Numerical Solutions to Partial Differential Equations

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Céa Lemma — an Abstract Error Estimate Theorem

变分问题

1. Consider the variational problem of the form

$$\begin{cases} \text{Find } \ u \in \mathbb{V} \text{ such that} \\ a(u, v) = f(v), \quad \forall v \in \mathbb{V}. \end{cases} \tag{7.1.1}$$

协调FEM

2. Consider the conforming finite element method of the form

$$\begin{cases} \text{Find } \ u_h \in \mathbb{V}_h \subset \mathbb{V} \text{ such that} \\ a(u_h, v_h) = f(v_h), \quad \forall v_h \in \mathbb{V}_h. \end{cases} \tag{7.1.2}$$

如何估计误差？

3. The problem: how to estimate the error $\|u - u_h\|$?
4. The method used for FDM is not an ideal framework for FEM.
5. The standard approach for the error estimations of a finite element solution is to use an abstract error estimate to reduce the problem to a function approximation problem.
FEM误差估计的标准方法：使用抽象的误差估计(Cea引理)把它化为函数逼近问题.

Céa Lemma — an Abstract Error Estimate Theorem

Theorem 7.1

Let $\mathbb{V}$ be a Hilbert space, $\mathbb{V}_h$ be a linear subspace of $\mathbb{V}$. Let the bilinear form $a(\cdot,\ \cdot)$ and the linear form $f(\cdot)$ satisfy the conditions of the Lax-Milgram lemma (see Theorem 5.1). *Let $u \in \mathbb{V}$ be the solution to the variational problem, and $u_h \in \mathbb{V}_h$ satisfy the equation*

$$a(u_h, v_h) = f(v_h), \quad \forall v_h \in \mathbb{V}_h. \tag{7.1.2}$$

Then, there exist a constant C independent of $\mathbb{V}_h$, such that

$$\boxed{\|u - u_h\| \le C \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|,} \tag{7.1.3}$$

where $\|\cdot\|$ is the norm of $\mathbb{V}$.

Cea引理表明: uh is quasi-optimal in the sense that the error ||u-uh||_V is proportional to the best it can be using the subspace Vh.

Proof of the Céa Lemma

1. Since u and u_h satisfy the equations, and $\mathbb{V}_h \subset \mathbb{V}$, we have

$$a(u-u_h,\, w_h) = a(u,\, w_h) - a(u_h,\, w_h) = f(w_h) - f(w_h) = 0, \quad \forall w_h \in \mathbb{V}_h. \tag{7.1.4}$$

2. In particular, taking $w_h = u_h - v_h$ leads to

$$\boxed{a(u-u_h,\, u_h - v_h) = 0.} \qquad \text{(*1)}$$

3. The $\mathbb{V}$-ellipticity $\Rightarrow$ $\alpha \|u - u_h\|^2 \le a(u-u_h,\, u-u_h).$ (*2)

4. The boundedness $\Rightarrow a(u-u_h,\, u-v_h) \le M\|u-u_h\|\|u-v_h\|.$

5. Hence, $\alpha\|u-u_h\|^2 \underset{(*1\text{-}*2)}{\le} a(u-u_h,\, u-v_h) \le M\|u-u_h\|\|u-v_h\|.$

6. Take $C = M/\alpha$, we have

$$\boxed{\|u-u_h\| \le C\|u-v_h\|, \quad \forall v_h \in \mathbb{V}_h.}$$

7. The conclusion of the theorem follows. ■

Remarks on the Céa Lemma

1. The Céa lemma reduces the error estimation problem of $\|u - u_h\|$ to the optimal approximation problem of $\inf_{v_h \in \mathbb{V}_h} \|u - v_h\|$. Cea引理: 把FEM误差估计化为函数逼近问题.
2. Error of the finite element solution $\|u - u_h\|$ is of the same order as the optimal approximation error $\inf_{v_h \in \mathbb{V}_h} \|u - v_h\|$.
3. Suppose the $\mathbb{V}_h$-interpolation function $\Pi_h u$ of u is well defined in the finite element function space $\mathbb{V}_h$, then,

$$\boxed{\|u - u_h\| \leq C \inf_{v_h \in \mathbb{V}_h} \|u - v_h\| \leq C \|u - \Pi_h u\|.} \tag{7.2.1}$$

FEM误差估计可进一步化为插值误差估计问题.

4. Therefore, the error estimation problem of $\|u - u_h\|$ can be further reduced to the error estimation problem for the $\mathbb{V}_h$-interpolation error $\|u - \Pi_h u\|$.

For Symmetric $a(\cdot,\cdot)$, u_h Is a Orthogonal Projection of u on $\mathbb{V}_h$

V范数 & 能量范数

1. If the $\mathbb{V}$-elliptic bounded bilinear form $a(\cdot,\ \cdot)$ is symmetric, then, $a(\cdot,\ \cdot)$ defines an inner product on $\mathbb{V}$, with the induced norm equivalent to the $\mathbb{V}$-norm.

