Numerical Solutions to Partial Differential Equations

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School of Mathematical Sciences

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Discretization of Boundary Conditions

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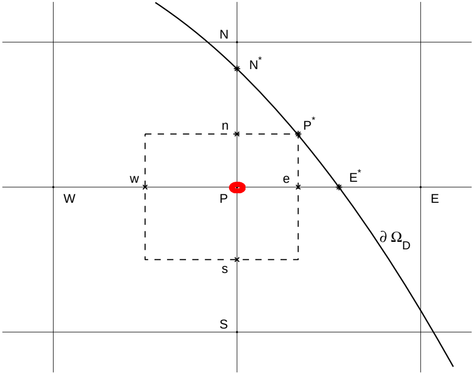
On boundary nodes and irregular interior nodes, we usually need to construct different finite difference approximation schemes to cope with the boundary conditions.

Remember that the set of irregular interior nodes is given by $\tilde{J}_\Omega = \{\mathbf{j} \in J \setminus J_D : D_{L_h}(\mathbf{j}) \not\subset J\}$, that is $\tilde{J}_\Omega$ is the set of all such interior node which has at least one neighboring node not located in $\bar{\Omega}$.

For simplicity, we take the standard 5-point difference scheme for the 2-D Poisson equation $-\triangle u = f$ as an example to see how the boundary conditions are handled.

Discretization of the Dirichlet Boundary Condition

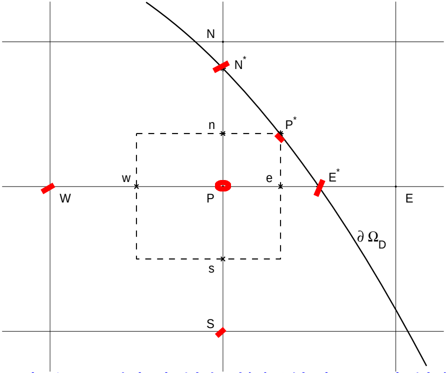
Since N, E are not in J, P is a irregular interior node, on which we need to construct a difference equation using the Dirichlet boundary condition on the nearby points N^*, P^* and/or E^*. The simplest way is to apply interpolations.



Discretization of the Dirichlet Boundary Condition

1 Difference equations on P derived by interpolations:

- Zero order: $U_P = U_{P^*}$ with truncation error $O(h)$;
- First order: $U_P = \frac{h_x U_{E^*} + h_x^* U_W}{h_x + h_x^*}$ or $U_P = \frac{h_y U_{N^*} + h_y^* U_S}{h_y + h_y^*}$, with truncation error $O(h^2)$;

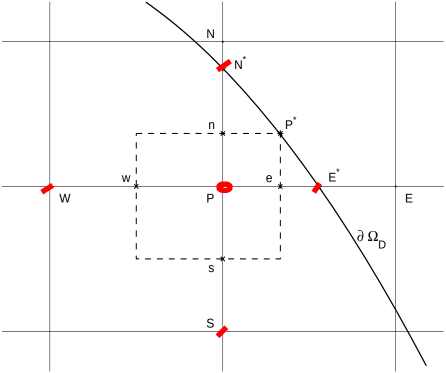


用边界点和正则内点处解的插值表示P点处的解

Discretization of the Dirichlet Boundary Condition

2 Difference equations on P can be derived by extrapolations and the standard 5-point difference scheme:

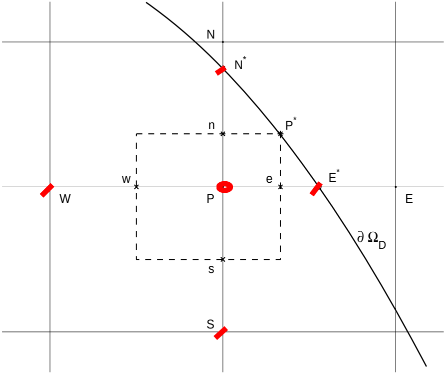
- The grid function values on the ghost nodes N and E can be given by second order extrapolations using the grid function values on S, P, N^* and W, P, E^* respectively (see Exercise 1.3).



