Numerical Solutions to Partial Differential Equations

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Things to Check for the Overall Performance of a Scheme

Under the CFL condition,

1. Local truncation error and consistency;
2. Dissipation and global error on the amplitude;
3. Dispersion and relative error on the phase angle (dispersion relation: $\omega(k) = -ak$, $\omega_h(k)\tau = \arg\lambda_k = -ak\tau(1 + \cdots)$);
4. The error on the group speed ($C(k) = \omega'(k)$, $C_h(k) = \omega_h'(k)$), for large k;
5. Spurious modes must be under control.

Vertical Fourier Modes and Amplification Factors

1. Separation of variables of difference solutions: $U_j^m = \lambda^m \mu^j$.
2. Standard Fourier modes: $U_j^m = \lambda_k^m e^{\mathrm{i}kjh}$. $(\mu_k = e^{\mathrm{i}kh})$
3. Vertical Fourier modes: $U_j^m = e^{\pm \mathrm{i}amk\tau} \mu_k^j$. $(\lambda_k = e^{\pm \mathrm{i}ak\tau})$
4. Fourier mode solutions of the advection equation: $e^{\mp \mathrm{i}k(x-at)}$.
5. The amplification factors of the advection equation: $e^{\mp \mathrm{i}kh}$.
6. Substitute the vertical Fourier mode $U_j^m = e^{\pm \mathrm{i}amk\tau} \mu_k^j$ into a difference scheme to get the amplification factor $\mu_k^{\pm}$.
7. For real solution modes, we expect $\mu_k^{\pm} \approx e^{\mp \mathrm{i}kh}$.

Strong Damping Effect of the Lax-Wendroff Scheme to Spurious Modes

In general, If the CFL condition is satisfied, the Lax-Wendroff scheme has very strong damping effect to the reversely propagating waves, so the additional numerical boundary conditions will not cause too much error pollution to the numerical results.

Amplification Factor of Vertical Fourier Mode of the Leap-frog Scheme

1. The leap-frog scheme: $U_j^{m+1} = U_j^{m-1} - \nu(U_{j+1}^m - U_{j-1}^m)$.
2. Substitute $U_j^m = \lambda_k^m \mu_k^j = e^{\pm \mathrm{i} amk\tau} \mu_k^j$ into the leap-frog scheme yields the characteristic equation:
$$\nu \lambda_k \mu_k^2 - (1 - \lambda_k^2)\mu_k - \nu \lambda_k = 0.$$
3. The amplification factor of the vertical Fourier mode:
$$\mu_k = \frac{(1-\lambda_k^2) \pm \sqrt{(1-\lambda_k^2)^2 + 4\nu^2 \lambda_k^2}}{2\nu\lambda_k}.$$
$(\mu_k^{\pm} = \pm(1 - (\frac{\sin\theta}{\nu})^2)^{1/2} - \mathrm{i}\frac{\sin\theta}{\nu},\ \theta = \pm ak\tau = \pm\nu kh)$.
4. $\lambda_k^{\pm} = e^{\pm \mathrm{i} ak\tau} \approx 1 \pm \mathrm{i} ak\tau = 1 \pm \mathrm{i}\nu kh.$

Amplification Factor of Vertical Fourier Mode of the Leap-frog Scheme

4. For the two real solution modes $U_j^{\pm m} = \left(\lambda_k^{\pm}\right)^m \left(\mu_k^{r\pm}\right)^j$:

$$\mu_k^{r\pm} = \mu_k^r(\lambda_k^{\pm}) \approx \begin{cases} 1 - \mathrm{i}kh \approx e^{-\mathrm{i}kh}, \\ 1 + \mathrm{i}kh \approx e^{\mathrm{i}kh}, \end{cases}$$

5. For the two spurious solution modes $V_j^{\pm m} = \left(\lambda_k^{\pm}\right)^m \left(\mu_k^{s\pm}\right)^j$:

$$\mu_k^{s\pm} = \mu_k^s(\lambda_k^{\pm}) \approx \begin{cases} -(1 + \mathrm{i}kh) \approx -e^{\mathrm{i}kh}; \\ -(1 - \mathrm{i}kh) \approx -e^{-\mathrm{i}kh}. \end{cases}$$

