Numerical Solutions to Partial Differential Equations

numpde_lecture_21_c8_2.pdf

School of Mathematical Sciences
Peking University

Mixed BVP of Poisson Equation on Polygonal Region in $\mathbb{R}^2$

Ch8 FEM误差控制与自适应方法P265

- Consider the boundary value problem of the Poisson equation

2D Poisson方程的BVP

$$\begin{cases} -\triangle u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega_0, \qquad \dfrac{\partial u}{\partial \nu} = g, \quad x \in \partial\Omega_1, \end{cases} \tag{8.1.1}$$

多边形区域

where Ω is a polygonal region in $\mathbb{R}^2$, $\partial\Omega_0$ is a relative closed subset in $\partial\Omega$ with positive 1-dimensional measure,

$$\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1, \quad \partial\Omega_0 \cap \partial\Omega_1 = \emptyset,$$

$$f \in \mathbb{L}^2(\Omega),\ g \in \mathbb{L}^2(\partial\Omega_1).$$

Mixed BVP of Poisson Equation on Polygonal Region in $\mathbb{R}^2$

标准弱形式

- consider the standard weak form of the problem:

$$\begin{cases} \text{Find} \quad u \in \mathbb{V} \text{ such that} \\ \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f \, v \, dx + \int_{\partial\Omega_1} g \, v \, ds, \quad \forall v \in \mathbb{V}, \end{cases} \tag{8.1.2}$$

where $\mathbb{V} = \{v \in \mathbb{H}^1(\Omega) : v|_{\partial\Omega_0} = 0\}$;

考虑协调FEM

- consider the conforming finite element method based on a family of regular class C^0 type (1) Lagrange triangular elements.

R.A. Verfuert, A review of a posteriori error estimation and adaptive mesh-refinement techniques, Chichester: Wiley-Teubner, 1996.

A Theorem on the Relation of Residual and Error of a FE Solution

1. Define the residual operator $R : \mathbb{V} \to \mathbb{V}^*$ of the problem by

残量算子

$$R(v)(w) \overset{\text{def}}{=} \int_\Omega f\, w\, dx + \int_{\partial\Omega_1} g\, w\, ds - \int_\Omega \nabla v \cdot \nabla w\, dx, \quad \forall w \in \mathbb{V}.$$

2. The dual norm of the residual of a finite element solution u_h:

残量的对偶范数

$$\|R(u_h)\|_{\mathbb{V}^*} = \sup_{\substack{w \in \mathbb{V} \\ \|w\|_{1,2,\Omega}=1}} \left\{ \int_\Omega fw\, dx + \int_{\partial\Omega_1} gw\, ds - \int_\Omega \nabla u_h \cdot \nabla w\, dx \right\}$$

Theorem 8.1 (残量范数与FE解H1范数下误差的之间关系)

Let $u \in \mathbb{V}$, $u_h \in \mathbb{V}_h$ be the weak solution and the finite element solution of the problem respectively. Then, there exists a constant $C(\Omega)$, which depends only on Ω, such that

残量与解误差之间的关系

$$\boxed{\|R(u_h)\|_{\mathbb{V}^*} \le \|u - u_h\|_{1,2,\Omega} \le C(\Omega)\, \|R(u_h)\|_{\mathbb{V}^*}.} \quad (8.1.3)$$

Remarks on Residual Dual Norm Estimation

1. We hope to develop a formula, which is easily computed and involves only available data such as f, g, u_h and geometric parameters of the triangulation and thus is usually called an a posteriori error estimator, to evaluate the dual norm of the residual. 我们希望得到一个易于计算的公式，它只涉及可用的数据，如f，g，uh和三角网的几何参数，因此通常称为后验误差估计子，用于估计残差的对偶范数.

2. Recall that in the a priori error estimates, the polynomial invariant interpolation operator plays an important role. For example, write w as $(w - \Pi_h w) + \Pi_h w$ can have some advantage. 在先验误差估计中，多项式不变的插值算子扮演了重要的角色.

3. However, the Lagrange nodal type interpolation operators require the function to be at least in $\mathbb{C}^0$. 但是，Lagrange节点型插值算子需要函数至少是C0的.

