7.1 Diagonalization of Symmetric Matrices

Symmetric matrix : defined as matrix A such that $A^T = A$

- matrix is = to its transpose

Remember that similar $\neq$ symmetric

If a symmetric matrix has distinct eigenvalues , it is guaranteed it will <u>also</u> have orthogonal eigenvectors

- when these vectors are normalized , they form orthonormal basis

Symmetric matrices are always diagonalizable

$A = QDQ^T$ where D is the eigenvalue matrix and Q is the eigenvector (orthonormal matrix)

Theorem 1: If A is symmetric , then any two eigenvectors from different eigenspaces are orthogonal

Theorem 2: An $n \times n$ matrix is orthogonally diagonalizable iff A is symmetric

If we observe an λ_1 with multiplicity ≥ 2, we must make all the respective eigenvectors orthonormal, so we use gram - schmidt

Spectral Theorem for symmetric matrices: an $n \times n$ symmetric matrix A has the following properties:

- A has n real eigenvalues
- Dimension of the eigenspace for each λ equals the multiplicity of λ as a root of the characteristic equation
- Eigenspaces are mutually orthogonal, in the case that eigenvectors for different λ are orthogonal
- A is orthogonally diagonalizable

The set of eigenvalues of a matrix A is often referred to as the spectrum of A

Spectral Decomposition:

$A = PDP^T$, P must be orthonormal

$$A = PDP^T = [u_1 \; \dots \; u_n] \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix}$$

$$= [\lambda_1 u_1 \; \dots \; \lambda_n u_n] \begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix}$$

$$= \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \dots \lambda_n u_n u_n^T$$