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# Numerical Stokes Equation

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**Abstract**

This report investigates numerical method to solve the two-dimensional Stockes equation. Through MAC scheme, we formulate the Stokes equation into a saddle point problem. However, the formulated equation is undetermined. Therefore, we introduce V-Cycle multi-grid method and choose Distributive Gauss Seidel Iteration and inexact Uzawa Iteration as smoother. Besides, we use V-cycle with Gauss-Seidel smoother to solve the subproblem of Uzawa iteration. By a thorough search of parameters, we find that V-cycle based on DGS and inexact Uzawa based on V-cycle methods outperform MATLAB's built-in linear solver when $N = 2048$.

# Contents

# 1 Introduction

Stokes flow is a type of fluid flow where advective inertial forces are small compared with viscous forces. The Reynolds number is low. For 2-dimensional Stokes Equation in the region $\Omega = (0,1)^2$, the stokes equation can be formulated as

$$\begin{aligned} \Delta \vec{u} + \nabla p &= \vec{F}, \\ \nabla \cdot \vec{u} &= 0. \end{aligned} \tag{1.1}$$

where $\vec{u} = (u, v)$ is the velocity, $p$ is the pressure, $\vec{F} = (f, g)$ is the external force.

The boundary condition is

$$\begin{aligned} &\left.\frac{\partial u}{\partial \nu}\right|_{y=0} = b, \quad \left.\frac{\partial u}{\partial \nu}\right|_{y=1} = t, \\ &\left.\frac{\partial v}{\partial \nu}\right|_{x=0} = l, \quad \left.\frac{\partial v}{\partial \nu}\right|_{x=1} = r, \\ &u|_{x=0} = u|_{x=1} = 0, \quad v|_{y=0} = v|_{y=1} = 0, \end{aligned} \tag{1.2}$$

In the problem, the external force will be

$$\begin{aligned} f(x,y) &= -4\pi^2(2\cos(2\pi x) - 1)\sin(2\pi y) + x^2, \\ g(x,y) &= 4\pi^2(2\cos(2\pi y) - 1)\sin(2\pi x). \end{aligned} \tag{1.3}$$

and the real solution is

$$\begin{aligned} u(x,y) &= (1 - \cos(2\pi x))\sin(2\pi y), \\ v(x,y) &= -(1 - \cos(2\pi y))\sin(2\pi x), \\ p(x,y) &= \frac{x^3}{3} - \frac{1}{12}. \end{aligned} \tag{1.4}$$

The boundary terms are:

$$\begin{aligned} b(x) &= -\left.\frac{\partial u}{\partial y}\right|_{y=0} = -2\pi(1 - \cos(2\pi x)), \\ t(x) &= \left.\frac{\partial u}{\partial y}\right|_{y=1} = 2\pi(1 - \cos(2\pi x)), \\ l(x) &= -\left.\frac{\partial v}{\partial x}\right|_{x=0} = 2\pi(1 - \cos(2\pi y)), \\ r(x) &= \left.\frac{\partial v}{\partial x}\right|_{x=1} = -2\pi(1 - \cos(2\pi y)). \end{aligned} \tag{1.5}$$

# 2 Numerical Algorithm

## 2.1 MAC scheme

We discretize the equation in the following sense[Harlow and Welch, 1965], the numerical solution $U, V, P$ can be defined as

$$
\begin{aligned}
u_{i,j-\frac{1}{2}} &\approx u(ih,(j-\frac{1}{2})h), \quad 0 \leq i \leq N, 1 \leq j \leq N, \\
v_{i-\frac{1}{2},j} &\approx u((i-\frac{1}{2})h, jh), \quad 1 \leq i \leq N, 0 \leq j \leq N, \\
p_{i-\frac{1}{2},j-\frac{1}{2}} &\approx p((i-\frac{1}{2})h,(j-\frac{1}{2})h), \quad 1 \leq i \leq N, 1 \leq j \leq N,
\end{aligned}
\tag{2.1}
$$

and the boundary condition can be written as

$$
\begin{aligned}
f_{i,j-\frac{1}{2}} &= f(ih,(j-\frac{1}{2})h), \quad 1 \leq i \leq N-1, 1 \leq j \leq N, \\
g_{i-\frac{1}{2},j} &= g((i-\frac{1}{2})h, jh), \quad 1 \leq i \leq N, 1 \leq j \leq N-1, \\
b_i &= b(ih), \quad t_i = t(ih), \quad 1 \leq i \leq N-1, \\
l_j &= l(jh), \quad r_j = r(jh), \quad 1 \leq j \leq N-1.
\end{aligned}
\tag{2.2}
$$

Owing to the boundary condition, we have $u_{0,j+\frac{1}{2}} = u_{N,j+\frac{1}{2}} = v_{i+\frac{1}{2},0} = v_{i+\frac{1}{2},N} = 0$. We can write $U, V, P$ compactly, such that $U \in \mathbb{R}^{(N-1)\times N}$, $V \in \mathbb{R}^{N\times N-1}$ and $P \in \mathbb{R}^{N\times N}$.

We define $T_{N-1} \in \mathbb{R}^{(N-1)\times(N-1)}$ with entries

$$
T_{N-1} = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix},
\tag{2.3}
$$

$S_{N-1} \in \mathbb{R}^{(N-1)\times N}$ with entries

$$
S_{N-1} = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}.
\tag{2.4}
$$

$R_{N-1} \in \mathbb{R}^{N\times N}$ with entries

$$
R_{N-1} = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{bmatrix},
\tag{2.5}
$$

### 2.1.1 Discretization of $U$

For $1 \leq i \leq N-1, 2 \leq j \leq N-1$, we have

$$
-\frac{1}{h^2}(u_{i+1,j-\frac{1}{2}} + u_{i-1,j-\frac{1}{2}} + u_{i,j+\frac{1}{2}} + u_{i,j-\frac{3}{2}} - 4u_{i,j-\frac{1}{2}}) + \frac{1}{h}(p_{i+\frac{1}{2},j-\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}) = f_{i,j-\frac{1}{2}}.
\tag{2.6}
$$

Stacking 2.6 with $1 \le i \le N-1$, we have

$$\frac{1}{h^2}\left(T_{N-1}U_{,.j} + 2U_{,.j} - U_{,.j-1} - U_{,.j+1}\right) + \frac{1}{h}S_{N-1}P_{.,j} = F_{.,j}. \tag{2.7}$$

For $1 \le i \le N-1, j = 1$, we have

$$-\frac{1}{h^2}(u_{i+1,j-\frac{1}{2}} + u_{i-1,j-\frac{1}{2}} + u_{i,j+\frac{1}{2}} - 3u_{i,j-\frac{1}{2}} + hb_i) + \frac{1}{h}(p_{i+\frac{1}{2},j-\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}) = f_{i,j-\frac{1}{2}}. \tag{2.8}$$

Stacking 2.8 with $1 \le i \le N-1$, we have

$$\frac{1}{h^2}\left(T_{N-1}U_{,.j} + U_{,.j} - U_{,.j+1} - hb\right) + \frac{1}{h}S_{N-1}P_{.,j} = F_{.,j}. \tag{2.9}$$

For $1 \le i \le N-1, j = N$, we have

$$-\frac{1}{h^2}(u_{i+1,j-\frac{1}{2}} + u_{i-1,j-\frac{1}{2}} + u_{i,j-\frac{3}{2}} - 3u_{i,j-\frac{1}{2}} + ht_i) + \frac{1}{h}(p_{i+\frac{1}{2},j-\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}) = f_{i,j-\frac{1}{2}}. \tag{2.10}$$

Stacking 2.10 with $1 \le i \le N-1$, we have

$$\frac{1}{h^2}\left(T_{N-1}U_{,.j} + U_{,.j} - U_{,.j-1} - ht\right) + \frac{1}{h}S_{N-1}P_{.,j} = F_{.,j}. \tag{2.11}$$

The above equation can be stacked into an equation

$$\frac{1}{h^2}(R_{N-1} \otimes I_{N-1} + I_N \otimes T_{N-1})\mathrm{Vec}(U) + \frac{1}{h}(I_N \otimes S_{N-1})\mathrm{Vec}(P) = \mathrm{Vec}(F) + \frac{1}{h}(e_N^{(1)} \otimes b + e_N^{(N)} \otimes t) \tag{2.12}$$

### 2.1.2 Discretization of $V$

For $2 \le i \le N-1, 1 \le j \le N-1$, we have

$$-\frac{1}{h^2}(v_{i-\frac{1}{2},j+1} + v_{i-\frac{1}{2},j-1} + v_{i+\frac{1}{2},j} + v_{i-\frac{3}{2},j} - 4v_{i-\frac{1}{2},j}) + \frac{1}{h}(p_{i-\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}) = g_{i-\frac{1}{2},j}. \tag{2.13}$$

Stacking 2.13 with $1 \le j \le N-1$, we have

$$\frac{1}{h^2}\left(T_{N-1}V_{i,.}^T + 2V_{i,.}^T - V_{i+1,.}^T - V_{i-1,.}^T\right) + (S_{N-1} \otimes (e_N^{(i)})^T)\mathrm{Vec}(P) = G_{i,.}^T. \tag{2.14}$$

For $i = 1, 1 \le j \le N-1$, we have

$$-\frac{1}{h^2}(v_{i-\frac{1}{2},j+1} + v_{i-\frac{1}{2},j-1} + v_{i+\frac{1}{2},j} - 3v_{i-\frac{1}{2},j} + hl_j) + \frac{1}{h}(p_{i-\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}) = g_{i-\frac{1}{2},j}. \tag{2.15}$$

Stacking 2.15 with $1 \le j \le N-1$, we have

$$\frac{1}{h^2}\left(T_{N-1}V_{i,.}^T + V_{i,.}^T - V_{i+1,.}^T - hl\right) + (S_{N-1} \otimes (e_N^{(i)})^T)\mathrm{Vec}(P) = G_{i,.}^T. \tag{2.16}$$

For $i = N, 1 \le j \le N-1$, we have

$$-\frac{1}{h^2}(v_{i-\frac{1}{2},j+1} + v_{i-\frac{1}{2},j-1} + v_{i-\frac{3}{2},j} - 3v_{i-\frac{1}{2},j} + hr_j) + \frac{1}{h}(p_{i-\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}) = g_{i-\frac{1}{2},j}. \tag{2.17}$$

Stacking 2.17 with $1 \le j \le N-1$, we have

$$\frac{1}{h^2}\left(T_{N-1}V_{i,\cdot}^T + 2V_{i,\cdot}^T - V_{i+1,\cdot}^T - V_{i-1,\cdot}^T - hr\right) + (S_{N-1} \otimes (e_N^{(i)})^T)\mathrm{Vec}(P) = G_{i,\cdot}^T. \tag{2.18}$$

We can combine 2.14, 2.16, 2.18 together in the following matrix form:

$$\frac{1}{h^2}(R_{N-1} \otimes I_{N-1} + I_N \otimes T_{N-1})\mathrm{Vec}(V^T) + \frac{1}{h}((S_{N-1} \otimes (e_N^{(i)})^T)_{i=1}^N)\mathrm{Vec}(P) = \mathrm{Vec}(G^T) + \frac{1}{h}(e_N^{(1)} \otimes l + e_N^{(N)} \otimes r) \tag{2.19}$$

We define

$$\begin{aligned} B_1 &= I_N \otimes S_{N-1} \\ B_2 &= (S_{N-1} \otimes (e_N^{(i)})^T)_{i=1}^N \end{aligned} \tag{2.20}$$

### 2.1.3 Discretization of $\nabla \cdot \vec{u} = 0$

$\nabla \cdot \vec{u} = 0$. For $1 \le i \le N$ and $1 \le j \le N$, we have

$$\frac{1}{h}(u_{i,j-\frac{1}{2}} - u_{i-1,j-\frac{1}{2}}) + \frac{1}{h}(v_{i-\frac{1}{2},j} - v_{i-\frac{1}{2},j-1}) = 0. \tag{2.21}$$

We can stack the equation as

$$\sum_{j=1}^{N} e_N^{(j)} \otimes \begin{bmatrix} U_{1,j} - U_{0,j} \\ \vdots \\ U_{N,j} - U_{N-1,j} \end{bmatrix} + \sum_{i=1}^{N} \begin{bmatrix} V_{i,1} - V_{i,0} \\ \vdots \\ V_{i,N} - V_{i,N-1} \end{bmatrix} \otimes e_N^{(i)} = 0. \tag{2.22}$$

i.e.

$$B_1^T\mathrm{Vec}(U) + B_2^T\mathrm{Vec}(V^T) = 0 \tag{2.23}$$

### 2.1.4 Summary of the discretization

We write

$$\begin{aligned} \vec{\mathbf{A}} &= \frac{1}{h^2}\begin{bmatrix} R_{N-1} \otimes I_{N-1} + I_N \otimes T_{N-1} & \\ & R_{N-1} \otimes I_{N-1} + I_N \otimes T_{N-1} \end{bmatrix}, \\ \vec{\mathbf{B}} &= \frac{1}{h}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \ \vec{\mathbf{U}} = \begin{bmatrix} \mathrm{Vec}(U) \\ \mathrm{Vec}(V^T) \end{bmatrix}, \ \vec{\mathbf{P}} = \mathrm{Vec}(P), \\ \vec{\mathbf{F}} &= \begin{bmatrix} \mathrm{Vec}(F) + \frac{1}{h}(e_N^{(1)} \otimes b + e_N^{(N)} \otimes t) \\ \mathrm{Vec}(G^T) + \frac{1}{h}((e_N^{(1)} \otimes l + e_N^{(N)} \otimes r) \end{bmatrix}. \end{aligned} \tag{2.24}$$

then the above relation can be written as

$$\begin{bmatrix} \vec{\mathbf{A}} & \vec{\mathbf{B}} \\ \vec{\mathbf{B}}^T & 0 \end{bmatrix} \begin{bmatrix} \vec{\mathbf{U}} \\ \vec{\mathbf{P}} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{F}} \\ 0 \end{bmatrix}. \tag{2.25}$$

## 2.2 Transfer operator

### 2.2.1 Restriction

At the $u$- and $v$ grid points, we consider six points restrictions, and at $p$-grid points, a four-point cell-centered restriction. In stencil notations, the restriction operators can be represented as

$$R^u_{h/2,h} = R^f_{h/2,h} = \frac{1}{8}\begin{bmatrix} 1 & 2 & 1 \\ & * & \\ 1 & 2 & 1 \end{bmatrix}, \quad R^g_{h/2,h} = R^v_{h/2,h} = \frac{1}{8}\begin{bmatrix} 1 & & 1 \\ 2 & * & 2 \\ 1 & & 1 \end{bmatrix}, \quad R^p_{h/2,h} = \frac{1}{4}\begin{bmatrix} 1 & & 1 \\ & * & \\ 1 & & 1 \end{bmatrix}. \tag{2.26}$$

If we define the following matrix

$$R^{(1)}_N = I_N \otimes \begin{bmatrix} 1 & 1 \end{bmatrix}, \tag{2.27}$$

and

$$R^{(2)}_N = \begin{bmatrix} 0 & R^{(1)}_{N-1} \end{bmatrix} + \begin{bmatrix} R^{(1)}_{N-1} & 0 \end{bmatrix}. \tag{2.28}$$

we can write the restrction operator in the following sense

$$\begin{aligned} R^u_{h/2,h} = R^v_{h/2,h} = R^f_{h/2,h} = R^g_{h/2,h} &= \frac{1}{8} R^{(1)}_N \otimes R^{(2)}_N \in \mathbb{R}^{N(N-1)\times 2N(2N-1)}. \\ R^p_{h/2,h} &= \frac{1}{4} R^{(1)}_N \otimes R^{(1)}_N \in \mathbb{R}^{N^2\times 4N^2}. \end{aligned} \tag{2.29}$$

i.e.

$$\begin{aligned} \mathrm{Vec}(U^h) &= R^u_{h/2,h}\mathrm{Vec}(U^{h/2}) \\ \mathrm{Vec}(F^h) &= R^f_{h/2,h}\mathrm{Vec}(F^{h/2}) \\ \mathrm{Vec}(G^h) &= R^g_{h/2,h}\mathrm{Vec}(G^{h/2}) \\ \mathrm{Vec}((V^h)^T) &= R^v_{h/2,h}\mathrm{Vec}((V^{h/2})^T) \\ \mathrm{Vec}(P^h) &= R^P_{h/2,h}\mathrm{Vec}(P^{h/2}) \end{aligned} \tag{2.30}$$

Hence, for $\vec{\mathbf{U}}$ and $\vec{\mathbf{F}}$, we can write

$$R^{\vec{\mathbf{U}}}_{h/2,h} = R^{\vec{\mathbf{F}}}_{h/2,h} = \begin{bmatrix} R^u_{h/2,h} & \\ & R^u_{h/2,h} \end{bmatrix}. \tag{2.31}$$

### 2.2.2 Prolongation

For $u$, we define a prolongation method that matches the transpose of the the restriction matrix.

$$U^{h/2}_{2i,2j-1} = U^{h/2}_{2i,2j} = U^h_{i,j}, \quad U^{h/2}_{2i-1,2j-1} = U^{h/2}_{2i-1,2j} = \frac{1}{2}U^h_{i-1,j} + \frac{1}{2}U^h_{i,j}. \tag{2.32}$$

Similarly, for $v$, we have

$$V^{h/2}_{2i-1,2j} = V^{h/2}_{2i,2j} = V^h_{i,j}, \quad V^{h/2}_{2i-1,2j-1} = V^{h/2}_{2i,2j-1} = \frac{1}{2}V^h_{i-1,j} + \frac{1}{2}V^h_{i,j}. \tag{2.33}$$

Therefore, we have

$$R^u_{h,h/2} = R^v_{h,h/2} = 4(R^u_{h/2,h})^T \tag{2.34}$$

For $p$, we have

$$P^{h/2}_{2i,2j} = P^{h/2}_{2i,2j-1} = P^{h/2}_{2i-1,2j} = P^{h/2}_{2i-1,2j-1} = P^h_{i,j}. \tag{2.35}$$

which indicates

$$R^p_{h,h/2} = (R^{(1)}_N)^T \otimes (R^{(1)}_N)^T = 4(R^p_{h/2,h})^T \tag{2.36}$$

## 2.3 DGS iteration

### 2.3.1 DGS-1

This section implements DGS in the paper [Wang and Chen, 2013]. We modify the problem 2.25 in the following form

$$\begin{bmatrix} \vec{\mathbf{A}} & \vec{\mathbf{B}} \\ \vec{\mathbf{B}}^T & 0 \end{bmatrix} \begin{bmatrix} \vec{\mathbf{U}} \\ \vec{\mathbf{P}} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{F}} \\ \vec{\mathbf{S}} \end{bmatrix}. \tag{2.37}$$

For $k = 0$, assume the initial values are $\vec{\mathbf{U}}^{(0)} = \begin{bmatrix} \mathrm{Vec}(U^{(0)}) \\ \mathrm{Vec}((V^{(0)})^T) \end{bmatrix}$ and $\vec{\mathbf{P}}^{(0)} = \mathrm{Vec}(\vec{\mathbf{P}}^{(0)})$. $\vec{\mathbf{A}} = D_{\vec{\mathbf{A}}} - L_{\vec{\mathbf{A}}} - U_{\vec{\mathbf{A}}}$ defined in the traditional Gauss-Seidel iteration.

