Homework 7 for "Algorithms for Big-Data Analysis"

Note: Please write up your solutions independently. If you get significant help from others, write down the source of references. A formal mathematical proof for all your claims is required.

1. Derive the dual optimization problem for

$$\begin{aligned} \min_{w,b,\xi} \quad & \frac{1}{2}\|w\|_2^2 + C_1 \sum_{i=1}^{n} \xi_i + C_2 \sum_{i=1}^{n} \xi_i^2 \\ \text{s.t.} \quad & y_i \cdot (x_i \cdot w + b) \geq 1 - \xi_i, \forall i = 1, \ldots, n \\ & \xi_i \geq 0, \forall i = 1, \ldots, n \end{aligned}$$

2. Properties of Submodular Functions

   (a) Prove that any non-negative submodular function is also subadditive, i.e. if $F : 2^X \to \mathbb{R}_+$ is submodular then $F(S \cup T) \leq F(S) + F(T)$ for any $S, T \subseteq X$. Here, $\mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$.

   (b) Prove that a function $F : 2^X \to \mathbb{R}_+$ is submodular if and only if for any $S, T \subseteq X$, the marginal contribution function $F_S(T) = F(S \cup T) - F(S)$ is subadditive. (If the statement is not true, please either add a condition to make it correct or give a counterexample.)

3. Consider a graph (V, E), where V is the set of nodes and E is the set of edges. Let S be a subset of V and $V \backslash S$ be the complement of S. Define $f(S)$ be the number of edges $e = (u, v)$ such that $u \in S$ and $v \in V \backslash S$. Prove that $f(S)$ is submodular.

4. Exercise 3.4 in `http://incompleteideas.net/book/RLbook2020.pdf`

5. Exercise 3.23 in `http://incompleteideas.net/book/RLbook2020.pdf`

Homework 7

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1 Problem 1

The Lagrange function can be written as

$$L(w, \xi, \lambda, \mu) = \frac{1}{2}\|w\|_2^2 + \sum_{i=1}^{n} C_1\xi_i + C_2\xi_i^2 - \lambda_i(y_i(x_i \cdot w + b) - 1 + \xi_i) - \mu_i\xi_i$$

By taking the derivative

$$\begin{aligned}
\frac{\partial L}{\partial w} &= w - \sum_{i=1}^{n} \lambda_i y_i x_i = 0 \\
\frac{\partial L}{\partial \xi_i} &= C_1 + 2C_2\xi_i - \lambda_i - \mu_i = 0 \\
\frac{\partial L}{\partial b} &= \sum_{i=1}^{n} \lambda_i y_i = 0
\end{aligned}$$

we have

$$L(w^*, \xi^*, \lambda, \mu) = -\frac{1}{2}\|\sum_{i=1}^{n} \lambda_i y_i x_i\|_2^2 + \sum_{i=1}^{n} \lambda_i - \frac{(\lambda_i + \mu_i - C_1)^2}{4C_2}$$

Hence, the dual problem can be formulated as

$$\begin{aligned}
\min_{\lambda,\mu} \quad & \frac{1}{2}\|\sum_{i=1}^{n} \lambda_i y_i x_i\|_2^2 - \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \frac{(\lambda_i + \mu_i - C_1)^2}{4C_2} \\
\text{s.t.} \quad & \lambda_i, \mu_i \geq 0, \sum_{i=1}^{n} \lambda_i y_i = 0
\end{aligned}$$

2 Problem 2

2.1 (a)

For any $S, T \subset 2^X$, according to the definition:

$$F(S) + F(T) \geq F(S \cup T) + F(S \cap T) \geq F(S \cup T)$$

hence f is subadditive.

2.2 (b)

- Sufficiency: Since $F_S(T)$ is subadditive. We have

  $$F_{A\cap B}((A - B) \cup (B - A)) \leq F_{A\cap B}(A - B) + F_{A\cap B}(B - A)$$

  Hence,

  $$F(A \cup B) - F(A \cap B) \leq F(A) + F(B) - 2F(A \cap B)$$

  Thus, F is submodular.

