Numerical Solutions to Partial Differential Equations

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7.4.3节 数值积分引起的相容性误差

仅以一个例子说明如何利用前面的理论估计数值积分引起的相容性误差

Approximate $a(u, v)$ by $a_h(u, v)$ Using Numerical Quadratures

多边形区域 一族正则仿射等价FE剖分

1. Let Ω be a polygonal region, and $\mathfrak{T}_h(\Omega)$ be a family of regular affine equivalent finite element triangulations of Ω.

仿射等价映射

2. $F_K : \hat{x} \in \hat{K} \to B_K\hat{x} + \underbrace{b_K}_{=x} \in K$: the corresponding affine equivalent mappings.

考虑双线性形式 嵌入定理Th5.5==〉a_{ij} 在Ω的闭包上在连续

3. Let $a_{ij} \in \mathbb{W}^{1,\infty}(\Omega)$, $i, j = 1, \dots, n$, and
$$a(u, v) = \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_i u\, \partial_j v\, dx = \sum_{K\in\mathfrak{T}_h(\Omega)} \sum_{i,j=1}^n \int_K a_{ij}\partial_i u\, \partial_j v\, dx.$$

数值积分节点

4. $\{\hat{b}_l\}_{l=1}^L$, $\{b_{l,K} = F_K(\hat{b}_l)\}_{l=1}^L$: quadrature nodes on $\hat{K}$, K.

数值积分权

5. $\{\hat{\omega}_l\}_{l=1}^L$, $\{\omega_{l,K} = \det(B_K)\hat{\omega}_l\}_{l=1}^L$, quadrature weights on $\hat{K}$, K.

Approximate $a(u,\ v)$ by $a_h(u,\ v)$ Using Numerical Quadratures

6. Approximate $a(u, v)$ by numerical quadrature

近似的双线性形式

$$a_h(u,\ v) \stackrel{\text{def}}{=} \sum_{K\in\mathfrak{T}_h(\Omega)} \sum_{i,j=1}^{n} \sum_{l=1}^{L} \omega_{l,K}\, a_{ij}(b_{l,K})\partial_i u(b_{l,K})\, \partial_j v(b_{l,K}).$$

7. Denote the errors of integrals and numerical integrals of φ and $\hat{\varphi}$ on K and $\hat{K}$ by

数值积分误差

$$E_K(\varphi) \stackrel{\text{def}}{=} \int_K \varphi(x)\, dx - \sum_{l=1}^{L} \omega_{l,K}\, \varphi(b_{l,K}),$$

$$\hat{E}(\hat{\varphi}) \stackrel{\text{def}}{=} \int_{\hat{K}} \hat{\varphi}(\hat{x})\, d\hat{x} - \sum_{l=1}^{L} \hat{\omega}_l\, \hat{\varphi}(\hat{b}_l).$$

Error on K of a Numerical Quadrature with Algebraic Accuracy $2k-2$

Lemma 7.2

Let $a_{ij} \in \mathbb{W}^{k,\infty}(\Omega)$, $i, j = 1, \ldots, n$, for some integer $k \geq 1$. Let the reference finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ and the numerical quadrature satisfy

$$\hat{P} = \mathbb{P}_k(\hat{K}) \quad \textit{and} \quad \hat{E}(\hat{\varphi}) = 0, \ \forall \hat{\varphi} \in \mathbb{P}_{2k-2}(\hat{K}). \qquad (*)$$

Then, there exists a constant C independent of K and h, such that

$$\boxed{|E_K(a_{ij}\partial_i \tilde{v}\, \partial_j \tilde{w})| \leq C\, h_K^k\, \|\tilde{v}\|_{k,K}\, |\tilde{w}|_{1,K}, \quad \forall \tilde{v} \in \mathbb{P}_k(K),\ \forall \tilde{w} \in \mathbb{P}_k(K). \qquad (7.4.12)}$$

