Numerical Solutions to Partial Differential Equations

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Relations Between the Errors in $\mathbb{L}^2$ and $\mathbb{H}^1$ norms

L2和H1模下误差之间关系

1. It follows from the interpolation error estimates on regular affine family of finite element function spaces (see Th7.7) that

$$\boxed{\|v - \Pi_h v\|_{m,\Omega} \leq C\, h^{k+1-m}\, |v|_{k+1,\Omega}, \quad m = 0, 1.} \tag{7.3.5}$$

2. In other words, under the same conditions, the error of the finite element interpolation in the $\mathbb{L}^2(\Omega)$-norm is one order higher than that in the $\mathbb{H}^1(\Omega)$-norm.

对C0类仿射等价FE族或能够嵌入到某个仿射族的C0类FE所构造的FE函数空间，可以在Th7.9中取l=1，由此有常用的FE函数空间上的插值误差估计(7.3.5). **==>**一般地, **L2**模下的误差比其在**H1**模下的误差高一阶.

The Relations Between the Errors in $\mathbb{L}^2$ and $\mathbb{H}^1$ norms

7.3.2节 Aubin-Nitsche技巧与L2模下误差估计

奥宾-尼歇尔技巧

3. By the Céa lemma, the error of the finite element solution u_h in $\mathbb{H}^1(\Omega)$-norm is optimal. However, the error in $\mathbb{L}^2(\Omega)$-norm thus obtained

$$\|u-u_h\|_{0,\Omega} \underset{\text{范数定义}}{\leq} \|u-u_h\|_{1,\Omega} \leq C\|u-\Pi_h u\|_{1,\Omega},$$

is obviously not optimal.

该不等式给出的结果是：FE解的L2模误差阶和H1下的插值误差同阶，低于L2模插值误差阶，故不是最优的/丰满的[它比插值误差低一阶]

4. Under certain additional conditions, optimal $\mathbb{L}^2(\Omega)$-norm error estimate for FE solutions can be obtained by applying the Aubin-Nitsche technique based on the dual variational problem.

Aubin-Nitsche引理（有时称为Aubin-Nitsche技术，技巧或估计）是有限元理论的一个基本结果（在边界元分析中也经常如此），它通过对偶论证（duality argument）证明了方法在弱于能量范数的范数下的超收敛性。在最简单的情况下（二阶边值问题），与H1中的自然估计相比，给出了一个L2中的改进的估计。这种改进的收敛估计源于伴随问题的正则性定理和Galerkin方法的最优收敛性。短文[*BIT Numer. Math.* **44:** 287–290, 2004]在抽象的层次上证明这些超收敛结果族的关键是能量空间包含到我们寻找改进估计的空间中的紧性。

Dual Variational Problem and Relations of Errors in $\mathbb{L}^2$ and $\mathbb{H}^1$ norms

1. Consider the variational problem
$$\begin{cases} \text{Find} \quad u \in \mathbb{V} \text{ such that} \\ a(u,v) = f(v), \quad \forall v \in \mathbb{V}, \end{cases} \tag{7.1.1}$$
where $\mathbb{V} \subset \mathbb{H}^1(\Omega)$, the bilinear form $a(\cdot,\ \cdot)$ and the linear form $f(\cdot)$ satisfy the conditions of the Lax-Milgram lemma.

2. Let $\mathbb{V}_h$ be a closed linear subspace of $\mathbb{V}$, and $u_h \in \mathbb{V}_h$ satisfy the equation
$$a(u_h, v_h) = f(v_h), \quad \forall v_h \in \mathbb{V}_h. \tag{7.1.2}$$

3. Define the dual variational problem: (7.1.1)的adjoint问题

(7.1.1)的对偶变分问题
$$\boxed{\begin{cases} \text{Find} \quad \varphi \in \mathbb{V} \text{ such that} \\ a(v,\varphi) = (u - u_h,\ v), \quad \forall v \in \mathbb{V}, \end{cases}} \tag{7.3.10}$$
where $(\cdot,\ \cdot)$ is the $\mathbb{L}^2(\Omega)$ inner product.

