Numerical Solutions to Partial Differential Equations

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Chapter 6 FEM for elliptic BVP

School of Mathematical Sciences
Peking University

FEM已成为当前求PDE数值解的一个重要方法．它属于变分方法的范畴，是古典变分方法(Ritz-Galerkin方法)与分片多项式插值结合的产物．这种结合不仅使FEM保持了原有变分法的优点，而且还兼顾了FDM的灵活性，使得古典变分法的不足之处得到了充分的弥补．

Variational Problems of the Dirichlet BVP of the Poisson Equation

1. For the homogeneous Dirichlet BVP of the Poisson equation

$$\begin{cases} -\triangle u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

2. The weak form w.r.t. the virtual work principle:

虚功原理 →

$$\boxed{\begin{cases} \text{Find} \quad u \in \mathbb{H}_0^1(\Omega), \quad \text{such that} \\ a(u, v) = (f, v), \quad \forall v \in \mathbb{H}_0^1(\Omega), \end{cases}} \tag{6.1.1}$$

where $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx$, $(f, v) = \int_\Omega fv \, dx$.

3. The weak form w.r.t. the minimum potential energy principle:

最小势能原理 →

$$\boxed{\begin{cases} \text{Find} \quad u \in \mathbb{H}_0^1(\Omega), \quad \text{such that} \\ J(u) = \min_{v \in \mathbb{H}_0^1(\Omega)} J(v), \end{cases}} \tag{6.1.2}$$

where $J(v) = \frac{1}{2} a(v, v) - (f, v)$.

虚功原理: 对于一个静态平衡的系统，所有外力的作用，经过虚位移，所作的虚功，总和等于零.它又称虚位移原理, 在力学中与最小能量原理同属变分原理. 在动力学里, 也有一个对应的原理, 叫做达朗伯特原理，它是拉格朗日力学的理论基础. 最小势能原理: 在一个保守系统的所有可能位移场中, 真实位移场的总势能取最小值. 当一个体系的势能最小时，系统会处于稳定平衡状态.

Use Finite Dimensional Trial, Test and Admissible Function Spaces

试探函数空间 & 检验函数空间

1. Replace the trial and test function spaces by appropriate finite dimensional subspaces, say $\mathbb{V}_h(0) \subset \mathbb{H}_0^1(\Omega)$, we are led to the discrete problem:

虚功原理 →

$$\boxed{\begin{cases} \text{Find} \quad u_h \in \mathbb{V}_h(0) \quad \text{such that} \\ a(u_h,\ v_h) = (f,\ v_h), \quad \forall v_h \in \mathbb{V}_h(0), \end{cases}} \tag{6.1.3}$$

Such an approach is called the Galerkin method.

2. Replace the admissible function space by an appropriate finite dimensional subspace, say $\mathbb{V}_h(0) \subset \mathbb{H}_0^1(\Omega)$, we are led to the discrete problem:

极小势能原理 →

$$\boxed{\begin{cases} \text{Find} \quad u_h \in \mathbb{V}_h(0) \quad \text{such that} \\ J(u_h) = \min_{v_h \in \mathbb{V}_h(0)} J(v_h). \end{cases}} \tag{6.1.4}$$

Such an approach is called the Ritz method.

3. Two methods lead to an equivalent system of linear algebraic equations.

当a(u,v)是对称时，上述两个方法才给出等价的线性代数方程组。Galerkin方法更具有一般性, Ritz方法要求a(u,v)对称.

Derivation of Algebraic Equations of the Galerkin Method

Let $\{\varphi_i\}_{i=1}^{N_h}$ be a set of basis functions of $\mathbb{V}_h(0)$, let

$$u_h = \sum_{j=1}^{N_h} u_j \varphi_j, \quad v_h = \sum_{i=1}^{N_h} v_i \varphi_i,$$

then, the Galerkin method leads to

$$\begin{cases} \text{Find } \ \mathbf{u}_h = (u_1, \ldots, u_{N_h})^T \in \mathbb{R}^{N_h} \ \text{ such that} \\ \sum_{i,j=1}^{N_h} a(\varphi_j, \varphi_i) u_j v_i = \sum_{i=1}^{N_h} (f, \varphi_i) v_i, \quad \forall \mathbf{v}_h = (v_1, \ldots, v_{N_h})^T \in \mathbb{R}^{N_h}, \end{cases}$$

which is equivalent to $\boxed{\sum_{j=1}^{N_h} a(\varphi_j, \varphi_i) u_j = (f, \varphi_i), \ i = 1, 2, \cdots, N_h.}$ (6.1.5)

vh取为基函数

刚度矩阵
荷载向量
位移向量

- The stiffness matrix: $K = (k_{ij}) = (a(\varphi_j, \varphi_i))$; the external load vector: $\mathbf{f}_h = (f_i) = ((f, \varphi_i))$; the displacement vector: $\mathbf{u}_h$; the linear algebraic equation: $\boxed{K\,\mathbf{u}_h = \mathbf{f}_h}$. (6.1.6)

