Chapter 6 FEM for elliptic BVP

Numerical Solutions to Partial Differential Equations

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General Abstract Definition of a Finite Element

Definition 6.1

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Ciarlet的FE的定义(1978)

A triple (K, P_K, Σ_K) is called a finite element, if

1. $K \subset \mathbb{R}^n$, called an element, is a closed set with non-empty interior and a Lipschitz continuous boundary;
2. $P_K : K \to \mathbb{R}$ is a finite dimensional function space consisting of sufficiently smooth functions defined on the element K;
3. Σ_K is a set of linearly independent linear functionals $\{\varphi_i\}_{i=1}^N$ defined on $\mathbb{C}^\infty(K)$, which are called the degrees of freedom of the finite element and form a dual basis corresponding to a "normalized" basis of P_K, meaning that there exists a unique basis $\{p_i\}_{i=1}^N$ of P_K such that $\varphi_i(p_j) = \delta_{ij}$.

自由度集

\phi_i和pi可设想为在代数意义下的一组对偶基.

自由度集合就是惟一地确定空间Pk中的一个函数的那些参数。例如k=1时，Pk取为P1(次数不超过1的多项式集合)，此时自由度集合由单元K的顶点上的函数值组成, 记号为{批(Ai)}, 其中p(x)是P1中函数.

P.G. Ciarlet, The Finite Element Method for Elliptic Problems, SIAM, 2002

Type (k) n-Simplex — The Simplest Class of Lagrange Finite Elements

6.2.3节

型(k) n单纯形

n-单纯形

1. $K^n = \{\mathbf{x} = \sum_{i=1}^{n+1} \lambda_i \mathbf{a}_i : 0 \le \lambda_i \le 1, 1 \le i \le n+1, \sum_{i=1}^{n+1} \lambda_i = 1\}$ is the convex hull of vertices $\mathbf{a}_j = (a_{ij})_{i=1}^n$, $j = 1, \ldots, n+1$, with (6.2.7)

凸包

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n+1} \\ a_{21} & a_{22} & \cdots & a_{2,n+1} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,n+1} \\ 1 & 1 & \cdots & 1 \end{pmatrix} = (\tilde{\mathbf{a}}_1, \cdots, \tilde{\mathbf{a}}_{n+1}) \quad \text{non-singular.}$$

即向量a1a2,...,a1a(n+1)线性无关

1D长度坐标，2D面积坐标的推广-->

重心坐标

2. Denote $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1})^T$, $\tilde{\mathbf{x}} = (x_1, x_2, \ldots, x_n, 1)^T$, then, $A\boldsymbol{\lambda} = \tilde{\mathbf{x}}$. $\boldsymbol{\lambda}(\mathbf{x}) = A^{-1}\tilde{\mathbf{x}}$ is called barycentric coordinates.
3. By $\boldsymbol{\lambda}(\mathbf{a}_j) = A^{-1}\tilde{\mathbf{a}}_j$, and $\tilde{\mathbf{a}}_j$ is the jth column of A, we have $\lambda_i(\mathbf{a}_j) = \delta_{ij}$, $1 \le i, j \le n+1$. (6.2.8)

Type (k) n-Simplex — $P_K = \mathbb{P}_k(K)$, $\Sigma_K = K_k^n$, the Principal Lattice

型(k) n单纯形

1. $P_K = \mathbb{P}_k(K)$: polynomials of degree no greater than k for the n variables $x_1, x_2, \ldots, x_n$ defined on K. $\dim \mathbb{P}_k(K) = C_{n+k}^n$.
证明见Lect-chap6-01.pdf

2. For $k = 1$, $\dim \mathbb{P}_1(K) = n + 1$. Since $\boxed{\lambda_i(\mathbf{x}) \in \mathbb{P}_1(K)}$ and $\boxed{\lambda_i(\mathbf{a}_j) = \delta_{ij}}$, if we take $\Sigma_K = \{p(\mathbf{a}_i), 1 \le i \le n+1\}$, then, the barycentric coordinates $\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}), \ldots, \lambda_{n+1}(\mathbf{x})$ form the normalized dual basis of $\mathbb{P}_1(K)$ with respect to Σ_K.
重心坐标与{ p(ai)}对偶

主格点

3. In general, for $k \ge 1$, the principal lattice

$$K_k^n = \left\{ \mathbf{x} \in \sum_{i=1}^{n+1} \lambda_i \mathbf{a}_i : \sum_{i=1}^{n+1} \lambda_i = 1,\ \lambda_i \in \left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\right\},\ 1 \le i \le n+1 \right\},$$

离散点集合

form a dual basis of $\mathbb{P}_k(K^n)$.

举例：textbookP220
见lect-chap6-02-appendix01.pptx

P.G. Ciarlet, 有限元素法的数值分析，上海科技出版社，1978, P37.