- Necessity:
  Since F_S is additive if and only if

  $$F(A \cup C) + F(B \cup C) \geq F(C) + F(A \cup B \cup C)$$

  Owing to the submodularity of F, we have

  $$F(A \cup C) + F(B \cup C) \geq F(C \cup (A \cap B)) + F(A \cup B \cup C)$$

  Hence, **we need an additional condition**: F is monotone. The results follows from the monotone condition

  $$F(C \cup (A \cap B)) \geq F(C)$$

2.3 Counterexample

For a given finite, non-empty set X, we take an element $a \in X$, and define the function $F(A), A \subset X$ as

$$F(A) = \begin{cases} |A|, & \text{if not } a \in A \\ 0 & \text{if } a \in A. \end{cases} \tag{1}$$

For any $A \subset B \subset X$, and $x \in X/B$, if $a \in A$, then

$$F(B \cup \{s\}) - F(B) = F(A \cup \{s\}) - F(A) = 0 \tag{2}$$

the inequality holds, and if $a = s$

$$F(B \cup \{s\}) - F(B) = -F(B) = -|B| \leq -|A| = F(A \cup \{s\}) - F(A) \tag{3}$$

the inequality holds as well. For other cases, we have

$$F(B \cup \{s\}) - F(B) \leq 1 = F(A \cup \{s\}) - F(A) \tag{4}$$

However, if we take $S = T = \{a\}, R = b \neq a$, we have

$$F_R(S) = F_R(T) = F(R \cup T) - F(R) = -1, \quad F_R(S \cup T) = F(S \cup R \cup T) - F(R) = -1 \tag{5}$$

Hence, $F_R(S) + F_R(T) < F_R(S \cup T)$, which means F_R is not subadditive.

3 Problem 3

Denote the number of edges connecting set A and B is $L(A, B)$. For any set $S, T \subset V$, denote sets $S - T, T - S, S \cap T, V - S \cup T$ as A, B, C, D separately.

Since we have

$$\begin{aligned} f(S) &= L(A, D) + L(A, B) + L(C, D) + L(C, B) \\ f(T) &= L(A, C) + L(A, B) + L(B, D) + L(C, D) \\ f(S \cap T) &= L(A, C) + L(D, C) + L(B, C) \\ f(S \cup T) &= L(A, D) + L(B, D) + L(C, D) \end{aligned}$$

Clearly, we have

$$f(S) + f(T) - f(S \cap T) - f(S \cup T) = 2L(A, B) \geq 0$$

Hence, f is submodular.

4 Problem 4

According to the the transition graph of the finite MDP

s	a	s'	r	$p(s', r \mid s, a)$
high	search	high	r_{search}	α
high	search	low	r_{search}	$1-\alpha$
low	search	high	-3	$1-\beta$
low	search	low	r_{search}	β
high	wait	high	r_{wait}	1
low	wait	low	r_{wait}	1
low	recharge	high	0	1

5 Problem 5

According to the definition, the Bellman equation can be written as

$$
\begin{aligned}
q_\star(\text{high}, \text{search}) &= \alpha(r_{\text{search}} + \gamma \max_{a'} q_\star(\text{high}, a') && \text{(6a)}\\
&\quad + (1-\alpha)(r_{\text{search}} + \gamma \max_{a'} q_\star(\text{low}, a')) && \text{(6b)}\\
q_\star(\text{high}, \text{wait}) &= r_{\text{wait}} + \gamma \max_{a'} q_\star(\text{high}, a') && \text{(6c)}\\
q_\star(\text{low}, \text{search}) &= (1-\beta)(-3 + \gamma \max_{a'} q_\star(\text{high}, a')) && \text{(6d)}\\
&\quad + \beta(r_{\text{search}} + \gamma \max_{a'} q_\star(\text{low}, a')) && \text{(6e)}\\
q_\star(\text{low}, \text{wait}) &= r_{\text{wait}} + \gamma \max_{a'} q_\star(\text{low}, a') && \text{(6f)}\\
q_\star(\text{low}, \text{recharge}) &= \gamma \max_{a'} q_\star(\text{high}, a') && \text{(6g)}
\end{aligned}
$$

where

$$
\begin{aligned}
\max_{a'} q_\star(\text{high}, a') &= \max\{q_\star(\text{high}, \text{search}), q_\star(\text{high}, \text{wait})\} && \text{(7a)}\\
\max_{a'} q_\star(\text{low}, a') &= \max\{q_\star(\text{low}, \text{search}), q_\star(\text{low}, \text{wait}), q_\star(\text{low}, \text{recharge})\} && \text{(7b)}
\end{aligned}
$$