3.1 Introduction to Determinants

If $\det A \neq 0$, then A is an invertible matrix

Very Simply:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det A = ad - bc$$

What about 2x2, or 3x3, or nxn? How do we compute the determinante of a matrix like THAT?

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with +/- alternating where $a_{11}, a_{12}, \ldots, a_{1n}$ are the first row entries of A

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

A_{ij} is the submatrix of A formed by deleting the i^{th} row and j^{th} column

Find $\det A$ for $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

$$= 1 \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0$$

$$= 1(4 \cdot 0 - (-1 \cdot -2)) - 5(0)$$

$$= 1(-2) = \boxed{-2}$$

Cofactor Expansion

Cofactor : C_{ij}, is the number given by $(-1)^{i+j} \det A_{ij}$

$\rightarrow$ $\det A = a_{11} C_{11} + a_{12} C_{12} + \dots a_{1n} C_{1n}$

Definition : The determinant of any $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column

Given a specific matrix with a good amount of zeroes you can strategically find the determinant by selecting the row / column which has the most zeroes

ONE MORE COOL TRICK:

- If A is triangular, then det A is a product of the entries along the main diagonal

$$A = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{vmatrix} \rightarrow \det(A) = 1 \cdot 4 \cdot (-2) = -8$$