B 9.1 (Matrix Methods for Linear Systems)

Say we have an equation of the form:

$$x_1' = -4x_1 + 2x_2$$
$$x_2' = 4x_1 - 4x_2$$

We can express the system as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -4 & 2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We call this a linear homogeneous system in normal form, where we can express this system as $x' = Ax$, where A is the coefficient matrix and x is the solution vector

Oftentimes if we have higher order differential functions of this form,

$$2x'' + 6x - 2y = 0$$
$$y'' + 2y - 2x = 0,$$

We introduce the following notations for the lower order derivatives

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= x' \end{aligned}\right\rangle$$ Second derivative of x, or x'', is just x_2'

$$\left.\begin{aligned} x_3 &= y \\ x_4 &= y' \end{aligned}\right\rangle$$ Second derivative of y, or y'', is just x_4'

Refer to the example below:

① $2x'' + 6x - 2y = 0$

② $y'' + 2y - 2x = 0$

Introduce the notation :

$$\begin{cases} x_1 = x \\ x_2 = x' \\ x_3 = y \\ x_4 = y' \end{cases}$$

So, we can rewrite equations as :

$$\begin{aligned} 2x_2' + 6x_1 - 2x_3 &= 0 \\ x_4' + 2x_3 - 2x_1 &= 0 \end{aligned} \xrightarrow[\text{normal form}]{} \begin{cases} x_1' = x_2 \\ x_2' = -3x_1 + x_3 \\ x_3' = x_4 \\ x_4' = 2x_1 - 2x_3 \end{cases}$$

In matrix notation, this forms

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Whenever we do a swap such as $x_1 = y'$ or something similar, recall that our main goal is to always have our differentials on the left hand side, variables on the right

B9.5 (Homogenous Linear Systems with Linear Coefficients)

- So, how do we actually go about finding a general solution to

  $x'(t) = Ax(t)$

- Let $A = [a_{ij}]$ be an $n \times n$ constant matrix. The eigenvalues of A are those numbers (real or complex) numbers r for which $(A - rI)u = 0$ has at least one nontrivial (real or complex) solution to u

- The corresponding nontrivial solutions u are called the eigenvectors of A associated with r

- **Theorem 5:** Suppose the $n \times n$ constant matrix A has n linearly independent eigenvectors $u_1, u_2, u_3, \dots u_n$, and let r_i be the eigenvalue corresponding to u_i. Then, $\{e^{r_1 t}u_1, e^{r_2 t}u_2, e^{r_3 t}u_3, \dots e^{r_n t}u_n\}$ is a fundamental solution set and $X(t) = [e^{r_1 t}u_1 \;\; e^{r_2 t}u_2 \; \dots \; e^{r_n t}u_n]$ is a fundamental matrix on $(-\infty, \infty)$ for homogenous system $x' = Ax$.

Thus, our general solution of $x' = Ax$ is: $x(t) = c_1 e^{r_1 t}u_1 + c_2 e^{r_2 t}u_2 + \dots c_n e^{r_n t}u_n$

Theorem 6: If $r_1, \dots r_m$ are distinct eigenvalues of the matrix A and u_i is an eigenvector associated with r_i, then $u_1, \dots u_m$ are linearly independent

- When solving for eigenvalues, always have left side "alone"

B 9.6 (Complex Eigenvalues)

- Now suppose that when we find the eigenvalues for $x'(t) = A\,x(t)$ we end up with $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ with associated eigenvectors $z_1 = a + ib$, $z_2 = a - ib$, then two linearly independent real vector solutions to $x'(t) = A\,x(t)$ are :

$$e^{\alpha t}\cos(\beta t)a - e^{\alpha t}\sin(\beta t)\,b$$
$$e^{\alpha t}\sin(\beta t)a + e^{\alpha t}\cos(\beta t)\,b$$

Recall that a is the <u>real</u> portion of the eigenvector, b is the <u>imaginary</u> portion

Example : Find the eigenvectors given that $\lambda_1 = -1 + 2i$, $\lambda_2 = -1 - 2i$.

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 0 \end{bmatrix}$$

$\rightarrow$ $A - \lambda_1 I = \begin{bmatrix} -2-(-1+2i) & -1 \\ 5 & -(-1+2i) \end{bmatrix} = \begin{bmatrix} -1+2i & -1 \\ 5 & 1-2i \end{bmatrix}$, we can choose that $V_1 = \begin{bmatrix} 1 \\ (-1+2i) \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \end{bmatrix}$