正交投影算子

2. Denote $\mathbf{P}_h : \mathbb{V} \to \mathbb{V}_h$ as the orthogonal projection operator induced by the inner product $a(\cdot,\ \cdot)$. Then,
$$a(u - \mathbf{P}_h u, v_h) = 0, \quad \forall v_h \in \mathbb{V}_h.$$

3. Therefore, the finite element solution $u_h = \mathbf{P}_h u$, *i.e.* it is the orthogonal projection of u on $\mathbb{V}_h$ with respect to the inner product $a(\cdot,\ \cdot)$. FEM解可用正交投影算子表示出来.

Céa Lemma for Symmetric $a(\cdot,\cdot)$

Corollary 7.1

Under the conditions of the Céa Lemma, if the bilinear form $a(\cdot,\ \cdot)$ is still symmetric, then, the solution u_h is the orthogonal projection, which is induced by the inner product $a(\cdot,\ \cdot)$, of the solution u on the subspace $\mathbb{V}_h$, meaning $u_h = \mathbf{P}_h u$.

Furthermore, we have

能量范数下
的误差估计

$$\boxed{a(u-u_h, u-u_h) = \inf_{v_h\in\mathbb{V}_h} a(u-v_h, u-v_h).}$$

与(7.2.1)对应，此时常数为1.

The proof follows the same lines as the proof of the Céa lemma. The only difference here is that $\alpha = M = 1$.

||u-uh||_E^2=a(u-uh,u-uh)=a(u-uh, u-vh+vh-uh)=(u-uh, u-vh)\leq ||u-uh||_E * ||u-vh||_E.
用了a(u-uh, vh-uh)=0

Céa Lemma in the Form of Orthogonal Projection Error Estimate

Denote $\tilde{P}_h : \mathbb{V} \to \mathbb{V}_h$ as the orthogonal projection operator induced by the inner product $(\cdot,\cdot)_{\mathbb{V}}$ of $\mathbb{V}$, then,

$$\|u - \tilde{P}_h u\| = \|(I - \tilde{P}_h)u\| = \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|.$$

Therefore, as a corollary of the Céa lemma, we have

Corollary 7.2

Let $\mathbb{V}$ be a Hilbert space, and $\mathbb{V}_h$ be a linear subspace of $\mathbb{V}$. Let $a(\cdot,\ \cdot)$ be a symmetric bilinear form on $\mathbb{V}$ satisfying the conditions of the Lax-Milgram lemma. Let P_h and $\tilde{P}_h$ be the orthogonal projection operators from $\mathbb{V}$ to $\mathbb{V}_h$ induced by the inner products $a(\cdot,\ \cdot)$ and $(\cdot,\cdot)_{\mathbb{V}}$ respectively. Then, we have

$$\boxed{\|I - \tilde{P}_h\| \underset{(*)}{\leq} \|I - P_h\| \overset{(7.1.3)}{\leq} \frac{M}{\alpha}\|I - \tilde{P}_h\|.} \tag{7.1.5}$$

(*) ||(I-P_h)u||^2=||(I-\tilde{P}_h)u +(\tilde{P}_h-P_h)u ||^2=||(I-\tilde{P}_h)u ||^2 +||(\tilde{P}_h-P_h)u ||^2 \geq ||(I-\tilde{P}_h)u ||^2.

1-D Example on Linear Interpolation Error Estimation for $\mathbb{H}^2$ Functions

例7.1 插值误差估计 (P236)

1. $\hat{\Omega} = (0,\ 1)$, $\Omega = (b,\ b+h)$, $h > 0$.

线性坐标变换

2. $F : \hat{x} \in [0,\ 1] \to [b,\ b+h]$, $\underbrace{F(\hat{x})}_{=x} = h\hat{x} + b$: an invertible affine mapping from $\overline{\hat{\Omega}}$ to $\overline{\Omega}$.

线性插值算子

3. $\hat{\Pi} : \mathbb{C}([0,\ 1]) \to \mathbb{P}_1([0,\ 1])$: the interpolation operator with $\hat{\Pi}\hat{v}(0) = \hat{v}(0)$, $\hat{\Pi}\hat{v}(1) = \hat{v}(1)$.

线性插值算子

4. $\Pi : \mathbb{C}([b,\ b+h]) \to \mathbb{P}_1([b,\ b+h])$: the interpolation operator with $\Pi v(b) = v(b)$, $\Pi v(b+h) = v(b+h)$.

