1.1 Systems of Linear Equations

Writing Systems

$$x_1 + x_2 = 5$$
$$3x_1 - 2x_2 = 10$$

$$\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$$

Coefficient matrix

$$\left[\begin{array}{cc:c} 1 & 1 & 5 \\ 3 & -2 & 10 \end{array}\right]$$

Augmented matrix

Solving via elimination:

$$\begin{array}{r} -3(x_1 + x_2 = 5) \\ + \; 3x_1 - 2x_2 = 10 \\ \hline 0x_1 - 5x_2 = -5 \end{array} \longrightarrow x_2 = 1$$

$$x_1 + x_2 = 5$$
$$x_1 = 5 - 1 = 4$$

So the solution to this set of linear equations is:

$$(x_1, x_2) = (4, 1)$$

Solving with row operations

$$\left[\begin{array}{cc:c} 1 & 1 & 5 \\ 3 & -2 & 10 \end{array}\right] \xrightarrow{R_2 = 3R_1 - R_2} \left[\begin{array}{cc:c} 1 & 1 & 5 \\ 0 & 5 & 5 \end{array}\right].$$

$$\xrightarrow{R_2 = \frac{R_2}{5}} \left[\begin{array}{cc:c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 = R_1 - R_2} \left[\begin{array}{cc:c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

We can definitely read off that $x_1 = 4$ and $x_2 = 1$.

This is the same exact solution we got from solving this system using elimination!

Terminology

Linear Equation: An equation that can be written as
$a_1x_1 + a_2x_2 + \ldots\ldots a_nx_n = b$ where $a_1, a_2, \ldots, a_n$ are real or complex numbers

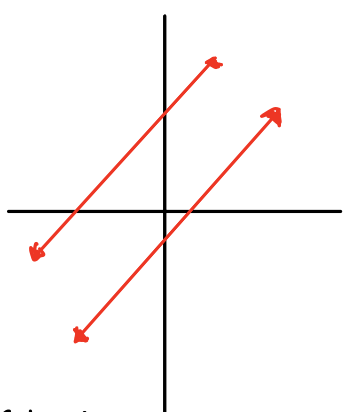
Systems of Linear Equations - a collection of 2 + linear equations using the same variables

Solution: A list of numbers (s_1, s_2, s_3) that makes each equation in the system true when substituted for $x_1, x_2, x_3 \ldots$ respectively

Solution Set: The set of all possible solutions to a system

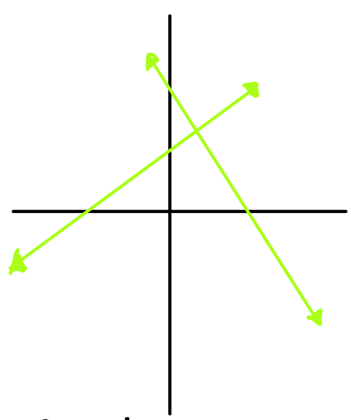
Types of Systems

Inconsistent



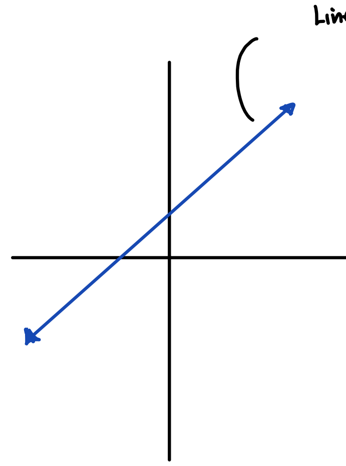
NO Solution

Consistent



ONE SOLUTION

Two identical Lines



∞ly many solutions

Solving Systems Using Augmented Matrices and Row Operations

Row Operations

Replacement : Replace 1 row by multiplying by the sum of itself and a multiple of another row

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 1 & 0 & 9 \end{array}\right] \rightarrow R_2 = -2R_2 + R_1 \left[\begin{array}{cc|c} 2 & 4 & 8 \\ 0 & 4 & -10 \end{array}\right]$$

Interchange : Interchange (swap) two rows

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 1 & 0 & 9 \end{array}\right] \rightarrow \begin{array}{l} R_1 = R_2 \\ R_2 = R_1 \end{array} \left[\begin{array}{cc|c} 1 & 0 & 9 \\ 2 & 4 & 8 \end{array}\right]$$

Scaling : Multiply by a row by a non-zero constant

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 1 & 0 & 9 \end{array}\right] \rightarrow R_1 = \tfrac{1}{2} R_1 \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 0 & 9 \end{array}\right]$$

When do we use these row operations?

- When we want to see if a system is consistent or inconsistent
- When we want to reduce a matrix to Row-Echelon Form or Reduced Row Echelon Form

Example: Show if the following system is consistent. If so, is the solution unique?

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3/2 & 1 & 1/12 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array}\right]$$

The bottom most row has: $0x_1 + 0x_2 + 0x_3 = 15$, which is NOT Possible. As a result, we can conclude this system is inconsistent