6. The leap-frog scheme may have no damp to some high frequency vertical spurious modes (In fact, whenever $|\sin \nu kh| \leq |\nu|$ holds. This happens when $|\nu|$ is close to 1. On the other hand, if $|\nu| < \pi/(1+\pi)$, the leap-frog scheme has strong damp to high frequency vertical spurious modes. So, in applications, consider to use $|\nu| \leq 3/4$);

Special Cares Needed to Limit Spurious Modes of the Leap-frog Scheme

7. General solution of the leap-frog scheme can be written as

$$U_j^m = \sum_k (\lambda_k^+)^m \left[R_k^+ (\mu_k^{r+})^j + S_k^+ (\mu_k^{s+})^j \right] + (\lambda_k^-)^m \left[R_k^- (\mu_k^{r-})^j + S_k^- (\mu_k^{r-})^j \right].$$

8. Start up scheme at $m = 0$, the data on $m = 1$ and numerical outflow boundary conditions should be properly provided and coupled, so that $S_k^+ = h.o.t$, $S_k^- = h.o.t.$, for all k.
9. The zero order extrapolation numerical boundary condition on the outflow boundary is a non-reflection boundary condition, which means no obvious new grid size spurious Fourier modes will be produced at the boundary.
10. It is important that U^1 contains as less as possible the spurious component, while has a real component with reasonable approximate accuracy.

Conservation Law and Its Cell Average Form

3.3节
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1. Differential form of 1D first order scalar conservation law:
$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(x,t,u(x,t))}{\partial x} = 0, \qquad x \in I \subset \mathbb{R}, \quad t > 0, \tag{3.3.1}$$
2. Integral form of 1D first order scalar conservation law:
$$\int_{x_l}^{x_r} u(x,t_a)\,dx = \int_{x_l}^{x_r} u(x,t_b)\,dx - \left[\int_{t_b}^{t_a} f(x_r,t,u(x_r,t))\,dt \right.$$
$$\left. - \int_{t_b}^{t_a} f(x_l,t,u(x_l,t))\,dt\right], \quad \forall x_l < x_r,\ 0 \le t_b < t_a. \tag{3.3.2}$$
3. $\bar{u}_j^m$: the integral average of u on $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ at t_m;
4. $\bar{f}_{j-\frac{1}{2}}^{m+\frac{1}{2}}$: the integral average of the flux f on (t_m, t_{m+1}) at $x_{j-\frac{1}{2}}$;
5. $\bar{u}_j^{m+1} = \bar{u}_j^m - \frac{\tau_m}{h_j}\left[\bar{f}_{j+\frac{1}{2}}^{m+\frac{1}{2}} - \bar{f}_{j-\frac{1}{2}}^{m+\frac{1}{2}}\right], \forall\, j,\ \forall\, m \ge 0.$ (3.3.3)

Numerical Flux and Discrete Conservation Law

1. $U_j^m \sim \bar{u}_j^m$, numerical flux $F_{j\pm\frac{1}{2}}^{m+\frac{1}{2}} \sim \bar{f}_{j\pm\frac{1}{2}}^{m+\frac{1}{2}}$;
2. Numerical flux is of the following (translation invariant) form:

$$F_{j+\frac{1}{2}}^{m+\frac{1}{2}} = F\left([x_{j-p}, U_{j-p}^m, U_{j-p}^{m+1}], \cdots, [x_{j+q}, U_{j+q}^m, U_{j+q}^{m+1}]; t_m, t_{m+1}\right), \tag{3.3.5}$$

3. Conservative difference scheme for the conservation law:

$$U_j^{m+1} = U_j^m - \frac{\tau_m}{h_j}\left[F_{j+\frac{1}{2}}^{m+\frac{1}{2}} - F_{j-\frac{1}{2}}^{m+\frac{1}{2}}\right];$$

