Numerical Solutions to Partial Differential Equations

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n-D 1st Order Linear Hyperbolic Partial Differential Equation

Chapter 3, P87

1. Scalar case ($u \in \mathbb{R}^1$), $$u_t + \sum_{i=1}^{n} a_i\, u_{x_i} + b\, u = \psi_0, \qquad (3.1.1)$$
   where a_i, b, ψ_0 are real functions of $x = (x_1, \ldots, x_n)$ and t.
2. Vector case ($\mathbf{u} = (u_1, \cdots, u_p)^T \in \mathbb{R}^p$),
   $$\mathbf{u}_t + \sum_{i=1}^{n} A_i\, \mathbf{u}_{x_i} + B\, \mathbf{u} = \boldsymbol{\psi}_0, \qquad (3.1.2)$$
   where A_i, $B \in \mathbb{R}^{p\times p}$, $\boldsymbol{\psi}_0 \in \mathbb{R}^p$ are real functions of t and $x = (x_1, \ldots, x_n)$, and $\forall \boldsymbol{\alpha} \in \mathbb{R}^n$, $A(x,t) = \sum_{i=1}^{n} \alpha_i A_i(x,t)$ is real diagonalizable, *i.e.* $A(x,t)$ has p linearly independent eigenvectors corresponding to real eigenvalues.
3. If $A(x,t) = \sum_{i=1}^{n} \alpha_i A_i(x,t)$ has p mutually different real eigenvalues, the system is called strictly hyperbolic.

n-D 2nd Order Linear Hyperbolic Partial Differential Equation

- A general 2nd order scalar equation ($u \in \mathbb{R}^1$),

$$u_{tt} + 2\sum_{i=1}^{n} a_i\, u_{x_i t} + b_0\, u_t - \sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j} + \sum_{i=1}^{n} b_i\, u_{x_i} + cu = \psi_0, \tag{3.1.3}$$

  where a_i, a_{ij}, b_i, c and ψ_0 are real functions $x = (x_1, \ldots, x_n)$ and t, (a_{ij}) is real symmetric positive definite.

- Define $v = u$, $v_0 = u_t$, $v_i = u_{x_i}$, then the above 2nd order scalar equation transforms into a first order linear system of partial differential equations for $\mathbf{v} = (v, v_0, v_1, \cdots, v_n)^T$

$$A\mathbf{v}_t + \sum_{i=1}^{n} A_i \mathbf{v}_{x_i} + B\mathbf{v} = \psi_0, \tag{3.1.4}$$

n-D 2nd Order Scalar Transforms to n-D 1st Order System ($p = n + 2$)

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & a_{n1} & \cdots & a_{nn} \end{bmatrix}, \qquad A_i = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 2a_i & -a_{1i} & \cdots & -a_{ni} \\ 0 & -a_{1i} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -a_{ni} & & & \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ c & b_0 & b_1 & \cdots & b_n \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \boldsymbol{\psi}_0 = \begin{bmatrix} 0 \\ \psi_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

n-D 2nd Order Scalar Transforms to n-D 1st Order System ($p = n + 2$)

1. Let $R^T A R = I$ (A is real symmetric positive definite);
2. Introduce a new variable $\mathbf{w} = R^{-1}\mathbf{v}$;
3. Denote $\hat{A}_i = R^T A_i R$, $\hat{B} = R^T B R$, $\hat{\psi}_0 = R^T \psi$;
4. $\hat{A}(x,t) = \sum_{i=1}^n \alpha_i \hat{A}_i(x,t)$ is a real symmetric matrix and thus is real diagonizable for all real α_i, $i = 1, \ldots, n$;
5. The 2nd order scalar equation now transforms into a 1st order linear hyperbolic system of partial differential equations for $\mathbf{w} \in \mathbb{R}^{(n+2)}$:

$$\mathbf{w}_t + \sum_{i=1}^n \hat{A}_i \mathbf{w}_{x_i} + \hat{B}\mathbf{w} = \hat{\psi}_0. \qquad (3.1.5)$$

Standard Form of *n*-D 1st Order Linear Hyperbolic Equations

1. The standard form of 1st order linear hyperbolic equation:

$$u_t + \sum_{i=1}^{n} a_i\, u_{x_i} = \psi, \quad (\psi = \psi_0 - b\, u). \qquad (3.1.6)$$

2. The standard form of 1st order linear hyperbolic system:

$$\mathbf{u}_t + \sum_{i=1}^{n} A_i\, \mathbf{u}_{x_i} = \boldsymbol{\psi}, \quad (\boldsymbol{\psi} = \boldsymbol{\psi}_0 - B\, \mathbf{u}); \qquad (3.1.7)$$

3. The equation (system) is said to be homogeneous, if $\psi = 0$;

Standard Form of *n*-D 1st Order Linear Hyperbolic Equations

4. In general, a higher order linear hyperbolic equation (system of equations) can always be transformed into a first order linear hyperbolic system of equations.

