Numerical Solutions to Partial Differential Equations

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A Result on Affine Equivalent Family of Polynomial Invariant Operators

7.2.3节

Theorem 7.5 (多项式不变算子的误差估计)

Let nonnegative integers k, m and $p, q \in [1, \infty]$ *be such that* $\mathbb{W}^{k+1,p}(\hat{\Omega}) \hookrightarrow \mathbb{W}^{m,q}(\hat{\Omega})$. *Let the bounded linear operator* $\hat{\Pi} \in \mathfrak{L}(\mathbb{W}^{k+1,p}(\hat{\Omega});\ \mathbb{W}^{m,q}(\hat{\Omega}))$ *be* $\mathbb{P}_k(\hat{\Omega})$ *invariant, meaning*

Polynomial preserving operator

$$\boxed{\hat{\Pi}\hat{w} = \hat{w}, \qquad \forall\, \hat{w} \in \mathbb{P}_k(\hat{\Omega}).} \qquad (*)$$

Let $\Omega = F(\hat{\Omega})$ *be an arbitrary open set which is affine equivalent to* $\hat{\Omega}$, *where* $F(\hat{x}) = B\hat{x} + b$. *Let the linear operator* $\Pi_\Omega \in \mathfrak{L}(\mathbb{W}^{k+1,p}(\Omega);\ \mathbb{W}^{m,q}(\Omega))$ *be defined by*

$$\boxed{\Pi_\Omega v = \left(\hat{\Pi}\,(v \circ F)\right) \circ F^{-1}.}$$

Then there exists a constant $C = C(\hat{\Pi}, \hat{\Omega}, k, m, n)$ *independent of* Ω, *such that, for all* $v \in \mathbb{W}^{k+1,p}(\Omega)$,

$$\boxed{|v - \Pi_\Omega v|_{m,q,\Omega} \le C\,(\mathrm{meas}(\Omega))^{(1/q-1/p)}\,\frac{h_\Omega^{k+1}}{\rho_\Omega^m}|v|_{k+1,p,\Omega}.} \qquad (7.2.10)$$

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Proof of Error Estimate for Affine-Family of Polynomial Invariant Operators

1. By the polynomial $\mathbb{P}_k(\hat{\Omega})$ invariant of $\hat{\Pi}$, we have
$$\hat{v} - \hat{\Pi}\hat{v} \overset{(*)}{=} (I - \hat{\Pi})(\hat{v} + \hat{w}), \qquad \forall \hat{v} \in \mathbb{W}^{k+1,p}(\hat{\Omega}), \ \forall \hat{w} \in \mathbb{P}_k(\hat{\Omega}).$$

2. By $\mathbb{W}^{k+1,p}(\hat{\Omega}) \hookrightarrow \mathbb{W}^{m,q}(\hat{\Omega})$, $\hat{\Pi} \in \mathfrak{L}(\mathbb{W}^{k+1,p}(\hat{\Omega}); \mathbb{W}^{m,q}(\hat{\Omega}))$, and the semi-norm $|\cdot|_{k+1,p,\hat{\Omega}}$ is an equivalent quotient norm: Th 7.2
$$|\hat{v} - \hat{\Pi}\hat{v}|_{m,q,\hat{\Omega}} \le \|I - \hat{\Pi}\| \inf_{\hat{w} \in \mathbb{P}_k(\hat{\Omega})} \|\hat{v} + \hat{w}\|_{k+1,p,\hat{\Omega}} \underset{\text{算子有界}}{\overset{(7.2.2)}{\le}} C(\hat{\Pi}, \hat{\Omega})|\hat{v}|_{k+1,p,\hat{\Omega}}. \qquad \text{(A1)}$$

3. Since $v - \Pi_\Omega v = (\hat{v} - \hat{\Pi}\hat{v}) \circ F^{-1}$, by Theorem 7.3,[(7.2.7-8)] we have

仿射变换
前后半范
数关系

$$|v - \Pi v|_{m,q,\Omega} \underset{(7.2.8)}{\le} C(m,n)\|B^{-1}\|^m \det(B)^{1/q} |\hat{v} - \hat{\Pi}\hat{v}|_{m,q,\hat{\Omega}}, \qquad \text{(A2)}$$
$$|\hat{v}|_{k+1,p,\hat{\Omega}} \overset{(7.2.7)}{\le} C(k+1,n)\|B\|^{k+1} \det(B)^{-1/p} |v|_{k+1,p,\Omega}. \qquad \text{(A3)}$$

4. The estimate follows as a consequence of the above three inequalities and Theorem 7.4.[(7.2.9)] ■

Def: 称X嵌入(连续地)到Y, 如果X包含于Y, X到Y具有连续内射即存在C>0,
s.t. 对X中任何元素x, x的Y范数不超过x的X范数的C倍: ||x||_Y\leq C||x||_X.
嵌入: Sobolev空间这函数的较"低"阶范数可以被较"高"阶范数控制.