1-D Example on Linear Interpolation Error Estimation for $\mathbb{H}^2$ Functions

5. Let $u \in \mathbb{H}^2(\Omega)$, denote $\hat{u}(\hat{x}) = u \circ F(\hat{x}) = u(h\hat{x} + b)$, then, it can be shown $\hat{u} \in \mathbb{H}^2(\hat{\Omega})$, thus, $\hat{u} \in \mathbb{C}([0,\ 1])$.==>可以将插值算子作用于\hat{u}
   嵌入定理

P1不变的插值算子

6. $\hat{\Pi}$ **is** $\mathbb{P}_1([0,1])$ **invariant**: $\hat{\Pi}\hat{w} = \hat{w}$, $\forall \hat{w} \in \mathbb{P}_1([0,1])$, thus,
$$\|(I - \hat{\Pi})\hat{u}\|_{0,\hat{\Omega}} = \|(I - \hat{\Pi})(\hat{u} + \hat{w})\|_{0,\hat{\Omega}} \leq \|I - \hat{\Pi}\| \, \|\hat{u} + \hat{w}\|_{2,\hat{\Omega}},$$
where $\|I - \hat{\Pi}\|$ is the norm of $I - \hat{\Pi} : \mathbb{H}^2(\hat{\Omega}) \to \mathbb{L}^2(\hat{\Omega})$.

★ This shows that $I - \hat{\Pi} \in \mathfrak{L}(\mathbb{H}^2(0,1)/\mathbb{P}_1([0,1]); \mathbb{L}^2(0,1))$, and

$$\boxed{(1) \quad \|\hat{u} - \hat{\Pi}\hat{u}\|_{0,\hat{\Omega}} \leq \|I - \hat{\Pi}\| \inf_{\hat{w} \in \mathbb{P}_1(\hat{\Omega})} \|\hat{u} + \hat{w}\|_{2,\hat{\Omega}},}$$

不变插值算子的插值误差

hat{u}在商空间里的范数

where $\inf_{\hat{w} \in \mathbb{P}_1(\hat{\Omega})} \|\hat{u} + \hat{w}\|_{2,\hat{\Omega}}$ is the norm of $\hat{u}$ in the quotient space $\mathbb{H}^2(0,1)/\mathbb{P}_1([0,1])$. Sobolev空间的多项式商空间H2/P1的元素为 H2中元素v的等价类 dot{v}={w ∈ H2: w-v ∈ P1}. 见7.2.1节

设V是域K上的一个向量空间，且N是V的一个子空间。定义在V上定义一个等价类，如果x-y属于N 则令x~y。即如果其中一个加上N中一个元素得到另一个，则与y 相关. x 的所在等价类通常记作[x]=x+N.
商空间V/N(读作V模N)定义为V/~，V在等价~下所有等价类集合.
可以定义等价类上的数乘与加法，和范数(见上面)；商空间X/M关于此范数是完备的，所以是一个巴拿赫空间.

1-D Example on Linear Interpolation Error Estimation for $\mathbb{H}^2$ Functions

见7.2.1节

★ It can be shown that, $\exists$ const. $C(\hat{\Omega}) > 0$ s.t.

$$(2)\quad |\hat{u}|_{2,\hat{\Omega}} \le \inf_{\hat{w}\in\mathbb{P}_1(\hat{\Omega})} \|\hat{u}+\hat{w}\|_{2,\hat{\Omega}} \underset{(7.2.2)}{\le} C(\hat{\Omega})|\hat{u}|_{2,\hat{\Omega}}.$$

H2的半范数是Sobolev空间的多项式商空间H2/P1(见7.2.1节) 的等价范数

★ It follows from the chain rule that $\hat{u}''(\hat{x}) = h^2 u''(x)$.

★ By a change of the integral variable, and $dx = hd\hat{x}$, we obtain

变换前后的半范数和范数之间的关系

$$(3)\quad \hat{u} \in \mathbb{H}^2(\hat{\Omega}),\ \text{and}\ |\hat{u}|^2_{2,\hat{\Omega}} = h^3|u|^2_{2,\Omega};$$

$$(4)\quad \|u-\Pi u\|^2_{0,\Omega} = h\|\hat{u}-\hat{\Pi}\hat{u}\|^2_{0,\hat{\Omega}}.$$

1-D Example on Linear Interpolation Error Estimation for $\mathbb{H}^2$ Functions

- The conclusion (1) says that the $\mathbb{L}^2$ norm of the error of a $\mathbb{P}_1$ invariant interpolation can be bounded by the quotient norm of the function in $\mathbb{H}^2(0,1)/\mathbb{P}_1([0,1])$.

- The conclusion (2) says that the semi norm $|\cdot|_{2,(0,1)}$ is an equivalent norm of the quotient space $\mathbb{H}^2(0,1)/\mathbb{P}_1([0,1])$.

例7.1中的(2)是指：H2的半范数是Sobolev空间的多项式商空间H2/P1(见7.2.1节) 的等价范数

- The conclusions (3) and (4) present the relations between the semi-norms of Sobolev spaces defined on affine-equivalent open sets.