1. Given $\vec{\mathbf{U}}^{(k)}$ and $\vec{\mathbf{P}}^{(k)}$, we use Gauss-Seidel iteration to update velocity $\vec{\mathbf{U}}^{(k+\frac{1}{2})}$,

$$\vec{\mathbf{U}}^{(k+\frac{1}{2})} = \vec{\mathbf{U}}^{(k)} + (D_{\vec{\mathbf{A}}} - L_{\vec{\mathbf{A}}})^{-1}(F - \vec{\mathbf{A}}\vec{\mathbf{U}}^{(k)} - \vec{\mathbf{B}}\vec{\mathbf{P}}^{(k)}). \tag{2.38}$$

2. Update the $\vec{\mathbf{P}}^{(k)}$

$$\Delta\vec{\mathbf{P}} = \vec{\mathbf{R}}(\vec{\mathbf{S}} - \vec{\mathbf{B}}^T\vec{\mathbf{U}}^{(k+\frac{1}{2})}). \tag{2.39}$$

   where $\vec{\mathbf{R}}$ is approximate inverse of the $\vec{\mathbf{B}}^T\vec{\mathbf{B}}$, we adopt R as the diagonal inverse of $\vec{\mathbf{B}}^T\vec{\mathbf{B}}$

3. Finally, we update $\vec{\mathbf{U}}^{(k+1)}$ and $\vec{\mathbf{P}}^{(k+1)}$ by

$$\begin{aligned} \vec{\mathbf{U}}^{(k+1)} &= \vec{\mathbf{U}}^{(k+\frac{1}{2})} + \vec{\mathbf{B}}\Delta\vec{\mathbf{P}} \\ \vec{\mathbf{P}}^{(k+1)} &= \vec{\mathbf{P}}^{(k)} - \vec{\mathbf{B}}^T\vec{\mathbf{B}}\Delta\vec{\mathbf{P}} \end{aligned} \tag{2.40}$$

We need to explicitly calculate $\vec{\mathbf{B}}^T\vec{\mathbf{B}}$. Actually,

$$\begin{aligned} \vec{\mathbf{B}}^T\vec{\mathbf{B}} &= (I_N \otimes S_{N-1})^T\, I_N \otimes S_{N-1} + \sum_{i=1}^{N} S^T_{N-1}S_{N-1} \otimes e^{(i)}_N (e^{(i)}_N)^T \\ &= I_N \otimes (S^T_{N-1}S_{N-1}) + (S^T_{N-1}S_{N-1}) \otimes I_N \end{aligned} \tag{2.41}$$

i.e.

$$\vec{\mathbf{B}}^T\vec{\mathbf{B}} = \frac{1}{h^2}\begin{bmatrix} S^T_{N-1}S_{N-1} + I_N & -I_N & & & \\ -I_N & S^T_{N-1}S_{N-1} + 2I_N & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & S^T_{N-1}S_{N-1} + 2I_N & -I_N \\ & & \ddots & -I_N & S^T_{N-1}S_{N-1} + I_N \end{bmatrix}. \tag{2.42}$$

Hence, $\mathcal{R} = \mathrm{Vec}(R')$, if we let $R' \in \mathbb{R}^{N\times N}$ inner unit be 1/4, edge unit be 1/3 and point unit be 1/2.

### 2.3.2 DGS-2

In the V-cycle process, the equation 2.25 can be rewritten as

$$\begin{bmatrix} \vec{\mathbf{A}} & \vec{\mathbf{B}} \\ \vec{\mathbf{B}}^T & 0 \end{bmatrix} \begin{bmatrix} \vec{\mathbf{U}} \\ \vec{\mathbf{P}} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{F}} \\ \vec{\mathbf{D}} \end{bmatrix}. \tag{2.43}$$

where $D$ is the error introduced by the V-cycle process. The notation is defined similarly in the last section, we summarize the DGS iteration in the following:

1. Given $\vec{\mathbf{U}}^{(k)}$ and $\vec{\mathbf{P}}^{(k)}$, we use Gauss-Seidel Iteration to update velocity $\vec{\mathbf{U}}^{(k+\frac{1}{2})}$,
$$\vec{\mathbf{U}}^{(k+\frac{1}{2})} = \vec{\mathbf{U}}^{(k)} + (D_{\vec{\mathbf{A}}} - L_{\vec{\mathbf{A}}})^{-1}(F - \vec{\mathbf{A}}\vec{\mathbf{U}}^{(k)} - \vec{\mathbf{B}}\vec{\mathbf{P}}^{(k)}). \tag{2.44}$$
where $\vec{\mathbf{D}}$ comes from the restriction and prolongation process.

2. For inner grid points, $((i-1)h,(j-1)h),(ih,(j-1)h),(ih,jh),((i-1)h,jh)$, we calculate the residual of the divergence.
$$r_{i,j} = -d(i,j) - \frac{u_{i,j-\frac{1}{2}} - u_{i-1,j-\frac{1}{2}}}{h} - \frac{u_{i-\frac{1}{2},j} - u_{i-\frac{1}{2},j-1}}{h} \tag{2.45}$$
Hence, $\mathrm{Vec}(r_{ij}) = \vec{\mathbf{D}} - \vec{\mathbf{B}}^T\vec{\mathbf{U}}^{k+\frac{1}{2}}$.

3. Update the speed of the inner grid points
$$\begin{aligned} u^{k+1}_{i-1.j-\frac{1}{2}} &= u^{k+\frac{1}{2}}_{i-1.j-\frac{1}{2}} - \delta, \quad u^{k+1}_{i.j-\frac{1}{2}} = u^{k+\frac{1}{2}}_{i.j-\frac{1}{2}} + \delta \\ v^{k+1}_{i-\frac{1}{2}.j-1} &= v^{k+\frac{1}{2}}_{i-\frac{1}{2}.j-1} + \delta, \quad v^{k+1}_{i-\frac{1}{2},j} = v^{k+\frac{1}{2}}_{i-\frac{1}{2},j} + \delta \end{aligned} \tag{2.46}$$
and update the pressure of the inner grid points
$$\begin{aligned} p^{k+1}_{i-\frac{1}{2},j-\frac{1}{2}} &= p^{k}_{i-\frac{1}{2},j-\frac{1}{2}} + r_{i,j} \\ p^{k+1}_{i+\frac{1}{2},j-\frac{1}{2}} &= p^{k}_{i+\frac{1}{2},j-\frac{1}{2}} - r_{i,j}/4 \\ p^{k+1}_{i+\frac{1}{2},j-\frac{1}{2}} &= p^{k}_{i-\frac{3}{2},j-\frac{1}{2}} - r_{i,j}/4 \\ p^{k+1}_{i+\frac{1}{2},j-\frac{1}{2}} &= p^{k}_{i-\frac{1}{2},j+\frac{1}{2}} - r_{i,j}/4 \\ p^{k+1}_{i+\frac{1}{2},j-\frac{1}{2}} &= p^{k}_{i-\frac{1}{2},j-\frac{3}{2}} - r_{i,j}/4 \end{aligned} \tag{2.47}$$

4. For the boundary points $(i,N)$, we calculate $r_{i,N}$ in the same way. Define $\delta = r_{i,N}h/3$. The speed can be updated as
$$\begin{aligned} u^{k+1}_{i-1.N-\frac{1}{2}} &= u^{k+\frac{1}{2}}_{i-1.N-\frac{1}{2}} - \delta, \quad u^{k+1}_{i.N-\frac{1}{2}} = u^{k+\frac{1}{2}}_{i.N-\frac{1}{2}} + \delta \\ v^{k+1}_{i-\frac{1}{2}.N-1} &= v^{k+\frac{1}{2}}_{i-\frac{1}{2}.N-1} + \delta, \end{aligned} \tag{2.48}$$
and the pressure can be updated as
$$\begin{aligned} p^{k+1}_{i-\frac{1}{2},N-\frac{1}{2}} &= p^{k}_{i-\frac{1}{2},N-\frac{1}{2}} + r_{i,N} \\ p^{k+1}_{i+\frac{1}{2},N-\frac{1}{2}} &= p^{k}_{i+\frac{1}{2},N-\frac{1}{2}} - r_{i,N}/3 \\ p^{k+1}_{i+\frac{1}{2},N-\frac{1}{2}} &= p^{k}_{i-\frac{3}{2},N-\frac{1}{2}} - r_{i,N}/3 \\ p^{k+1}_{i+\frac{1}{2},N-\frac{1}{2}} &= p^{k}_{i-\frac{1}{2},N-\frac{3}{2}} - r_{i,N}/3 \end{aligned} \tag{2.49}$$

5. For the corner points $(1,1)$, we calculate $r_{i,N}$ in the same way. Define $\delta = r_{i,N}h/2$. The speed can be updated as

$$\begin{aligned} u^{k+1}_{1.\frac{1}{2}} &= u^{k+\frac{1}{2}}_{1.\frac{1}{2}} + \delta \\ v^{k+1}_{\frac{1}{2},1} &= v^{k+\frac{1}{2}}_{\frac{1}{2},1} + \delta \end{aligned} \tag{2.50}$$

and the pressure can be updated as

$$\begin{aligned} p^{k+1}_{\frac{1}{2},\frac{1}{2}} &= p^{k}_{\frac{1}{2},\frac{1}{2}} + r_{1,1} \\ p^{k+1}_{\frac{3}{2},\frac{1}{2}} &= p^{k}_{\frac{3}{2},\frac{1}{2}} - r_{1,1}/2 \\ p^{k+1}_{\frac{1}{2},\frac{3}{2}} &= p^{k}_{\frac{1}{2},\frac{3}{2}} - r_{i,1}/2 \end{aligned} \tag{2.51}$$

### 2.3.3 Difference of two DGSs

We can write DGS-2 in the following form

1. Given $\vec{\mathbf{U}}^{(k)}$ and $\vec{\mathbf{P}}^{(k)}$, we use Gauss-Seidel iteration to update velocity $\vec{\mathbf{U}}^{(k+\frac{1}{2})}$,

$$\vec{\mathbf{U}}^{(k+\frac{1}{2})} = \vec{\mathbf{U}}^{(k)} + (D_{\vec{\mathbf{A}}} - L_{\vec{\mathbf{A}}})^{-1}(F - \vec{\mathbf{A}}\vec{\mathbf{U}}^{(k)} - \vec{\mathbf{B}}\vec{\mathbf{P}}^{(k)}). \tag{2.52}$$

2. Update the $\vec{\mathbf{P}}^{(k)}$

$$\Delta\vec{\mathbf{P}} = \vec{\mathbf{R}}(\vec{\mathbf{S}} - \vec{\mathbf{B}}^T\vec{\mathbf{U}}^{(k+\frac{1}{2})}). \tag{2.53}$$

   where $\vec{\mathbf{R}}$ is approximate inverse of the $\vec{\mathbf{B}}^T\vec{\mathbf{B}}$, we adopt $\vec{\mathbf{R}}$ as the diagonal inverse of $\vec{\mathbf{B}}^T\vec{\mathbf{B}}$.
   From the update process, it is not difficult to realize that $\mathrm{Vec}(\delta_{ij}) = \Delta\vec{\mathbf{P}}$.

3. Finally, we update $\vec{\mathbf{U}}^{(k+1)}$ and $\vec{\mathbf{P}}^{(k+1)}$ by

$$\begin{aligned} \vec{\mathbf{U}}^{(k+1)} &= \vec{\mathbf{U}}^{(k+\frac{1}{2})} + \vec{\mathbf{B}}\Delta\vec{\mathbf{P}} \\ \vec{\mathbf{P}}^{(k+1)} &= \vec{\mathbf{P}}^{(k)} - (\vec{\mathbf{B}}^T\vec{\mathbf{B}} + \mathrm{diag}(\mathrm{Vec}(D')))\Delta\vec{\mathbf{P}} \end{aligned} \tag{2.54}$$

   where $D' \in \mathbb{R}^{N\times N}$ with 0 on its inner unit, 1 on its edge unit and 2 on its corner unit.

### 2.3.4 V-cycle process

Based on the configuration of the DGS process, we define a modified V-cycle process.
First, we set a stopping criterion $\epsilon$ for the V-Cycle. Then, we utilize three parameters $v_1, v_2$ and $L$ for the V-Cycle.
In the specific setting, suppose $N = 2^n$. We denote $k$-th grid to be the grid of size $2^{k-n}$, and the V-cycle process have $L$-th layer in total. We denote $\vec{\mathbf{A}}^{(k)}, \vec{\mathbf{B}}^{(k)}$ to be $\vec{\mathbf{A}}, \vec{\mathbf{B}}$ in the $k$-th grid and $\vec{\mathbf{F}}^{(k)}$ to be the residual in the $k$-th grid. Since we set initial value to be zero vector, $\vec{\mathbf{F}}^{(0)} = \vec{\mathbf{F}}$. Hence, on the $k$-th grid, we need to solve

$$\vec{\mathbf{A}}^{(k)}\vec{\mathbf{U}}^{(k)} + \vec{\mathbf{B}}^{(k)}\vec{\mathbf{P}}^{(k)} = \vec{\mathbf{F}}^{(k)}, \quad (\vec{\mathbf{B}}^{(k)})^T\vec{\mathbf{U}}^{(k)} = \vec{\mathbf{D}}^{(k)}. \tag{2.55}$$

### Elaboration of V-cycle

1. We start with $k = 0$. While $k < L$, on the $k$-th grid, we set the initial value $\vec{\mathbf{U}}^{(k),0} = 0, \vec{\mathbf{P}}^{(k),0} = 0$ and apply DGS $v_1$ times to get an approximate solution $\vec{\mathbf{U}}^{(k),v_1}$ and $\vec{\mathbf{P}}^{(k),v_1}$. Then, we compute the residual $r_k$

$$r_k = h^2(\vec{\mathbf{F}}^{(k)} - \vec{\mathbf{A}}^{(k)}\vec{\mathbf{U}}^{(k),v_1} - \vec{\mathbf{B}}^{(k)}\vec{\mathbf{P}}^{(k),v_1}). \tag{2.56}$$

   If $k = 0$ and $\|r_h\| < \epsilon\|r_0\|$, we stop the algorithm. Otherwise, we restrict $r_k$ to $(k+1)$-th grid

$$\vec{\mathbf{F}}^{(k+1)} = R^F_{2^k h, 2^{k+1} h} r_k, \tag{2.57}$$

   For the DGS iteration, we need additionally use $\vec{\mathbf{D}}^{(k)}$ improve performance. We define

$$\begin{aligned} &\vec{\mathbf{D}}^{(0)} = 0 \\ &\vec{\mathbf{D}}^{(k+1)} = R^p_{2^k h, 2^{k+1} h}(\vec{\mathbf{D}}^{(k)} - \vec{\mathbf{B}}^{(k)}\vec{\mathbf{U}}^{(k),v_1}) \end{aligned} \tag{2.58}$$

   Then we replace $k$ by $k+1$ until $k = L$.

2. For $k = L$, we set the initial value $\vec{\mathbf{U}}^{(k),0} = 0, \vec{\mathbf{P}}^{(k),0} = 0$ and apply DGS $v_1$ times to get an approximate solution $\vec{\mathbf{U}}^{(k),v_1}$ and $\vec{\mathbf{P}}^{(k),v_1}$ of Equation 2.55. We let $\vec{\mathbf{U}}_{(n-l),0} = \vec{\mathbf{U}}^{(n-l),v_1}$ and $\vec{\mathbf{P}}_{(n-l),0} = \vec{\mathbf{P}}^{(n-l),v_1}$.