5. An equation (system) is said to be nonlinear, if at least one of the coefficients depends on the unknown or the right hand side term is a nonlinear function of the unknown.

An Example of a Balance Law

Substance balance in flowing fluid in a 1D pipe.

1. $u(x, t)$: substance density, measured by mass per unit length.
2. $f(x, t, u)$: mass flux, measured by mass per unit time.
3. $\psi(x, t, u)$: the density of the mass source (or sink), measured by mass per unit length per unit time.
4. Balance law (integral form), for given $x_l < x_r$ and $t_b < t_a$,

$$\int_{x_l}^{x_r} u(x, t_a)\, dx = \int_{x_l}^{x_r} u(x, t_b)\, dx + \int_{t_b}^{t_a} f(x_l, t, u(x_l, t))\, dt$$
$$- \int_{t_b}^{t_a} f(x_r, t, u(x_r, t))\, dt + \int_{t_b}^{t_a} \int_{x_l}^{x_r} \psi(x, t, u(x, t))\, dx\, dt.$$

An Example of a Balance Law

5. Suppose u and f are sufficiently smooth, we are led to

$$\int_{t_b}^{t_a}\int_{x_l}^{x_r}\left[u(x,t)_t + f(x,t,u(x,t))_x - \psi(x,t,u(x,t))\right] dx\,dt = 0,$$

for all given $x_l < x_r$ and $t_b < t_a$;

6. or equivalently:

$$u(x,t)_t + f(x,t,u(x,t))_x = \psi(x,t,u(x,t)),$$

this is the balance law in differential form.

Advection Equation and Advection-Diffusion Equation

The simplest example is the 1D advection equation:

$$u_t + a\, u_x = 0,$$

in which the flowing velocity of the fluid is a constant a, the flux is simply $f(x, t, u) = f(u) = a\, u$, and the source term is zero.

Another simple example is the 1D advection-diffusion equation:

$$u_t + a\, u_x = \sigma\, u_{xx},$$

in which diffusion as well as advection is considered, and the flux is of the form $f(x, t, u, u_x) = a\, u - \sigma\, u_x$, where σ is the diffusion parameter.

Note: Advection equation is also called convection equation, advection-diffusion equation is also called convection-diffusion equation.

Characteristics and Riemann Invariants

In hyperbolic equations (systems), the information propagates at finite characteristic speeds.

Consider a 1D constant-coefficient hyperbolic system

$$\mathbf{u}_t + A\,\mathbf{u}_x = 0.$$

1. $AR = R\Lambda$, $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_p)$ with $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_p$.
2. Define $\mathbf{w} = R^{-1}\mathbf{u}$, the system is transformed to
$$\mathbf{w}_t + \Lambda\,\mathbf{w}_x = 0.$$
3. p families of characteristics given by p characteristic equations
$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \lambda_i, \qquad i = 1, \cdots, p.$$

Riemann Invariants and General Solutions of Initial Value Problems

4. w_i is a constant on each characteristics of the ith family:

$$\frac{\mathrm{d}w_i(x_i(t),t)}{\mathrm{d}t} = [w_{i,t} + \lambda_i\, w_{i,x}]\,(x_i(t),t) = 0.$$

5. $w_i(x,t) = w_i(x - \lambda_i t, 0)$, $i = 1, \cdots, p$.

6. w_i, $i = 1, \cdots, p$, are called the Riemann invariants of the system with respect to the characteristics of $(\lambda_i, \boldsymbol{\xi}^i)$, with $A\boldsymbol{\xi}^i = \lambda_i \boldsymbol{\xi}^i$, $i = 1, \cdots, p$.

Riemann Invariants and General Solutions of Initial Value Problems

7. Denote $\boldsymbol{\xi}^i$ the ith column of R, and $\boldsymbol{\eta}^i$ the ith row of R^{-1}.

8. By definition $\boldsymbol{\eta}^i \mathbf{u} = w_i$, in particular $w_i(x, 0) = \boldsymbol{\eta}^i \mathbf{u}^0(x)$.