Dual Variational Problem and Relations of Errors in $\mathbb{L}^2$ and $\mathbb{H}^1$ norms

对偶变分问题， L2和H1模下误差之间关系

Lemma 7.1

Let $\varphi \in \mathbb{V}$ be the solution of the dual variational problem, and let $\varphi_h \in \mathbb{V}_h$ satisfy the equation

(7.1.1)的对偶变分问题(7.3.10)

$$a(v_h, \varphi_h) = (u - u_h, v_h), \quad \forall v_h \in \mathbb{V}_h. \tag{7.3.11}$$

Then, we have

$$\boxed{\|u - u_h\|_{0,\Omega}^2 \leq M\|u - u_h\|_{1,\Omega}\|\varphi - \varphi_h\|_{1,\Omega}.} \tag{7.3.12}$$

L2模下的误差　　H1模下的误差

Dual Variational Problem and Relations of Errors in $\mathbb{L}^2$ and $\mathbb{H}^1$ norms

Proof:

(7.1.1)的对偶变分问题(7.3.10)

Take $v = u - u_h$ in the dual variational equation, and by the facts that $a(u - u_h,\ v_h) = 0$, $\forall v_h \in \mathbb{V}_h$ and $a(\cdot,\ \cdot)$ is bounded, we are lead to

$$\|u - u_h\|^2_{0,\Omega} \overset{(7.3.10)}{=} a(u - u_h,\ \varphi) = a(u - u_h,\ \varphi - \varphi_h)$$
$$\leq M\|u - u_h\|_{1,\Omega}\underbrace{\|\varphi - \varphi_h\|_{1,\Omega}}_{\text{尼歇尔技巧参数}}. \quad \blacksquare$$

An Optimal Error Estimate in $\mathbb{L}^2$-Norm

Theorem 7.12

Let the space dimension $n \leq 3$. Assume that the solution φ of the dual variational problem (7.3.10) is in $\mathbb{H}^2(\Omega) \cap \mathbb{V}$, and satisfies

(7.1.1)的对偶变分问题(7.3.10)的解满足

$$\|\varphi\|_{2,\Omega} \leq C \|u - u_h\|_{0,\Omega}. \tag{7.3.13}$$

Let $\{(K, P_K, \Sigma_K)\}_{K \in \bigcup_{h>0} \mathfrak{T}_h(\Omega)}$ be a family of regular class $\mathbb{C}^0$ type (1) Lagrange affine equivalent finite elements. Then, the $\mathbb{L}^2(\Omega)$-norm error of the finite element solutions of the variational problem (7.1.1) satisfy

L2模下的误差

$$\boxed{\|u - u_h\|_{0,\Omega} \leq C h \|u - u_h\|_{1,\Omega}.} \tag{7.3.14}$$

(7.1.1)的解满足

Furthermore, if the solution u of the variational problem (7.1.1) is in $\mathbb{H}^2(\Omega) \cap \mathbb{V}$, then

L2模下的误差

$$\boxed{\|u - u_h\|_{0,\Omega} \leq C h^2 |u|_{2,\Omega},} \tag{7.3.15}$$

Here C in the three inequalities represent generally different constants which are independent of h.

Proof of the Optimal Error Estimate in $\mathbb{L}^2$-Norm

Th5.5(P191)

1. By the Sobolve embedding theorem, $\mathbb{W}^{m+s,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^s(\bar{\Omega})$, $\forall\, s \geq 0$, if $m > n/p$. In particular, $\mathbb{H}^2(\Omega) \hookrightarrow \mathbb{C}(\bar{\Omega})$, if $n \leq 3$.