Proof of the Lemma — Key: Bounds the Error in Semi-Norms

1. $a \in \mathbb{W}^{k,\infty}(K)$ and $v,\ w \in \mathbb{P}_{k-1}(K) \Rightarrow \hat{a} \in \mathbb{W}^{k,\infty}(\hat{K})$ and $\hat{v}$, $\hat{w} \in \mathbb{P}_{k-1}(\hat{K})$, and we have $E_K(a\,v\,w) = \det(B_K)\,\hat{E}(\hat{a}\,\hat{v}\,\hat{w})$.
   变换前后的数值积分误差
2. For a given $\hat{w} \in \mathbb{P}_{k-1}(\hat{K})$ and an arbitrary $\hat{\varphi} \in \mathbb{W}^{k,\infty}(\hat{K})$, $|\hat{E}(\hat{\varphi}\,\hat{w})| \leq \hat{C}\,\|\hat{\varphi}\,\hat{w}\|_{0,\infty,\hat{K}} \leq \hat{C}\,\|\hat{\varphi}\|_{0,\infty,\hat{K}}\,\|\hat{w}\|_{0,\infty,\hat{K}}$.
3. Since norms on the finite dimensional space $\mathbb{P}_{k-1}(\hat{K})$ are equivalent, we have
   L∞范数与L2范数
   $$|\hat{E}(\hat{\varphi}\,\hat{w})| \leq \hat{C}\,\|\hat{\varphi}\|_{0,\infty,\hat{K}}\,\|\hat{w}\|_{0,\hat{K}} \leq \hat{C}\,\|\hat{\varphi}\|_{k,\infty,\hat{K}}\,\|\hat{w}\|_{0,\hat{K}}.$$
   Sobolev范数定义
4. Therefore, for a fixed $\hat{w} \in \mathbb{P}_{k-1}(\hat{K})$, $\hat{E}(\cdot\,\hat{w})$ is a bounded linear form on $\mathbb{W}^{k,\infty}(\hat{K})$ with its norm $\leq \hat{C}\,\|\hat{w}\|_{0,\hat{K}}$.

Proof of the Lemma — Key: Bounds the Error by Semi-Norms

5. In addition, $\hat{E}(\hat{\varphi}\,\hat{w}) \overset{(*)}{=} 0$, $\forall \hat{\varphi} \in \mathbb{P}_{k-1}(\hat{K})$. 已知条件(*) Consequently, by the Bramble-Hilbert lemma (see Theorem 7.15), we have

$$|\hat{E}(\hat{\varphi}\,\hat{w})| \le \hat{C}|\hat{\varphi}|_{k,\infty,\hat{K}}\|\hat{w}\|_{0,\hat{K}}, \quad \forall \hat{\varphi} \in \mathbb{W}^{k,\infty}(\hat{K}),\ \forall \hat{w} \in \mathbb{P}_{k-1}(\hat{K}). \qquad \text{(I)}$$

6. On the other hand, for $\hat{a} \in \mathbb{W}^{k,\infty}(\hat{K})$ and $\hat{v} \in \mathbb{P}_{k-1}(\hat{K})$, by the chain rule of the derivatives of composition functions and the equivalence of norms in the finite dimensional space $\mathbb{P}_{k-1}(\hat{K})$, we have

$$|\hat{a}\hat{v}|_{k,\infty,\hat{K}} \le \hat{C}\sum_{j=0}^{k-1}|\hat{a}|_{k-j,\infty,\hat{K}}\,|\hat{v}|_{j,\infty,\hat{K}} \le \hat{C}\sum_{j=0}^{k-1}|\hat{a}|_{k-j,\infty,\hat{K}}\,|\hat{v}|_{j,\hat{K}}. \qquad \text{(II)}$$

Proof of the Lemma — Key: Bounds the Error in Semi-Norms

7. ⑤ and ⑥ yield that, $\forall \hat{a} \in \mathbb{W}^{k,\infty}(\hat{K})$,

$$|\hat{E}(\hat{a}\,\hat{v}\,\hat{w})| \overset{(I,II)}{\leq} \hat{C}\left(\sum_{j=0}^{k-1} |\hat{a}|_{k-j,\infty,\hat{K}} |\hat{v}|_{j,\hat{K}}\right) \|\hat{w}\|_{0,\hat{K}}, \quad \forall \hat{v},\ \hat{w} \in \mathbb{P}_{k-1}(\hat{K}).$$