Discretization of the Dirichlet Boundary Condition

Difference equations on P can also be derived by the Taylor series expansions and the partial differential equation to be solved:

- Express u_W, u_{E^*}, u_S, u_{N^*} by the Taylor expansions of u at P. Express u_x, u_y, u_{xx}, u_{yy} on P in terms of u_W, u_{E^*}, u_S, u_{N^*} and u_P. Substitute these approximation values into the differential equation (see Exercise 1.4).

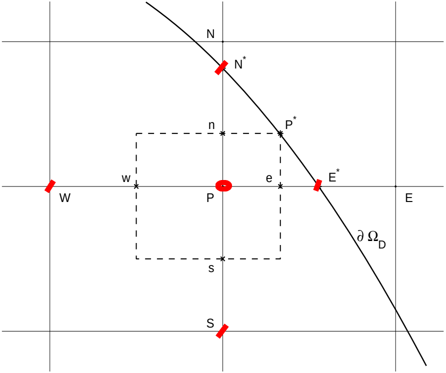


Discretization of the Dirichlet Boundary Condition

2a Finite difference schemes with nonuniform grid spacing:
a difference equation on P using the values of U on the nodes N^*, S, W, E^* and P with truncation error $O(h)$:

$$-\left\{\frac{2}{h_x+h_x^*}\left(\frac{U_{E^*}-U_P}{h_x^*}-\frac{U_P-U_W}{h_x}\right)+\frac{2}{h_y+h_y^*}\left(\frac{U_{N^*}-U_P}{h_y^*}-\frac{U_P-U_S}{h_y}\right)\right\}=f_P. \quad (1.3.23)$$

Shortcoming:
nonsymmetric.

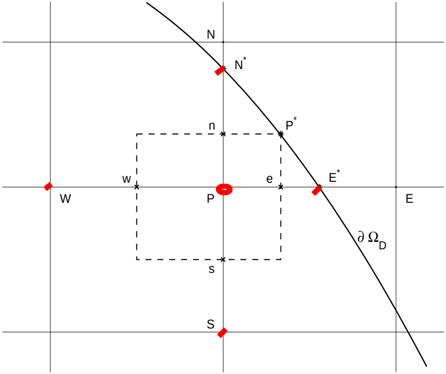


Discretization of the Dirichlet Boundary Condition

2b Symmetric finite difference schemes with nonuniform grid spacing: a difference equation on P using the values of U on the nodes N^*, S, W, E^* and P with truncation error $O(1)$:

$$-\left\{\frac{1}{h_x}\left(\frac{U_{E^*}-U_P}{h_x^*}-\frac{U_P-U_W}{h_x}\right)+\frac{1}{h_y}\left(\frac{U_{N^*}-U_P}{h_y^*}-\frac{U_P-U_S}{h_y}\right)\right\}=f_P. \qquad (1.3.24)$$

It can be shown: the global error is $O(h^2)$.



Discretization of the Dirichlet Boundary Condition

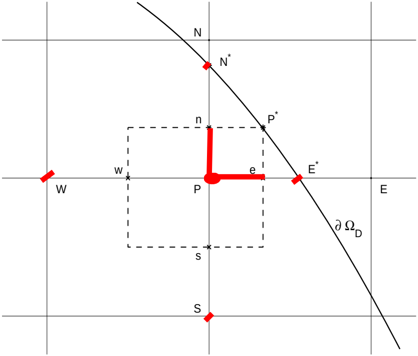
3 Construct a finite difference equation on P based on the integral form of the partial differential equation $-\int_{\partial V_P} \frac{\partial u}{\partial \nu}\, ds = \int_{V_P} f\, dx$: (1.3.25)

$$-\left(\frac{U_W - U_P}{h_x} + \frac{U_{E^*} - U_P}{h_x^*}\right)\frac{h_y + \phi h_y^*}{2}$$

$$-\left(\frac{U_S - U_P}{h_y} + \frac{U_{N^*} - U_P}{h_y^*}\right)\frac{h_x + \theta h_x^*}{2} = f_P \frac{(h_x + \theta h_x^*)(h_y + \phi h_y^*)}{4}, \quad (1.3.26)$$

