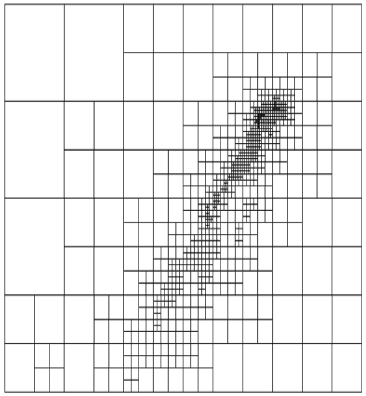
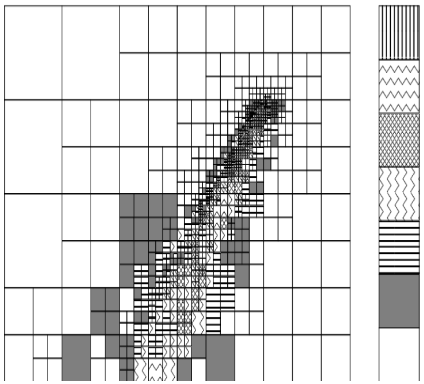
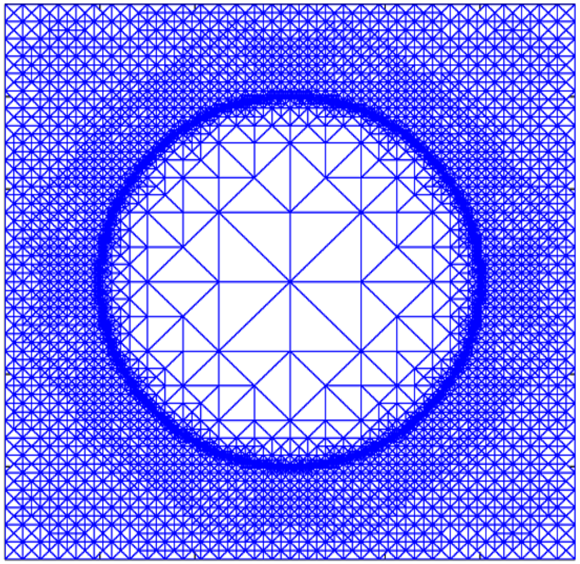
Adaptive FEM

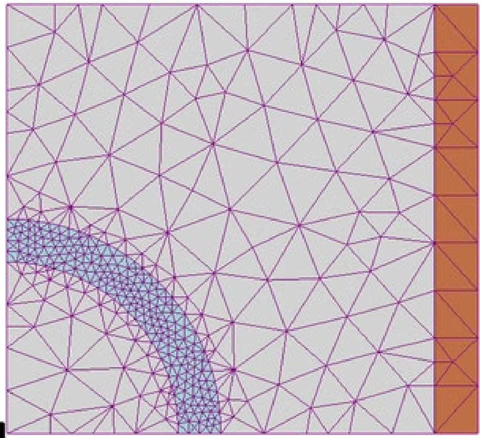
THz

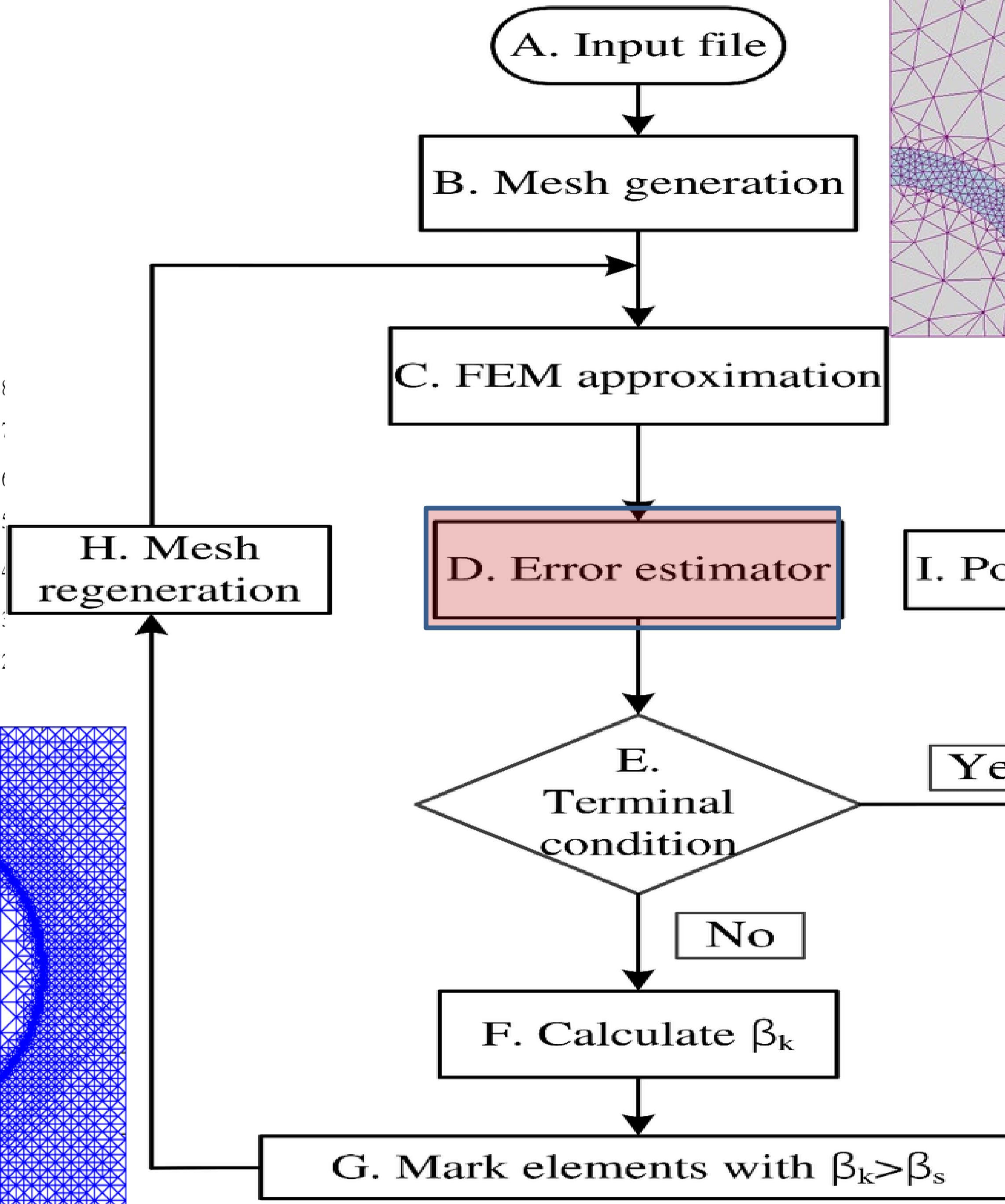
2019-12-25











Flow chart of adaptive finite-element mesh refinement.

Algorithm 1.1 (General adaptive algorithm). *Given: the data of a partial differential equation and a tolerance ε.*

Sought: a numerical solution with an error less than ε.

(1) *Construct an initial coarse mesh $\mathcal{T}_0$ representing sufficiently well the geometry and data of the problem; set $k = 0$.*

(2) *Solve the discrete problem associated with $\mathcal{T}_k$.*

(3) *For every element K in $\mathcal{T}_k$ compute the a posteriori error indicator.*

(4) *If the estimated global error is less than ε* stop, *otherwise decide which elements have to be refined and construct the next mesh $\mathcal{T}_{k+1}$. Increase k by 1 and return to step (2).*

The above algorithm is best suited for stationary problems. For transient calculations, some changes have to be made:

- the accuracy of the computed numerical solution has to be estimated every few time-steps,
- the refinement process in space should be coupled with a time-step control,
- a partial coarsening of the mesh might be necessary,
- occasionally, a complete re-meshing could be desirable.

R. Verfurth, *A Posteriori Error Estimation Techniques for Finite Element Methods*, Oxford University Press, 2013

In both stationary and transient problems, the refinement and unrefinement process may also be coupled with or replaced by a moving-point technique which keeps the number of grid-points constant but changes their relative location.

In order to make Algorithm 1.1 operative, we must specify:

- a discretisation method,
- a solution method for the discrete problems,
- an error indicator which furnishes the a posteriori error estimate,
- a refinement strategy which determines which elements have to be refined or coarsened and how this has to be done.

R. Verfurth, *A Posteriori Error Estimation Techniques for Finite Element Methods*, Oxford University Press, 2013

- In chapter I we will describe various possibilities for a posteriori error estimation. In order to keep the presentation as simple as possible we will consider a simple model problem: the 2D Poisson equation, problem (1.1) (p. 4), discretised by continuous linear or bi-linear finite elements, equation (1.3) (p. 5). We will review several a posteriori error estimates and show that, in a certain sense, they are all equivalent and yield lower and upper bounds for the error of the finite element discretisation. The estimates can roughly be classified as follows.

✓ *Residual estimates*: Estimate the error of the computed numerical solution by a suitable norm of its residual with respect to the strong form of the differential equation (Sections 1.4 (p. 10), 1.5 (p. 17), and 1.6 (p. 20)).

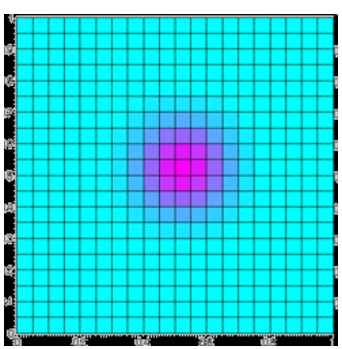
R. Verfurth, *A Posteriori Error Estimation Techniques for Finite Element Methods*, Oxford University Press, 2013

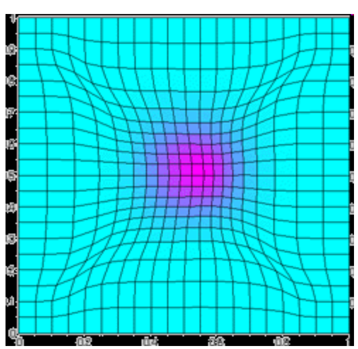
- ✓ *Solution of auxiliary local problems*: On small patches of elements, solve auxiliary discrete problems similar to but simpler than the original problem and use appropriate norms of the local solutions for the error estimation (Section 1.7 (p. 25)).
- ✓ *Hierarchical error estimates*: Evaluate the residual of the computed finite element solution w.r.t. another finite element space corresponding to higher order elements or to a refined grid (Section 1.8 (p. 31)).
- ✓ *Averaging methods*: Use some local extrapolate or average of the gradient of the computed numerical solution for error estimation (Section 1.9 (p. 36)).
- ✓ *Equilibrated residuals*: Evaluate approximately a dual variational problem posed on a function space with a weaker topology (Section 1.10 (p. 41)).
- ✓ *Dual weighted residuals*: Approximately solve a dual variational problem with the residual of the primal problem as RHS (Section 1.11 (p. 45)).
- ✓ *Hyper-circle method*: Use a functional analytic relationship of *H*1- and *H*(div)-spaces in order to evaluate the residual of the numerical solution with respect to the original primal variational problem (Section 1.12 (p. 48)).