2. Thus, by applying the error estimates for finite element solutions in $\mathbb{H}^1(\Omega)$ norm (see Th7.10 with $k = 1$ and $s = 0$) to the dual variational problem (7.3.10), we obtain

$$\underline{\|\varphi - \varphi_h\|_{1,\Omega} \overset{(7.3.8)}{\leq} Ch|\varphi|_{2,\Omega}}.$$

奥宾-尼歇尔技巧参数

Proof of the Optimal Error Estimate in $\mathbb{L}^2$-Norm

3. Therefore, by the $\overset{\text{Lem7.1 (7.3.12)}}{\text{Lem7.1}}$ on the dual problem and $\|\varphi\|_{2,\Omega} \underset{\text{即(7.3.13)}}{\leq} C\,\|u-u_h\|_{0,\Omega}$, we have

$$\|u-u_h\|_{0,\Omega} \leq C\,h\,\|u-u_h\|_{1,\Omega}. \qquad (7.3.14)$$

4. Applying again Th7.10 with $k \underset{(7.3.7)}{=} 1$ and $s = 0$ to $\|u-u_h\|_{1,\Omega}$, we get

$$\|u-u_h\|_{0,\Omega} \leq C\,h^2\,|u|_{2,\Omega}. \qquad (7.3.15) \quad \blacksquare$$

Remarks on the Optimal Error Estimate in $\mathbb{L}^2$-Norm

提升一阶L2模误差估计的关键是

(7.3.13)

1. The key to increase the $\mathbb{L}^2$-norm error estimate by an order is $\|\varphi\|_{2,\Omega} \leq C\,\|u - u_h\|_{0,\Omega}$, which does hold, if the coefficients of the second order elliptic operator are sufficiently smooth, 2阶椭圆型算子的系数是充分光滑的 and Ω is a convex polygonal region or a region with 区域是凸多边形的, 或有充分光滑的边界 sufficiently smooth boundary.

2. In the general case, if we have $\|\varphi - \varphi_h\|_{1,\Omega} \propto h^{\alpha}\|u - u_h\|_{0,\Omega}$ and $\|u - u_h\|_{1,\Omega} \propto h^{\alpha}$, then,
$$\|u - u_h\|_{0,\Omega} \propto h^{2\alpha}.$$

符号 " ∝ " 表示成正比例；y∝x (读作"y正比于x")

3. Generally, we expect the convergence rate of finite element solutions in the $\mathbb{L}^2$-norm is twice of that in the $\mathbb{H}^1$-norm.
一般来说，我们期望L2范数的有限元解的收敛速度是H1范数的两倍.

文献中也有讨论连续模(L^\infty)下的有限元解的误差估计[一致收敛]，参见姜礼尚等FEM的书.
也讨论非FEM的整数阶误差估计，参见王&许P128

Nonconformity and Consistency Error

7.4节 非协调性与相容性误差 p253

FEM的协调性经常会被破坏，因此需要相应地推广抽象的误差估计

- The conformity of finite element methods is often broken, so it is necessary to extend abstract error estimates accordingly.

数值积分会破坏FEM的协调性，并引入相容性误差

- Numerical quadratures break the conformity and introduce consistency error.

First Strang Lemma — Abstract Error Estimate Including Consistency Error

Theorem 7.13 (第一Strang引理)

Let $\mathbb{V}_h \subset \mathbb{V}$, and let the bilinear form $a_h(\cdot,\cdot)$ defined on $\mathbb{V}_h \times \mathbb{V}_h$ be uniform $\mathbb{V}_h$-elliptic, i.e. *there exists a constant $\hat{\alpha} > 0$ independent of h such that*

$$a_h(v_h, v_h) \geq \hat{\alpha}\|v_h\|^2, \quad \forall v_h \in \mathbb{V}_h. \tag{7.4.1}$$

Then, there exists a constant C independent of h such that

$$\|u-u_h\| \leq C\Big(\underbrace{\inf_{v_h\in\mathbb{V}_h} \Big\{\|u-v_h\|}_{\text{对应于Cea引理}} + \underbrace{\sup_{w_h\in\mathbb{V}_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|}\Big\}}_{\text{数值积分误差}} + \underbrace{\sup_{w_h\in\mathbb{V}_h} \frac{|f(w_h) - f_h(w_h)|}{\|w_h\|}}_{\text{数值积分误差}} \Big). \tag{7.4.2}$$

Proof of the First Strang Lemma

1. Since $\mathbb{V}_h \subset \mathbb{V}$ and $a(u, v) = f(v)$, $\forall v \in \mathbb{V}$, we have
$$a(u - v_h,\, u_h - v_h) + a(v_h,\, u_h - v_h) - f(u_h - v_h) = 0, \quad \forall v_h \in \mathbb{V}_h. \qquad (*1)$$