$\Sigma_K = K_k^n$, the Principal Lattice, Form a Dual Basis of $P_K = \mathbb{P}_k(K^n)$

Theorem 6.1

For $k = 0$, denote $K_0^n = \left\{ \frac{1}{n+1} \sum_{i=1}^{n+1} \mathbf{a}_i \right\}$; for $k \geq 1$, denote

$$K_k^n = \left\{ \mathbf{x} \in \sum_{i=1}^{n+1} \lambda_i \mathbf{a}_i : \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \in \left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\right\}, 1 \leq i \leq n+1 \right\},$$

离散点集合

k阶主格点

and call them the kth order principal lattice of the n-simplex $K^n \subset \mathbb{R}^n$. Then, the degrees of freedom given by $\Sigma_k^n = \{p(x) : x \in K_k^n\}$ form a dual basis of $\mathbb{P}_k(K^n)$, and are called the kth order principal degrees of freedom of the n-simplex K^n.

k阶主格点自由度集

Theorem 2.2.1 次数不超过k的多项式可以被其在k阶主格点集上所有点处的值唯一确定. 见P.G. Ciarlet, SIAM, 2002, P70；证明见石和王, FEM, 科学出版社, 2010, P22

R. A. Nicolaides, SIAM J. Numer. Anal., 9(3), 435-445,1972

Proof of the Principal Lattice Form a Dual Basis of $P_K = \mathbb{P}_k(K^n)$

The key points to the proof:

Pk的维数等于k阶主格点集合的元素个数

- There are exactly $\dim \mathbb{P}_k(K^n) = C_{n+k}^n$ points in K_k^n.
- If $p \in \mathbb{P}_k(K^n)$ satisfies $p(\mathbf{x}) = 0$ on K_k^n, then, $p(\mathbf{x}) \equiv 0$.

如果p在n单纯形Kn的k阶主格点处取值为0，则它恒等于0

Proof:

, set of principal lattice of K

K的主格点集合与多重指标间有1-1对应关系

1. $\alpha_i \triangleq k\lambda_i$, $i = 1, 2, \ldots, n$ is 1–1 to the multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i \leq k$. $\sharp K_k^n = \dim \mathbb{P}_k(K^n)$.

   集合元素个数
2. For $n = 1$, the conclusion of the theorem obviously holds for all $k \geq 0$. We will prove by the principle of induction. 归纳原理
3. Assume that, for all space dimensions less than n (≥ 2), the conclusion of the theorem holds for all $k \geq 0$.

Proof of the Principal Lattice Form a Dual Basis of $P_K = \mathbb{P}_k(K^n)$

由于坐标x和形心坐标λ的关系, tilde{x}=(x,1)

4. Since $\tilde{\mathbf{x}} = A\boldsymbol{\lambda}$, $p \in \mathbb{P}_k(K^n)$ can be written as $p(\mathbf{x}) = \sum_{|\alpha| \le k} a_\alpha \lambda_1^{\alpha_1}(\mathbf{x}) \cdots \lambda_{n+1}^{\alpha_{n+1}}(\mathbf{x})$, and in particular, written as

$$p(x) = \sum_{i=0}^{k} \left[p_{k-i}(\lambda_1(x), \ldots, \lambda_n(x)) \prod_{j=1}^{i} \left(\lambda_{n+1}(x) - \frac{j-1}{k} \right) \right], \qquad (*1)$$

形如1D的Newtown型插值多项式基函数

where $p_{k-i}(\lambda_1, \ldots, \lambda_n)$ is a polynomial of $\lambda_1, \ldots, \lambda_n$ of degree no greater than $k - i$.

Proof of the Principal Lattice Form a Dual Basis of $P_K = \mathbb{P}_k(K^n)$

5. Let $\hat{\lambda}_j = \frac{k}{k-i}\lambda_j$, $\hat{\mathbf{a}}_j = \frac{k-i}{k}\mathbf{a}_j + \frac{i}{k}\mathbf{a}_{n+1}$, $j = 1, 2, \ldots, n$, then

$$\tilde{K}^{n-1}_{k-i} = \left\{\mathbf{x} \in \sum_{j=1}^{n} \lambda_j \mathbf{a}_j + \frac{i}{k}\mathbf{a}_{n+1} : \sum_{j=1}^{n} \lambda_j = 1 - \frac{i}{k}, \lambda_j \in \left\{0, \frac{1}{k}, \ldots, \frac{k-i}{k}\right\}, 1 \le j \le n\right\},$$

(*2)

$$\Leftrightarrow \quad \tilde{K}^{n-1}_{k-i} = \left\{\mathbf{x} \in \sum_{j=1}^{n} \hat{\lambda}_j \hat{\mathbf{a}}_j : \sum_{j=1}^{n} \hat{\lambda}_j = 1, \hat{\lambda}_j \in \left\{0, \frac{1}{k-i}, \ldots, 1\right\}, 1 \le j \le n\right\}.$$

k-i 阶主格点

6. $\tilde{K}^{n-1}_{k-i}$ is the $(k-i)$th order principal lattice of the $(n-1)$-simplex $\boxed{K^{n-1}_{i,k} = \{\mathbf{x} \in K^n : \lambda_{n+1}(\mathbf{x}) = \frac{i}{k}\}}$. (*3)

Proof of $\Sigma_K = K_k^n$ Form a Dual Basis of $P_K = \mathbb{P}_k(K^n)$ — continue

如果p在n单纯形Kn的k阶主格点处取值为0，则由(*2)知

7. $p(x) = 0$ on $K_k^n \underset{(*2)}{\Rightarrow} p(x) = 0$ on $\tilde{K}_{k-i}^{n-1}$.

