1.4 The Matrix Equation $Ax = b$

Linear Combinations as the product of a matrix + vector

If A is an $m \times n$ matrix with columns $a_1, a_2, \dots, a_n$ and if $x \in \mathbb{R}^n$, then Ax is the linear combination of the columns of A using the corresponding entries in x as weights

$$Ax = \overbrace{[\, a_1 \quad a_2 \quad a_3 \dots a_n \,]}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = a_1 x_1 + a_2 x_2 + \dots a_n x_n$$

Ex:

1) Find Ax if $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

$$Ax = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}$$

Equivalencies

Augmented matrix: $\left[\begin{array}{ccc|c} 2 & 3 & -1 & 3 \\ 0 & 2 & 3 & 4 \end{array}\right]$

Vector Equation: $x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Matrix Equation: $Ax = b$ $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Existence of Solutions

Let A be an $m \times n$ matrix. Then these statements are Logically Equivalent:

1) For each $b \in \mathbb{R}^m$, $Ax = b$ has a solution

2) Each $b \in \mathbb{R}^m$ is a linear combination of the column of A

3) The columns of A span $\mathbb{R}^m$

4) A has pivot position in each row (A must be a coefficient matrix)