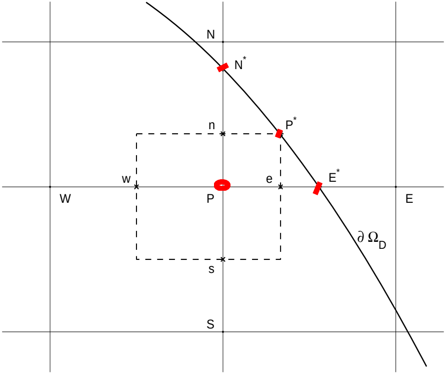
where $\theta h_x^*/2$, $\phi h_y^*/2$ are the lengthes of the line segments $\overline{Pe}$ and $\overline{Pn}$.

($O(h)$, nonsymmetric)



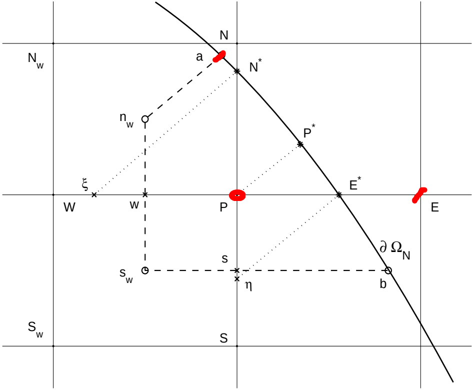
Extension of the Dirichlet Boundary Condition Nodes J_D

Add all of the Dirichlet boundary points used in the equations on the irregular interior nodes concerning the curved Dirichlet boundary, such as E^*, N^* and P^*, into the set J_D to form an extended set of Dirichlet boundary nodes, still denoted by J_D.



Discretization of the Neumman Boundary Condition

4 Since N, E are not in J, P is a irregular interior node, on which we need to construct a difference equation using the Nuemman boundary condition on the nearby points N^*, P^* and/or E^*. The simplest way is again to apply interpolations.

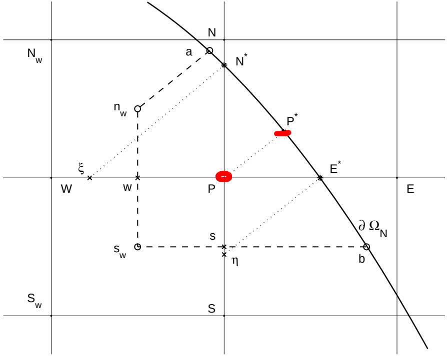


Discretization of the Neumman Boundary Condition

Let P^* be the closest point to P on $\partial\Omega_N$, and α be the angle between the x-axis and the out normal to $\partial\Omega_N$ at the point P^*.

$\partial_\nu u(P^*) \sim \nabla u(P)\cdot \nu_{P^*}$, a zero order extrapolation to the out normal, leads to a difference equation on P with local truncation error $O(h)$:

$$\frac{U_P - U_W}{h_x}\cos\alpha + \frac{U_P - U_S}{h_y}\sin\alpha = g(P^*). \qquad (1.3.27)$$



用边界点和正则内点处解的插值表示P*点处法向导数

Discretization of the Neumman Boundary Condition

We can combine the nonuniform grid spacing difference equations

$$-\left\{\frac{2}{h_x+h_x^*}\left(\frac{U_{E^*}-U_P}{h_x^*}-\frac{U_P-U_W}{h_x}\right)+\frac{2}{h_y+h_y^*}\left(\frac{U_{N^*}-U_P}{h_y^*}-\frac{U_P-U_S}{h_y}\right)\right\}=f_P, \quad (1.3.23)$$

非对称

or

$$-\left\{\frac{1}{h_x}\left(\frac{U_{E^*}-U_P}{h_x^*}-\frac{U_P-U_W}{h_x}\right)+\frac{1}{h_y}\left(\frac{U_{N^*}-U_P}{h_y^*}-\frac{U_P-U_S}{h_y}\right)\right\}=f_P, \quad (1.3.24)$$