8. By the relations of the semi-norms on K and $\hat{K}$, this yields

$$|E_K(avw)| \leq Ch_K^k\left(\sum_{j=0}^{k-1} |a|_{k-j,\infty,K} |v|_{j,K}\right) \|w\|_{0,K}, \quad \forall v, w \in \mathbb{P}_{k-1}(K).$$

9. Thus, the lemma $\underset{(7.4.12)}{\text{follows}}$ by taking $a = a_{ij}$, $v = \partial_i \tilde{v}$ and $w = \partial_j \tilde{w}$. ■

Consistency Error Estimate of Bilinear Forms

1. As a consequence of Lemma 7.2, for numerical quadrature with algebraic accuracy $2k-2$, we see that

$$|a(\Pi_h u,\, w_h) - a_h(\Pi_h u,\, w_h)| \leq Ch^k \Big(\sum_{K \in \mathfrak{T}_h(\Omega)} \|\Pi_h u\|_{k,K}^2 \Big)^{1/2} |w_h|_{1,\Omega}. \qquad \text{(A)}$$

2. On the other hand, by the interpolation error estimates of the finite element solutions and the Cauchy-Schwarz inequality, we have

$$\Big(\sum_{K \in \mathfrak{T}_h(\Omega)} \|\Pi_h u\|_{k,K}^2 \Big)^{1/2} \leq \|u\|_{k,\Omega} + \Big(\sum_{K \in \mathfrak{T}_h(\Omega)} \|u - \Pi_h u\|_{k,K}^2 \Big)^{1/2} \leq C\|u\|_{k+1,\Omega}. \qquad \text{(B)}$$

Consistency Error Estimate of Bilinear Forms

3. Consequently, we obtain the following consistency error estimate:

双线性形式的相容性误差估计

$$\boxed{\sup_{w_h \in \mathbb{V}_h} \frac{|a(\Pi_h u,\ w_h) - a_h(\Pi_h u,\ w_h)|}{\|w_h\|_{1,\Omega}} \overset{(A,B)}{\leq} C\, h^k\, \|u\|_{k+1,\Omega}.} \tag{7.4.13}$$

which is the same order as the interpolation error estimate.

Consistency Error Estimate of Linear Forms

对右端项(f,v)数值积分引起的相容性误差可以类似地得到

Similarly, the consistency error estimation can be carried out for the numerical integration of $f(v) = \int_\Omega fv\,dx$. However, to reach accuracy of order h^k, the algebraic accuracy of the numerical quadrature needs to be $2k-1$.

1. Consider the linear form $f(v) = \int_\Omega fv\,dx$, and numerical quadrature $f_h(v) = \sum_{K\in\mathfrak{T}_h(\Omega)} \sum_{l=1}^{L} \omega_{l,K} f(b_{l,K}) v(b_{l,K})$.
2. Assume $\mathbb{H}^k(\Omega) \hookrightarrow \mathbb{C}(\bar{\Omega})$, and the finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ and the numerical quadrature satisfy

$$\hat{P} = \mathbb{P}_k(\hat{K}), \quad \text{and} \quad \hat{E}(\hat{\varphi}) = 0, \quad \forall \hat{\varphi} \in \mathbb{P}_{2k-1}(\hat{K}).$$

Consistency Error Estimate of Linear Forms

3. Then, follows a similar argument as above, we can obtain the error estimate (see Exercise 7.9)

线性形式的相容性误差估计 P262

$$\boxed{\sup_{w_h \in \mathbb{V}_h} \frac{|f(w_h) - f_h(w_h)|}{\|w_h\|_{1,\Omega}} \le C\, h^k |f|_{k,\Omega}.}$$

Consistency Error Estimate of Linear Forms

P260 There is another approach, in which, to reach the accuracy of the same order h^k, the algebraic accuracy of the numerical quadrature needed goes back to $2k-2$. In fact, we have the following result:

1. Let $q \geq 2$ and assume $kq > n$ (so $\mathbb{W}^{k,q}(\Omega) \hookrightarrow \mathbb{C}(\bar{\Omega})$).
2. The reference finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ and the numerical quadrature satisfy

$$\hat{P} = \mathbb{P}_k(\hat{K}), \quad \text{and} \quad \hat{E}(\hat{\varphi}) = 0, \ \forall \hat{\varphi} \in \mathbb{P}_{2k-2}(\hat{K}).$$

Consistency Error Estimate of Linear Forms

3. Introduce a $\mathbb{P}_0(\hat{K})$ invariant orthogonal projection operator $\hat{\Pi} : \mathbb{L}^2(\hat{K}) \to \mathbb{P}_1(\hat{K})$ induced by the $\mathbb{L}^2(\hat{K})$ inner product.