2. Since $a_h(u_h,\, v_h) = f_h(v_h)$, $\forall v_h \in \mathbb{V}_h$, we have
$$a_h(u_h - v_h,\, u_h - v_h) = f_h(u_h - v_h) - a_h(v_h,\, u_h - v_h), \quad \forall v_h \in \mathbb{V}_h. \qquad (*2)$$

3. Therefore, by the uniform $\mathbb{V}_h$-ellipticity of $a_h(\cdot,\cdot)$ on $\mathbb{V}_h$, we have
$$\begin{aligned}\hat{\alpha}\|v_h - u_h\|^2 &\overset{(7.4.1)}{\leq} a_h(u_h - v_h,\, u_h - v_h)\\ &\overset{(*1,*2)}{=} a(u - v_h, u_h - v_h) + \underline{\{a(v_h,\, u_h - v_h) - a_h(v_h,\, u_h - v_h)\}}\\ &\quad + \underline{\{f_h(u_h - v_h) - f(u_h - v_h)\}}.\end{aligned}$$

Proof of the First Strang Lemma

4. Hence, by the boundedness of the bilinear form $a(\cdot, \cdot)$, and $|f_h(u_h - v_h) - f(u_h - v_h)| \leq \sup_{w_h \in \mathbb{V}_h} \frac{|f(w_h) - f_h(w_h)|}{\|w_h\|} \|u_h - v_h\|$

$$|a(v_h, u_h - v_h) - a_h(v_h, u_h - v_h)| \leq \sup_{w_h \in \mathbb{V}_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|} \|u_h - v_h\|,$$

we have

$$\hat{\alpha}\|u_h - v_h\| \leq M\|u - v_h\| + \sup_{w_h \in \mathbb{V}_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|} + \sup_{w_h \in \mathbb{V}_h} \frac{|f(w_h) - f_h(w_h)|}{\|w_h\|}.$$

5. Since $\|u - u_h\| \leq \|u - v_h\| + \|u_h - v_h\|$, the conclusion of the theorem follows for $C = \max\{\hat{\alpha}^{-1}, 1 + \hat{\alpha}^{-1} M\}$. ■

Use of Non-Conforming Finite Element Function Spaces

非协调有限元

1. The conformity will be broken, if a non-conforming finite element is used to construct the finite element function spaces.
2. In such a case, $\mathbb{V}_h \not\subseteq \mathbb{V}$, therefore, $\|\cdot\|$, $f(\cdot)$ and $a(\cdot,\cdot)$ must be extended to $\|\cdot\|_h$, $f_h(\cdot)$ and $a_h(\cdot,\cdot)$ defined on $\mathbb{V}+\mathbb{V}_h$.
   需要把范数，线性泛函和双线性形式延拓为V+Vh上的||*||h等
3. For example, if $\mathbb{V}=\mathbb{H}_0^1(\Omega)$ and $a(u,v)=\int_\Omega \nabla u\cdot\nabla v\,dx$, we may define

$$\underbrace{v_h \mapsto \|v_h\|_h}_{\text{离散范数}} := \Big(\sum_{K\in\mathfrak{T}_h(\Omega)} |v_h|_{1,K}^2\Big)^{1/2},$$

$$\underbrace{(u_h, v_h) \mapsto a_h(u_h, v_h)}_{\text{离散双线性形式}} := \sum_{K\in\mathfrak{T}_h(\Omega)} \int_K \nabla u_h\cdot\nabla v_h\,dx.$$

Error Bound of Non-Conforming Finite Element solution

非协调性有限元解的误差界

The following abstract error estimate again bounds the error of the finite element solution in the non-conforming finite element function spaces by

- the approximation error of the finite element function space;
  有限元函数空间的逼近误差
- and the consistency error of the approximation functionals $a_h(\cdot, \cdot)$ and $f_h(\cdot)$.
  近似泛函的相容误差

Second Strang Lemma — Abstract Error Estimate for Non-Conforming FE

Theorem 7.14 (第二Strang引理)