4. Discrete conservation law: for all $j_l < j_r$, $k > l \geq 0$,

$$\sum_{j=j_l}^{j_r} h_j U_j^k = \sum_{j=j_l}^{j_r} h_j U_j^l - \left[\sum_{m=l}^{k-1} \tau_m F_{j_r+\frac{1}{2}}^{m+\frac{1}{2}} - \sum_{m=l}^{k-1} \tau_m F_{j_l-\frac{1}{2}}^{m+\frac{1}{2}}\right]. \tag{3.3.6}$$

Definition 3.2

A difference scheme is said to be a consistent conservative scheme for the conservation law

$$\int_{x_l}^{x_r} u(x,t_a)\,dx = \int_{x_l}^{x_r} u(x,t_b)\,dx - \left[\int_{t_b}^{t_a} f(x_r,t,u(x_r,t))\,dt - \int_{t_b}^{t_a} f(x_l,t,u(x_l,t))\,dt\right], \ \ \forall x_l < x_r,\ 0 \le t_b < t_a.$$

if its numerical flux is of the form

$$F_{j+\frac{1}{2}}^{m+\frac{1}{2}} = F\left([x_{j-p}, U_{j-p}^m, U_{j-p}^{m+1}], \cdots, [x_{j+q}, U_{j+q}^m, U_{j+q}^{m+1}]; t_m, t_{m+1}\right),$$

and

(1) $F([x,u,u],\cdots,[x,u,u];t,t) = f(x,t,u),\ \forall\,(x,t,u)$;

(2) F is continuous, and is Lipschitz continuous with respect to the unknown U.

Conservative is Crucial to a Difference Scheme for a Conservation Law

例 3.4

1. Consider the initial value problem of the Burgers equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0, \quad u(x,0) = \begin{cases} 1, & x < 0; \\ 0, & x \geq 0; \end{cases}$$

2. The upwind scheme based on the equivalent nonconservative equation $u_t + uu_x = 0$:

$$U_j^{m+1} = \begin{cases} U_j^m - \frac{\tau}{h} U_j^m \left(U_j^m - U_{j-1}^m \right), & U_j^m \geq 0, \\ U_j^m - \frac{\tau}{h} U_j^m \left(U_{j+1}^m - U_j^m \right), & U_j^m < 0. \end{cases}$$

3. Truncation error $O(\tau + h)$, and stable, in fact satisfies the maximum principle, if CFL condition holds.

Conservative is Crucial to a Difference Scheme for a Conservation Law

4. Numerical solution:

$$U_j^m = U_j^0 = \begin{cases} 1, & j < 0; \\ 0, & j \geq 0, \end{cases} \quad \forall m \geq 0.$$

5. Numerical solution converges to $\tilde{u}(x,t) = u(x,0)$, $\forall t > 0$, $\forall x$.

6. $0 = s[\tilde{u}] \neq [f] = 1/2$, Rankine-Hugoniot jump condition is not satisfied, therefore, $\tilde{u}$ is not a weak solution.

Conservative is Crucial to a Difference Scheme for a Conservation Law

For a nonlinear conservation law

7. Non-conservative schemes can converge to a function, which is not a weak solution.

8. Consistency and stability of a finite difference scheme do not necessarily lead to the convergence of the scheme.

9. On the other hand, it can be shown, for a consistent conservative scheme to a conservation law, if the numerical solutions converge in certain specified sense, then the limit function must be a weak solution to the conservation law.

Upwind Numerical Flux and Conservative Upwind Scheme

The finite volume method is a natural way to establish conservative difference schemes. For simplicity, we consider $u_t + f(u)_x = 0$ with f smooth, and uniform grids in tensor product form.