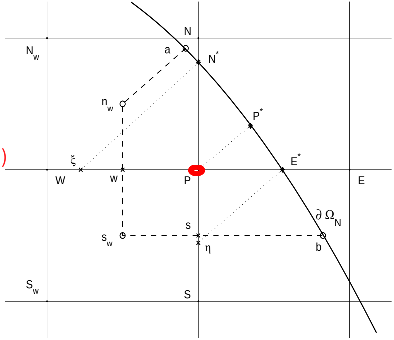
对称

on the irregular interior node P, and add in the difference equations for the new unknowns U_{N^*} and U_{E^*} by making use of the boundary conditions. Say

$$\frac{U_{N^*}-U_\xi}{|\overline{\xi N^*}|}=g(N^*),\ O(h),$$

and

$$\frac{U_{E^*}-U_\eta}{|\overline{\eta E^*}|}=g(E^*),\ O(h). \quad (1.3.28)$$

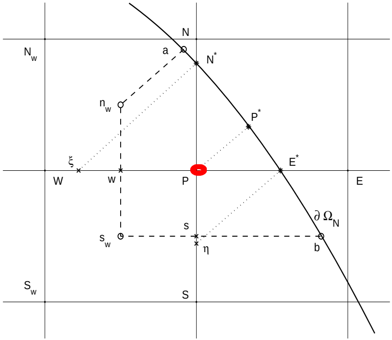


Discretization of the Neumman Boundary Condition

The numerical method based on the integral form of the Poisson equation $-\int_{\partial V_P} \frac{\partial u}{\partial \nu}\, ds = \int_{V_P} f\, dx$ with V_P being the domain enclosed by the broken line segments, where $\overline{an_w} \perp \overline{PN_W}$, leads to an asymmetric scheme on the irregular interior node P

$$-\frac{U_{N_W} - U_P}{|\overline{N_W P}|}|\overline{an_w}| - \frac{U_W - U_P}{h_x} h_y - \frac{U_S - U_P}{h_y}|\overline{s_w b}| - g(P^*)|\widetilde{ab}| = f(P)|V_P|. \quad (1.3.29)$$

The local truncation error is $O(h)$, since numerical quadrature is not centered.



More Emphasis on Global Properties

In dealing with the BCs, in comparison to the local truncation error, more attention should be paid on the important global features: symmetry, maximum principle, conservation, etc.

so that the finite difference scheme can

- have good stability and higher order of global convergence;
- inherit as much as possible the important global properties from the analytical solution;
- be solved by applying fast solvers.

More Emphasis on Global Properties

Truncation Error, Consistency, Stability and Convergence

第*1.3.3*节（*Page 13*）

Consider the BVP of a PDE

$$\begin{cases} -Lu(x) = f(x), & \forall x \in \Omega, \\ Gu(x) = g(x), & \forall x \in \partial\Omega \end{cases} \tag{1.3.13}$$

and its finite difference equation on a rectangular grid with spacing h

$$-L_h U_{\mathbf{j}} = f_{\mathbf{j}}, \forall j \in J. \tag{1.3.14}$$

Notice, if $\mathbf{j}$ is not a regular interior node, then L_h and $f_{\mathbf{j}}$ may depend on G, g, L and f.

Denote $\bar{L}u(x) = Lu(x)$, if $x \in \Omega$, and $\bar{L}u(x) = Gu(x)$, if $x \in \partial\Omega$.

Truncation Error

Definition

Suppose that the solution u to the problem is sufficiently smooth. Let

$$T_{\mathbf{j}}(u) = L_h u_{\mathbf{j}} - (\bar{L}u)_{\mathbf{j}}, \quad \forall j \in J. \qquad (1.3.15)$$

Define $T_{\mathbf{j}}(u)$ as the local truncation error of the finite difference operator L_h approximating to the differential operator $\bar{L}$.

The grid function $T_h(u) = \{T_{\mathbf{j}}(u)\}_{\mathbf{j} \in J}$ is called the truncation error of the finite difference equation approximating to the problem.

注: 该定义的前提是右端函数f,g的"离散"没有误差!

Remark 1: Briefly speaking, the truncation error measures the difference between the difference operator and the differential operator on smooth functions.

Remark 2: $T_h(u)$ can also be viewed as a piece-wise constant function defined on Ω via the control volumes.