4. Here, we need to introduce a polynomial invariant interpolation operator for functions in $\mathbb{H}^1$. 这里需要引入一个H1中函数的多项式不变的插值算子

Notations on a Family of Regular Triangular Triangulations $\{\mathfrak{T}_h(\Omega)\}_{h>0}$

1. $\mathcal{E}(K)$, $\mathcal{N}(K)$: the sets of all edges and vertices of $K \in \mathfrak{T}_h(\Omega)$.
   单元K的所有边和顶点集合
2. Denote $\mathcal{E}_h := \bigcup_{K\in\mathfrak{T}_h(\Omega)} \mathcal{E}(K)$, $\mathcal{N}_h := \bigcup_{K\in\mathfrak{T}_h(\Omega)} \mathcal{N}(K)$.
   区域三角剖分中的所有边和顶点集
3. $\mathcal{N}(E)$: the sets of all vertices of an edge $E \in \mathcal{E}_h$.
   边E的所有顶点集合
4. $\mathcal{E}_{h,i} := \left\{ E \in \mathcal{E}_h : \overset{\circ}{E} \subset \partial\Omega_i \right\}$, $\mathcal{N}_{h,i} := \mathcal{N}_h \cap \partial\Omega_i$, $i = 0, 1$.
   区域三角剖分中的在区域边界上的所有边 / 顶点的集合，其中i=0,1为第一类和第二类边界
5. $\mathcal{E}_{h,\Omega} = \mathcal{E}_h \setminus (\mathcal{E}_{h,0} \cup \mathcal{E}_{h,1})$, $\mathcal{N}_{h,\Omega} = \mathcal{N}_h \setminus (\mathcal{N}_{h,0} \cup \mathcal{N}_{h,1})$.
   区域三角剖分中的在区域内所有边 / 顶点的集合

Notations on a Family of Regular Triangular Triangulations $\{\mathfrak{T}_h(\Omega)\}_{h>0}$

6. $\omega_K := \bigcup_{\mathcal{E}(K)\cap\mathcal{E}(K')\neq\emptyset} K'$, $\omega_E := \bigcup_{E\in\mathcal{E}(K')} K'$, $\omega_x := \bigcup_{x\in\mathcal{N}(K')} K'$.
   与单元K共边的所有单元集合　以边E为边的所有单元集合　以x为顶点的所有单元集合
7. $\tilde{\omega}_K := \bigcup_{\mathcal{N}(K)\cap\mathcal{N}(K')\neq\emptyset} K'$, $\tilde{\omega}_E := \bigcup_{\mathcal{N}(E)\cap\mathcal{N}(K')\neq\emptyset} K'$.
   与K有公共顶点的所有单元集合　与E有公共顶点的所有边的集合
8. The corresponding finite element function space:

FE函数空间

$$\mathbb{V}_h = \{v \in \mathbb{C}(\bar{\Omega}) : v|_K \in \mathbb{P}_1(K), \forall K \in \mathfrak{T}_h(\Omega), v(x) = 0, \forall x \in \mathcal{N}_{h,0}\}.$$

区域三角剖分中的在区域第一类边界上的所有顶点的集合

The Clément Interpolation Operator $I_h : \mathbb{V} \to \mathbb{V}_h$

非光滑函数的插值【Clement插值】

Definition 8.1

Nh: 区域三角剖分中的所有顶点集合

For any $v \in \mathbb{V}$ and $x \in \mathcal{N}_h$, denote $\pi_x v$ as the $\mathbb{L}^2(\omega_x)$ projection of v on $\mathbb{P}_1(\omega_x)$, meaning $\pi_x v \in \mathbb{P}_1(\omega_x)$ satisfies

ω_x: 以x为顶点的所有单元集合, 也称宏单元

$$\int_{\omega_x} v\, p\, dx = \int_{\omega_x} (\pi_x v)\, p\, dx, \quad \forall p \in \mathbb{P}_1(\omega_x).$$

The Clément interpolation operator $I_h : \mathbb{V} \to \mathbb{V}_h$ is defined by

拟插值算子

$$I_h v(x) \overset{\text{def}}{=} (\pi_x v)(x), \ \ \forall x \in \mathcal{N}_{h,\Omega} \cup \mathcal{N}_{h,1}; \qquad I_h v(x) = 0, \ \ \forall x \in \mathcal{N}_{h,0}.$$