1. Rewrite the conservation law in the form $u_t + f'(u)u_x = 0$. (3.3.7)
2. The numerical characteristic on the cell interface $x_{j+\frac{1}{2}}$:

$$a^m_{j+\frac{1}{2}} = \begin{cases} \dfrac{f(U^m_{j+1}) - f(U^m_j)}{U^m_{j+1} - U^m_j}, & \text{if } \ U^m_{j+1} \neq U^m_j; \\ 0, & \text{if } \ U^m_{j+1} = U^m_j, \end{cases} \tag{3.3.8}$$

3. CFL condition: $\max_{u\in\mathcal{U}} |f'(u)|\, \tau \leq h$, ($\mathcal{U}$: the solution set). (3.3.9)

Upwind Numerical Flux and Conservative Upwind Scheme

4. Define the numerical flux by (the method of characteristics)

$$F_{j+\frac{1}{2}}^{m+\frac{1}{2}} = \begin{cases} f(U_j^m), & \text{if } a_{j+\frac{1}{2}}^m \geq 0; \\ f(U_{j+1}^m), & \text{if } a_{j+\frac{1}{2}}^m \leq 0. \end{cases} \tag{3.3.10}$$

This leads to the upwind scheme for the scalar conservation law:

$$U_j^{m+1} = U_j^m - \frac{\tau}{2h}\Big\{\Big[\Big(1+\mathrm{sign}\Big(a_{j+\frac{1}{2}}^m\Big)\Big) f(U_j^m) + \Big(1-\mathrm{sign}\Big(a_{j+\frac{1}{2}}^m\Big)\Big) f(U_{j+1}^m)\Big] - \Big[\Big(1+\mathrm{sign}\Big(a_{j-\frac{1}{2}}^m\Big)\Big) f(U_{j-1}^m) + \Big(1-\mathrm{sign}\Big(a_{j-\frac{1}{2}}^m\Big)\Big) f(U_j^m)\Big]\Big\}. \tag{3.3.11}$$

Upwind Numerical Flux and Conservative Upwind Scheme

5. For $f(u) = au$ ($a =$constant), the scheme is upwind.
6. The scheme is a consistent conservative difference scheme.
7. The integral averages $\bar{u}_j^{m+1}$ and $\bar{u}_j^m$ are calculated by the middle point quadrature rule.
8. The integral averages of the flux $\bar{f}_{j\pm\frac{1}{2}}^{m+\frac{1}{2}}$ are calculated by a one point (generally not the middle point) quadrature rule.
9. Consequently, the truncation error of the upwind scheme is generally only first order.

Middle Point Quadrature and a Conservative Lax-Wendroff Scheme

To achieve higher order accuracy, the middle point quadrature rule can be used to calculate the integral averages of the flux, so that

$$F_{j+\frac{1}{2}}^{m+\frac{1}{2}} = f(U_{j+\frac{1}{2}}^{m+\frac{1}{2}})$$

with a properly defined finite difference operator $R_{j+\frac{1}{2}}$ satisfying

$$U_{j+\frac{1}{2}}^{m+\frac{1}{2}} = R_{j+\frac{1}{2}}(U^m), \quad u_{j+\frac{1}{2}}^{m+\frac{1}{2}} = R_{j+\frac{1}{2}}(u^m) + O(\tau^2 + h^2).$$

1. The two steps Richtmyer scheme:

$$U_{j+\frac{1}{2}}^{m+\frac{1}{2}} = \frac{1}{2}\left(U_j^m + U_{j+1}^m\right) - \frac{\tau}{2h}\left[f(U_{j+1}^m) - f(U_j^m)\right] \triangleq R_{j+\frac{1}{2}}(U^m), \quad (3.3.12)$$

$$U_j^{m+1} = U_j^m - \frac{\tau}{h}\left[f(U_{j+\frac{1}{2}}^{m+\frac{1}{2}}) - f(U_{j-\frac{1}{2}}^{m+\frac{1}{2}})\right]. \quad (3.3.13)$$