1-D Example on Linear Interpolation Error Estimation for $\mathbb{H}^2$ Functions

★ The combination of (4) and (1) yields

$$\boxed{\|u-\Pi u\|_{0,\Omega} \leq h^{\frac{1}{2}} \|I-\hat{\Pi}\| \inf_{\hat{w}\in\mathbb{P}_1(\hat{\Omega})} \|\hat{u}+\hat{w}\|_{2,\hat{\Omega}}}$$

★ This together with (2) and (3) lead to the expected interpolation error estimate:

L2范数下的H2函数的插值误差估计

$$\boxed{\|u-\Pi u\|_{0,\Omega} \leq \|I-\hat{\Pi}\| C(\hat{\Omega}) |u|_{2,\Omega} h^2, \quad \forall u \in \mathbb{H}^2(\Omega).}$$

H1半范数下的H2函数的插值误差估计|(I-\Pi)u|_1\leq C*h* |u|_2，见习题7中题4(p262).

A Framework for Interpolation Error Estimation of Affine Equivalent FEs

例子7.1的提示：仿射等价FE的插值误差估计的一个框架

1. The polynomial quotient spaces of a Sobolev space and their equivalent quotient norms ((2) in the example);
   Sobolev空间的多项式商空间(见7.2.1节) & 它们的等价范数(例7.1中的（2))
2. The relations between the semi-norms of Sobolev spaces defined on affine-equivalent open sets ((3), (4) in the exmample); 定义在仿射等价开集上的Sobolev空间的半范数之间的关系
3. The abstract error estimates for the polynomial invariant operators ((1) in the example);
   多项式不变算子的抽象的误差估计
4. To estimate the constants appeared in the relations of the Sobolev semi-norms by means of the geometric parameters of the corresponding affine-equivalent open sets.
   用仿射等价开集的几何参数(见下页)来估计相应的Sobolev空间半范数关系中的常数

A Framework for Interpolation Error Estimation of Affine Equivalent FEs

仿射等价FE的插值误差估计的一个框架

- the change of integral variable will introduce the Jacobian determinant $\det\left(\frac{\partial F(\hat{x})}{\partial \hat{x}}\right)$; 积分变量变换---〉仿射变换的Jacobi行列式
- in high dimensions, the Jacobi determinant represents the ratio of the volumes $|\Omega|/|\hat{\Omega}|$; Jacobi行列式代表体积比
- the chain rule for the mth derivative will produce h^m.
- h actually represents the ratio of the lengths in the directions of corresponding directional derivatives of the regions $\Omega = F(\hat{\Omega})$ and $\hat{\Omega}$. 这里的h是由变换F(hat(x))联系的区域沿求导方向的尺度之比

The related technique is often referred to as the scaling technique.
尺度技术/比例缩小技术

下面就是在这个框架下简要介绍椭圆型BVP弱解的基本函数空间Sobolev空间上的多项式插值误差估计理论. 详见FEM专著[P. G. Ciarlet, The Finite Element Method for Elliptic Problems, SIAM, 2002].

Polynomial Quotient Spaces

7.2.1节　多项式商空间　&　等价商范数

多项式商空间 W/P

1. The quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$, in which a vector $\dot{v}$ is the equivalent class of $v \in \mathbb{W}^{k+1,p}(\Omega)$ in the sense that

$$\dot{v} = \{w \in \mathbb{W}^{k+1,p}(\Omega) : (w - v) \in \mathbb{P}_k(\Omega)\}.$$

Sobolev空间的多项式商空间W/P的元素为 W中元素v的等价类dot{v}.

范数

2. The quotient norm of a vector $\dot{v}$ is defined by

$$\dot{v} \in \mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega) \to \|\dot{v}\|_{k+1,p,\Omega} \overset{\text{def}}{:=} \inf_{w\in\mathbb{P}_k(\Omega)} \|v+w\|_{k+1,p,\Omega}.$$

设V是域K上的一个向量空间，且N是V的一个子空间。定义在V上定义一个等价类，如果x-y属于N　则令x~y。即如果其中一个加上N中一个元素得到另一个，则与y 相关. x 的所在等价类通常记作[x]=x+N.
商空间V/N(读作V模N)定义为V/~，V在等价~下所有等价类集合.
可以定义等价类上的数乘与加法，和范数(见上面)；商空间X/M关于此范数是完备的，所以是一个巴拿赫空间.

Polynomial Quotient Spaces

完备的赋范空间

3. The quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$ is a Banach space.

半范数

4. $\dot{v} \in \mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega) \to |\dot{v}|_{k+1,p,\Omega} \stackrel{\text{def}}{=} |v|_{k+1,p,\Omega}$ is a semi-norm of the quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$, and obviously $|\dot{v}|_{k+1,p,\Omega} \leq \|\dot{v}\|_{k+1,p,\Omega}$.