3. We start with $k = L$. On the $k$-th level, we lift $\vec{\mathbf{U}}_{(k)}$, $\vec{\mathbf{P}}_{(k)}$ to $(k-1)$-th level, and update

$$\vec{\mathbf{U}}_{(k-1),0} = \vec{\mathbf{U}}^{(k-1),v_1} + R^U_{2^k h, 2^{k-1} h}\vec{\mathbf{U}}_{(k)}, \quad \vec{\mathbf{P}}_{(k-1),0} = \vec{\mathbf{P}}^{(k-1),v_1} + R^p_{2^k h, 2^{k-1} h}\vec{\mathbf{P}}_{(k)}. \tag{2.59}$$

   Then, we use $\vec{\mathbf{U}}_{(k-1),0}$ and $\vec{\mathbf{P}}_{(k-1),0}$ as initial value, run DGS $v_2$ times to get the approximate solution $\vec{\mathbf{U}}_{(k-1),v_2}$ and $\vec{\mathbf{P}}_{(k-1),v_2}$ to 2.55. Then we replace $k$ by $k-1$ until $k = 0$.

4. Go to Step 1.

## 2.4 Uzawa iteration

### 2.4.1 Exact Uzawa iteration

We summarize the iteration formula below:

1. $k = 0, \vec{\mathbf{P}}_0 = 0$
2. Solve $\vec{\mathbf{A}}\vec{\mathbf{U}}_{k+1} = \vec{\mathbf{F}} - \vec{\mathbf{B}}\vec{\mathbf{P}}_k$
3. Update the pressure $\vec{\mathbf{P}}_{k+1} = \vec{\mathbf{P}}_k + \alpha(\vec{\mathbf{B}}^T\vec{\mathbf{U}}_{k+1})$
4. Convergence test, else go to Step 2.

**Selection of $\alpha$** Since

$$\vec{\mathbf{P}}_{k+1} = (I - \alpha\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}})\vec{\mathbf{P}}_k + \alpha\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{F}} \tag{2.60}$$

**Lemma 1.** $\arg\min_{\alpha>0} \rho(I - \alpha\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}})$ is given by

$$\alpha_* = \frac{2}{\lambda_{min}(\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}}) + \lambda_{max}(\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}})}. \tag{2.61}$$

*Proof.* Assuming the eigenvalue and corresponding eigenvector of $\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}}$ is $\lambda$ and $x$, then we can easily derive that

$$(I - \alpha\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}})x = (1 - \alpha\lambda)x \tag{2.62}$$

Hence,

$$\begin{aligned}
\arg\min_{\alpha>0} \rho(I - \alpha\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}}) &= \arg\min_{\alpha>0} \max_{\lambda_{\min}(\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}}) \le \lambda \le \lambda_{\max}(\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}})} |1 - \alpha\lambda| \\
&= \arg\min_{\alpha>0} \max\{|1 - \alpha\lambda_{\min}(\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}})|, |1 - \alpha\lambda_{\max}(\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}})|\} \\
&= \frac{2}{\lambda_{min}(\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}}) + \lambda_{max}(\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}})}.
\end{aligned} \tag{2.63}$$

□

**Lemma 2.** $\lambda_{min}(\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}}) = 0$

*Proof.* From the construction, $\vec{\mathbf{B}}\mathbb{1} = 0$. Hence $\lambda_{min}(\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}}) = 0$. □

**Lemma 3.** The eigenvalue of $\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}}$ is either 0 or 1.

*Proof.* [1] From the construction of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, $\vec{\mathbf{B}}^T\vec{\mathbf{A}}\vec{\mathbf{B}} = B_1^T A^{-1} B_1 + B_2^T A^{-1} B_2$. Since $A = R_{N-1} \otimes I_{N-1} + I_N \otimes T_{N-1}$, we can determine $A$'s eigenvalue from $T_{N-1}$'s and $R_{N-1}$'s.

Since $T_{N-1}$ has spectral $\{\lambda_i^{(1)}\}_{i=1}^{N-1}$ where $\lambda_i^{(1)} = 4\sin^2\frac{i\pi}{2N}$. The corresponding eigenvector for $\lambda_i^{(1)}$ is

$$x_i^{(1)} = \left[\sin\frac{i\pi}{N}, \sin\frac{2i\pi}{N}, \dots \sin\frac{(N-1)i\pi}{N}\right]^T. \tag{2.64}$$

$R_{N-1}$ has spectral $\{\lambda_n^{(2)}\}_{j=1}^{N}$ where $\lambda_n^{(2)} = 4\sin^2\frac{(j-1)\pi}{2N}$. The corresponding eigenvector for $\lambda_n^{(2)}$ is

$$x_j^{(2)} = \left[\cos\frac{(j-1)\pi}{2N}, \cos\frac{3(j-1)\pi}{2N}, \dots \cos\frac{(2N-1)(j-1)\pi}{2N}\right]^T. \tag{2.65}$$

Define $y_{ij} = x_i^{(1)} \otimes x_j^{(2)} / \|x_i^{(1)} \otimes x_j^{(2)}\|$. Then

$$\begin{aligned} Ay_{ij} &= (\lambda_i^{(1)} + \lambda_j^{(2)})y_{ij} \\ \text{by} \quad (A \otimes B)(C \otimes D) &= (AC) \otimes (BD) \end{aligned} \tag{2.66}$$

[1] The proof is modified from online resources

hence, $y_{ij}$ is the eigenvector of $A$, whose eigenvalue is defined as $\lambda_{ij}$. To explicitly write $y_{ij}$, we need to calculate $\|x_i^{(1)} \otimes x_j^{(2)}\|_2$.
Since

$$\begin{aligned}
\|x_i^{(1)}\|_2^2 &= \sum_{k=1}^{N-1} \sin^2 \frac{ki\pi}{N} \\
&= \sum_{k=1}^{N-1} \frac{1-\cos\frac{2ki\pi}{N}}{2} = \frac{N-1}{2} - \frac{1}{2}(-\frac{1}{2} + \frac{\sin(2N-1)i\pi/N}{2\sin i\pi/N}) = \frac{N}{2} \\
||x_j^{(2)}||_2^2 &= \sum_{k=1}^{N} \cos^2 \frac{(2k-1)(j-1)\pi}{2N} \\
&= \sum_{k=1}^{N} \frac{1+\cos\frac{(2k-1)(j-1)\pi}{N}}{2} \\
&= \begin{cases} N, & j=1 \\ \frac{N}{2}, & j\neq 1 \end{cases}
\end{aligned} \tag{2.67}$$

We get

$$\|x_i^{(1)} \otimes x_j^{(2)}\|_2 = \begin{cases} \frac{N}{\sqrt{2}}, & j=1 \\ \frac{N}{2}, & j \neq 1 \end{cases} \tag{2.68}$$

Thus, we can write

$$A^{-1} = \sum_{i=1}^{N-1} \sum_{j=1}^{N} \lambda_{ij}^{-1} y_{ij} (y_{ij})^T \tag{2.69}$$

Based on the construction of $B_1, B_2$

$$\begin{aligned}
(B_1)^T y_{ij} =& \xi_{ij}^{-1} \sum_{k=1}^{N} e_N^{(k)} \otimes \begin{bmatrix} (y_{ij})_{0,k} - (y_{ij})_{1,k} \\ \vdots \\ (y_{ij})_{N-1,k} - (y_{ij})_{N,k} \end{bmatrix} \\
=& -2\sin\frac{i\pi}{2N} \xi_{ij}^{-1} x_j^{(2)} \otimes x_{i+1}^{(2)}.
\end{aligned} \tag{2.70}$$

$$\begin{aligned}
(B_2)^T y_{ij} =& \xi_{ij}^{-1} \sum_{k=1}^{N} \begin{bmatrix} (y_{ij})_{0,k} - (y_{ij})_{1,k} \\ \vdots \\ (y_{ij})_{N-1,k} - (y_{ij})_{N,k} \end{bmatrix} \otimes e_N^{(k)} \\
=& -2\sin\frac{i\pi}{2N} \xi_{ij}^{-1} x_{i+1}^{(2)} \otimes x_j^{(2)}.
\end{aligned} \tag{2.71}$$

Define

$$X^{i,j} = (x_{i+1}^{(2)} (x_{i+1}^{(2)})^T) \otimes (x_{j+1}^{(2)} (x_{j+1}^{(2)})^T) \tag{2.72}$$

where

$$\begin{aligned}
\sum_{j=1}^{N-1} (x_{j+1}^{(2)} (x_{j+1}^{(2)})^T)_{kl} &= \sum_{j=1}^{N-1} \cos\frac{j\pi(2k-1)}{2N} \cos\frac{j\pi(2l-1)}{2N} \\
&= \frac{1}{2} \sum_{j=1}^{N-1} \cos\frac{(k+l-1)j\pi}{N} + \cos\frac{(k-l)j\pi}{N}
\end{aligned} \tag{2.73}$$

Since, $2|N$, we have

$$\sum_{n=1}^{N-1} \cos\frac{kn\pi}{N} = \begin{cases} 0, & 2|k+1 \\ -1, & k \neq 0, 2|k \\ N-1, & k=0 \end{cases} \tag{2.74}$$

Therefore,

$$\sum_{j=1}^{N-1} x_{j+1}^{(2)}(x_{j+1}^{(2)})^T = \frac{N}{2}I_N - \frac{1}{2}\mathbb{1}_N\mathbb{1}_N^T. \tag{2.75}$$

Then, we have

$$\begin{aligned}
&\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}} = B_1^TA^{-1}B_1 + B_2^TA^{-1}B_2 \\
=&\sum_{i=1}^{N-1}\sum_{j=1}^{N} 4\sin^2\frac{i\pi}{2N}\lambda_{i,j}^{-1}\xi_{i,j}^{-2}((x_{i+1}^{(2)}(x_{i+1}^{(2)})^T)\otimes((x_j^{(2)})^Tx_j^{(2)}) + (x_j^{(2)}(x_j^{(2)})^T)\otimes((x_{i+1}^{(2)})^Tx_{i+1}^{(2)})) \\
=&\frac{4}{N^2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}(\frac{4\sin\frac{i\pi}{2N}}{4\sin\frac{i\pi}{2N}+4\sin\frac{j\pi}{2N}}X^{i,j} + \frac{4\sin\frac{j\pi}{2N}}{4\sin\frac{i\pi}{2N}+4\sin\frac{j\pi}{2N}}X^{i,j}) + \frac{2}{N^2}\sum_{i=1}^{N-1}X^{0,i}+X^{i,0} \\
=&\frac{4}{N^2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}X^{i,j} + \frac{2}{N^2}\sum_{i=1}^{N-1}X^{0,i}+X^{i,0} \\
=&\frac{1}{N^2}(NI_N - \mathbb{1}_N\mathbb{1}_N^T)\otimes(NI_N - \mathbb{1}_N\mathbb{1}_N^T) + \frac{1}{N^2}\mathbb{1}_N\mathbb{1}_N^T\otimes(NI_N - \mathbb{1}_N\mathbb{1}_N^T) + \frac{1}{N^2}(NI_N - \mathbb{1}_N\mathbb{1}_N^T)\otimes\mathbb{1}_N\mathbb{1}_N^T \\
=&I_N\otimes I_N - \frac{1}{N^2}\mathbb{1}_N\mathbb{1}_N^T\otimes\mathbb{1}_N\mathbb{1}_N^T = I_{N^2} - \frac{1}{N^2}\mathbb{1}_{N^2}\mathbb{1}_{N^2}^T
\end{aligned} \tag{2.76}$$

Hence,

$$(\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}})^2 = I_{N^2} - 2\frac{1}{N^2}\mathbb{1}_{N^2}\mathbb{1}_{N^2}^T + (\frac{1}{N^2}\mathbb{1}_{N^2}\mathbb{1}_{N^2}^T)^2 = I_{N^2} - \frac{1}{N^2}\mathbb{1}_{N^2}\mathbb{1}_{N^2}^T = \vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}} \tag{2.77}$$

By, Lemma 2, all of $\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}}$'s eigenvalue are 0s or 1s. □

However, if we adopt $\alpha_\star = 2$, the algorithm behaves significantly poor. For $N = 64, 128, 256$, we use exact Uzawa iteration as smoother. In the Table 5, We need only compare $\alpha = 1$ and $\alpha = 2$, it is obvious that setting $\alpha = 2$ deteriorates convergence dramatically, V-cycle cannot converge to desired accuracy even after 1000 iterations. Empirically, we find that when $\alpha = 2$, the convergence is not guaranteed. However, for $\alpha = 1$, we have the following theorem:

**Theorem 1.** For $\alpha = 1$, if $\vec{\mathbf{F}} \in \text{range}(\vec{\mathbf{B}})$, then the exact Uzawa Iteration will converge in at most 2 iteration.

*Proof.* Namely, If $\vec{\mathbf{P}}_0 = 0$, $\vec{\mathbf{P}}_1 = \vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{F}}$, $\vec{\mathbf{U}}_2 = \vec{\mathbf{A}}^{-1}(\vec{\mathbf{F}} - \vec{\mathbf{B}}\vec{\mathbf{P}}_1)$, $\vec{\mathbf{P}}_2 = (I - \vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}})\vec{\mathbf{P}}_1 + \vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{F}}$. Then $\vec{\mathbf{U}}_2$ and $\vec{\mathbf{P}}_2$ are the exact solution to 2.25.
Assume $\vec{\mathbf{F}} = \vec{\mathbf{B}}\mathbf{K}$. Owing to 2.77, we have

$$\begin{aligned}
\vec{\mathbf{B}}^T\vec{\mathbf{U}}_2 &= \vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{F}} - \vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}}\vec{\mathbf{P}}_1 = \vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}}\mathbf{K} - (\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}})^2\mathbf{K} = 0 \\
\vec{\mathbf{A}}\vec{\mathbf{U}}_2 + \vec{\mathbf{B}}\vec{\mathbf{P}}_2 &= \vec{\mathbf{F}} - \vec{\mathbf{B}}\vec{\mathbf{P}}_1 + \vec{\mathbf{B}}(I - \vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}})\vec{\mathbf{P}}_1 + \vec{\mathbf{B}}\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{F}} \\
&= \vec{\mathbf{F}} - \vec{\mathbf{B}}\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}}\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{F}} + \vec{\mathbf{B}}\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{F}} \\
&= \vec{\mathbf{F}} + \vec{\mathbf{B}}(\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}} - (\vec{\mathbf{B}}^T\vec{\mathbf{A}}^{-1}\vec{\mathbf{B}})^2)\mathbf{K} = \vec{\mathbf{F}}
\end{aligned} \tag{2.78}$$

□

According to Theorem 1, Uzawa iteration ($\alpha = 1$) is not suitable as a smoother, since it will convergnce to the real solution in 2 iteration, which is validated in Section 3.2.2, thus making V-cycle structure futile. The inexact variant behaves similarly to the exact version, as Section 3.2.4 illustrates.

### 2.4.2 Inexact Uzawa iteration

According to [Elman and Golub, 1994], instead solving $\vec{\mathbf{A}}\vec{\mathbf{U}}_{k+1} = \vec{\mathbf{F}} - \vec{\mathbf{B}}\vec{\mathbf{P}}_k$, we find an approximate solution $\tilde{\vec{\mathbf{U}}}_{k+1}$, s.t.

$$\|\vec{\mathbf{A}}\tilde{\vec{\mathbf{U}}}_{k+1} - (\vec{\mathbf{F}} - \vec{\mathbf{B}}\vec{\mathbf{P}}_k)\|_2 \leq \tau \|\vec{\mathbf{B}}^T \tilde{\vec{\mathbf{U}}}_k\| \tag{2.79}$$

where $\tau$ is a predetermined parameter. Note that the quantity $\|\vec{\mathbf{B}}^T \tilde{\vec{\mathbf{U}}}_k\|_2$ appearing in the right-hand side of this expression is the residual of the equation $\vec{\mathbf{B}}^T\vec{\mathbf{U}} = 0$.
However, as a smoother, we do not use $\tau$, instead, we ues Gauss-Seidel to iterate $v$ times to get an approximate soluiton to $\vec{\mathbf{A}}\vec{\mathbf{U}}_{k+1} = \vec{\mathbf{F}} - \vec{\mathbf{B}}\vec{\mathbf{P}}_k$.

### 2.4.3 V-cycle multigrid

We use the Uzawa iterations (exact and inexact) as smoother. Still just in the cases of DGS, we introduce the error of divergence. To solve Equation 2.37, we modify the Uzawa itertion to the following form:

1. $k = 0, \vec{\mathbf{P}}_0 = 0$
2. Solve $\vec{\mathbf{A}}\vec{\mathbf{U}}_{k+1} = \vec{\mathbf{F}} - \vec{\mathbf{B}}\vec{\mathbf{P}}_k$
3. Update the pressure $\vec{\mathbf{P}}_{k+1} = \vec{\mathbf{P}}_k + \alpha(\vec{\mathbf{B}}^T\vec{\mathbf{U}}_{k+1} - \vec{\mathbf{S}})$
4. Convergence test, else go to Step 2.

The exact configuration is the same as Section 2.2. Hence, we will not restate the V-cycle procedures again.