9. By definition $\mathbf{u}(x, t) = R\mathbf{w}(x, t)$, thus, by 5 and 8,

$$\mathbf{u}(x, t) = \sum_{i=1}^{p} \boldsymbol{\xi}^i w_i(x, t) = \sum_{i=1}^{p} \boldsymbol{\xi}^i \boldsymbol{\eta}^i \mathbf{u}^0(x - \lambda_i t).$$

Domains of Dependance, Influence and Determination

Since $\mathbf{u}(x,t) = \sum_{i=1}^{p} \boldsymbol{\xi}^i \boldsymbol{\eta}^i \mathbf{u}^0(x - \lambda_i t)$, we define

1. domain of dependance of a set $\Omega_T \subset \mathbb{R} \times \mathbb{R}_+$:
   $D(\Omega_T) = \{y \in \mathbb{R} : y = x - \lambda_i t, \;\; i = 1, \cdots, p, \;\; \forall (x,t) \in \Omega_T\}$.
2. domain of influence of a set $\Omega \subset \mathbb{R}$:
   $I(\Omega) = \{(x,t) \in \mathbb{R} \times \mathbb{R}_+ : \; \exists\, 1 \le i \le p, \; s.t. \; x - \lambda_i t \in \Omega\}$,
3. domain of determination of a set $\Omega \subset \mathbb{R}$:
   $K(\Omega) = \{(x,t) \in \mathbb{R} \times \mathbb{R}_+ : x_i = x - \lambda_i t \in \Omega, \;\; i = 1, \cdots, p\}$.
4. We can also define $D(\Omega_T, t_0)$, $I(\Omega, t_0)$ and $K(\Omega, t_0)$ by replacing $\lambda_i t$ by $\lambda_i(t - t_0)$ in the above definitions.

Boundary Conditions in Initial-Boundary Value Problems

For an initial-boundary value problem defined on $(0,1)\times\mathbb{R}_+$, suppose $\lambda_1 < \ldots < \lambda_l < 0 < \lambda_r < \lambda_p$ ($l = r-1$ or $l = r-2$).

By the characteristics of the system, boundary conditions should be imposed in the following way:

1. $p-r+1$ linearly independent boundary conditions on the left boundary 0, since $\{w_i\}_{i=r}^{p}$ picks up information from left.
2. l linearly independent boundary conditions on the right boundary 1, since $\{w_i\}_{i=1}^{l}$ picks up information from right.

Boundary Conditions in Initial-Boundary Value Problems

3. The $p - r + 1$ boundary conditions on the left boundary 0 must contain sufficient information for $\{w_i\}_{i=r}^{p}$.

4. The l boundary conditions on the right boundary 1 must contain sufficient information for $\{w_i\}_{i=1}^{l}$.

5. The domains of dependance, influence and determination can also be defined for initial-boundary value problems.

The Characteristic Method for the Advection Equation

1. Advection equation: $u_t(x,t) + a(x,t)\, u_x(x,t) = 0$, $x \in I = (x_l, x_r) \subset \mathbb{R}$, $t > 0$;
2. Characteristics: $x'(t) = a(x,t)$, $x \in I$, $t > 0$;
3. Solution is a constant on a characteristic curve:
$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial u}{\partial t} + a\,\frac{\partial u}{\partial x} = 0.$$
4. Suppose $a(x_l, t) > 0$ and $a(x_r, t) > 0$ for all $t \geq 0$.

The Characteristic Method for the Advection Equation

5. Initial-boundary conditions $u(x, 0) = u^0(x),\ u(x_l, t) = u_l(t)$.

6. Take $x_l \leq x_1 < \cdots < x_N < x_r$, and $0 \leq t^1 < \cdots < t^M$.

7. Solving the characteristic equation with initial conditions: $x(0) = x_i,\ i = 1, \cdots, N$, and $x(t^m) = x_l,\ m = 1, \cdots, M$, to obtain characteristic curves;

8. On these (approximate) characteristic curves, set $U_i(x, t) := u^0(x_i),\ i = 1, \cdots, N$, and $U^m(x, t) := u(x_l, t^m) = u_l(t^m),\ m = 1, \cdots, M$ respectively.