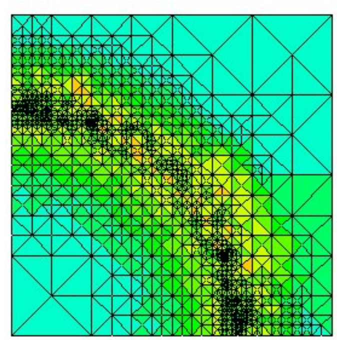
R. Verfurth, *A Posteriori Error Estimation Techniques for Finite Element Methods*, Oxford University Press, 2013

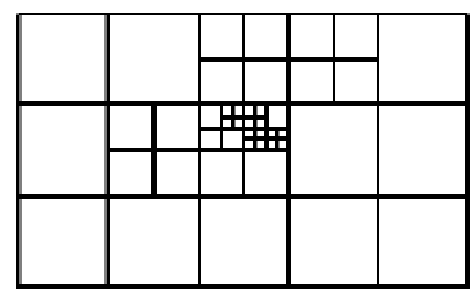
自适应FEM分类

- 在有限元方法中，自适应方法分为
  - p方法:不同单元的逼近可以不同
  - h方法: 局部细分或加细/粗化或合并方法(local refinement/coarsening)
  - r方法: 移动网格方法(moving mesh method) /网格重分布方法









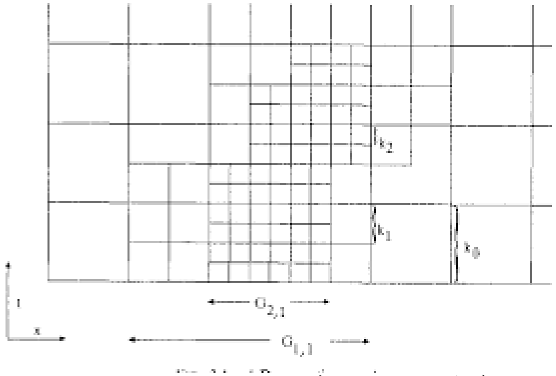
p方法

- 网格单元固定(大小和位置)
- 每个单元中解的逼近阶可以变化，即不同单元的形状函数或有限元基函数可以不同
- 有时也被看作“spectral”方法，由于是通过改进解的谱逼近来得到“adaptivity”
- p——有限元基函数次数

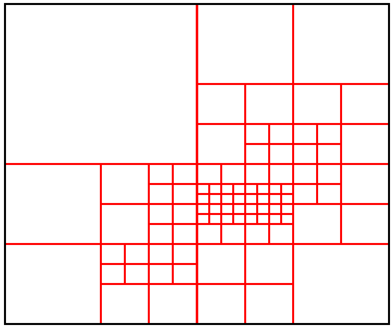
$$u_h(x)=\sum_{k=1}^{3}u^{(k)}\phi_k(x) \quad u_h(x)=\sum_{k=1}^{5}u^{(k)}\phi_k(x) \quad u_h(x)=\sum_{k=1}^{3}u^{(k)}\phi_k(x)$$

C.T. Tsai, Ph.D. thesis, Washington Univ. St. Louis, 1971.

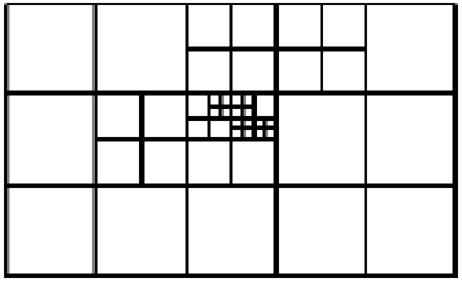
I. Babuska et al., SIAM J. Numer. Anal., 18 (1981); SIAM Rev., 36(1994)

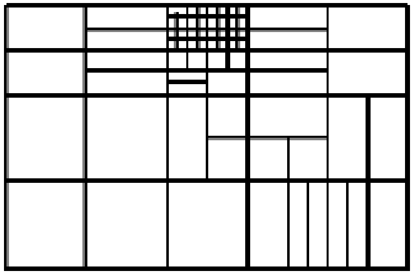


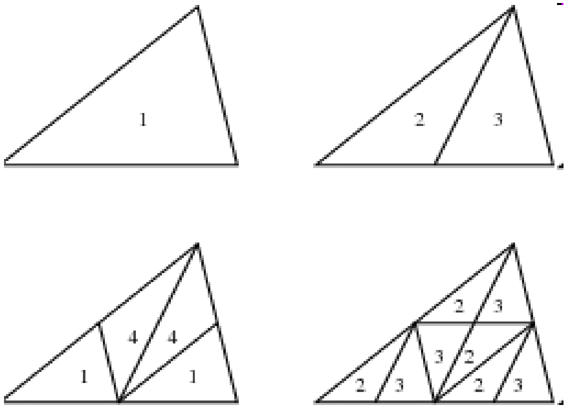
h方法



- 用几个小单元代替某大单元以提高解的逼近
- 不论采用什么策略，传统的“一对一”的邻居准则将被打破：对应某边，大单元有多个小单元邻居。【可用三角形单元细分大的邻居来避免这种不规则性】







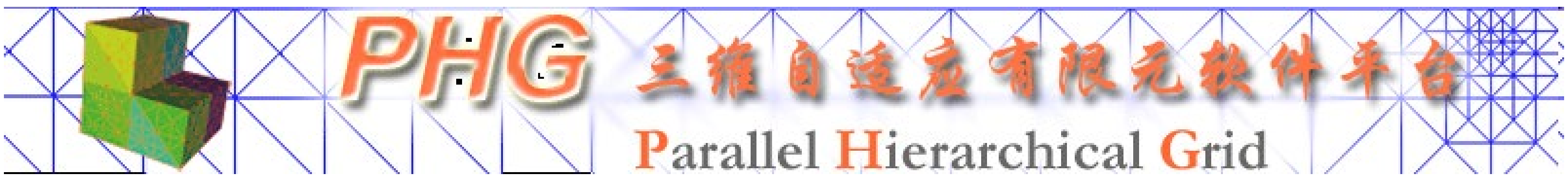
Turner MJ, Clough RW, Martin HC, Topp LJ. Stiffness and deflection analysis of complex structures. J Aero Sci 1956;23:805-23.

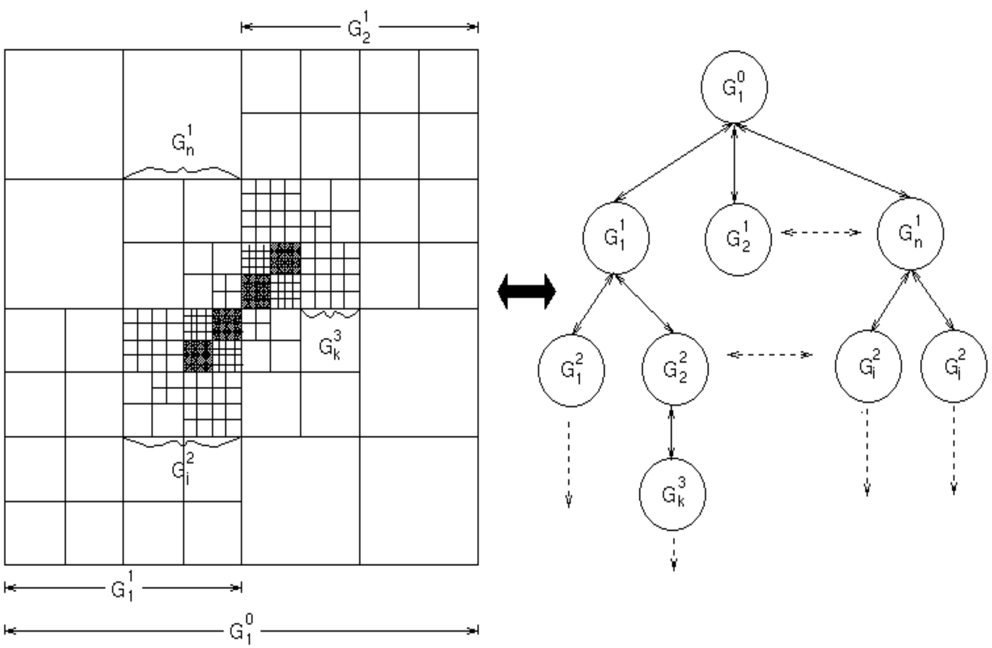
h方法(续)

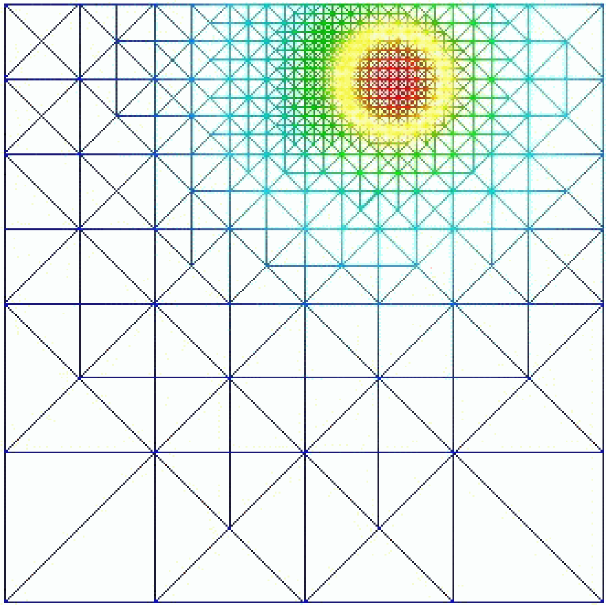
- 网格自动加细过程中，单元的加密或粗化由下列控制：
  - 单元的误差估计值：如果某大单元【某些小单元】中的误差大于【小于】用户设定的容限，则细分【合并或粗化】单元
  - 方向误差指示器：用于各向异性加密【粗化】
- h——单元或网格的大小

M.J. Berger, Ph.D. thesis, Stanford Univ, 1982
J.T. Oden & L. Demkowicz, A review of local mesh refinement techniques and corresponding data structures in h-type adaptive finite element methods, TICOM Report 88-02, Austin, 1988

h方法(续)