Error Estimates of Affine Family Finite Element Interpolations

Theorem 7.6 (仿射族FE插值的误差估计) Th7.5的应用

Let $(\hat{K}, \hat{P}, \hat{\Sigma})$ be a finite element, let s be the highest order of the partial derivatives appeared in the set of the degrees of freedom $\hat{\Sigma}$. Let nonnegative integers k and m, and $p, q \in [1, \infty]$ be such that

s=0: Lagrange FE;
s=1: Hermite FE

$$\mathbb{W}^{k+1,p}(\hat{K}) \hookrightarrow \mathbb{C}^s(\hat{K}), \quad \text{k+1>s+n/p} \tag{7.2.11}$$

$$\mathbb{W}^{k+1,p}(\hat{K}) \hookrightarrow \mathbb{W}^{m,q}(\hat{K}), \quad \text{k+1<m+n/p} \quad \text{k+1=m+n/p} \tag{7.2.12}$$

$$\mathbb{P}_k(\hat{K}) \subset \hat{P} \subset \mathbb{W}^{m,q}(\hat{K}). \tag{7.2.13}$$

Then, there exists a constant $C(\hat{K}, \hat{P}, \hat{\Sigma})$ such that, for all finite elements (K, P, Σ) which are affine equivalent to $(\hat{K}, \hat{P}, \hat{\Sigma})$, and for all $v \in \mathbb{W}^{k+1,p}(K)$,

$$\boxed{|v - \Pi_K v|_{m,q,K} \le C(\hat{K}, \hat{P}, \hat{\Sigma})\,(\mathrm{meas}(K))^{(1/q-1/p)}\,\frac{h_K^{k+1}}{\rho_K^m}|v|_{k+1,p,K}.} \tag{7.2.14}$$

该不等式与Th7.5中的(7.2.10)对应, Ω换成K; 算子换成插值算子

Proof of Error Estimates for Affine-Family of Finite Element Interpolations

1. By the error estimates for affine-family of polynomial invariant operators (see Theorem 7.5), we only need to verify that the corresponding finite element interpolation operators

仅需验证 $\boxed{\hat{\Pi} = \hat{\Pi}_{\hat{K}} \in \mathfrak{L}(\mathbb{W}^{k+1,p}(\hat{K}); \mathbb{W}^{m,q}(\hat{K}))}.$ (A*)

且有界

2. Let $\{\hat{w}_i\}_{i=1}^N$ be a set of basis of $\hat{P}$, and $\{\hat{\varphi}_i\}_{i=1}^N \subset \widehat{\Sigma}$ be the corresponding dual basis.

3. $\{\hat{\varphi}_i\}_{i=1}^N$ are also bounded linear functionals on $\mathbb{C}^s(\hat{K})$, since s is the highest order of partial derivatives in $\widehat{\Sigma}$.

s=0: Lagrange FE;
s=1: Hermite FE

Proof of Error Estimates for Affine-Family of Finite Element Interpolations

4. By $\hat{\Pi}\hat{v} = \sum_{i=1}^{N} \hat{\varphi}_i(\hat{v})\hat{w}_i \in \hat{P} \underset{(7.2.13)}{\subset} \mathbb{W}^{m,q}(\hat{K})$, and 即(A*)成立
$\mathbb{W}^{k+1,p}(\hat{K}) \underset{(7.2.11)}{\hookrightarrow} \mathbb{C}^s(\hat{K})$, we have, for all $\hat{v} \in \mathbb{W}^{k+1,p}(\hat{K})$,

$$\|\hat{\Pi}\hat{v}\|_{m,q,\hat{K}} \le \sum_{i=1}^{N} |\hat{\varphi}_i(\hat{v})| \, \|\hat{w}_i\|_{m,q,\hat{K}} \overset{\text{phi\_i 有界线性泛函}}{\le}$$

$$C \left(\sum_{i=1}^{N} \|\hat{w}_i\|_{m,q,\hat{K}} \right) \|\hat{v}\|_{s,\infty,\hat{K}} \underset{(7.2.11)}{\le} C_1 \|\hat{v}\|_{k+1,p,\hat{K}}. \quad \blacksquare$$

即算子有界

Relations Between Geometric Parameters of a Finite Element K

- Denote $\sigma_n = \mathrm{meas}\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$, we have

  直径为1的球的测度

$$\sigma_n \rho_K^n \leq \mathrm{meas}\,(K) \leq h_K^n.$$

在FE误差分析中，如果有限元满足一定的几何条件，例如下面给出的正则条件，收敛速度可以由单元K的直径h_K的幂刻画.

- In the finite element error analysis, the convergence rate can be characterized by the powers of the element K's diameter h_K, if the finite elements satisfy certain geometric conditions, for example, the regularity condition given below.

Regular Family of Finite Element Triangulations

一族正则或正规的FE剖分

Definition 7.1 (P244)

$\{\mathfrak{T}_h(\Omega)\}_{h>0}$ is said to be a regular family of finite element triangulations of Ω, if

(i) there exists a constant σ independent of h such that

$$\boxed{h_K \le \sigma\, \rho_K, \qquad \forall K \in \bigcup_{h>0} \mathfrak{T}_h(\Omega);} \tag{7.2.15}$$

(ii) 0 is the accumulation point of the parameter h.
0是参数h的聚点

(7.2.15)中第一格条件意味着：单元K不能任意薄(thin)，或等价地， K的角不能任意小，常数1/sigma是三角剖分中三角形最小角的度量.