The Characteristic Method for the Advection Equation

In particular, if $a(x,t) \equiv a$ is a constant, then, we have

9. for $a > 0$,
$$u(x,t) = \begin{cases} u^0(x - at), & x - at \geq x_l, \\ u_l\left(t - \dfrac{x - x_l}{a}\right), & x - at < x_l. \end{cases}$$

10. if $a < 0$,
$$u(x,t) = \begin{cases} u^0(x - at), & x - at \leq x_r, \\ u_r\left(t - \dfrac{x - x_r}{a}\right), & x - at > x_r. \end{cases}$$

Inhomogeneous Advection Equation and Characteristic Method

1. Advection equation: $u_t(x,t) + a(x,t)\, u_x(x,t) = \psi(x,t)$, $x \in I = (x_l, x_r) \subset \mathbb{R}$, $t > 0$;
2. Characteristics: $x'(t) = a(x,t)$, $x \in I$, $t > 0$;
3. On a characteristic curve, the solution satisfies:

$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial u}{\partial t} + a\,\frac{\partial u}{\partial x} = \psi(x(t),t).$$

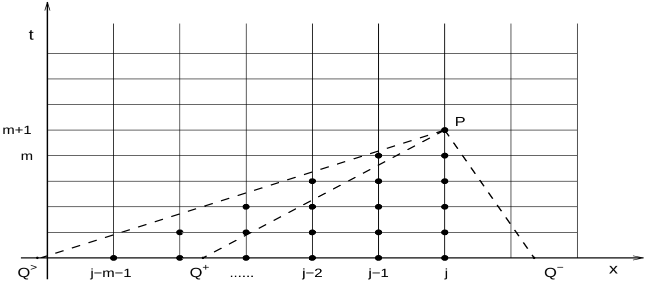
4. Suppose $a(x_l,t) > 0$ and $a(x_r,t) > 0$ for all $t \geq 0$.

Inhomogeneous Advection Equation and Characteristic Method

5. Initial-boundary conditions $u(x,0) = u^0(x),\ u(x_l,t) = u_l(t)$.
6. Take $x_l \le x_1 < \cdots < x_N < x_r$, and $0 \le t^1 < \cdots < t^M$.
7. Solving the characteristic equation with initial conditions: $x_i(0) = x_i,\ i = 1, \cdots, N$, and $x^m(t^m) = x_l,\ m = 1, \cdots, M$, to obtain characteristic curves;
8. Along those (approximate) characteristic curves, solving ODE system $\dot{U}_i(t) = \psi(x_i(t), t)$ and $\dot{U}^m(t) = \psi(x^m(t), t)$ with initial conditions $U_i(0) := u^0(x_i),\ i = 1, \cdots, N$, and $U^m(t^m) := u(x_l, t^m) = u_l(t^m),\ m = 1, \cdots, M$ respectively.

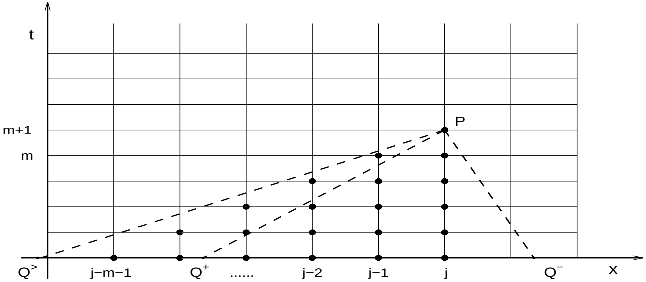
Domain of Dependance of a Difference Scheme

1. Advection equation: $u_t(x,t) + a\,u_x(x,t) = 0$, ($a =$ constant).
2. The simplest scheme: $\dfrac{U_j^{m+1} - U_j^m}{\tau} + a\,\dfrac{U_j^m - U_{j-1}^m}{h} = 0;$
3. or equivalently: $U_j^{m+1} = (1-\nu)U_j^m + \nu U_{j-1}^m$, ($\nu = a\tau/h$).
4. For $P = (x_j, t_{m+1})$, $D(P) = Q = x_j - a\,t_{m+1}$, the domain of dependance of the scheme $D_h(P) = \{x_{j-m-1}, \cdots, x_{j-1}, x_j\}$.



The CFL Condition of a Difference Scheme

- Simple observation: if $Q = Q^{>} < x_{j-m-1}$, or $Q = Q^{-} > x_j$, then U_j^{m+1} can not properly approximate $u(x_j, t_{m+1})$.
- **CFL condition**: a necessary condition for a numerical scheme to converge is that the domain of dependence of the the real solution is contained, at least in the sense of limit, in the domains of dependence of the numerical scheme.