### 2.4.4 Solver in inexact iteration

Practically, instead of solving 2.79, we find an approximate solution $\tilde{\vec{\mathbf{U}}}_{k+1}$, s.t.

$$\|\vec{\mathbf{A}}\tilde{\vec{\mathbf{U}}}_{k+1} - (\vec{\mathbf{F}} - \vec{\mathbf{B}}\vec{\mathbf{P}}_k)\|_2 \leq \tau_0 \tag{2.80}$$

**CG** Each iteration, we use zero vector as initial value, and diag$(A)$ as preconditioner.

**V-cycle multigrid** Instead of using Uzawa smoother, here, we use V-cycle multigrid method to solve Equation 2.79. The exact configuration is the same as Section 2.2, except for the fact that we do not need to introduce $\vec{\mathbf{D}}, \vec{\mathbf{P}}$. Suppose $\tilde{\vec{\mathbf{F}}} = \vec{\mathbf{F}} - \vec{\mathbf{B}}\vec{\mathbf{P}}$. We want to solve

$$\vec{\mathbf{A}}\vec{\mathbf{U}} = \tilde{\vec{\mathbf{F}}} \tag{2.81}$$

**Elaboration of V-cycle**

1. We start with $k = 0$. While $k < L$, on the $k$-th grid, we set the initial value $\vec{\mathbf{U}}^{(k),0} = 0$, and apply Gauss-Seidel $v_1$ times to get an approximate solution $\vec{\mathbf{U}}^{(k),v_1}$. Then, we compute the residual $r_k$

$$r_k = h^2(\tilde{\vec{\mathbf{F}}}^{(k)} - \vec{\mathbf{A}}^{(k)}\vec{\mathbf{U}}^{(k),v_1}). \tag{2.82}$$

   If $k = 0$ and $\|r_h\| < \epsilon$, we stop the algorithm. Otherwise, we restrict $r_k$ to $(k+1)$-th grid

$$\tilde{\vec{\mathbf{F}}}^{(k+1)} = R^F_{2^k h, 2^{k+1}h} r_k, \tag{2.83}$$

   Then we replace $k$ by $k+1$ until $k = L$.

2. For $k = L$, we set the initial value $\vec{\mathbf{U}}^{(k),0} = 0$, and apply Gauss-Seidel $v_1$ times to get an approximate solution $\vec{\mathbf{U}}^{(k),v_1}$ of Equation 2.81. We let $\vec{\mathbf{U}}_{(n-l),0} = \vec{\mathbf{U}}^{(n-l),v_1}$.

3. We start with $k = L$. On the $k$-th level, we lift $\vec{\mathbf{U}}_{(k)}$ to $(k-1)$-th level, and update

$$\vec{\mathbf{U}}_{(k-1),0} = \vec{\mathbf{U}}^{(k-1),v_1} + R^U_{2^k h, 2^{k-1}h}\vec{\mathbf{U}}_{(k)} \tag{2.84}$$

   Then, we use $\vec{\mathbf{U}}_{(k-1),0}$ as initial value, run Gauss-Seidel $v_2$ times to get the approximate solution $\vec{\mathbf{U}}_{(k-1),v_2}$ to Equation 2.81.

4. Go to Step 1.

# 3 Numerical Experiment

In the experiments, we set $v_1 = v_2 = v$, and define $L_0 = n - L$. **Note that we measure the whole period of time, not just the V-cycle process.** The stopping criterions in the slides are not practical. In order to rule out the influence of $N$, we use

$$\|r_h\|_2 \leq \|r_0\|_2 \epsilon \tag{3.1}$$

In the table, "u_error" means $e_h$, "res" means $\frac{\|r_h\|_2}{\|r_0\|_2}$.

## 3.1 DGS

### 3.1.1 DGS-1

To get the result, please execute "V_cycle_DGS_1_test.m", which utilizes function "V_cycle.m". For $n = 6, 7, 8, 9$, we set $v = 2, 4, 6, 8, 16, 32$, $L_0 = 1, 2, 3$. For $n = 10, 11$, we set $v = 2, 4, 8$, $L_0 = 1, 2$. The results are presented in the Table 1, Table 2.

**Observation**

1. DGS-1 does not cannot converge to the desired accuracy $10^{-8}$ when $N$ is small, even after 10000 V-cycle iterations.

2. Depths matters. For larger $N$, deeper V-cycle converges faster.

3. We cannot talk about the influence of different $v$ under small $N$ since different scale of residuals. If we focus on the cases where $N = 1024$, $L = 8$, we find that $v = 4, 8$ seems to behave better.

### 3.1.2 DGS-2

To get the result, please execute "V_cycle_DGS_2_test.m", which utilizes function "V_cycle.m". For $n = 6, 7, 8, 9$, we set $v = 2, 4, 6, 8, 16, 32$, $L_0 = 1, 2, 3$. For $n = 10, 11$, we set $v = 2, 4, 8$, $L_0 = 1, 2,$. The results are presented in the Table 3, Table 4.

#### Observation

1. Still, Depths matters. For large $N$, deeper V-cycle converges faster. We should set $L = N - 1$ or $N - 2$.
2. Based on the experiments when $N \geq 256$, $v = 4$ seems outperforms others.

**Visualization** We visualize the error of $U, V, P$ under different $N$ in Figure 1$\sim$ 6. Here, we set $v = 4$, $L_0 = 1$. Please execute "Plot_DGS.m".

## 3.2 Uzawa

We find that exact Uzawa iteration is not suitable for multigrid smoother, since each iteration is computationally onerous, and it only takes a few (two in the exact scenario) iterations to converge to the real solution.

### 3.2.1 Exact Uzawa as smoother

To get the result, please execute "V_cycle_Uzawa_test.m", which utilizes function "V_cycle.m". For $n = 6$, we set $v = 2, 4, 8$, $L = 1, 2, 4$, $\alpha = 0.5, 1, 1.5, 2$. For $n = 7, 8, 9$, we set $v = 2, 3, 4$, $L = 1, 2, 4$, $\alpha = 0.5, 1, 1.5$. For $N = 10, 11$, we set $v = 2, 3$, $L = 1, 2$, $\alpha = 1$. The results are summarized in the Table 5 and Table 6.

#### Observation

1. Aligned with our theoretical prediction, when $\alpha = 1$, even we set $v = 2, L = 2$, the predetermined threshold is achieved before one V-cycle iteration. Hence, more layers or more iterations will be futile. Hence, the optimal choice is that we set $v = 2$.
2. To validate $\alpha = 1$ is the best choice, we set $\alpha = 0.5, 1, 1.5$. We find that when $\alpha \neq 1$, the V-cycle structure functions, which deteriorate performance.

### 3.2.2 Inexact Uzawa as smoother

To get the result, please execute "V_cycle_inexact_Uzawa_test.m", which utilizes function "V_cycle.m".
For $n = 6$, we set $v = 2, 4, 8, 16, 32, 128$, $L_0 = 1, 2, 4$, $\alpha = 0.5, 1, 1.5, 2$. We notice that $\alpha$ should be 1. As for $v$, we shsould search within $4 \sim 50$. Besides, shallower V-cycle needs larger $v$ to converge. Hence, we should set $L_0 = 1, 2$. For $n = 7, 8, 9$, we set $v = 4, 8, 16, 32$, $L_0 = 1, 2$, $\alpha = 1$. For $N = 10, 11$, we set $v = 4, 6, 8, 10, 12, 14, 16$, $L_0 = 1$, $\alpha = 1$. The results are summarized in the Table 8, Table 9, and Table 10.
Here, $v$ means the number of each batch of Gauss-Seidel iterations.

#### Observation

1. Still, deep V-cycle performs better, shallower V-cycle requires more GS iteration to converge. And $\alpha$ should be set to 1.

2. When we set $L_0 = 1$, we should set $v = 4, 6, 8$. It takes about 50-60s to converge to the solution.

### 3.2.3 Inexact Uzawa based on built-in solver

This part just provide a baseline for the Uzawa iteration when $N = 2048$. It takes about 60s.
Please refer to "Inexact_Uzawa_builtin_test.m", which utilize "Inexact_Uzawa_builtin.m".

### 3.2.4 Inexact Uzawa based on CG

To get the result, please execute "Inexact_Uzawa_CG_test.m", which utilizes function "inexact_Uzawa_CG".
For $n = 6, 7, 8$, we set $\tau = 1.6 \times 10^{-8}, 4 \times 10^{-9}, 10^{-9}$, $L = 1, 2, 4$, $\alpha = 0.5, 1, 1.5, 2$. The results are summarized in the Table 11. NOte that when $N \geq 9$, PCG with diagonal preconditional cannot converge.

### 3.2.5 Inexact Uzawa iteration based on V-cycle

To get the result, please execute "inexact_Uzawa_V_cycle_test.m" under different parameters.

For $n = 6, 7, 8, 9$, we set $v = 2, 4, 6$. $L_0 = 1, 2$, $\alpha = 1$, $\tau = 16 \times 10^{-8}, 4 \times 10^{-8}, 1 \times 10^{-8}$. We perform extensive search to find the potimal parameters. For $n = 10, 11$, we set $v = 1, 2, 3, 4$. $L_0 = 1$, $\alpha = 1$, $\tau = 10^{-8}, 0.5 \times 10^{-8}$. The result is in Table 12, Table 13, Table 14. Here, $v$ means the number of each batch of Gauss-Seidel iterations.

#### Observation

1. We find that as usual, deeper V-cycle behaves better, and the selection of $v$ matters.

2. The V-cycle uses Gauss-Seidel iteration as smoother, which is better than conjugate gradient smoother.

3. We find that the optimal parameters are $L_0 = 1$, $v = 3$, $\tau = 10^{-8}$ for the V-cycle. In this setting, it takes 30s to solve $N = 2048$, which is obviously better than the built-in solver.

**Visualization** We visualize the error of $U, V, P$ under different $N$ in Figure 7$\sim$ 12. Here, we set $v = 3$, $L_0 = 1$, $\alpha = 1$, $\tau = 10^{-8}$. Please execute "Plot_Uzawa.m".

We find that different methods (DGS-2 and Uzawa based on V-cycle) produce similar results, and different scales of grid have similar shape of error. In particular, $P$ is smooth inside the region, and have some fluctuations on the boundary.

# 4 Conclusion

In the section, we present a succinct summary of what we have done and a blueprint of what we can improve.

Through MAC scheme, we formulate the Stokes equation into a saddle point problem. However, the equation itself is undetermined. Therefore, we introduce V-Cycle multi-grid method and choose Distributive Gauss Seidel Iteration and Uzawa Iteration as smoother.

For DGS, we consider two types of DGS, and find out that DGS in the sildes performs better. In the meantime, we find that we need to introduce the error of divergence can improve performace. In general, deeper V-cycle performs better, shallower V-cycle requires more iterations of smoothers, which slows down the convergence. And the optimal iteration as smoother is $4 \sim 8$. The most effective group of parameters takes approximately 50s (46.63s) to solve $N = 2048$.

For exact Uzawa, we find that it is not suitable as a smoother, since only after 2 iterations will the Uzawa iteration converge, the optimal $v$ is obviously 2. The optimal $\alpha$ is also 1, as our Theorem 1 predicts. However, inexact version of Uzawa is effective as smoother, which takes about 1 minute for $N = 2048$.

Most of the computational work occurs in solving $\vec{\mathbf{A}}\vec{\mathbf{U}} = \vec{\mathbf{F}} - \vec{\mathbf{B}}\vec{\mathbf{P}}$. We find that it is not effective to use PCG with diagnoal preconditional to solve the equation, hence, we turn to V-cycle multigrid method (Gauss-Seidel iteration as smoother). For $N = 1024, 2048$, we conducted extensive parameter search in Table 14. Finally, we find that $L_0 = 1, \alpha = 1, v = 3, \tau = 10^{-8}$ is the optimal choice. The most effective group of parameters takes approximately 30s (32.21s) to solve $N = 2048$, which is faster than the built-in linear solver 61.91s.

However, we find that PCG with diagonal preconditioner cannot solve the equation effectively, we hope more designed preconditioner can be designed to solve this problem. Since preconditioned Uzawa is not related to our problem, we do not investigates this problem in this report.

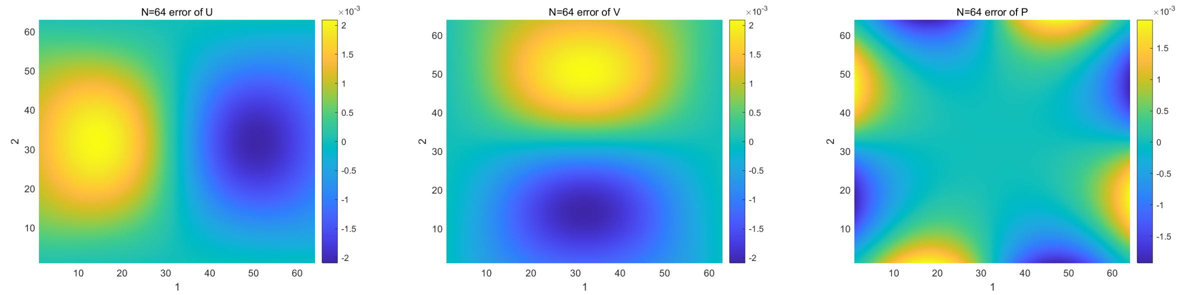
# References


[Elman and Golub, 1994] Elman, H. C. and Golub, G. H. (1994). Inexact and preconditioned uzawa algorithms for saddle point problems. *SIAM J. Numer. Anal.*, 31(6):1645–1661.

[Harlow and Welch, 1965] Harlow, F. H. and Welch, J. E. (1965). Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface. *The physics of fluids*, 8(12):2182–2189.

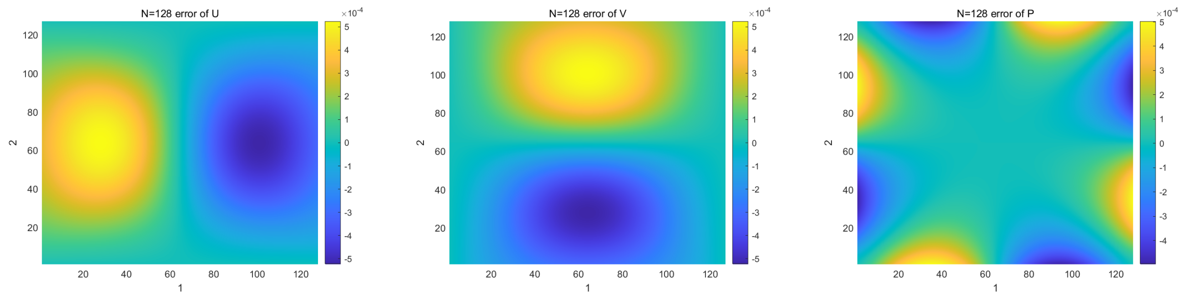
[Wang and Chen, 2013] Wang, M. and Chen, L. (2013). Multigrid methods for the stokes equations using distributive gauss—seidel relaxations based on the least squares commutator. *J. Sci. Comput.*, 56(2):409–431.