5. In fact, $|\dot{v}|_{k+1,p,\Omega} = |v|_{k+1,p,\Omega}$ is an equivalent norm of the quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$. 商空间的半范数是与其范数等价的，见下页定理

Semi-norm $|\dot{v}|_{k+1,p,\Omega}$ is an equivalent norm of $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$

Theorem 7.2 (商空间的等价模定理)

There exists a constant $C(\Omega)$ such that

$$\boxed{\|\dot{v}\|_{k+1,p,\Omega} \le C(\Omega)|\dot{v}|_{k+1,p,\Omega}, \qquad \forall \dot{v} \in \mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega).} \quad (7.2.2)$$

商空间的半范数是与其范数等价的

Proof:

fi是P_k的共轭空间(线性泛函)的一组基

1. Let $\{p_i\}_{i=1}^N$ be a basis of $\mathbb{P}_k(\Omega)$, and f_i, $i = 1, \ldots, N$, be the corresponding dual basis, meaning $f_i(p_j) = \delta_{ij}$.

2. Thus, for any $w \in \mathbb{P}_k(\Omega)$, $f_i(w) = 0$, $i = 1, \ldots, N \Leftrightarrow w = 0$. $(*)$

Hahn-Banach延拓或扩张定理=>

3. Extend f_i, $i = 1, \ldots, N$, to a set of bounded linear functionals defined on $\mathbb{W}^{k+1,p}(\Omega)$, which satisfy $(*)$.

Hahn-Banach延拓或扩张定理：设X为实线性空间，M为它的线性子空间， p是X上的次可加正齐性泛函, f0是M上的线性泛函，则存在X上的线性泛函f, s.t. f(x)=f0(x), for all x in M; 如果f0(x)\leq p(x), for all x in M， 则可使f满足 f(x)\leq p(x), for all x in M.
一般泛函分析教科书中的X常取为赋范线性空间，p则取为空间的范数.这样，哈恩—巴拿赫定理就变为线性泛函的保持范数不变的可延拓定理。从选择公理可以推出哈恩-巴拿赫定理。然而，反过来不成立。注意超滤子引理比选择公理更弱，但从它也可以推出哈恩-巴拿赫定理（反过来则不行）。实际上，哈恩-巴拿赫定理还可以用比超滤子引理更弱的假设来证明。

Semi-norm $|v|_{k+1,p,\Omega}$ Is an equivalent Norm of $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$

④ 仅需证明 that there exists a constant $C(\Omega)$ such that

$$\|v\|_{k+1,p,\Omega} \le C(\Omega)(|v|_{k+1,p,\Omega} + \sum_{i=1}^{N} |f_i(v)|), \forall v \in \mathbb{W}^{k+1,p}(\Omega). \quad (7.2.3)$$

如果(7.2.3)已经得证, 则由此可推得(7.2.2). 事实上，

⑤ For $v \in \mathbb{W}^{k+1,p}(\Omega)$, define $\tilde{w} = -\sum_{j=1}^{N} f_j(v)p_j$, then, $f_i(v+\tilde{w}) = 0$, $i = 1, \ldots, N$, consequently,

$$\inf_{w\in\mathbb{P}_k(\Omega)} \|v+w\|_{k+1,p,\Omega} \le \|v+\tilde{w}\|_{k+1,p,\Omega} \underset{(7.2.3)}{\le} C(\Omega)|v|_{k+1,p,\Omega}.$$

采用反证法证明(7.2.3)：

Suppose ④ doesn't hold. Then,

⑥ there exists a sequence $\{v_j\}_{j=1}^{\infty}$ in $\mathbb{W}^{k+1,p}(\Omega)$ s.t.

$$\|v_j\|_{k+1,p,\Omega} = 1, \forall j \ge 1 \text{ and } \lim_{j\to\infty}(|v_j|_{k+1,p,\Omega} + \sum_{i=1}^{N} |f_i(v_j)|) = 0. \quad (7.2.4)$$

希望抽取出它的一个收敛子列，一方面证明其极限函数为0，另一方面又证明其范数为1，进而矛盾！

紧嵌入定理 ⑦ $\mathbb{W}^{k+1,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,p}(\Omega)$, $1 \le p < \infty$; $\mathbb{W}^{k+1,\infty}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^k(\bar{\Omega})$.

Kondrasov-Rellich定理 ($1\le p<\infty$)。
Rellich定理：H1紧嵌入L2

Ascoli定理 ($p=\infty$)

Semi-norm $|\dot{v}|_{k+1,p,\Omega}$ Is an equivalent Norm of $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$

8. So, there exist a subsequence of $\{v_j\}_{j=1}^{\infty}$, denoted again as $\{v_j\}_{j=1}^{\infty}$, and a function $v \in \mathbb{W}^{k,p}(\Omega)$, such that

$$\lim_{j\to\infty} \|v_j - v\|_{k,p,\Omega} = 0. \tag{7.2.5}$$

9. $\overset{(7.2.4)}{⑥}$ implies $\{v_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $\mathbb{W}^{k+1,p}(\Omega)$.