在区域内所有顶点+在第2类边界上的所有顶点

在第1类边界上的所有顶点处

经典的插值要求被插值函数v(x)光滑，例如在三角形K上过其三个顶点的一次插值Pi(v)可以用面积坐标表示。此时要求v(x)是C0(\bar{K})，同时有误差估计|v-Pi(v)|_{m,K}\leq c*h^{2-m} |v|_{2,T},m\in [0,2].
若v是L1函数，如何构造v的连续的分片多项式插值，并具有与经典插值相同的误差阶。Clement提出了这种所谓局部正则化插值【王烈衡&许学军，P136】

The Clément Interpolation Operator $I_h : \mathbb{V} \to \mathbb{V}_h$

1. The Clément interpolation operator is well defined on $\mathbb{L}^1(\Omega)$.
2. If $v \in \mathbb{P}_1(\omega_x)$, then $(\pi_x)v(x) = v(x)$, $\forall x \in \omega_x$.
   ω_x: 以x为顶点的所有单元集合, 也称宏单元
3. If $v \in \mathbb{P}_1(\tilde{\omega}_K)$, then $I_h v(x) = v(x)$, $\forall x \in K$.
   与K有公共顶点的所有单元集合
4. It is in the above sense that the Clément interpolation operator is polynomial (more precisely $\mathbb{P}_1$) invariant.

Error Estimates of the Clément Interpolation Operator I_h

Clement插值算子Ih的局部误差估计

Lemma 8.1

There exist constants $C_1(\theta_{\min})$ and $C_2(\theta_{\min})$, which depend only on the smallest angle $\theta_{\min}$ of the triangular elements in the triangulation $\mathfrak{T}_h(\Omega)$, such that, for any given $K \in \mathfrak{T}_h(\Omega)$, $E \in \mathcal{E}_h$ and $v \in \mathbb{V}$,

区域三角剖分，所有边集合

$$\boxed{\begin{aligned} \|v - I_h v\|_{0,2,K} &\leq C_1(\theta_{\min})\, h_K\, |v|_{1,2,\tilde{\omega}_K}, \\ \|v - I_h v\|_{0,E} := \|v - I_h v\|_{0,2,E} &\leq C_2(\theta_{\min})\, h_K^{1/2}\, |v|_{1,2,\tilde{\omega}_E}. \end{aligned}}$$

与K有公共顶点的所有单元集合

与E有公共顶点的所有边集合

Error Estimates of the Clément Interpolation Operator I_h

- More general properties and proofs on the Clément interpolation operator may be found in [8, 31].

缩放技术

- The basic ingredients of the proof are the scaling techniques (which include the polynomial quotient space and equivalent quotient norms, the relations of semi-norms on affine equivalent open sets), and the inverse inequality.

逆向不等式
Th7.8

An Upper Bound for the Dual Norm of the Residual $R(u_h)$

Lemma 8.2 (残量R(uh)的上界估计)

There exists a constant $C(\theta_{\min})$, where $\theta_{\min}$ is the smallest angle of the triangular elements in the triangulation $\mathfrak{T}_h(\Omega)$, such that

残量R(uh)(W)=

$$\int_\Omega f\, w\, dx + \int_{\partial\Omega_1} g\, w\, ds - \int_\Omega \nabla u_h \cdot \nabla w\, dx$$

$$\leq C(\theta_{\min})\|w\|_{1,2,\Omega}\Big\{ \sum_{K\in\mathfrak{T}_h(\Omega)} h_K^2 \|f\|_{0,2,K}^2 + \sum_{E\in\mathcal{E}_{h,1}} h_E \|g-\nu_E\cdot\nabla u_h\|_{0,E}^2 + \sum_{E\in\mathcal{E}_{h,\Omega}} h_E \|[\nu_E\cdot\nabla u_h]_E\|_{0,E}^2 \Big\}^{1/2}, \quad \forall w\in\mathbb{V}, \tag{8.1.5}$$

Proof of the Lemma — An Upper Bound for the Residual $R(u_h)$

where in the theorem, ν_E is an arbitrarily given unit normal of E if $E \in \mathcal{E}_{h,\Omega}$, and is the unit outward normal of Ω if $E \in \mathcal{E}_{h,1}$, $[\varphi]_E$ is the jump of φ across E in the direction of ν_E, *i.e.*
跳跃

$$[\varphi]_E(x) = \lim_{t \to 0+} \varphi(x + t\nu_E) - \lim_{t \to 0+} \varphi(x - t\nu_E), \quad \forall x \in E.$$