Derivation of Algebraic Equations of the Ritz Method

1. The Ritz method leads to a finite dimensional minimization problem, whose stationary points satisfy the equation given by the Galerkin method, and vice versa.

P208 2. It follows from the symmetry of $a(\cdot, \cdot)$ and the Poincaré-Friedrichs inequality (see Theorem 5.4) P190 that stiffness matrix K is a symmetric positive definite matrix, and thus the linear system has a unique solution, which is a minima of the Ritz problem.

3. So, the Ritz method also leads to $K\,\mathbf{u}_h = \mathbf{f}_h$. (6.1.6)

The Key Is to Construct Finite Dimensional Subspaces

There are many ways to construct finite dimensional subspaces for the Galerkin method and Ritz method. For example

1. For $\Omega = (0, 1) \times (0, 1)$, the functions
$$\varphi_{mn}(x, y) = \sin(m\pi x)\sin(n\pi y), \quad m, n \geq 1,$$
which are the complete family of the eigenfunctions $\{\varphi_i\}_{i=1}^{\infty}$ of the corresponding eigenvalue problem
$$\begin{cases} -\triangle u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$
and form a set of basis of $\mathbb{H}_0^1(\Omega)$.
2. Define $\mathbb{V}_N = \text{span}\{\varphi_{mn} : m \leq N, n \leq N\}$, the corresponding numerical method is called the spectral method.
3. Finite element method is a systematic way to construct subspaces for more general domains.

Jie Shen, Tao Tang and Li-Lian Wang, Spectral Methods: Algorithms, Analysis and Applications (Springer Series in Computational Mathematics, V. 41), Springer, Aug. 2011

Construction of a Finite Element Function Space for $\mathbb{H}_0^1([0,1]^2)$

6.2.1节
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1. The Dirichlet boundary value problem of the Poisson equation

$$-\triangle u = f, \quad \forall x \in \Omega = (0,1)^2, \qquad u = 0, \quad \forall x \in \partial\Omega. \qquad (6.1.1)$$

2. We need to construct a finite element subspace of $\mathbb{H}_0^1((0,1)^2)$.

3. Firstly, introduce a triangulation $\mathfrak{T}_h(\Omega)$ on the domain $\overline{\Omega}$:

Triangular element $\{T_i\}_{i=1}^M$;

任两个单元的内部不相交 $\mathring{T}_i \cap \mathring{T}_j = \emptyset$, $1 \le i \ne j \le M$;

If $T_i \cap T_j \ne \emptyset$: it must be a common edge or vertex;

$h = \max_i \mathrm{diam}(T_i)$;

Nodes $\{A_i\}_{i=1}^N$, which is globally numbered.

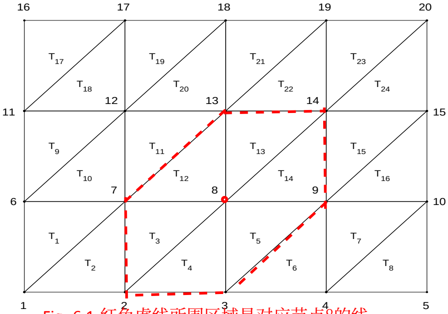


Fig. 6.1 红色虚线所围区域是对应节点8的线性基函数的支集.

Construction of a Finite Element Function Space for $\mathbb{H}_0^1((0,1)^2)$

4. Secondly, define a finite element function space, which is a subspace of $\mathbb{H}^1((0,1)^2)$, on the triangulation 三角剖分 $\mathfrak{T}_h(\Omega)$:

连续的分片线性函数

$$\mathbb{V}_h = \{u \in \mathbb{C}(\overline{\Omega}) : u|_{T_i} \in \mathbb{P}_1(T_i),\ \forall T_i \in \mathfrak{T}_h(\Omega)\}. \quad (6.2.1)$$

5. Then, define finite element trial and test function spaces, 试探函数空间 & 检验函数空间 which are subspaces of $\mathbb{H}_0^1((0,1)^2)$:

$$\mathbb{V}_h(0) = \{u \in \mathbb{V}_h : u(A_i) = 0,\ \forall A_i \in \partial\Omega\}. \quad (6.2.2)$$

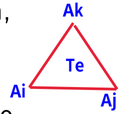
6. A function $u \in \mathbb{V}_h$ is uniquely determined by $\{u(A_i)\}_{i=1}^N$.