Middle Point Quadrature and a Conservative Lax-Wendroff Scheme

2. Let the integral averages $\bar{u}_j^{m+1}$ and $\bar{u}_j^m$ be calculated by the 2nd order accurate middle point quadrature rule.
3. for piecewise constant $\bar{u}_j^m$, $\bar{u}(x_j,t) = \bar{u}_j^m$, $t_m \le t \le t_{m+\frac{1}{2}}$,
4. by Taylor expansion, $u_{j+\frac{1}{2}}^{m+\frac{1}{2}} = u_{j+\frac{1}{2}}^m - \frac{\tau}{2} f(u_{j+\frac{1}{2}}^m)_x + O(\tau^2)$.
5. $f(u_{j+\frac{1}{2}}^m)_x = \frac{f(u_{j+1}^m) - f(u_j^m)}{h} + O(h^2) \Rightarrow u_{j+\frac{1}{2}}^{m+\frac{1}{2}} = R_{j+\frac{1}{2}}(u^m) + O(\tau^2 + h^2)$.
6. Consequently, the truncation error of the scheme is 2nd order.
7. For $f(u) = au$ ($a =$constant), the scheme reduces to the Lax-Wendroff scheme.

Another Consistent Conservative Lax-Wendroff Scheme

There are many ways to extend the Lax-Wendroff scheme to a conservative scheme for the scalar conservation law.

For example, if f is smooth, by $u_t = -f(u)_x$ and $u_{tt} = -(f(u)_t)_x = (f'(u)f(u)_x)_x$, we can establish the Lax-Wendroff scheme for the scalar conservation law

$$U_j^{m+1} = U_j^m - \frac{\tau}{2h}\left[f(U_{j+1}^m) - f(U_{j-1}^m)\right] + \frac{\tau^2}{2h^2}\left[a_{j+\frac{1}{2}}^m\left(f(U_{j+1}^m) - f(U_j^m)\right) - a_{j-\frac{1}{2}}^m\left(f(U_j^m) - f(U_{j-1}^m)\right)\right], \quad (3.3.14)$$

where $a_{j\pm\frac{1}{2}}^m = f'(\frac{1}{2}(U_j^m + U_{j\pm1}^m))$. It can be shown (see Exercise 3.10):

Another Consistent Conservative Lax-Wendroff Scheme

1. For $f(u) = au$ (constant a), the scheme reduces to the Lax-Wendroff scheme.

2. It is a consistent conservative scheme for the scalar conservation law.

3. The truncation error of the scheme is 2nd order.

Numerical Initial-Boundary Conditions To Maintain Global Conservation

3.3.2节
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1. An initial-boundary value problem of a conservation law ($f' > 0$):

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0,\quad x\in(0,1),\ t>0,\ \begin{cases} u(x,0)=u^0(x), & x\in(0,1); \\ u(0,t)=u_0(t), & t\geq 0. \end{cases} \quad (3.3.15\text{-}17)$$

2. Global conservation (or balance) of weak solutions:

$$\int_0^1 u(x,t)\,dx=\int_0^1 u(x,0)\,dx-\left[\int_0^t f(u(1,t))\,dt-\int_0^t f(u(0,t))\,dt\right]. \quad (3.3.18)$$

3. Conservative scheme: $U_j^{m+1}=U_j^m-\tau\,h^{-1}\left[F_{j+\frac{1}{2}}^m-F_{j-\frac{1}{2}}^m\right].$

Numerical Initial-Boundary Conditions To Maintain Global Conservation

4. Numerical initial and boundary conditions ($f' > 0$):

$$U_j^0 = \frac{1}{h}\int_{j-\frac{1}{2}}^{j+\frac{1}{2}} u^0(x)\,dx,\ \forall\, 1 \le j \le N;\ F_{\frac{1}{2}}^m = \frac{1}{\tau}\int_{t_m}^{t_{m+1}} f(u_0(t))\,dt,\ \forall\, m \ge 0. \tag{3.3.19}$$

5. We can also define the value of U_0^m on the ghost node x_0 by making use of the relation

$$F(U_0^m, U_1^m) = F_{\frac{1}{2}}^m = \frac{1}{\tau}\int_{t_m}^{t_{m+1}} f(u_0(t))\,dt. \tag{3.3.21}$$

