4.1 Vector Spaces and Subspaces

A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers) subject to ten axioms below. These must hold for all vectors u, v, and w in V for all scalars c and d.

1) Sum of u and v denoted as $u+v$
2) $u+v = v+u$
3) $(u+v)+w = u+(v+w)$
4) $\exists\ \vec{0}$ in V such that $0+\vec{u} = \vec{u}$
5) For each $u \in V$, $\exists\ -u$ in V such that $u+(-u) = \vec{0}$
6) Scalar multiples of u, denoted by cu, is $\in V$
7) $c(u+v) = cu+cv$
8) $(c+d)u = cu+du$
9) $c(du) = (cd)u$
10) $1u = u$

Subspace: A subspace of vector space V is a subset H of V that has three properties:

a) The zero vector $\in H$
b) H closed under addition, so $u \in H$, $s \in H \rightarrow u+s \in H$
c) H closed under multiplication for scalars, so for scalar c and vector $\vec{u}$, $c\vec{u} \in H$

How do we see if a certain H is a subspace? Well you must ensure that each of the three requirements / properties of subspaces are met:

Example: Given $v_1, v_2 \in V$, let $H = \text{span}\{v_1, v_2\}$. Show that H is a subspace of V

1) Zero vector: $0 = 0v_1 + 0v_2$

2) closed under addition: let $u = s_1 v_1 + s_2 v_2$, $w = t_1 v_1 + t_2 v_2$

$u + w = (s_1 v_1 + s_2 v_2) + (t_1 v_1 + t_2 v_2)$
$= (s_1 + t_1) v_1 + (s_2 + t_2) v_2$

3) closed under scale mult:

given scalar c,

$cu = c(s_1 v_1 + s_2 v_2)$
$= (cs_1) v_1 + (cs_2) v_2$

$\rightarrow$ H is a subspace of V

Theorem 1: If $v_1, \dots v_p$ are in vector space V, then $\text{span}\{v_1, \dots v_p\}$ is a subspace of V