Numerical Solutions to Partial Differential Equations

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Finite Element Method — a Method Based on Variational Problems

Finite Difference Method:

1. Based on PDE problem.
2. Introduce a grid (or mesh) on Ω. $^{-}$
3. Define grid function.
4. Approximate differential operators by difference operators.
5. PDE discretized into a finite algebraic equation.

Finite Element Method:

1. Based on variational problem, say $F(u) = \inf_{v \in \mathbb{X}} F(v)$.
   minimize Functional
2. Introduce a grid (or mesh) on $\overline{\Omega}$.
3. Establish finite dimensional subspaces $\mathbb{X}_h$ of $\mathbb{X}$.
4. Restrict the original problem on the subspaces, say $F(U_h) = \inf_{V_h \in \mathbb{X}_h} F(V_h)$.
5. PDE discretized into a finite algebraic equation.

Functional refers to a linear mapping from a vector space V into its field of scalars, i.e., it refers to an element of the dual space V*. It refers to a mapping from a space X into the real numbers, or sometimes into the complex numbers, for the purpose of establishing a calculus-like structure on X.

An Abstract Variational Form of Energy Minimization Problem

Many physics problems, such as minimum potential energy principle in elasticity, lead to an abstract variational problem:

$$\begin{cases} \text{Find } \ u \in \mathbb{U} \text{ such that} \\ J(u) = \inf\limits_{v \in \mathbb{U}} J(v), \end{cases} \tag{5.1.1}$$

where $\mathbb{U}$ is a nonempty closed subset of a Banach space $\mathbb{V}$, and $J : v \in \mathbb{U} \to \mathbb{R}$ is a functional. In many practical linear problems,

- $\mathbb{V}$ is a Hilbert space, $\mathbb{U}$ a closed linear subspace of $\mathbb{V}$;
- the functional J often has the form

$$J(v) = \frac{1}{2} a(v,\ v) - f(v), \tag{5.1.2}$$

- $a(\cdot,\ \cdot)$ and f are continuous bilinear and linear functionals.

最小势能原理: 在所有变形可能的位移场中, 真实的位移场使总势能泛函取最小值。
一个体系的势能最小时，系统会处于稳定平衡状态。

Find Solutions to a Functional Minimization Problem

Method 1 — Direct method of calculus of variations:

introduced by Zaremba and David Hilbert around 1900

1. Find a minimizing sequence, say, by gradient type methods;
2. Find a convergent subsequence of the minimizing sequence, say, by certain kind of compactness;
3. Show the limit is a minimizer, say, by lower semi-continuity of the functional.

Lower semi-continuous at x_0: lim inf_{x\rigntarrow x_0} f(x) \geq f(x_0), where \liminf is the limit inferior (of the function f at point x_{0}).Û. Dacorogna, Direct method in the calculus of variations, Springer, 2008.

Find Solutions to a Functional Minimization Problem

Method 2 — Solving the Euler-Lagrange equation:

developed by Euler and Lagrange in the 1750s.

1. Work out the corresponding Euler-Lagrange equation;
2. For smooth solutions, the Euler-Lagrange equation leads to classical partial differential equations;
3. In general, the Euler-Lagrange equation leads to another form of variational problems (weak form of classical partial differential equations).

Both methods involve the derivatives of the functional J.

Fréchet Derivatives of Maps on Banach Spaces

Let $\mathbb{X}$, $\mathbb{Y}$ be real normed linear spaces, Ω is an open set of $\mathbb{X}$. Let $F : \Omega \to \mathbb{Y}$ be a map, nonlinear in general.

Definition 5.1

F is said to be Fréchet differentiable at $x \in \Omega$, if there exists a linear map $A : \mathbb{X} \to \mathbb{Y}$ satisfying: for any $\varepsilon > 0$, there exists a $\delta > 0$, such that F-可微的

有的文献中会要求映射A是有界的。

$$\|F(x+z) - F(x) - Az\| \le \varepsilon \|z\|, \quad \forall z \in \mathbb{X} \ \text{ with } \ \|z\| \le \delta. \tag{5.1.3}$$

The map A is called the Fréchet derivative of F at x, denoted as $F'(x) = A$, or $dF(x) = A$. $F'(x)z = Az$ is called the Fréchet differential of F at x, or the first order variation. F-微商 F-微分

The Fréchet differential is an extension of total differential in the multidimensional calculus.

