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Numerical Solutions of Partial Differential Equations

偏微分方程数值解

——收敛性、相容性、稳定性概念

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2.2 Convergence

Definition 2.2.1 *A difference scheme $L_k^n u_k^n = G_k^n$ approximating the partial differential equation $\mathcal{L}v = F$ is a pointwise convergent scheme if for any x and t, as $(k\Delta x, (n+1)\Delta t)$ converges to (x, t), u_k^n converges to $v(x, t)$ as Δx and Δt converge to 0.*

Definition 2.2.2 *A difference scheme $L_k^n u_k^n = G_k^n$ approximating the partial differential equation $\mathcal{L}v = F$ is a convergent scheme at time t if, as $(n+1)\Delta t \to t$,*

$$\| \mathbf{u}^{n+1} - \mathbf{v}^{n+1} \| \to 0 \tag{2.2.11}$$

as $\Delta x \to 0$ and $\Delta t \to 0$.

Definition 2.2.3 *A difference scheme $L_k^n u_k^n = G_k^n$ approximating the partial differential equation $\mathcal{L}v = F$ is a convergent scheme of order (p, q) if for any t, as $(n+1)\Delta t$ converges to t,*

$$\| \mathbf{u}^{n+1} - \mathbf{v}^{n+1} \| = \mathcal{O}(\Delta x^p) + \mathcal{O}(\Delta t^q) \tag{2.2.12}$$

as Δx and Δt converge to 0.

2.3 Consistency

Definition 2.3.1 *The finite difference scheme $L_k^n u_k^n = G_k^n$ is pointwise consistent with the partial differential equation $\mathcal{L}v = F$ at point (x, t) if for any smooth function $\phi = \phi(x, t)$,*

$$(\mathcal{L}\phi - F)|_k^n - \left[L_k^n \phi(k\Delta x, n\Delta t) - G_k^n\right] \to 0 \tag{2.3.1}$$

as $\Delta x,\ \Delta t \to 0$ and $(k\Delta x, (n+1)\Delta t) \to (x, t)$.

Definition 2.3.2 *The difference scheme (2.3.4) is consistent with the partial differential equation in a norm $\|\cdot\|$ if the solution of the partial differential equation, v, satisfies*

$$\mathbf{v}^{n+1} = Q\mathbf{v}^n + \Delta t \mathbf{G}^n + \Delta t \boldsymbol{\tau}^n, \tag{2.3.5}$$

and

$$\| \boldsymbol{\tau}^n \| \to 0$$

as $\Delta x,\ \Delta t \to 0$, where $\mathbf{v}^n$ denotes the vector whose kth component is $v(k\Delta x, n\Delta t)$.

2.3 Consistency

Definition 2.3.3 *The difference scheme (2.3.4) is said to be accurate of order (p, q) to the given partial differential equation if*

$$\| \tau^n \| = \mathcal{O}(\Delta x^p) + \mathcal{O}(\Delta t^q). \tag{2.3.6}$$

We refer to τ^n or $\| \tau^n \|$ as the **truncation error**.

Remark 1: Of course, it is easy to see that if a scheme is accurate of order (p, q), $p,\ q \geq 1$, then it is a consistent scheme. Also, it is easy to see that if a scheme is either consistent or accurate of order (p, q), the scheme is pointwise consistent.

2.4 Stability

We define stability for a two level difference scheme of the form

$$\mathbf{u}^{n+1} = Q\mathbf{u}^n, \; n \geq 0, \tag{2.4.1}$$

which will generally be a difference scheme for solving a given initial–value problem on $\mathbb{R}$ which includes a homogeneous linear partial differential equation.

Definition 2.4.1 *The difference scheme (2.4.1) is said to be stable with respect to the norm* $\|\cdot\|$ *if there exist positive constants* Δx_0 *and* Δt_0*, and non-negative constants* K *and* β *so that*

$$\| \mathbf{u}^{n+1} \| \leq K e^{\beta t} \| \mathbf{u}^0 \|, \tag{2.4.2}$$

for $0 \leq t = (n+1)\Delta t, \; 0 < \Delta x \leq \Delta x_0$ *and* $0 < \Delta t \leq \Delta t_0$.

2.4 Stability

Remark 1: Notice that as with the definitions of convergence and consistency, the definition of stability is given in terms of a norm. As was also the case with convergence and consistency, this norm may differ depending on the situation. Also notice that the definition of stability does indeed allow the solution to grow. We should notice that *the solution can grow with time, not with the number of time steps.*

Remark 2: We also notice that stability is defined for a homogeneous difference scheme. As we shall see in Section 2.5, stability of the homogeneous equation, along with the correct consistency, is enough to prove convergence of the nonhomogeneous difference scheme. All of the contributions of the nonhomogeneous term will be contained in the truncation term τ^n. In fact, *when we discuss stability of a nonhomogeneous difference scheme, such as difference scheme (2.3.14), we consider the stability of the associated homogeneous scheme.*

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Remark 3: And finally, we warn the reader that there are a variety of definitions of stability in the literature. Definition 2.4.1 happens to be one of the stronger definitions. One common definition is to require that condition (2.4.2) hold only for $(n+1)\Delta t \leq T$ for any T (where K and β depend on T). Another, more common, definition that is used is one that does not allow for exponential growth. Inequality (2.4.2) is replaced by

$$\| \mathbf{u}^{n+1} \| \leq K \| \mathbf{u}^0 \|, \tag{2.4.3}$$

(with or without the restriction $(n+1)\Delta t \leq T$.) Clearly, inequality (2.4.3) implies inequality (2.4.2). Also, inequality (2.4.2) along with the restriction $(n+1)\Delta t \leq T$ implies inequality (2.4.3). This latter definition of stability

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implies that the solutions to the difference equation must be bounded. Using the fact that the iterations must be bounded is a much nicer concept with which to work. However, very soon after a definition using inequality (2.4.3) is given, it must be expanded to include more general situations. We have merely included these more general situations in our first definition. But, when it is convenient (as it often is) to prove inequality (2.4.3) instead of (2.4.2), we will do so realizing that it is sufficient. *When we want to use stability based on inequality (2.4.3), we will refer to it as "Definition 2.4.1-(2.4.3)".*

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Numerical Partial Differential Equations: Finite Difference Methods