基函数的自然选取

7. Basis $\{\varphi_i\}_{i=1}^N$ of $\mathbb{V}_h$: $\varphi_i(A_j) = \delta_{ij}$, $i = 1, 2, \ldots, N$.
8. $k_{ij} = a(\varphi_j, \varphi_i) \neq 0$, iff $A_i \cup A_j \subset T_e$ for some $1 \leq e \leq M$.
9. $\mathrm{supp}(\varphi_i)$ is small $\Rightarrow$ the stiffness matrix K is sparse.

注：(6.2.2)中的表达式理解为对应于两个线性空间的和

Assemble the Global Stiffness Matrix K from the Element One K^e



1. Denote $a^e(u,v) = \int_{T_e} \nabla u \cdot \nabla v \, dx$, by the definition, then,
   $k_{ij} = a(\varphi_j, \varphi_i) = \sum_{e=1}^{M} a^e(\varphi_j, \varphi_i) = \sum_{e=1}^{M} k_{ij}^e$.
2. $k_{ij}^e = a^e(\varphi_j, \varphi_i) \neq 0$, iff $A_i \cup A_j \subset T_e$. For most e, $k_{ij}^e = 0$.
3. It is inefficient to calculate k_{ij} by scanning i, j node by node.
   通常不采用扫描节点i,j方式计算K的元素，而是通过扫描单元的方式计算.
4. Element T_e with nodes $\{A_\alpha^e\}_{\alpha=1}^3 \Leftrightarrow$ the global nodes $A_{en(\alpha,e)}$.

定义在三角形单元Te上的线性函数可以用面积坐标表示，或者P1(Te)的基可用面积坐标表示.

面积坐标/形心坐标

5. Area coordinates $\lambda^e(A) = (\lambda_1^e(A), \lambda_2^e(A), \lambda_3^e(A))^T$ for $A \in T_e$,
   $\lambda_\alpha^e(A) = |\triangle A A_\beta^e A_\gamma^e| / |\triangle A_\alpha^e A_\beta^e A_\gamma^e| \in \mathbb{P}_1(T_e)$, $\lambda_\alpha^e(A_\beta^e) = \delta_{\alpha\beta}$.
6. $\varphi_{en(\alpha,e)}|_{T_e}(A) = \lambda_\alpha^e(A)$, $\forall A \in T_e$.

en(i,e)：单元e的第i个节点的整体编号(对三角形单元i=1,2,3).
cd(i,nd)：nd为节点的整体编号，i为节点nd的坐标的第i个分量.

Algorithm for Assembling Global K and $\mathbf{f}_h$

7. Define the element stiffness matrix

单元刚度矩阵

$$K^e = (k^e_{\alpha\beta}), \quad k^e_{\alpha\beta} \triangleq a^e(\lambda^e_\alpha, \lambda^e_\beta) = \int_{T_e} \nabla\lambda^e_\alpha \cdot \nabla\lambda^e_\beta \, dx,$$

8. Then, $k_{ij} = \displaystyle\sum_{\substack{en(\alpha,e)=i\in T_e \\ en(\beta,e)=j\in T_e}} k^e_{\alpha\beta}$ can be assembled element wise. 逐单元荷组装/装配

9. The external load vector $\mathbf{f}_h = (f_i)$ can also be assembled by scanning through elements 扫描单元荷组装

荷载向量

$$f_i = \sum_{en(\alpha,e)=i\in T_e} \int_{T_e} f\,\lambda^e_\alpha \, dx = \sum_{en(\alpha,e)=i\in T_e} f^e_\alpha.$$

en(i,e)：单元e的第i个节点的整体编号(对三角形单元i=1,2,3).
cd(i,nd)：nd为节点的整体编号，i为节点nd的坐标的第i个分量.