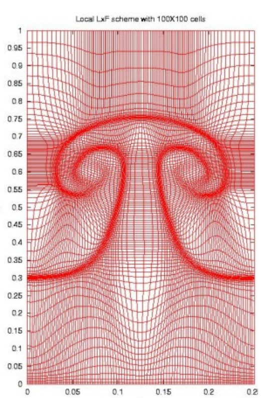


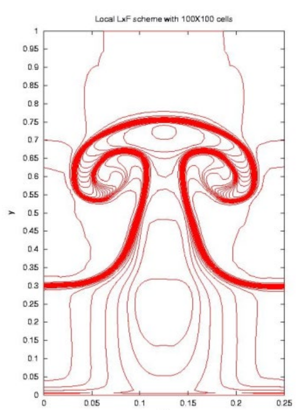
http://lsec.cc.ac.cn/phg/

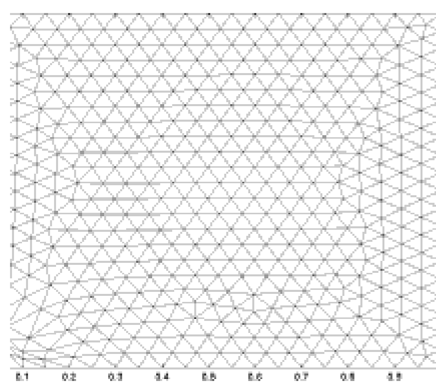
一些大单元被细分成几个较小的单元，形成多个层次的网格

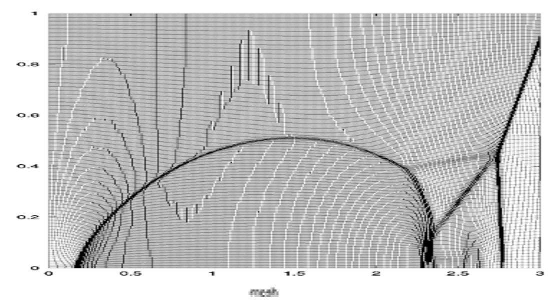
自适应移动网格方法

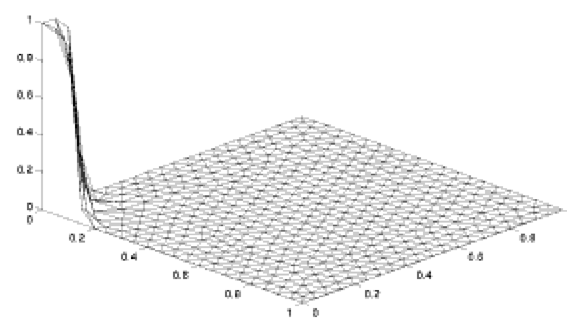
- **移动网格方法**: 网格点数和有限元基函数固定，网格点位置随“时间”变化，网格的拓扑关系可以发生变化，也可以不发生变化
- r——redistribution





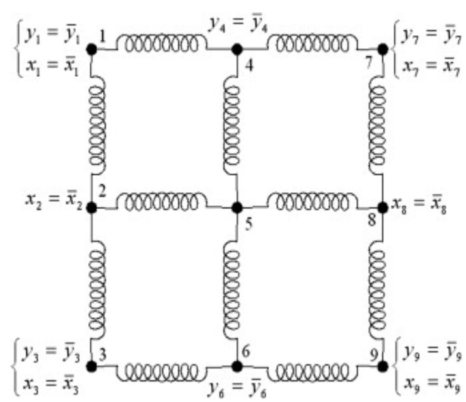






自适应移动网格方法

- 等分布原理(C. de Boor, In Lecture Notes in Mathematics, Vol.363, Springer-Verlag, 1973): 解ODE边值问题时选择网格点使得解的误差的一些度量在每个子区间内相等
- 基于弹簧模型的动态方法(J.T. Batina, AIAA 89-0115): 将网格各节点之间的连线看作弹簧，其刚度与节点间距离成反比
- 基于变分原理的网格生成(A. Winslow, JCP, 1973)
- ……
- 计算流体力学中Lagrange方法



C.J. Budd, W. Huang & R.D. Russell, Acta Numerica 18 (2009), 111-241.
W. Huang and R. D. Russell, Adaptive Moving Mesh Methods, Springer, 2011.
T. Tang & J.C. Xu, Adaptive Computations: Theory and Algorithms, Science Pub., 2007.

Suppose that u is Lipschitz in $[0,1]$. Consider the approximation

$$u_N(x) := u(x_{i-1}), \text{ for all } x_{i-1} \leq x < x_i.$$

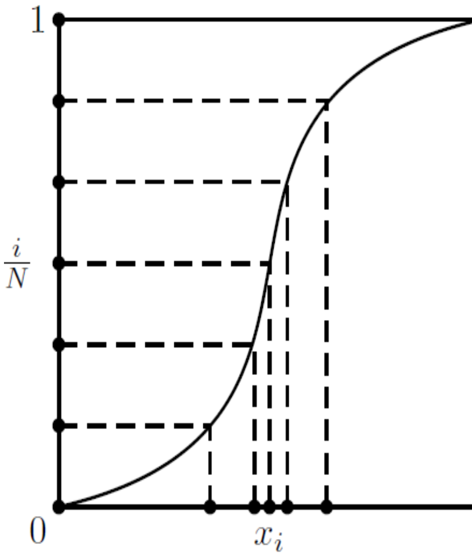
If the grid is quasi-uniform in the sense that $h_i = x_i - x_{i-1} \leq C/N$ for $i = 1, \cdots N$, then it is easy to show that

$$\|u - u_N\|_\infty \leq CN^{-1}\|u'\|_\infty \tag{1}$$

We can achieve the same convergent rate N^{-1} with less smoothness of the function. Suppose $\|u'\|_{L^1} \neq 0$. Let us define a grid distribution function

$$F(x) := \frac{1}{\|u'\|_{L^1}} \int_0^x |u'(t)|\, dt.$$

Then $F : [0,1] \to [0,1]$ is a non-decreasing function. Let $y_i = i/N, i = 0, \cdots, N$ be a uniform grid. We choose x_i such that $F(x_i) = y_i$, see Fig. 1 for an illustration.



A grid distribution function

Then

$$\int_{x_{i-1}}^{x_i} |u'(t)|\, dt = F(y_i) - F(y_{i-1}) = N^{-1}, \tag{2}$$

and

$$|u(x) - u(x_{i-1})| \leq \int_{x_{i-1}}^{x_i} |u'(t)|\, dt \leq N^{-1}\|u'\|_{L^1},$$

which leads to the estimate

$$\|u - u_N\|_\infty \leq CN^{-1}\|u'\|_{L^1}. \tag{3}$$

We use the following example to illustrate the advantage of (3) over (1). Let us consider the function $u(x) = x^r$ with $r \in (0,1)$. Then $u' \notin L^\infty(\Omega)$ but $u' \in L^1(\Omega)$. Therefore we cannot obtain optimal convergent rate on quasi-uniform grids while we could on the correctly adapted grid. For this simple example, one can easily compute when

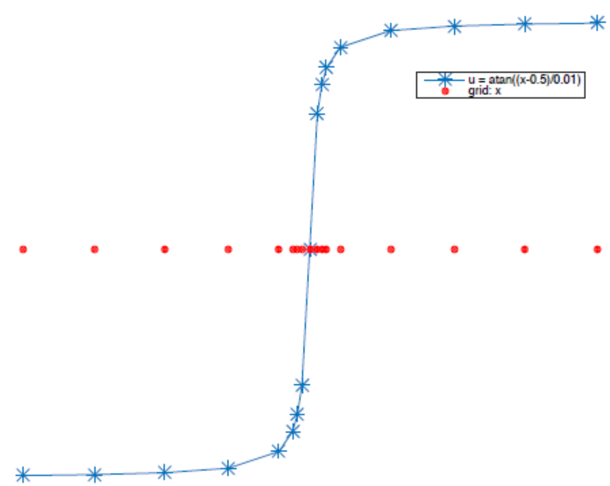
$$x_i = \left(\frac{i}{N}\right)^{1/r}, \quad \text{for all } 0 \le i \le N,$$

estimate (3) will hold on the grid $\mathcal{T}_N = \{x_i\}_{i=0}^N$ which has higher density of grid points near the singularity of u.

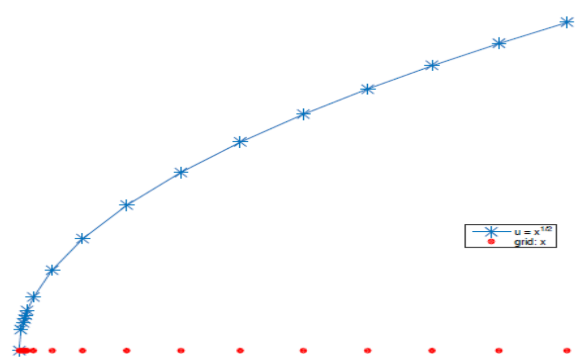
In (2), we choose a grid such that a upper bound of the error is equidistributed. This is instrumental for adaptive finite element methods on solving PDEs.

A possible MATLAB code is given below.

```
function x = equidistribution(M,x)
h = diff(x);
F = [0; cumsum(h.*M)];
F = F/F(end);
y = (0:1/(length(x)-1):1)';
x = interp1(F,x,y);
```



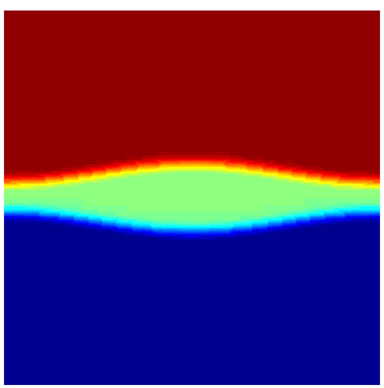
(b) $u(x) = \arctan((x-0.5)/0.01)$

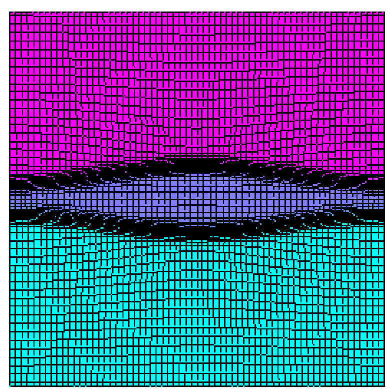


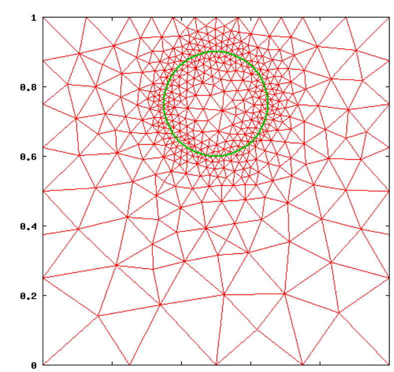
(a) $u(x) = x^{1/2}$

自适应移动网格方法

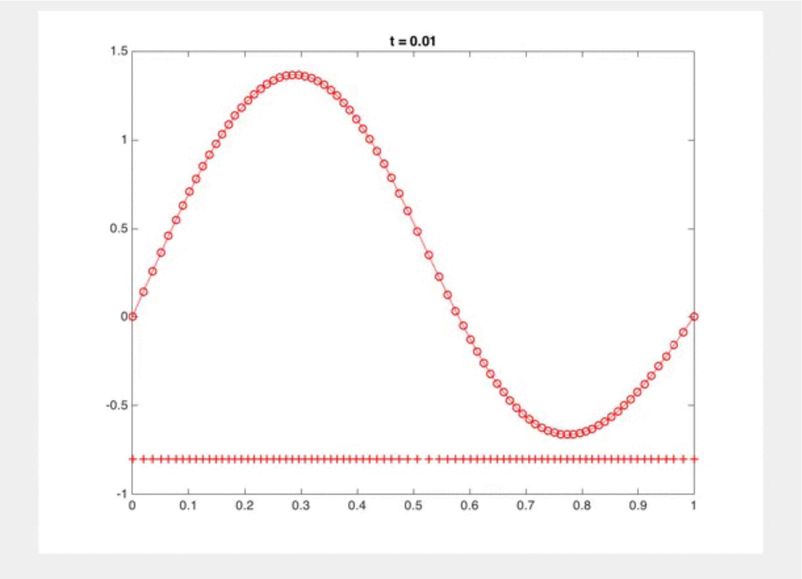
- 两个非耦合的步骤：
  - 网格迭代重分布
    - 迭代离散的网格方程一步
    - 重映解：用高分辨方法将解从老网格插值到新网格
  - 在固定网格上解控制方程
    - 用高分辨方法解控制方程