4. Rewrite the error as $E(fw_h) = E(\underline{f(w_h - \Pi w_h)}) + E(\underline{f\Pi w_h})$.

5. Then, we have

线性形式的相容性误差估计 P260

$$\boxed{\sup_{w_h \in \mathbb{V}_h} \frac{|f(w_h) - f_h(w_h)|}{\|w_h\|_{1,\Omega}} \leq C\, h^k \|f\|_{k,q,\Omega}.}$$

前面仅仅就形如(7.1.1)的变分问题讨论了有限元解的误差估计。对于其它形式的变分问题，也可以类似地得到误差估计，例如参加第7.5节(P261).

Summary of the a Priori Finite Element Error Estimates

有限元解的先验误差估计

The a priori finite element error estimates basically consist of the following main parts FE的先验误差估计基本上由下面几个主要部分组成

1. Abstract error estimate — transform the problem to the errors of subspace approximation and consistency of discrete operators. 抽象误差估计：问题化为子空间逼近的误差 和离散算子的相容性误差

2. Error estimates of subspace approximation — polynomial invariant interpolation operators, polynomial invariant projection operators, polynomial quotient spaces, relations of semi-norms on affine equivalent open sets, etc..

   子空间逼近的误差估计：多项式不变插值算子，多项式不变投影算子、多项式商空间，仿射等价开集上的半范数间的关系等

3. Error estimates of consistency of discrete operators — polynomial invariant interpolation or projection operators and linear, bilinear forms, polynomial quotient spaces, relations of semi-norms on affine equivalent open sets, etc..

   离散算子的相容性的误差估计：多项式不变插值或投影算子、线性形式和双线性形式，多项式商空间，仿射等价开集上的半范数间的关系等

Scaling Technique is the Key

Remark:

An important technique: scaling 缩放技术是关键

- polynomial invariant + polynomial quotient space — show the required inequality in semi-norms in the function spaces on the reference finite element;
- the relations of semi-norms on affine equivalent open sets — bring out the power of h (scaling). Th7.3

仿射等价开集上的半范数间的关系--带来h的幂

h^α appeared in the error estimates can usually be efficiently derived by the scaling technique.

A Priori and A Posteriori Error Estimates of Finite Element Solutions

有限元解的先验和后验误差估计

1. A Priori Error Estimate: error bounds given by known information on the solution of the variational problem and the finite element function spaces. For example, for the second order elliptic problems, the error estimate is given by $\|u - u_h\|_{1,\Omega} \leq C\, h^k\, |u|_{k+1,\Omega}$ (see Theorem 7.10).

   先验误差估计：误差界由变分问题的解和FE函数空间的信息给出

2. A Posteriori Error Estimate: error bounds given by information on the numerical solutions obtained on the finite element function spaces.

   后验误差估计：误差界由FE函数空间中得到的数值解的信息给出

A Priori and A Posteriori Error Estimates of Finite Element Solutions

有限元解的先验和后验误差估计

Remarks:

- In applications, $|u|_{k+1,\Omega}$ is generally not known a priori.

先验误差估计不能给出网格应该如何分布以便平衡开销和精度的提示

- A priori error estimate doesn't give a clue on how the mesh should be distributed to balance the cost and accuracy.