Algorithm for Assembling Global K and $\mathbf{f}_h$

Page 212　　形成刚度矩阵K和荷载向量f

Algorithm 6.1: $K = (k(i,j)) := 0$; $\mathbf{f} = (f(i)) := 0$;

for $e = 1 : M$

单元刚度矩阵　$K^e = (k^e(\alpha, \beta))$; % *calculate the element stiffness matrix*

单元荷载　$\mathbf{f}^e = (f^e(\alpha))$; % *calculate the element external load vector*

组装为总刚度矩阵　$k(en(\alpha, e), en(\beta, e)) := k(en(\alpha, e), en(\beta, e)) + k^e(\alpha, \beta)$;

组装为总荷载　$f(en(\alpha, e)) := f(en(\alpha, e)) + f^e(\alpha)$;

end

en(i,e)：单元e的第i个节点的整体编号(对三角形单元i=1,2,3).
cd(i,nd)：nd为节点的整体编号，i为节点nd的坐标的第i个分量.

Calculations of K^e and $\mathbf{f}^e$ Are Carried Out on a Reference Element

参考三角形及其3个顶点

1. The standard reference triangle
$T_s = \{\hat{x} = (\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2 : \hat{x}_1 \geq 0, \ \hat{x}_2 \geq 0 \text{ and } \ \hat{x}_1 + \hat{x}_2 \leq 1\}$,
with $A_1^s = (0, 0)^T$, $A_2^s = (1, 0)^T$ and $A_3^s = (0, 1)^T$.
2. For T_e with $A_1^e = (x_1^1, x_2^1)^T$, $A_2^e = (x_1^2, x_2^2)^T$, $A_3^e = (x_1^3, x_2^3)^T$, define $A_e = (A_2^e - A_1^e, \ A_3^e - A_1^e)$, $a_e = A_1^e$.

Te和Ts之间的仿射变换

3. $x = L_e(\hat{x}) := A_e\hat{x} + a_e : \ T_s \to T_e$ is an affine map.

Te和Ts的面积坐标之间的关系

4. The area coordinates of T_e: $\lambda_\alpha^e(x) = \lambda_\alpha^s(L_e^{-1}(x))$, since it is an affine function of x, and $\lambda_\alpha^s(L_e^{-1}(A_\beta^e)) = \lambda_\alpha^s(A_\beta^s) = \delta_{\alpha\beta}$.

计算梯度

5. $\nabla\lambda^e(x) = \nabla\lambda^s(\hat{x})\nabla L_e^{-1}(x) = \nabla\lambda^s(\hat{x})A_e^{-1}$.

==>**借助于仿射变换(线性变换+平移,即y=Ax+b),可以将单元刚度矩阵和荷载向量的计算统一在标准三角形Ts上进行.**

Calculations of K^e and $\mathbf{f}^e$ Are Carried Out on a Reference Element

P213 ❻ Change of integral variable $\hat{x} = L_e^{-1}(x) := A_e^{-1}x - A_e^{-1}A_1^e$,

$$K^e = \int_{T_e} \nabla\lambda^e(x)\,(\nabla\lambda^e(x))^T dx = \int_{T_s} \nabla\lambda^s(\hat{x})A_e^{-1}(\nabla\lambda^s(\hat{x})A_e^{-1})^T \det A_e d\hat{x},$$

$$\mathbf{f}^e = \int_{T_e} f(x)\lambda^e(x)\,dx = \det A_e \int_{T_s} f(L_e(\hat{x}))\lambda^s(\hat{x})\,d\hat{x}. \qquad (6.2.4)$$

P212 ❼ $\lambda_1^s(\hat{x}_1,\hat{x}_2) = 1 - \hat{x}_1 - \hat{x}_2$, $\lambda_2^s(\hat{x}_1,\hat{x}_2) = \hat{x}_1$, $\lambda_3^s(\hat{x}_1,\hat{x}_2) = \hat{x}_2$, so

$$\nabla\lambda^s(\hat{x}) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_e^{-1} = \frac{1}{\det A_e}\begin{pmatrix} x_2^3 - x_2^1 & x_1^1 - x_1^3 \\ x_2^1 - x_2^2 & x_1^2 - x_1^1 \end{pmatrix}.$$

==>借助于**仿射变换(线性变换+平移,即y=Ae*x+b)**,可以将单元刚度矩阵和荷载向量的计算统一在标准三角形Ts上进行.