# 5 Appendix



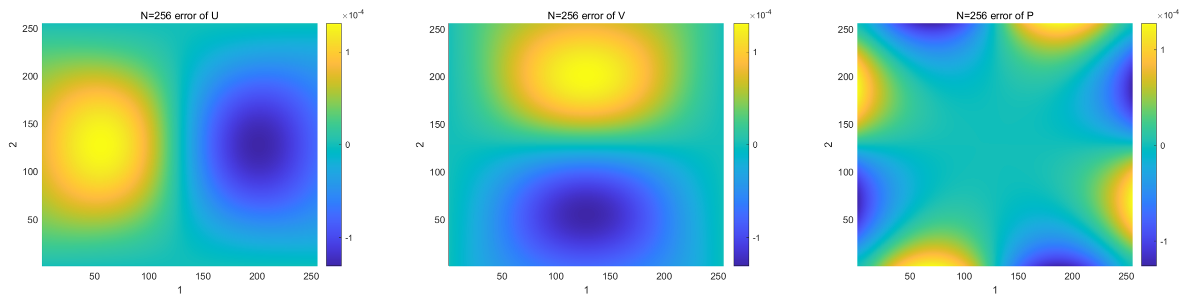

(a) U (b) V (c) P

Figure 1: Distributive Gauss-Seidel, N=64



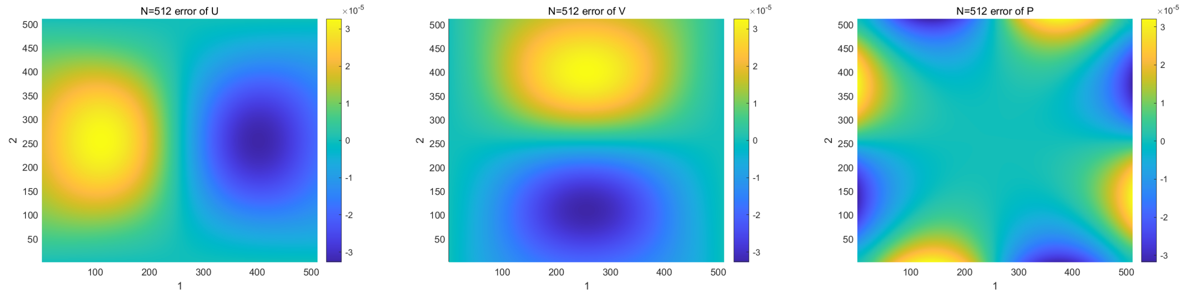

(a) U (b) V (c) P

Figure 2: Distributive Gauss-Seidel, N=128



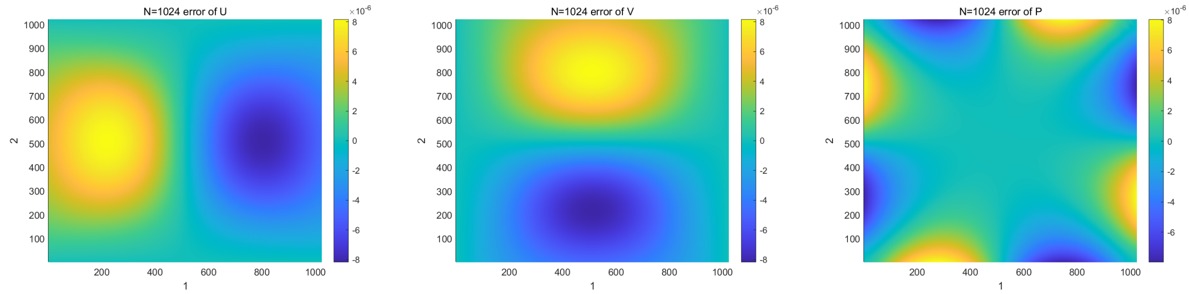

(a) U (b) V (c) P

Figure 3: Distributive Gauss-Seidel, N=256



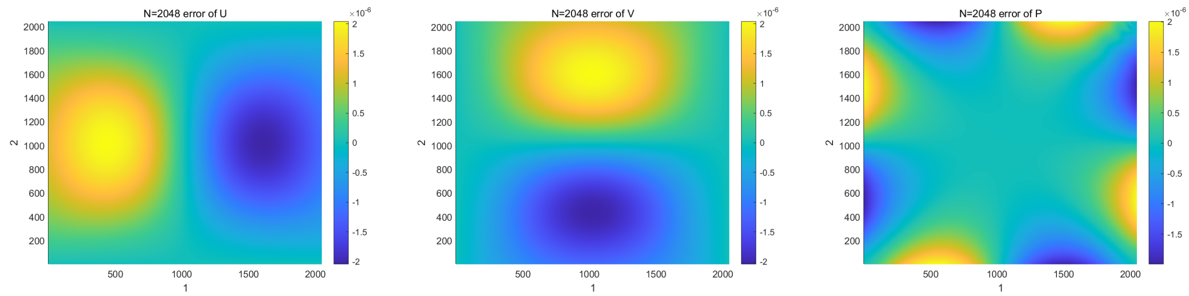

(a) U (b) V (c) P

Figure 4: Distributive Gauss-Seidel, N=512



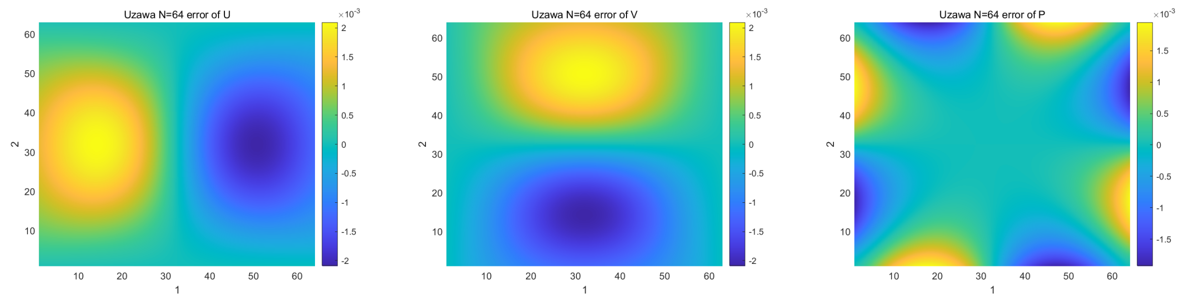

(a) U (b) V (c) P

Figure 5: Distributive Gauss-Seidel, N=1024



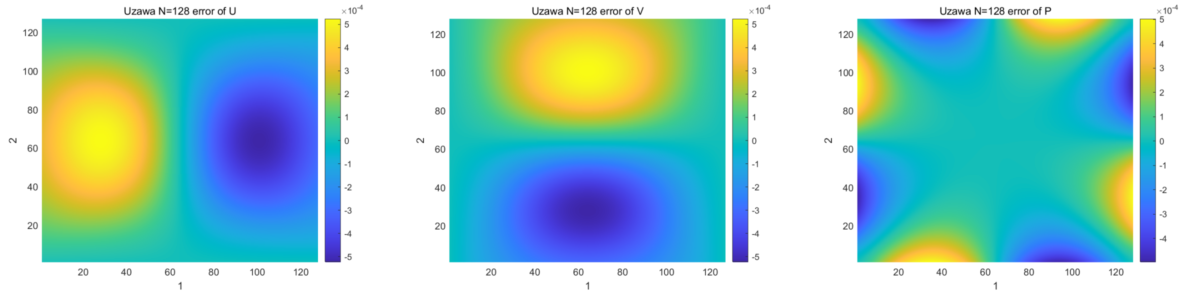

(a) U (b) V (c) P

Figure 6: Distributive Gauss-Seidel, N=2048



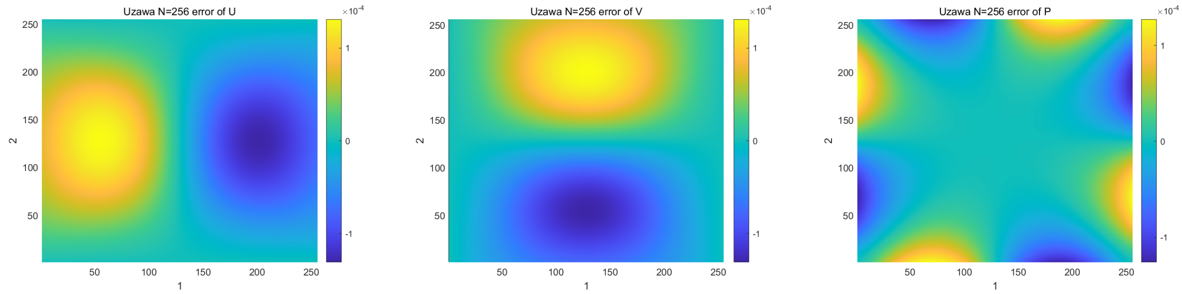

(a) U (b) V (c) P

Figure 7: Inexact Uzawa based on V-cycle, N=64



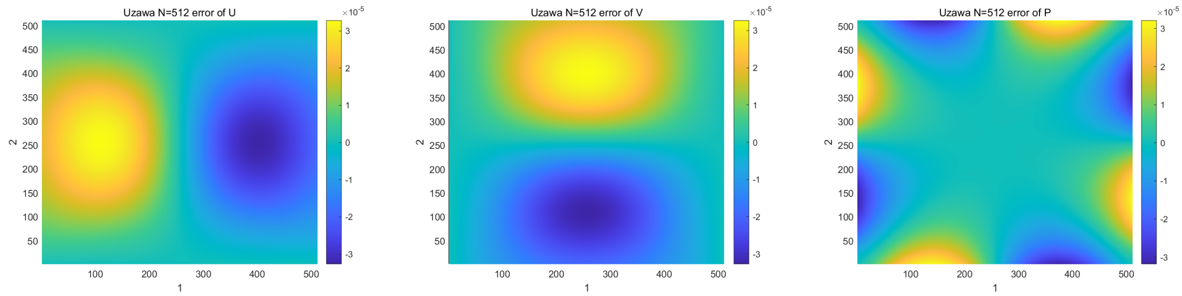

(a) U (b) V (c) P

Figure 8: Inexact Uzawa based on V-cycle, N=128



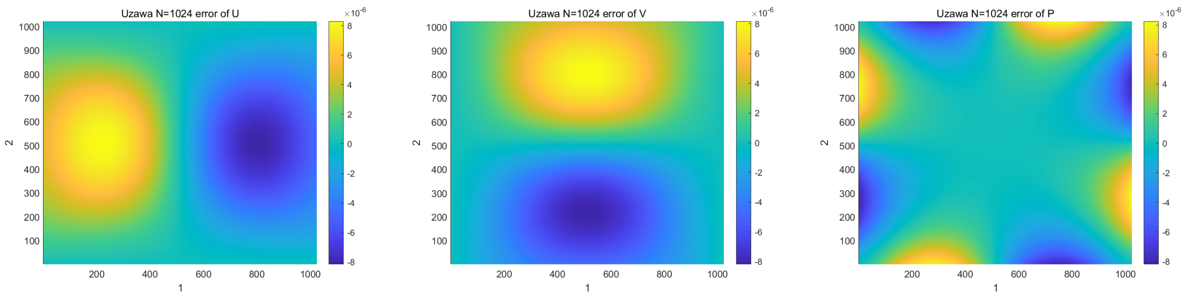

(a) U (b) V (c) P

Figure 9: Inexact Uzawa based on V-cycle, N=256



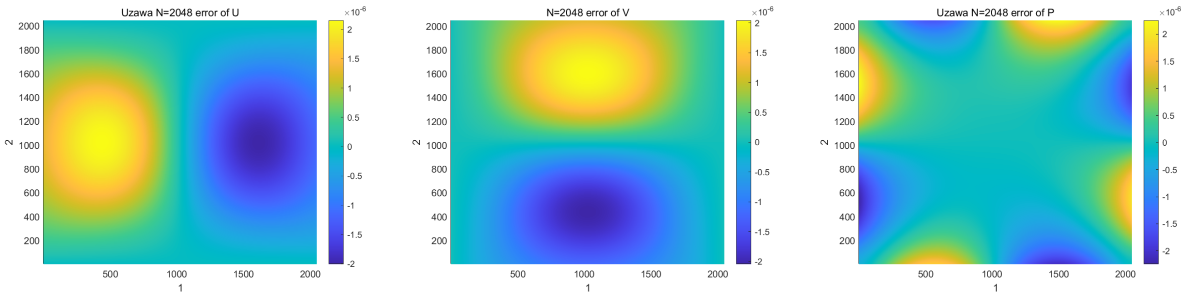

(a) U (b) V (c) P

Figure 10: Inexact Uzawa based on V-cycle, N=512




(a) U (b) V (c) P

Figure 11: Inexact Uzawa based on V-cycle, N=1024




(a) U (b) V (c) P

Figure 12: Inexact Uzawa based on V-cycle, N=2048

Table 1: DGS_1 Part 1

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 64 | 1 | 2 | 100 | 0.82 | 3.61E-06 | 0.0015 | 0.000692 |
| | | 4 | 100 | 0.62 | 2.48E-06 | 0.0015 | 0.000672 |
| | | 8 | 100 | 0.8 | 1.61E-06 | 0.0015 | 0.000661 |
| | | 16 | 100 | 0.97 | 1.14E-06 | 0.0015 | 0.000657 |
| | | 32 | 100 | 1.51 | 6.89E-07 | 0.0015 | 0.000654 |
| | 2 | 2 | 100 | 0.22 | 3.58E-06 | 0.0015 | 0.000692 |
| | | 4 | 100 | 0.3 | 2.48E-06 | 0.0015 | 0.000672 |
| | | 8 | 100 | 0.48 | 1.61E-06 | 0.0015 | 0.000661 |
| | | 16 | 100 | 0.82 | 1.14E-06 | 0.0015 | 0.000657 |
| | | 32 | 100 | 1.51 | 6.89E-07 | 0.0015 | 0.000654 |
| | 3 | 2 | 100 | 0.23 | 3.52E-06 | 0.0015 | 0.00069 |
| | | 4 | 100 | 0.33 | 2.45E-06 | 0.0015 | 0.000671 |
| | | 8 | 100 | 0.44 | 1.61E-06 | 0.0015 | 0.000661 |
| | | 16 | 100 | 0.79 | 1.14E-06 | 0.0015 | 0.000657 |
| | | 32 | 100 | 1.44 | 6.88E-07 | 0.0015 | 0.000654 |
| 128 | 1 | 2 | 100 | 0.86 | 5.44E-07 | 0.000374 | 0.000177 |
| | | 4 | 100 | 1.24 | 4.54E-07 | 0.000374 | 0.000173 |
| | | 8 | 100 | 1.95 | 3.77E-07 | 0.000374 | 0.00017 |
| | | 16 | 100 | 3.4 | 2.94E-07 | 0.000374 | 0.000168 |
| | | 32 | 100 | 6.25 | 2.02E-07 | 0.000374 | 0.000165 |
| | 2 | 2 | 100 | 0.83 | 5.40E-07 | 0.000374 | 0.000177 |
| | | 4 | 100 | 1.16 | 4.53E-07 | 0.000374 | 0.000173 |
| | | 8 | 100 | 1.91 | 3.77E-07 | 0.000374 | 0.00017 |
| | | 16 | 100 | 3.33 | 2.94E-07 | 0.000374 | 0.000168 |
| | | 32 | 100 | 6.29 | 2.02E-07 | 0.000374 | 0.000165 |
| | 3 | 2 | 100 | 0.8 | 5.39E-07 | 0.000374 | 0.000177 |
| | | 4 | 100 | 1.16 | 4.50E-07 | 0.000374 | 0.000173 |
| | | 8 | 100 | 1.89 | 3.76E-07 | 0.000374 | 0.00017 |
| | | 16 | 100 | 3.24 | 2.94E-07 | 0.000374 | 0.000168 |
| | | 32 | 100 | 6.08 | 2.02E-07 | 0.000374 | 0.000165 |
| 256 | 1 | 2 | 100 | 3.75 | 7.94E-08 | 9.34E-05 | 4.56E-05 |
| | | 4 | 100 | 5.26 | 6.81E-08 | 9.34E-05 | 4.44E-05 |
| | | 8 | 100 | 8.27 | 5.96E-08 | 9.34E-05 | 4.36E-05 |
| | | 16 | 100 | 17 | 5.30E-08 | 9.34E-05 | 4.30E-05 |
| | | 32 | 100 | 27.91 | 4.65E-08 | 9.34E-05 | 4.25E-05 |
| | 2 | 2 | 100 | 4.16 | 7.91E-08 | 9.34E-05 | 4.56E-05 |
| | | 4 | 100 | 5.28 | 6.80E-08 | 9.34E-05 | 4.44E-05 |
| | | 8 | 100 | 8.3 | 5.96E-08 | 9.34E-05 | 4.36E-05 |
| | | 16 | 100 | 14.32 | 5.30E-08 | 9.34E-05 | 4.30E-05 |
| | | 32 | 100 | 26.81 | 4.65E-08 | 9.34E-05 | 4.25E-05 |
| | 3 | 2 | 100 | 3.65 | 8.00E-08 | 9.34E-05 | 4.57E-05 |
| | | 4 | 100 | 5.19 | 6.82E-08 | 9.34E-05 | 4.44E-05 |
| | | 8 | 100 | 8.42 | 5.96E-08 | 9.34E-05 | 4.36E-05 |
| | | 16 | 100 | 14.46 | 5.30E-08 | 9.34E-05 | 4.30E-05 |
| | | 32 | 100 | 26.71 | 4.65E-08 | 9.34E-05 | 4.25E-05 |

Table 2: DGS_1 Part 2

| $N$ | $L_0$ | $v$ | iter | time | res | u_error | p_error |
|---|---|---|---|---|---|---|---|
| 512 | 1 | 2 | 100 | 22.36 | 1.28E-08 | 2.33E-05 | 1.21E-05 |
| | | 4 | 100 | 30.78 | 1.09E-08 | 2.33E-05 | 1.16E-05 |
| | | 8 | 78 | 38.77 | 9.98E-09 | 2.33E-05 | 1.14E-05 |
| | | 16 | 49 | 42.04 | 9.99E-09 | 2.33E-05 | 1.14E-05 |
| | | 32 | 37 | 58.54 | 9.96E-09 | 2.33E-05 | 1.14E-05 |
| | 2 | 2 | 100 | 21.74 | 1.28E-08 | 2.33E-05 | 1.21E-05 |
| | | 4 | 100 | 31.46 | 1.09E-08 | 2.33E-05 | 1.16E-05 |
| | | 8 | 78 | 39.02 | 9.98E-09 | 2.33E-05 | 1.14E-05 |
| | | 16 | 49 | 41.76 | 9.98E-09 | 2.33E-05 | 1.14E-05 |
| | | 32 | 37 | 58.68 | 9.96E-09 | 2.33E-05 | 1.14E-05 |
| | 3 | 2 | 100 | 22.24 | 1.29E-08 | 2.33E-05 | 1.22E-05 |
| | | 4 | 100 | 30.97 | 1.09E-08 | 2.33E-05 | 1.16E-05 |
| | | 8 | 78 | 39.32 | 9.98E-09 | 2.33E-05 | 1.14E-05 |
| | | 16 | 49 | 41.83 | 9.99E-09 | 2.33E-05 | 1.14E-05 |
| | | 32 | 37 | 58.64 | 9.96E-09 | 2.33E-05 | 1.14E-05 |
| 1024 | 1 | 2 | 11 | 12.75 | 8.70E-09 | 5.84E-06 | 4.77E-06 |
| | | 4 | 6 | 10.37 | 7.96E-09 | 5.84E-06 | 4.68E-06 |
| | | 8 | 5 | 13.72 | 6.93E-09 | 5.84E-06 | 4.26E-06 |
| | 2 | 2 | 18 | 19.3 | 8.94E-09 | 5.90E-06 | 5.12E-06 |
| | | 4 | 9 | 14.6 | 9.91E-09 | 5.93E-06 | 5.54E-06 |
| | | 8 | 5 | 13.75 | 7.53E-09 | 5.85E-06 | 4.50E-06 |
| 2048 | 1 | 2 | 11 | 52.66 | 3.76E-09 | 1.46E-06 | 1.55E-06 |
| | | 4 | 6 | 43.57 | 2.06E-09 | 1.46E-06 | 1.50E-06 |
| | | 8 | 4 | 48.45 | 6.82E-09 | 1.44E-06 | 1.94E-06 |
| | 2 | 2 | 17 | 78.84 | 8.23E-09 | 2.25E-06 | 5.80E-06 |
| | | 4 | 9 | 62 | 5.27E-09 | 1.81E-06 | 3.74E-06 |
| | | 8 | 5 | 58.31 | 2.71E-09 | 1.53E-06 | 1.98E-06 |