Proof:

1. Since u_h is the finite element solution, we have

残量 $$R(u_h)(v_h) := \int_\Omega f v_h \, dx + \int_{\partial\Omega_1} g v_h \, ds - \int_\Omega \nabla u_h \cdot \nabla v_h \, dx = 0, \quad \forall v_h \in \mathbb{V}_h.$$

In particular, $R(u_h)(w) = R(u_h)(w - I_h w)$, for all $w \in \mathbb{V}$.

Proof of the Lemma — An Upper Bound for the Residual $R(u_h)$

2. Applying the Green's formula on every element K, denoting ν_K as the unit exterior normal of ∂K, and noticing $u_h|_K \in \mathbb{P}_1(K)$ and thus $\triangle u_h = 0$ on each element, we obtain

$$\int_\Omega \nabla u_h \cdot \nabla v \, dx = \sum_{K \in \mathfrak{T}_h(\Omega)} \int_K \nabla u_h \cdot \nabla v \, dx$$

$$\overset{\text{Green公式}}{=} \sum_{K \in \mathfrak{T}_h(\Omega)} \left\{ -\int_K \triangle u_h \, v \, dx + \int_{\partial K} \nu_K \cdot \nabla u_h \, v \, dx \right\}$$

$$= \underbrace{\sum_{E \in \mathcal{E}_{h,1}}}_{\text{区域三角剖分中的在区域第2类边界上的所有边}} \int_E \nu_E \cdot \nabla u_h \, v \, ds + \underbrace{\sum_{E \in \mathcal{E}_{h,\Omega}}}_{\text{区域三角剖分中的在区域内的所有边}} \int_E [\nu_E \cdot \nabla u_h]_E \, v \, ds, \quad \forall v \in \mathbb{V}. \qquad (*1)$$

Proof of the Lemma — An Upper Bound for the Residual $R(u_h)$

3. Thus, recall $R(u_h)(w) = R(u_h)(w - I_h w)$, we have

(*1)

$$\int_\Omega f\, w\, dx + \int_{\partial\Omega_1} g\, w\, ds - \int_\Omega \nabla u_h \cdot \nabla w\, dx = \sum_{K\in\mathfrak{T}_h(\Omega)} \int_K f\,(w - I_h w)\, dx$$

$$+ \sum_{E\in\mathcal{E}_{h,1}} \int_E (g - \nu_E\cdot\nabla u_h)(w - I_h w)\, ds - \sum_{E\in\mathcal{E}_{h,\Omega}} \int_E [\nu_E\cdot\nabla u_h]_E\,(w - I_h w)\, ds,$$

第2类边界上的所有边　　　　在区域内的所有边

...... $\leq \sum(A1) + \sum(A2) + \sum(A3)$

Proof of the Lemma — An Upper Bound for the Residual $R(u_h)$

4. By the Cauchy-Schwarz inequality and Lemma 8.1, we have

(A1) $$\int_K f\,(w - I_h w)\,dx \le \|f\|_{0,2,K}\|w - I_h w\|_{0,2,K} \le C_1\, h_K\, \|f\|_{0,2,K}\|w\|_{1,2,\tilde{\omega}_K},$$

(A2) $$\int_E (g - \nu_E \cdot \nabla u_h)(w - I_h w)ds \le \|g - \nu_E \cdot \nabla u_h\|_{0,E}\|w - I_h w\|_{0,E} \le C_2 h_E^{1/2}\|g - \nu_E \cdot \nabla u_h\|_{0,E}\|w\|_{1,2,\tilde{\omega}_E},$$

(A3) $$\int_E [\nu_E \cdot \nabla u_h]_E\,(w - I_h w)\,ds \le \|[\nu_E \cdot \nabla u_h]_E\|_{0,E}\|w - I_h w\|_{0,E} \le C_2 h_E^{1/2}\|[\nu_E \cdot \nabla u_h]_E\|_{0,E}\|w\|_{1,2,\tilde{\omega}_E}.$$