Calculations of K^e and $\mathbf{f}^e$ in Terms of λ^s and A^e

8. The area of T_s is $1/2$, hence, the element stiffness matrix is

$$K^e = \frac{1}{2\det A_e}\begin{pmatrix} x_2^2 - x_2^3 & x_1^3 - x_1^2 \\ x_2^3 - x_2^1 & x_1^1 - x_1^3 \\ x_2^1 - x_2^2 & x_1^2 - x_1^1 \end{pmatrix}\begin{pmatrix} x_2^2 - x_2^3 & x_2^3 - x_2^1 & x_2^1 - x_2^2 \\ x_1^3 - x_1^2 & x_1^1 - x_1^3 & x_1^2 - x_1^1 \end{pmatrix}. \qquad (6.2.3)$$

9. In general, it is necessary to apply a numerical quadrature to the calculation of the element external load vector $\mathbf{f}^e$.

10. If f is a constant on T_e, then

$$\mathbf{f}^e = \frac{1}{6} f(T_e) \det A_e\,(1,\,1,\,1)^T = \frac{1}{3} f(T_e)\,|T_e|\,(1,\,1,\,1)^T. \qquad (6.2.5)$$

Extension of the Example to More General Boundary Conditions

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- For a Dirichlet boundary condition $u(x) = u_0(x) \neq 0$, on $\partial\Omega$, FE trial function space $\mathbb{V}_h(0)$ should be replaced with
试探函数空间

$$\mathbb{V}_h(u_0) = \{u \in \mathbb{V}_h : u(A_i) = u_0(A_i), \ \forall A_i \in \partial\Omega\}.$$

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- For a more general mixed type boundary condition
混合型BC

$$\begin{cases} u(x) = u_0(x), & \forall x \in \partial\Omega_0, \\ \dfrac{\partial u}{\partial \nu} + bu = g, & \forall x \in \partial\Omega_1, \end{cases}$$

We need to

1. add contributions of $\int_{\partial\Omega_1} buv\,dx$ and $\int_{\partial\Omega_1} gv\,dx$ to K and $\mathbf{f}$ by scanning through edges on $\partial\Omega_1$;

Extension of the Example to More General Boundary Conditions

2. Set finite element trial function space:

试探函数空间

(1) $\mathbb{V}_h(u_0; \partial\Omega_0) = \{u \in \mathbb{V}_h : u(A_i) = u_0(A_i), \ \forall A_i \in \partial\Omega_0\},$

if $\partial\Omega_0 \neq \emptyset$ (mixed boundary condition);

注：该表达式实际理解为对应于两个线性空间的和

试探函数空间

(2) $\mathbb{V}_h$, if $\partial\Omega_0 = \emptyset$ but $b > 0$ (the 3rd type boundary condition);

试探函数空间

(3) $\mathbb{V}_h(0; A_i) = \{u \in \mathbb{V}_h : u(A_i) = 0, \ \text{on a specified node } A_i \in \overline{\Omega}\},$

if $\partial\Omega_0 = \emptyset$ and $b = 0$ (pure Neumann boundary condition).

Note: In the case of pure Neumann BC, the solution is unique up to an additive constant. $\mathbb{V}_h(0; A_i)$ removes such uncertainty, so the solution in $\mathbb{V}_h(0; A_i)$ is unique. Likewise, let l be a non-zero linear functional on $\mathbb{V}_h$, then we may take $\mathbb{V}_h(0; l) = \{u \in \mathbb{V}_h : l(u) = 0\}$.

Summary of the Typical Example on FEM

1. 有限元剖分/三角剖分 Introduce a finite element partition (triangulation) $\mathcal{T}_h$ to the region $\overline{\Omega}$, such as the triangular partition shown above.

2. 试探函数空间 & 检验函数空间 Establish finite element trial and test function spaces on $\mathcal{T}_h(\Omega)$, such as continuous piecewise affine function spaces satisfy appropriate BCs shown above.

3. 基函数 形函数 Select a set of basis functions, known as the shape functions, for example, the area coordinates on the triangular element.

4. 刚度矩阵 荷载向量 Calculate the element stiffness matrixes K^e and element external load vector $\mathbf{f}_h^e$, and form the global stiffness matrix K and external load vector $\mathbf{f}_h$.

Some General Remarks on the Implementation of FEM

Arrays used in the algorithm: FEM实现中的数组

1. $en(\alpha, e)$: assigns a global node number to a node with the local node number α on the eth element.

第一类BC的区域边界上单元边的端点

2. $edg0(\alpha, edg)$: assigns a global node number to a node with the local node number α on the edgth edge on $\partial\Omega_0$. $edg1(\alpha, edg)$, $edg2(\alpha, edg)$ are similar arrays with respect to Neumann and Robin type boundaries.