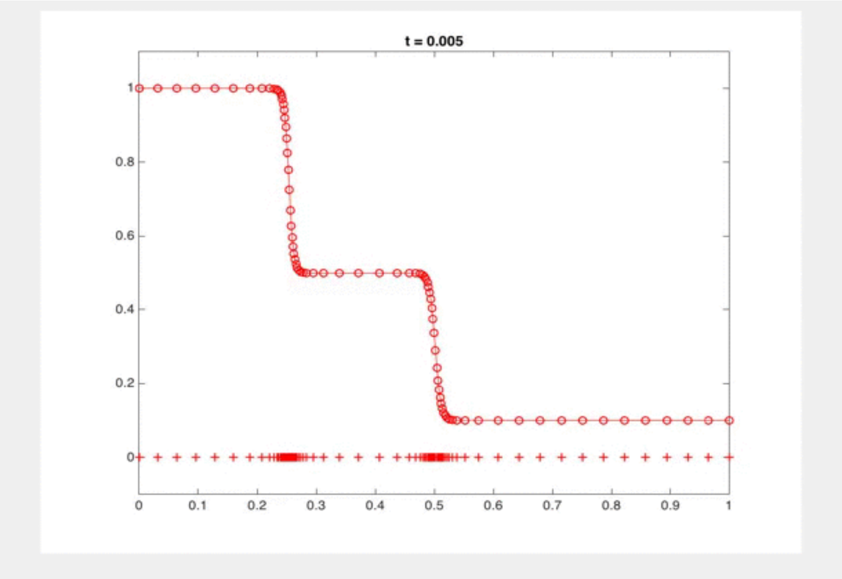






H.Z. Tang & T. Tang, SIAM J. Numer. Anal., 41(2003), 487-515

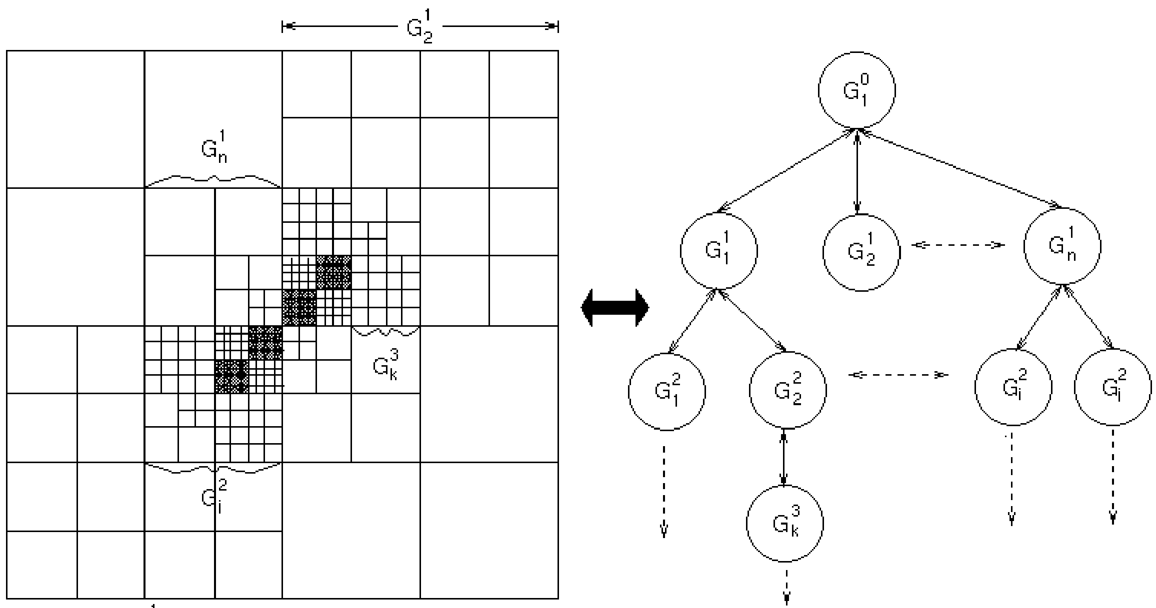




https://whuang.ku.edu/research/movingmesh.html

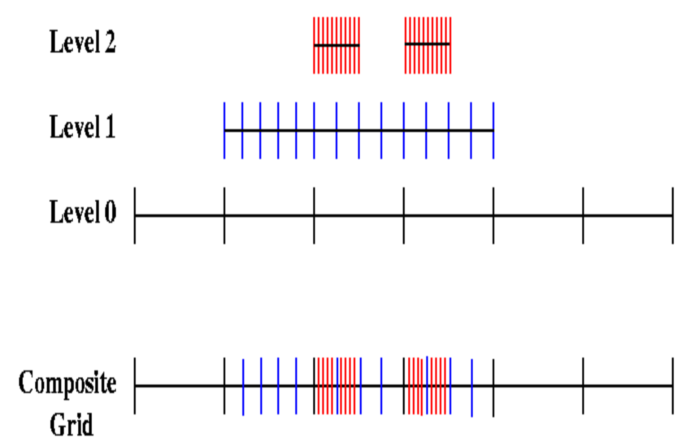
structured AMR

- M.J. Berger and J. Oliger, Adaptive mesh refinement for hyperbolic partial differential equations, *Journal of Computational Physics*, 53(3), 1984, 484-512



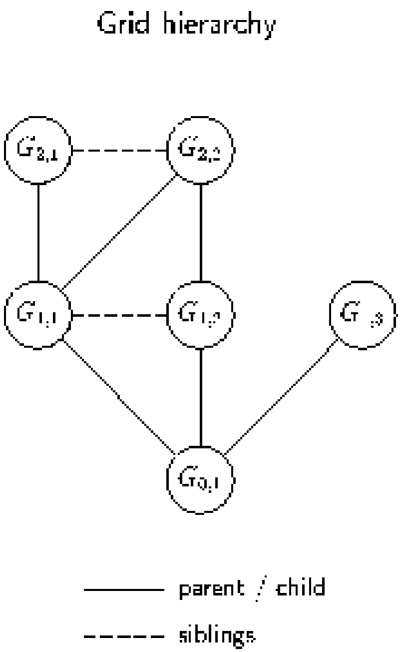
Grid Structure and associated Data Structure

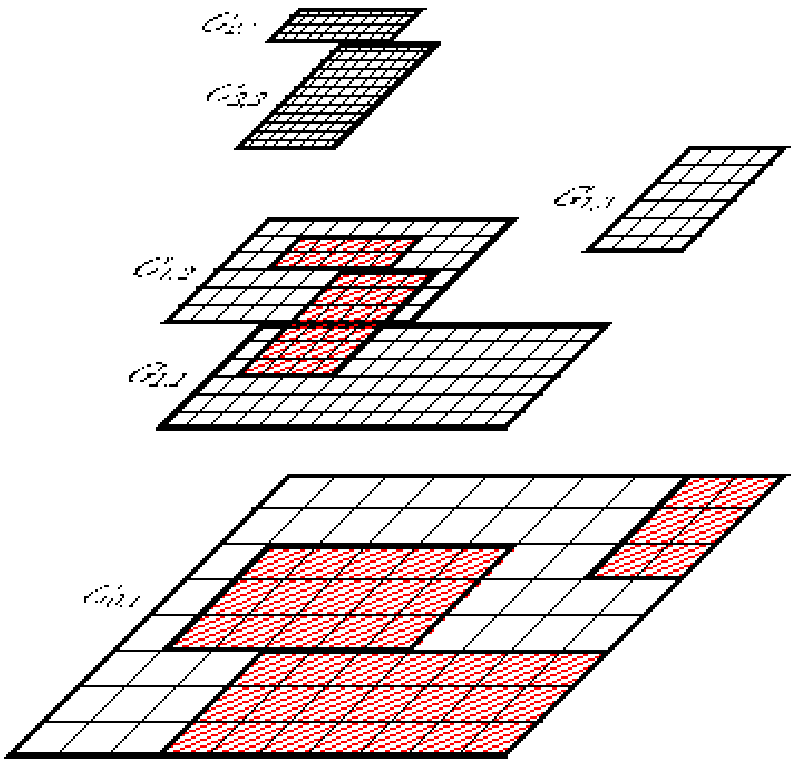
structured AMR

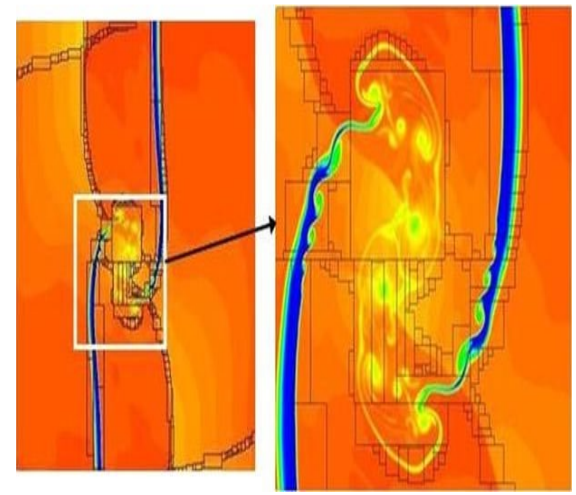


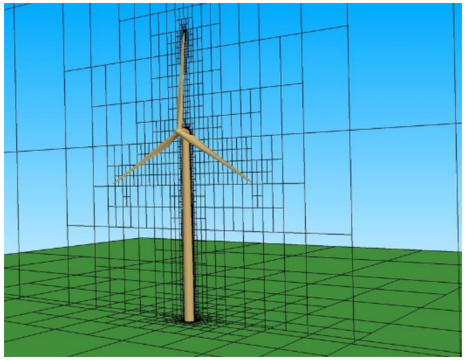
structured AMR

- M.J. Berger and J. Oliger, Adaptive mesh refinement for hyperbolic partial differential equations, *Journal of Computational Physics*, 53(3), 1984, 484-512