Higher Order Fréchet Derivatives

Definition 5.1b

If for any $z \in \mathbb{X}$, $F'(x)z$ is Fréchet differentiable at $x \in \Omega$, F is said to be second order Fréchet differentiable at $x \in \Omega$.

The second order Fréchet derivative of F at x is a $\mathbb{X} \times \mathbb{X} \to \mathbb{Y}$ bilinear form, denoted as $F''(x)$ or $d^2F(x)$.

$F''(x)(z, y) = d^2F(x)(z, y) = (F'(x)z)'y$ is called the second order Fréchet differential of F at x, or the second order variation.

Recursively, we can define the mth order Fréchet derivative of F at x by $d^mF(x) \triangleq d(d^{m-1}F(x))$, and the mth order Fréchet differential (or the mth order variation) $d^mF(x)(z_1, \ldots, z_m)$.

The mth order Fréchet derivative $d^mF(x)$ is said to be bounded, if $d^mF(x)(z_1, \ldots, z_m) : \mathbb{X}^m \to \mathbb{Y}$ is a bounded m linear map.

Gâteaux Derivatives — An Extension of Directional Derivatives

Definition 5.2

F is said to be Gâteaux differentiable at $x \in \Omega$ in the direction $z \in \mathbb{X}$, if the following limit exists:

$$DF(x; z) = \lim_{t \to 0} \frac{F(x + tz) - F(x)}{t}. \tag{5.1.4}$$

$DF(x; z)$ is called the Gâteaux differential of F at x in the direction $z \in \mathbb{X}$. if the map $DF(x; z)$ is linear with respect to z, *i.e.* there exists a linear map $A : \mathbb{X} \to \mathbb{Y}$ such that $DF(x; z) = Az$, then the map A is called the Gâteaux derivative of F at x, and is denoted as $DF(x) = A$.

Gâteaux Derivatives — An Extension of Directional Derivatives

- The Gâteaux derivative is an extension of the directional directives in the multidimensional calculus;
- Fréchet differentiable implies Gâteaux differentiable, the inverse is not true in general.

Higher Order Gâteaux Derivatives

Definition 5.2b

If for a given $z \in \mathbb{X}$, $DF(x; z)$ is Gâteaux differentiable at $x \in \Omega$ in the direction $y \in \mathbb{X}$, then the corresponding differential is called the second order mixed Gâteaux differential of F at x in the directions z and y, and is denoted as $D^2F(x; z, y)$.

If $D^2F(x; z, y)$ is bilinear with respect to (z, y), then the bilinear form $D^2F(x)$, with $D^2F(x)(z, y) \triangleq D^2F(x; z, y)$, is called the second order Gâteaux derivative of F at x.

We can recursively define the mth order mixed Gâteaux differential $D^mF(x; z_1, \ldots, z_m) \triangleq D(D^{m-1}F)(x; z_1, \ldots, z_{m-1}; z_m)$, and the mth order Gâteaux derivative $D^mF(x) \triangleq D(D^{m-1}F)(x)$.

Higher Order Gâteaux Derivatives — Commutability

1. If the Gâteaux differential $DF(\cdot)$ of F exists in a neighborhood of x and is continuous at x, then, the Fréchet differential of F at x exists and $\boxed{dF(x)z = DF(x)z = \frac{d}{dt}F(x+tz)\Big|_{t=0}.}$ (5.1.5)

   用G微分计算F微分

   (notice that $F(x+z) - F(x) = \int_0^1 \frac{d}{dt}F(x+tz)\,dt$).

2. In general, $D^2F(x;z,y) \neq D^2F(x;y,z)$, *i.e.* the map is not necessarily symmetric with respect to (y,z).

   (counter examples can be found in multi-dimensional calculus).

计算F微分常常是方便的: 引入实参数t, 暂固定z, 则F(x+tz)是一个R到Y(F的值域)的映射: 由微分运算的链式法则可以导出公式(5.1.5).