Numerical Initial-Boundary Conditions To Maintain Global Conservation

6. Global property of the numerical solution:

$$h\sum_{j=1}^{N} U_j^0 = \int_0^1 u^0(x)\,dx, \quad \tau \sum_{l=m}^{m+k} F_{\frac{1}{2}}^l = \int_{t_m}^{t_{m+k}} f(u_0(t))\,dt.$$

7. Numerical boundary condition on outflow boundary — construction of upwind numerical flux:

   (1) 1st-order upwind numerical flux $F_{N+\frac{1}{2}}^m = f(U_N^m)$,

   (may also be viewed as resulted from the zero order extrapolation $U_{N+1}^m = U_N^m$);

   (2) 2nd-order upwind numerical flux $\hat{F}_{N+\frac{1}{2}}^m = \hat{F}(U_{N-1}^m, U_N^m)$.

   (say the Beam-Warming Flux; may also be viewed as resulted from the 1st-order extrapolation.)

Advection-Diffusion Equation — A Model Problem

3.4节
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- An initial value problem of a 1D constant-coefficient advection-diffusion equation ($a > 0$, $c > 0$): $u_t + au_x = cu_{xx}$, (3.4.1)
  $x \in \mathbb{R}$, $t > 0$; $\quad u(x, 0) = u^0(x)$, $x \in \mathbb{R}$.
- By a change of variables $y = x - at$ and $v(y, t) \triangleq u(y + at, t)$,
  $v_t = cv_{yy}$, $\quad y \in \mathbb{R}$, $\ t > 0$; $v(x, 0) = u^0(x)$, $x \in \mathbb{R}$.

Characteristic global properties of the solution u:

1. There is a characteristic speed as in the advection equation, which plays an important role to the solution, especially when $|a| \gg c$ (advection dominant).
2. Along the characteristic, the solution behaves like a parabolic solution (dissipation and smoothing).

Central Explicit Scheme for the Advection-Diffusion Equation

The simplest difference scheme is the central explicit scheme

$$\frac{U_j^{m+1} - U_j^m}{\tau} + a\frac{U_{j+1}^m - U_{j-1}^m}{2h} = c\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{h^2}.$$

or equivalently ($\nu = a\tau/h$, $\mu = c\tau/h^2$)

$$U_j^{m+1} = (\mu - \frac{1}{2}\nu)U_{j+1}^m + (1 - 2\mu)U_j^m + (\mu + \frac{1}{2}\nu)U_{j-1}^m,$$

1. Condition for the maximum principle: $\mu = \frac{c\tau}{h^2} \leq \frac{1}{2}$, $h \leq \frac{2c}{a}$.
2. Amplification factor $\lambda_k = 1 - 2\mu(1 - \cos kh) - \mathrm{i}\nu \sin kh$.

Note: L^2-stable $\Leftrightarrow |\lambda_k| \leq 1 + O(\tau)$; L^2-strongly stable $\Leftrightarrow |\lambda_k| \leq 1$.

$\mathbb{L}^2$ Strong Stability Conditions of the Central Explicit Scheme

3. $|\lambda_k|^2 = 1 - (1 - \cos kh)[4\mu - 4\mu^2(1 - \cos kh) - \nu^2(1 + \cos kh)]$.
4. Strongly stable $\Leftrightarrow 4\mu - 4\mu^2(1 - \cos kh) - \nu^2(1 + \cos kh) \geq 0$
$\Leftrightarrow 4\mu - 2\nu^2 + (\nu^2 - 4\mu^2)(1 - \cos kh) \geq 0$, if $\cos kh \neq 1$.
5. $\mathbb{L}^2$ strongly stable $\Leftrightarrow 4\mu - 2\nu^2 + (\nu^2 - 4\mu^2)(1 - \cos kh) \geq 0$.
6. Take $k = \pi h^{-1}$, we are lead to a necessary condition:
$4\mu(1 - 2\mu) = 4\mu - 2\nu^2 + 2(\nu^2 - 4\mu^2) \geq 0 \Leftrightarrow \mu \leq \frac{1}{2}$.