Let the bilinear form $a_h(\cdot, \cdot)$ *be uniformly bounded on* $(\mathbb{V}+\mathbb{V}_h)\times(\mathbb{V}+\mathbb{V}_h)$*, and be uniformly* $\mathbb{V}_h$*-elliptic*, i.e. *there exist constants* $\hat{M}$ *and* $\hat{\alpha}>0$ *independent of* h *such that*

$$\boxed{\begin{aligned} |a_h(u_h, v_h)| &\le \hat{M}\|u_h\|_h\,\|v_h\|_h, \quad \forall u_h,\ v_h\in\mathbb{V}+\mathbb{V}_h, \qquad (7.4.3)\\ a_h(v_h, v_h) &\ge \hat{\alpha}\|v_h\|_h^2, \quad \forall v_h\in\mathbb{V}_h. \qquad (7.4.4)\end{aligned}}$$

Then, the error of the solution u_h *of the corresponding approximation variational problem with respect to the solution* u *of the original variational problem satisfies*

$$\boxed{\|u-u_h\|_h \cong \left(\inf_{v_h\in\mathbb{V}_h}\|u-v_h\|_h + \sup_{w_h\in\mathbb{V}_h}\frac{|a_h(u, w_h)-f_h(w_h)|}{\|w_h\|_h}\right).} \qquad (7.4.5)$$

Here $A_h(u)\cong B_h(u)$ *means that there exist positive constants* C_1 *and* C_2 *independent of* u *and* h *s.t.* $C_1B_h(u)\le A_h(u)\le C_2B_h(u)$*, for all* $h>0$ *sufficiently small.*

Proof of the Second Strang Lemma

(Step 1)

1. Since $a_h(u_h, v_h) = f_h(v_h)$, $\forall v_h \in \mathbb{V}_h$, we have

$$a_h(u_h - v_h,\ u_h - v_h) = f_h(u_h - v_h) - a_h(v_h,\ u_h - v_h), \quad \forall v_h \in \mathbb{V}_h. \qquad (*3)$$

2. Therefore, by the uniform $\mathbb{V}_h$-ellipticity of $a_h(\cdot,\cdot)$ on $\mathbb{V}_h$, we have

$$\hat{\alpha}\|v_h - u_h\|^2 \overset{(7.4.4)}{\leq} a_h(u_h - v_h,\ u_h - v_h)$$
$$\overset{(*3)}{=} a_h(u - v_h, u_h - v_h) + \{f_h(u_h - v_h) - a_h(u,\ u_h - v_h)\}.$$

3. Thus, the uniform boundedness of $a_h(\cdot,\ \cdot)$ and

$$\|u - u_h\|_h \leq \|u - v_h\|_h + \|u_h - v_h\|_h$$

led to "$\leq$" part of the theorem for $C_2 = \max\{\hat{\alpha}^{-1}, 1 + \hat{\alpha}^{-1}\hat{M}\}$.

Proof of the Second Strang Lemma

(Step 2)

4. On the other hand, it follows from the uniform boundedness of $a_h(\cdot,\,\cdot)$ that

$$a_h(\underline{u,}\, w_h) - f_h(w_h) = a_h(\underline{u-}\, u_h, w_h) \overset{(7.4.3)}{\leq} \hat{M}\|u-u_h\|_h\|w_h\|_h, \quad \forall w_h \in \mathbb{V}_h.$$

5. Thus, by the arbitrariness of w_h, we have

$$\|u - u_h\|_h \geq \hat{M}^{-1} \sup_{w_h \in \mathbb{V}_h} \frac{|a_h(u,\, w_h) - f_h(w_h)|}{\|w_h\|_h}.$$

6. This together with $\|u - u_h\|_h \geq \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_h$ yield the "$\geq$" part of theorem with $C_1 = \dfrac{1}{2}\min\{\hat{M}^{-1}, 1\}$. ∎

In General $\mathbb{V}_h \not\subseteq \mathbb{V}$ for Non-Polygon Ω

当区域不是多边形的时，网格剖分也会导致非协调性

1. If Ω is not a polygonal region, the region Ω_h covered by a finite element triangulation is generally not equal to Ω, this will also lead to nonconformity $\mathbb{V}_h \not\subseteq \mathbb{V}$.