再结合(7.2.5)，W空间的完备知，{vj}在W^{k+1,p}中收敛。

10. Therefore, v in $\underset{(7.2.5)}{⑧}$ is actually a function in $\mathbb{W}^{k+1,p}(\Omega)$.

11. Again, it follows from $\underset{(7.2.4)}{⑥}$ that

$$|\partial^\alpha v|_{0,p,\Omega} = \lim_{j\to\infty} |\partial^\alpha v_j|_{0,p,\Omega} \underset{(7.2.4)}{=} 0, \quad \forall \alpha, \quad |\alpha| = k+1,$$

Semi-norm $|v|_{k+1,p,\Omega}$ Is an equivalent Norm of $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$

Th5.2 区域Ω是一个连通的开域，u的所有m+1阶的广义偏导数均为0， 则u是Ω上的一个次数不超过m的多项式.

12. By Theorem 5.2, 11 implies $v \in \mathbb{P}_k(\Omega)$.
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13. On the other hand, it follows from 6 that
(7.2.4)

$$f_i(v) = \lim_{j\to\infty} f_i(v_j) = 0, \quad i = 1, \ldots, N,$$

14. Therefore, by 2, we have $v = 0$.

矛盾

15. On the other hand, since v_j converges to v in $\mathbb{W}^{k+1,p}(\Omega)$, by 6, we have $\|v\|_{k+1,p,\Omega} = \lim_{j\to\infty} \|v_j\|_{k+1,p,\Omega} = 1$.
(7.2.4)

16. The contradiction of 14 and 15 completes the proof. ■

Relations of Semi-norms on Open Sets Related by $F(\hat{x}) = h\hat{x} + b \in \mathbb{R}^n$

7.2.2节　仿射等价开集上Sobolev空间半范数之间的关系

特例：特殊的仿射变换

1. Let $F : \hat{x} \in \mathbb{R}^n \to \underbrace{F(\hat{x}) = h\hat{x} + b}_{=x} \in \mathbb{R}^n$, and $\Omega = F(\hat{\Omega})$,
   $\Rightarrow \quad \mathrm{diam}(\Omega)/\mathrm{diam}(\hat{\Omega}) = h$.
2. Then, $\partial^\alpha v(x) = h^{-|\alpha|}\partial^\alpha \hat{v}(\hat{x})$, and $dx = |\det(B)|\, d\hat{x} = h^n\, d\hat{x}$.
3. Therefore, by a change of integral variable, we have

积分变量变换

$$\boxed{|v|_{m,p,\Omega} = h^{-m}|\det(B)|^{1/p}|\hat{v}|_{m,p,\hat{\Omega}} = h^{-m+n/p}|\hat{v}|_{m,p,\hat{\Omega}}.} \qquad (7.2.6)$$

变换前后的半范数之比

4. $|v|_{m,p,\Omega}/|\hat{v}|_{m,p,\hat{\Omega}} \propto h^{-m+n/p}$,

符号 “ ∝ ” 表示成正比例；y∝x (读作"y正比于x")

Affine Equivalent Open Sets Related by $F(\hat{x}) = B\hat{x} + b \in \mathbb{R}^n$

一般的仿射变换

Let $\Omega = F(\hat{\Omega})$ be affine equivalent open set in $\mathbb{R}^n$ with 仿射等价开集

仿射变换 $$F : \hat{x} \in \mathbb{R}^n \to \underbrace{F(\hat{x}) \overset{\text{def}}{=} B\hat{x} + b}_{=x} \in \mathbb{R}^n,$$

For $v \in \mathbb{W}^{m,p}(\Omega)$ and $\hat{v}(\hat{x}) = v(F(\hat{x}))$, the Sobolev semi-norms $|v|_{m,p,\Omega}$ and $|\hat{v}|_{m,p,\hat{\Omega}}$ have a similar relation for general B, *i.e.*

$$\underbrace{|v|_{m,p,\Omega}/|\hat{v}|_{m,p,\hat{\Omega}}}_{\text{变换前后的半范数之比}} \propto h^{-m+n/p},$$

符号 “ ∝ ” 表示成正比例；y∝x (读作"y正比于x")

where $h = \mathrm{diam}(\Omega)/\mathrm{diam}(\hat{\Omega})$.