为了证p(x)=0, 只要证明(*1)右端每一项p(k-i)为0

如果p(x)是pk(x), n-1单纯形上次数不超过k的多项式，则

8. $p(x) = p_k(\lambda_1(x), \ldots, \lambda_n(x)) \in \mathbb{P}_k(K_{0,k}^{n-1})$, $p(x) = 0$ on $\tilde{K}_k^{n-1}$, (*3)
by the induction assumption 归纳假设 $\Rightarrow p_k(\lambda_1(x), \ldots, \lambda_n(x)) \equiv 0 \Rightarrow$

9. $p(x) = \lambda_{n+1}(x) p_{k-1}(\lambda_1(x), \ldots, \lambda_n(x)) \in \mathbb{P}_k(K_{1,k}^{n-1})$, (*3) thus,
$p_{k-1} = 0$ on $\tilde{K}_{k-1}^{n-1}$ (*2) $\Rightarrow$ 归纳假设 $p_{k-1}(\lambda_1(x), \ldots, \lambda_n(x)) \equiv 0$. Similarly,

10. $p_{k-i}(\lambda_1(x), \ldots, \lambda_n(x)) \equiv 0$, $2 \le i \le k$, $\underset{(*1)}{\Rightarrow} p(x) \equiv 0$. ■

Type (k) n-Simplex Finite Elements

型(k) n单纯形

P222

1. A finite element (K, P_K, Σ_K) is called a type (k) n-simplex, if K is a n-simplex, $P_K = \mathbb{P}_k(K)$, and Σ_K is the kth order principal degrees of freedom Σ_k^n of K.
   K的k阶主格点自由度集合

2. Type (k) n-simplex finite elements are an affine family.
   即与一给定的有限元仿射等价

   规范化的对偶基

3. The normalized dual basis of $\mathbb{P}_k(K)$ corresponding to Σ_k^n of the n-simplex K can be easily expressed in barycentric coordinates. Pk的与n-单纯形的主格点自由度对偶的基可以很容易地用重心坐标表示

Type (k) n-Simplex Finite Elements

4. For example, for the type (2) n-simplex, the normalized dual basis of $\mathbb{P}_2(K)$ corresponding to Σ_2^n is given by 规范化的对偶基

$$\lambda_i(x)(2\lambda_i(x)-1),\ \ i=1,2,\ldots,n+1;\ \ 4\lambda_i(x)\lambda_j(x),\ 1\le i<j\le n+1.$$

5. In fact, denoting $\mathbf{a}_{ij}=(\mathbf{a}_i+\mathbf{a}_j)/2$, we have 次数不超过2的多项式可以表示为

$$p(x)=\sum_{i=1}^{n+1}\lambda_i(x)(2\lambda_i(x)-1)p(\mathbf{a}_i)+\sum_{1\le i<j\le n+1}4\lambda_i(x)\lambda_j(x)p(\mathbf{a}_{ij}),\quad \forall p\in\mathbb{P}_2(K).$$

Type (k) n-Rectangle — Another Class of Lagrange Finite Elements

另一类Lagrange有限元

n-长方形/正2n-面体

1. $K^n = [X_{11}, X_{12}] \times [X_{21}, X_{22}] \times \cdots \times [X_{n1}, X_{n2}]$ is a n-rectangle.
2. $P_K \triangleq \mathbb{Q}_k(K^n) = \left\{ p(\mathbf{x}) : p(\mathbf{x}) = \sum_{\substack{\alpha_i \leq k \\ 1 \leq i \leq n}} p_{\alpha_1 \cdots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \right\}$.
3. $\dim \mathbb{Q}_k(K^n) = (k+1)^n$.
4. Let $h_i = X_{i2} - X_{i1}$, $1 \leq i \leq n$, define the kth order principal lattice of the n-rectangle K^n:

$$\bar{K}_k^n = \left\{ \mathbf{x} = (X_{11} + \frac{i_1}{k} h_1, \cdots, X_{n1} + \frac{i_n}{k} h_n)^T \in \mathbb{R}^n : i_j \in \{0, 1, \ldots, k\},\ 1 \leq j \leq n \right\},$$

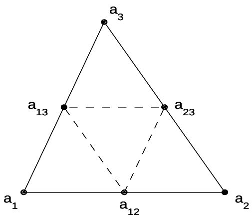
集合元素个数

5. $\sharp \bar{K}_k^n = (k+1)^n$.
6. If $p \in \mathbb{Q}_k(K^n)$ and $p = 0$ on $\bar{K}_k^n$, then, $p \equiv 0$.
7. $\bar{\Sigma}_k^n = \{p(\mathbf{x}) : \mathbf{x} \in \bar{K}_k^n\}$ form a dual basis of $\mathbb{Q}_k(K^n)$.