Point-Wise Consistency of L_h

Definition

The difference operator L_h is said to be consistent with the differential operator L on Ω, if for all sufficiently smooth solutions u, we have

$$\lim_{h\to 0} T_{\mathbf{j}}(u) = 0, \quad \forall \mathbf{j} \in \overset{\circ}{J}_{\Omega}. \qquad (1.3.16)$$

The difference operator L_h is said to be consistent with the differential operator L on the boundary $\partial\Omega$, if for all sufficiently smooth u, we have

$$\lim_{h\to 0} T_{\mathbf{j}}(u) = 0, \quad \forall \mathbf{j} \in J \setminus \overset{\circ}{J}_{\Omega}. \qquad (1)$$

(1.3.17)

Remark: Briefly speaking, this is the point-wise consistency of L_h to $\bar{L}$. In fact, for u sufficiently smooth, in the above definition $\lim_{h\to 0} \| T_h(u)\|_\infty = 0$ in either l^∞ or L^∞ are equivalent.

Consistency and accuracy in the norm $\|\cdot\|$

Definition

The finite difference equation $L_h U = \bar{f}_h$ is said to be consistent in the norm $\|\cdot\|$ with the boundary value problem of the differential equation $\bar{L}u = \bar{f}$, if, for all sufficiently smooth u, we have

$$\lim_{h \to 0} \| T_h(u) \| = 0. \tag{1.3.18}$$

The truncation error is said to be of order p, or order p accurate, if the convergent rate above is of $O(h^p)$, *i.e.* $\| T_h(u) \| = O(h^p)$.

Remark: Here $T_h(u)$ is viewed as a piece- wise constant function defined on Ω via the control volumes. The norms in the above definition is the corresponding function norm.

Stability in the norm $\|\cdot\|$

Definition

The difference equation $L_h U = f$ is said to be stable or have stability in the norm $\|\cdot\|$, if there exists a constant K independent of the grid size h such that, for arbitrary grid functions f^1 and f^2, the corresponding solutions U^1 and U^2 to the equation satisfy

$$\|U^1 - U^2\| \leq K\|f^1 - f^2\|, \quad \forall h > 0. \tag{1.3.19}$$

The stability implies the uniform well-posedness of the difference equation, more precisely, it has a unique solution which depends uniformly (with respect to h) Lipschitz continuously on the right hand side (source terms and boundary conditions).

Convergence in the norm $\|\cdot\|$

Definition

The difference equation $L_h U = \bar{f}$ is said to converge in the norm $\|\cdot\|$ to the boundary value problem $\bar{L}u = \bar{f}$, or convergent, if, for any given $\bar{f}$ (f and g) so that the problem $\bar{L}u = \bar{f}$ is well posed, the error $e_h = \{e_{\mathbf{j}}\}_{\mathbf{j}\in J} \triangleq \{U_{\mathbf{j}} - u(x_{\mathbf{j}})\}_{\mathbf{j}\in J}$ of the finite difference approximation solution U satisfies

$$\lim_{h\to 0} \|e_h\| = 0. \tag{1.3.20}$$

Furthermore, if $\|e_h\| = O(h^p)$, then the difference equation is said to converge in order p, or order p convergent.

Stability + Consistency ⇒ Convergence

Since $-L_h e_{\mathbf{j}} = -(L_h U_{\mathbf{j}} - L_h u_{\mathbf{j}})$, the stability of the difference operator L_h yields

(1.3.21)

$$\|U - u\| = \|e_h\| \le K\|L_h U - L_h u\| \le K(\|L_h U - \bar{L}u\| + \|\bar{L}u - L_h u\|).$$

1. $\|L_h U - \bar{L}u\|$ is the residual of the algebraic equation $L_h U = \bar{f}$, which is 0 when U is the solution of the difference equation;
2. $L_h u - \bar{L}u = T_h$ is the truncation error;
3. If U is the finite difference solution, then, the stability implies (1.3.21)中第一项为0 $\|U - u\| = \|e_h\| \le K\|T_h\|$. (1.3.22)
4. Stability + Consistency ⇒ Convergence.