Regular Affine Equivalent Family of Finite Element Triangulations

- Let $\{\mathfrak{T}_h(\Omega)\}_{h>0}$ be a regular family of finite element triangulations of Ω. 一族正则的或正规的FE剖分
- Let $\{(K, P_K, \Sigma_K)\}_{K \in \bigcup_{h>0} \mathfrak{T}_h(\Omega)}$ be a family of finite elements, each of which is affine equivalent to the reference finite element $(\hat{K},\ \hat{P},\ \hat{\Sigma})$. 其中的每个都与参考FE仿射等价

Then, the finite elements $\{(K, P_K, \Sigma_K)\}_{K \in \bigcup_{h>0} \mathfrak{T}_h(\Omega)}$ are called a regular affine equivalent family of finite elements.

正则的仿射等价族

Error Estimates of Regular Affine Family of Finite Element Interpolations

Theorem 7.7 (P245) (正则仿射族FE插值的误差估计) <== Def7.1 & Th7.6

Let $\{\mathfrak{T}_h(\Omega)\}_{h>0}$ be a regular family of finite element triangulations of Ω, let $\{(K, P_K, \Sigma_K)\}_{K\in\bigcup_{h>0}\mathfrak{T}_h(\Omega)}$ be a regular affine equivalent family of finite elements with respect to the reference finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$, which satisfies the conditions (7.2.11)–(7.2.13) of Theorem 7.6 (on interpolation error of affine equivalent FEs). Then, there exists a constant $C = C(\hat{K}, \hat{P}, \hat{\Sigma})$ s.t.

$$\|v - \Pi_K v\|_{m,q,K} \leq C\,(\mathrm{meas}\,(K))^{(1/q-1/p)}\,\sigma^m\,h_K^{k+1-m}\,|v|_{k+1,p,K},$$
$$\forall v \in \mathbb{W}^{k+1,p}(K),\quad \forall K \in \bigcup_{h>0} \mathfrak{T}_h(\Omega), \tag{7.2.16}$$

where σ is the constant in the definition of regularity. (7.2.14)+(7.2.15)==>(7.2.16)

Proof of Error Estimates for Regular Affine Family of Finite Elements

1. By the error estimates for affine equivalent family of finite element interpolation operators (Theorem 7.6), we have that, for all $v \in \mathbb{W}^{k+1,p}(K)$,

$$|v - \Pi_K v|_{m,q,K} \overset{(7.2.14)}{\leq} C(\hat{K}, \hat{P}, \widehat{\Sigma})\,(\mathrm{meas}(K))^{(1/q-1/p)} \frac{h_K^{k+1}}{\rho_K^m} |v|_{k+1,p,K}.$$

2. By the regularity of the triangulation, we have

$$\rho_K^{-1} \underset{(7.2.15)}{\leq} \sigma h_K^{-1}, \quad \forall K \in \bigcup_{h>0} \mathfrak{T}_h(\Omega).$$

3. Thus, the conclusion of the theorem follows. ■

7.2.4节 FE函数的反估计

Sobolev空间的函数的较"低"阶范数可以被较"高"阶范数控制.

嵌入定理 **Theorem** 5.5 (P191)

Let Ω be a bounded connected domain with a Lipschitz continuous boundary $\partial\Omega$, then

$$\mathbb{W}^{m+k,p}(\Omega) \hookrightarrow \mathbb{W}^{k,q}(\Omega), \ \forall\, 1 \le q \le \frac{np}{n-mp}, \ k \ge 0, \quad \text{if } \ m < n/p;$$

$$\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,q}(\Omega), \ \forall\, 1 \le q < \frac{np}{n-mp}, \ k \ge 0, \quad \text{if } \ m < n/p;$$

$$\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,q}(\Omega), \ \forall\, 1 \le q < \infty, \ k \ge 0, \quad \text{if } \ m = n/p;$$

$$\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^{k}(\overline{\Omega}), \quad \forall\, k \ge 0, \quad \text{if } \ m > n/p.$$

对通常的函数，低阶模可用高阶模来估计，如Poincare-Friedrichs不等式 & 嵌入定理; 但反之不行，即一般不能用低阶模去估计高阶模，例如函数值本身一致有界的函数列，其导数可以无界。 然而对有限元空间中的函数，即分片多项式函数，却可以用低阶模去估计高阶模。这是有限元空间中的函数的特有性质，即所谓的反不等式，很有用处。

7.2.4节 FE函数的反估计

Remark:

By the theorem (see Theorem 5.5), a "lower" order norm of a Sobolev function can be bounded by its "higher" order norms.

嵌入定理

Sobolev空间函数的较"低"阶范数可以被较"高"阶范数控制.

Generally speaking, the reverse inequalities do not hold. However, we can prove the inverse inequalities on certain finite element function spaces.

一般地讲，反过来的不等式不成立. 但是，我们能在适当的FE函数空间里证明相反的不等式.