Higher Order Gâteaux Derivatives — Commutability

3. If the mth order Gâteaux differential $D^m F(\cdot)$ is a uniformly bounded m linear map in a neighborhood of x_0 and is uniformly continuous with respect to x, then $D^m F(\cdot)$ is indeed symmetric with respect to $(z_1, \ldots, z_m)$,

   in addition the mth order Fréchet differential exists and

   $\boxed{F^{(m)}(x_0) = d^m F(x_0) = D^m F(x_0)}$ with

用G微分计算F微分

$$F^{(m)}(x)(z_1, \ldots, z_m) = \frac{d}{dt_m}\Big[\cdots\Big[\frac{d}{dt_1} F(x + t_1 z_1 + \cdots + t_m z_m)\Big|_{t_1=0}\Big]\cdots\Big]\Big|_{t_m=0}.$$

A Necessary Condition for a Functional to Attain an Extremum at x

Let $F : \mathbb{X} \to \mathbb{R}$ be Fréchet differentiable, and F attains a local extremum at x. Then

1. For fixed $z \in \mathbb{X}$, $f(t) \triangleq F(x + tz)$, as a differentiable function of $t \in \mathbb{R}$, attains a same type of local extremum at $t = 0$.
2. Hence, $F'(x)z = f'(0) = 0$, $\forall z \in \mathbb{X}$.
3. Therefore, a necessary condition for a Fréchet differentiable functional F to attain a local extremum at x is

$$F'(x)z = 0, \quad \forall z \in \mathbb{X}, \qquad (5.1.6)$$

即F'(x)=0

which is called the weak form (or variational form) of the Euler-Lagrange equation $F'(x) = 0$ of the extremum problem.

称使(5.1.6)成立的点为驻点. 极值点必为驻点, 反之不一定真.

A Typical Example on Energy Minimization Problem

例5.1 讨论当a(u,v)对称时变分问题存在唯一解.

(5.1.2)

1. $J(v) = \frac{1}{2}\, a(v,\ v) - f(v)$. $a(\cdot,\cdot)$ symmetric, a, f continuous.
2. $t^{-1}(J(u+tv) - J(u)) = a(u,v) - f(v) + \frac{t}{2}a(v,v)$.

(5.1.4) (Since $a(u+tv,u+tv) = a(u,u) + t(a(u,v) + a(v,u)) + t^2 a(v,v)$ and $f(u+tv) = f(u) + tf(v)$.)

3. Gâteaux differential $DJ(u)v = a(u,v) - f(v)$.
4. Continuity $\overset{(5.1.5)}{\Rightarrow}$ Fréchet differential $J'(u)v = a(u,v) - f(v)$.
5. $t^{-1}(J'(u+tw,v) - J'(u,v)) = a(w,v)$.

(5.1.4)

6. $J''(u)(v,w) = a(w,v)$. $J^{(k)}(u) = 0$, for all $k \geq 3$.

A Typical Example on Energy Minimization Problem

7. Suppose that $u \in \mathbb{U}$ satisfies $J'(u)v = 0, \ \forall v \in \mathbb{U}$. Then (5.1.6)
8. $J(u+tv) = J(u) + tJ'(u)v + \frac{t^2}{2}J''(u)(v,v) = J(u) + \frac{t^2}{2}a(v,v)$.
9. If, in addition, $\exists$ const. $\alpha > 0$, s.t. $a(v,v) \geq \alpha\|v\|^2$, $\forall v \in \mathbb{U}$, then $J(u+tv) \geq J(u) + \frac{1}{2}\alpha t^2\|v\|^2 \geq$ J(u)

Under the conditions that $a(\cdot,\cdot)$ is a symmetric, continuous and uniformly elliptic bilinear form, and f is a continuous linear form,

$$u \textbf{ is the unique minimum of } J \Leftrightarrow J'(u) = 0.$$

问题: 当a(u,v)不对称时变分问题是否存在唯一解? 【Lax-Milgram引理】

Abstract Variational Problem Corresponding to the Virtual Work Principle

Various forms of variational principles, such as the virtual work principle in elasticity, etc., lead to the following abstract variational problem:

$$\begin{cases} \text{Find } \ u \in \mathbb{V} \text{ such that} \\ A(u)v = 0, \quad \forall v \in \mathbb{V}, \end{cases} \qquad (5.1.8)$$

where $A \in \mathfrak{L}(\mathbb{V}; \mathbb{V}^*)$, *i.e.* $A(\cdot)$ is a linear map from $\mathbb{V}$ to its dual space $\mathbb{V}^*$.