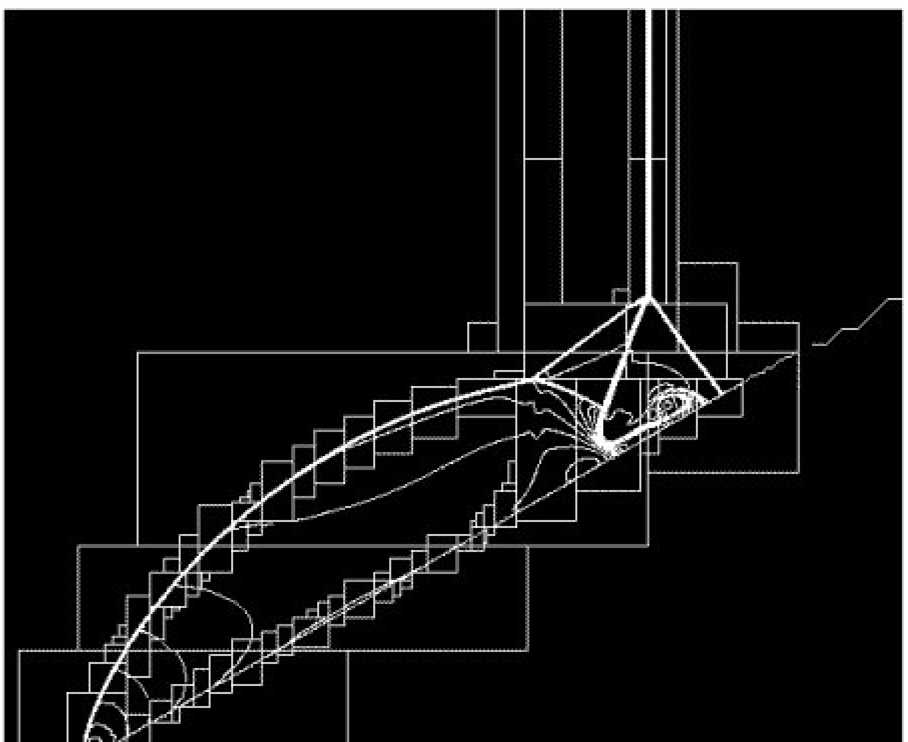


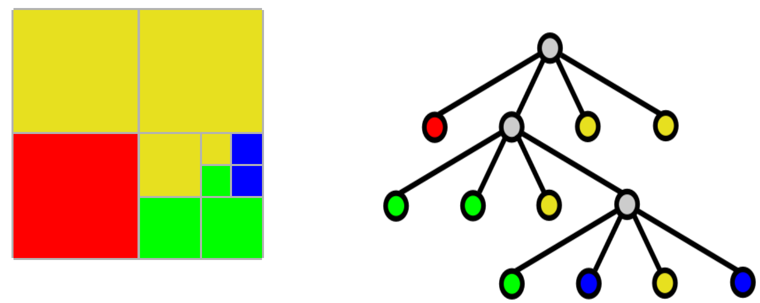


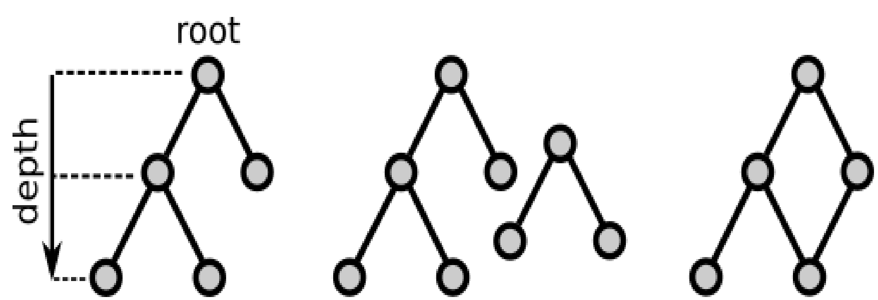




The block structured refinement strategy creates a hierarchy of properly nested subgrids.
http://amroc.sourceforge.net/html/amr.htm



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- https://www.math.uzh.ch/conferences/fileadmin/conferencesHomepages/zss12/nochetto1.pdf …… nochetto6.pdf

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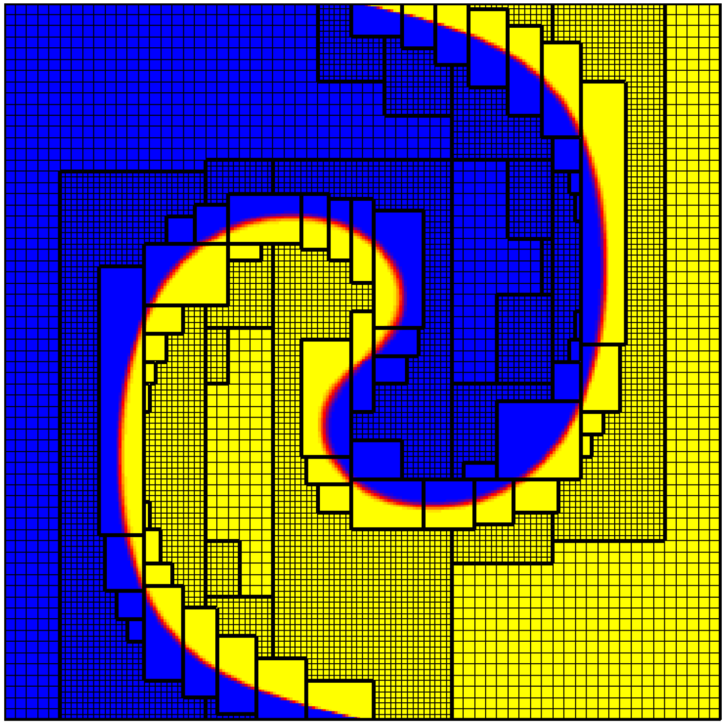
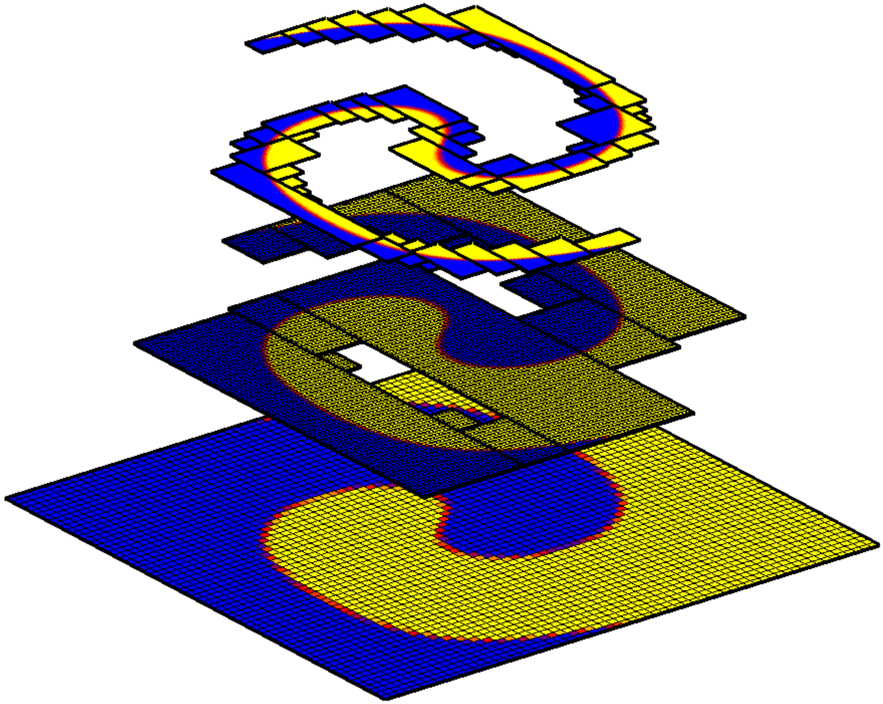
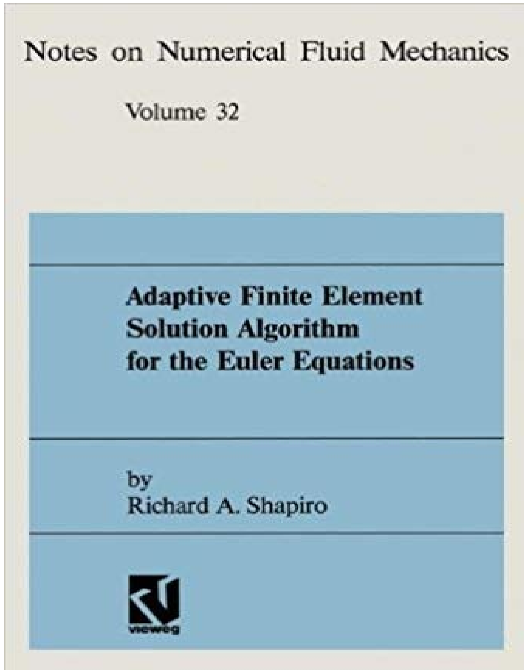
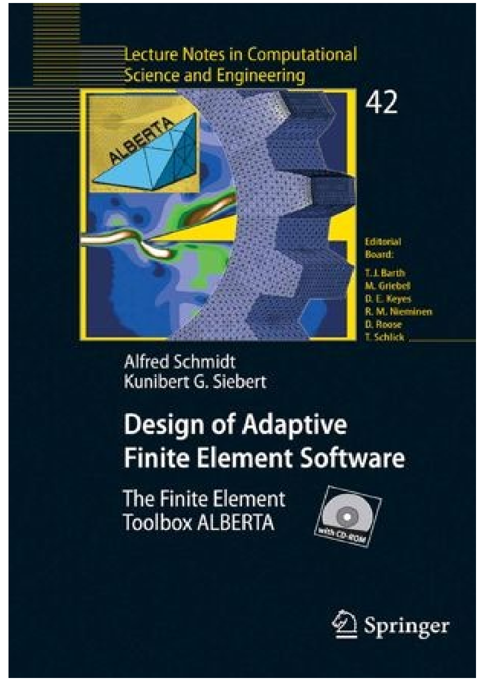
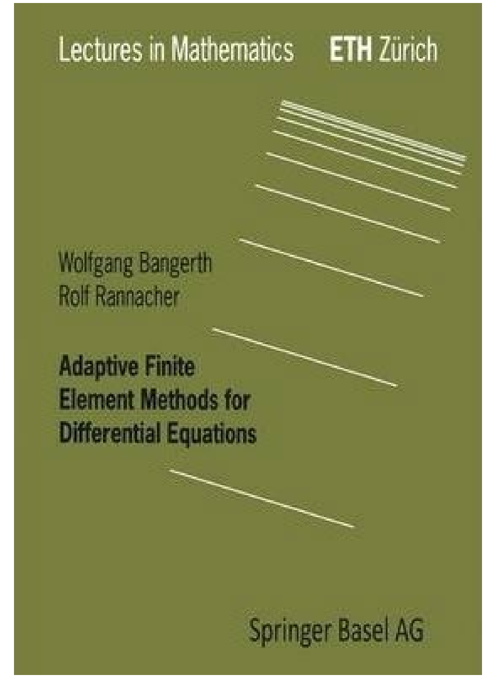
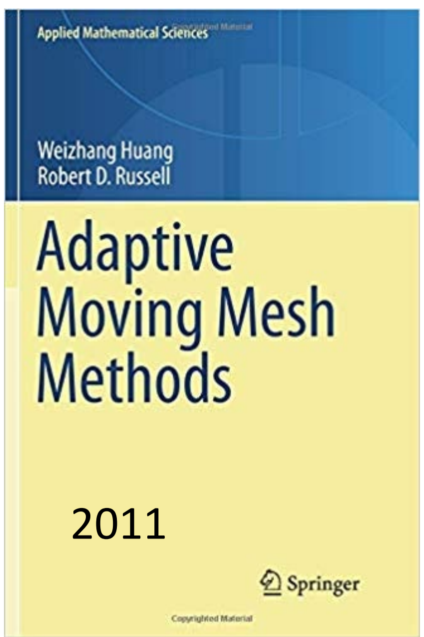
 