对一般的非协调情况, 从Strang引理可知，为了得到FE解的误差估计，除了需要插值误差估计外，还要估计相容性误差

2. For a general case when there is nonconformity, it follows from the first and second Strang lemmas that, to obtain the **error estimates** for finite element solutions, in addition to the **interpolation error** estimates, the **consistency errors** of the approximate bilinear forms $a_h(\cdot,\cdot)$ and linear forms $f_h(\cdot)$ must also be properly estimated.

通常需要近似算子是一致连续和稳定的

3. It is usually required that the **approximate operators are uniformly continuous and stable** (*i.e.* the approximate bilinear forms $a_h(\cdot,\cdot)$ are uniformly bounded and uniformly $\mathbb{V}_h$-elliptic).

The Basic Tools for the Analysis of Consistency Error

7.4.2节 Bramble-Hilbert 引理和双线性引理

分析相容误差的基本工具仍是多项式商空间中的等价商范数和Sobolev半范数的关系

1. The basic tools for the analysis are still the equivalent quotient norms in the polynomial quotient spaces and relations between the Sobolev semi-norms on affine equivalent open sets.

多项式不变算子的误差估计在插值误差估计中扮演了很重要的角色

2. Error estimates on polynomial invariant operators play a very important role in the interpolation error estimates. The following two lemmas are the counterparts in the consistency error estimates for polynomial vanishing linear and bilinear forms.

下面两个引理是多项式消失的线性形式和双线性形式的相容误差估计中的对应物

The Bramble-Hilbert Lemma —

— An Abstract Estimate on Polynomial Vanishing Linear Forms

Theorem 7.15 (Bramble-Hilbert 引理)

Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary. For some $p \in [1, \infty]$ and some integer $k \geq 0$, let the bounded linear form f defined on $\mathbb{W}^{k+1,p}(\Omega)$ be such that

$$f(w) = 0, \qquad \forall w \in \mathbb{P}_k(\Omega). \tag{7.4.6}$$

Then, there exists a constant $C(\Omega)$ such that

$$\boxed{|f(v)| \leq C(\Omega) \, \|f\|^*_{k+1,p,\Omega} |v|_{k+1,p,\Omega},} \tag{7.4.7}$$

where $\|\cdot\|^*_{k+1,p,\Omega}$ is the norm on the dual space of $\mathbb{W}^{k+1,p}(\Omega)$.

It is an abstract theoretical tool for studying the approximation error of functions in Sobolev spaces (cf. also Approximation of functions; Sobolev space) by algebraic polynomials.

J.H. Bramble, S.R. Hilbert, "Bounds for a class of linear functionals with applications to Hermite interpolation" Numer. Math. , 16 (1971) pp. 362 -369.

J.H. Bramble, S.R. Hilbert, "Estimation of linear functionals on Sobolev spaces with application to Fourier transforms and spline interpolation" SIAM J. Numer. Anal. , 7 (1970) pp. 112 -124

Proof of the Bramble-Hilbert Lemma

1. For any $v \in \mathbb{W}^{k+1,p}(\Omega)$, it follows from $f(w) = 0$, $\forall w \in \mathbb{P}_k(\Omega)$, that
$$|f(v)| = |f(v+w)| \leq \|f\|^*_{k+1,p,\Omega} \|v+w\|_{k+1,p,\Omega}, \quad \forall w \in \mathbb{P}_k(\Omega),$$
2. Thus,
$$|f(v)| \leq \|f\|^*_{k+1,p,\Omega} \inf_{w\in\mathbb{P}_k} \|v + w\|_{k+1,p,\Omega}.$$
商空间的范数
3. Since $|\cdot|_{k+1,p,\Omega}$ is an equivalent quotient norm in the polynomial quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$ (see Theorem 7.2), the conclusion of the theorem follows. ■
(7.2.2)

The Bilinear Lemma —
— An Abstract Estimate on Polynomial Vanishing Bilinear Forms

Theorem 7.16 (双线性引理)

Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary. For some $p, q \in [1, \infty]$, some integers $k, l \geq 0$ and a subspace $\mathbb{W}$ which satisfies the inclusion relation $\mathbb{P}_l(\Omega) \subset \mathbb{W} \subset \mathbb{W}^{l+1,q}(\Omega)$ and is endowed with the norm $\|\cdot\|_{l+1,q,\Omega}$, let the bounded bilinear form b defined on $\mathbb{W}^{k+1,p}(\Omega) \times \mathbb{W}$ be such that

$$b(r,\ w) \ = \ 0, \qquad \forall r \in \mathbb{P}_k(\Omega),\ \ \forall w \in \mathbb{W}, \tag{7.4.8}$$

$$b(v,\ r) \ = \ 0, \qquad \forall v \in \mathbb{W}^{k+1,p}(\Omega),\ \ \forall r \in \mathbb{P}_l(\Omega). \tag{7.4.9}$$

Then, there exists a constant $C(\Omega)$ such that

$$\boxed{|b(v,w)| \leq C(\Omega)\|b\||v|_{k+1,p,\Omega}|w|_{l+1,q,\Omega}, \quad \forall v \in \mathbb{W}^{k+1,p}(\Omega),\ \forall w \in \mathbb{W}} \tag{7.4.10}$$

where $\|b\|$ is the norm of the bilinear form b on $\mathbb{W}^{k+1,p}(\Omega) \times \mathbb{W}$.

Proof of the Bilinear Lemma

1. For any given $w \in \mathbb{W}$, $b(\cdot,\ w)$, regarded as a bounded linear form defined on $\mathbb{W}^{k+1,p}(\Omega)$ satisfies the conditions of the Bramble-Hilbert lemma.

2. Hence, there exists a constant $C_1(\Omega)$ such that

$$|b(v,\ w)| \underset{(7.4.7)}{\leq} C_1(\Omega)\, \underbrace{\|b(\cdot,\ w)\|^*_{k+1,p,\Omega}}_{\text{估算该范数是下面的主要任务}}\, |v|_{k+1,p,\Omega}, \quad \forall v \in \mathbb{W}^{k+1,p}(\Omega). \qquad (7.4.11)$$

3. On the other hand, since for any $v \in \mathbb{W}^{k+1,p}(\Omega)$, $b(v,\ r) = 0$, $\forall r \in \mathbb{P}_l(\Omega)$, we have

$$|b(v,w)| \underset{(7.4.9)}{=} |b(v,w+r)| \overset{\text{有界}}{\leq} \|b\| \|v\|_{k+1,p,\Omega} \|w+r\|_{l+1,q,\Omega}, \quad \forall r \in \mathbb{P}_l. \quad (*1)$$

Proof of the Bilinear Lemma

4. Since $|\cdot|_{l+1,p,\Omega}$ is an equivalent quotient norm in the polynomial quotient space $\mathbb{W}^{l+1,p}(\Omega)/\mathbb{P}_l(\Omega)$, $\exists$ const. $C_2(\Omega)$ s.t. $\inf_{r\in\mathbb{P}_l} \|w+r\|_{l+1,q,\Omega} \overset{(7.2.2)}{\leq} C_2(\Omega)|w|_{l+1,q,\Omega}$.

(*1)

5. Therefore, for any $v \in \mathbb{W}^{k+1,p}(\Omega)$, we have
$$|b(v,w)| \leq C_2(\Omega)\|b\|\|v\|_{k+1,p,\Omega}|w|_{l+1,q,\Omega}, \quad \forall w \in \mathbb{W}. \qquad (*2)$$

6. This implies
$$\|b(\cdot,w)\|^*_{k+1,p,\Omega} \overset{(7.4.11),\ (*2)}{=} \sup_{v\in\mathbb{W}^{k+1,p,\Omega}} \frac{|b(v,\ w)|}{\|v\|_{k+1,p,\Omega}} \leq C_2(\Omega)\|b\||w|_{l+1,q,\Omega}.$$

7. Combining this with ② (see (7.4.11)) leads to the conclusion of the theorem. ==>(7.4.10) ■

Thank You!