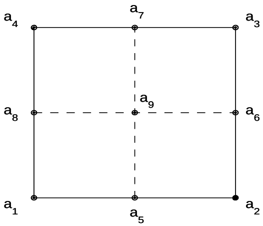
Type (k) n-Rectangle Lagrange Finite Elements

型(k) n-矩形 拉格朗日FE

1. Type (k) n-rectangle finite element: K a n-rectangle K^n, $P_K = \mathbb{Q}_k(K^n)$, and $\Sigma_K = \bar{\Sigma}_k^n = \{p(\mathbf{x}) : \mathbf{x} \in \bar{K}_k^n\}$.
2. The type (k) n-rectangles form a particular subset of an affine family.
3. Figures (a): a type (2) triangle; (b): a type (2) rectangle.



(a) 型(2) 三角形



(b) 型(2) 矩形

Incomplete Type (k) n-Simplex and Type (k) n-Rectangle

1. Finite elements can be obtained by removing some of the principal degrees of freedom and the corresponding dual basis functions from a type (k) n-simplex or a type (k) n-rectangle.
移去某些主格点自由度和对应的基函数

2. For example, by removing the nodal degree of freedom a_9 and its corresponding basis function

$$16(h_1h_2)^{-1}(x_1 - X_{11})(x_1 - X_{12})(x_2 - X_{21})(x_2 - X_{22})$$

from the type (2) rectangle, we obtain a finite element called the type $(2)'$ rectangle, or incomplete biquadratic rectangle.

型(2)' 矩形　　不完全双二次矩形

Isoparametric Family Given by a Type (k) Simplex or Rectangle

FE等参族

1. Isoparametric families of finite elements can be constructed by a complete or incomplete type (k) n-simplex or n-rectangle.

考虑参考FE

2. Let the reference finite element (K, P_K, Σ_K) be a complete or incomplete type (k) n-simplex or type (k) n-rectangle.
3. Let $\{\hat{p}_i\}_{i=1}^N$ be the dual basis of P_K corresponding to the kth order principal degrees of freedom of K.
4. Then, for any given invertible map $F : K \to F(K) \in \mathbb{R}^n$, the maps

$$\boxed{\begin{cases} \mathbf{x} = F(\hat{\mathbf{x}}) := \sum_{i=1}^N \mathbf{a}_i \hat{p}_i(\hat{\mathbf{x}}), \\ u = \sum_{i=1}^N u_i \hat{p}_i(\hat{\mathbf{x}}), \end{cases} \quad \hat{\mathbf{x}} \in K,}$$

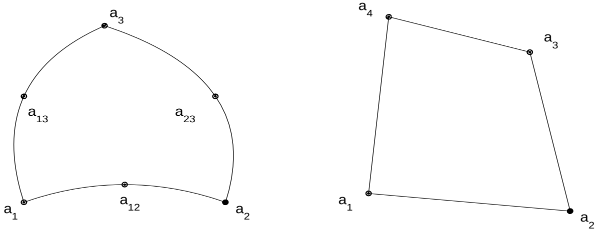
define a finite element $(F(K), P_{F(K)}, \Sigma_{F(K)})$, where $P_{F(K)} = \text{span}\{\hat{p}_i \circ F^{-1}, 1 \le i \le N\}$, $\Sigma_{F(K)} = \{\mathbf{a}_i, 1 \le i \le N\}$.

5. The finite elements so defined form an isoparametrically equivalent family with (K, P_K, Σ_K) as a reference FE.

这样定义的FE形成一个等参FE等价族

Isoparametric Family Given by a Type (k) Simplex or Rectangle

6. Computations are on the reference finite element, it is unnecessary to calculate F^{-1}.
7. $u(\mathbf{x})$ is implicitly expressed by the same set of parameters.
8. Figures (a): type (2) curved triangle; (b): type (1) quadrilateral.



(a) 型(2)曲三角形 (b) 型(1)四边形

Finite Element of Class $\mathbb{C}^k$

Ck有限元

1. A finite element is said to be of class $\mathbb{C}^k$, if all functions in a finite element function space, which is composed of such type of finite elements, are in $\mathbb{C}^k(\overline{\Omega})$.

前面定义的FE都是C0有限元

2. The Lagrange finite elements introduced above are all of class $\mathbb{C}^0$, since the face value of a finite element function is completely determined by its nodal values on the face.
3. For 2nd order elliptic problems, finite elements of class $\mathbb{C}^0$ are sufficient, since the underlying function space is $H^1(\Omega)$.
4. For 4th order elliptic problems, we need finite elements of class $\mathbb{C}^1$ to construct a conforming finite element function space.
5. To construct finite element of class $\mathbb{C}^k$, $k \geq 1$, we need to use Hermite finite elements.