3. $cd(i, nd)$: assigns the ith component of the spatial coordinates to a node with the global node number nd.

en(i,e)：单元e的第i个节点的整体编号(对三角形单元i=1,2,3).
cd(i,nd)：nd为节点的整体编号，i为节点nd的坐标的第i个分量.
edg0,edg1...指出边的类型(内部网格边，区域边界上的网格边的几种类型(如第i类型BC)).
单元的共边的邻居单元.

Some General Remarks on the Implementation of FEM

Arrays used in the algorithm:

4. In iterative methods for solving $K\mathbf{u}_h = \mathbf{f}_h$, it is not necessary to form the global stiffness matrix K, since it always appears in the form $K\mathbf{v}_h = \sum_{e\in\mathcal{T}_h} K^e\mathbf{v}_h^e$. In such cases, we may need:

5. $et(i,\tau)$: assigns the global element number to the τth local element of the ith global node. And $edgrt(i,\tau)$, etc.

Three Basic Ingredients in a Finite Element Function Space

6.2.2节 FEM的关键是构造适当的有限元函数空间, 有限元函数空间的构造有三个基本要求:

(FEM 1) 基本要求1 Introduce a ~~finite element~~ triangulation $\mathcal{T}_h$ on the region $\overline{\Omega}$, which divides the region $\overline{\Omega}$ into finite numbers of subsets K, generally called ~~finite~~ element, 单元 such that

- ($\mathcal{T}_h$1) $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} K$;
- ($\mathcal{T}_h$2) each ~~finite~~ element $K \in \mathcal{T}_h$ is a closed set with a nonempty interior set $\mathring{K}$; K的内部
- ($\mathcal{T}_h$3) $\mathring{K}_1 \cap \mathring{K}_2 = \emptyset$, for any two different ~~finite~~ elements $K_1, K_2 \in \mathcal{T}_h$;
- ($\mathcal{T}_h$4) every ~~finite~~ element $K \in \mathcal{T}_h$ has a Lipschitz continuous boundary.

P.G. Ciarlet, The Finite Element Method for Elliptic Problems, SIAM, 2002

Three Basic Ingredients in a Finite Element Function Space

(FEM 2) 基本要求2 Introduce on each ~~finite~~ element $K \in \mathcal{T}_h$ a function space P_K which consists of some polynomials or other functions having certain approximation properties and at the same time easily manipulated analytically and numerically;

(FEM 3) **基本要求3** The finite element function space $\mathbb{V}_h$ has a set of "normalized" 规范化基函数 basis functions which are easily computed, and each basis function has a "small" support.

Generally speaking, a finite element is not just a subset K, it includes also the finite dimensional function space P_K defined on K and the corresponding "normalized" basis functions 规范化基函数.

General Abstract Definition of a Finite Element

Definition 6.1

A triple (K, P_K, Σ_K) is called a finite element, if

Ciarlet的FE的定义(1978)

1. $K \subset \mathbb{R}^n$, called an element, is a closed set with non-empty interior and a Lipschitz continuous boundary;
2. $P_K : K \to \mathbb{R}$ is a finite dimensional function space consisting of sufficiently smooth functions defined on the element K;
3. Σ_K is a set of linearly independent linear functionals $\{\varphi_i\}_{i=1}^N$ defined on $\mathbb{C}^\infty(K)$, which are called the degrees of freedom of the finite element and form a dual basis corresponding to a "normalized" basis of P_K, meaning that there exists a unique basis $\{p_i\}_{i=1}^N$ of P_K such that $\varphi_i(p_j) = \delta_{ij}$.

自由度集合

自由度集合就是惟一地确定空间Pk中的一个函数的那些参数．例如k=1时，Pk取为P1(次数不超过1的多项式集合)，此时自由度集合由单元K的顶点上的函数值组成, 记号为{p(Ai)}, 其中p(x)是P1中函数.

\phi_i和pi可设想为在代数意义下的一组对偶基.