The CFL Condition is a Necessary Condition for Stability

The CFL condition is often used as a necessary condition for the stability of finite difference schemes for hyperbolic differential equations.

In general, the CFL condition is not a sufficient condition for the stability of a numerical method.

CFL Condition is not a Sufficient Condition for Stability

Consider the finite difference scheme of the advection equation

$$\frac{U_j^{m+1} - U_j^m}{\tau} + a\,\frac{U_{j+1}^m - U_{j-1}^m}{2h} = 0, \quad a \in R^1 \setminus \{0\}\ .$$

The CFL condition for the scheme is $|\nu| = |a|\tau/h \le 1.$

The Fourier modes solution $U_j^m = \lambda_k^m e^{\mathrm{i}\frac{kj\pi}{N}}$ with amplification factor $\lambda_k = 1 - \mathrm{i}\,\nu \sin\frac{k\pi}{N}$. Since $|\lambda_k| > 1$ for all $\nu \neq 0$, the scheme is always unstable, whether the CFL condition is satisfied or not.

Note: For initial value problems or initial-boundary value problems with periodic boundary conditions of a constant coefficient advection equation, a necessary and sufficient condition for L^2-stability is the so called von Neumann condition: $|\lambda_k| \le 1 + K\tau$, for all $-N+1 \le k \le N$.

The Upwind Scheme for the Advection Equation

For the advection equation $u_t(x,t) + a(x,t)\,u_x(x,t) = 0$, by the CFL condition, the simplest difference scheme is

$$U_j^{m+1} = \begin{cases} U_j^m - \nu_j^m \triangle_+ U_j^m, & \text{if } \ a_j^m \le 0, \\ U_j^m - \nu_j^m \triangle_- U_j^m, & \text{if } \ a_j^m \ge 0, \end{cases}$$

where $\nu_j^m = a_j^m\,\tau/h$ satisfies $|\nu_j^m| \le 1$, or equivalently

$$U_j^{m+1} = \begin{cases} (1+\nu_j^m)U_j^m - \nu_j^m U_{j+1}^m, & \text{if } \ a_j^m \le 0, \\ (1-\nu_j^m)U_j^m + \nu_j^m U_{j-1}^m, & \text{if } \ a_j^m \ge 0, \end{cases}$$

which may be viewed as obtained by using

- $\triangle_{+t}/\triangle t$ to approximate $\partial/\partial t$;
- $\triangle_{+x}/\triangle x$ $(a<0)$ or $\triangle_{-x}/\triangle x$ $(a>0)$ to approximate $\partial/\partial x$.

The Upwind Scheme for the Advection Equation

Characteristic method + Linear interpolation $\Rightarrow$ Upwind scheme.

Let $a > 0$ be a constant, assume $\nu = a\tau/h \leq 1$.

1. By the characteristic method: $u(x_j, t_{m+1}) = u(x_j - a\,\tau, t_m)$;
2. By linear interpolation:
$$u(x_j - a\,\tau, t_m) \approx \frac{x_j - (x_j - a\tau)}{h} u_{j-1}^m + \frac{(x_j - a\tau) - x_{j-1}}{h} u_j^m;$$
3. Hence $u(x_j, t_{m+1}) \approx \nu u_{j-1}^m + (1-\nu) u_j^m$;
4. This leads to the scheme: $U_j^{m+1} = (1-\nu)U_j^m + \nu U_{j-1}^m$, which is exactly the upwind scheme.

The Truncation Error of the Upwind Scheme $O(\tau + h)$

By Taylor series expansion, we have

$$
\begin{aligned}
T_j^m :=& \frac{u_j^{m+1} - u_j^m}{\tau} + a\,\frac{u_j^m - u_{j-1}^m}{h} - \left[\frac{\partial u}{\partial t} + a\,\frac{\partial u}{\partial x}\right]_j^m \\
=& \frac{1}{2}\left[\tau u_{tt} - a\,h\,u_{xx}\right]_j^m + \frac{1}{6}\left[\tau^2 u_{ttt} + a\,h^2\,u_{xxx}\right]_j^m + \cdots \\
=& -\left[\frac{1}{2} a\,h\,(1-\nu)u_{xx} + \frac{1}{6} a\,h^2(1-\nu^2)u_{xxx} + \cdots\right]_j^m, \text{ if } a > 0.
\end{aligned}
$$

Note, here a is a constant, so $u_{tt} = a^2 u_{xx}$ and $\nu = a\tau/h$, and $\tau u_{tt} = ah\nu u_{xx}$. Similarly $\tau^k \partial_t^{k+1} u = (-1)^{k+1} a h^k \nu^k \partial_x^{k+1} u$.

