

## B9.1 (Matrix methods for Linear Systems)

Say we have an equation of the form:

$$x_1' = -4x_1 + 2x_2$$

$$x_2' = 4x_1 - 4x_2$$

We can express the system as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -4 & 2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We call this a linear homogeneous system in normal form,

where we can express this system as  $x' = Ax$ , where  $A$  is

the coefficient matrix and  $x$  is the solution vector

Oftentimes if we have higher order differential functions of this form,

$$2x'' + 6x - 2y = 0$$

$$y'' + 2y - 2x = 0,$$

We introduce the following notation for the lower order derivatives

$$\begin{array}{l} x_1 = x \\ x_2 = x' \end{array} \quad \begin{array}{l} > \\ > \end{array} \quad \begin{array}{l} \text{Second derivative of } x, \text{ or } x'', \text{ is just } x_2' \end{array}$$

$$\begin{array}{l} x_3 = y \\ x_4 = y' \end{array} \quad \begin{array}{l} > \\ > \end{array} \quad \begin{array}{l} \text{Second derivative of } y, \text{ or } y'', \text{ is just } x_4' \end{array}$$

Refer to the example below:

$$\textcircled{1} \quad 2x'' + 6x - 2y = 0$$

$$\textcircled{2} \quad y'' + 2y - 2x = 0$$

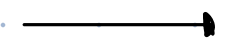
Introduce the  
notation :

$$\begin{cases} x_1 = x \\ x_2 = x' \\ x_3 = y \\ x_4 = y' \end{cases}$$

So, we can rewrite equations as :

$$2x_2' + 6x_1 - 2x_3 = 0$$

$$x_4' + 2x_3 - 2x_1 = 0$$



normal form

$$\begin{cases} x_1' = x_2 \\ x_2' = -3x_1 + x_3 \\ x_3' = x_4 \\ x_4' = 2x_1 - 2x_3 \end{cases}$$

In matrix notation, this forms

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Whenever we do a swap such as  $x_1 = y'$  or something similar,

recall that our main goal is to always have our differentials on the

left hand side, variables on the right

## B9.5 (Homogenous Linear Systems with Linear Coefficients)

- So, now do we actually go about finding a general solution to

$$x'(t) = Ax(t)$$

- Let  $A = [a_{ij}]$  be an  $n \times n$  constant matrix. The eigenvalues of  $A$  are those numbers (real or complex) numbers  $r$  for which  $(A - rI)u = 0$  has at least one nontrivial (real or complex) solution to  $u$ .

- The corresponding nontrivial solutions  $u$  are called the eigenvectors of  $A$  associated with  $r$ .

- Theorem 5: Suppose the  $n \times n$  constant matrix  $A$  has  $n$  linearly independent eigenvectors  $u_1, u_2, u_3, \dots, u_n$ , and let  $r_i$  be the eigenvalue corresponding to  $u_i$ . Then,  $\{e^{r_1 t} u_1, e^{r_2 t} u_2, e^{r_3 t} u_3, \dots, e^{r_n t} u_n\}$  is a fundamental solution set and  $X(t) = [e^{r_1 t} u_1 \ e^{r_2 t} u_2 \ \dots \ e^{r_n t} u_n]$  is a fundamental matrix on  $(-\infty, \infty)$  for homogenous system  $x' = Ax$ .

Thus, our general solution of  $x' = Ax$  is:  $x(t) = c_1 e^{r_1 t} u_1 + c_2 e^{r_2 t} u_2 + \dots + c_n e^{r_n t} u_n$

- Theorem 6: If  $r_1, \dots, r_m$  are distinct eigenvalues of the matrix  $A$  and  $u_i$  is an eigenvector associated with  $r_i$ , then  $u_1, \dots, u_m$  are linearly independent.

- When solving for eigenvalues, always have left side "alone"

## B 9.6 (Complex Eigenvalues)

- Now suppose that when we find the eigenvalues for  $x'(t) = Ax(t)$  we end up with  $r_1 = \alpha + \beta i$  and  $r_2 = \alpha - \beta i$  with associated eigenvectors  $z_1 = a + ib$ ,  $z_2 = a - ib$ , then two linearly

independent real vector solutions to  $x'(t) = Ax(t)$  are:

$$e^{\alpha t} \cos(\beta t) a - e^{\alpha t} \sin(\beta t) b$$

$$e^{\alpha t} \sin(\beta t) a + e^{\alpha t} \cos(\beta t) b$$

Recall that  $a$  is the real portion of the eigenvector,  $b$  is the imaginary portion

Example: Find the eigenvectors given that  $\lambda_1 = -1 + 2i$ ,  $\lambda_2 = -1 - 2i$ .

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 0 \end{bmatrix}$$

$$\rightarrow A - \lambda_1 I = \begin{bmatrix} -2 - (-1 + 2i) & -1 \\ 5 & -(-1 + 2i) \end{bmatrix} = \begin{bmatrix} -1 + 2i & -1 \\ 5 & 1 - 2i \end{bmatrix}, \text{ we can}$$

$$\text{choose that } v_1 = \begin{bmatrix} 1 \\ (-1 + 2i) \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$