Table 3: DGS_2 Part 1

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 64 | 1 | 2 | 11 | 0.09 | 8.94E-09 | 0.0015 | 0.000653 |
| | | 4 | 7 | 0.04 | 2.28E-09 | 0.0015 | 0.000653 |
| | | 8 | 5 | 0.05 | 4.26E-10 | 0.0015 | 0.000653 |
| | | 16 | 4 | 0.06 | 1.55E-09 | 0.0015 | 0.000653 |
| | | 32 | 4 | 0.09 | 1.01E-10 | 0.0015 | 0.000653 |
| | 2 | 2 | 15 | 0.04 | 3.46E-09 | 0.0015 | 0.000653 |
| | | 4 | 8 | 0.04 | 1.75E-09 | 0.0015 | 0.000653 |
| | | 8 | 5 | 0.04 | 5.47E-10 | 0.0015 | 0.000653 |
| | | 16 | 4 | 0.05 | 1.55E-09 | 0.0015 | 0.000653 |
| | | 32 | 4 | 0.08 | 1.01E-10 | 0.0015 | 0.000653 |
| | 3 | 2 | 56 | 0.13 | 8.80E-09 | 0.0015 | 0.000653 |
| | | 4 | 28 | 0.1 | 6.43E-09 | 0.0015 | 0.000653 |
| | | 8 | 14 | 0.09 | 6.98E-09 | 0.0015 | 0.000653 |
| | | 16 | 7 | 0.08 | 7.66E-09 | 0.0015 | 0.000653 |
| | | 32 | 4 | 0.07 | 1.53E-09 | 0.0015 | 0.000653 |
| 128 | 1 | 2 | 11 | 0.14 | 5.76E-09 | 0.000374 | 0.000163 |
| | | 4 | 6 | 0.11 | 3.20E-09 | 0.000374 | 0.000163 |
| | | 8 | 5 | 0.12 | 3.00E-10 | 0.000374 | 0.000163 |
| | | 16 | 4 | 0.18 | 1.10E-09 | 0.000374 | 0.000163 |
| | | 32 | 4 | 0.34 | 7.03E-11 | 0.000374 | 0.000163 |
| | 2 | 2 | 14 | 0.12 | 7.29E-09 | 0.000374 | 0.000163 |
| | | 4 | 8 | 0.12 | 1.90E-09 | 0.000374 | 0.000163 |
| | | 8 | 5 | 0.14 | 3.88E-10 | 0.000374 | 0.000163 |
| | | 16 | 4 | 0.18 | 1.10E-09 | 0.000374 | 0.000163 |
| | | 32 | 4 | 0.31 | 7.03E-11 | 0.000374 | 0.000163 |
| | 3 | 2 | 55 | 0.36 | 8.60E-09 | 0.000374 | 0.000163 |
| | | 4 | 27 | 0.33 | 6.83E-09 | 0.000374 | 0.000163 |
| | | 8 | 14 | 0.31 | 4.47E-09 | 0.000374 | 0.000163 |
| | | 16 | 7 | 0.28 | 4.97E-09 | 0.000374 | 0.000163 |
| | | 32 | 4 | 0.27 | 1.01E-09 | 0.000374 | 0.000163 |
| 256 | 1 | 2 | 11 | 0.49 | 3.75E-09 | 9.34E-05 | 4.08E-05 |
| | | 4 | 6 | 0.35 | 9.94E-09 | 9.34E-05 | 4.08E-05 |
| | | 8 | 5 | 0.48 | 2.12E-10 | 9.34E-05 | 4.08E-05 |
| | | 16 | 4 | 0.68 | 7.84E-10 | 9.34E-05 | 4.08E-05 |
| | | 32 | 4 | 1.22 | 4.92E-11 | 9.34E-05 | 4.08E-05 |
| | 2 | 2 | 14 | 0.49 | 5.31E-09 | 9.34E-05 | 4.08E-05 |
| | | 4 | 7 | 0.37 | 7.38E-09 | 9.34E-05 | 4.08E-05 |
| | | 8 | 5 | 0.48 | 2.75E-10 | 9.34E-05 | 4.08E-05 |
| | | 16 | 4 | 0.66 | 7.84E-10 | 9.34E-05 | 4.08E-05 |
| | | 32 | 4 | 1.23 | 4.92E-11 | 9.34E-05 | 4.08E-05 |
| | 3 | 2 | 54 | 1.59 | 9.58E-09 | 9.34E-05 | 4.08E-05 |
| | | 4 | 26 | 1.19 | 8.02E-09 | 9.34E-05 | 4.09E-05 |
| | | 8 | 13 | 1.05 | 8.79E-09 | 9.34E-05 | 4.09E-05 |
| | | 16 | 7 | 1.08 | 3.50E-09 | 9.34E-05 | 4.08E-05 |
| | | 32 | 4 | 1.24 | 7.12E-10 | 9.34E-05 | 4.08E-05 |

Table 4: DGS_2 Part 2

| $N$ | $L_0$ | $v$ | iter | time | res | u_error | p_error |
|---|---|---|---|---|---|---|---|
| 512 | 1 | 2 | 12 | 2.61 | 2.52E-09 | 2.33E-05 | 1.02E-05 |
| | | 4 | 6 | 2.23 | 7.41E-09 | 2.33E-05 | 1.02E-05 |
| | | 8 | 4 | 2.64 | 8.44E-09 | 2.33E-05 | 9.91E-06 |
| | | 16 | 4 | 4.88 | 5.60E-10 | 2.33E-05 | 1.02E-05 |
| | | 32 | 3 | 8.25 | 8.14E-09 | 2.33E-05 | 9.79E-06 |
| | 2 | 2 | 14 | 4.31 | 3.87E-09 | 2.34E-05 | 1.02E-05 |
| | | 4 | 7 | 2.83 | 5.68E-09 | 2.34E-05 | 1.03E-05 |
| | | 8 | 4 | 2.89 | 9.43E-09 | 2.33E-05 | 9.97E-06 |
| | | 16 | 4 | 5.16 | 5.60E-10 | 2.33E-05 | 1.02E-05 |
| | | 32 | 3 | 7.6 | 8.14E-09 | 2.33E-05 | 9.79E-06 |
| | 3 | 2 | 54 | 11.09 | 8.53E-09 | 2.34E-05 | 1.03E-05 |
| | | 4 | 25 | 7.89 | 9.55E-09 | 2.34E-05 | 1.08E-05 |
| | | 8 | 13 | 7.34 | 6.27E-09 | 2.34E-05 | 1.05E-05 |
| | | 16 | 7 | 7.57 | 2.50E-09 | 2.34E-05 | 1.03E-05 |
| | | 32 | 4 | 8.91 | 5.08E-10 | 2.33E-05 | 1.02E-05 |
| 1024 | 1 | 2 | 12 | 12.58 | 4.05E-09 | 5.84E-06 | 2.55E-06 |
| | | 4 | 6 | 10.47 | 2.55E-09 | 5.84E-06 | 2.55E-06 |
| | | 8 | 4 | 12.57 | 6.01E-09 | 5.80E-06 | 2.29E-06 |
| | 2 | 2 | 13 | 13.27 | 8.25E-09 | 5.92E-06 | 4.30E-06 |
| | | 4 | 7 | 11.94 | 3.96E-09 | 5.87E-06 | 2.98E-06 |
| | | 8 | 4 | 12.58 | 6.72E-09 | 5.82E-06 | 2.57E-06 |
| 2048 | 1 | 2 | 12 | 53.85 | 4.28E-09 | 1.46E-06 | 6.38E-07 |
| | | 4 | 6 | 46.63 | 1.87E-09 | 1.46E-06 | 6.37E-07 |
| | | 8 | 4 | 55.32 | 4.28E-09 | 1.42E-06 | 5.45E-07 |
| | 2 | 2 | 13 | 57.2 | 6.23E-09 | 1.69E-06 | 3.64E-06 |
| | | 4 | 7 | 51.55 | 2.63E-09 | 1.52E-06 | 1.72E-06 |
| | | 8 | 4 | 54.88 | 4.79E-09 | 1.46E-06 | 1.32E-06 |

Table 5: Uzawa as smoother Part 1

| $N$ | $\alpha$ | $L$ | $v$ | iter | time | res | u_error | p_error |
|---|---|---|---|---|---|---|---|---|
| 64 | 0.5 | 1 | 2 | 15 | 0.33 | 9.78E-09 | 0.0015 | 6.53E-04 |
| | | | 4 | 6 | 0.27 | 1.85E-09 | 0.0015 | 6.53E-04 |
| | | | 8 | 2 | 0.22 | 1.99E-09 | 0.0015 | 6.53E-04 |
| | | 2 | 2 | 17 | 0.41 | 8.09E-09 | 0.0015 | 6.53E-04 |
| | | | 4 | 6 | 0.30 | 2.20E-09 | 0.0015 | 6.53E-04 |
| | | | 8 | 2 | 0.24 | 2.00E-09 | 0.0015 | 6.53E-04 |
| | | 4 | 2 | 17 | 0.43 | 7.92E-09 | 0.0015 | 6.53E-04 |
| | | | 4 | 6 | 0.31 | 2.20E-09 | 0.0015 | 6.53E-04 |
| | | | 8 | 2 | 0.25 | 2.00E-09 | 0.0015 | 6.53E-04 |
| | 1 | 1 | 2 | 0 | 0.03 | 6.72E-14 | 0.0015 | 6.53E-04 |
| | | | 4 | 0 | 0.04 | 6.18E-14 | 0.0015 | 6.53E-04 |
| | | | 8 | 0 | 0.09 | 6.22E-14 | 0.0015 | 6.53E-04 |
| | | 2 | 2 | 0 | 0.04 | 6.72E-14 | 0.0015 | 6.53E-04 |
| | | | 4 | 0 | 0.04 | 6.18E-14 | 0.0015 | 6.53E-04 |
| | | | 8 | 0 | 0.07 | 6.22E-14 | 0.0015 | 6.53E-04 |
| | | 4 | 2 | 0 | 0.03 | 6.72E-14 | 0.0015 | 6.53E-04 |
| | | | 4 | 0 | 0.04 | 6.18E-14 | 0.0015 | 6.53E-04 |
| | | | 8 | 0 | 0.07 | 6.22E-14 | 0.0015 | 6.53E-04 |
| | 1.5 | 1 | 2 | 12 | 0.24 | 5.97E-09 | 0.0015 | 6.53E-04 |
| | | | 4 | 6 | 0.25 | 3.71E-09 | 0.0015 | 6.53E-04 |
| | | | 8 | 2 | 0.22 | 5.93E-09 | 0.0015 | 6.53E-04 |
| | | 2 | 2 | 24 | 0.56 | 8.21E-09 | 0.0015 | 6.53E-04 |
| | | | 4 | 6 | 0.30 | 7.17E-09 | 0.0015 | 6.53E-04 |
| | | | 8 | 2 | 0.25 | 6.01E-09 | 0.0015 | 6.53E-04 |
| | | 4 | 2 | 29 | 0.71 | 7.47E-09 | 0.0015 | 6.53E-04 |
| | | | 4 | 6 | 0.31 | 7.22E-09 | 0.0015 | 6.53E-04 |
| | | | 8 | 2 | 0.27 | 6.01E-09 | 0.0015 | 6.53E-04 |
| | 2 | 1 | 2 | 1000 | 19.42 | 0.0137 | 0.0278 | 9.45E-02 |
| | | | 4 | 1000 | 41.80 | 0.0137 | 0.0278 | 9.45E-02 |
| | | | 8 | 1000 | 73.34 | 0.0137 | 0.0278 | 9.45E-02 |
| | | 2 | 2 | 1000 | 24.31 | 0.0137 | 0.0278 | 9.45E-02 |
| | | | 4 | 1000 | 46.84 | 0.0137 | 0.0278 | 9.45E-02 |
| | | | 8 | 1000 | 90.66 | 0.0137 | 0.0278 | 9.45E-02 |
| | | 4 | 2 | 1000 | 24.89 | 0.0137 | 0.0278 | 9.45E-02 |
| | | | 4 | 1000 | 47.74 | 0.0137 | 0.0278 | 9.45E-02 |
| | | | 8 | 1000 | 90.75 | 0.0137 | 0.0278 | 9.45E-02 |

Table 6: Uzawa as smoother Part 2

| $N$ | $\alpha$ | $L$ | $v$ | iter | time | res | u_error | p_error |
|---|---|---|---|---|---|---|---|---|
| 128 | 0.5 | 1 | 2 | 15 | 1.81 | 4.99E-09 | 0.000374 | 0.000163 |
| | | | 3 | 8 | 1.34 | 5.50E-09 | 0.000374 | 0.000163 |
| | | | 4 | 5 | 1.3 | 7.74E-09 | 0.000374 | 0.000163 |
| | | 2 | 2 | 16 | 2.3 | 8.46E-09 | 0.000374 | 0.000163 |
| | | | 3 | 8 | 1.54 | 8.23E-09 | 0.000374 | 0.000163 |
| | | | 4 | 5 | 1.35 | 8.93E-09 | 0.000374 | 0.000163 |
| | | 4 | 2 | 16 | 2.13 | 8.26E-09 | 0.000374 | 0.000163 |
| | | | 3 | 8 | 1.61 | 8.18E-09 | 0.000374 | 0.000163 |
| | | | 4 | 5 | 1.39 | 8.92E-09 | 0.000374 | 0.000163 |
| | 1 | 1 | 2 | 0 | 0.09 | 1.06E-13 | 0.000374 | 0.000163 |
| | | | 3 | 0 | 0.13 | 1.04E-13 | 0.000374 | 0.000163 |
| | | | 4 | 0 | 0.2 | 1.02E-13 | 0.000374 | 0.000163 |
| | | 2 | 2 | 0 | 0.13 | 1.06E-13 | 0.000374 | 0.000163 |
| | | | 3 | 0 | 0.14 | 1.04E-13 | 0.000374 | 0.000163 |
| | | | 4 | 0 | 0.18 | 1.02E-13 | 0.000374 | 0.000163 |
| | | 4 | 2 | 0 | 0.09 | 1.06E-13 | 0.000374 | 0.000163 |
| | | | 3 | 0 | 0.13 | 1.04E-13 | 0.000374 | 0.000163 |
| | | | 4 | 0 | 0.18 | 1.02E-13 | 0.000374 | 0.000163 |
| | 1.5 | 1 | 2 | 12 | 1.32 | 3.05E-09 | 0.000374 | 0.000163 |
| | | | 3 | 10 | 1.72 | 9.66E-09 | 0.000374 | 0.000163 |
| | | | 4 | 6 | 1.39 | 1.89E-09 | 0.000374 | 0.000163 |
| | | 2 | 2 | 23 | 2.84 | 7.65E-09 | 0.000374 | 0.000163 |
| | | | 3 | 10 | 1.9 | 3.33E-09 | 0.000374 | 0.000163 |
| | | | 4 | 6 | 1.58 | 3.67E-09 | 0.000374 | 0.000163 |
| | | 4 | 2 | 28 | 3.58 | 6.77E-09 | 0.000374 | 0.000163 |
| | | | 3 | 10 | 1.99 | 3.09E-09 | 0.000374 | 0.000163 |
| | | | 4 | 6 | 1.68 | 3.70E-09 | 0.000374 | 0.000163 |
| 256 | 0.5 | 1 | 2 | 14 | 7.91 | 5.76E-09 | 9.34E-05 | 4.08E-05 |
| | | | 3 | 8 | 7.05 | 2.78E-09 | 9.34E-05 | 4.08E-05 |
| | | | 4 | 5 | 6.25 | 3.91E-09 | 9.34E-05 | 4.08E-05 |
| | | 2 | 2 | 15 | 9.97 | 8.85E-09 | 9.34E-05 | 4.08E-05 |
| | | | 3 | 8 | 8.24 | 4.15E-09 | 9.34E-05 | 4.08E-05 |
| | | | 4 | 5 | 7.34 | 4.51E-09 | 9.34E-05 | 4.08E-05 |
| | | 4 | 2 | 15 | 11.19 | 8.62E-09 | 9.34E-05 | 4.08E-05 |
| | | | 3 | 8 | 8.59 | 4.13E-09 | 9.34E-05 | 4.08E-05 |
| | | | 4 | 5 | 7.61 | 4.50E-09 | 9.34E-05 | 4.08E-05 |
| | 1 | 1 | 2 | 0 | 0.5 | 2.13E-13 | 9.34E-05 | 4.08E-05 |
| | | | 3 | 0 | 0.69 | 2.13E-13 | 9.34E-05 | 4.08E-05 |
| | | | 4 | 0 | 0.94 | 2.11E-13 | 9.34E-05 | 4.08E-05 |
| | | 2 | 2 | 0 | 0.51 | 2.13E-13 | 9.34E-05 | 4.08E-05 |
| | | | 3 | 0 | 0.75 | 2.13E-13 | 9.34E-05 | 4.08E-05 |
| | | | 4 | 0 | 0.95 | 2.11E-13 | 9.34E-05 | 4.08E-05 |
| | | 4 | 2 | 0 | 0.52 | 2.13E-13 | 9.34E-05 | 4.08E-05 |
| | | | 3 | 0 | 0.68 | 2.13E-13 | 9.34E-05 | 4.08E-05 |
| | | | 4 | 0 | 0.95 | 2.11E-13 | 9.34E-05 | 4.08E-05 |
| | 1.5 | 1 | 2 | 11 | 6.34 | 5.32E-09 | 9.34E-05 | 4.08E-05 |
| | | | 3 | 10 | 8.73 | 4.88E-09 | 9.34E-05 | 4.08E-05 |
| | | | 4 | 5 | 6.2 | 8.78E-09 | 9.34E-05 | 4.08E-05 |
| | | 2 | 2 | 22 | 14.38 | 6.98E-09 | 9.34E-05 | 4.08E-05 |
| | | | 3 | 9 | 9.21 | 6.33E-09 | 9.34E-05 | 4.08E-05 |
| | | | 4 | 6 | 8.41 | 1.86E-09 | 9.34E-05 | 4.08E-05 |
| | | 4 | 2 | 26 | 17.47 | 9.37E-09 | 9.34E-05 | 4.08E-05 |
| | | | 3 | 9 | 9.48 | 5.92E-09 | 9.34E-05 | 4.08E-05 |
| | | | 4 | 6 | 8.76 | 1.87E-09 | 9.34E-05 | 4.08E-05 |