https://math.boisestate.edu/~calhoun/www_personal/research/amr_software/









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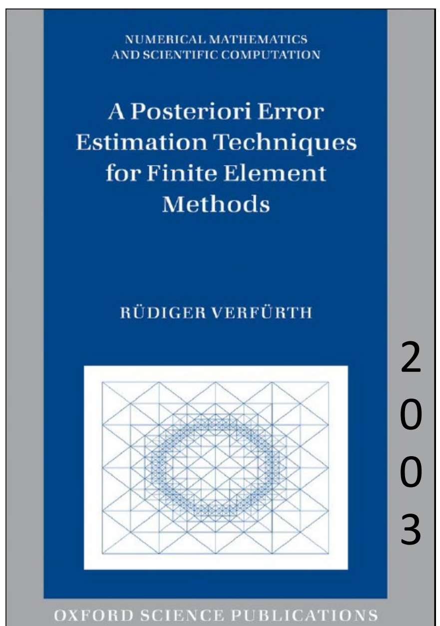

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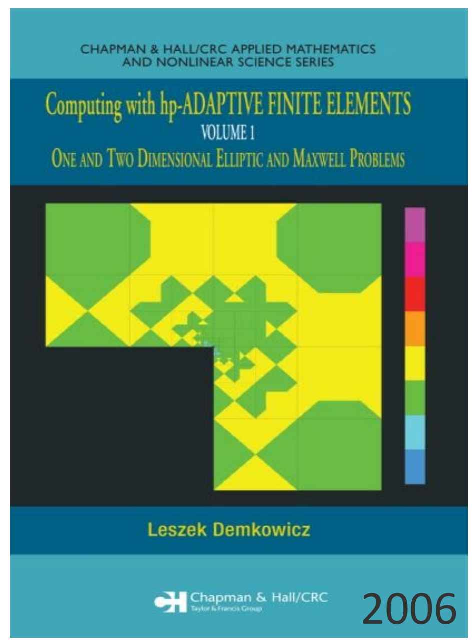

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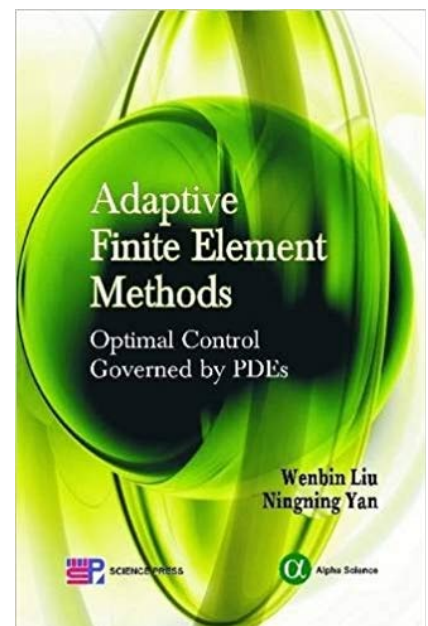
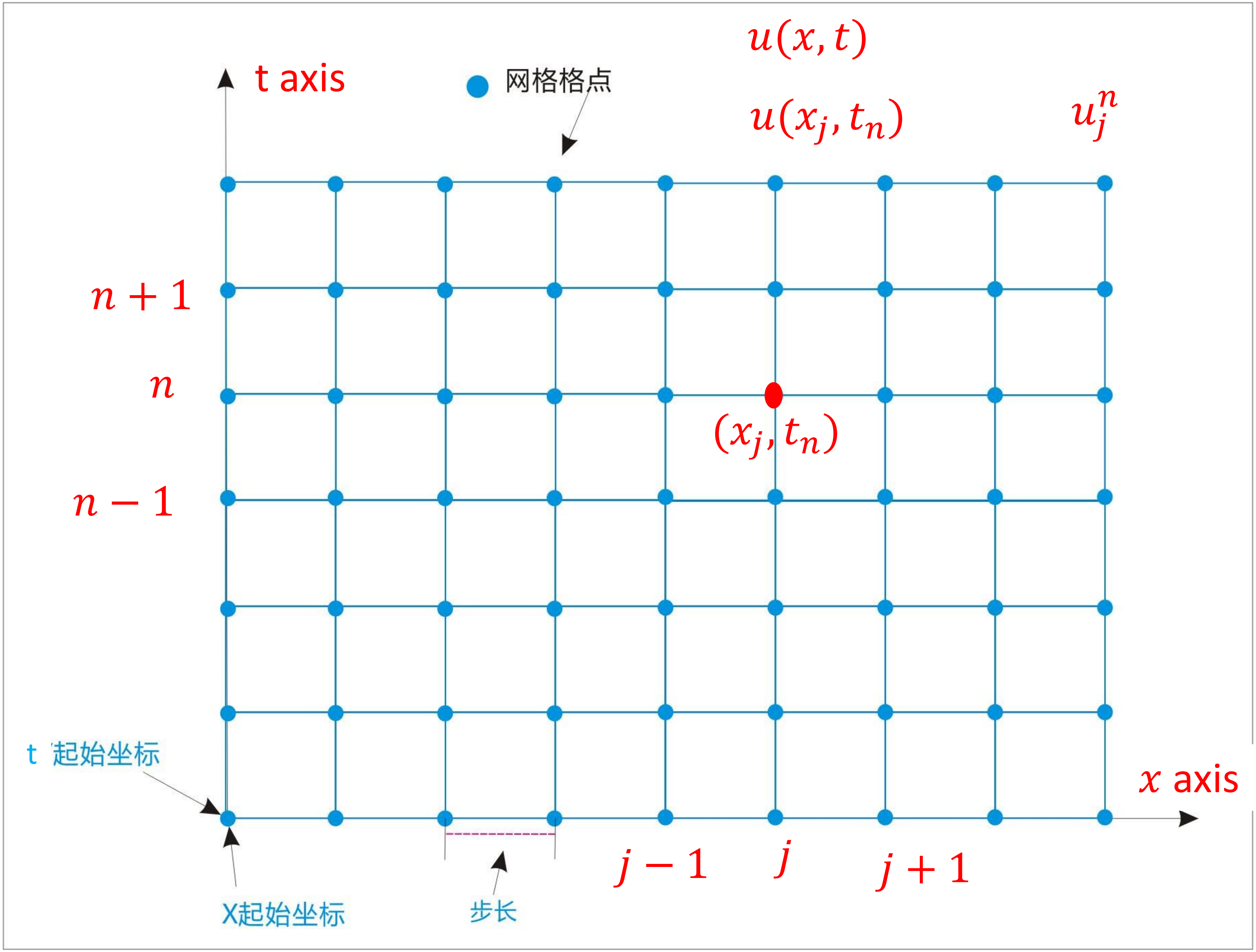
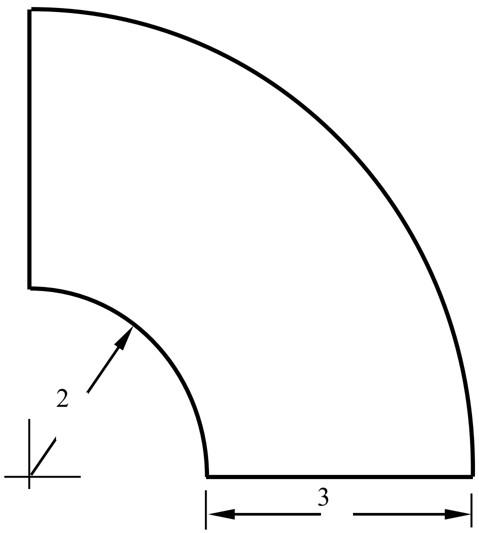
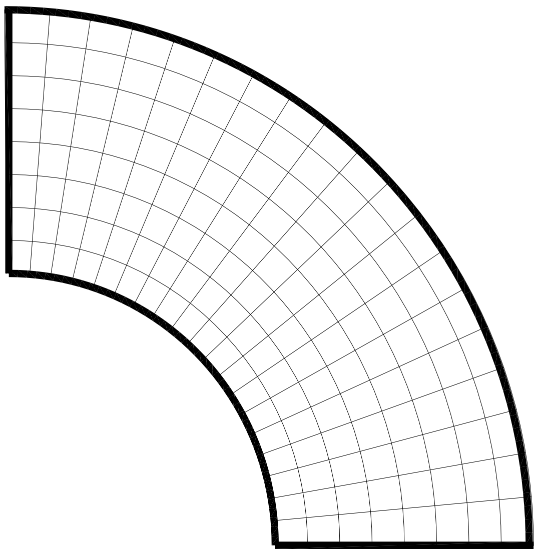