Relations of Semi-norms on Open Sets Related by $F(\hat{x}) = B\hat{x} + b$

Theorem 7.3 (半范数间的关系)

Let Ω *and* $\hat{\Omega}$ *be two affine equivalent open sets in* $\mathbb{R}^n$. *Let* $v \in \mathbb{W}^{m,p}(\Omega)$ *for some* $p \in [1, \infty]$ *and nonnegative integer* m. *Then,* $\hat{v} = v \circ F \in \mathbb{W}^{m,p}(\hat{\Omega})$, *and there exists a constant* $C = C(m, n)$ *such that*

$$\boxed{|\hat{v}|_{m,p,\hat{\Omega}} \leq C\|B\|^m \,|\det(B)|^{-1/p}\, |v|_{m,p,\Omega},} \quad (7.2.7)$$

where B *is the matrix in the affine mapping* F, $\|\cdot\|$ *represents the operator norms induced from the Euclidian norm of* $\mathbb{R}^n$. *Similarly, we also have*

$$\boxed{|v|_{m,p,\Omega} \leq C\|B^{-1}\|^m \,|\det(B)|^{1/p}\, |\hat{v}|_{m,p,\hat{\Omega}}.} \quad (7.2.8)$$

Proof of $|\hat{v}|_{m,p,\hat{\Omega}} \leq C(n,m)\|B\|^m\,|\det(B)|^{-1/p}\,|v|_{m,p,\Omega}$ (7.2.7)

记号

1. Let $\xi_i = (\xi_{i1}, \ldots, \xi_{in})^T \in \mathbb{R}^n$, $i = 1, \cdots, m$, be unit vectors, $D = (\partial_1, \ldots, \partial_n)$, $D^m\hat{v}(\hat{x})(\xi_1, \ldots, \xi_m) \triangleq (\prod_{i=1}^m D \cdot \xi_i)\,\hat{v}(\hat{x})$.
   n维向量的点积

Step 1

2. Assume $v \in \mathbb{C}^m(\bar{\Omega})$, therefore, $\hat{v} \in \mathbb{C}^m(\overline{\hat{\Omega}})$ also. We have

$$|\partial^\alpha \hat{v}(\hat{x})| \leq \|D^m\hat{v}(\hat{x})\| := \sup_{\substack{\|\xi_i\|=1 \\ 1\leq i\leq m}} |D^m\hat{v}(\hat{x})(\xi_1, \ldots, \xi_m)|, \quad \forall|\alpha| = m.$$

3. Let $C_1(m,n)$ be the cardinal number of α, then

$$|\hat{v}|_{m,p,\hat{\Omega}} = \left(\int_{\hat{\Omega}} \sum_{|\alpha|=m} |\partial^\alpha \hat{v}(\hat{x})|^p d\hat{x}\right)^{1/p} \leq C_1(m,n)\left(\int_{\hat{\Omega}} \|D^m\hat{v}(\hat{x})\|^p d\hat{x}\right)^{1/p}.$$

P241: C1(m.n)=C^n_{n+m}-C^n_{n+m-1}=(n/m)*C^n_{n+m-1}; α的基数，即n维空间m重指标α的个数, C1(m,n)=sup_{1≤p} (card{α∈N^n, |α|=m})^{1/p}, Page 118 of Ciarlet's book

4. On the other hand, by the chain rule of differentiations for composition of functions,

$$(D\cdot\xi)\hat{v}(\hat{x}) = D(v\circ F(\hat{x}))\,\xi = Dv(x)\frac{\partial F(\hat{x})}{\partial \hat{x}}\,\xi = (D\cdot B\xi)v(x).$$

5. Therefore, $\left(\prod_{i=1}^m D\cdot\xi_i\right)\hat{v}(\hat{x}) = \left(\prod_{i=1}^m D\cdot B\xi_i\right)v(x)$, *i.e.*

$$D^m\hat{v}(\hat{x})(\xi_1,\ldots,\xi_m) = D^m v(x)(B\xi_1,\ldots,B\xi_m).$$

6. Consequently, $\|D^m\hat{v}(\hat{x})\| \leq \|B\|^m\,\|D^m v(x)\|$.

7. Thus, by a change of integral variable, we obtain

$$\int_{\hat{\Omega}} \|D^m\hat{v}(\hat{x})\|^p d\hat{x} \leq \|B\|^{mp}\,\left|\det\left(B^{-1}\right)\right| \int_{\Omega} \|D^m v(x)\|^p dx.$$

Proof of $|\hat{v}|_{m,p,\hat{\Omega}} \leq C(n,m)\|B\|^m \,|\det(B)|^{-1/p}\, |v|_{m,p,\Omega}$ (7.2.7)

8. For any given $\eta_i \in \mathbb{R}^n$ with $\|\eta_i\| = 1$, $1 \leq i \leq m$, we have

$$D^m v(x)(\eta_1, \ldots, \eta_m) = \left[\prod_{i=1}^{m} \sum_{j=1}^{n} \eta_{ij}\partial_j\right] v(x) = \sum_{j_1,\cdots,j_m=1}^{n} \left[\prod_{i=1}^{m} \eta_{ij_i}\partial_{j_i}\right] v(x).$$