The Truncation Error of the Upwind Scheme $O(\tau + h)$

Similarly, we have

$$
\begin{aligned}
T_j^m := & \frac{u_j^{m+1} - u_j^m}{\tau} + a\,\frac{u_{j+1}^m - u_j^m}{h} - \left[\frac{\partial u}{\partial t} + a\,\frac{\partial u}{\partial x}\right]_j^m \\
= & \frac{1}{2}\left[\tau\, u_{tt} + a\,h\,u_{xx}\right]_j^m + \frac{1}{6}\left[\tau^2 u_{ttt} + a\,h^2\,u_{xxx}\right]_j^m + \cdots \\
= & \left[\frac{1}{2} a\,h\,(1+\nu) u_{xx} + \frac{1}{6} a\,h^2 (1-\nu^2) u_{xxx} + \cdots\right]_j^m, \text{ if } a < 0.
\end{aligned}
$$

Note, here a is a constant, so $u_{tt} = a^2 u_{xx}$ and $\nu = a\tau/h$, and $\tau u_{tt} = ah\nu u_{xx}$. Similarly $\tau^k \partial_t^{k+1} u = (-1)^{k+1} a h^k \nu^k \partial_x^{k+1} u$.

The Stability and Convergence of the Upwind Scheme

The error $e_j^m = U_j^m - u_j^m$ satisfies the following error equation

$$e_j^{m+1} = \begin{cases} (1-|\nu|)e_j^m + |\nu|e_{j+1}^m - \tau T_j^m, & \text{if } \ a < 0, \\ (1-|\nu|)e_j^m + |\nu|e_{j-1}^m - \tau T_j^m, & \text{if } \ a > 0. \end{cases}$$

If the CFL condition is satisfied, *i.e.* $|\nu| \le 1$, then, we have

$$E^{m+1} := \max_j |e_j^{m+1}| \le E^m + \tau \max_j |T_j^m| \le E^0 + t_{\max} \max_{m,j} |T_j^m|,$$
$$\forall (m+1)\tau \le t_{\max}.$$

The Solutions to Nonlinear Hyperbolic Equations are Generally not Smooth

Therefore, if the CFL condition is satisfied, the upwind scheme is stable in $\mathbb{L}^\infty$ norm, and its convergence rate is $O(\tau + h)$, **if** the solution u is sufficiently smooth.

However, in hyperbolic problems, discontinuities are commonly seen in the physical solutions. In fact, the discontinuity creation and propagation is a focus of investigation on nonlinear problems.

Nonlinear Advection Equation and Blow Up of Its Classical Solution

1. Nonlinear advection equation:
   $u_t(x,t) + a(u(x,t))\, u_x(x,t) = 0;$
2. Characteristic equation: $x'(t) = a(u(x,t))$, $t > 0$;
3. Solution is a constant on a characteristic curve:

$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial u}{\partial t} + a\,\frac{\partial u}{\partial x} = 0.$$

4. On a characteristic curve, $a(u(x,t))$ is a constant.
5. Characteristic curves are straight lines (with different slopes).
6. Suppose for $\xi_1 < \xi_2$, $a_1 := a(u^0(\xi_1)) > a_2 := a(u^0(\xi_2))$, then, at time $\bar{t} = (\xi_2 - \xi_1)/(a_1 - a_2) > 0$, the two corresponding characteristic lines intersect: $(\xi_1 + a_1\bar{t}, \bar{t}) = (\xi_2 + a_2\bar{t}, \bar{t})$.
7. In such a case, the classical solution blows up.

Weak Solutions for Nonlinear Conservation Laws

Since discontinuities are inevitable, we should take it into consideration by introducing the concept of weak solutions.