- In an energy minimization problem, a necessary condition for $u \in \mathbb{U}$ to be a minimizer is that $J'(u)v = 0$, $\forall v \in \mathbb{U}$. (5.1.6)
- In the case when $a(\cdot,\cdot)$ is uniformly elliptic, the two problems are equivalent.

(5.1.1) (5.1.8)

(5.1.7)

Lax-Milgram Lemma — Existence and Uniqueness of a Solution

Theorem 5.1

Let $\mathbb{V}$ be a Hilbert space. let $a(\cdot,\ \cdot) : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ be a continuous bilinear form satisfying the $\mathbb{V}$-elliptic condition (also known as the coerciveness condition):

$$\exists \alpha > 0, \quad \text{such that } a(u,u) \geq \alpha \|u\|^2, \quad \forall u \in \mathbb{V}, \tag{5.1.9}$$

$f : \mathbb{V} \to \mathbb{R}$ be a continuous linear form. Then, the abstract variational problem

$$\begin{cases} \text{Find } \ u \in \mathbb{V} \text{ such that} \\ a(u,v) = f(v), \quad \forall v \in \mathbb{V}, \end{cases} \tag{5.1.10}$$

has a unique solution.

注: 这里a(u,v)不必对称. 此时不再对应到极小问题(5.1.1).

Proof of the Lax-Milgram Lemma

1. Continuity of $a(\cdot,\cdot) \Rightarrow \exists$ const. $M > 0$ such that

Banach空间中线性泛函和双线性形式连续等价于有界

$$a(u,v) \le M\|u\|\|v\|, \quad \forall u,\ v \in \mathbb{V}. \tag{5.1.11}$$

2. $v \in \mathbb{V} \to a(u,v)$ continuous linear $\Rightarrow \exists!\ A(u) \in \mathbb{V}^*$ such that

For fixed u

functional, so in V*

$$A(u)v = a(u,v), \quad \forall v \in \mathbb{V}. \tag{5.1.12}$$

3. $\|A\|_{\mathfrak{L}(\mathbb{V},\mathbb{V}^*)} = \sup_{u\in\mathbb{V},\|u\|=1} \sup_{v\in\mathbb{V},\|v\|=1} |A(u)v| \le M.$ (5.1.13)

4. $\tau : \mathbb{V}^* \to \mathbb{V}$, the Riesz map: $f(v) = \langle \tau f,\ v\rangle, \forall v \in \mathbb{V}.$

Riesz表示定理：设f\in H*(H的对偶空间)，则恰有一个z_f\in H, s.t. f(x)=<x, z_f>, for all x\in H, and ||f||_{H*}=||z_f||_H.

Proof of the Lax-Milgram Lemma (Cont'd)

5. The abstract variational problem is equivalent to

$$\begin{cases} \text{Find } \ u \in \mathbb{V} \text{ such that} \\ \tau A(u) = \tau f. \end{cases} \tag{5.1.14}$$

6. Define $F : \mathbb{V} \to \mathbb{V}$ as $F(v) = v - \rho(\tau A(v) - \tau f)$.
7. Then, u is a solution $\Leftrightarrow$ $F(u) = u$. (*i.e.* u is a fix point of F.)
8. Since $\langle \tau A(v),\ v\rangle = A(v)v = a(v, v) \geq \alpha\|v\|^2$,
9. $\|\tau A(v)\| = \|A(v)\|^* \leq \|A\|_{\mathfrak{L}(\mathbb{V},\mathbb{V}^*)}\|v\| \leq M\|v\|$, and
10. $\|F(w + v) - F(w)\|^2 = \|v\|^2 - 2\rho\langle\tau A(v),\ v\rangle + \rho^2\|\tau A(v)\|^2$,

