3.3 Cramer's Rule, Volume, and Linear Transformation

Notation: For any $n \times n$ matrix A and any b in $\mathbb{R}^n$, let $A_i(b)$ be the matrix obtained from A by replacing column i by the vector b.

$$A_i(b) = \begin{bmatrix} a_1 & a_2 & \cdots & \underset{\substack{\uparrow \\ \text{column } i}}{b} & \cdots & a_n \end{bmatrix}$$

Theorem 7: Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any b in $\mathbb{R}^n$, the unique solution x of $Ax = b$ has entries given by:

$$x_i = \frac{\det A_i(b)}{\det A}, \quad i = 1, 2, \ldots n$$

so we can use determinants to solve linear systems

Example 1: Use cramer's rule to solve the system

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned} \longrightarrow A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix} \quad \det A = 2, \quad b = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$A_1(b) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix} \quad \det A_1(b) = 40 \rightarrow x_1 = \frac{40}{2} = 20$$

$$A_2(b) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix} \quad \det A_2(b) = 54 \rightarrow x_2 = \frac{54}{2} = 27$$

$$x_1 = 20$$
$$x_2 = 27$$

Another Formula for A^{-1}

The jth column of A^{-1} is a vector that satisfies $Ax = e_j$ where j is the jth column of I and the ith entry of x is the (i, j) entry of A^{-1}

Then by Cramer's Rule $\{(i,j) \text{ entry of } A\} = x_i = \dfrac{\det A_i(e_j)}{\det A}$

A cofactor expression down column i $A_i(e_j)$ shows that

$\det A_i(e_j) = (-1)^{i+j} \det A_{ji} = C_{ji}$ ← cofactor of A

$$A^{-1} = \frac{1}{\det A} \underbrace{\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \\ C_{1n} & C_{2n} & & C_{nn} \end{bmatrix}}_{\text{Adjugate of } A}$$

So, In general : Theorem 8 : If A is an invertible $n \times n$ matrix, then $A^{-1} = \dfrac{1}{\det A} \operatorname{adj} A$

Example 3 :

Find Inverse of $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$

$C_{11} = + \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2$, $C_{12} = - \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3$ $C_{13} = + \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$

$C_{21} = - \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 6$, $C_{22} = + \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7$ + ...

If we go on, we should get $\operatorname{adj} A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$

To get det A, multiply adj A by A: $(\det A\, I = \operatorname{adj} A \cdot A)$

$$\operatorname{adj} A \quad A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = 14I$$

so $\det A = 14$, and

$$A^{-1} = \frac{1}{14} \operatorname{adj} A = \begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix}$$

Theorem 9: If A is a 2×2 matrix, the area of the parallelogram determind by the columns of A is $|\det A|$.

If A is a 3×3 matrix, the volume of the parallelpiped determined by the columns of A is $|\det A|$

Theorem 10: Area of Transformations

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determind by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then $\{\text{area of } T(S)\} = |\det A| \{\text{Area of } S\}$

If T determind by a 3×3 matrix A, and if S is a parallelpiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \{\text{volume of } S\}$$