- Compared to the extrapolation technique (see § 1.5), the a posteriori local error estimator can be used to refine or coarsen the mesh wherever necessary locally. **细分/粗化**

R.Verfuerth,A review of a posteriori error estimation and adaptive mesh-refinement techniques,Wiley-Teubner,1996.
R. Verfuerth, A posteriori error estimation techniques for finite element methods-Oxford University Press, 2013.
https://www.ruhr-uni-bochum.de/num1/skriptenE.html

有限元解的残量和误差

Mixed BVP of Poisson Equation on Polygonal Region in $\mathbb{R}^2$

第8章 FEM误差控制与自适应方法P265

- Consider the boundary value problem of the Poisson equation

2D Poisson 方程的BVP

$$\begin{cases} -\triangle u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega_0, \qquad \dfrac{\partial u}{\partial \nu} = g, \quad x \in \partial\Omega_1, \end{cases} \tag{8.1.1}$$

多边形区域

where Ω is a polygonal region in $\mathbb{R}^2$, $\partial\Omega_0$ is a relatively closed subset in $\partial\Omega$ with positive 1-dimensional measure,

$$\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1, \quad \partial\Omega_0 \cap \partial\Omega_1 = \emptyset,$$

$$f \in \mathbb{L}^2(\Omega),\ g \in \mathbb{L}^2(\partial\Omega_1).$$

R.A. Verfuert, A review of a posteriori error estimation and adaptive mesh-refinement techniques, Chichester: Wiley-Teubner, 1996.

Mixed BVP of Poisson Equation on Polygonal Region in $\mathbb{R}^2$

标准弱形式

- consider the standard weak form of the problem:

$$\begin{cases} \text{Find} \quad u \in \mathbb{V} \text{ such that} \\ \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f\, v \, dx + \int_{\partial\Omega_1} g\, v \, ds, \quad \forall v \in \mathbb{V}, \end{cases} \tag{8.1.2}$$

where $\mathbb{V} = \left\{ v \in \mathbb{H}^1(\Omega) : v|_{\partial\Omega_0} = 0 \right\}$;

考虑协调 FEM

- consider the conforming finite element method based on a family of regular class C^0 type (1) Lagrange triangular elements.

A Theorem on the Relation of Residual and Error of a FE Solution

1. Define the residual operator $R : \mathbb{V} \to \mathbb{V}^*$ of the problem by

残量算子
$$R(v)(w) \stackrel{\text{def}}{=} \int_\Omega f\, w\, dx + \int_{\partial\Omega_1} g\, w\, ds - \int_\Omega \nabla v \cdot \nabla w\, dx, \quad \forall w \in \mathbb{V}.$$

2. The dual norm of the residual of a finite element solution u_h:

残量的对偶范数
$$\|R(u_h)\|_{\mathbb{V}^*} \stackrel{\text{def}}{=} \sup_{\substack{w \in \mathbb{V} \\ \|w\|_{1,2,\Omega}=1}} \left\{ \int_\Omega fw\, dx + \int_{\partial\Omega_1} gw\, ds - \int_\Omega \nabla u_h \cdot \nabla w\, dx \right\}$$

Theorem 8.1 (残量范数与FE解H1范数下误差的之间关系)

Let $u \in \mathbb{V}$, $u_h \in \mathbb{V}_h$ be the weak solution and the finite element solution of the problem respectively. Then, there exists a constant $C(\Omega)$, which depends only on Ω, such that

残量与解误差之间的关系
$$\boxed{\|R(u_h)\|_{\mathbb{V}^*} \le \|u - u_h\|_{1,2,\Omega} \le C(\Omega)\, \|R(u_h)\|_{\mathbb{V}^*}.} \qquad (8.1.3)$$

Proof of Residual Dual Norm $\cong$ Error of a FE Solution in $\mathbb{H}^1$-Norm

1. Since u is the weak solution of the problem, we have

(8.1.2)的左右各加一项

$$\int_\Omega \nabla(u - u_h)\cdot\nabla w\, dx = \int_\Omega fw\, dx + \int_{\partial\Omega_1} gw\, ds - \int_\Omega \nabla u_h\cdot\nabla w\, dx, \quad \forall w \in \mathbb{V}.$$

2. Hence, by the Cauchy-Schwarz inequality, we have

C-S不等式　　　半范数小于范数

$$\int_\Omega \nabla(u - u_h)\cdot\nabla w\, dx \le |u - u_h|_{1,2,\Omega}|w|_{1,2,\Omega} \le \|u - u_h\|_{1,2,\Omega}\|w\|_{1,2,\Omega}.$$

3. Thus, the first inequality follows directly from the definition.

Proof of Residual Dual Norm $\cong$ Error of a FE Solution in $\mathbb{H}^1$-Norm

