6.5 Least Squares Problem

- If $Ax=b$ is inconsistent, we can't find a solution since b is not in the column space of A

- Instead, we can find $\hat{b}$, which is in the column space of A, which is closest to b

- $\hat{b}$ is an orthogonal projection of b onto column space of A

- The orthogonal projection of b onto column space of A, is most often called $\hat{b}$, and is called the least squares solution

If $\exists$ $\hat{x}$ such that $A\hat{x} = \hat{b}$, we can find it through the following formula: $A^TAx = A^Tb$

- The matrix equation represents a system of equations called the normal equations for $Ax=b$

The solution to this equation is denoted $\hat{x}$, and represent the certain vector, which multiplied by A, gets us equal to $\hat{b}$, the $\text{proj}_{\text{col} A}\, b$

14) Let A be an $m \times n$ matrix. The following statements are logically equivalent:

a) The equation $Ax=b$ has a unique least squares solution for each $b \in \mathbb{R}^m$

b) The columns of A are linearly independent

c) A^TA is invertible

$\hat{x}$ is given by $(A^TA)^{-1}A^Tb$

In general, the process of finding $\hat{x}$ is to find two quantity :

- $A^T A$ and $A^T b$ and then use these to solve for $\hat{x}$
- Remember, to get b, we must multiply by the matrix A, this is our close approximation to b
- Least squares Error : $||b - \hat{b}|| = ||b - A\hat{x}||$

Theorem 15: Given an $m \times n$ matrix A with linearly independent columns, let $A = QR$ be a QR factorization of A. Then, for each b in $\mathbb{R}^m$, $Ax = b$ has a unique least squares solution, given by $\hat{x} = R^{-1} Q^T b$

If columns of A are orthogonal, this reduces to finding projection of b onto each one of the column vectors in A (this is our $\hat{b}$), and then we can plug and chug from there.