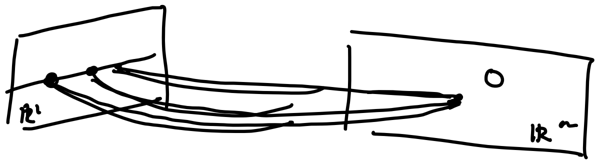
4.2 Null Spaces

The null space of A is the set of all solutions to $Ax = 0$.

Homogeneous

$$\text{Nul } A = \{ x \mid x \in \mathbb{R}^n \text{ and } Ax = 0 \}$$



Example :

Is $u = \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}$ in Nul A where $A = \begin{bmatrix} 3 & 5 & 0 \\ -2 & 2 & -8 \\ 0 & 4 & -6 \end{bmatrix}$?

$$\begin{bmatrix} 3 & 5 & 0 \\ -2 & 2 & -8 \\ 0 & 4 & -6 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$ So $A\vec{u} = \vec{0}$, yes

Theorem 2 :

The Null space of an $m \times n$ matrix A, Nul A, is a subspace of $\mathbb{R}^n$

1) Is $\vec{0} \in$ Nul A ?

Yes, $A\vec{0} = \vec{0}$ so $\vec{0} \in$ Nul $\vec{A}$

2) When $A\vec{u}$, $A\vec{v}$ $\in$ Nul A, is $A(\vec{u}+\vec{v}) \in$ Nul A

$A\vec{u} = 0$
$A\vec{v} = 0$

$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$

So, $A(\vec{u}+\vec{v}) \in$ Nul A

3) When $A\vec{u} \in$ Nul A, is $A(c\vec{u}) \in$ Nul A ?

$A\vec{u} = 0$, $A(c\vec{u}) \to c(A\vec{u}) = c(\vec{0}) = \vec{0}$

So $A(c\vec{u}) \in 0$

So, the Nul A is a subspace of A

Example : Find the spanning set for the null space of A

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$\hookrightarrow$ $A = \left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\hookrightarrow$ $x_1 = 2x_2 + x_4 - 3x_5$

$x_3 = -2x_4 + 2x_5$

$\hookrightarrow$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

$\hookrightarrow$ $x = x_2 \vec{u} + x_4 \vec{v} + x_5 \vec{w}$

- The spanning set produced with the method above is always be linearly independent

- When Nul A contains non zero vectors, the # of vectors in the spanning set for Nul A is equal to the number of free variables in the solution set of $Ax = 0$

Column Spaces

Column space of an $m \times n$ matrix is the set of all linear combinations of columns of A.

If $A = [a_1 \ a_2 \ .. \ a_n]$ then col A $=$ Span $[a_1 \ a_2 \ ... \ a_n]$

$$\text{Col } A = \{ b \mid b = Ax \text{ for some } x \in \mathbb{R}^n \}$$

If A is an $m \times n$ matrix, col A is A subspace of $\mathbb{R}^m$

You can check if a vector is in the column space by making an augmented matrix with A on the left, and the vector of A.

- If the system, after row reducing, is inconsistent, then $\vec{u}$ not $\in A$

Linear Transformations:

A linear transformation T from a vector space V to vector space W is a rule that assigns a unique vector $T(x)$ to each vector such that:

1) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \forall \ \vec{u}, \vec{v} \in V$

ii) $T(c\vec{u}) = cT\vec{u} \quad \forall \ \vec{u} \in V$ and $c \in \mathbb{R}$

Nul T is called the kernal $\{ u \in V \mid T(\vec{u}) = 0 \}$

Range of T is the set of all vectors $\in W$ of the form $T(\vec{x})$ for some $\vec{x} \in V$.
$\{ T(\vec{x}) \in W \mid \vec{x} \in V \}$