1.9 Matrix of a Linear Transformation

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $\exists$ a unique matrix A such that $T(x) = Ax$ for all x in $\mathbb{R}^n$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(e_j)$, where e_j is the jth column of the identity matrix in $\mathbb{R}^n$:

$$A = [T(e_1) \ \dots \ T(e_n)]$$

We call A the standard matrix of the transformation

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $b \in \mathbb{R}^m$ is the image of at least one x in $\mathbb{R}^n$

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each b in $\mathbb{R}^m$ is the image of at most one x in $\mathbb{R}^n$

Example: Let T be a linear transformation whose standard matrix is:

$$\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \leftarrow$$

Pivot in every row, so $Ax=b$ is consistent. So T maps $\mathbb{R}^4$ to $\mathbb{R}^3$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T one-to-one?

Since there is a free variable, each b is the image of > 1 x. So no, not one-to-one

Thm 11: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, T is one-to-one iff the equation $T(x) = 0$ has only the trivial solution (linear independence)

Theorem 12: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then,

a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$

b) T is one-to-one iff the columns of A are linear independent