7.2 Quadratic Forms

- A Quadratic form on $\mathbb{R}^n$ is a function Q defined on $\mathbb{R}^n$ whose values at vector x in $\mathbb{R}^n$ can be computed by an expression of the form: $Q(x) = x^T A x$ where A is an $n \times n$ symmetric matrix

Ex: $Q(x) = x^T I x = \|x\|^2$

From a $Q(x)$ equation, we can derive the A matrix (A is symmetric

- First note our variables, and create x^T and x
- For each one of the non cross product terms, their coefficients will go along our diagonal of A
- Then, in our A, we look at each cross product term, and split the value in half
- Each one of these split values will go to a corresponding place in the matrix

→ Let $Q(x) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$

→ $Q(x) = x^T A x$ → $x^T = [x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} 5 & -0.5 & 0 \\ -0.5 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$

Change of Variables:

Given an x variable vector in $\mathbb{R}^n$, a change of variable equation of the form $x = Py$ or $y = P^{-1}x$

$\rightarrow$ $x^TAx = (Py)^T A (Py)$

$= y^T P^T A P y$

A is symmetric, $P^TAP = D$, where D is a diagonal matrix with entries that are eigenvalues

$\rightarrow$ Quadratic form is $x^TAx = y^TDy$

- P matrix created from A's eigenvectors, which are orthonormal

Once you have solved for the general solution for $x^TAx = y^TDy$ where you have an expression only in terms of y variables and no cross product, then we use $x = Py$ to find the specific y that makes this hold, namely $y = P^{-1}x$ or P^Tx, and this happens when we have a specific x we want to express in terms of y.

If $Q > 0$ for all x except when $x = 0$ $\rightarrow$ positive definite   semi-def: $Q \geq 0$ always

$Q < 0$ for all x except $x = 0$, negative definite

Indefinite: both >0 and <0 values $\rightarrow$ Determine with eigenvalues $\rightarrow$ all >0 PD
all <0 ND
some both Indef.