

B4.6 Variation of Parameters

Let's consider the case of the non-homogeneous differential equation of the form:

$$y'' + P(x)y' + Q(x)y = g(x)$$

The homogeneous Equation $y'' + P(x)y' + Q(x)y = 0$ has 2 linearly independent solutions y_1 and y_2

We guess $y = u_1 y_1 + u_2 y_2$

$$\hookrightarrow u_1 = - \int \frac{y_2 g}{y_1 y_2' - y_2 y_1'} dx$$

$$u_2 = \int \frac{y_1 g}{y_1 y_2' - y_2 y_1'} dx$$

Example: $y'' + y = \sec(x) \rightarrow r^2 + 1 = 0 \rightarrow r = \pm i$. So, our homogeneous solution is: $C_1 \cos(x) + C_2 \sin(x)$. Here, $y_1 = \cos(x)$, $y_2 = \sin(x)$. So:

$$y_1' = -\sin(x), \quad y_2' = \cos(x)$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$\hookrightarrow u_1 = - \int \frac{\sin(x) \sec(x)}{\cos^2(x) + \sin^2(x)} dx \rightarrow - \int \tan(x) \rightarrow \ln |\cos(x)|$$

$$u_2 = \int \frac{\cos(x) \sec(x)}{\cos^2(x) + \sin^2(x)} dx \rightarrow x$$

So,

$$y = C_1 \cos(x) + C_2 \sin(x) + \cos(x) \ln |\cos(x)| + x \sin(x)$$