6.7 Inner Product Spaces

Main Goal: Generalize the dot product in $\mathbb{R}^n$ so we can define length, orthogonality, projection in abstract space

The dot product this far $(v_1 \cdot v_2)$ is referred to as the standard inner dot product

Formal definition: An inner product on vector space V is a function that satisfies the following properties for all v, u, and w in V with constant c

1) Symmetry: $\langle u, v \rangle = \langle v, u \rangle$

2) and 3): Linearity $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

$\langle cu, v \rangle = c \langle u, v \rangle$

4) positivity: $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ iff $u = 0$

The notation $\langle \ \rangle$ is the notation for <u>any</u> inner product

- takes in two inputs: arbitrary u and v vectors, and
- returns a real number (scalar)

Definition: An inner product space is a vector space with an inner product

- So take $V = \mathbb{R}^n$ with $\langle u, v \rangle = 4u_1v_1 + 5u_2v_2$
- when we are asked to show if the equation above defines an inner product space, we're essentially saying we must verify that the equation satisfies the axioms above for ANY of the vectors within the space

Standard Inner Product: $\langle u, v \rangle = u_1v_1 + u_2v_2$ (if we assume $\mathbb{R}^2$)

Cauchy - Schwarz Inequality: $|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$

Triangle Inequality: $\|u + v\| \leq \|u\| + \|v\|$

For any General Inner Product: we can say that

Length of u: $\|u\| = \sqrt{\langle u, u \rangle}$

Distance from two vectors u, v as: $\|u - v\| = \sqrt{\langle (u-v), (u-v) \rangle} = 0$

Orthogonal vectors if $\langle u, v \rangle = 0$

Projections : $\text{proj}_{\text{span of } u} v = \frac{\langle v, u \rangle}{\langle u, u \rangle} u$

Unit Vector : $\|u\| = 1$

We can still use Gram - Schmidt , but instead using some general inner products in place of the standard

So if we're given a subspace of $\mathbb{R}^n$ where $\{x_1, \dots x_p\}$ and take $v_1 = x_1$, we can now define v_2 with a specific inner product as : $v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$