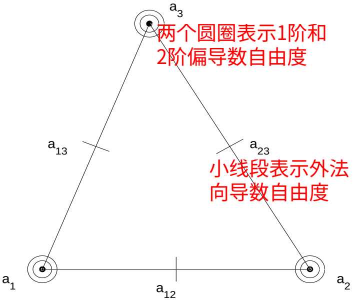
An Example of Hermite Finite Element — The Argyris Triangle

阿吉里斯三角形

1. $K \subset \mathbb{R}^2$: a triangle with vertices $\mathbf{a}_i$, $i = 1, 2, 3$; $P_K = \mathbb{P}_5(K)$;
$$\Sigma_K = \{p(\mathbf{a}_i), \partial_j p(\mathbf{a}_i), \partial^2_{jk} p(\mathbf{a}_i), 1 \le i \le 3, 1 \le j \le k \le 2; \quad \partial_\nu p(\mathbf{a}_{ij}), 1 \le i < j \le 3\}. \qquad (6.2.9)$$
2. $\dim \mathbb{P}_5(K) = C_7^2 = 21$, and $\sharp\Sigma_K = 21$.

唯一可解性

Need to show:
If $p \in \mathbb{P}_5(K)$,
$p = 0$ on Σ_K,
then, $p \equiv 0$.



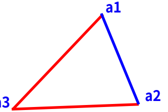
自由度统计：三角形的每个顶点处有1个函数值，两个1阶偏导数的值，和3个2阶偏导数的值；三角形的每个边的中点处有一个法向导数的值。3X(1+2+3+1)=3X7

Show $p \equiv 0$, if $p \in \mathbb{P}_5(K)$ and $p = 0$ on Σ_K for the Argyris triangle

阿吉里斯三角形元的唯一可解性

阿吉里斯三角形

1. Let t be the coordinate on the edge
   lambda_3是线性函数，且在a1,a2处取值为0，在a3处为1
   $K_{12}^1 = \{\mathbf{a}_1 + t(\mathbf{a}_2 - \mathbf{a}_1) : 0 \le t \le 1\} = \{\mathbf{x} \in K : \lambda_3(\mathbf{x}) = 0\}$.
2. 如果 $p \in \mathbb{P}_5(K)$, $p = 0$, $dp = 0$, $d^2p = 0$ on $\mathbf{a}_1$, $\mathbf{a}_2$ $\Rightarrow$ $q(t) = p|_{K_{12}^1} \in \mathbb{P}_5(K_{12}^1)$, $q = 0$, $q' = 0$, $q'' = 0$ on $\mathbf{a}_1$, $\mathbf{a}_2$ $\Rightarrow$ $q \equiv 0$.
   q(t)为一元5次多项式, 在t=0,1处满足6个条件
3. Similarly, $\partial_\nu p \in \mathbb{P}_4(K)$, $\partial_\nu p = 0$, on $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_{12}$, and as a function of t, $(\partial_\nu p)'_t = 0$ on $\mathbf{a}_1$, $\mathbf{a}_2$ $\Rightarrow$ $\partial_\nu p|_{K_{12}^1} \equiv 0$.
   一元4次多项式, 在t=0,1处满足5个条件
4. $p|_{K_{12}^1} \equiv 0$, $\partial_\nu p|_{K_{12}^1} \equiv 0$ $\Rightarrow$ $\nabla p|_{K_{12}^1} \equiv 0$ $\Rightarrow$ $p|_{\lambda_3=0} \equiv 0$ and $\partial_{\lambda_3} p|_{\lambda_3=0} \equiv 0$ $\Rightarrow$ λ_3^2 must be a factor of p.
5. Similarly, λ_1^2 and λ_2^2 must also be factors of p.
6. Thus $p = r\lambda_1^2\lambda_2^2\lambda_3^2$ $\Rightarrow$ $r \equiv 0$, since $p \in \mathbb{P}_5(K)$. ■



Show $p \equiv 0$, if $p \in \mathbb{P}_5(K)$ and $p = 0$ on Σ_K for the Argyris triangle

Remark:

阿吉里斯三角形

The Argyris triangle is a class $\mathbb{C}^1$ finite element, since the values of the function and its first order derivatives on an edge are completely determined by the nodal degrees of freedom there.

The Argyris triangles are not an affine family, since the normals are not affine invariant. 仿射变换不保持方向

"It is relatively difficult to construct differentiable (C1) finite elements. Most C1-elements were constructed in the early 1970s, cf. [5]. The most famous C1 element is the Argyris P5-triangle [2]. The element was extended to the full C1-P5 space, known as the Morgan-Scott Pk-triangles, for all k ≥ 5 [12]. In the other direction, we have the Bell reduced P5-triangle [3]." http://www.math.udel.edu/~szhang/research/p/c.pdf
张上游，四维空间上的C1单纯形有限元族，计算数学 2016, 38(3) 309-324
http://www.computmath.com/Jwk_jssx/CN/article/showZhaiYao.do?id=13002

Another Type (5) Hermite Triangle — an Affine Equivalent Family

1. $K \subset \mathbb{R}^2$: a triangle with vertices $\mathbf{a}_i$, $i = 1, 2, 3$; $P_K = \mathbb{P}_5(K)$;

$$\Sigma'_K = \Big\{ p(\mathbf{a}_i),\ \partial_{\xi_{ij}} p(\mathbf{a}_i),\ \partial^2_{\xi_{ij}\xi_{ik}} p(\mathbf{a}_i), 1 \le i \le 3,\ 1 \le j \le k \le 3, \\ i \notin \{j,k\}; \partial_{\eta_{ijk}} p(\mathbf{a}_{ij}),\ 1 \le i < j \le 3,\ k \notin \{i,j\} \Big\}, \qquad (6.2.10)$$

where $\xi_{ij} = \mathbf{a}_j - \mathbf{a}_i$, $\eta_{ijk} = \mathbf{a}_{ij} - \mathbf{a}_k$.