Definition 3.1 P101

$u \in \mathbb{L}^1_{loc}(\mathbb{R} \times \mathbb{R}_+)$ is called a weak solution to the following 1D initial value problem of the conservation law

$$\begin{aligned} u_t(x,t) + f(u(x,t))_x &= 0, \\ u(x,0) &= u^0(x), \end{aligned}$$

if u satisfies the initial condition, and for all fixed $0 \le t_b < t_a$ and $-\infty < x_l < x_r < \infty$, the following equation holds

$$\int_{x_l}^{x_r} u(x,t_a)dx = \int_{x_l}^{x_r} u(x,t_b)dx + \int_{t_b}^{t_a} f(u(x_l,t))dt - \int_{t_b}^{t_a} f(u(x_r,t))dt.$$

Shock Speed and Rankine-Hugoniot Jump Condition

1. Suppose there is an isolated discontinuity (*i.e.* a shock) $x(t)$ propagating at speed $s(t)$ (*i.e.* $x'(t) = s(t)$);
2. By the integral form conservation law, for $x_1 = x(t_1)$ and $x_1 + \triangle x = x(t_1 + \triangle t)$, we have
$$\int_{x_1}^{x_1+\triangle x} u(x, t_1 + \triangle t)dx - \int_{x_1}^{x_1+\triangle x} u(x, t_1)dx$$
$$= \int_{t_1}^{t_1+\triangle t} f(u(x_1, t))dt - \int_{t_1}^{t_1+\triangle t} f(u(x_1 + \triangle x, t))dt.$$
3. Denote $u_l = u(x_1 - 0, t_1)$, $u_r = u(x_1 + 0, t_1)$, then, we have
$$(u_l - u_r)\triangle x = (f(u_l) - f(u_r))\triangle t + O(\triangle t^2).$$
4. Since $x'(t) = s(t)$, we are led to the Rankine-Hugoniot jump condition: $s[u] = [f]$.

Weak Solutions of Non-viscous Burger's Equation

Consider the initial value problem of the Burgers equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0; \qquad u(x,0) = \begin{cases} 1, & x < 0; \\ 0, & x \geq 0. \end{cases}$$

- By the Rankine-Hugoniot jump condition,

$$s = \frac{f(u_r) - f(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{0 - \frac{1}{2}}{0 - 1} = \frac{1}{2}.$$

- Hence the weak solution of the problem is given by

$$u(x,t) = \begin{cases} 1, & x < t/2; \\ 0, & x \geq t/2. \end{cases}$$

Weak Solutions of Non-viscous Burger's Equation

Consider the initial value problem of the equation

$$\frac{\partial u^2}{\partial t} + \frac{2}{3}\frac{\partial u^3}{\partial x} = 0; \qquad u(x,0) = \begin{cases} 1, & x < 0; \\ 0, & x \geq 0. \end{cases}$$

- Define $w = u^2$, by the Rankine-Hugoniot jump condition,
$$s = \frac{f(w_r) - f(w_l)}{w_r - w_l} = \frac{\frac{2}{3}w_r^{3/2} - \frac{2}{3}w_l^{3/2}}{w_r - w_l} = \frac{0 - \frac{2}{3}}{0 - 1} = \frac{2}{3}.$$
- Hence the weak solution of the problem is given by
$$u(x,t) = \begin{cases} 1, & x < 2t/3; \\ 0, & x \geq 2t/3. \end{cases}$$

Weak Solutions of Non-viscous Burger's Equation

For smooth u, both the Burger's equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$

and the equation

$$\frac{\partial u^2}{\partial t} + \frac{2}{3}\frac{\partial u^3}{\partial x} = 0$$

are equivalent to the first order nonlinear hyperbolic equation

$$u_t + uu_x = 0.$$

Key Ingredients of a Conservation Law

The key ingredients of a conservation law are

- the conservative quantities and the corresponding fluxes;
- the integral form equations based on physical conservation laws;
- Entropy condition — to distinguish the physical solution from the others.

Remark: For a nonlinear problem, the weak solution is generally not unique even for a conservation law (see Exercise 3.4).

$\mathbb{L}^2$ Is a Better Space to Investigate Hyperbolic Equations

- The upwind scheme of the advection equation satisfies the maximum principle, if CFL condition is satisfied;
- However, for general hyperbolic equations (systems), the solution do not satisfy the maximum principle.
- Hyperbolic equations (systems) are often used to characterize the propagation and evolution of waves.
- Fourier modes are waves of various frequencies propagating at their own characteristic speeds.

It is natural to apply the Fourier method to analyze the $\mathbb{L}^2$ stability and accuracy of finite difference schemes for hyperbolic equations (systems).

习题 3：1，3 Page 151

Thank You!