($\because$ $F(w + v) = w + v - \rho(\tau A(w + v) - \tau f) = F(w) + v - \rho\tau A(v)$,)

Proof of the Lax-Milgram Lemma (continue)

10. therefore, for any given $\rho \in (0,\, 2\alpha/M^2)$, we have

$$\|F(w+v) - F(w)\|^2 \leq (1 - 2\rho\alpha + \rho^2 M^2)\|v\|^2 < \|v\|^2,$$

11. $F : \mathbb{V} \to \mathbb{V}$ is a contractive map, for $\rho \in (0,\, 2\alpha/M^2)$.
12. ~~In addition, if~~ $\cancel{\|v\| > (2\alpha - M^2\rho)^{-1}\|f\|}$~~, then~~ $\cancel{\|F(v)\| < \|v\|}$~~.~~
13. By the contractive-mapping principle, F has a unique fixed point in $\mathbb{V}$. ■

Remark: In applications, the Hilbert space $\mathbb{V}$ in the variational problem usually consists of functions with derivatives in some weaker sense. Sobolev spaces are important in studying variational forms of PDE and the finite element method.

Let V be an Hilbert space, a(·, ·) a bilinear form on V , that is both continuous and coercive.

Lemma 2.3 (*Lax-Milgram*). *Given a continuous linear form L on V, there exists a unique $u \in V$ such that*

$$a(u, v) = L(v), \tag{2.13}$$

and it holds that $\|u\|_V \leq \|L\|/\alpha$.

some constants $c > 0$ and $\alpha > 0$:

$$\begin{cases} \text{(i)} & |a(u,v)| \leq c\|u\|_V\|v\|_V, & \text{for all } u, v \text{ in } V, \\ \text{(ii)} & a(u,u) \geq \alpha\|u\|_V^2, & \text{for all } u \text{ in } V. \end{cases} \tag{2.12}$$

Proof of lemma 2.13. Taking $v = u$ in (2.13) we obtain that

$$\alpha\|u\|_V^2 \leq a(u,u) \leq L(u) \leq \|L\|\|u\|, \tag{2.18}$$

so that $\|u\|_V \leq \|L\|/\alpha$. The uniqueness of u follows. It remains to prove the existence property. By the Riesz theorem, there exists $w \in V$ and $B(u) \in V$ such that

$$L(v) = (w, v)_V \text{ and } \mathcal{A}u(v) = (B(u), v)_V, \quad \text{for all } v \in V. \tag{2.19}$$

It is easily checked that $u \mapsto B(u)$ is linear continuous so we may write $B(u) = Bu$ with $B \in L(V)$. Therefore (2.13) is equivalent to

$$Bu = w. \tag{2.20}$$

This is equivalent to find a fixed point of the affine mapping $V \to V$, $\mathcal{T}u := u - \varepsilon(Bu - w)$ for some $\varepsilon > 0$. Observe that

$$(Bu, u)_V = a(u,u) \geq \alpha\|u\|_V^2. \tag{2.21}$$

Therefore, taking u and u' in V and setting $u := u'' - u'$:

$$\begin{aligned} \|\mathcal{T}u'' - \mathcal{T}u'\|_V^2 &= \|u\|^2 - 2\varepsilon(Bu, u)_V + \varepsilon^2\|Bu\|^2 \\ &\leq \left(1 - 2\varepsilon\alpha + \varepsilon^2\|B\|^2\right)\|u\|^2. \end{aligned} \tag{2.22}$$

So, $\mathcal{T}$ is, when $\varepsilon < 2\alpha$, a contractive mapping and has therefore a unique fixed point. □

http://www.cmap.polytechnique.fr/~bonnans/notes/co-edp/ch2-3.pdf

Definition of Generalized Derivatives for Functions in $\mathbb{L}^1_{loc}(\Omega)$