9. Since, $|\eta_{ij}| \leq 1$, $1 \leq i \leq m$, $1 \leq j \leq n$, we have

$$\|D^m v(x)\| \leq n^m \max_{|\alpha|=m} |\partial^\alpha v(x)| \leq n^m \Big(\sum_{|\alpha|=m} |\partial^\alpha v(x)|^p \Big)^{1/p}.$$

Proof of $|\hat{v}|_{m,p,\hat{\Omega}} \leq C(n,m)\|B\|^m\,|\det(B)|^{-1/p}\,|v|_{m,p,\Omega}$ (7.2.7)

⑩ By ③, ⑦ and ⑨, the inequality (7.2.7) hold for $v \in \mathbb{C}^m(\bar{\Omega})$.

Step 2 ⑪ For $1 \leq p < \infty$, $\mathbb{C}^m(\bar{\Omega})$ is dense in $\mathbb{W}^{m,p}(\Omega)$, so the inequality (7.2.7) also holds for all $v \in \mathbb{W}^{m,p}(\Omega)$.

除了稠密性外，还要用到：线性算子L: C^m(\bar{Ω})-->W^{m,p}(\hat{Ω})关于W^{m,p}(Ω)和W^{m,p}(\hat{Ω})的范数连续，映射L唯一延拓到W^{m,p}(Ω)的定义。

Step 3 ⑫ If $p = \infty$, since the inequality (7.2.7) holds uniformly for $1 \leq q < \infty$, and for the bounded domain Ω, it holds

$$\|w\|_{0,\infty,\Omega} = \lim_{q\to\infty} \|w\|_{0,q,\Omega}, \qquad \forall w \in \mathbb{L}^\infty(\Omega),$$

the inequality (7.2.7) holds also for $v \in \mathbb{W}^{m,\infty}(\Omega)$. ■

Ω有界时, W^{m,∞}函数属于W^{m,p}, p≤∞。
\hat{v}属于W^{m,p}(\hat{Ω})，p≤∞，且半范数|\partial^α \hat{v}|_{0.p,\hat{Ω}}的上界不依赖p，故\partial^α \hat{v}属于L^∞, |α|≤m。因此函数\hat{v}属于W^{m,∞}.
-----细节见Page 119 of Ciarlet's book

Bound $\|B\|$ and $\|B^{-1}\|$ by the Interior and Exterior Diameters

利用几何参数来估计仿射变换矩阵B及其逆的范数

1. Denote the exterior and interior diameters of a region Ω as

$$\begin{cases} h_\Omega := \text{diam}\,(\Omega), & \text{Ω的直径} \\ \rho_\Omega := \sup\{\text{diam}\,(S) : S \subset \Omega \ \text{ is a } n\text{-dimensional ball}\}. \end{cases}$$

Theorem 7.4

Let Ω and $\hat{\Omega}$ be two affine-equivalent open sets in $\mathbb{R}^n$, let $F(\hat{x}) = B\hat{x} + b$ be the invertible affine mapping, and $\Omega = F(\hat{\Omega})$. Then,

$$\|B\| \le \frac{h}{\hat{\rho}}, \quad \text{and} \quad \|B^{-1}\| \le \frac{\hat{h}}{\rho}, \tag{7.2.9}$$

where $h = h_\Omega$, $\hat{h} = h_{\hat{\Omega}}$, $\rho = \rho_\Omega$, $\hat{\rho} = \rho_{\hat{\Omega}}$.

Proof of $\|B\| \le \frac{h}{\hat{\rho}}$ and the Geometric Meaning of $\det(B)$

1. By the definition of $\|B\|$, we have
$$\boxed{\|B\| = \frac{1}{\hat{\rho}} \sup_{\|\xi\|=\hat{\rho}} \|B\xi\|.}$$
2. Let the vectors $\hat{x}$, $\hat{y} \in \overline{\hat{\Omega}}$ be such that $\|\hat{y} - \hat{x}\| = \hat{\rho}$, then, we have $x = F(\hat{x}) \in \bar{\Omega}$, $y = F(\hat{y}) \in \bar{\Omega}$.
3. Therefore, $\|B(\hat{y} - \hat{x})\| = \|F(\hat{y}) - F(\hat{x})\| \le h \ \Rightarrow \|B\| \le \frac{h}{\hat{\rho}}$. ■

行列式有明显的几何意义：

The determinant $\det(B)$ also has an obvious geometric meaning:

$$\boxed{|\det(B)| = \frac{\mathrm{meas}(\Omega)}{\mathrm{meas}(\hat{\Omega})}} \quad \text{and} \quad \boxed{|\det(B^{-1})| = \frac{\mathrm{meas}(\hat{\Omega})}{\mathrm{meas}(\Omega)}}.$$

Thank You!