4. On the other hand, by the Poincaré-Friedrichs inequality (see Exercise 5.6), $\exists$ constant $\gamma_0(\Omega) > 0$, s.t.

$$\gamma_0 \|v\|_{1,2,\Omega} \leq |v|_{1,2,\Omega}, \quad \forall v \in \mathbb{V}.$$

5. Thus, by taking $w = u - u_h$ in ①, the second inequality follows for $C(\Omega) = \gamma_0^{-2}$. ■

Remarks on Residual Dual Norm Estimation

残量的对偶范数的估计

1. We hope to develop a formula, which is easily computed and involves only available data such as f, g, u_h and geometric parameters of the triangulation and thus is usually called an a posteriori error estimator, to evaluate the dual norm of the residual. 我们希望得到一个易于计算的公式，它只涉及可用的数据，如f，g，uh和三角网的几何参数，因此通常称为后验误差估计子，用于估计残差的对偶范数.
2. Recall that in the a priori error estimates, the polynomial invariant interpolation operator plays an important role. For example, writing w as $(w - \Pi_h w) + \Pi_h w$ can have some advantage. 回顾，在先验误差估计中，多项式不变插值算子扮演了重要角色. l例如可以把w写为
3. However, the Lagrange nodal type interpolation operators require the function to be at least in $\mathbb{C}^0$. Lagrange 节点型插值算子要求函数至少是C0的
4. Here, we need to introduce a polynomial invariant interpolation operator for functions in $\mathbb{H}^1$.

这里需要引入一个H1函数的多项式不变插值算子

Notations on a Family of Regular Triangular Triangulations $\{\mathfrak{T}_h(\Omega)\}_{h>0}$

1. $\mathcal{E}(K), \mathcal{N}(K)$: the sets of all edges and vertices of $K \in \mathfrak{T}_h(\Omega)$.
   单元K的所有边和顶点集合
2. Denote $\mathcal{E}_h := \bigcup_{K\in\mathfrak{T}_h(\Omega)} \mathcal{E}(K)$, $\mathcal{N}_h := \bigcup_{K\in\mathfrak{T}_h(\Omega)} \mathcal{N}(K)$.
   区域三角剖分中的所有边和顶点集
3. $\mathcal{N}(E)$: the sets of all vertices of an edge $E \in \mathcal{E}_h$.
   边E的所有顶点集合
4. $\mathcal{E}_{h,i} := \left\{ E \in \mathcal{E}_h : \overset{\circ}{E} \subset \partial\Omega_i \right\}$, $\mathcal{N}_{h,i} := \mathcal{N}_h \cap \partial\Omega_i$, $i = 0, 1$.
   区域三角剖分中的在区域边界上的所有边 / 顶点的集合，其中i=0,1为第一类和第二类边界
5. $\mathcal{E}_{h,\Omega} = \mathcal{E}_h \setminus (\mathcal{E}_{h,0} \cup \mathcal{E}_{h,1})$, $\mathcal{N}_{h,\Omega} = \mathcal{N}_h \setminus (\mathcal{N}_{h,0} \cup \mathcal{N}_{h,1})$.
   区域三角剖分中的在区域内所有边 / 顶点的集合

Notations on a Family of Regular Triangular Triangulations $\{\mathfrak{T}_h(\Omega)\}_{h>0}$

6. $\omega_K := \bigcup_{\mathcal{E}(K)\cap\mathcal{E}(K')\neq\emptyset} K'$, $\omega_E := \bigcup_{E\in\mathcal{E}(K')} K'$, $\omega_x := \bigcup_{x\in\mathcal{N}(K')} K'$.

   与单元K共边的所有单元集合　以边E为边的所有单元集合　以x为顶点的所有单元集合

7. $\tilde{\omega}_K := \bigcup_{\mathcal{N}(K)\cap\mathcal{N}(K')\neq\emptyset} K'$, $\tilde{\omega}_E := \bigcup_{\mathcal{N}(E)\cap\mathcal{N}(K')\neq\emptyset} K'$.