2. $\dim \mathbb{P}_5(K) = C_7^2 = 21$, and $\sharp\Sigma_K = 21$.
3. Let Π_K and Π'_K be the $P_K = \mathbb{P}_5(K)$ interpolation operators defined by Σ_K and Σ'_K respectively, then,

$$\Pi_K v = \Pi'_K v, \quad \forall v \in \mathbb{P}_5(K), \qquad \text{(or equivalently } \Pi'_K \Pi_K v = \Pi_K v, \ \ \forall K \in C^\infty(K)\text{)}. \qquad \textbf{(6.2.11)}$$

自由度统计：三角形的每个顶点处有1个函数值，两个1阶导数的值，和3个2阶导数的值；三角形的每个边的中点处有一个导数的值。3X(1+2+3+1)=3X7 例如三个顶点处的2阶导数对应的指标 i=1: 1212, 1213,1313; i=2: 2121, 2123, 2323; i=3: 3131, 3132, 3232

Finite Elements Embedded into an Affine Equivalent Family

Definition 6.6

Let (K, P_K, Σ_K) and (K, P_K, Σ'_K) be finite elements, and the latter is in an affine family. The former is said to embed into the affine family of the latter, if the two finite elements satisfy

前者嵌入到后者的仿射族，如果

$$\Pi_K v = \Pi'_K v, \quad \forall v \in P_K, \quad \text{(or equivalently } \Pi'_K \Pi_K v = \Pi_K v, \quad \forall K \in C^\infty(K)\text{)}. \tag{6.2.11}$$

Remak 1: On the finite element function space consisting of Argyris triangles, one can still compute the global stiffness matrix by working on reference finite element using the degrees of freedom Σ'_K and the corresponding dual basis functions expressed in barycentric coordinates.

Remak 2: Such an embedding property is useful in the error analysis of finite element solutions, when a finite element which is not in an affine equivalent family is used in constructing the finite element function space.

A Class $\mathbb{C}^1$ Type (3) Hermite FE — Bogner-Fox-Schmit Rectangle

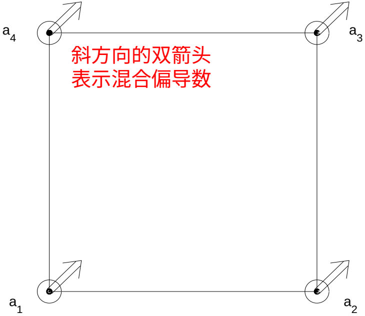
1. $K \subset \mathbb{R}^2$: a rectangle with vertices $\{\mathbf{a}_i\}_{i=1}^4$; $P_K = \mathbb{Q}_3(K)$;
$$\Sigma_K = \left\{ p(\mathbf{a}_i), \partial_j p(\mathbf{a}_i), \partial_{12}^2 p(\mathbf{a}_i), \ 1 \le i \le 4, \ j = 1, 2 \right\}, \qquad (6.2.12)$$

维数/个数

2. $\dim \mathbb{Q}_3(K) = (3+1)^2 = 16$, and $\sharp\Sigma_K = 16$. √

唯一可解性

Easily verified:
If $p \in \mathbb{Q}_3(K)$,
$p = 0$ on Σ_K,
then, $p \equiv 0$.



An Example of Finite Element Equations of Elliptic Problems

1. The weak form w.r.t. the homogeneous Dirichlet boundary value problem of the Poisson equation:

$$\boxed{\begin{cases} \text{Find } \ u \in \mathbb{H}_0^1(\Omega), \ \text{ such that} \\ a(u, v) = (f, v), \quad \forall v \in \mathbb{H}_0^1(\Omega), \end{cases}}$$

where $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx$, $(f, v) = \int_\Omega fv \, dx$.

2. Let $\mathbb{V}_h(0) \subset \mathbb{H}_0^1(\Omega)$ be a finite element space, then

$$\boxed{\begin{cases} \text{Find } \ u_h \in \mathbb{V}_h(0) \ \text{ such that} \\ a(u_h, v_h) = (f, v_h), \quad \forall v_h \in \mathbb{V}_h(0), \end{cases}}$$

is called the finite element problem of the original problem.

An Example of Finite Element Equations of Elliptic Problems

3. Let $\{\varphi_i\}_{i=1}^{N_h}$ be a set of basis functions of $\mathbb{V}_h(0)$. Denote
$$\boxed{u_h = \sum_{j=1}^{N_h} u_j \varphi_j, \quad \mathbf{u}_h = (u_1, \ldots, u_{N_h})^T}$$
4. $K\mathbf{u}_h = \mathbf{f}_h$ is called the finite element equation of the original problem, where $K = (k_{ij}) = (a(\varphi_j, \varphi_i))$ is the stiffness matrix, $\mathbf{f}_h = (f_i) = ((f, \varphi_i))$ is the external load vector.
5. In general, a finite element method discretizes a problem of partial differential equations to a finite dimensional algebraic problem.
6. In particular, a finite element equation derived from a linear problem is usually linear.