P.G. Ciarlet, The Finite Element Method for Elliptic Problems, SIAM, 2002, P78

An Additional Requirement on the Partition

In applications, an element K is usually taken to be

1. a triangle in $\mathbb{R}^2$; a tetrahedron in $\mathbb{R}^3$; a n simplex in $\mathbb{R}^n$;
   三角形、四面体、n-单纯形
2. a rectangle or parallelogram in $\mathbb{R}^2$; a cuboid or a parallelepiped or more generally a convex hexahedron in $\mathbb{R}^3$; a parallelepiped or more generally a convex $2n$ polyhedron in $\mathbb{R}^n$;
   矩形或平行四边形；长方体或平行六面体或者更一般的凸六面体；平行六面体或更一般的凸2n多面体
3. a triangle with curved edges or a tetrahedron with curved faces, etc..
   曲边三角形、曲面四面体

An Additional Requirement on the Partition

构造有限元空间的第3个基本要求：易得支撑小的基函数

When a region $\overline{\Omega}$ is partitioned into a finite element triangulation $\mathcal{T}_h$ with such elements, to ensure that (FEM 3) holds, the adjacent elements are required to satisfy the following compatibility condition:

($\mathcal{T}_h$5) For any pair of $K_1, K_2 \in \mathcal{T}_h$, if $K_1 \bigcap K_2 \neq \emptyset$, then, there must exists an $0 \leq i \leq n-1$, such that $K_1 \bigcap K_2$ is exactly a common i dimensional face of K_1 and K_2. P217

公共的i维面

n=2时, i=1 维的边，i=0的顶点

Function Space P_K Usually Consists of Polynomials

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型(k)n单纯形

1. The finite element of the n-simplex of type (k): K is a n-simplex, $P_K = \mathbb{P}_k(K)$, which is the space of all polynomials of degree no greater than k defined on K.

   分片仿射三角形单元(型(1) 2单纯形, 或...)

   For example, the piecewise affine triangular element (2-simplex of type (1), or type (1) 2-simplex, or type(1) triangle).

型(k)n矩形

2. The finite element of n-rectangle of type (k) (abbreviated as the n-k element): K is a n-rectangle, $P_K = \mathbb{Q}_k(K)$, which is the space of all polynomials of degree no greater than k with respect to each one of the n variables.

   双线性元(型(1) 2矩形, 或...)

   For example, the bilinear element (the 2-rectangle of type (1), or type (1) 2-rectangle, or 2-1 rectangle); etc..

Nodal Degrees of Freedom Σ_K

P217 节点型自由度

The degrees of freedom in the nodal form:

$$\begin{cases} \varphi_i^0 : \quad p \to p\,(a_i^0), & \text{Lagrange FE, if contains point values only} \\ \varphi_{ij}^1 : \quad p \to \partial_{\nu_{ij}^1} p\,(a_i^1), & \text{Hermite FE, if contains at least} \\ \varphi_{ijk}^2 : \quad p \to \partial^2_{\nu_{ij}^2 \nu_{ik}^2} p\,(a_i^2), & \text{one of the derivatives} \end{cases}$$

其中p(x)是P_K中函数.

where the points $a_i^s \in K$, $s = 0, 1, 2$ are called nodes, $\nu_{ij}^s \in \mathbb{R}^n$, $s = 1, 2$ are specified nonzero vectors.

Integral Degrees of Freedom Σ_K

积分型自由度

The degrees of freedom in the integral form:

$$\psi_i^s : p \to \frac{1}{\mathrm{meas}_s(K_i^s)} \int_{K_i^s} p(x)\, dx,$$

where K_i^s, $s = 0, 1, \ldots, n$ are s-dimensional faces of the element K, and $\mathrm{meas}_s(K_i^s)$ is the s-dimensional Lebesgue measure of K_i^s.

For example, if $s = n$, then the corresponding degree of freedom is the average of the element integral.
单元积分平均值

P_K Interpolation for a Given Finite Element (K, P_K, Σ_K)

Definition 6.2

Let (K, P_K, Σ_K) be a finite element, and let $\{\varphi_i\}_{i=1}^N$ be its degrees of freedom and $\{p_i\}_{i=1}^N \in P_K$ be the corresponding dual basis, satisfying $\varphi_i(p_j) = \delta_{ij}$. Define the P_K interpolation operator $\Pi_K : \mathbb{C}^\infty(K) \to P_K$ by

插值算子

$$\boxed{\Pi_K(v) = \sum_{i=1}^N \varphi_i(v)\, p_i,} \quad \forall v \in \mathbb{C}^\infty(K),$$

局部的插值算子

插值函数 Here $\Pi_K(v)$ is the P_K-interpolant of the function v.

In applications, it is often necessary to extend the domain of the definition of the P_K interpolation operator, for example, to extend the domain of the definition of a Lagrange finite element to $\mathbb{C}(K)$.

PK插值算子是FEM的特征属性,其分析性质不依赖于基的选取. PK插值算子的定义域作适当的拓展,例如对Lagrange有限元,可将插值算子的定义域拓广至C(K).

