2.2) The Inverse of a Matrix

I_n refers to the identity matrix

$A I_n = A \qquad AA^{-1} = I_n$

$I_n A = A \qquad A^{-1} A = I_n$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, If $\underbrace{ad - bc}_{\det A} \neq 0$, A is invertible

Theorem 4 in words: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad-bc \neq 0$, then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. If $ad-bc = 0$, A is NOT invertible

Practice: Find A^{-1} if $A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$ and verify it's the Inverse

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}, \quad A^{-1} = \frac{1}{8}\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/8 \\ 0 & 1/4 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1/2 & -1/8 \\ 0 & 1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Theorem 5 in Text:

If A is an invertible $n \times m$ matrix, then for each b in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$

Example: Solve the system using the inverse matrix:

$$\begin{aligned} 3x_1 + 4x_2 &= 3 \\ 5x_1 + 6x_2 &= 7 \end{aligned} \rightarrow \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & +2 \\ 5/2 & 3/2 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & +2 \\ 5/2 & 3/2 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Provable Facts About Invertible Matrices

- If A is invertible then A^{-1} is invertible and $(A^{-1})^{-1} = A$
- If A and B are $n \times n$ matrices, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$
- If A is an Invertible matrix, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$

Elementary Matrices

An elementary row operation is a matrix obtained by performing
one row operation on an identity matrix

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Row operations: Swapping rows, Multiplying rows, adding rows together

Invertibility

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n and any sequence of elementary row operations that reduces A to I_n also transforms I_n to A^{-1}

A) assume A is invertible. Since $Ax = b$ has unique sol for every b (Thm 5), then A has a pivot in every row. Since A is square, the pivots are in the diagonal, which implies RREF of A is I_n $A \sim I_n$

B) Now assume $A \sim I_n$. Then each step to row reduce A corresponds to left multiplication by an elem. matrix $E_1, E_2, \dots E_p$ such that $A \sim E_1 A \sim E_2(E_1 A) \sim E_3(E_2 E_1 A)] \sim \dots E_p(E_{p-1} \dots E_1 A) = I_n$. Since product of Invertible matrices is invertible, then $(E_p \dots E_1)^{-1} (E_p \dots E_1) A = (E_p \dots E_1)^{-1} I_n$, so $A = (E_p \dots E_1)^{-1}$. Therefore A is invertible since it's the inverse of an invertible matrix

Example: Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = 3R_1 + R_2$

$R_3 = 2R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 3 & -8 & 2 & 0 & -1 \end{array}\right] \xrightarrow{R_3 = 3R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$