The Convergence Theorem

Theorem

Suppose that the finite difference approximation equation $L_h U = \bar{f}$ of the boundary value problem of partial differential equation $\bar{L}u = \bar{f}$ is consistent and stable. Suppose the solution u of the problem $\bar{L}u = \bar{f}$ is sufficiently smooth. Then the corresponding finite difference equation must converge, and the convergent order is at least the order of the truncation error, i.e. $\|T_h\| = O(h^p)$ *implies* $\|e_h\| = O(h^p)$.

Note the additional condition that the solution u is sufficiently smooth, which guarantees that the truncation error for this specified function converges to zero in the expected rate.

The Problem and Notations for the Maximum Principle

第*1.4*节（*Page 19*）

1. $\Omega \subset \mathbb{R}^n$: a connected region;
2. $J = J_\Omega \cup J_D$: grid nodes with grid spacing h;
3. Boundary value problem of linear difference equations:

$$\begin{cases} -L_h U_{\mathbf{j}} = f_{\mathbf{j}}, & \forall \mathbf{j} \in J_\Omega, \\ U_{\mathbf{j}} = g_{\mathbf{j}}, & \forall \mathbf{j} \in J_D, \end{cases} \tag{1.4.1}$$

4. L_h has the following form on J_Ω:

$$L_h U_{\mathbf{j}} = \sum_{\mathbf{i} \in J \setminus \{\mathbf{j}\}} c_{\mathbf{ij}} U_{\mathbf{i}} - c_{\mathbf{j}} U_{\mathbf{j}}, \quad \forall \mathbf{j} \in J_\Omega. \tag{1.4.2}$$

邻点集

5. $D_{L_h}(\mathbf{j}) = \{\mathbf{i} \in J \setminus \{\mathbf{j}\} : c_{\mathbf{ij}} \neq 0\}$: the set of neighboring nodes of $\mathbf{j}$ with respect to L_h.

Connection and J_D connection of J with respect to L_h

Definition

A grid J is said to be connected with respect to the difference operator L_h, if for any given nodes $\mathbf{j} \in J_\Omega$ and $\mathbf{i} \in J$, there exists a set of interior nodes $\{\mathbf{j}_k\}_{k=1}^m \subset J_\Omega$ such that

$$\mathbf{j}_0 = \mathbf{j}, \ \ \mathbf{i} \in D_{L_h}(\mathbf{j}_m), \ \ \mathbf{j}_{k+1} \in D_{L_h}(\mathbf{j}_k), \quad \forall k = 0, 1, \ldots, m-1. \qquad (1.4.3)$$

邻点集

Suppose $J_D \neq \emptyset$, a grid J is said to be J_D connected with respect to the difference operator L_h, if for any given interior node $\mathbf{j} \in J_\Omega$ there exists a Dirichlet boundary node $\mathbf{i} \in J_D$ and a set of interior nodes $\{\mathbf{j}_k\}_{k=0}^m \subset J_\Omega$ such that the above inclusion relations hold.

The Maximum Principle

Theorem

Suppose $L_h U_{\mathbf{j}} = \sum_{\mathbf{i} \in J \setminus \{\mathbf{j}\}} c_{\mathbf{ij}} U_{\mathbf{i}} - c_{\mathbf{j}} U_{\mathbf{j}}$, $\forall \mathbf{j} \in J_\Omega$; *J and* L_h *satisfy*

(1) $J_D \neq \emptyset$, *and J is* J_D *connected with respect to* L_h;

(2) $c_{\mathbf{j}} > 0,\ c_{\mathbf{ij}} > 0$, $\forall \mathbf{i} \in D_{L_h}(\mathbf{j})$, *and* $c_{\mathbf{j}} \geq \sum_{\mathbf{i} \in D_{L_h}(\mathbf{j})} c_{\mathbf{ij}}$.