Table 7: Uzawa as smoother Part 3

| $N$ | $\alpha$ | $L_0$ | $v$ | iter | time | res | u_error | p_error |
|---|---|---|---|---|---|---|---|---|
| 512 | 0.5 | 1 | 2 | 13 | 39.82 | 6.61E-09 | 2.34E-05 | 1.02E-05 |
| | | | 3 | 7 | 36.15 | 5.93E-09 | 2.34E-05 | 1.02E-05 |
| | | | 4 | 5 | 33.32 | 1.96E-09 | 2.33E-05 | 1.02E-05 |
| | | 2 | 2 | 14 | 50.27 | 9.25E-09 | 2.34E-05 | 1.02E-05 |
| | | | 3 | 7 | 39.08 | 8.44E-09 | 2.34E-05 | 1.02E-05 |
| | | | 4 | 5 | 38.93 | 2.26E-09 | 2.33E-05 | 1.02E-05 |
| | | 4 | 2 | 14 | 51.39 | 9.01E-09 | 2.34E-05 | 1.02E-05 |
| | | | 3 | 7 | 40.34 | 8.39E-09 | 2.34E-05 | 1.02E-05 |
| | | | 4 | 5 | 39.58 | 2.26E-09 | 2.33E-05 | 1.02E-05 |
| | 1 | 1 | 2 | 0 | 2.57 | 4.23E-13 | 2.33E-05 | 1.02E-05 |
| | | | 3 | 0 | 3.74 | 4.24E-13 | 2.33E-05 | 1.02E-05 |
| | | | 4 | 0 | 4.92 | 4.17E-13 | 2.33E-05 | 1.02E-05 |
| | | 2 | 2 | 0 | 2.71 | 4.23E-13 | 2.33E-05 | 1.02E-05 |
| | | | 3 | 0 | 3.72 | 4.24E-13 | 2.33E-05 | 1.02E-05 |
| | | | 4 | 0 | 4.9 | 4.17E-13 | 2.33E-05 | 1.02E-05 |
| | | 4 | 2 | 0 | 2.53 | 4.23E-13 | 2.33E-05 | 1.02E-05 |
| | | | 3 | 0 | 3.69 | 4.24E-13 | 2.33E-05 | 1.02E-05 |
| | | | 4 | 0 | 4.94 | 4.17E-13 | 2.33E-05 | 1.02E-05 |
| | 1.5 | 1 | 2 | 10 | 31.12 | 7.76E-09 | 2.33E-05 | 1.02E-05 |
| | | | 3 | 9 | 41.76 | 8.26E-09 | 2.33E-05 | 1.02E-05 |
| | | | 4 | 5 | 33.18 | 4.41E-09 | 2.33E-05 | 1.02E-05 |
| | | 2 | 2 | 21 | 73.21 | 6.30E-09 | 2.33E-05 | 1.02E-05 |
| | | | 3 | 9 | 49.35 | 3.20E-09 | 2.33E-05 | 1.02E-05 |
| | | | 4 | 5 | 38.29 | 7.54E-09 | 2.33E-05 | 1.02E-05 |
| | | 4 | 2 | 25 | 90.13 | 7.78E-09 | 2.33E-05 | 1.02E-05 |
| | | | 3 | 9 | 50.83 | 2.99E-09 | 2.33E-05 | 1.02E-05 |
| | | | 4 | 5 | 39.63 | 7.59E-09 | 2.33E-05 | 1.02E-05 |
| 1024 | 1 | 1 | 2 | 0 | 12.12 | 8.51E-13 | 5.84E-06 | 2.55E-06 |
| | | | 3 | 0 | 18.74 | 8.51E-13 | 5.84E-06 | 2.55E-06 |
| | | 2 | 2 | 0 | 13.17 | 8.51E-13 | 5.84E-06 | 2.55E-06 |
| | | | 3 | 0 | 18.27 | 8.51E-13 | 5.84E-06 | 2.55E-06 |
| 2048 | 1 | 1 | 2 | 0 | 73.41 | 1.71E-12 | 1.46E-06 | 6.38E-07 |
| | | | 3 | 0 | 99.04 | 1.71E-12 | 1.46E-06 | 6.38E-07 |
| | | 2 | 2 | 0 | 63.98 | 1.71E-12 | 1.46E-06 | 6.38E-07 |
| | | | 3 | 0 | 96.9 | 1.71E-12 | 1.46E-06 | 6.38E-07 |

Table 8: Inexact Uzawa as smoother Part 1

| $N$ | $\alpha$ | $L_0$ | $v$ | iter | time | res | u_error | p_error |
|---|---|---|---|---|---|---|---|---|
| 64 | 0.5 | 1 | 2 | 1000 | 2.15 | 4.15E+116 | 1.51E+117 | 7.95E+117 |
| | | | 4 | 32 | 0.08 | 7.40E-09 | 0.0015 | 0.000653 |
| | | | 8 | 19 | 0.07 | 7.59E-09 | 0.0015 | 0.000653 |
| | | | 16 | 19 | 0.1 | 6.98E-09 | 0.0015 | 0.000653 |
| | | | 32 | 19 | 0.18 | 6.21E-09 | 0.0015 | 0.000653 |
| | | | 128 | 18 | 0.57 | 9.59E-09 | 0.0015 | 0.000653 |
| | | 2 | 2 | 1000 | 1.3 | 1.05E+82 | 3.30E+82 | 1.47E+83 |
| | | | 4 | 1000 | 1.95 | 1.44E+87 | 9.03E+87 | 2.99E+88 |
| | | | 8 | 1000 | 2.7 | 0.0216 | 0.12 | 0.479 |
| | | | 16 | 19 | 0.09 | 7.44E-09 | 0.0015 | 0.000653 |
| | | | 32 | 19 | 0.17 | 6.12E-09 | 0.0015 | 0.000653 |
| | | | 128 | 18 | 0.52 | 9.51E-09 | 0.0015 | 0.000653 |
| | | 4 | 2 | 412 | 0.5 | 7.79E-09 | 0.0015 | 0.000653 |
| | | | 4 | 202 | 0.31 | 8.58E-09 | 0.0015 | 0.000653 |
| | | | 8 | 102 | 0.24 | 5.85E-09 | 0.0015 | 0.000653 |
| | | | 16 | 49 | 0.2 | 9.30E-09 | 0.0015 | 0.000653 |
| | | | 32 | 26 | 0.19 | 8.03E-09 | 0.0015 | 0.000653 |
| | | | 128 | 18 | 0.5 | 8.20E-09 | 0.0015 | 0.000653 |
| | 1 | 1 | 2 | 1000 | 1.39 | 4.35E+40 | 3.36E+40 | 5.68E+41 |
| | | | 4 | 10 | 0.03 | 7.42E-09 | 0.0015 | 0.000653 |
| | | | 8 | 8 | 0.03 | 5.57E-09 | 0.0015 | 0.000653 |
| | | | 16 | 7 | 0.04 | 2.39E-09 | 0.0015 | 0.000653 |
| | | | 32 | 6 | 0.06 | 5.63E-09 | 0.0015 | 0.000653 |
| | | | 128 | 5 | 0.17 | 2.18E-09 | 0.0015 | 0.000653 |
| | | 2 | 2 | 959 | 1.31 | NaN | NaN | NaN |
| | | | 4 | 945 | 1.72 | NaN | NaN | NaN |
| | | | 8 | 86 | 0.25 | 8.76E-09 | 0.0015 | 0.000653 |
| | | | 16 | 7 | 0.04 | 4.24E-09 | 0.0015 | 0.000653 |
| | | | 32 | 6 | 0.06 | 5.63E-09 | 0.0015 | 0.000653 |
| | | | 128 | 5 | 0.16 | 2.18E-09 | 0.0015 | 0.000653 |
| | | 4 | 2 | 430 | 0.52 | 8.24E-09 | 0.0015 | 0.000653 |
| | | | 4 | 218 | 0.4 | 6.50E-09 | 0.0015 | 0.000653 |
| | | | 8 | 117 | 0.3 | 9.35E-09 | 0.0015 | 0.000653 |
| | | | 16 | 156 | 0.66 | 9.60E-09 | 0.0015 | 0.000653 |
| | | | 32 | 164 | 1.2 | 9.66E-09 | 0.0015 | 0.000653 |
| | | | 128 | 31 | 0.82 | 7.69E-09 | 0.0015 | 0.000653 |

Table 9: Inexact Uzawa as smoother Part 2

| $N$ | $\alpha$ | $L_0$ | $v$ | iter | time | res | u_error | p_error |
|---|---|---|---|---|---|---|---|---|
| 64 | 1.5 | 1 | 2 | 145 | 0.22 | 8.31E-09 | 0.0015 | 0.000653 |
| | | | 4 | 26 | 0.05 | 7.21E-09 | 0.0015 | 0.000653 |
| | | | 8 | 21 | 0.06 | 6.12E-09 | 0.0015 | 0.000653 |
| | | | 16 | 21 | 0.1 | 5.46E-09 | 0.0015 | 0.000653 |
| | | | 32 | 21 | 0.19 | 5.45E-09 | 0.0015 | 0.000653 |
| | | | 128 | 20 | 0.59 | 6.82E-09 | 0.0015 | 0.000653 |
| | | 2 | 2 | 651 | 0.84 | NaN | NaN | NaN |
| | | | 4 | 1000 | 1.81 | 1.74E+287 | 9.05E+286 | 2.18E+288 |
| | | | 8 | 22 | 0.07 | 4.97E-09 | 0.0015 | 0.000653 |
| | | | 16 | 21 | 0.1 | 6.80E-09 | 0.0015 | 0.000653 |
| | | | 32 | 21 | 0.17 | 5.76E-09 | 0.0015 | 0.000653 |
| | | | 128 | 20 | 0.61 | 7.02E-09 | 0.0015 | 0.000653 |
| | | 4 | 2 | 1000 | 1.23 | 2.08E+52 | 8.71E+50 | 4.17E+52 |
| | | | 4 | 1000 | 1.71 | 4.04E+95 | 4.13E+94 | 1.36E+96 |
| | | | 8 | 1000 | 2.42 | 4.33E+91 | 5.70E+90 | 1.81E+92 |
| | | | 16 | 1000 | 4.26 | 3.18E+109 | 8.52E+108 | 1.93E+110 |
| | | | 32 | 1000 | 7.31 | 8.26E+115 | 4.40E+115 | 7.18E+116 |
| | | | 128 | 20 | 0.56 | 8.66E-09 | 0.0015 | 0.000653 |
| | 2 | 1 | 2 | 1000 | 1.56 | 1.88E+176 | 7.52E+173 | 1.06E+176 |
| | | | 4 | 1000 | 1.99 | 2.77E+69 | 2.02E+67 | 2.05E+69 |
| | | | 8 | 1000 | 2.96 | 1.11E+17 | 1.44E+15 | 1.06E+17 |
| | | | 16 | 1000 | 4.73 | 83 | 1.49 | 90 |
| | | | 32 | 1000 | 8.12 | 0.026 | 0.00164 | 0.0308 |
| | | | 128 | 1000 | 31.81 | 0.00841 | 0.00151 | 0.00746 |
| | | 2 | 2 | 590 | 1.07 | NaN | NaN | NaN |
| | | | 4 | 1000 | 1.86 | 5.30E+76 | 3.85E+74 | 3.94E+76 |
| | | | 8 | 1000 | 2.91 | 2.28E+17 | 3.35E+15 | 2.29E+17 |
| | | | 16 | 1000 | 4.83 | 86.2 | 1.52 | 92.8 |
| | | | 32 | 1000 | 9.04 | 0.0271 | 0.00164 | 0.0312 |
| | | | 128 | 1000 | 30.22 | 0.00858 | 0.00151 | 0.00741 |
| | | 4 | 2 | 1000 | 1.15 | 1.38E+159 | 5.41E+156 | 7.69E+158 |
| | | | 4 | 1000 | 1.82 | 8.11E+118 | 2.39E+117 | 9.61E+118 |
| | | | 8 | 1000 | 2.53 | 1.19E+184 | 8.32E+182 | 2.74E+184 |
| | | | 16 | 1000 | 4.25 | 8.75E+184 | 1.87E+184 | 3.30E+185 |
| | | | 32 | 1000 | 7.79 | 4.99E+183 | 2.16E+183 | 2.63E+184 |
| | | | 128 | 1000 | 27.13 | 0.00646 | 0.00151 | 0.007 |

Table 10: Inexact Uzawa as smoother Part 3

| $N$ | $\alpha$ | $L_0$ | $v$ | iter | time | res | u_error | p_error |
|---|---|---|---|---|---|---|---|---|
| 128 | 1 | 1 | 4 | 11 | 0.11 | 4.28E-09 | 0.000374 | 0.000163 |
| | | | 8 | 8 | 0.11 | 2.43E-09 | 0.000374 | 0.000163 |
| | | | 16 | 7 | 0.14 | 1.51E-09 | 0.000374 | 0.000163 |
| | | | 32 | 6 | 0.23 | 2.87E-09 | 0.000374 | 0.000163 |
| | | 2 | 4 | 710 | 5.13 | NaN | NaN | NaN |
| | | | 8 | 1000 | 10.71 | 2.27E+15 | 3.43E+15 | 6.25E+16 |
| | | | 16 | 8 | 0.17 | 1.23E-09 | 0.000374 | 0.000163 |
| | | | 32 | 6 | 0.23 | 2.87E-09 | 0.000374 | 0.000163 |
| 256 | 1 | 1 | 4 | 12 | 0.47 | 5.84E-09 | 9.34E-05 | 4.08E-05 |
| | | | 8 | 7 | 0.43 | 9.14E-09 | 9.34E-05 | 4.08E-05 |
| | | | 16 | 6 | 0.58 | 8.29E-09 | 9.34E-05 | 4.08E-05 |
| | | | 32 | 6 | 1.02 | 1.29E-09 | 9.34E-05 | 4.08E-05 |
| | | 2 | 4 | 591 | 20.97 | NaN | NaN | NaN |
| | | | 8 | 1000 | 52.28 | 4.06E+86 | 7.54E+86 | 1.62E+88 |
| | | | 16 | 8 | 0.75 | 2.70E-09 | 9.34E-05 | 4.08E-05 |
| | | | 32 | 6 | 1 | 1.29E-09 | 9.34E-05 | 4.08E-05 |
| 512 | 1 | 1 | 4 | 13 | 2.84 | 8.86E-09 | 2.33E-05 | 1.02E-05 |
| | | | 8 | 8 | 2.61 | 2.32E-09 | 2.33E-05 | 1.02E-05 |
| | | | 16 | 6 | 3.24 | 4.25E-09 | 2.33E-05 | 1.02E-05 |
| | | | 32 | 6 | 6.7 | 5.97E-10 | 2.33E-05 | 1.02E-05 |
| | | 2 | 4 | 518 | 101.13 | NaN | NaN | NaN |
| | | | 8 | 1000 | 277.35 | 1.27E+149 | 2.99E+149 | 7.37E+150 |
| | | | 16 | 8 | 4.13 | 4.96E-09 | 2.33E-05 | 1.02E-05 |
| | | | 32 | 6 | 5.52 | 5.97E-10 | 2.33E-05 | 1.02E-05 |
| 1024 | 1 | 1 | 4 | 15 | 15.44 | 5.74E-09 | 5.84E-06 | 2.55E-06 |
| | | | 6 | 10 | 12.76 | 1.76E-09 | 5.84E-06 | 2.55E-06 |
| | | | 8 | 8 | 12.56 | 7.59E-09 | 5.84E-06 | 2.57E-06 |
| | | | 10 | 8 | 13.82 | 1.37E-09 | 5.84E-06 | 2.55E-06 |
| | | | 12 | 7 | 13.91 | 2.57E-09 | 5.84E-06 | 2.55E-06 |
| | | | 14 | 7 | 15.05 | 7.63E-10 | 5.84E-06 | 2.55E-06 |
| | | | 16 | 6 | 14.42 | 5.95E-09 | 5.84E-06 | 2.54E-06 |
| 2048 | 1 | 1 | 4 | 17 | 68.28 | 5.03E-09 | 1.46E-06 | 6.57E-07 |
| | | | 6 | 10 | 52.24 | 7.03E-09 | 1.46E-06 | 6.54E-07 |
| | | | 8 | 9 | 56.62 | 3.25E-09 | 1.46E-06 | 6.28E-07 |
| | | | 10 | 8 | 58.33 | 3.34E-09 | 1.46E-06 | 6.48E-07 |
| | | | 12 | 7 | 56.22 | 5.43E-09 | 1.46E-06 | 6.39E-07 |
| | | | 14 | 7 | 62.6 | 1.72E-09 | 1.46E-06 | 6.41E-07 |
| | | | 16 | 7 | 68.72 | 8.85E-10 | 1.46E-06 | 6.41E-07 |