教材的“7.5节补充与注记”中提及反估计，课件中提及引理8.1的证明要用它，但都没具体使用或详细介绍。书[R. Verfurth: *A Posteriori Error Estimation Techniques for Finite Element Methods*, OUP, 2013] 的第3.6节(第112页)：所有后验误差估计的下界都依赖于加权Lp范数(适当的局部截止函数作为权重函数)的逆/反估计。这些逆估计中常数的显式的且强的(sharp)界对于计算后验误差下界中的乘性常数是至关重要的。逆估计通常通过转换到适当的参考元并利用有限维空间上范数的等价性来证明。这一证明相当简单，可以洞察常数对单元几何结构的依赖性。然而，它没有提供关于多项式次数依赖的信息(可通过应用降维参数和借助一维Legendre多项式的性质来获得这些信息)。

Family of Quasi-Uniform Finite Element Triangulations

Recall, $\{\mathfrak{T}_h(\Omega)\}_{h>0}$ is a regular family of FE triangulations of Ω, if
一族正则的或正规的FE剖分

(i) there exists a constant σ independent of h such that

$$h_K \leq \sigma\, \rho_K, \qquad \forall K \in \bigcup_{h>0} \mathfrak{T}_h(\Omega); \tag{7.2.15}$$

0是参数h的聚点

(ii) 0 is the accumulation point of the parameter h.

Definition 7.2 (P245)

一族正则的或正规的FE剖分

Let $\{\mathfrak{T}_h(\Omega)\}_{h>0}$ be a regular family of finite element triangulations of Ω. If there exists a constant γ such that

$$\max_{K'\in\mathfrak{T}_h(\Omega)} h_{K'} \leq \gamma\, h_K, \qquad \forall K \in \mathfrak{T}_h(\Omega),\ \ \forall h > 0, \tag{7.2.17}$$

then, $\{\mathfrak{T}_h(\Omega)\}_{h>0}$ is called a family of quasi-uniform finite element triangulations of Ω.

一族拟一致的正则的FE剖分

王&许P123的拟一致条件(也称为"反假设", 因为h\geq h_K)：存在常数nu>0, s.t. h\leq nu * h_K, 对任意K, h趋于0，其中h_K是K的直径, h等于max_K {h_K}

Theorem 7.8 (P245)

Let $\{\mathfrak{T}_h(\Omega)\}_{h>0}$ *be a family of quasi-uniform finite element triangulations of a bounded open set* Ω *in* $\mathbb{R}^n$, $\mathbb{V}_h(\Omega)$ *be the FE function spaces on* $\mathfrak{T}_h(\Omega)$, *where the FEs* (K, P_K, Σ_K) *are affine equivalent to the reference FE* $(\hat{K}, \hat{P}, \hat{\Sigma})$. *Let integers* $1 \le l \le m$ *and* $p,\ q \in [1, \infty]$ *satisfy* $\underline{\mathbb{P}_{l-1}(\hat{K}) \subset \hat{P} \subset \mathbb{W}^{l,p}(\hat{K}) \cap \mathbb{W}^{m,q}(\hat{K})}$. (7.2.18)

Then, there exists a constant $C(\sigma, \gamma, l, m)$, *where* σ *and* γ *are the regularity and quasi-uniform constants, such that, for all* $q < \infty$,

$$\Big(\underbrace{\sum_{K\in\mathfrak{T}_h(\Omega)} |v|^q_{m,q,K}}_{\text{高阶模}}\Big)^{1/q} \le C(\sigma,\gamma,l,m)\, h^{l-m-s}\Big(\underbrace{\sum_{K\in\mathfrak{T}_h(\Omega)} |v|^p_{l,p,K}}_{\text{低阶模}}\Big)^{1/p}, \quad \forall v \in \mathbb{V}_h(\Omega), \tag{7.2.19}$$

where $s = \max\{0, n\,(1/p - 1/q)\}$; *and for* $q = \infty$,

$$\max_{K\in\mathfrak{T}_h(\Omega)} \{|v|_{m,\infty,K}\} \le C(\sigma,\gamma,l,m)\, h^{l-m-n/p}\Big(\sum_{K\in\mathfrak{T}_h(\Omega)} |v|^p_{l,p,K}\Big)^{1/p}, \quad \forall v \in \mathbb{V}_h(\Omega). \tag{7.2.20}$$

对某些FE，函数的较"高"阶范数也可以被较"低"阶范数控制.

Proof of the Inverse Inequalities of Finite Element Functions

Th7.3: 半范数间的关系；Th7.4: 利用几何参数来估计仿射变换矩阵B及其逆的范数

1. It follows from the relations of semi-norms on affine equivalent open sets (see Th7.3 & Th7.4) and the regularity of the triangulation that, there exists a constant C_0, which depends only on σ, γ, l, m, such that

由Th7.3-7.4和剖分的正则性==>

(7.2.7)--> $$|\hat{v}_K|_{l,p,\hat{K}} \leq C_0\, h_K^{l-n/p} |v|_{l,p,K}, \quad (7.2.21)$$

(7.2.8)--> $$|v|_{m,q,K} \leq C_0\, h_K^{-m+n/q} |\hat{v}_K|_{m,q,\hat{K}}, \quad (7.2.22)$$

where $\hat{v}_K = v \circ F_K$, $F_K : K \to \hat{K}$ are the corresponding affine mappings.

Proof of the Inverse Inequalities of Finite Element Functions

2. It follows from the equivalent norm on the polynomial quotient space (see Theorem 7.2) that, there exists a constant $C_1 = C(\hat{K})$ such that the quotient norm $\|\dot{w}\|_{l,p,\hat{K}}$ of the quotient space $\hat{P}/\mathbb{P}_{l-1}(\hat{K})$ satisfies

多项式商空间的等价范数

(7.2.2)-->
$$\boxed{\|\dot{w}\|_{l,p,\hat{K}} \leq C_1 \, |w|_{l,p,\hat{K}}, \quad \forall w \in \hat{P},} \tag{7.2.23}$$

where $\dot{w}$ is the equivalent class of w in the quotient space $\hat{P}/\mathbb{P}_{l-1}(\hat{K})$. (In fact, we have 直和 $\hat{P} = \hat{P}/\mathbb{P}_{l-1}(\hat{K}) \oplus \mathbb{P}_{l-1}(\hat{K})$).