Suppose the grid function U satisfies $L_h U_{\mathbf{j}} \geq 0$, $\forall \mathbf{j} \in J_\Omega$. *Then,* (1.4.4)

$$M_\Omega \triangleq \max_{\mathbf{i} \in J_\Omega} U_{\mathbf{i}} \leq \max \left\{ \max_{\mathbf{i} \in J_D} U_{\mathbf{i}},\ 0 \right\}. \tag{1.4.5}$$

Furthermore, if J and L_h *satisfy* (3): *J is connected with respect to* L_h; *and there exists interior node* $\mathbf{j} \in J_\Omega$ *such that*

$$U_{\mathbf{j}} = \max_{\mathbf{i} \in J} U_{\mathbf{i}} \geq 0. \tag{1.4.6}$$

Then, U must be a constant on J.

Proof of The Maximum Principle

Assume for some $\mathbf{j} \in J_\Omega$, $U_{\mathbf{j}} = M_\Omega > M_D \triangleq \max_{\mathbf{i} \in J_D} U_{\mathbf{i}}$, $M_\Omega > 0$.

By the J_D connection, there exist $\mathbf{i} \in J_D$ and $\{\mathbf{j}_k\}_{k=0}^m \subset J_\Omega$ such that the inclusion relation $\mathbf{j}_{k+1} \in D_{L_h}(\mathbf{j}_k)$ hold.

It follows from the conditions on L_h and the condition (2) that

$$U_{\mathbf{j}} \leq \sum_{\mathbf{i} \in D_{L_h}(\mathbf{j})} \frac{c_{\mathbf{ij}}}{c_{\mathbf{j}}} \max_{\hat{\mathbf{i}} \in D_{L_h}(\mathbf{j})} U_{\hat{\mathbf{i}}} \leq \sum_{\mathbf{i} \in D_{L_h}(\mathbf{j})} \frac{c_{\mathbf{ij}}}{c_{\mathbf{j}}} M_\Omega.$$

Since $U_{\mathbf{j}} = M_\Omega \geq 0$ and $\sum_{\mathbf{i} \in D_{L_h}(\mathbf{j})} \frac{c_{\mathbf{ij}}}{c_{\mathbf{j}}} \leq 1$, this implies that the equalities must all hold, which can be true only if $L_h U_{\mathbf{j}} = 0$ as well as $U_{\hat{\mathbf{i}}} = U_{\mathbf{j}} = M_\Omega$ for all $\hat{\mathbf{i}} \in D_{L_h}(\mathbf{j})$.

Proof of The Maximum Principle

Similarly, we have $U_{\mathbf{j}_k} = M_\Omega$, $k = 1, 2, \ldots, m$ and $U_{\mathbf{i}} = M_\Omega$. But this contradicts the assumption $U_{\mathbf{i}} \leq M_D < M_\Omega$.

The same argument also leads to the conclusion that $U_{\mathbf{i}} = U_{\mathbf{j}}$ for all $\mathbf{i} \in J$, *i.e.* U is a constant on J, provided that the condition (3) and relation $U_{\mathbf{j}} = \max_{\mathbf{i} \in J} U_{\mathbf{i}} \geq 0$ hold. □

Remark 1: For the 5 point scheme of $-\Delta$, the condition (2) of the maximum principle holds. If $-\Delta$ is replaced by $-(\Delta + b_1\partial_x + b_2\partial_y + c)$ with $c < 0$, the conclusion still holds if ∂_x and ∂_y are approximated by central difference operators $(2h_x)^{-1}\triangle_{0x}$ and $(2h_y)^{-1}\triangle_{0y}$ respectively.

Remark 2: For uniformly elliptic operators with $c < 0$, one can always construct consistent finite difference operators so that the condition (2) of the maximum principle holds for sufficiently small h, noticing that the second order difference operator has a factor of $O(h^{-2})$, while the first order difference operator has a factor of $O(h^{-1})$.

The Maximum Principle

Apply the maximum principle to $-U$, we have 最小值原理

Corollary

Suppose J and L_h satisfy the conditions (1) and (2) in Theorem 1.2. Suppose that the grid function U satisfies

$$L_h U_{\mathbf{j}} \le 0, \qquad \forall \mathbf{j} \in J_\Omega. \tag{1.4.7}$$

Then U can not take nonpositive minima on a interior node, i.e.

$$m_\Omega \triangleq \min_{\mathbf{i}\in J_\Omega} U_{\mathbf{i}} \ge \min\left\{\min_{\mathbf{i}\in J_D} U_{\mathbf{i}},\, 0\right\}. \tag{1.4.8}$$

If L_h satisfies further **(3)***: J is connected with respect to the operator L_h, and there exists an interior node $\mathbf{j} \in J_\Omega$ such that*

$$U_{\mathbf{j}} = \min_{\mathbf{i}\in J} U_{\mathbf{i}} \le 0,$$

Then, U must be a constant grid function on J.