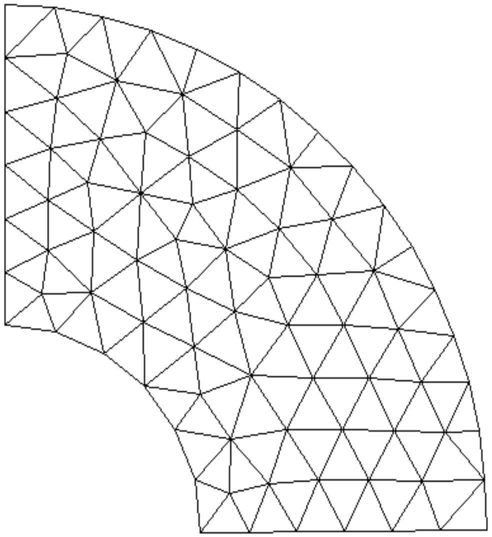
t axis
网格格点
$u(x,t)$
$u(x_j,t_n)$
u_j^n
$n+1$
n
$n-1$
(x_j,t_n)
t 起始坐标
x axis
$j-1$
j
$j+1$
X起始坐标
步长

Meshes for 2D domain



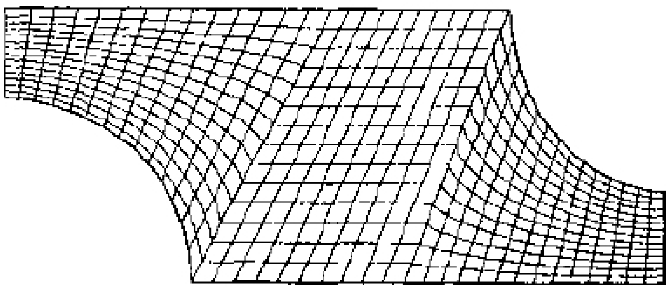


正交四边形网格

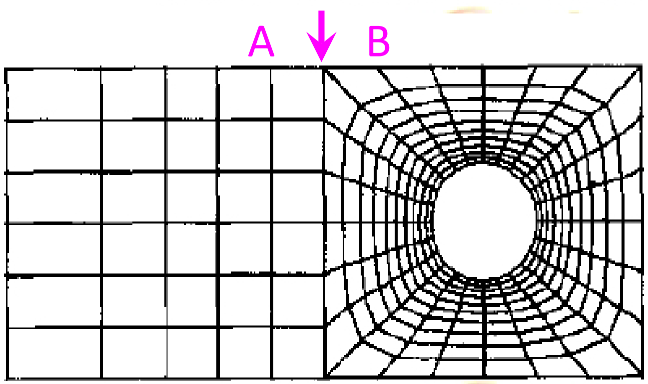


三角形网格

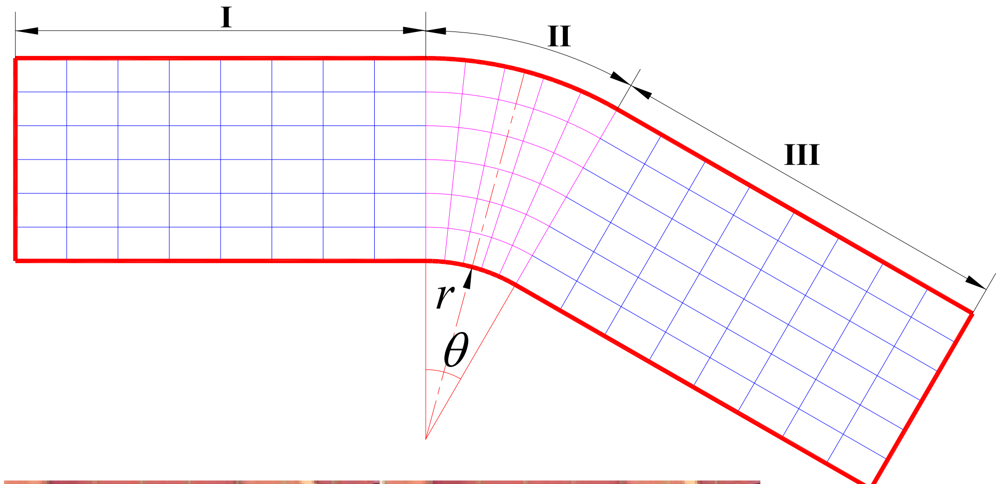
Meshes for 2D domain



单域结构化网格



分区块结构化网格



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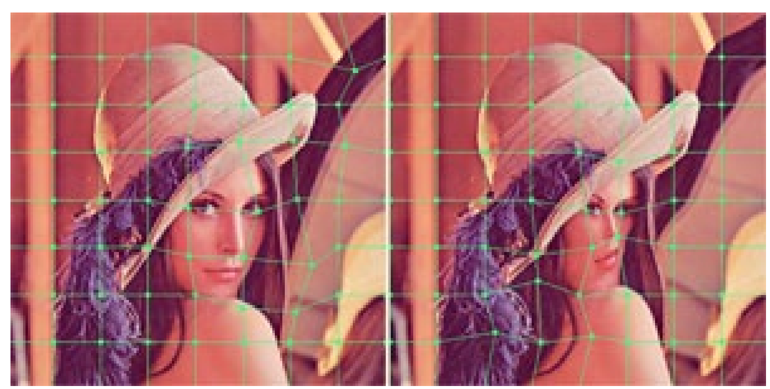
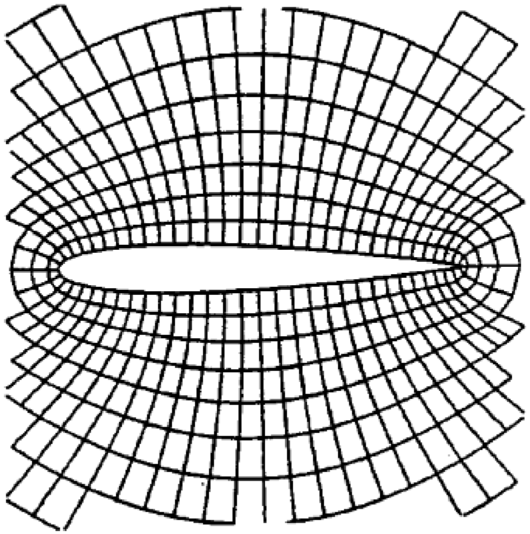
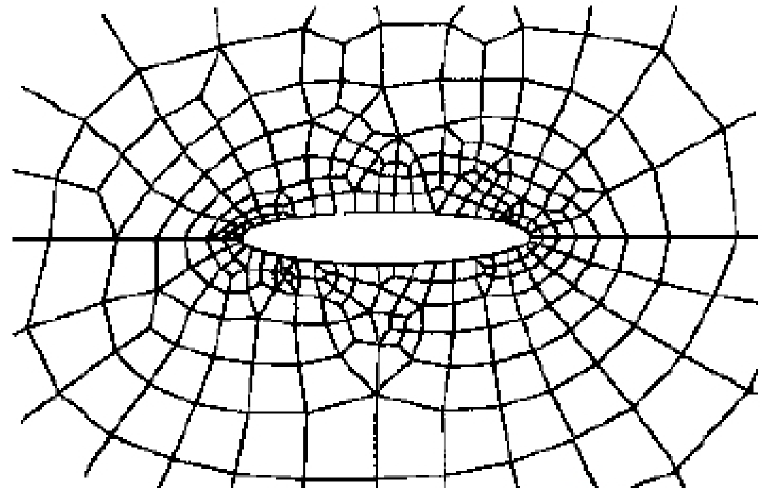


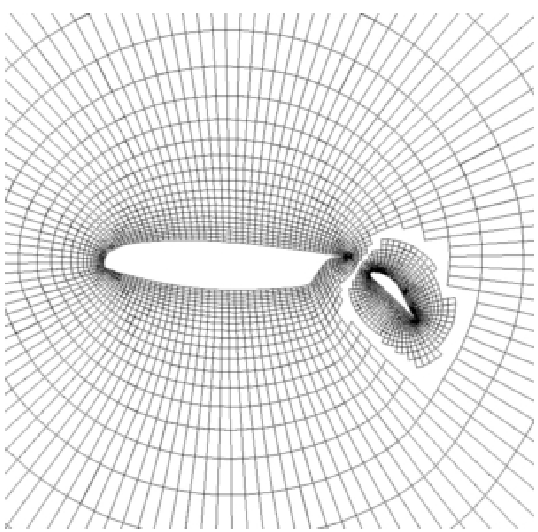
Meshes for 2D domain

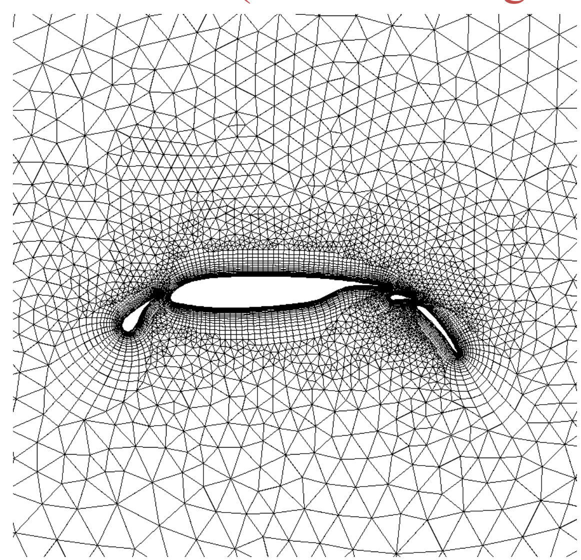


结构化网格(structured grid)

