6.3 Orthogonal Decomposition and Best Approximation

Theorem 8: Orthogonal Decomposition: Let W be a subspace of $\mathbb{R}^n$.

Then each y in $\mathbb{R}^n$ can be written uniquely in the form

$$y = \hat{y} + z$$

where $\hat{y}$ is in W and z is in $W^{\perp}$. In fact, if $\{u_1, \dots u_p\}$ is any orthogonal basis of W, then

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \dots \frac{y \cdot u_p}{u_p \cdot u_p} u_p$$

and $z = y - \hat{y}$

Theorem 9: Best Approximation Theorem

Let W be a subspace of $\mathbb{R}^n$, let y be any vector in $\mathbb{R}^n$, and $\hat{y}$ be the orthogonal projectis y onto W. Then, $\hat{y}$ is the closest point in W to y, in the sense that

$\|y - \hat{y}\| < \|y - v\|$ for all v in W distinct from $\hat{y}$

- We call $\hat{y}$ the best approximation of y based on elements in W
- The dimension of a subspace and its orthogonal complement equals the dimension of the entire space, so $W + W^{\perp} = \mathbb{R}^n$, where W is a subspace of $\mathbb{R}^n$