$\mathbb{L}^2$ Strong Stability Conditions of the Central Explicit Scheme

7. Let $kh \to 0 \Rightarrow \nu^2 \leq 2\mu \Leftrightarrow \tau \leq \frac{2c}{a^2}$, a necessary condition.
8. $g(\xi) = 4\mu - 2\nu^2 + (\nu^2 - 4\mu^2)\xi$ is monotonic with respect to ξ, and $g(0) \geq 0$, $g(2) \geq 0$, if the two necessary conditions hold.
9. $g(\xi) \geq 0$, $\forall \xi \in [0, 2]$, if the two necessary conditions hold.
10. $\mathbb{L}^2$ strongly stable $\Leftrightarrow \mu \leq \frac{1}{2}$ and $\tau \leq \frac{2c}{a^2}$.

Péclet Number and the Stability of the Central Explicit Scheme

Under the condition $\mu \leq 1/2$,

11. the maximum principal holds if and only if the Péclet number $\frac{|a|h}{c}$ satisfies $\frac{|a|h}{c} \leq 2$.
12. $\frac{|a|h}{c} \leq 2 \Rightarrow \tau \leq h^2/(2c) \leq 4c^2/(2ca^2) = 2c/a^2 \Rightarrow$ the second condition for $\mathbb{L}^2$ strong stability holds automatically.
13. When $\frac{|a|h}{c} > 2$, the scheme does not satisfy the maximum principle, and the condition $\tau \leq \frac{2c}{a^2}$ substantially restricts the time step size for the scheme to be $\mathbb{L}^2$ strongly stable (see Example 4.1).

Note: The truncation error of the Central Explicit Scheme is $O(\tau + h^2)$.

Modified Central Explicit Scheme for the Advection-Diffusion Equation

By $u_t = -au_x + cu_{xx}$, $u_{tt} = a^2 u_{xx} - 2ac\partial_x^3 u + c^2 \partial_x^4 u$, for $c \ll |a|$:

$$\begin{aligned} u_j^{m+1} &= \left[u + \tau u_t + \frac{\tau^2}{2} u_{tt} \right]_j^m + O(\tau^3) \\ &= \left[u - a\tau u_x + \left(c + \frac{a^2\tau}{2} \right) \tau u_{xx} \right]_j^m + O(c\tau^2 + \tau^3). \end{aligned}$$

This leads to the modified central explicit scheme

$$\frac{U_j^{m+1} - U_j^m}{\tau} + a\frac{U_{j+1}^m - U_{j-1}^m}{2h} = \left(c + \frac{1}{2} a^2 \tau \right) \frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{h^2}.$$

Modified Central Explicit Scheme for the Advection-Diffusion Equation

1. Local truncation error $O(c\tau + \tau^2 + h^2)$.
2. By the results of the central explicit scheme (with $\tilde{c} = c + \frac{1}{2}a^2\tau$):

$$\begin{cases} \left(c + \dfrac{1}{2}a^2\tau\right)\dfrac{\tau}{h^2} \leq \dfrac{1}{2}, \quad h \leq \dfrac{2c + a^2\tau}{|a|}, & \Leftrightarrow \text{maximum principle;} \\ \left(c + \dfrac{1}{2}a^2\tau\right)\dfrac{\tau}{h^2} \leq \dfrac{1}{2}, \quad \tau \leq \dfrac{2c + a^2\tau}{a^2}, & \Leftrightarrow \mathbb{L}^2 \text{ strong stability.} \end{cases}$$

3. Compared with the central explicit scheme, to satisfy the maximum principle, the condition on the spatial grid size is relaxed, but the condition on the grid ratio is more restricted.