Let $u \in \mathbb{C}^m(\Omega)$, then, for any $\phi \in \mathbb{C}_0^\infty(\Omega)$, it follows from the Green's formula that

$$\int_\Omega (\partial^\alpha u)\,\phi\,dx = (-1)^{|\alpha|}\int_\Omega u\,(\partial^\alpha \phi)\,dx.$$

Definition 5.3

Let $u \in \mathbb{L}^1_{loc}(\Omega)$, if there exists $v_\alpha \in \mathbb{L}^1_{loc}(\Omega)$ such that

$$\int_\Omega v_\alpha\,\phi\,dx = (-1)^{|\alpha|}\int_\Omega u\,(\partial^\alpha \phi)\,dx, \qquad \forall \phi \in \mathbb{C}_0^\infty(\Omega),$$

then v_α is called a $|\alpha|$th order generalized partial derivative (or weak partial derivative) of u with respect to the multi-index α, and is denoted as $\partial^\alpha u = v_\alpha$.

任何可积函数是局部可积的, 反之不真. 例f(x)=1在R上局部可积, 但不可积.

An Important Property of Generalized Derivatives

The concept of the generalized derivatives are obviously an extension of that of the classical derivatives.

In addition, the generalized derivatives also inherit some important properties of the classical derivatives. In particular, we have

Theorem 5.2

Let $\Omega \subset \mathbb{R}^n$ be a connected open set. Let all of the generalized partial derivatives of order $|\alpha| = m + 1$ of u are zero, then, u is a polynomial of degree no greater than m on Ω.

An Important Property of Generalized Derivatives

Remark: Two functions in $\mathbb{L}^1_{loc}(\Omega)$ are considered to be the same (or in the same equivalent class of functions), if they are different only on a set of zero measure.

The theorem above is understood in the sense that there exists a representative in the equivalent class of u such that the conclusion holds.

Definition of the Sobolev Spaces

Definition 5.4

Let m be a nonnegative integer, let $1 \leq p \leq \infty$, define

$$\mathbb{W}^{m,p}(\Omega) = \{u \in \mathbb{L}^p(\Omega) : \partial^\alpha u \in \mathbb{L}^p(\Omega),\ \forall \alpha \text{ s.t. } 0 \leq |\alpha| \leq m\},$$

where $\mathbb{L}^p(\Omega)$ is the Banach space consists of all Lebesgue p integrable functions on Ω with norm $\|\cdot\|_{0,p,\Omega}$. Then, the set $\mathbb{W}^{m,p}(\Omega)$ endowed with the following norm

$$\|u\|_{m,p,\Omega} = \Big(\sum_{0\leq|\alpha|\leq m} \|\partial^\alpha u\|_{0,p,\Omega}^p\Big)^{1/p}, \quad 1 \leq p < \infty;$$

$$\|u\|_{m,\infty,\Omega} = \max_{0\leq|\alpha|\leq m} \|\partial^\alpha u\|_{0,\infty,\Omega}$$

is a normed linear space, and is called a Sobolev space, denoted again as $\mathbb{W}^{m,p}(\Omega)$.

Some Basic Inequalities of $\mathbb{L}^p(\Omega)$ Functions

The following inequalities are very important for analysis in the Sobolev spaces.

Minkowski inequality: For any $1 \leq p \leq \infty$ and f, $g \in \mathbb{L}^p(\Omega)$,

$$\|f + g\|_{0,p,\Omega} \leq \|f\|_{0,p,\Omega} + \|g\|_{0,p,\Omega}.$$

Hölder inequality: Let $1 \leq p$, $q \leq \infty$ satisfy $1/p + 1/q = 1$, then, for any $f \in \mathbb{L}^p(\Omega)$ and $g \in \mathbb{L}^q(\Omega)$, we have $f \cdot g \in \mathbb{L}^1(\Omega)$, and

$$\|f \cdot g\|_{0,1,\Omega} \leq \|f\|_{0,p,\Omega} \|g\|_{0,q,\Omega}.$$

Cauchy-Schwarz inequality: In particular, for $p = q = 2$, it follows from the Hölder inequality that

$$\|f \cdot g\|_{0,1,\Omega} \leq \|f\|_{0,2,\Omega} \|g\|_{0,2,\Omega}.$$

Some important Facts of Sobolev Spaces

- $\mathbb{W}^{m,p}(\Omega)$ is a Banach space.
- If $p = 2$, $\mathbb{W}^{m,p}(\Omega)$ is a Hilbert space, denoted as $\mathbb{H}^m(\Omega)$, and its norm is often denoted as $\|\cdot\|_{m,\Omega}$.