Some Numerical Methods for Solving Finite Element Equations

1. For the Dirichlet boundary value problem of the Poisson equation, the finite element equation is usually a symmetric positive definite linear algebraic equation.
2. Classical numerical methods include: Cholesky decomposition, the Gauss-Seidel iterative method, the successive over relaxation iterative method, the conjugate gradient method, the preconditioned conjugate gradient method, etc..
3. In solving large scale symmetric positive definite finite element equations, the preconditioned conjugate method with the incomplete Cholesky decomposition method serving as a preconditioner (ICCG) is a highly recommended method.

The Multigrid Method for Solving Finite Element Equations

1. Observation: for classical iterative methods, (a): the highest frequency modes of the initial error decay very fast; (b): the smaller the grid size, the slower the final convergence speed.
2. Observation: after a very limited number of iterations, the error $\delta u_h^{(k)} = u_h - u_h^{(k)}$ of the finite element solution and the residual $r_h^{(k)} = \sum_{i=1}^{N_h} r_i^{(k)} \varphi_i$, where $(r_1^{(k)}, \cdots, r_{N_h}^{(k)})^T = \mathbf{r}_h^{(k)} = \mathbf{f}_h - K\mathbf{u}_h^{(k)}$ will become very smooth.
3. To increase the efficiency of the computation, one could consider to reduce the residual error on a coarser grid.
4. A typical two-grid method consists of the following 5 parts: pre-smoothing, restriction, coarse grid correction, prolongation and post-smoothing.

预磨光、限制、粗网格校正、延拓和后磨光。
http://www.mgnet.org

The Multigrid Method for Solving Finite Element Equations

预磨光、限制、粗网格校正、延拓和后磨光。

Pre-smoothing Perform a few iterations using the Gauss-Seidel, SOR etc., to smooth out the residual and obtain an approximate solution $u_h^{(k)}$ on the fine grid;

Restriction Calculate the residual and restrict the information on to the coarse grid by, say, interpolation, projection or integral average, etc.;

Coarse grid correction Solve the error equation on the coarse grid;

Prolongation Inject the correction solution defined on the coarse grid to the fine grid by, say, interpolation, etc., and added it to $u_h^{(k)}$ to obtain a better approximation;

Post-smoothing Perform a few more smoothing iterations to diminishing the high frequency errors possibly introduced in the prolongation step.

William L. Briggs, Van Emden Henson, Steve F. McCormick, A Multigrid Tutorial, 2nd Edition, SIAM,2000 [多重网格方法的初级教程，自学学习极好的资料]
https://www.researchgate.net/publication/264929445_A_Multigrid_Tutorial_2nd_edition_with_corrections

The Domain Decomposition Method for Solving PDEs

大规模问题的计算: DDM

1. In numerically solving large scale partial differential equations, the domain decomposition method is a type of highly efficient iterative methods, which are particularly suitable for parallel computation.
2. Divide the domain Ω into subdomains Ω_i, $i = 1, 2, \dots, M$, with or without overlapping.
3. Decompose the problem into subproblems defined on the subdomains Ω_i.
4. Improve the current approximate solution iteratively using the information exchanged between the subdomains.
5. The process could be coupled with some postprocessing

http://www.ddm.org
https://www.image.ucar.edu/public/TOY/2008/focus4/Presentations/TALKKeyes.pdf
https://www.image.ucar.edu/public/TOY/2008/focus4/Presentations/

Mixed Finite Element Problem

P203------->P229

抽象的变分问题(5.3.10)--------->(6.3.1)

Typical mixed finite element problem:

$$\begin{cases} \text{Find } \ \mathbf{p}_h \in \mathbb{X}_h, \ u_h \in \mathbb{Y}_h \text{ such that} \\ a(\mathbf{p}_h, \mathbf{q}_h) + b(\mathbf{q}_h, u_h) = G(\mathbf{q}_h), \quad \forall \mathbf{q}_h \in \mathbb{X}_h, \\ \qquad\qquad\qquad\quad b(\mathbf{p}_h, v_h) = F(v_h), \quad \forall v_h \in \mathbb{Y}_h. \end{cases} \tag{6.3.1}$$

Existence Theorem of Mixed Finite Element Problem

Theorem 6.2 P229

Th5.15(Brezzi定理)

(**Brezzi**) *Let* $a(\mathbf{p}, \mathbf{q})$ *and* $b(\mathbf{q}, u)$ *be bounded bilinear forms on* $\mathbb{X} \times \mathbb{X}$ *and* $\mathbb{X} \times \mathbb{Y}$ *respectively, let* $G(\mathbf{q})$ *and* $F(v)$ *be bounded linear forms on* $\mathbb{X}$ *and* $\mathbb{Y}$ *respectively. Denote* $\mathbb{V}_{h0} = \{\mathbf{p}_h \in \mathbb{X}_h : b(\mathbf{p}_h, v_h) = 0, \ \ \forall v_h \in \mathbb{Y}_h\}$*. Suppose*