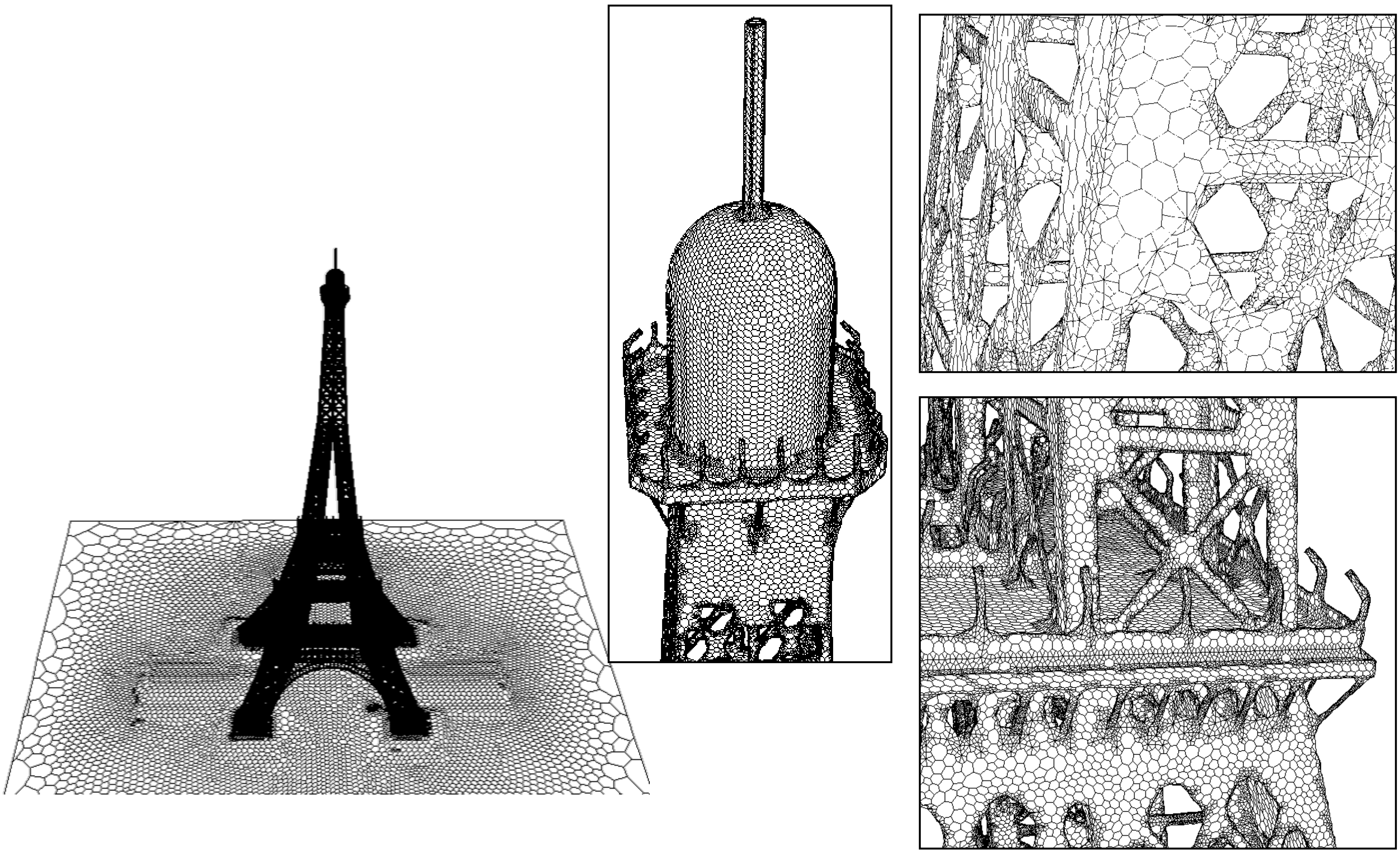


非结构化网格(unstructured grid)

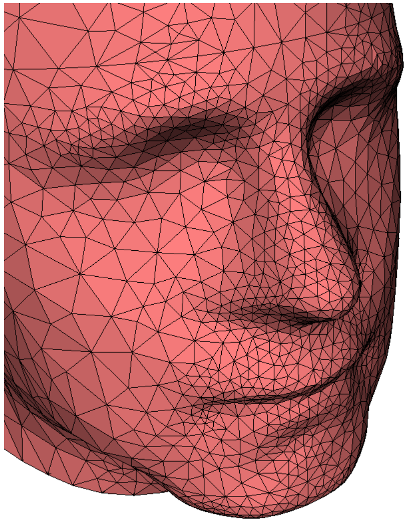




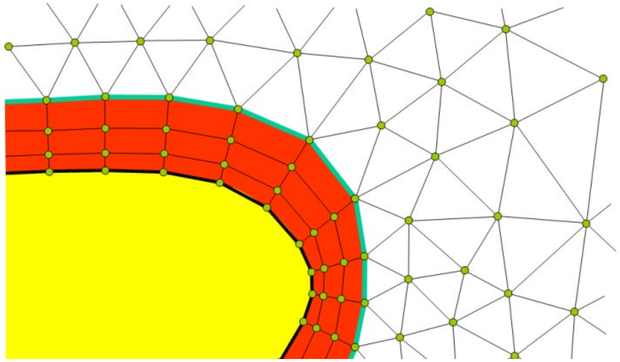
Meshes for 3D domain



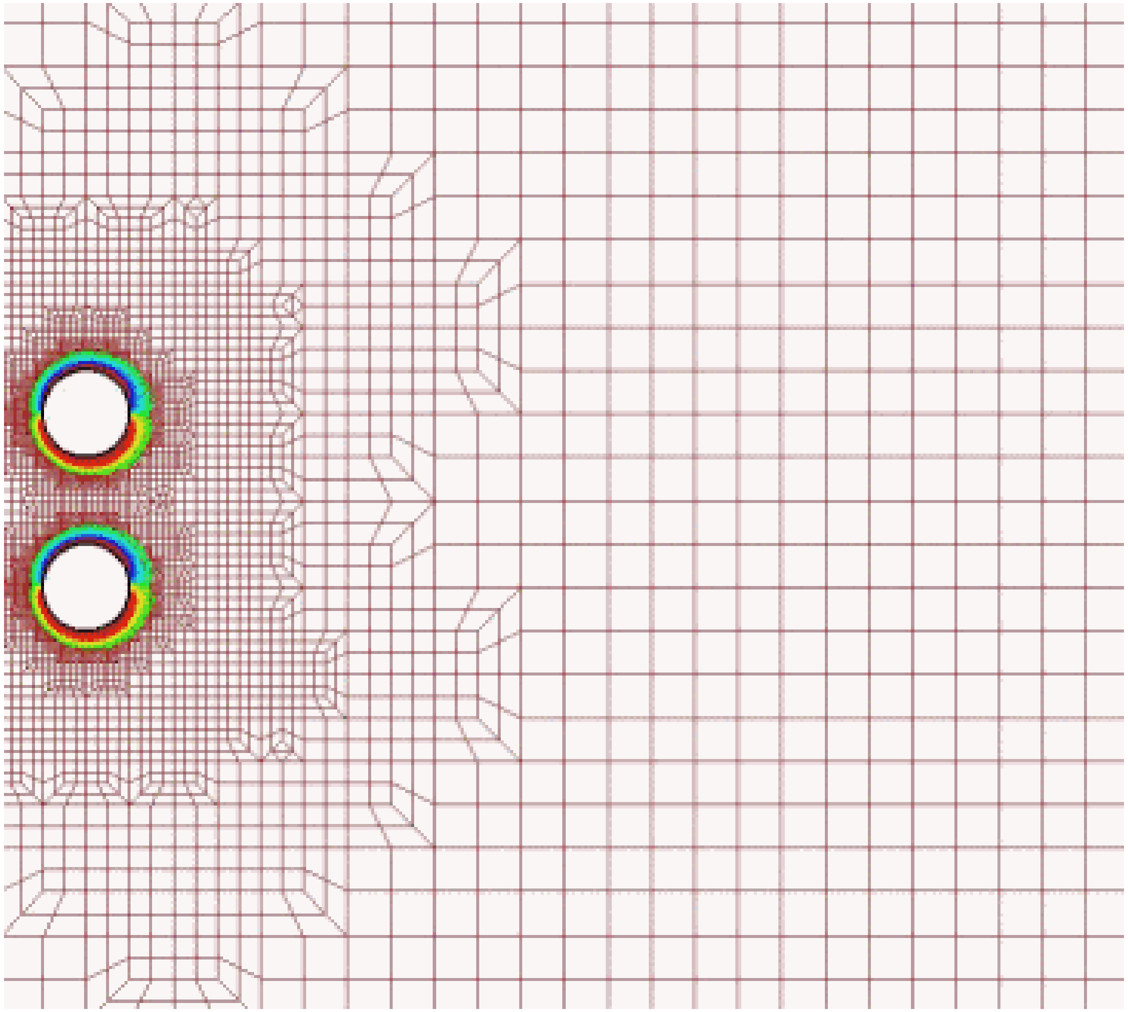
Meshes for 2D domain



非结构化网格



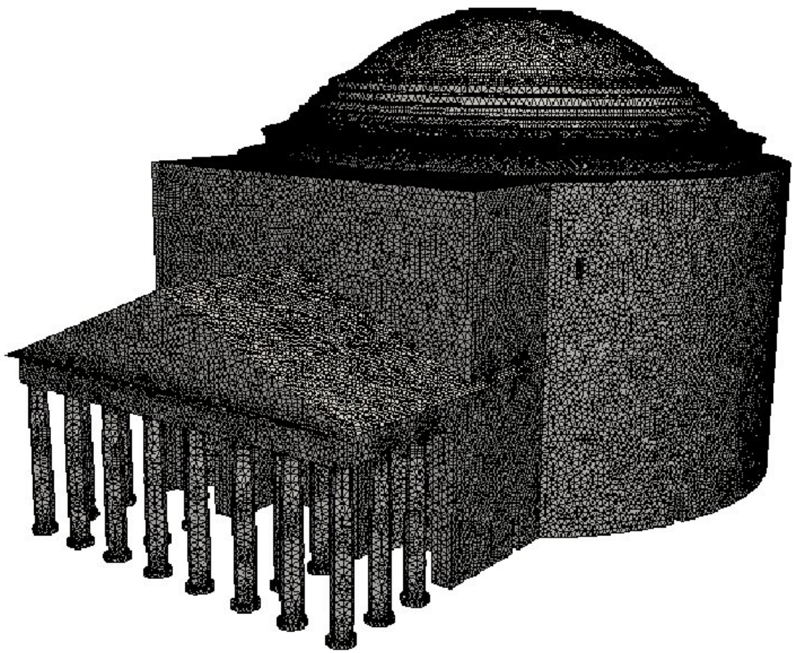
结构化/非结构化混合网格

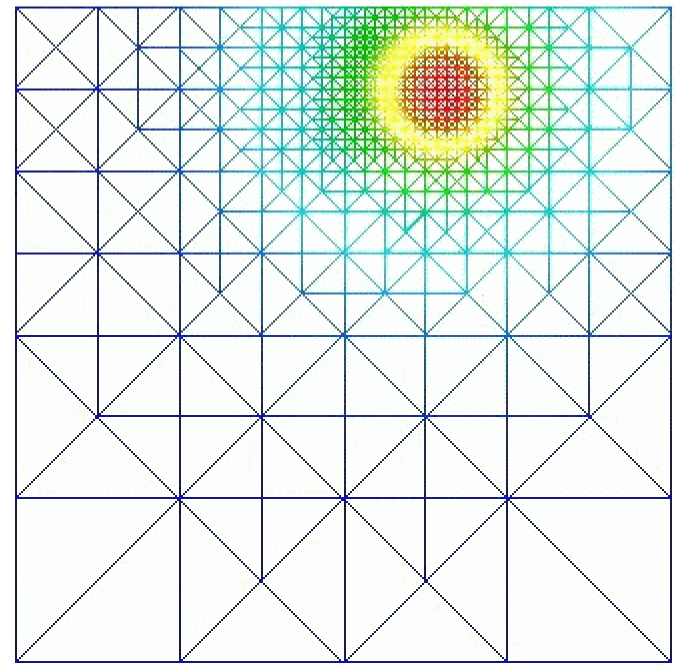


自适应网格

PHG 三维自适应有限元软件平台

Parallel Hierarchical Grid





三维结构分析 PHG-Solid

热传导方程　　自适应网格

http://lsec.cc.ac.cn/phg/application.htm#"