Theorem 5.3 Sobolev空间的理论中的基本结果

If the boundary $\partial\Omega$ of the domain Ω is Lipschitz continuous, then, for $1 \le p < \infty$, $\mathbb{C}^\infty(\overline{\Omega})$ is dense in $\mathbb{W}^{m,p}(\Omega)$.

- $\mathbb{W}^{m,p}(\Omega)$ is a closure of $\mathbb{C}^\infty(\overline{\Omega})$ w.r.t the norm $\|\cdot\|_{m,p}$.

Def：设A,B是距离空间中的两个集合，如果对A中任意元素x，总存在B中序列y_n使得当n趋于无穷时{y_n}的极限是x, 则称B在A中稠密.

A subset A of a topological space X is dense for which the closure is the entire space X. If U⊂X, a set A⊂X is called dense in U if A∩U is a dense set in the subspace topology of U. When U is open this is equivalent to the requirement that the closure (in X) of A contains U. 例C^∞(Ω) 是H^m(Ω)的子集, 且在H^m中稠密.

Some important Facts of Sobolev Spaces

Definition 5.4b

The closure of $\mathbb{C}_0^\infty(\Omega)$ w.r.t. the norm $\|\cdot\|_{m,p}$ is a subspace of the Sobolev space $\mathbb{W}^{m,p}(\Omega)$, and is denoted as $\mathbb{W}_0^{m,p}(\Omega)$.

p=2

- $\mathbb{H}_0^m(\Omega) \triangleq \mathbb{W}_0^{m,2}(\Omega)$ is a Hilbert space.

Poincaré-Friedrichs Inequality

Theorem 5.4

Let the domain Ω *be of finite width,* i.e. *it is located between two parallel hyperplanes. Then, there exist a constant* $K(n,m,d,p)$*, which depends only on the space dimension* n*, the order* m *of the partial derivatives, the distance* d *between the two hyperplanes and the Sobolev index* $1 \leq p < \infty$*, such that*

$$\boxed{|u|_{m,p} \leq \|u\|_{m,p} \leq K(n,m,d,p)|u|_{m,p},} \quad \forall u \in W_0^{m,p}(\Omega), \tag{5.2.3}$$

where

$$|u|_{m,p} = \Big(\sum_{|\alpha|=m} \|\partial^\alpha u\|_{0,p,\Omega}^p \Big)^{1/p}, \quad 1 \leq p < \infty$$

is a semi-norm of the Sobolev space $\mathbb{W}^{m,p}(\Omega)$*. The inequality is usually called the Poincaré-Friedrichs inequality.*

由范数和半范数定义知，(5.2.3)的左边不等号是显然的

Proof of the Poincaré-Friedrichs Inequality

1. Assume the domain Ω is between $x_n = 0$ and $x_n = d$.
2. Denote $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1})$. For any given $u \in \mathbb{C}_0^\infty(\Omega)$, we have $u(x) = \int_0^{x_n} \frac{d}{dt} u(x', t)\, dt$.
3. For $p' = \frac{p}{p-1}$, by the Hölder inequality,

$$|u(x)| = \left| \int_0^{x_n} \partial_n u(x', t)\, dt \right| \le \left(\int_0^{x_n} 1^{p'} \right)^{1/p'} \left(\int_0^{x_n} |\partial_n u(x', t)|^p \right)^{1/p} \quad (*)$$

Proof of the Poincaré-Friedrichs Inequality

4.

$$\|u\|_{0,p,\Omega}^p = \int_{R^{n-1}} \int_0^d |u(x)|^p dx_n dx'$$

$$\underset{(*)}{\leq} \int_0^d x_n^{p-1} dx_n \int_{R^{n-1}} \int_0^d |\partial_n u(x', t)|^p dt\, dx' \leq (d^p/p)|u|_{1,p,\Omega}^p.$$

5. $\|u\|_{1,p,\Omega} = \|u\|_{0,p,\Omega} + |u|_{1,p,\Omega} \leq K(d,p)|u|_{1,p,\Omega}$, $\forall u \in \mathbb{C}_0^\infty(\Omega)$.

