B9.1 (Matrix Methods for Linear Systems)

Say we have an equation of the form:

$$x_1' = -4x_1 + 2x_2$$
$$x_2' = 4x_1 - 4x_2$$

We can express the system as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -4 & 2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We call this a linear homogeneous system in normal form,

where we can express this system as $x' = Ax$, where A is

the coefficient matrix and x is the solution vector

Oftentimes if we have higher order differential functions of this form,

$$2x'' + 6x - 2y = 0$$
$$y'' + 2y - 2x = 0,$$

We introduce the following notation for the lower order derivatives

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= x' \end{aligned}\right\rangle$$ Second derivative of x, or x'', is just x_2'

$$\left.\begin{aligned} x_3 &= y \\ x_4 &= y' \end{aligned}\right\rangle$$ Second derivative of y, or y'', is just x_4'

Refer to the example below:

① $2x'' + 6x - 2y = 0$

② $y'' + 2y - 2x = 0$

Introduce the notation : $\begin{cases} x_1 = x \\ x_2 = x' \\ x_3 = y \\ x_4 = y' \end{cases}$

So, we can rewrite equations as:

$$2x_2' + 6x_1 - 2x_3 = 0$$
$$x_4' + 2x_2 - 2x_1 = 0$$

$\xrightarrow{\hspace{3em}}$ normal form

$$\begin{cases} x_1' = x_2 \\ x_2' = -3x_1 + x_3 \\ x_3' = x_4 \\ x_4' = 2x_1 - 2x_3 \end{cases}$$

In matrix notation, this forms

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Whenever we do a swap such as $x_1 = y'$ or something similar, recall that our main goal is to always have our differentials on the left hand side, variables on the right