P.G. Ciarlet, The Finite Element Method for Elliptic Problems, SIAM, 2002, P81

The P_K Interpolation Operator Is Independent of the Choice of Basis

Definition 6.3

Let two finite elements (K, P_K, Σ_K) and (L, P_L, Σ_L) satisfy

$$K = L, \quad P_K = P_L, \text{ and } \Pi_K = \Pi_L,$$

where Π_K and Π_L are respectively P_K and P_L interpolation operators, then the two finite elements are said to be equivalent.

两个有限元等价

Compatibility Conditions for P_K and Σ_K on Adjacent Elements

1. $\mathcal{T}_h$: a finite element triangulation of Ω; $\{(K, P_K, \Sigma_K)\}_{K\in\mathcal{T}_h}$: a given set of corresponding finite elements.
2. $\mathbb{V}_h = \{v : \bigcup_{K\in\mathcal{T}_h} K \to \mathbb{R} : v|_K \in P_K\}$: FE function space.
3. Compatibility conditions are required to assure $\mathbb{V}_h$ satisfies (FEM 3), as well as a subspace of $\mathbb{V}$.
构造有限元空间的第3个基本要求：易得支撑小的基函数

例如, 对多面体单元和节点型自由度

For example, for polyhedron elements and nodal degrees of freedom, if $K_1 \bigcap K_2 \neq \emptyset$, then, we require that a point $a_i^s \in K_1 \bigcap K_2$ is a node of K_1, if and only if it is also the same type of node of K_2.

$\mathbb{V}_h$ Interpolation Operator and $\mathbb{V}_h$ Interpolation

Denote $\Sigma_h = \bigcup_{K \in \mathcal{T}_h} \Sigma_K$ as the degrees of freedom of the finite element function space $\mathbb{V}_h$.

Definition 6.4

Define the $\mathbb{V}_h$ interpolation operator $\Pi_h : \mathbb{C}^\infty(\overline{\Omega}) \to \mathbb{V}_h$ by

整体的插值算子

$$\boxed{\Pi_h(v)|_K = \underset{\text{Def 6.2}}{\Pi_K}(v|_K), \quad \forall v \in \mathbb{C}^\infty(\overline{\Omega}),}$$

and define $\Pi_h(v)$ as the $\mathbb{V}_h$ interpolant of v.

In applications, similar as for the P_K interpolation operator, the domain of definition of the $\mathbb{V}_h$ interpolation operator is often extended to meet certain requirements.

P.G. Ciarlet, The Finite Element Method for Elliptic Problems, SIAM, 2002, P94-95

Reference Finite Element $(\hat{K}, \hat{P}, \hat{\Sigma})$ and Its Isoparametric Equivalent Family

Definition 6.5

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两个有限元等参等价，仿射等价

Let $\hat{K},\ K \in \mathbb{R}^n$, $(\hat{K},\ \hat{P},\ \hat{\Sigma})$ and $(K,\ P_K,\ \Sigma_K)$ be two finite elements. Suppose that there exists a sufficiently smooth invertible map $F_K : \hat{K} \to K$, such that

$$\begin{cases} F_K(\hat{K}) = K; \\ p_i = \hat{p}_i \circ F_K^{-1}, \quad i = 1, \ldots, N; \\ \varphi_i(p) = \hat{\varphi}_i(p \circ F_K), \quad \forall p \in P_K, \quad i = 1, \ldots, N, \end{cases}$$

复合运算

where $\{\hat{\varphi}_i\}_{i=1}^N$ and $\{\varphi_i\}_{i=1}^N$ are the basis of the degrees of freedom spaces $\hat{\Sigma}$ and Σ_K respectively, $\{\hat{p}_i\}_{i=1}^N$ and $\{p_i\}_{i=1}^N$ are the corresponding dual basis of $\hat{P}$ and P_K respectively. Then, the two finite elements are said to be isoparametrically equivalent. In particular, if F_K is an affine mapping, the two finite elements are said to be affine-equivalent.

P.G. Ciarlet, The Finite Element Method for Elliptic Problems, SIAM, 2002, P84

An Isoparametric (Affine) Family of Finite Elements

If all finite elements in a family are isoparametrically (affine-) equivalent to a given reference finite element, then we call the family an isoparametric (affine) family. 等参(仿射)族

For example, the finite elements with triangular elements and piecewise linear function space used in the previous subsection, *i.e.* finite elements of 2-simplex of type (1), are an affine family.

Thank You!