') := v(x + \triangle x, x') - v(x, x')$; **(1.3.3)**
2. $1^{st}$-order backward: $\triangle_{-x} v(x, x') := v(x, x') - v(x - \triangle x, x')$; **(1.3.4)**
3. $1^{st}$-order central: on one grid step

$$\delta_x v(x, x') := v(x + \frac{1}{2}\triangle x, x') - v(x - \frac{1}{2}\triangle x, x'), \qquad (1.3.5)$$

   and on two grid steps

$$\begin{aligned} \triangle_{0x} v(x, x') &:= \frac{1}{2}\left(\triangle_{+x} + \triangle_{-x}\right) v(x, x') \\ &= \frac{1}{2}\left[v(x + \triangle x, x') - v(x - \triangle x, x')\right] \end{aligned} \qquad (1.3.6)$$

4. 2nd order central: $\delta_x^2 v(x, x') = \delta_x(\delta_x v(x, x')) =$
   $v(x + \triangle x, x') - 2v(x, x') + v(x - \triangle x, x')$. **(1.3.7)**

# A FD Scheme for the Steady State Convection-Diffusion Equation

$$-\nabla\cdot(a(x,y)\nabla u(x,y))+\nabla\cdot(u(x,y)\,\mathbf{v}(x,y)) = f(x,y),\ \forall(x,y)\in\Omega, \tag{1.3.8}$$

Substitute the differential operators by difference operators:

1. $(au_x)_x|_{i,j} \sim \delta_x(a_{i,j}\delta_x u_{i,j})/(\triangle x)^2$: where
$$\delta_x(a_{i,j}\delta_x u_{i,j}) = a_{i+\frac{1}{2},j}(u_{i+1,j}-u_{i,j}) - a_{i-\frac{1}{2},j}(u_{i,j}-u_{i-1,j});$$
2. $(au_y)_y|_{i,j} \sim \delta_y(a_{i,j}\delta_y u_{i,j})/(\triangle y)^2$: where
$$\delta_y(a_{i,j}\delta_y u_{i,j}) = a_{i,j+\frac{1}{2}}(u_{i,j+1}-u_{i,j}) - a_{i,j-\frac{1}{2}}(u_{i,j}-u_{i,j-1});$$
3. $(uv^1)_x|_{i,j}2\triangle x \sim \triangle_{0x}(uv^1)_{i,j} = (uv^1)_{i+1,j} - (uv^1)_{i-1,j};$
4. $(uv^2)_y|_{i,j}2\triangle y \sim \triangle_{0y}(uv^2)_{i,j} = (uv^2)_{i,j+1} - (uv^2)_{i,j-1};$

# A FD Scheme for the Steady State Convection-Diffusion Equation

we are lead to the following finite difference scheme for the steady state convection-diffusion equation:

$$
\begin{aligned}
&-\frac{a_{i+\frac{1}{2},j}(U_{i+1,j}-U_{i,j})-a_{i-\frac{1}{2},j}(U_{i,j}-U_{i-1,j})}{(\triangle x)^2}\\
&\qquad-\frac{a_{i,j+\frac{1}{2}}(U_{i,j+1}-U_{i,j})-a_{i,j-\frac{1}{2}}(U_{i,j}-U_{i,j-1})}{(\triangle y)^2}\\
&\quad+\frac{(Uv^1)_{i+1,j}-(Uv^1)_{i-1,j}}{2\triangle x}+\frac{(Uv^2)_{i,j+1}-(Uv^2)_{i,j-1}}{2\triangle y}=f_{i,j}.
\end{aligned}
\tag{1.3.9}
$$

# A Finite Volume Scheme for the Steady State Convection-Diffusion Equation in Conservation Form

$$\int_{\partial\omega} (a(x,y)\nabla u(x,y) - u(x,y)\mathbf{v}(x,y)) \cdot \nu(x,y)\, ds + \int_{\omega} f(x,y)\, dxdy = 0. \quad \text{(1.3.10)}$$

Take a proper control volume $\omega$ and substitute the differential operators by appropriate difference operators, and integrals by appropriate numerical quadratures:

1. for the index $(i,j) \in J_\Omega$, taking the control volume $\omega_{i,j} =$ **(1.3.11)**
$$\left\{ (x,y) \in \Omega \cap \{[(i-\frac{1}{2})h_x, (i+\frac{1}{2})h_x) \times [(j-\frac{1}{2})h_y, (j+\frac{1}{2})h_y)\} \right\};$$
2. Applying the middle point quadrature on $\omega_{i,j}$ as well as on its four edges;
3. $\partial_\nu u(x_{i+\frac{1}{2}}, y_j) \quad \sim \quad (u_{i+1,j} - u_{i,j})/h_x$, etc.;

we are lead to the following finite volume scheme for the steady state convection-diffusion equation:

$$
\begin{aligned}
&-\frac{a_{i+\frac{1}{2},j}(U_{i+1,j}-U_{i,j})-a_{i-\frac{1}{2},j}(U_{i,j}-U_{i-1,j})}{(\triangle x)^2}\\
&-\frac{a_{i,j+\frac{1}{2}}(U_{i,j+1}-U_{i,j})-a_{i,j-\frac{1}{2}}(U_{i,j}-U_{i,j-1})}{(\triangle y)^2}\\
&+\frac{(U_{i+1,j}+U_{i,j})v^1_{i+\frac{1}{2},j}-(U_{i,j}+U_{i-1,j})v^1_{i-\frac{1}{2},j}}{2\triangle x}\\
&+\frac{(U_{i,j+1}+U_{i,j})v^2_{i,j+\frac{1}{2}}-(U_{i,j}+U_{i,j-1})v^2_{i,j-\frac{1}{2}}}{2\triangle y}=f_{i,j},
\end{aligned}
\tag{1.3.12}
$$

which is also called a conservative finite difference scheme.

# A Finite Volume Scheme for Partial Differential Equations in Conservation Form

Finite volume methods:

1. control volume;
2. numerical flux;
3. conservative form.

# More General Finite Difference Schemes

***Page 13***

In more general case, say for triangular grid, hexagon grid, nonuniform grid, unstructured grid, and even grid less situations, in principle, we could still establish a finite difference scheme by

1. Taking proper neighboring nodes $J(P)$;
2. Approximating $Lu(P)$ by $L_h U_P := \sum_{i \in J(P)} c_i(P) U(Q_i)$;
3. Determining the weights $c_i(P)$ according to certain requirements, say the order of the local truncation error, local conservative property, discrete maximum principle, etc..

习题 1：1，3 *See Page 29*

# Thank You!