3. On the other hand, $|w|_{m,q,\hat{K}} = 0$, $\forall w \in \mathbb{P}_{l-1}(\hat{K})$, since $l \leq m$. (*1)
次数不超过l-1多项式w的m次偏导数为0

4. Let $\{\hat{w}_i\}_{i=1}^{M} \subset \hat{P}$ be a basis of $\hat{P}$ with $\{\hat{w}_i\}_{i=1}^{L} \subset \mathbb{P}_{l-1}(\hat{K})$ being a basis of $\mathbb{P}_{l-1}(\hat{K})$. (*2)

线性空间V的子空间M和N的直和: 如果和M+N的每个向量a的分解式a=b+c, b ∈ M, c ∈ N是唯一的. M⊕N

5. Let $\{\varphi_i\}_{i=1}^M \subset \hat{\Sigma}$ be the corresponding dual basis of $\{\hat{w}_i\}_{i=1}^M$, (*3)
meaning $\varphi_i(\hat{w}_j) = \delta_{ij}$, $i, j = 1, \ldots, M$.

6. $\|\dot{w}\|_{m,q,\hat{K}} := |\dot{w}|_{m,q,\hat{K}} + \sum_{i=L+1}^M |\varphi_i(w)|$ defines a norm on $\hat{P}/\mathbb{P}_{l-1}(\hat{K})$, since $w \in \hat{P}$, $\|\dot{w}\|_{m,q,\hat{K}} = 0 \Rightarrow \varphi_i(w) = 0$,
(7.2.18)-->次数不超过l-1多项式w的m次偏导数为0
$i = L+1, \ldots, M \underset{直和}{\Rightarrow} w \in \mathbb{P}_{l-1}(\hat{K})$.

7. Since any two norms on a finite dimensional space are equivalent, there exists a constant $C_2 = C(l, m)$ such that

$$|w|_{m,q,\hat{K}} \underset{定义}{=} |\dot{w}|_{m,q,\hat{K}} \leq \|\dot{w}\|_{m,q,\hat{K}} \leq C_2 \|\dot{w}\|_{l,p,\hat{K}} \underset{(7.2.23)}{\overset{商空间范数等价}{\leq}} C_1 C_2 |w|_{l,p,\hat{K}}, \quad \forall w \in \hat{P}. \tag{7.2.24}$$

8. By ①, ⑦, see (7.2.21), (7.2.22) and (7.2.24), and the quasi-uniformness of the triangulation, we have

$$\boxed{|v|_{m,q,K} \leq C\, h^{l-m-n(1/p-1/q)} |v|_{l,p,K}, \quad \forall v \in P_K, \ \forall K \in \mathfrak{T}_h(\Omega).} \tag{7.2.25}$$

此时，函数的较"高"阶半范数也可以被较"低"阶半范数控制.

Proof of the Inverse Inequalities of Finite Element Functions

(7.2.25) ➡第1种情况是q=∞

9. Hence, the conclusion of the theorem for $q = \infty$ follows. 即(7.2.20)

(7.2.19)的证明的三种情况

10. For $p \leq q < \infty$, the conclusion of the theorem follows as a consequence of ⑧ (7.2.25) and the Jensen's inequality P247或王烈衡&许学军书P127

$$\Big(\sum_{K \in \mathfrak{T}_h(\Omega)} |v|^q_{l,p,K} \Big)^{1/q} \leq \Big(\sum_{K \in \mathfrak{T}_h(\Omega)} |v|^p_{l,p,K} \Big)^{1/p}.$$

11. If $q < p < \infty$, it follows from the Hölder's inequality that

$$\Big(\sum_{K \in \mathfrak{T}_h(\Omega)} |v|^q_{l,p,K} \Big)^{1/q} \leq C(\mathfrak{T}_h(\Omega))^{(1/q-1/p)} \Big(\sum_{K \in \mathfrak{T}_h(\Omega)} |v|^p_{l,p,K} \Big)^{1/p}, \tag{7.2.26}$$

where $C(\mathfrak{T}_h(\Omega))$ is the $\#\mathfrak{T}_h(\Omega)$.即元素的个数

Proof of the Inverse Inequalities of Finite Element Functions

12. Since the triangulations are quasi-uniform, there exists a constant $C_3 = C(\sigma, \gamma)$ such that $C(\mathfrak{T}_h(\Omega)) \leq C_3\, h^{-n}$.