Proof of the Lemma — An Upper Bound for the Residual $R(u_h)$

5. The number of element in ω_x, $\sharp\omega_x \leq C_3 = 2\pi/\theta_{\min}$, $\forall x \in \mathcal{N}_h$.
6. Each element K has three vertices, each edge E has two vertices.
7. Therefore, $\sharp\tilde{\omega}_K \leq 3C_3$, $\forall K \in \mathfrak{T}_h$, $\sharp\tilde{\omega}_E \leq 2C_3$, $\forall E \in \mathcal{E}_h$.
8. Thus, we have

$$\Big[\sum_{K\in\mathfrak{T}_h(\Omega)} \|w\|^2_{1,2,\tilde{\omega}_K} + \sum_{E\in\mathcal{E}_{h,\Omega}\cup\mathcal{E}_{h,1}} \|w\|^2_{1,2,\tilde{\omega}_E}\Big]^{1/2} \leq \sqrt{5\,C_3}\,\|w\|_{1,2,\Omega}. \tag{8.1.6}$$

9. The conclusion of the lemma follows as a consequence of ③, ④ and ⑧ with $C(\theta_{\min}) = \sqrt{5C_3}\max\{C_1(\theta_{\min}), C_2(\theta_{\min})\}$. ■

A Theorem on the a Posteriori Error Estimate

Theorem 8.2

~~*As a corollary of Theorem 8.1 and Lemma 8.2,*~~ *we have the following a posteriori error estimate of the finite element solution:*

$$\|u-u_h\|_{1,2,\Omega} \overset{(8.1.3),(8.1.5)}{\leq} C\Big\{\sum_{K\in\mathfrak{T}_h(\Omega)} h_K^2\,\|f\|_{0,2,K}^2 +$$

右端大括号内的项与(8.1.5)右端大括号中一致

$$\sum_{E\in\mathcal{E}_{h,1}} h_E\,\|g-\nu_E\cdot\nabla u_h\|_{0,E}^2 + \sum_{E\in\mathcal{E}_{h,\Omega}} h_E\,\|[\nu_E\cdot\nabla u_h]_E\|_{0,E}^2\Big\}^{1/2}, \quad (8.1.7)$$

where $C = C(\theta_{\min})\,C(\Omega)$ is a constant depending only on Ω and the smallest angle $\theta_{\min}$ of the triangulation $\mathfrak{T}_h(\Omega)$.

- The righthand side term above essentially gives an upper bound estimate for the $\mathbb{V}^*$-norm of the residual $R(u_h)$, which can be directly used as a posteriori error estimator for the upper bound of the error of the finite element solution.

(8.1.7)的右端本质上给出的是残量在V的对偶空间V*中的范数的上界估计， 可以将其用作后验误差估计子来估计FEM解的误差上界。

A Practical a Posteriori Error Estimator

1. For convenience of analysis and practical computations, f and g are usually replaced by some approximation functions, say by $f_K = \frac{1}{|K|}\int_K f\,dx$ and $g_E = h_E^{-1}\int_E g\,ds$.
2. A practical a posteriori error estimator of residual type:

$$\eta_{R,K} \stackrel{\text{def}}{=} \Big\{ h_K^2 \|f_K\|_{0,2,K}^2 + \sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_{h,1}} h_E \|g_E - \nu_E\cdot\nabla u_h\|_{0,E}^2 + \frac{1}{2}\sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_{h,\Omega}} h_E \|[\nu_E\cdot\nabla u_h]_E\|_{0,E}^2 \Big\}^{1/2}. \quad (8.1.8)$$

实用的残量型后验误差估计子

3. In applications, f_K and g_E can be further replaced by the numerical quadratures of the corresponding integrals.

Error Estimate Based on the Practical a Posteriori Error Estimator

Theorem 8.3 P270

For the constant $C = C(\theta_{\min})\, C(\Omega)$ in Theorem 8.2, the following a posteriori error estimate holds:

$$\|u - u_h\|_{1,2,\Omega} \le C \Big\{ \sum_{K \in \mathfrak{T}_h(\Omega)} \eta_{R,K}^2 + \sum_{K \in \mathfrak{T}_h(\Omega)} h_K^2 \, \|f - f_K\|_{0,2,K}^2 + \sum_{E \in \mathcal{E}_{h,1}} h_E \, \|g - g_E\|_{0,E}^2 \Big\}^{1/2}. \tag{8.1.9}$$

Remark: Generally speaking, for h sufficiently small, the first term on the righthand side represents the leading part of the error. Therefore, in practical computations, $\eta_{R,K}$ alone is often used to estimate the local error, particularly in a mesh adaptive process.