   与K有公共顶点的所有单元集合　与E有公共顶点的所有边的集合

8. The corresponding finite element function space:

FE函数空间

$$\mathbb{V}_h = \{v \in \mathbb{C}(\bar{\Omega}) : v|_K \in \mathbb{P}_1(K), \forall K \in \mathfrak{T}_h(\Omega), v(x) = 0, \forall x \in \mathcal{N}_{h,0}\}.$$

The Clément Interpolation Operator $I_h : \mathbb{V} \to \mathbb{V}_h$

非光滑函数的插值【Clement插值】

Definition 8.1

Nh: 区域三角剖分中的所有顶点集合

For any $v \in \mathbb{V}$ and $x \in \mathcal{N}_h$, denote $\pi_x v$ as the $\mathbb{L}^2(\omega_x)$ projection of v on $\mathbb{P}_1(\omega_x)$, meaning $\pi_x v \in \mathbb{P}_1(\omega_x)$ satisfies

ω_x: 以x为顶点的所有单元集合, 也称宏单元

$$\int_{\omega_x} v\, p\, dx = \int_{\omega_x} (\pi_x v)\, p\, dx, \quad \forall p \in \mathbb{P}_1(\omega_x).$$

The Clément interpolation operator $I_h : \mathbb{V} \to \mathbb{V}_h$ is defined by

拟插值算子

$$I_h v(x) = (\pi_x v)(x), \ \ \forall x \in \mathcal{N}_{h,\Omega} \cup \mathcal{N}_{h,1}; \qquad I_h v(x) = 0, \ \ \forall x \in \mathcal{N}_{h,0}.$$

在区域内所有顶点+在区域第2类边界上的所有顶点

在区域第1类边界上的所有顶点处

经典的插值要求被插值函数v(x)光滑，例如在三角形K上过其三个顶点的一次插值Pi(v)可以用面积坐标表示。此时要求v(x)是C0(\bar{K})，同时有误差估计|v-Pi(v)|_{m,K}\leq c*h^{2-m} |v|_{2,T},m\in [0,2].
若v是L1函数，如何构造v的连续的分片多项式插值，并具有与经典插值相同的误差阶。Clement提出了这种所谓局部正则化插值【王烈衡&许学军，P136】

The Clément Interpolation Operator $I_h : \mathbb{V} \to \mathbb{V}_h$

1. The Clément interpolation operator is well defined on $\mathbb{L}^1(\Omega)$.
2. If $v \in \mathbb{P}_1(\omega_x)$, then $(\pi_x)v(x) = v(x)$, $\forall x \in \omega_x$.
   ω_x: 以x为顶点的所有单元集合, 也称宏单元
3. If $v \in \mathbb{P}_1(\tilde{\omega}_K)$, then $I_h v(x) = v(x)$, $\forall x \in K$.
   与K有公共顶点的所有单元集合
4. It is in the above sense that the Clément interpolation operator is polynomial (more precisely $\mathbb{P}_1$) invariant.

Error Estimates of the Clément Interpolation Operator I_h

Clement插值算子Ih的误差估计

Lemma 8.1

There exist constants $C_1(\theta_{\min})$ and $C_2(\theta_{\min})$, which depend only on the smallest angle $\theta_{\min}$ of the triangular elements in the triangulation $\mathfrak{T}_h(\Omega)$, such that, for any given $K \in \mathfrak{T}_h(\Omega)$, $E \in \mathcal{E}_h$ and $v \in \mathbb{V}$,

区域三角剖分，所有边集合

$$\boxed{\begin{aligned} \|v - I_h v\|_{0,2,K} &\le C_1(\theta_{\min})\, h_K\, |v|_{1,2,\tilde{\omega}_K}, \\ \|v - I_h v\|_{0,E} := \|v - I_h v\|_{0,2,E} &\le C_2(\theta_{\min})\, h_K^{1/2}\, |v|_{1,2,\tilde{\omega}_E}. \end{aligned}}$$

与K有公共顶点的所有单元集合

与E有公共顶点的所有边集合

Error Estimates of the Clément Interpolation Operator I_h

- More general properties and proofs on the Clément interpolation operator may be found in [8, 31].

缩放技术

- The basic ingredients of the proof are the scaling techniques (which include the polynomial quotient space and equivalent quotient norms, the relations of semi-norms on affine equivalent open sets), and the inverse inequality. 逆向不等式 Th7.8

Thank You!