13. Hence, the conclusion of the theorem for $q < p < \infty$ follows from ⑧ and ⑪, see (7.2.25) and (7.2.26).

14. For $q < p = \infty$, the conclusion of the theorem follows as a consequence of ⑧, see (7.2.25), and

$$\Big(\sum_{K \in \mathfrak{T}_h(\Omega)} |v|^q_{l,\infty,K} \Big)^{1/q} \leq C(\mathfrak{T}_h(\Omega))^{1/q} \max_{K \in \mathfrak{T}_h(\Omega)} |v|_{l,\infty,K}. \quad \blacksquare$$

教材的“7.5节补充与注记”中提及反估计，课件中提及引理8.1的证明要用它，但都没具体使用或详细介绍。书[R. Verfurth: *A Posteriori Error Estimation Techniques for Finite Element Methods*, OUP, 2013] 的第3.6节(第112页)：所有后验误差估计的下界都依赖于加权Lp范数(适当的局部截止函数作为权重函数)的逆/反估计。

7.3节 多边形区域上2阶问题FE解的误差估计

We Restrict Ourselves to the Simplest Ideal Situations

1. The second order linear elliptic problems defined on polygonal domains in $\mathbb{R}^n$.

   多角区域/多边形区域

   有的要求凸的多角区域/多边形区域，见姜礼尚等书

2. The conditions in the Céa lemma are satisfied.

3. The domain Ω is exactly triangulated into polygonal finite elements.

   多角/多边形有限元

4. The Dirichlet boundary $\partial\Omega_0$ consists exactly of some $(n-1)$-dimensional faces of the corresponding triangulation.

5. For further simplification, consider only the homogeneous Dirichlet boundary value problems, so that the function space of the corresponding variational problem is $\mathbb{V} = \mathbb{H}_0^1(\Omega)$.

6. Use only class $\mathbb{C}^0$ conforming finite elements, so that the finite element function spaces satisfy $\mathbb{V}_h \subset \mathbb{V}$.

   C0协调有限元

Interpolation Error Estimates on Finite Element Function Spaces

Theorem 7.9 (P248)

Let $\{(K, P_K, \Sigma_K)\}_{K\in\bigcup_{h>0}\mathfrak{T}_h(\Omega)}$ *be a family of regular affine* 一族正则的仿射等价有限元 *equivalent finite elements with* $(\hat{K}, \hat{P}, \hat{\Sigma})$ *being the reference finite element. Assume that there exist nonnegative integers* k, l *such that* $\mathbb{P}_k(\hat{K}) \subset \hat{P} \subset \mathbb{H}^l(\hat{K}), \qquad \mathbb{H}^{k+1}(\hat{K}) \hookrightarrow \mathbb{C}^s(\hat{K}),$ (7.3.1) (7.3.2)
where s *is the highest order of the partial derivatives appeared in the set of degrees of freedom* $\hat{\Sigma}$. *Then, there exists a constant* C *independent of* h *such that, for all* $v \in \mathbb{H}^{k+1}(\Omega) \cap \mathbb{V}$,

$$\|v - \Pi_h v\|_{m,\Omega} \le C\, h^{k+1-m}\, |v|_{k+1,\Omega}, \quad 0 \le m \le \min\{1,\, l\}, \tag{7.3.3}$$

对C0的FE，只能属于H1，此时有范数||*||0 和||*||1, 因而，限于C0有限元时，该不等式只对m=0,1成立

$$\Big(\sum_{K\in\mathfrak{T}_h(\Omega)} \|v - \Pi_h v\|_{m,K}^2\Big)^{1/2} \le C h^{k+1-m}\, |v|_{k+1,\Omega},\ 2 \le m \le \min\{k+1,\, l\}, \tag{7.3.4}$$

where Π_h *is the* $\mathbb{V}_h$ *interpolation operator.*

Proof of Interpolation Error Estimates on Finite Element Function Spaces

1. For $p = q = 2$, $\mathbb{P}_k(\hat{K}) \subset \hat{P} \subset \mathbb{H}^l(\hat{K})$, $\mathbb{H}^{k+1}(\hat{K}) \hookrightarrow \mathbb{C}^s(\hat{K}) \Rightarrow$ 嵌入关系式 P244
   the conditions (7.2.11)–(7.2.13) of the theorem on error estimates of affine family finite element interpolations (see Th7.6) hold for $m \leq \min\{k+1, l\}$.

Th7.6 仿射族有限元插值的误差估计

Th7.7 正则仿射族有限元插值的误差估计

2. Thus, by the theorem on error estimates of regular affine family finite element interpolations (see Th7.7), we have

$$\|v - \Pi_K v\|_{m,K} \leq C\, h_K^{k+1-m}\, |v|_{k+1,K}, \quad 0 \leq m \leq \min\{k+1, l\}.$$

Proof of Interpolation Error Estimates on Finite Element Function Spaces

3. Since, by definition, $(\Pi_h v)|_K = \Pi_K(v|_K)$, $h_K \le h$, $\forall K \in \mathfrak{T}_h(\Omega)$, this leads to

$$\Big(\sum_{K\in\mathfrak{T}_h(\Omega)} \|v-\Pi_h v\|^2_{m,K}\Big)^{1/2} \le Ch^{k+1-m}\,|v|_{k+1,\Omega},\ 0 \le m \le \min\{k+1, l\},$$

==〉 (7.3.4)

注: 对C0元，m=0,1时，可进一步写为(7.3.3), 见下面.