一般地说，当h充分小时，上式右端的第1项往往反映了误差的主要部分，因此实际计算中，常用\eta_R,K来估计局部误差，特别时在作网格自适应时

Reliability of an a Posteriori Error Estimator

8.2节 后验误差估计子的可靠性和有效性

1. The a posteriori error estimators given in Theorem 8.2 and 8.3 provide upper bounds for the error of the finite element solution u_h in the $\mathbb{V}$-norm.
   定理8.2和8.3中的后验误差估计子给出了有限元解误差的上界
2. Such a property is called the reliability of the a posteriori error estimator. 后验误差估计子的可靠性
3. In general, the reliability of an a posteriori error estimator can be understood in the sense of a constant times
   一般地，可以在相差一个常数倍的意义下理解后验误差估计

Reliability of an a Posteriori Error Estimator

Definition 8.2

Let u and u_h be the solution and the finite element solution of the variational problem. Let η_h be an a posteriori error estimator. If there exists a constant $\widehat{C}$ independent of h such that 后验误差估计子

$$\boxed{\|u - u_h\|_{1,2,\Omega} \leq \widehat{C}\,\eta_h.}$$

Then, the a posteriori error estimator η_h is said to be reliable, or has reliability. 后验误差估计子是可靠的

1. Reliability guarantees the accuracy.
2. To avoid mesh being unnecessarily refined and have the computational cost under control, efficiency is required.
为了避免网格被不必要地细分，和有可控的计算开销，需要有效性

Efficiency of an a Posteriori Error Estimator

Definition 8.3

Let u and u_h be the solution and the finite element solution of the variational problem. Let η_h be an a posteriori error estimator. If, for any given $h_0 > 0$, there exists a constant $\widetilde{C}(h_0)$ such that

$$\boxed{\widetilde{C}(h_0)^{-1} \|u - u_h\|_{1,2,\Omega} \leq \eta_h \leq \widetilde{C}(h_0) \|u - u_h\|_{1,2,\Omega}, \quad \forall h \in (0,\ h_0),} \tag{8.2.1}$$

Then, the a posteriori error estimator η_h is said to be efficient, or has efficiency. In addition, if the constant $\widetilde{C}(h_0)$ is such that

后验误差估计子是有效的

$$\boxed{\lim_{h_0 \to 0+} \widetilde{C}(h_0) = 1,}$$

Then, the a posteriori error estimator η_h is said to be asymptotically exact.

后验误差估计子是渐近精确的

a Posteriori Local Error Estimator and the Local Error

1. In applications, to efficiently control the error, we hope to refine the mesh only on the regions where the local error is relatively large. 为了有效地控制误差，我们希望只对局部误差较大的区域进行网格细化

2. Therefore, in addition to a good estimate of the global error of a finite element solution, what we expect more on an a posteriori error estimator is that it can efficiently evaluate the local error. 因此，除了能很好地估计有限元解的整体误差外，我们更期望后验误差估计子，它能有效地估计局部误差。

a Posteriori Local Error Estimator and the Local Error

3. Recall $\int_\Omega \nabla(u-u_h)\cdot\nabla w\,dx = R(u_h)(w)$, and the a posteriori local error estimator of residual type is given as

$$\eta_{R,K} = \Big\{ h_K^2 \,\|f_K\|_{0,2,K}^2 + \sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_{h,1}} h_E \,\|g_E - \nu_E\cdot\nabla u_h\|_{0,E}^2 + \frac{1}{2}\sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_{h,\Omega}} h_E \,\|[\nu_E\cdot\nabla u_h]_E\|_{0,E}^2 \Big\}^{1/2}. \qquad (8.1.8)$$

4. We hope, by choosing proper test functions w, to establish relationship between the local error of $u-u_h$ and the three terms in $\eta_{R,K}$. 我们希望通过适当选择试验函数w建立u-uh的局部误差和eta中三项的关系

