

7.4 Singular Value Decomposition

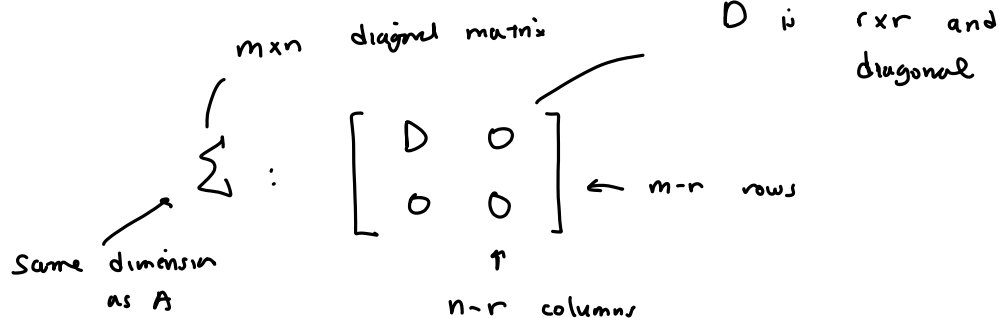
- Let A be an $n \times n$ matrix. Then, $A^T A$ is symmetric and can be orthogonally diagonalized
- Let $\{v_1, \dots, v_n\}$ be an orthonormal basis for \mathbb{R}^n , consisting of eigenvectors of $A^T A$ and $\lambda_1, \dots, \lambda_n$ be its eigenvalues
- Then, $\|Av_i\|^2 = (Av_i)^T(Av_i) = v_i^T A^T A v_i = v_i^T \lambda_i v_i = \lambda_i v_i^T v_i = \lambda_i$ is a unit vector
- So, all of these eigenvalues are non-negative
- The singular values of A are the square roots of the eigenvalues of $A^T A$ denoted $\sigma_1, \dots, \sigma_n$ arranged in decreasing order
- All the length of Av_1, \dots, Av_n

Theorem 9: Suppose $\{v_1, \dots, v_n\}$ is an orthonormal basis of \mathbb{R}^n consisting of the eigenvectors of $A^T A$, arranged so that the corresponding eigenvalues of $A^T A$ satisfy $\lambda_1 \geq \dots \geq \lambda_n$ and suppose A has r non zero singular values. Then, $\{Av_1, \dots, Av_r\}$ is an orthogonal basis for $\text{Col } A$ and $\text{rank } A = r$

Theorem 10: The Singular Value Decomposition

- Let A be an $n \times n$ matrix with rank r , then \exists an $m \times n$ matrix Σ as below, for which the diagonal entries in D are the first r singular values of A , $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$, and \exists an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that:

$$A = U \Sigma V^T$$



The steps of SVD for a matrix A :

1) Orthogonally diagonalize $A^T A$: Find all the eigenvalues and associated eigenvectors. D is the singular value matrix

2) Set up V and Σ

a) V is the set of the NORMALIZED eigenvectors

3) Construct U

a) The first R columns of U are normalized vectors of

Av_1, \dots, Av_r where v 's are the eigenvectors

If A is an $n \times n$ matrix with n nonsingular values, then A has n nonzero singular values