对高次导数相继应用该不等式:

6. $|u|_{m,p} \leq \|u\|_{m,p} \leq K(n,m,d,p)|u|_{m,p}$, $\forall u \in \mathbb{C}_0^\infty(\Omega)$.

7. For $u \in \mathbb{W}_0^{m,p}(\Omega)$, recall that $\mathbb{C}_0^\infty(\Omega)$ is dense in $\mathbb{W}_0^{m,p}(\Omega)$. ■

(5) used the inequality $a^p + b^p \leq (a+b)^p$ for $a, b \geq 0$ and $p \geq 1$; while (6) used induction.

Embedding Operator and Embedding Relation of Banach Spaces

嵌入定理深刻地刻划Sobolev空间之间或Sobolev空间与其它函数空间之间的关系.

1. $\mathbb{X}$, $\mathbb{Y}$: Banach spaces with norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$.

X嵌入(连续地)到Y 的定义

2. If $x \in \mathbb{X} \Rightarrow x \in \mathbb{Y}$, & $\exists$ const. $C > 0$ independent of x s.t. $\|x\|_{\mathbb{Y}} \leq C\|x\|_{\mathbb{X}}$, $\forall x \in \mathbb{X}$, then the identity map $I : \mathbb{X} \to \mathbb{Y}$, $I\,x = x$ is called an embedding operator, and the corresponding embedding relation is denoted by $\mathbb{X} \hookrightarrow \mathbb{Y}$.

3. The embedding operator $I : \mathbb{X} \to \mathbb{Y}$ is a bounded linear map.

4. If, in addition, I is happened to be a compact map, then, the corresponding embedding is called a compact embedding, and is denoted by $\mathbb{X} \overset{c}{\hookrightarrow} \mathbb{Y}$.

Some embedding relations exist in Sobolev spaces, which play an very important role in the theory of partial differential equations and finite element analysis.

设X,Y是赋范线性空间, T是X到Y的连续算子. 如果T把定义域中任何有界集映射成Y中的列紧集, 则称T是紧算子或全连续算子.

紧算子是一类重要的有界算子, 它最接近于有限维空间上的线性算子.

设A是度量空间X中的无穷集, 如果A中的任一无穷子集必有一个收敛的点列, 就称A是X中的列紧集。

The Sobolev Embedding Theorem

在近代PDE理论研究中起着重要的作用.

Theorem 5.5

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Let Ω be a bounded connected domain with a Lipschitz continuous boundary $\partial\Omega$, then

$$\mathbb{W}^{m+k,p}(\Omega) \hookrightarrow \mathbb{W}^{k,q}(\Omega), \ \forall\, 1 \le q \le \frac{np}{n-mp}, \ k \ge 0, \quad \text{if } \ m < n/p;$$
$$\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,q}(\Omega), \ \forall\, 1 \le q < \frac{np}{n-mp}, \ k \ge 0, \quad \text{if } \ m < n/p;$$
$$\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,q}(\Omega), \ \forall\, 1 \le q < \infty, \ k \ge 0, \quad \text{if } \ m = n/p;$$
$$\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^{k}(\overline{\Omega}), \quad \forall\, k \ge 0, \quad \text{if } \ m > n/p.$$

Rem: Sobolev空间这函数的较"低"阶范数可以被较"高"阶范数控制. 一般地讲, 反之不真.

Remark: The last embedding relation implies that for every u in $\mathbb{W}^{m+k,p}(\Omega)$, there is a $\tilde{u} \in \mathbb{C}^{k}(\overline{\Omega})$ such that $u - \tilde{u} = 0$ almost everywhere.

参考余德浩&汤华中的书的Chap5中的特例.

习题 5：2，3，6. Page 204

Thank You!