Relate Terms in $\eta_{R,K}$ to the Local Errors of $u - u_h$

1. Notice that f_K is piecewise constant, $\forall w \in \mathbb{V}$, we have

$$\int_K f_K\,(f_K w)\,dx = |f_K|^2 \int_K w\,dx = |K|^{-1}\left(\int_K w\,dx\right)\|f_K\|_{0,2,K}^2,$$

2. If we take a positive $w \in \mathbb{H}_0^1(K)$, called a bubble function on K, then, the above equation will establish a relation between $\|f_K\|_{0,2,K}^2$ and the local error of $(u - u_h)|_K$ through

$$\underline{\int_K \nabla(u-u_h)\cdot\nabla(f_K w)\,dx = R(u_h)(f_K w)} \stackrel{\text{def}}{=} \int_K f\,(f_K w)\,dx$$

$$= \int_K f_K\,(f_K w)\,dx + \int_K (f - f_K)\,(f_K w)\,dx.$$

Relate Terms in $\eta_{R,K}$ to the Local Errors of $u - u_h$

3. Similarly, by taking proper bubble functions, we can also establish the relationship between the local error of $u - u_h$ and the terms

$$\|g_E - \nu_E \cdot \nabla u_h\|_{0,E}^2, \qquad \|[\nu_E \cdot \nabla u_h]_E\|_{0,E}^2,$$

which are also piecewise constant functions.

Triangular Element Bubble Functions and Edge Bubble Functions

1. Let $\lambda_{K,i}$, $i = 1, 2, 3$ be the area coordinates of $K \in \mathfrak{T}_h(\Omega)$, define the triangular bubble function $\mathfrak{b}_K$ as
泡函数

$$\mathfrak{b}_K(x) = \begin{cases} 27\, \lambda_{K,1}(x)\, \lambda_{K,2}(x)\, \lambda_{K,3}(x), & \forall x \in K; \\ 0, & \forall x \in \Omega \setminus K. \end{cases}$$

Triangular Element Bubble Functions and Edge Bubble Functions

2. For a given edge $E \in \mathcal{E}_{h,\Omega}$, let $\omega_E = K_1 \cup K_2$, let $\lambda_{K_i,j}$, $j = 1, 2, 3$ be the area coordinates of K_i, denote the vertex of K_i which is not on E as the third vertex of K_i, define the edge bubble function $\mathfrak{b}_E$ as

Ki的不在边E上的顶点表示为Ki的第三个顶点

边泡函数

$$\mathfrak{b}_E(x) = \begin{cases} 4\,\lambda_{K_i,1}(x)\,\lambda_{K_i,2}(x), & \forall x \in K_i, \quad i = 1, 2; \\ 0, & \forall x \in \Omega \setminus \omega_E. \end{cases}$$

Triangular Element Bubble Functions and Edge Bubble Functions

3. For a given edge $E \in \mathcal{E}_{h,\partial\Omega}$, let $\omega_E = K'$, denote the vertex of K' which is not on E as the third vertex of K', define the edge bubble function $\mathfrak{b}_E$ as K'的不在边E上的顶点表示为K'的第三个顶点

$$\mathfrak{b}_E(x) = \begin{cases} 4\,\lambda_{K',1}(x)\,\lambda_{K',2}(x), & \forall x \in K'; \\ 0, & \forall x \in \Omega \setminus K'. \end{cases}$$

Properties of the Bubble Functions

Lemma 8.3

(1) *For any given $K \in \mathfrak{T}_h(\Omega)$ and $E \in \mathcal{E}_h$, the bubble functions $\mathfrak{b}_K$ and $\mathfrak{b}_E$ have the following properties:*

单元/边泡函数的性质

$$\operatorname{supp} \mathfrak{b}_K \subset K, \quad 0 \leq \mathfrak{b}_K \leq 1, \quad \max_{x \in K} \mathfrak{b}_K(x) = 1; \qquad (*1)$$

$$\operatorname{supp} \mathfrak{b}_E \subset \omega_E, \quad 0 \leq \mathfrak{b}_E \leq 1, \quad \max_{x \in E} \mathfrak{b}_E(x) = 1; \qquad (*2)$$

$$\int_E \mathfrak{b}_E \, ds = \frac{2}{3} h_E; \qquad (*3)$$

Properties of the Bubble Functions

Lemma 8.3(续)

