7.4 Singular Value Decomposition

- Let A be an $n \times n$ matrix. Then, A^TA is symmetric and can be orthogonally diagonalized

- Let $\{v_1, \dots v_n\}$ be an orthonormal basis for $\mathbb{R}^n$, consisting of eigenvectors of A^TA and $\lambda_1, \dots \lambda_n$ be its eigenvalues

- Then, $\|Av_i\|^2 = (Av_i)^T(Av_i) = v_i^T A^T A v_i = v_i^T \lambda_i v_i = \lambda_i$ v_i is a unit vector

- So, all of these eigenvalues are non-negative

- The singular values of A are the square roots of the eigenvalues of A^TA denoted $\sigma_1, \dots \sigma_n$ arranged in decreasing order

- All the lengths of $Av_1, \dots Av_n$

Theorem 9: Suppose $\{v_1, \dots v_n\}$ is an orthonormal basis of $\mathbb{R}^n$ consisting of the eigenvectors of A^TA, arranged so that the corresponding eigenvalues of A^TA satisfy $\lambda_1 \geq \dots \geq \lambda_n$ and suppose A has r nonzero singular values. Then, $\{Av_1, \dots Av_r\}$ is an orthogonal basis for Col A and rank $A = r$

Theorem 10: The Singular Value Decomposition

- Let A be an $n \times n$ matrix with rank r, then $\exists$ an $m \times n$ matrix Σ as below, for which the diagonal entries in D are the first r singular values of A, $\sigma_1 \geq \sigma_2 \geq \sigma_r > 0$, and $\exists$ an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that:

$$A = U \Sigma V^T$$

$m \times n$ diagonal matrix

D is $r \times r$ and diagonal

$$\Sigma : \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \leftarrow m-r \text{ rows}$$

$\uparrow$ $n-r$ columns

Same dimension as A

The steps of SVD for a matrix A:

1) Orthogonally diagonalize A^TA: Find all the eigenvalues and associated eigenvectors. D is the singular value matrix

2) Set up V and Σ

a) V is the set of the NORMALIZED eigenvectors

3) Construct U

a) The first R columns of U are normalized vectors of $Av_1, \dots Av_r$ where v's are the eigenvectors

If A is an nxn matrix with n nonsingular values, then A has n nonzero singular values