Stability Conditions for the Modified Central Explicit Scheme

4. The second condition for the $\mathbb{L}^2$ strong stability holds automatically.

5. The necessary and sufficient stability conditions:

$$\begin{cases} \left(c + \frac{1}{2}a^2\tau\right)\frac{\tau}{h^2} \le \frac{1}{2}, \quad h \le \frac{2c + a^2\tau}{a}, & \Leftrightarrow \text{maximum principle;} \\ \left(c + \frac{1}{2}a^2\tau\right)\frac{\tau}{h^2} \le \frac{1}{2}, & \Leftrightarrow \mathbb{L}^2 \text{ strong stability.} \end{cases}$$

The Upwind Scheme for the Advection-Diffusion Equation

The upwind scheme ($a > 0$)

$$\frac{U_j^{m+1} - U_j^m}{\tau} + a\frac{U_j^m - U_{j-1}^m}{h} = c\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{h^2},$$

or equivalently (compare the central explicit scheme)

$$\frac{U_j^{m+1} - U_j^m}{\tau} + a\frac{U_{j+1}^m - U_{j-1}^m}{2h} = \left(c + \frac{1}{2}ah\right)\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{h^2}.$$

The Upwind Scheme for the Advection-Diffusion Equation

1. Local truncation error $O(\tau + h)$.
2. By the results of the central explicit scheme (with $\tilde{c} = c + \frac{1}{2}ah$):

$$\begin{cases} (2c + ah)\dfrac{\tau}{h^2} \leq 1, \quad h \leq \dfrac{2c + ah}{a}, & \Leftrightarrow \text{maximum principle;} \\ (2c + ah)\dfrac{\tau}{h^2} \leq 1, \quad \tau \leq \dfrac{2c + ah}{a^2}, & \Leftrightarrow \mathbb{L}^2 \text{ strong stability.} \end{cases}$$

3. The second condition for the maximum principle holds automatically.

The Stability Condition for the Upwind Scheme

4. The second condition for the $\mathbb{L}^2$ strong stability is a consequence of the first, since

$$\frac{a^2\tau}{2c+ah} = \frac{a^2h^2}{2c+ah}\,\frac{\tau}{h^2} \le \frac{(2c+ah)^2}{2c+ah}\,\frac{\tau}{h^2} = (2c+ah)\frac{\tau}{h^2},$$

5. The necessary and sufficient stability condition:

$$\begin{cases} (2c+ah)\dfrac{\tau}{h^2} \le 1, & \Leftrightarrow \text{maximum principle;} \\ (2c+ah)\dfrac{\tau}{h^2} \le 1, & \Leftrightarrow \mathbb{L}^2 \text{ strong stability.} \end{cases}$$

A Crank-Nicolson Type Scheme for the Advection-Diffusion Equation

The Crank-Nicolson scheme for the advection-diffusion equation:

$$\frac{U_j^{m+1}-U_j^m}{\tau}+\frac{a}{2}\left[\frac{U_{j+1}^m-U_{j-1}^m}{2h}+\frac{U_{j+1}^{m+1}-U_{j-1}^{m+1}}{2h}\right]$$
$$=\frac{c}{2}\left[\frac{U_{j+1}^m-2U_j^m+U_{j-1}^m}{h^2}+\frac{U_{j+1}^{m+1}-2U_j^{m+1}+U_{j-1}^{m+1}}{h^2}\right].$$

or

$$\begin{aligned}(1+\mu)U_j^{m+1}=(1-\mu)U_j^{m+1}&+\frac{1}{2}(\mu-\frac{\nu}{2})\left(U_{j+1}^m+U_{j+1}^{m+1}\right)\\&+\frac{1}{2}(\mu+\frac{\nu}{2})\left(U_{j-1}^m+U_{j-1}^{m+1}\right).\end{aligned}$$

A Crank-Nicolson Type Scheme for the Advection-Diffusion Equation

1. Amplification factor: $\lambda_k = \dfrac{1 - \mu(1 - \cos kh) - \mathrm{i}\frac{1}{2}\nu \sin kh}{1 + \mu(1 - \cos kh) + \mathrm{i}\frac{1}{2}\nu \sin kh}$.

2. $|\lambda_k| \leq 1$, for all k, unconditionally $\mathbb{L}^2$ strongly stable.

3. Condition for the maximum principle $\mu \leq 1$, $h \leq \dfrac{2c}{|a|}$, *i.e.* the condition on μ is the same as for the diffusion equation, provided h is less than the Péclet number $\frac{|a|h}{c} \leq 2$.

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Thank You!