(2) *there exists a constant* $\hat{c}_i$, $i = 1, \ldots, 6$, *which depends only on the smallest angle of the triangular triangulation* $\mathfrak{T}_h(\Omega)$, *such that*

$$\hat{c}_1\, h_K^2 \le \int_K \mathfrak{b}_K\, dx = \frac{9}{20}\, |K| \le \hat{c}_2\, h_K^2; \qquad (*4)$$

单元/边泡函数的性质

$$\hat{c}_3\, h_E^2 \le \int_{K'} \mathfrak{b}_E\, dx = \frac{1}{3}\, |K'| \le \hat{c}_4\, h_E^2, \quad \forall K' \subset \omega_E; \qquad (*5)$$

$$\|\nabla \mathfrak{b}_K\|_{0,2,K} \le \hat{c}_5\, h_K^{-1} \|\mathfrak{b}_K\|_{0,2,K}; \qquad (*6)$$

$$\|\nabla \mathfrak{b}_E\|_{0,2,K'} \le \hat{c}_6\, h_E^{-1} \|\mathfrak{b}_E\|_{0,2,K'}, \quad \forall K' \subset \omega_E. \qquad (*7)$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|f_K\|_{0,2,K}$

定理8.4的证明的第一部分

P274

1. For any given $K \in \mathfrak{T}_h(\Omega)$, set $w_K := f_K \mathfrak{b}_K$. Then, by the properties of $\mathfrak{b}_K$ (see Lemma 8.3), we have

$$\int_K f_K \, w_K \, dx \overset{(*4)}{=} \frac{9}{20} |K| \, |f_K|^2 = \frac{9}{20} \|f_K\|^2_{0,2,K}. \qquad (8.2.3)$$

2. Since $\operatorname{supp} w_K \subset K$, it follows

$$\cancel{\int_{\partial\Omega_1} g \, w_K \, ds} - \int_\Omega \nabla u_h \cdot \nabla w_K \, dx \overset{\text{uh为次数不超过1的多项式}}{=\!=} -\nabla u_h|_K \int_K \nabla w_K \, dx = 0. \qquad (*8)$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|f_K\|_{0,2,K}$

3. Thus, by $\int_\Omega \nabla(u-u_h)\cdot\nabla w_K\,dx = R(u_h)(w_K)$, we obtain

$$\begin{aligned}\int_K f_K\,w_K\,dx &= \int_K f\,w_K\,dx \overset{-}{+} \int_K (f-f_K)\,w_K\,dx\\ &\overset{(8.1.4)}{\underset{(*8)}{=}} \int_K \nabla(u-u_h)\cdot\nabla w_K\,dx \overset{-}{+} \int_K (f-f_K)\,w_K\,dx\\ &\le \|u-u_h\|_{1,2,K}\|\nabla w_K\|_{0,2,K} + \|f-f_K\|_{0,2,K}\|w_K\|_{0,2,K}. \qquad (*9)\end{aligned}$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|f_K\|_{0,2,K}$

4. On the other hand, since f_K is a constant, by the properties of $\mathfrak{b}_K$ (see Lemma 8.3), we have

$$\|w_K\|_{0,2,K} \overset{\text{WK定义}}{=} |f_K| \, \|\mathfrak{b}_K\|_{0,2,K} \leq |f_K| \left(\int_K \mathfrak{b}_K \, dx \right)^{1/2} \overset{(*4)}{=} \sqrt{\frac{9}{20}} \, \|f_K\|_{0,2,K};$$

$$\|\nabla w_K\|_{0,2,K} \overset{(*6)}{\leq} \hat{c}_5 \, h_K^{-1} \|w_K\|_{0,2,K}.$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|f_K\|_{0,2,K}$

5. Combining the three inequalities obtained in ③ and ④ with $\int_K f_K\, w_K\, dx = \frac{9}{20}\, \|f_K\|^2_{0,2,K}$ (see (8.2.3)) leads to

$$\|f_K\|_{0,2,K} \leq \sqrt{\frac{20}{9}}\, \hat{c}_5\, h_K^{-1} \|u - u_h\|_{1,2,K} + \sqrt{\frac{20}{9}}\, \|f - f_K\|_{0,2,K}. \tag{8.2.4}$$

Similar techniques can be applied to estimate the other terms in $\eta_{R,K}$.

习题 8：4. Page 283

Thank You!