The Existence Theorem

Theorem 1.3

Suppose the grid J and the linear operator L_h satisfy the conditions (1) and (2) of the maximum principle. Then, the difference equation

$$\begin{cases} -L_h U_{\mathbf{j}} = f_{\mathbf{j}}, & \forall \mathbf{j} \in J_\Omega, \\ U_{\mathbf{j}} = g_{\mathbf{j}}, & \forall \mathbf{j} \in J_D, \end{cases} \tag{1.4.1}$$

has a unique solution (It exists and is unique).

Proof of the Existence Theorem

We only need to show that

$$L_h U_{\mathbf{j}} = 0,\ \forall \mathbf{j} \in J \quad \Rightarrow \quad U_{\mathbf{j}} = 0,\ \forall \mathbf{j} \in J.$$

In fact, by the maximum principle $L_h U \geq 0$ implies $U \leq 0$, and by the corollary of the maximum principle, $L_h U \leq 0$ implies $U \geq 0$, thus $U \equiv 0$ on J . $\square$

$(-L_h)^{-1}$ is a Positive Operator

Consider the discrete problem

$$\begin{cases} -L_h U_{\mathbf{j}} = f_{\mathbf{j}}, & \forall \mathbf{j} \in J_\Omega, \\ U_{\mathbf{j}} = g_{\mathbf{j}}, & \forall \mathbf{j} \in J_D. \end{cases}$$

Corollary

Suppose the grid J and the linear operator L_h satisfy the conditions (1) and (2) of the maximum principle. Then,

$$f_{\mathbf{j}} \geq 0, \forall \mathbf{j} \in J_\Omega, \; g_{\mathbf{j}} \geq 0, \forall \mathbf{j} \in J_D, \quad \Rightarrow \quad U_{\mathbf{j}} \geq 0, \forall \mathbf{j} \in J;$$

and

$$f_{\mathbf{j}} \leq 0, \forall \mathbf{j} \in J_\Omega, \; g_{\mathbf{j}} \leq 0, \forall \mathbf{j} \in J_D, \quad \Rightarrow \quad U_{\mathbf{j}} \leq 0, \forall \mathbf{j} \in J;$$

$(-L_h)^{-1}$ is a Positive Operator

The corollary says that $(-L_h)^{-1}$ is a positive operator, *i.e.* $(-L_h)^{-1} \geq 0$. In other words, every element of the matrix $(-L_h)^{-1}$ is nonnegative.

In fact, the matrix $-L_h$ is a M matrix, *i.e.* the diagonal elements of A are all positive, the off-diagonal elements are all nonpositive, and elements of A^{-1} are all nonnegative.

Comparison Theorem and Stability

Theorem

Suppose the grid J and the linear operator L_h satisfy the conditions (1) and (2) of the maximum principle. Let the grid function U be the solution to the linear difference equation

$$\begin{cases} -L_h U_{\mathbf{j}} = f_{\mathbf{j}}, & \forall \mathbf{j} \in J_\Omega, \\ U_{\mathbf{j}} = g_{\mathbf{j}}, & \forall \mathbf{j} \in J_D. \end{cases}$$

Let Φ be a nonnegative grid function defined on J satisfying

$$L_h \Phi_{\mathbf{j}} \geq 1, \qquad \forall \mathbf{j} \in J_\Omega.$$

Then, one has $\max\limits_{\mathbf{j}\in J_\Omega} |U_{\mathbf{j}}| \leq \max\limits_{\mathbf{j}\in J_D} |U_{\mathbf{j}}| + \max\limits_{\mathbf{j}\in J_D} \Phi_{\mathbf{j}} \max\limits_{\mathbf{j}\in J_\Omega} |f_{\mathbf{j}}|.$

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Thank You!