对m=0,1 4. For class $\mathbb{C}^0$ finite elements, and $0 \le m \le \min\{1, l\}$, we have

对C0的FE，只能属于H1，此时有范数||*||0 和||*||1, 因而，限于C0有限元时，该等式只对m=0,1成立

$$\|v - \Pi_h v\|_{m,\Omega} = \Big(\sum_{K\in\mathfrak{T}_h(\Omega)} \|v - \Pi_h v\|^2_{m,K}\Big)^{1/2}$$

==〉 (7.3.3) ∎

Application of General Result to 2nd Order Problem ($n \leq 3$)

1. For class $\mathbb{C}^0$ affine equivalent finite elements or finite elements which embed into an affine family, for example, the complete or incomplete type (k) n-simplex, type (k) n-rectangle, etc., by taking $l = 1$, we get the error estimates

$$\boxed{\|v-\Pi_h v\|_{m,\Omega} \leq C\, h^{k+1-m} |v|_{k+1,\Omega}, \quad m = 0, 1, \ \forall v \in \mathbb{H}^{k+1}(\Omega) \cap \mathbb{V}.} \qquad (7.3.5)$$

2. In particular, for type (1) Lagrange finite elements, we have

$$\boxed{\|v-\Pi_h v\|_{m,\Omega} \leq C\, h^{2-m} |v|_{2,\Omega}, \qquad m = 0, 1, \ \forall v \in \mathbb{H}^{2}(\Omega) \cap \mathbb{V}.} \qquad (7.3.6)$$

3. In general, the finite element interpolation error in the $\mathbb{L}^2(\Omega)$ norm is an order higher than that in the $\mathbb{H}^1(\Omega)$ norm.

对C0类仿射等价FE族或能够嵌入到某个仿射族的C0类FE所构造的FE函数空间，可以在Th7.9中取l=1，由此有常用的FE函数空间上的插值误差估计(7.3.5). 特别地, 如果取型(1)的Lagrange有限元，可得插值误差估计(7.3.6). ==>一般地, L2模下的误差比其在H1模下的误差高一阶.

Error Estimates for Regular Affine Family Finite Element Solutions

7.3.1节 H1模(或范数)下的误差估计

Theorem 7.10 (H1模或范数下的误差估计P249)

一族正则的仿射等价有限元

Let $\{(K, P_K, \Sigma_K)\}_{K\in\bigcup_{h>0}\mathfrak{T}_h(\Omega)}$ *be a family of regular affine equivalent finite elements with* $(\hat{K}, \hat{P}, \hat{\Sigma})$ *being the reference finite element. Assume that there exists an integer* $k \geq 1$ *such that*

$$\mathbb{P}_k(\hat{K}) \subset \hat{P} \subset \mathbb{H}^1(\hat{K}),$$
$$\mathbb{H}^{k+1}(\hat{K}) \hookrightarrow \mathbb{C}^s(\hat{K}),$$

where s *is the highest order of the partial derivatives appeared in the set of the degrees of freedom* $\widehat{\Sigma}$*. Assume that the solution* u *of the variational problem is in* $\mathbb{H}^{k+1}(\Omega) \cap \mathbb{V}$*. Then, there exists a constant* C *independent of* h *such that*

由Cea引理和插值误差估计==〉

$$\boxed{\|u - u_h\|_{1,\Omega} \leq C\, h^k\, |u|_{k+1,\Omega},} \quad (7.3.7)$$

与插值误差同阶

where $u_h \in \mathbb{V}_h \subset \mathbb{V}$ *is a finite element solution of the variational problem.*

Proof of Error Estimates for Regular Affine Family Finite Element Solutions

1. By the Céa lemma (see Th7.1), we have

$$\|u - u_h\|_{1,\Omega} \overset{(7.1.3)}{\leq} C \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_{1,\Omega} \leq C \, \|u - \Pi_h u\|_{1,\Omega}.$$

2. By the error estimates for regular affine family finite element interpolations (see Th7.9, in particular (7.3.5) P249), since $u \in \mathbb{H}^{k+1}(\Omega) \cap \mathbb{V}$, we have

$$\|u - \Pi_h u\|_{1,\Omega} \leq C \, h^k \, |u|_{k+1,\Omega}.$$

3. Thus, the conclusion of the theorem follows. ∎

Proof of Error Estimates for Regular Affine Family Finite Element Solutions

Remark:

If the solution u of the variational problem has higher regularity, *i.e.* smoother, then, we can obtain finite element solutions with higher order of accuracy by selecting finite elements containing higher order polynomial function spaces.

如果变分问题的解u具有更高的正则性，即光滑性，则可以通过选择含有高阶多项式函数空间的有限元来获得具有高精度的有限元解。

Application of the General Result to $u \in \mathbb{H}^2(\Omega) \cap \mathbb{V}$ and $n \leq 3$

1. By the Sobolve embedding theorem, $\mathbb{W}^{m+s,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^s(\overline{\Omega})$, $\forall\ s \geq 0$, if $m > n/p$. Th5.5 P191
2. Here, we have $m = 2$, $n \leq 3$ and $p = 2$, so $2 > 3/2$ implies $\mathbb{H}^2(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^0(\overline{\Omega})$.
3. 考虑C0型(1) Lagrange FE For class $\mathbb{C}^0$ type (1) Lagrange finite elements, we have $s = 0$, $k = 1$ and $\hat{P} = \mathbb{P}_1(\hat{K})$.
4. Thus, the conditions of Theorem 7.10: $\mathbb{P}_k(\hat{K}) \subset \hat{P} \subset \mathbb{H}^1(\hat{K})$ and $\mathbb{H}^{k+1}(\hat{K}) \hookrightarrow \mathbb{C}^s(\hat{K})$ are satisfied for $k = 1$, $s = 0$.
5. As a consequence, we have $\|u - u_h\|_{1,\Omega} \leq C\, h\, |u|_{2,\Omega}$. (7.3.8)
   得到了丰满的误差估计(阶和插值误差, 见(7.3.6)相同)
6. FE解的误差估计是最优/丰满的[它与插值误差同阶] In this case, the error estimate for the finite element solution is optimal, meaning that the error estimate of the finite element 有限元解的误差估计与有限元函数空间中实解插值的误差估计阶数相同。 solution is of the same order as that of the interpolation of the real solution in the finite element function space.

