B4.5 Non homogenous Equations : The Method of Undetermined Coefficients

- Suppose we have $y'' - 2y' - 3y = 3e^{2t}$

- Up to this point, we're used to having constants as our coefficients, and this kind of equation set equal to 0.

If we call y_h the homogeneous solution to $y'' - 2y' - 3y = 0$ and y_p the particular solution to $y'' - 2y' - 3y = 3e^{2t}$, then we refer to y_g as the general solution for the equation

$y_g = y_h + y_p$

1) Solve the homogenous Equation
   - Use methods from B4.2 to B4.4

2) Look at the RHS and find a solution by using an undetermined coefficient
   - we go about guessing what the undetermined coefficient is

3) Solve for the general solution
   - Substitute back into part 2) and use that to solve for y_p

What kind of guess should we use for part 2?

RHS:	e^{rt}	Ae^{rt}
	$\sin(rt)$ or $\cos(rt)$	$A\sin(rt) + B\cos(rt)$
	Degree n polynomial	$A_0 + A_1 t + \dots A_n t^n$

Note that when we have the case when we observe the multiplication between 2 non-homogenous terms, we only need to reference 1 part of the coefficient

If we observe the non homogenous contains part of the homogenous, we multiply by a factor of t.

Example: $y'' - 9y = te^t + \sin(2t)$

$\rightarrow \quad y_g = y_n + y_{p1} + y_{p1}$

$r^2 - 9 = 0 \rightarrow (r+3)(r-3) \rightarrow y_n = C_1 e^{-3t} + C_2 e^{3t}$

Let $y_{p_1} = (At + B) \cdot e^t$, $y'_{p_1} = Ae^t + (At+B)e^t$

$y''_{p_1} = Ae^t + Ae^t + (At+B)e^t = 2Ae^t + (At+B)e^t$

So, $y''_{p_1} - 9y_{p_1}'' = 2Ae^t + (At+B)e^t - 9((At+B)e^t)$

$= 2Ae^t - 8(At+B)e^t = te^t$

$= 2Ae^t - 8Ate^t - 8Be^t = te^t$

$-8Ate^t + 2Ae^t - 8Be^t = te^t$

$\hookrightarrow$

$-8A = 1 \leftarrow A = \frac{-1}{8}$ $\quad 2(-\frac{1}{8}) - 8(B) = 0$

$8B = \frac{-2}{8} \rightarrow B = -\frac{1}{32}$

So $y_{p_1} = \left(-\frac{1}{8}t - \frac{1}{32}\right)e^{t}$

Let $y_{p_2}(t) = E\sin(2t) + D\cos(2t)$

So, $y'_{p_2}(t) = 2E\cos(2t) - 2D\sin(2t)$

$y''_{p_2}(t) = -4E\sin(2t) - 4D\cos(2t)$

$y''_{p_2} - 9y_{p_2} = -4E\sin(2t) - 4D\cos(2t) - 9(E\sin(2t) + D\cos(2t))$

$= -13E\sin(2t) - 13D\cos(2t) \qquad = \sin(2t)$

$\rightarrow \quad E = -\frac{1}{13}, \quad D = 0$

So $y_{p_2} = -\frac{1}{13}\sin(2t)$

Thus, the general solution is: $\boxed{C_1e^{3t} + C_2e^{-3t} - \left(\frac{1}{8}t + \frac{1}{32}\right)e^{t} - \frac{1}{13}\sin(2t)}$

A simpler Example:

$y'' + 5y' + 6y = t^2$

$\hookrightarrow$ $r^2 + 5r + 6 = 0$

$\hookrightarrow (r+2)(r+3) = 0 \rightarrow r = -2, -3 \quad$ so $\quad y_h = C_1e^{-3t} + C_2e^{-2t}$

$y_p = At^2 + Bt + C$

$\hookrightarrow y'_p = 2At + B$

$y''_p = 2A$

$y'' + 5y' + 6y = 2A + 5(2At + B) + 6(At^2 + Bt + C) = t^2$

$\hookrightarrow 2A + 10At + 5B + 6At^2 + 6Bt + 6C = t^2$

$6At^2 + (10A + 6B)t + (2A + 5B + 6C) = t^2$

$\hookrightarrow A = \frac{1}{6}, \quad B = -\frac{10}{36}, \quad C = -\frac{38}{216}$

so

$\boxed{y_g = C_1e^{-2t} + C_2e^{-3t} + \frac{1}{6}t^2 - \frac{10}{36}t - \frac{38}{216}}$