$(1)'$ *there exists* $\alpha_h > 0$*, such that*

$$\boxed{a(\mathbf{p}_h, \mathbf{p}_h) \geq \alpha_h \|\mathbf{p}_h\|_{\mathbb{X}}^2, \quad \forall \mathbf{p}_h \in \mathbb{V}_{h0},}$$

$(2)'$ *there exists* $\beta_h > 0$*, such that*

$$\boxed{\sup_{0 \neq \mathbf{p}_h \in \mathbb{X}_h} \frac{b(\mathbf{p}_h, v_h)}{\|\mathbf{p}_h\|_{\mathbb{X}}} \geq \beta_h \|v_h\|_{\mathbb{Y}}, \quad \forall v_h \in \mathbb{Y}_h.}$$

Then, the mixed finite element problem has a unique solution.

Finite Element Function Spaces $\mathbb{X}_h$ and $\mathbb{Y}_h$ Must be Properly Coupled

1. Condition $(2)'$: Babuška-Brezzi condition or B-B condition.
2. To guarantee the convergence, the constants α_h and β_h are usually required to be independent of h.
3. The B-B condition imposes restrictions on the choice of finite element function spaces.
4. Let $\dim(\mathbb{X}_h) = N$, $\dim(\mathbb{Y}_h) = M$, and $\{\varphi_i\}_{i=1}^N$ and $\{\psi_j\}_{j=1}^M$ be the normalized bases of $\mathbb{X}_h$ and $\mathbb{Y}_h$ respectively.

Finite Element Function Spaces $\mathbb{X}_h$ and $\mathbb{Y}_h$ Must be Properly Coupled

5. A necessary condition for the finite element equation

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p}_h \\ u_h \end{pmatrix} = \begin{pmatrix} \mathbf{g}_h \\ \mathbf{f}_h \end{pmatrix}. \tag{6.3.2}$$

   to have no more than one solution is $\text{rank}(B) = M \leq N$. (6.3.3)

6. If A is positive definite, then,
   $\text{rank}(B) = M \leq N \quad \Leftrightarrow \quad$ B-B condition holds.

Conforming: $\mathbb{V}_h \subset \mathbb{V}$; Non-Conforming: $\mathbb{V}_h \not\subset \mathbb{V}$

非协调FEM例子

An example of the non-conforming finite element method.

1. Consider the variational problem on a polygon region $\Omega \subset \mathbb{R}^2$:
   2D多边形区域

$$\boxed{\begin{cases} \text{Find} \quad u \in \mathbb{H}_0^1(\Omega), \quad \text{such that} \\ \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega fv \, dx, \quad \forall v \in \mathbb{H}_0^1(\Omega). \end{cases}}$$

2. K: triangle with vertices $\{\mathbf{a}_i\}_{i=1}^3$, $P_K = \mathbb{P}_1(K)$, $\Sigma_K = \{\mathbf{a}_{ij}\}$.

3. $\tilde{\mathbb{V}}_h = \{u|_{T_i} \in \mathbb{P}_1, \ \forall\ T_i \in \mathfrak{T}_h(\Omega), \text{ continuous on } Q_i \in \mathbb{Q}_h\}$, (6.3.4)
   $\tilde{\mathbb{V}}_h(0) = \left\{u \in \tilde{\mathbb{V}}_h : u(Q_i) = 0, \ \forall Q_i \in \mathbb{Q}_h \cap \partial\Omega\right\}$, where (6.3.5)
   $\mathbb{Q}_h = \Sigma_K = \{\mathbf{a}_{ij}\}$ is the set of middle points of edges in $\mathfrak{T}_h(\Omega)$.

如果FEM中有限元函数空间都是变分问题中的基本函数空间的子空间, 有限元问题中的泛函就是变分问题中的泛函, 称这样的FEM为协调FEM. 相应的有限元为协调FEM. (P230)

Conforming: $\mathbb{V}_h \subset \mathbb{V}$; Non-Conforming: $\mathbb{V}_h \not\subset \mathbb{V}$

4. $a_h(u, v) = \sum_{T \in \mathfrak{T}_h(\Omega)} \int_T \nabla u \cdot \nabla v \, dx.$ (6.3.6)

5. The nonconforming finite element problem:

一个典型的非协调FE问题的提法

$$\boxed{\begin{cases} \text{Find} \quad u_h \in \tilde{\mathbb{V}}_h(0) \quad \text{such} \\ a(u_h, v_h) = (f, v_h), \quad \forall v_h \in \tilde{\mathbb{V}}_h(0). \end{cases}} \qquad (6.3.7)$$

6. Provide a lot of convenience, accompanied by additional difficulties. 非协调FEM提供了很多便利，例如对高阶PDE， 协调FEM的构造和使用往往比非协调的FEM要困难得多，此时非协调FEM备受青睐。 当然, 它也会带来了额外的困难.

(1) 用数值积分代替积分也是一种常见的带来非协调性的做法.
(2) 区域无法作严格的有限元剖分时也会引入非协调性.

Thank You!