Finite Element Solution u Is Not in $\mathbb{H}^2(\Omega) \cap \mathbb{V}$ in General

1. For general variational problems, the condition $u \in \mathbb{H}^2(\Omega)$ is not always satisfied. For example, the Poisson equation on a 星形区域 star shaped domain.

在最小/弱的假设下，仍可以证明收敛性，即使得不到FE解的逼近阶.

2. Under the minimal assumption that the solution $u \in \mathbb{H}^1(\Omega) \cap \mathbb{V}$, we can still prove the convergence of the finite element solutions, even though the order of the approximation accuracy is no longer available.

Convergence of Finite Element Solutions when u is only in $\mathbb{H}^1(\Omega)\cap\mathbb{V}$

Theorem 7.11 (非光滑解的收敛性)

Let $\{(K, P_K, \Sigma_K)\}_{K\in\bigcup_{h>0}\mathfrak{T}_h(\Omega)}$ be a family of regular class $\mathbb{C}^0$ affine equivalent finite elements with the reference finite element $(\hat{K},\ \hat{P},\ \hat{\Sigma})$ satisfying: $\mathbb{P}_1(\hat{K})\subset\hat{P}\subset\mathbb{H}^1(\hat{K})$, and there is no partial derivatives of order greater than or equal to 2 in $\hat{\Sigma}$. Then,

自由度集合hat{Sigma}中不含大于等于二阶的偏导数

$$\lim_{h\to 0}\|u-u_h\|_{1,\Omega}=0. \tag{7.3.9}$$

Proof of $\lim_{h\to 0}\|u-u_h\|_{1,\Omega}=0$ when u is only in $\mathbb{H}^1(\Omega)\cap\mathbb{V}$

1. Take $k=1$, $m=1$, $q=2$, $p=\infty$, $s=0$ or 1 accordingly. We have, see conditions (7.2.11)–(7.2.13) in Th7.6, Th7.7,

函数空间的包含关系

$$\mathbb{W}^{2,\infty}(\hat{K}) \hookrightarrow \mathbb{C}^s(\hat{K}), \quad s=0,\,1$$
$$\mathbb{W}^{2,\infty}(\hat{K}) \hookrightarrow \mathbb{H}^1(\hat{K}),$$
$$\mathbb{P}_k(\hat{K}) \subset \hat{P} \subset \mathbb{H}^1(\hat{K}).$$

2. Thus, by the interpolation error estimates on regular affine equivalent family of finite elements (see Th7.7), for all $v\in \mathbb{W}^{2,\infty}(\Omega)\cap\mathbb{V}$, we have

正则的仿射等价族

$$\|v-\Pi_h v\|_{1,\Omega}=\Big\{\sum_{K\in\mathfrak{T}(\Omega)}\|v-\Pi_K v\|_{1,K}^2\Big\}^{1/2}\le Ch(\mathrm{meas}\,(\Omega))^{1/2}|v|_{2,\infty,\Omega}.$$

This implies $$\lim_{h\to 0}\|v-\Pi_h v\|_{1,\Omega}=0,\quad \forall v\in\mathbb{W}^{2,\infty}(\Omega)\cap\mathbb{V}.$$

Proof of $\lim_{h\to 0} \|u - u_h\|_{1,\Omega} = 0$ when u is only in $\mathbb{H}^1(\Omega) \cap \mathbb{V}$

3. Since $u \in \mathbb{H}^1(\Omega) \cap \mathbb{V}$ and $\mathbb{W}^{2,\infty}(\Omega) \cap \mathbb{V}$ is dense in $\mathbb{H}^1(\Omega) \cap \mathbb{V}$, for any given $\varepsilon > 0$, there exists a $v_\varepsilon \in \mathbb{W}^{2,\infty}(\Omega) \cap \mathbb{V}$ such that
$$\|u - v_\varepsilon\|_{1,\Omega} < \varepsilon/2.$$
4. For a fixed v_ε, by ②, there exists an $h(\varepsilon) > 0$ such that $\|v_\varepsilon - \Pi_h v_\varepsilon\|_{1,\Omega} < \varepsilon/2$, if $0 < h < h(\varepsilon)$.
5. Consequently
$$\|u - \Pi_h v_\varepsilon\|_{1,\Omega} \leq \|u - v_\varepsilon\|_{1,\Omega} + \|v_\varepsilon - \Pi_h v_\varepsilon\|_{1,\Omega} < \varepsilon, \quad \forall h \in (0,\ h(\varepsilon)).$$
6. Therefore, we conclude $\displaystyle\lim_{h\to 0} \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_{1,\Omega} = 0.$
7. The conclusion of the theorem follows now from the Céa lemma. ■

Thank You!