Table 11: Inexact Uzawa based on CG

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 64 | 0.5 | 1.60E-08 | 100 | 2.16 | 1.15E-08 | 0.0015 | 0.000653 |
| | | 4.00E-09 | 19 | 0.47 | 6.67E-09 | 0.0015 | 0.000653 |
| | | 1.00E-09 | 19 | 0.46 | 6.03E-09 | 0.0015 | 0.000653 |
| | 1 | 1.60E-08 | 100 | 1.88 | 1.15E-08 | 0.0015 | 0.000653 |
| | | 4.00E-09 | 2 | 0.05 | 4.72E-09 | 0.0015 | 0.000653 |
| | | 1.00E-09 | 2 | 0.05 | 1.14E-09 | 0.0015 | 0.000653 |
| | 1.5 | 1.60E-08 | 100 | 1.95 | 1.16E-08 | 0.0015 | 0.000653 |
| | | 4.00E-09 | 20 | 0.46 | 9.64E-09 | 0.0015 | 0.000653 |
| | | 1.00E-09 | 20 | 0.52 | 9.03E-09 | 0.0015 | 0.000653 |
| | 2 | 1.60E-08 | 100 | 2.61 | 0.00626 | 0.0278 | 0.0945 |
| | | 4.00E-09 | 100 | 4.03 | 0.00626 | 0.0278 | 0.0945 |
| | | 1.00E-09 | 100 | 2.94 | 0.00626 | 0.0278 | 0.0945 |
| 128 | 0.5 | 1.60E-08 | 100 | 21.7 | 1.52E-08 | 0.000374 | 0.000163 |
| | | 4.00E-09 | 18 | 4.93 | 9.77E-09 | 0.000374 | 0.000163 |
| | | 1.00E-09 | 18 | 4.97 | 9.09E-09 | 0.000374 | 0.000163 |
| | 1 | 1.60E-08 | 100 | 20.46 | 1.52E-08 | 0.000374 | 0.000163 |
| | | 4.00E-09 | 2 | 0.54 | 4.76E-09 | 0.000374 | 0.000163 |
| | | 1.00E-09 | 2 | 0.55 | 1.15E-09 | 0.000374 | 0.000163 |
| | 1.5 | 1.60E-08 | 100 | 21.48 | 1.50E-08 | 0.000374 | 0.000163 |
| | | 4.00E-09 | 20 | 5.56 | 8.05E-09 | 0.000374 | 0.000163 |
| | | 1.00E-09 | 20 | 5.35 | 6.96E-09 | 0.000374 | 0.000163 |
| | 2 | 1.60E-08 | 100 | 27.99 | 0.00474 | 0.0278 | 0.0945 |
| | | 4.00E-09 | 100 | 29.18 | 0.00474 | 0.0278 | 0.0945 |
| | | 1.00E-09 | 100 | 30.25 | 0.00474 | 0.0278 | 0.0945 |
| 256 | 0.5 | 1.60E-08 | 100 | 152.35 | 1.53E-08 | 9.34E-05 | 4.08E-05 |
| | | 4.00E-09 | 18 | 34 | 7.61E-09 | 9.34E-05 | 4.08E-05 |
| | | 1.00E-09 | 18 | 35.48 | 6.69E-09 | 9.34E-05 | 4.08E-05 |
| | 1 | 1.60E-08 | 100 | 146.47 | 1.53E-08 | 9.34E-05 | 4.08E-05 |
| | | 4.00E-09 | 2 | 3.6 | 4.61E-09 | 9.34E-05 | 4.08E-05 |
| | | 1.00E-09 | 2 | 3.94 | 1.17E-09 | 9.34E-05 | 4.08E-05 |
| | 1.5 | 1.60E-08 | 100 | 152.65 | 1.59E-08 | 9.34E-05 | 4.08E-05 |
| | | 4.00E-09 | 20 | 37.29 | 7.03E-09 | 9.34E-05 | 4.08E-05 |
| | | 1.00E-09 | 20 | 39.93 | 5.21E-09 | 9.34E-05 | 4.08E-05 |
| | 2 | 1.60E-08 | 100 | 203.78 | 0.00348 | 0.0278 | 0.0945 |
| | | 4.00E-09 | 100 | 213.12 | 0.00348 | 0.0278 | 0.0945 |
| | | 1.00E-09 | 100 | 221.27 | 0.00348 | 0.0278 | 0.0945 |

Table 12: Inexact Uzawa based on V-cycle Part 1

| $N$ | $\alpha$ | $L_0$ | $v$ | $\tau$ | iter | time | res | u_error | p_error |
|---|---|---|---|---|---|---|---|---|---|
| 64 | 1 | 1 | 2 | 1.60E-07 | 100 | 0.07 | 3.46E-08 | 0.0015 | 0.000653 |
| 64 | 1 | 1 | 2 | 4.00E-08 | 3 | 0.02 | 4.90E-09 | 0.0015 | 0.000653 |
| 64 | 1 | 1 | 2 | 1.00E-08 | 2 | 0.03 | 2.68E-09 | 0.0015 | 0.000653 |
| 64 | 1 | 1 | 4 | 1.60E-07 | 100 | 0.15 | 2.30E-07 | 0.0015 | 0.000653 |
| 64 | 1 | 1 | 4 | 4.00E-08 | 100 | 0.17 | 3.48E-08 | 0.0015 | 0.000653 |
| 64 | 1 | 1 | 4 | 1.00E-08 | 2 | 0.02 | 8.79E-09 | 0.0015 | 0.000653 |
| 64 | 1 | 1 | 6 | 1.60E-07 | 100 | 0.24 | 2.99E-07 | 0.0015 | 0.000653 |
| 64 | 1 | 1 | 6 | 4.00E-08 | 2 | 0.02 | 4.09E-09 | 0.0015 | 0.000653 |
| 64 | 1 | 1 | 6 | 1.00E-08 | 2 | 0.02 | 4.09E-09 | 0.0015 | 0.000653 |
| 64 | 1 | 2 | 2 | 1.60E-07 | 100 | 0.15 | 1.98E-07 | 0.0015 | 0.000653 |
| 64 | 1 | 2 | 2 | 4.00E-08 | 6 | 0.04 | 8.26E-09 | 0.0015 | 0.000653 |
| 64 | 1 | 2 | 2 | 1.00E-08 | 2 | 0.03 | 6.46E-09 | 0.0015 | 0.000653 |
| 64 | 1 | 2 | 4 | 1.60E-07 | 100 | 0.16 | 2.19E-07 | 0.0015 | 0.000653 |
| 64 | 1 | 2 | 4 | 4.00E-08 | 12 | 0.04 | 8.01E-09 | 0.0015 | 0.000653 |
| 64 | 1 | 2 | 4 | 1.00E-08 | 2 | 0.02 | 3.23E-09 | 0.0015 | 0.000653 |
| 64 | 1 | 2 | 6 | 1.60E-07 | 100 | 0.21 | 1.50E-07 | 0.0015 | 0.000653 |
| 64 | 1 | 2 | 6 | 4.00E-08 | 100 | 0.22 | 5.14E-08 | 0.0015 | 0.000653 |
| 64 | 1 | 2 | 6 | 1.00E-08 | 2 | 0.02 | 2.73E-09 | 0.0015 | 0.000653 |
| 128 | 1 | 1 | 2 | 1.60E-07 | 100 | 0.38 | 2.48E-08 | 0.000374 | 0.000163 |
| 128 | 1 | 1 | 2 | 4.00E-08 | 100 | 0.52 | 5.50E-08 | 0.000374 | 0.000163 |
| 128 | 1 | 1 | 2 | 1.00E-08 | 2 | 0.09 | 5.86E-09 | 0.000374 | 0.000163 |
| 128 | 1 | 1 | 4 | 1.60E-07 | 100 | 0.75 | 2.31E-07 | 0.000374 | 0.000163 |
| 128 | 1 | 1 | 4 | 4.00E-08 | 100 | 0.75 | 1.90E-08 | 0.000374 | 0.000163 |
| 128 | 1 | 1 | 4 | 1.00E-08 | 2 | 0.07 | 4.15E-09 | 0.000374 | 0.000163 |
| 128 | 1 | 1 | 6 | 1.60E-07 | 100 | 0.94 | 2.58E-07 | 0.000374 | 0.000163 |
| 128 | 1 | 1 | 6 | 4.00E-08 | 100 | 0.93 | 7.02E-08 | 0.000374 | 0.000163 |
| 128 | 1 | 1 | 6 | 1.00E-08 | 2 | 0.09 | 1.91E-09 | 0.000374 | 0.000163 |
| 128 | 1 | 2 | 2 | 1.60E-07 | 100 | 0.56 | 1.72E-07 | 0.000374 | 0.000164 |
| 128 | 1 | 2 | 2 | 4.00E-08 | 4 | 0.14 | 6.25E-09 | 0.000374 | 0.000163 |
| 128 | 1 | 2 | 2 | 1.00E-08 | 2 | 0.16 | 9.30E-09 | 0.000374 | 0.000163 |
| 128 | 1 | 2 | 4 | 1.60E-07 | 100 | 0.81 | 1.60E-07 | 0.000374 | 0.000163 |
| 128 | 1 | 2 | 4 | 4.00E-08 | 30 | 0.25 | 9.29E-09 | 0.000374 | 0.000163 |
| 128 | 1 | 2 | 4 | 1.00E-08 | 3 | 0.1 | 4.46E-09 | 0.000374 | 0.000163 |
| 128 | 1 | 2 | 6 | 1.60E-07 | 100 | 0.91 | 1.65E-07 | 0.000374 | 0.000163 |
| 128 | 1 | 2 | 6 | 4.00E-08 | 100 | 0.96 | 3.06E-08 | 0.000374 | 0.000163 |
| 128 | 1 | 2 | 6 | 1.00E-08 | 2 | 0.09 | 7.03E-09 | 0.000374 | 0.000163 |

Table 13: Inexact Uzawa based on V-cycle Part 2

| $N$ | $\alpha$ | $L_0$ | $v$ | $\tau$ | iter | time | res | u_error | p_error |
|---|---|---|---|---|---|---|---|---|---|
| 256 | 1 | 1 | 2 | 1.60E-07 | 100 | 2.41 | 9.56E-08 | 9.34E-05 | 4.08E-05 |
| | | | | 4.00E-08 | 3 | 0.38 | 5.36E-09 | 9.34E-05 | 4.08E-05 |
| | | | | 1.00E-08 | 2 | 0.38 | 2.24E-09 | 9.34E-05 | 4.08E-05 |
| | | | 4 | 1.60E-07 | 100 | 3.37 | 2.22E-07 | 9.34E-05 | 4.13E-05 |
| | | | | 4.00E-08 | 100 | 3.42 | 2.95E-08 | 9.34E-05 | 4.08E-05 |
| | | | | 1.00E-08 | 2 | 0.36 | 2.02E-09 | 9.34E-05 | 4.08E-05 |
| | | | 6 | 1.60E-07 | 100 | 4.28 | 2.34E-07 | 9.34E-05 | 4.31E-05 |
| | | | | 4.00E-08 | 100 | 4.36 | 3.11E-08 | 9.34E-05 | 4.08E-05 |
| | | | | 1.00E-08 | 2 | 0.39 | 9.21E-10 | 9.34E-05 | 4.08E-05 |
| | | 2 | 2 | 1.60E-07 | 100 | 2.54 | 1.10E-07 | 9.34E-05 | 4.08E-05 |
| | | | | 4.00E-08 | 7 | 0.59 | 5.74E-09 | 9.34E-05 | 4.08E-05 |
| | | | | 1.00E-08 | 3 | 0.54 | 4.60E-09 | 9.34E-05 | 4.08E-05 |
| | | | 4 | 1.60E-07 | 100 | 3.39 | 1.12E-07 | 9.34E-05 | 4.09E-05 |
| | | | | 4.00E-08 | 5 | 0.5 | 7.36E-09 | 9.34E-05 | 4.08E-05 |
| | | | | 1.00E-08 | 2 | 0.43 | 6.45E-09 | 9.34E-05 | 4.08E-05 |
| | | | 6 | 1.60E-07 | 100 | 4.38 | 1.85E-07 | 9.34E-05 | 4.13E-05 |
| | | | | 4.00E-08 | 100 | 4.37 | 2.64E-08 | 9.34E-05 | 4.08E-05 |
| | | | | 1.00E-08 | 3 | 0.45 | 6.40E-09 | 9.34E-05 | 4.08E-05 |
| 512 | 1 | 1 | 2 | 1.60E-07 | 100 | 12.99 | 1.11E-07 | 2.34E-05 | 1.02E-05 |
| | | | | 4.00E-08 | 100 | 13.89 | 6.49E-08 | 2.34E-05 | 1.04E-05 |
| | | | | 1.00E-08 | 2 | 2.04 | 4.77E-09 | 2.34E-05 | 1.02E-05 |
| | | | 4 | 1.60E-07 | 100 | 18.75 | 2.27E-07 | 2.35E-05 | 1.42E-05 |
| | | | | 4.00E-08 | 100 | 19.63 | 7.47E-08 | 2.34E-05 | 1.08E-05 |
| | | | | 1.00E-08 | 2 | 1.76 | 3.24E-09 | 2.34E-05 | 1.02E-05 |
| | | | 6 | 1.60E-07 | 100 | 24.43 | 8.23E-08 | 2.34E-05 | 1.01E-05 |
| | | | | 4.00E-08 | 100 | 24.73 | 2.49E-08 | 2.34E-05 | 1.02E-05 |
| | | | | 1.00E-08 | 3 | 2.4 | 5.51E-09 | 2.34E-05 | 1.02E-05 |
| | | 2 | 2 | 1.60E-07 | 100 | 14.14 | 8.57E-08 | 2.34E-05 | 1.01E-05 |
| | | | | 4.00E-08 | 6 | 2.99 | 8.73E-09 | 2.33E-05 | 1.07E-05 |
| | | | | 1.00E-08 | 2 | 2.93 | 6.84E-09 | 2.34E-05 | 1.02E-05 |
| | | | 4 | 1.60E-07 | 100 | 21.16 | 9.09E-08 | 2.34E-05 | 1.01E-05 |
| | | | | 4.00E-08 | 22 | 6.23 | 8.02E-09 | 2.34E-05 | 1.02E-05 |
| | | | | 1.00E-08 | 2 | 2.83 | 3.22E-09 | 2.34E-05 | 1.02E-05 |
| | | | 6 | 1.60E-07 | 100 | 28.14 | 1.32E-07 | 2.34E-05 | 1.04E-05 |
| | | | | 4.00E-08 | 2 | 2.26 | 6.69E-09 | 2.34E-05 | 1.02E-05 |
| | | | | 1.00E-08 | 2 | 2.23 | 6.69E-09 | 2.34E-05 | 1.02E-05 |

Table 14: Inexact Uzawa based on V-cycle Part 3

| $N$ | $\alpha$ | $L_0$ | $v$ | $\tau$ | iter | time | res | u_error | p_error |
|---|---|---|---|---|---|---|---|---|---|
| 1024 | 1 | 1 | 1 | 1.00E-08 | 3 | 14.03 | 9.12E-09 | 5.84E-06 | 2.55E-06 |
| | | | | 5.00E-09 | 2 | 14.55 | 8.51E-09 | 5.84E-06 | 2.55E-06 |
| | | | | 1.00E-09 | 2 | 15.92 | 1.57E-09 | 5.84E-06 | 2.55E-06 |
| | | | 2 | 1.00E-08 | 2 | 8.61 | 3.71E-09 | 5.84E-06 | 2.55E-06 |
| | | | | 5.00E-09 | 2 | 9.06 | 3.71E-09 | 5.84E-06 | 2.55E-06 |
| | | | | 1.00E-09 | 2 | 10.03 | 6.41E-10 | 5.84E-06 | 2.55E-06 |
| | | | 3 | 1.00E-08 | 2 | 7.84 | 3.07E-09 | 5.85E-06 | 2.56E-06 |
| | | | | 5.00E-09 | 2 | 7.89 | 3.07E-09 | 5.85E-06 | 2.56E-06 |
| | | | | 1.00E-09 | 2 | 8.45 | 3.07E-09 | 5.85E-06 | 2.56E-06 |
| | | | 4 | 1.00E-08 | 10 | 13.26 | 2.27E-08 | 5.85E-06 | 2.91E-06 |
| | | | | 5.00E-09 | 2 | 8.85 | 1.02E-09 | 5.84E-06 | 2.55E-06 |
| | | | | 1.00E-09 | 2 | 9.8 | 1.02E-09 | 5.84E-06 | 2.55E-06 |
| 2048 | 1 | 1 | 1 | 1.00E-08 | 3 | 56.37 | 8.87E-09 | 1.46E-06 | 6.38E-07 |
| | | | | 5.00E-09 | 2 | 58.47 | 8.25E-09 | 1.46E-06 | 6.38E-07 |
| | | | | 1.00E-09 | 2 | 62.64 | 1.52E-09 | 1.46E-06 | 6.38E-07 |
| | | | 2 | 1.00E-08 | 10 | 49.03 | 1.19E-08 | 1.60E-06 | 6.92E-07 |
| | | | | 5.00E-09 | 6 | 44.2 | 7.64E-09 | 1.51E-06 | 6.66E-07 |
| | | | | 1.00E-09 | 2 | 39.19 | 2.01E-09 | 1.46E-06 | 6.37E-07 |
| | | | 3 | 1.00E-08 | 2 | 32.48 | 2.14E-09 | 1.48E-06 | 6.65E-07 |
| | | | | 5.00E-09 | 2 | 32.31 | 2.14E-09 | 1.48E-06 | 6.65E-07 |
| | | | | 1.00E-09 | 2 | 35.56 | 2.14E-09 | 1.48E-06 | 6.65E-07 |
| | | | 4 | 1.00E-08 | 10 | 54.03 | 1.62E-08 | 1.74E-06 | 6.67E-07 |
| | | | | 5.00E-09 | 5 | 39.67 | 9.94E-09 | 1.58E-06 | 6.50E-07 |
| | | | | 1.00E-09 | 2 | 34.53 | 7.21E-10 | 1.46E-06 | 6.41E-07 |